Editorial of the Golden Fall 2024 issue 54, Vol 16 No 4

Laela Sagita, Issue 54 Editor, Vol 16 (4) MTRJ

Universitas PGRI Yogyakarta, Indonesia

Editorial by Laela Sagita

Mathematics Teaching-Research Journal's Golden Fall issue is here. With the return of The Problem Corner and 11 articles across various countries on mathematics education reveals diverse innovative approaches to developing both teacher and student competencies. These diverse approaches demonstrate that mathematics education is continuously evolving with innovative interventions and strategies aimed at enhancing students' understanding, motivation, and skills in various global contexts.

Taganap from the Philippines highlights an error analysis of pre-service mathematics teachers in solving verbal problems, emphasizing the importance of a deeper understanding of such problems. In Spain, **Muñoz-Escolano** explores the problem posing approach to train prospective teachers in developing proportional reasoning, which is highly relevant for improving mathematical teaching competencies. Meanwhile, **Sie** in Ghana examines the harmony between pre-service teachers' knowledge of fractions and their classroom practices, stressing the need for alignment between theory and practice.

In Indonesia, two notable studies emerge. **Miftah** develops student worksheets through an interactive case-based learning model assisted by the Cublend app, aiming to enhance students' mathematical literacy skills, while **Nugroho** investigates how augmented reality can motivate students in the mathematics classroom. Additionally, **Syawahid**, also from Indonesia, highlights the functional thinking skills of mathematically gifted students in figural and non-figural linear patterns.

Research in other countries also offers significant insights. **Yazgan-Sağ** from Turkey discusses prospective teachers' thoughts on educating mathematically gifted students, which is increasingly relevant in the context of inclusive education. In Bhutan, **Gembo Tshering** conducted action research intervention in teaching algebra to Grade 7 students, emphasizing the importance of action-based approaches in the classroom. In the USA, **Yu** encourages undergraduate students to explore multiple proofs of the multinomial theorem, promoting critical thinking in advanced mathematics. **López** from Mexico explores how high school students understand and apply trigonometric ratios in motion vectors, aiming to shift from instrumental to relational understanding. Finally, in Greece, **Polydoros** examines math anxiety in the virtual classroom during the COVID-19 pandemic and its relationship to academic achievement, providing critical insights into the impact of online learning on students.

We extend our gratitude to the authors who entrusted us with their manuscripts, thereby establishing the Mathematics Teaching-Research Journal as an international authority in mathematics education research. **Enjoy!**

Error Analysis of Pre-Service Mathematics Teachers in Solving Verbal Problems

Author: *Frinz Adrian O. Valdez, Eduard C. Taganap* (p.6)

The paper focuses on analyzing the errors made by pre-service mathematics teachers when solving verbal math problems, using Newman's Error Analysis framework. It highlights the persistent issue of low mathematics proficiency among these teachers in the Philippines, partly due to challenges in understanding English, the medium of instruction. The study identifies various types of errors across different stages of problem-solving, including reading, comprehension, transformation, process skills, and encoding errors. Key findings suggest that while the teachers perform relatively well in reading the problems, they struggle more with comprehension and transformation, which are crucial for effective problem-solving. The research emphasizes the need for improved teaching strategies, critical thinking, and practical applications of mathematical concepts to enhance the mathematical abilities of future educators. Additionally, it suggests a refresher program for graduating pre-service teachers to address potential knowledge gaps, especially those exacerbated by online learning during the pandemic.

Problem Posing in Mathematics Teacher Training: Developing Proportional Reasoning

Author: *María Burgos, Jorhan Chaverri, José M. Muñoz-Escolano* (p.35)

The paper focuses on a study examining how prospective mathematics teachers create problems to develop proportional reasoning. It explores their beliefs about what constitutes a good problem and the challenges they face in problem posing. The study highlights the importance of problem posing in teacher training and identifies gaps in teachers' understanding of proportionality across various mathematical contexts, such as arithmetic, geometric, and probabilistic. Using the Onto-semiotic Approach, the research analyzes participants' responses to uncover discrepancies between their beliefs and practices in problem posing. The findings emphasize the need for improved training in proportional reasoning, particularly in geometric and probabilistic contexts, and suggest that a good problem should have a clear statement, motivating context, and promote reasoning. The study calls for further research and interventions to enhance the competencies of prospective teachers in creating effective mathematical problems.

Harmony in Teaching: Unraveling the Interplay between Pre-Service Teachers' Mathematical Knowledge Fractions and Classroom Practices

Author: *Charles Kwabena Sie, Douglas Darko Agyei* (p.59)

The paper explores the relationship between pre-service teachers' Mathematical Knowledge for Teaching Fractions (MKTF) and their teaching practices. It examines how different domains of

MKTF influence various components of teaching effectiveness, particularly in the context of teaching fractions. The study involves 171 pre-service teachers from Ghana and uses regression analyses to assess the impact of MKTF on teaching practices. Key findings highlight the importance of specific MKTF domains, such as Knowledge of Content and Students (KCFS) and Content Knowledge for Teaching Fractions (CCKF), in improving teaching quality. The research emphasizes the need for enhanced teacher preparation and professional development to strengthen mathematical knowledge and teaching practices, ultimately aiming to improve instructional quality and student performance in mathematics. The study also notes the interdependence of MKTF domains and their collective influence on teaching practices.

Development of Prospective Teacher Student Worksheets Through Interactive Case-Based Learning Model Assisted by Cublend App to Improve Mathematical Literacy Skills

Author: *Ramdani Miftah, Lia Kurniawati, Kamal Fikri Musa* (p.76)

The paper presents details a study on enhancing mathematical literacy skills in prospective teacher students through the development of teaching materials based on the Inquiry-Based Collaborative Learning (ICBL) model, supported by the Cublend application. The study involves creating Learner Worksheets (LKPD) focused on arithmetic and geometric sequences, validated by experts and evaluated through student feedback. The findings indicate that these materials are valid, practical, and effective, significantly improving students' mathematical literacy skills, as evidenced by increased test scores. The research suggests that the ICBL-Cublend App model is a promising alternative for teacher professional development programs.

Figural and Non-Figural Linear Pattern: Case of Primary Mathematical Gifted Students' Functional Thinking

Author: *M. Syawahid, Nasrun, Rully Charitas Indra Prahmana* (p.94)

The authors explore the functional thinking abilities of mathematically gifted elementary students, focusing on their strategies for generalizing relationships in linear pattern tasks. The research highlights the students' use of various strategies, such as counting, multiplicative approaches, and contextual reasoning, to derive general forms and express relationships symbolically. The studies emphasize the advanced problem-solving skills of these students and suggest curriculum adjustments to better support their capabilities in functional thinking.

Thoughts of Prospective Mathematics Teachers on Educating Mathematically Gifted Students

Author: *Gönül Yazgan-Sağ* (p.116)

The article examines the perspectives of 40 prospective mathematics teachers on educating mathematically gifted students, highlighting their views on specialized education, differentiated curricula, and classroom practices. While many believe in the benefits of specialized schools for gifted students, concerns about socialization and isolation are noted. The teachers suggest strategies such as differentiated tasks and advanced materials to engage gifted students, but express challenges in implementing these in regular classrooms. The study underscores the need for enhanced teacher education programs to better equip teachers with the knowledge and skills necessary for effectively supporting gifted learners.

Teaching Algebra to a Grade 7 Student: Action Research Intervention

Author: *Gembo Tshering* (p.132)

The paper highlights the importance of responsive pedagogy, advocating for teacher professional development, metacognitive strategies, and culturally relevant teaching methods. It emphasizes creating supportive learning environments through personalized and differentiated instruction, ultimately aiming to foster a deeper conceptual understanding of algebra and encourage students to engage with mathematics as critical thinkers and lifelong learners.

Encouraging Undergraduate Students to Explore Multiple Proofs of the Multinomial Theorem

Author: *Christian Farkash, Michael Storm, Thomas Palmeri, Chunhui Yu* (p.154)

The paper explores various methods for proving the multinomial theorem, including combinatorial, induction, probability, and differential calculus approaches, and examines undergraduate students' attitudes toward these methods. A study conducted with math majors at Farmingdale State College revealed that while students appreciated multiple proof methods, they were most comfortable with combinatorial and induction proofs. The document emphasizes the educational benefits of exposing students to diverse proof techniques to enhance their understanding and problem-solving skills, and it also encourages further exploration of the binomial and multinomial theorems.

Trigonometric Ratios in High School Students: From Instrumental Understanding to Relational Understanding through their Application in Motion Vectors

Author: *Ivonne Alejandra Toledo-Nieto, José Antonio Juárez-López* (p.164)

The paper discusses various educational interventions and teaching strategies aimed at improving students' understanding of trigonometric concepts and vector addition. It includes a didactic sequence that integrates technology and collaborative learning, using tools like Google Maps and GeoGebra. The interventions focus on transitioning students from a superficial, procedural understanding to a deeper, relational comprehension of trigonometric ratios and their applications in motion vectors. The effectiveness of these strategies is evaluated through pretest-posttest assessments, with statistical analysis indicating significant improvements in students' understanding post-intervention.

The Problem Corner (p.183)

Ivan Retamoso

How to Motivate Students Using Augmented Reality in The Mathematics Classroom? An Experimental Study

Author: *Wanda Nugroho Yanuarto1, Elfis Suanto2, Ira Hapsari1, Aulia Nisa Khusnia* (p.191)

The paper is a compilation of research articles and studies focused on the use of augmented reality (AR) in mathematics education. It explores the development, implementation, and effectiveness of AR-based learning tools aimed at enhancing students' critical thinking, motivation, and understanding of mathematical concepts. The studies cover various topics, including the challenges educators face, the impact of AR on students' learning experiences, and its practical applications in teaching geometry and other mathematical areas. The research indicates that AR can significantly improve student engagement, problem-solving skills, and overall learning outcomes compared to traditional teaching methods. However, it also highlights challenges such as high costs, technical issues, and the need for improvements in user interface and graphics. Overall, the findings suggest that while AR has the potential to enhance learning, effective teaching practices are crucial for maximizing its benefits.

Math Anxiety in the Virtual Classroom during COVID-19 Pandemic and its Relationship to Academic Achievement

Author: *Georgios Polydoros* (p.213)

The paper explores the relationship between mathematics anxiety (MA) and academic performance, particularly focusing on gender differences among sixth-grade students during the COVID-19 pandemic. It highlights that girls tend to experience higher levels of MA, which negatively impacts their math test scores compared to boys. The research underscores the importance of addressing math anxiety in educational settings, especially for female students, and suggests targeted interventions to mitigate its effects. The study also discusses the challenges posed by emergency online education during the pandemic and its psychological effects on children.

Error Analysis of Pre-Service Mathematics Teachers in Solving Verbal Problems

Frinz Adrian O. Valdez[1], Eduard C. Taganap[2]

[1]Department of Science Education, Central Luzon State University, Philippines

[2] Department of Mathematics and Physics, Central Luzon State University, Philippines

valdez.frinz@clsu2.edu.ph, eduardtaganap@clsu.edu.ph

Abstract: Mathematics proficiency in the Philippines is a persistent concern, seen by many as a sign of an educational crisis. Teachers are responsible for improving learning outcomes in any discipline, including math. Thus, the study intended to conduct an error analysis of verbal problems among pre-service mathematics teachers. The researcher employed descriptive research and Newman's Error Analysis to suffice the research objectives. The findings revealed low levels of error in reading stage, but moderate levels of error in comprehension, transformation, process skills, and encoding stages. In addition, underlying factors contributing to these errors were incomplete solutions and answers, incorrect or incomplete processes, grammatical errors, conversion errors, and failure to indicate an answer. The researchers concluded that pre-service teachers are proficient in the early stage of problem solving, but challenged in properly constructing equations, the utility of operations, and interpreting their results.

Keywords: newman's error analysis, verbal problems, educators, mathematical operations

INTRODUCTION

Mathematics has been recognized as a discipline paramount to the acquisition of various abilities, such as logical reasoning, critical thinking, and problem-solving. Educators play a pivotal role in shaping students' understanding and appreciation of mathematics. Hence, it is vital to assess and correct the mathematical errors of pre-service mathematics teachers to improve their mathematical ability and to eventually become better educators.

In the Philippines, Filipino students were revealed to have low mathematics proficiency, scoring only 355 points while the average score was 472 points, as per the Program for International Student Assessment (PISA) 2022 International Report (OECD, 2023). Mumu et al. (2021) also revealed that mathematics achievement is adversely affected by growing instructional issues while Panthi et al. (2019) mentioned that theoretical problems in instruction are rampant among developing countries. Similarly, it was reported that the proficiency issue in mathematics was anchored to the lack of understanding the English language – the language used in teaching mathematics in the Philippines (Vera, 2021); thereby resulting to lack of understanding verbal concepts, ideas and principles in mathematics. Therefore, to attain proficiency in mathematics, it is necessary for students to comprehend mathematical language such as symbols (inequalities symbols, summation, integration, and approximation), operators (exponentiation, square root, and factorial), geometric shapes or figures (circle, sphere, and cube), and equations. In line with this, solving verbal problems can be a suitable assessment for understanding mathematical language.

In pursuit of alleviating mathematical proficiency among students, it is crucial that their educators also exhibit mathematics proficiency by mastering the content of the mathematics subjects. Further, University at Buffalo (2023) revealed desirable learning outcomes can be achieved through efficient teaching methods, such as mastery of the content of the course. Also, Gholami (2021) emphasized in their study the importance of mathematics teachers' proficiency to enhance learning outcomes. Meanwhile, pre-service teachers undeniably have skills and knowledge that are still not at the same level as experienced teachers. Also, it is important that these individuals are determined to be ready in practice considering the dynamic educational approach brought by the pandemic (Nasir et al., 2022). Nevertheless, Ocampo (2021) mentioned that the pedagogical competence of pre-service teachers in the Philippines has increased in the 21st century and the new normal modalities. However, there is still a lack of studies focusing on error analysis among pre-service mathematics teachers regarding their verbal problem-solving skills. Existing studies focus on error analysis of pre-service teachers in their English written texts (Cocjin, 2020), in their mathematical literacy (Khalo et al., 2015), and in estimation problems known as Fermi problem that aim at encouraging students to make educated guesses (Segura & Ferrando, 2021). Thus, there was still a lack of knowledge regarding error analysis among pre-service mathematics teachers regarding solving verbal problems.

This work assesses the content knowledge on verbal problem solving of the graduating pre-service mathematics teachers who experienced asynchronous online learning from 2020 to 2023 due to the COVID-19 pandemic. Asynchronous learning, is a student-centered learning approach where the students are expected to self-study the given modules and reading assignments. This study then sheds light to the errors of pre-service mathematics teachers in solving verbal problems. It utilized the Newman's Error Analysis to identify underlying factors and the level of reading, comprehension, transformation, process skills, and encoding errors of the respondents in solving verbal problems. In line with this, the identification of committed errors among the respondents may give direction to the academe to further enhance tertiary pedagogy and possible academic interventions.

Theoretical Framework

This study employed Newman's Error Analysis by Australian educator Newman (1977) to uncover respondents' mathematical errors and evaluate their attitudes toward mathematics as a course and discipline (Seng, 2020). Additionally, it helps discover students' mathematical knowledge, language gaps, and challenge them when they do not understand the subject. This strategy also helped teachers discover areas of misunderstanding and provided a foundation for thinking about students' arithmetic verbal problem issues (White, 2010). Newman's Error Analysis also suggested ways teachers might use efficient teaching methods to avoid them.

This analysis has five stages: reading, comprehension, transformation, process skill, and encoding (Triliana & Asih, 2019). In the reading stage, students should understand the question and recognize what is being asked. In the comprehension stage, students should be able to understand what is needed to solve the verbal problem. Students should choose the right mathematical operations to solve the problem in the transformation stage. While in the process skill stage, the students should use the mathematical operations with the correct procedure. Finally, the encoding stage involves students writing their final answers based on what is asked in the problem and accurate format, such as proper indication of units.

In pursuit of identifying the best instructional tactics, models, and material to use with students, it is necessary to recognize and analyze the mistakes they make (Ling, 2020). The process of finding and analyzing can be made more focused and organized by employing Newman's error analysis. Thus, educators and academic institutions throughout the globe have been utilizing this method in identifying the type of mistakes of the students.

Mathematics is challenging; thus, students are encouraged to approach problems systematically. Students must be able to recognize and understand problems that are similar to those that have been addressed and adapt problem-solving theory or method to the issue. Mathematics also helps students learn abstract concepts that improve problem-solving skills. Mathematically proficient students have fewer arithmetic problems. Knowledge of the subject helps students avoid many errors. Hence, Saleh et al. (2022) suggested that students must learn ideas and theorems before solving the problems. Meanwhile, Boonen et al. (2016) revealed that researchers and educational professionals have paid close attention to mathematical word problems, which are mathematical exercises that offer pertinent information on a subject as text rather than in the manner of mathematical language. Development of realistic mathematics problems is also being encouraged towards the objective (Agustina et al., 2021). Therefore, it is assumed that student's ability to solve arithmetic word problems successfully depends on both their ability to comprehend the word problem's text accurately and their ability to execute the necessary mathematical operations.

In addition, several studies that utilized Newman's Error Analysis also explore the factors that resulted in the errors in each stage. Patac and Patac (2015) said that errors in the problem-solving process among students are anchored to incorrect procedures, incomplete schema, and failure to show their solutions. Impulsive styles in solving mathematical problems often lead to inaccurate answers (Zamzam & Alfiana, 2017). Thus, this study further explores the factors that cause errors in solving verbal problems of PSMTs. Hence, Newman's Error Analysis was deemed suitable as it explored the errors committed by the PSMTs. Specifically, it served as a foundation as the study adapted the stages encapsulated in the theory, such as reading, comprehension, transformation, process skills, and encoding errors.

Literature Review

The Program for International Student Assessment 2018 results revealed that a mere 53% of learners from various nations involved in their conducted assessment were able to successfully tackle tasks that demanded more than straightforward deduction and the utilization of many sources of information for mathematics problem-solving (Klang et al., 2021). Hence, there can be a need for better strategies, approaches, and tools necessary for instruction. Similarly, Khatimah and Sugiman (2019) discovered that various problem-solving approaches can result in varied abilities for solving mathematical problems. For instance, Setyaningsih et al. (2018) revealed the learners' approach to addressing ratio and proportion issues will vary concerning their strategies.

The increasing importance of introducing and developing problem-solving among students can be observed considering its influence on their thoughts, knowledge, and skills (Sinaga et al., 2023). Students solve problems to learn important mathematical concepts and problem-solving techniques (Albay, 2019). During such activity, students must actively explore and think using mathematical principles, theories, and procedures (Xu & Qi, 2022). However, students can also be susceptible to errors as they conduct problem-solving. Ratnaningsih and Hidayat (2021) revealed in their study that lacking cognitive skills makes it hard for students to understand and use representations, convey logical reasoning, and explain their interpretations. More often than not, students struggle with explaining, arguing, and expressing reasons, which results to hesitancy, imprecision, failure to calculate, lack of problem comprehension, and rapid interpretation without examination that further causes errors.

Students often find solving mathematical word problems challenging due to the numerous mathematical principles involved. Furthermore, the predominant mistake identified in students' responses was their inability to perform calculations accurately (Haryanti et al., 2019). Thus, various studies have examined variables affecting errors during problem-solving. One of these studies was conducted by Abu Bakar et al. (2021), which revealed mathematics self-efficacy does not contribute significantly to mathematical problem-solving performance. Another study focused on language issues, lack of resources, mathematics pedagogical understanding, parent participation, encouragement and assistance, and lack of instruction and seminars contributed to poor problem-solving at the integrated school (Chirimbana et al., 2022). However, the analysis of errors and factors that contribute to inaccuracies in solving mathematical problems is crucial due to their significant impact on performance (Ratnaningsih & Hidayat, 2021). Meanwhile, Rushton (2018) mentioned that the pedagogy of mathematics education has predominantly relied on educators displaying accurately solved example problems as examples for students to emulate when completing their exercises. Thus, by integrating accurately solved exercises with error analysis, one can get enhanced mathematical comprehension.

An error analysis can assist educators in identifying common mistakes made by pupils and offering suitable remedies to address these faults. By identifying and comprehending the errors made by pupils, strategies can be developed to mitigate or rectify these faults effectively (Putri et al., 2023). Furthermore, the usage of Newman's Error Analysis has been practiced, this analysis is a technique used to analyze errors made by students when completing issues in the form of descriptive problems (Zamzam & Patricia, 2018). Specifically, Zulyanty & Mardia (2022) suggested that this method of analysis is typically used for exploring verbal problem-solving.

METHOD

This study utilized a quantitative-descriptive study considering the data that sufficed the research objectives are numerical in nature. Descriptive research refers to a research method that allows the researcher to measure actual behavior, which in this case was the errors in solving verbal mathematics problems. The study was undertaken in Central Luzon State University (CLSU), Philippines where the research respondents are 38 pre-service mathematics teachers who are graduating Filipino students, aged 20 to 23 years old, and have 480 hours of teaching experiences. Among them, 23 (61%) are female and 15 (41%) are male. They are selected considering that the study aimed to determine the performance among the country's future mathematics educators who experienced asynchronous learning. In addition, a purposive sampling technique was utilized for the selection of the respondents. Purposeful or purposive sampling is a sampling technique used for the identification and selection of cases that are data-rich and relevant to the research variables (Palinkas et al., 2015).

Research Instrument

To acquire this data, the researcher formulated a research questionnaire for the assessment of verbal problem solving. The number of items in this instrument consists of five (5) verbal problems that must be solved to determine the errors in answering the verbal problems by the respondents. The verbal problems consisted of digit problem, age problem, work problem, mixture problem, and rate problem. These problems are as follows.

Digit problem. The denominator of a certain fraction is 5 more than twice the numerator. If 8 is added to both terms, the resulting fraction is 3/5. Find the original fraction.

Age problem. The sum of Mary and John's ages is 32. Four years ago, Mary will be twice as old as John. What are their ages now.

Work problem. A farmer can plow the field in 8 ½ days. After working for 4 days, his son joins him and together they plow the field 3 more days. How many days will it require for the son to plow the field alone?

Mixture problem. Twelve liters of 30% salt solution and 15 liters of 35% salt solution are poured into a drum originally containing 36 liters of 16% salt solution. What is the percent concentration of salt in the mixture?

Rate problem. A jogger starts a course at a steady rate of 10kph. Eight minutes later, a second jogger starts to run the same course at 12kph. How long will it take the second jogger to catch the first?

Table 1 shows the five (5) indicators based on the error categories defined by Newman (1977) were utilized. The descriptions and scoring rubric were adapted from Rohmah and Sutiarso (2017).

It should be noted that the work of Rohmah and Sutiarso (2017) is also based from the Theory of Newman (1977).

No.	Indicators	Descriptions	Score
1.	Reading Errors	Identify the problem completely	2
		Identify the problem incompletely	1
		Didn't answer/ Incorrect answer	0
2.	Comprehension Errors	Write down what is known or given value completely	2
		Write down what is known or given value incompletely	1
		Didn't answer/ Incorrect answer	0
3.	Transformation Errors	Write down the formula or mathematical model correctly	2
		Write down the mathematical model but not complete	1
		Didn't answer/ Incorrect answer	0
4.	Process Skills Errors	Using a particular procedure right and the answer is correct	2
		Using a particular procedure right but the answer is wrong	1
		Didn't answer/ Incorrect answer	0
5.	Encoding Errors	The conclusion is rendered right	2
		The conclusion is given less precise	1
		Didn't answer/ Incorrect answer	0

Table 1: Newman's error analysis indicators and scoring rubric

To ensure the questionnaire's validity, the researcher assessed the suitability of its items for pre-service mathematics teachers. The items underwent a systematic review by mathematics experts, and researchers incorporated feedback and suggestions to make necessary revisions and edits. Following this validation process, the mathematics test was developed and subsequently subjected to a pilot test. Further, the researcher utilized descriptive statistics to determine the level of the committed errors of the respondents using weighted mean and standard deviation. The scores on problem-solving tests using Newman's Error Analysis were also transmuted into the qualitative description to rate the level of problem-solving performance of the respondents by establishing the norm which includes high $(0.00 - 0.67)$, moderate $(0.68 - 1.34)$, and low level $(1.35 - 2.00)$ as shown in Table 2.

Mean Percentage Score	Interpretation
1.35-2.00	Low Level
0.68-1.34	Moderate Level
0.00-0.67	High Level

Table 2: Qualitative description of errors in solving verbal problems

RESULTS

Level of Errors of Pre-Service Mathematics Teachers in Solving Verbal Problems

Reading errors

Table 3 shows the level of reading errors of the pre-service mathematics teachers in the different verbal mathematics problems. Notably, the higher the computed mean, the lower the level of the respondents' errors in solving verbal problems. Analogously, the lower the computed mean, the higher the level of the respondent's errors in problem solving. Based on the findings, the respondents had a low level of reading errors: digit problem solving with mean $(\bar{x})$ =1.95, and standard deviation $(\sigma) = 0.23$, age problem $(\bar{x} = 1.71, \sigma = 0.46)$, work problem $(\bar{x} = 1.89, \sigma = 0.39)$, mixture problem $(\bar{x} = 1.68, \sigma = 0.62)$, and rate problem $(\bar{x} = 1.63, \sigma = 0.77)$. The overall mean of 1.77 and standard deviation of 0.27 implied that they had generally low levels of reading errors in the selected verbal problems. In other words, majority of them were able to completely identify the question being asked. This result agrees with Singh et al. (2010) that students usually commit fewer errors in the reading stage as compared to the preceding stages.

Types of Verbal Problems	Frequency				Mean $(\bar{x})$	Standard Deviation (σ)	Interpretation
	2	1	0	Total			
Digit Problem	36	2	0	38	1.95	0.23	Low Level
Age Problem	27	11	0	38	1.71	0.46	Low Level
Work Problem	35	2	1	38	1.89	0.39	Low Level
Mixture Problem	29	6	3	38	1.68	0.62	Low Level
Rate Problem	31	0	7	38	1.63	0.77	Low Level
Overall					**1.77**	**0.27**	**Low Level**

Legend: High Level (0.00-0.67), Moderate Level (0.68-1.34), Low Level (1.35-2.00)

Table 3: Level of reading errors of pre-service mathematics teachers in solving verbal problems

Comprehension errors

Table 4 presents the level of comprehension errors of the pre-service mathematics teachers in the different verbal problem solving. Based on the findings, the respondents had a low level of error in digit problem ($\bar{x} = 1.39$, $\sigma = 0.82$) while moderate level in the rest of the verbal problems: age problem ($\bar{x} = 1.13, \sigma = 0.78$), work problem ($\bar{x} = 0.71, \sigma = 0.90$), mixture problem ($\bar{x} = 1.08, \sigma = 0.78$), and rate problem ($\bar{x} = 0.76, \sigma = 0.82$). In general, the respondents had moderate levels of comprehension errors in verbal mathematics problems as implied by the overall mean of 1.02 and standard deviation of 0.61. However, the study of Hijada and Dela Cruz (2022) yielded a contradicting result, illustrating that comprehension of students do not predict their performance in solving verbal problems. On the other hand, the findings of Kurshumlia and Vula (2019) were consistent with the results of the current study, which infers that comprehension has a positive influence on the enhancement of verbal mathematics problem-solving skills.

Types of Verbal Problems	Frequency				$\bar{x}$	σ	Interpretation
	2	1	0	Total			
Digit Problem	23	7	8	38	1.39	0.82	Low Level
Age Problem	14	15	9	38	1.13	0.78	Moderate Level
Work Problem	11	5	22	38	0.71	0.90	Moderate Level
Mixture Problem	13	15	10	38	1.08	0.78	Moderate Level
Rate Problem	9	11	18	38	0.76	0.82	Moderate Level
Overall					**1.02**	**0.61**	**Moderate Level**

Legend: High Level (0.00-0.67), Moderate Level (0.68-1.34), Low Level (1.35-2.00)
Table 4: Level of comprehension errors of pre-service mathematics teachers in solving verbal problems

Transformation errors

Table 5 shows the level of transformation errors committed by pre-service mathematics teachers in the given verbal problems. Results revealed that the respondents had a low level of transformation errors in digit problem ($\bar{x} = 1.71, \sigma = 0.61$), whereas they had a moderate level of comprehension error in age problem ($\bar{x} = 1.03, \sigma = 0.82$), work problem ($\bar{x} = 0.87, \sigma = 0.94$), mixture problem ($\bar{x} = 0.97, \sigma = 0.92$), and rate problem ($\bar{x} = 0.76, \sigma = 0.94$). In general, the respondents had moderate levels of transformation errors in verbal math problems as implied by the overall mean of 1.07 and standard deviation of 0.52. Similar to the findings of Singh et al. (2010), transformation errors among students are also at moderate levels and one of the stages where students are more likely to commit errors.

Types of Verbal Problems	Frequency				$\bar{x}$	σ	Interpretation
	2	1	0	Total			
Digit Problem	30	5	3	38	1.71	0.61	Low Level
Age Problem	13	13	12	38	1.03	0.82	Moderate Level
Work Problem	14	5	19	38	0.87	0.94	Moderate Level
Mixture Problem	15	7	16	38	0.97	0.92	Moderate Level
Rate Problem	13	3	22	38	0.76	0.94	Moderate Level
Overall					**1.07**	**0.53**	**Moderate Level**

Legend: High Level (0.00-0.67), Moderate Level (0.68-1.34), Low Level (1.35-2.00)

Table 5: Level of transformation errors of pre-service mathematics teachers in solving verbal problems

Process skills errors

Table 6 shows the level of process skills errors of pre-service teachers in the different verbal problem solving. The respondents had a low level of process skills errors in digit problem ($\bar{x} = 1.50, \sigma = 0.73$) and age problem ($\bar{x} = 1.39, \sigma = 0.79$). The least of the respondents had a high level of process skills errors in work problem ($\bar{x} = 0.61, \sigma = 0.82$) while several of them had a moderate level of said errors in mixture problem ($\bar{x} = 1.03, \sigma = 0.85$) and rate problem ($\bar{x} = 0.79, \sigma = 0.99$). The overall mean of 1.06 and standard deviation of 0.57 implied that the respondents had a moderate level of process skills errors. This is a consequence of not being able to transform the problem into correct mathematical equation. Zamzam and Alfiana (2017) revealed that students often commit errors in the stage of process skills due to forgetting the next step that they have to undertake, which further results in difficulty in the completion of problem solving.

Types of Verbal Problems	Frequency				$\bar{x}$	σ	Interpretation
	2	1	0	Total			
Digit Problem	24	9	5	38	1.50	0.73	Low Level
Age Problem	22	9	7	38	1.39	0.79	Low Level
Work Problem	8	7	23	38	0.61	0.82	High Level
Mixture Problem	14	11	13	38	1.03	0.85	Moderate Level
Rate Problem	15	0	23	38	0.79	0.99	Moderate Level
Overall					**1.06**	**0.57**	**Moderate Level**

Legend: High Level (0.00-0.67), Moderate Level (0.68-1.34), Low Level (1.35-2.00)

Table 6: Level of process skills errors of pre-service mathematics teachers in solving verbal problems

Encoding errors

Table 7 presented the level of encoding errors of the pre-service mathematics teachers in the different verbal problems provided. Findings showed that the respondents had low levels of encoding errors in digit problem ($\bar{x} = 1.39, \sigma = 0.92$) and age problem ($\bar{x} = 1.63, \sigma = 0.71$). Few of them had a high level of said error in work problem ($\bar{x} = 0.61, \sigma = 0.89$) and rate problem ($\bar{x} = 0.63, \sigma = 0.85$). Some of the respondents had moderate levels of such error in mixture problem ($\bar{x} = 1.03, \sigma = 0.82$). In general, respondents had moderate levels of encoding errors with an overall mean of 1.06 and a standard deviation of 0.54 in the provided verbal math problems. Moreover, Rohmah and Sutiarso (2018) revealed in a similar study that 19.57% of respondents are more likely to commit encoding errors.

Types of Verbal Problems	**Frequency**				$\bar{x}$	σ	**Interpretation**
	2	**1**	**0**	**Total**			
Digit Problem	26	1	11	38	1.39	0.92	Low Level
Age Problem	29	4	5	38	1.63	0.71	Low Level
Work Problem	10	3	25	38	0.61	0.89	High Level
Mixture Problem	13	13	12	38	1.03	0.82	Moderate Level
Rate Problem	9	6	23	38	0.63	0.85	High Level
Overall					**1.06**	**0.54**	**Moderate Level**

Legend: High Level (0.00-0.67), Moderate Level (0.68-1.34), Low Level (1.35-2.00)

Table 7: Level of encoding errors of pre-service mathematics teachers in solving verbal problems

Overall errors

Table 8 shows the overall findings, there was a low level of reading errors committed by the pre-service teachers, which infers their ability to accurately read and identify what is being asked in mathematical verbal problems. However, the errors committed in the preceding stages manifested moderate levels. For comprehension errors, a moderate level was revealed but entails the lowest mean. As a result, PSMTs are more likely to struggle with identifying the supplied values in verbal problems, which they can work on further by understanding the problem again. Meanwhile, there was a moderate level of mistake in the transformation stage, indicating that students did not select the correct operation and equation while answering problems. Moving on, there was also a moderate level of process skill errors committed by the respondents, indicating that some students struggled with appropriately employing the specified equations. Lastly, the researcher concluded that there was a moderate level of encoding errors, which infer that even though pre-service teachers were able to complete the process, their interpretation of the result was inaccurate. In general, the pre-service math teachers committed a moderate level of errors implying that they have gaps in their mathematical knowledge or understanding of concepts, and not developed effective problem-solving strategies.

Indicators	$\bar{x}$	σ	Interpretation
Reading Error	1.77	0.27	Low Level
Comprehension Error	1.02	0.61	Moderate Level
Transformation Error	1.07	0.53	Moderate Level
Process Skills Error	1.06	0.57	Moderate Level
Encoding Error	1.06	0.54	Moderate Level
Overall	**1.20**	**0.32**	**Moderate Level**

Legend: High Level (0.00-0.67), Moderate Level (0.68-1.34), Low Level (1.35-2.00)

Table 8: Overall errors of pre-service mathematics teachers in solving verbal problems

DISCUSSION

Reading errors

Common reading errors committed by the respondents were incomplete identification of the problem where the respondents failed to fully grasp or understand the main question or task presented in the reading material. Among the errors are misspelled words, which disrupt the flow of reading and cause confusion, making it harder for respondents to comprehend the content accurately. As a result, their responses or answers to the questions may be incorrect or insufficient.

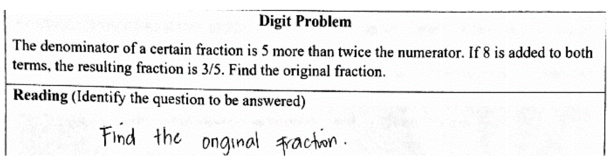
Specifically in the given digit problem, findings revealed that 36 (or 95%) of the pre-service teachers did not commit any reading error and almost were able to identify the question to be answered completely, and two of them (or 5%) only scored 1 which means they were able to identify the question to be answered incompletely. The digit problem states that “The denominator of a certain fraction is 5 more than twice the numerator. If 8 is added to both terms, the resulting fraction is 3/5. Find the original fraction”. As seen in the sample answer in Figure 1.1, Respondent 17 understood that the problem is asking to find the original fraction. Hence, a score of 2 was provided since the respondent was able to identify the question to be answered completely. Meanwhile, Respondent 25 encountered difficulty articulating the question as he/she forgot the crucial term "original." Consequently, he/she only received a score of 1 due to this oversight.

In the given age problem, 27 (or 71%) of them were able to identify the question being asked completely while 11 (or 29%) of them were not. The age problem goes, “The sum of Mary and John’s ages is 32. Four years ago, Mary will be twice as old as John. What are their ages now?”. In the sample answer (Figure 1.2), Respondent 9 was able to completely identify the question being asked, “What are the present ages of Mary and John” which was deemed to be a complete statement of the question being asked since the subjects and the timeline being referred to were specified. Hence, a score of 2 was provided to the respondent. On the other hand, Respondent 34 was not able to completely identify the question to be answered. In the same figure, Respondent 34 repeated exactly what was stated in the problem therefore a score of 1 was given. This is deemed incomplete as the answer did not specify whose ages were being asked unlike the answer provided by Respondent 9.

In the work problem, 35 (or 92%) of the respondents scored 2 or were able to completely identify the question being asked; two of them (or 5%) scored 1 or incompletely identified the question being asked; while one (or 3%) scored 0 or did not answer. Respondent 22 was able to identify entirely the question being asked and hence was provided a score of 2. On the other hand, as seen in Figure 1.3, Respondent 8 failed to clearly indicate that the question pertained specifically to "how many days it will require for the son to plow the field alone," rather than addressing the general timeframe for plowing the field. This lack of precision led to a score of 1 being assigned.

In the mixture problem, 29 (or 76%) of them had completely identified the question being asked; six (or 15%) were not; and three (or 8%) of them did not answer. As seen in the sample in Figure 1.4, Respondent 21 was able to completely identify the question, hence was provided a score of 2 while Respondent 13 had written a wrong word in his answer, writing “present” rather than “percent” which is why the respondent had been provided a score of 1 only. Finally, in a rate problem, findings showed that 31 (or 82%) of the respondents had specified the question completely while seven (or 18%) of them did not answer. As seen in Figure 1.5, Respondent 20 was able to identify the question completely and hence was provided a score of 2.

(1.1) *Digit Problem*

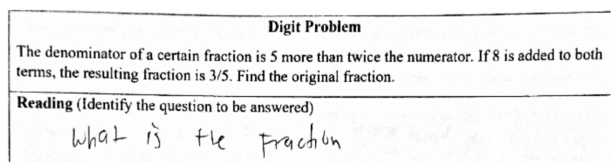

Digit Problem

The denominator of a certain fraction is 5 more than twice the numerator. If 8 is added to both terms, the resulting fraction is 3/5. Find the original fraction.

Reading (Identify the question to be answered)

Find the original fraction.

Respondent 17


Digit Problem

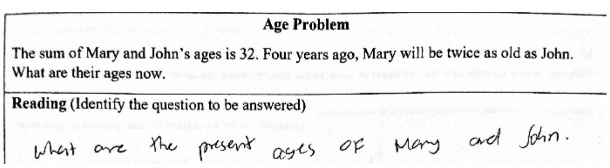
The denominator of a certain fraction is 5 more than twice the numerator. If 8 is added to both terms, the resulting fraction is 3/5. Find the original fraction.

Reading (Identify the question to be answered)

What is the Fraction

Respondent 25

(1.2) *Age Problem*

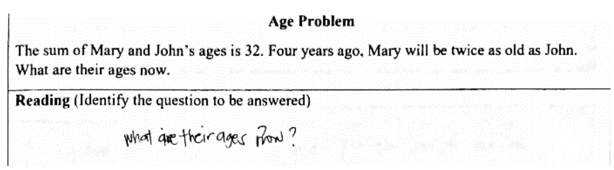

Age Problem

The sum of Mary and John’s ages is 32. Four years ago, Mary will be twice as old as John. What are their ages now.

Reading (Identify the question to be answered)

What are the present ages of Mary and John.

Respondent 9


Age Problem

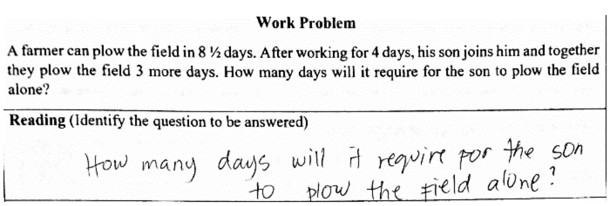
The sum of Mary and John’s ages is 32. Four years ago, Mary will be twice as old as John. What are their ages now.

Reading (Identify the question to be answered)

What are their ages now?

Respondent 34

(1.3) *Work Problem*

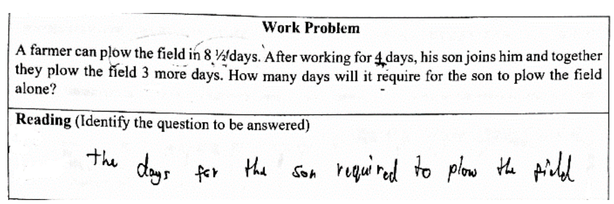

Work Problem

A farmer can plow the field in 8 ½ days. After working for 4 days, his son joins him and together they plow the field 3 more days. How many days will it require for the son to plow the field alone?

Reading (Identify the question to be answered)

How many days will it require for the son to plow the field alone?

Respondent 22


Work Problem

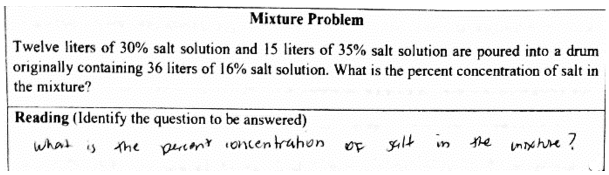
A farmer can plow the field in 8 ½ days. After working for 4 days, his son joins him and together they plow the field 3 more days. How many days will it require for the son to plow the field alone?

Reading (Identify the question to be answered)

The days for the son required to plow the field

Respondent 8

(1.4) *Mixture Problem*

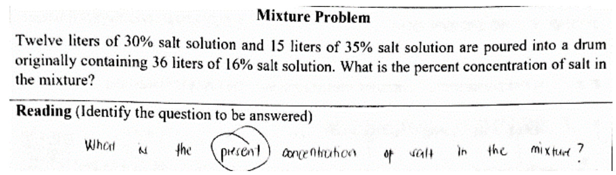

Mixture Problem

Twelve liters of 30% salt solution and 15 liters of 35% salt solution are poured into a drum originally containing 36 liters of 16% salt solution. What is the percent concentration of salt in the mixture?

Reading (Identify the question to be answered)

what is the percent concentration of salt in the mixture?

Respondent 21


Mixture Problem

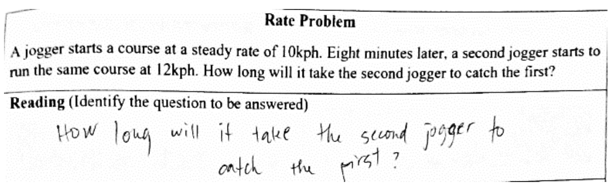
Twelve liters of 30% salt solution and 15 liters of 35% salt solution are poured into a drum originally containing 36 liters of 16% salt solution. What is the percent concentration of salt in the mixture?

Reading (Identify the question to be answered)

What is the percent concentration of salt in the mixture?

Respondent 13

(1.5) *Rate Problem*


Rate Problem

A jogger starts a course at a steady rate of 10kph. Eight minutes later, a second jogger starts to run the same course at 12kph. How long will it take the second jogger to catch the first?

Reading (Identify the question to be answered)

How long will it take the second jogger to catch the first?

Respondent 20

Figure 1: Reading errors in solving verbal problems – sample of correct and incorrect answers

Comprehension errors

The respondents found digit problem easier to comprehend than the rest of the problems. It should be noted that eight (or 21%) did not attempt to answer what the given are in the posted digit problem, and more than half (22 or 58%) did not attempt to determine the given in the posted work problem. A common comprehension error committed by the respondents was incompletely identifying the given values in the problems.

Specifically, 23 (or 61%) of the respondents did not commit any comprehension error in the digit problem while seven (or 18%) of them did, and another eight (or 21%) of them did not answer. Shown in Figure 2.1 is a sample of the answers provided by selected respondents. Respondent 4 was able to write down the given value completely and correctly, while Respondent 2 wrote down the given incompletely. The said respondent only wrote down the denominator and the numerator but failed to include the details about the sum of the provided values. Hence, Respondent 4 scored 2 while Respondent 2 scored 1.

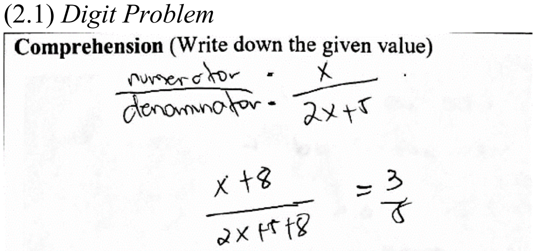
In the given age problem, 14 (or 37%) of the respondents did not commit any comprehension error, 15 (or 39%) of them did not completely write down the given value, while nine (or 24%) of them did not answer. In Figure 2.2, Respondent 10 was able to write down the given value completely and correctly. Individual ages of Mary and John, including both of their ages, were written down. On the other hand, Respondent 19 fell short of identifying the given value resulting an incomplete answer.

In the given work problem, 11 (or 29%) of the respondents had committed no comprehension error, five (or 13%) failed to completely write down the given, whereas 22 (or 58%) did not answer. As shown in Figure 2.3, Respondent 8 completely identified the given values, hence was provided a score of 2 while Respondent 3 failed to do so, hence was provided a score of only 1.

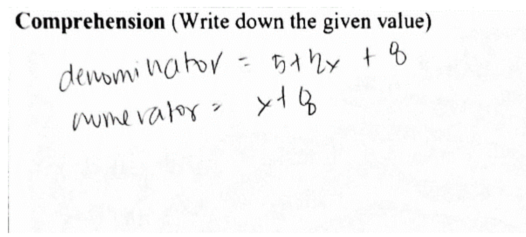
In the given mixture problem, 13 (or 34%) of the respondents were able to completely identify the given values, 15 (or 40%) of them failed to do so, while 10 (or 26%) of them had incorrect answers and did not answer. As presented in Figure 2.4, Respondent 2 completely and correctly identified the given values, while Respondent 11 inaccurately represented the given values, led him/her score marked as 0.

In the given rate problem, nine (or 24%) of the respondents did not commit any comprehension error, 11 (or 29%) of them incompletely identified the given values, while 18 (or 47%) did not answer. As seen in Figure 2.5, Respondent 16 completely and correctly specified the given whereas Respondent 24 failed to specify which jogger covered the indicated distance.

(2.1) *Digit Problem*

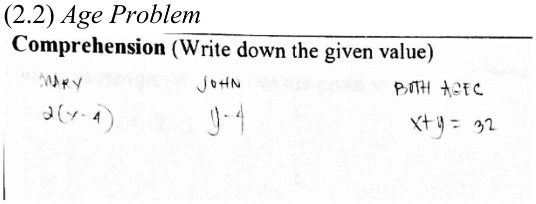


Respondent 4

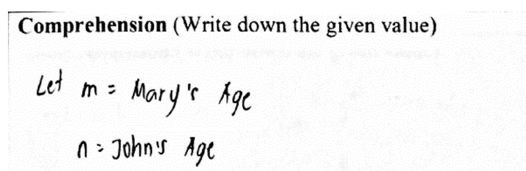


Respondent 2

(2.2) *Age Problem*

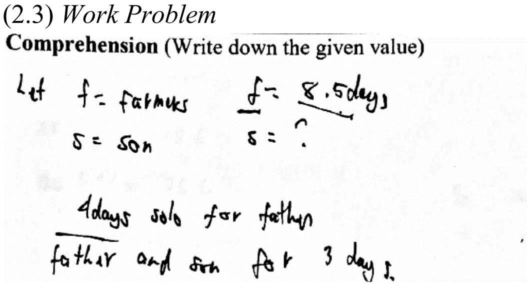


Respondent 10

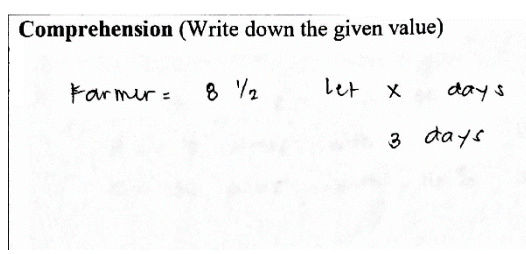


Respondent 19

(2.3) *Work Problem*

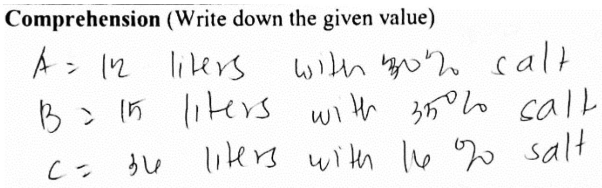


Respondent 8

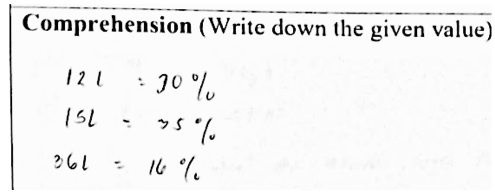


Respondent 3

(2.4) *Mixture Problem*

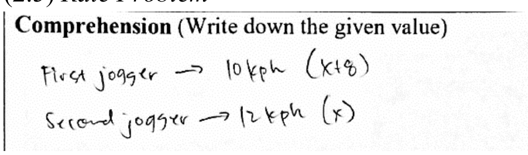


Respondent 2

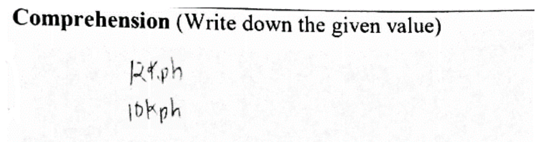


Respondent 11

(2.5) *Rate Problem*



Respondent 16



Respondent 24

Figure 2: Comprehension errors in solving verbal problems – sample of correct and incorrect answers

Transformation errors

The respondents found digit problem easier than the other problems. However, it is concerning that almost half of respondents did not write the mathematical equation properly specifically in rate problem. It should be noted that only three (or 8%) did not attempt to answer what is the corresponding mathematical equation of the digit problem, but in the given rate problem more than half (22 or 58%) did not attempt to determine what is the corresponding mathematical equation. A common transformation error committed by the respondents was incompletely and inaccurate transcription of mathematical equations, a noteworthy challenge in maintaining precision during problem-solving.

Specifically, 30 (or 79%) of the respondents had committed no transformation errors in the digit problem, five (or 13%) of them failed to do so, while three (or 8%) of them did not answer. As shown in Figure 3.1, Respondent 17 demonstrated proficiency by successfully identifying the formula and constructing the model or equation as required. In contrast, Respondent 35 encountered challenges in accurately translating the problem into its corresponding mathematical equation, resulting in an incorrect representation. Hence, receive a score of 0.

In the given age problem, 13 (or 34%) of the respondents did not commit any transformation error, 13 (or 34%) of them were not able to completely identify the formula and equation, while 12 (or 32%) of them did not attempt to answer or had incorrect mathematical equations. As presented in the sample answers in Figure 3.2, Respondent 28 successfully identified and accurately construct a model or equation, along with correctly articulating the relevant formula. Conversely, Respondent 26 faced a limitation as he/she failed to provide the complete formula, resulting in a score of 0.

In the given work problem, 14 (or 37%) had no transformation errors, five (or 13%) had some while 19 (or 50%) of them either did not write an answer (14 of 19) or had incorrect equations. As gleaned in the sample answers in Figure 3.3, Respondent 32 was able to write down the model or equation completely and correctly while Respondent 30 had incorrect construction of the equation.

The answer of Respondent 30 suggests that he/she did not understand that the addend "$3\left(\frac{1}{8\ 1/2}\right)$" means that the farmer having the ability to finish one task alone in $8\,\frac{1}{2}$ days only worked for 3 days alone.

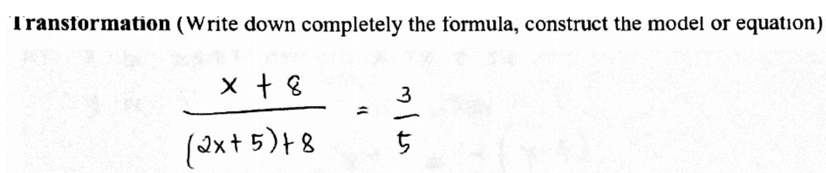
In the given mixture problem, 15 (or 40%) of the respondents did not commit any transformation error; seven (or 18%) had some errors; while 16 (or 42%) of them did not answer or had incorrect answers. Sample answers as shown in Figure 3.4, revealed that Respondent 12 was able to write down the equation and model completely and correctly, whereas Respondent 25 failed to do so. Some respondents similar to Respondent 25's answer did not define what variable x represents. In Respondent 25's answer, the equation "$(12)(30\%) + 15(35)\% + 36(16\%) = x$" suggests that x represents the amount of salt in the mixture. However, what the problem ask is the percent concentration of salt in the mixture. Another notable concern is the use of the symbol $\times$ for multiplication in the answer of Respondent 30. In his/her answer, "$[(.30 \times 12) + (.35 \times 15) + (.16 \times 36)] = x$" the symbol $\times$ may cause vagueness and misunderstanding since the variable x is also used in the equation.

Finally, 13 (or 34%) of the respondents committed no transformation error in the given rate problem, three (or 8%) had committed some, while 22 (or 58%) had incorrect answers or did not write any answer. In the presented sample answers in Figure 3.5, Respondent 13 was able to simply and correctly present the answer whereas Respondent 27 had written incorrect formulas and mathematical equations, hence had been scored 0.

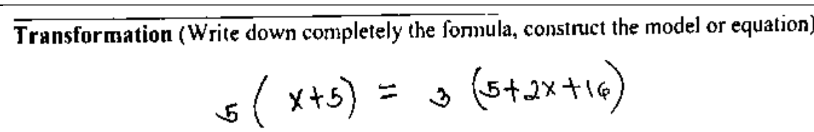
(3.1)

Digit Problem

Respondent 17



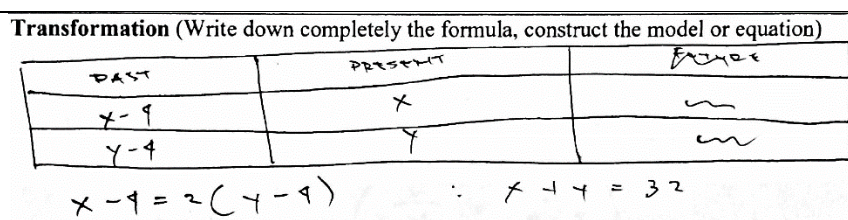
Respondent 35



(3.2)

Age Problem

Respondent 28



Respondent 26

Transformation (Write down completely the formula, construct the model or equation)

Past	Present	~~Future~~
$x-4$	x	Mary
$y-4$	y	John

(3.3)

Work Problem

Respondent 32

Transformation (Write down completely the formula, construct the model or equation)

$$4\left(\frac{1}{8.5}\right)+\left[3\left(\frac{1}{x}\right)+3\left(\frac{1}{8.5}\right)\right]=1$$

Respondent 30

Transformation (Write down completely the formula, construct the model or equation)

$$3\left(\frac{1}{8\frac{1}{2}}\right)+3\left(\frac{1}{x}+\frac{1}{8\frac{1}{2}}\right)=1$$

$$3\left(\frac{1}{17/2}\right)+3\left(\frac{1}{x}+\frac{1}{17/2}\right)=1$$

$$3\left(3/17\right)+3/x+\left(4/17\right)=1$$

(3.4)

Mixture Problem

Respondent 12

Transformation (Write down completely the formula, construct the model or equation)

$$12(0.3)+15(0.35)+36(0.16)=63(x)$$

Respondent 25

Transformation (Write down completely the formula, construct the model or equation)

~~(12+30%+15~~

$$(12)(30\%)+15(35\%)+36(16\%)=X$$

$$3.6+5.25+5.76=X$$

$$X=14.61$$

Respondent 30

Transformation (Write down completely the formula, construct the model or equation)

$$12+15+36=X$$

$$[(.30\times12)+(.35\times15)+(.16\times36)]=X$$

(3.5)

Respondent 13

Transformation (Write down completely the formula, construct the model or equation)

$$(x+8)(10)=(12)(x)$$

Rate Problem

Respondent 27

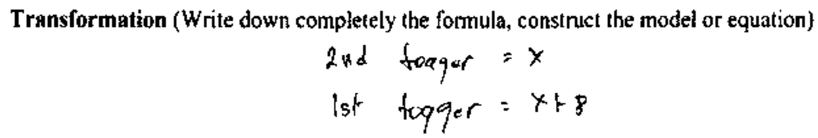


Figure 3: Transformation errors in solving verbal problems – sample of correct and incorrect answers

Process skills errors

The respondents found digit problem easier to comprehend than the rest of the problems. It should be noted that five (or 13%) did not attempt to answer or had incorrect procedures regarding the given digit problem, and more than half (23 or 61%) did not attempt or have incorrect answers in the posted work and rate problem. A common process skills error committed by the respondents was incorrect process leading to incorrect answers in the given problems.

Specifically, in the given digit problem, 24 (or 63%) had committed no process skills errors; nine (or 24%) had some; while five (or 13%) did not answer or had incorrectly used a procedure. As presented in Figure 4.1, Respondent 12 had correctly evaluated the provided problem, arriving at the correct answer, hence had scored 2 while Respondent 24 used the wrong evaluation process, arriving at incorrect answer, hence had scored 0.

In the given age problem, 22 (or 58%) did not commit any process skills errors; nine (or 24%) had committed some; whereas seven (or 18%) incorrectly used a process or did not answer. As seen in the sample answers in Figure 4.2, Respondent 32 had followed the process leading to correct answer whereas Respondent 5 had an incomplete process but arrived at the same correct answer. As seen in the figure, he/she did not write the proper cancellation of the constant on variable "j" which is dividing both sides by 3 to determine the value of the variable.

In the given work problem, eight (or 21%) scored perfect or had committed no process skills errors; seven (or 18%) only scored 1 while 23 (or 61%) did not answer or had written the wrong answer. As shown in the sample answer in Figure 4.3, Respondent 17 had correctly and completely evaluated the problem, thus scored 2 while Respondent 25 had performed with unfinished solution leading him/her to not determine the value of "x".

In the given mixture problem, 14 (or 37%) had correctly used a procedure, and thus were scored 2; 11 (or 29%) had incompletely employed a process; whereas another 13 (or 34%) did not answer or had incorrectly used a process. As apparent from the sample answer in Figure 4.4, Respondent 22 had a simple yet complete procedure with a correct answer. Respondent 24 on the other hand, had failed to convert his answer to percentage which was what is asked in the provided problem. Hence, the respondent scored 1 only.

Finally, in the given rate problem, 15 (or 39%) had committed no process skills errors while 23 (or 61%) of them had either answered incorrectly or gave a blank answer. In Figure 4.5, Respondent 5 have correct process and answer. The rest of the respondents did not answer.

(4.1)
Digit Problem

Process Skills (Perform the indicated problem, and evaluate using the given value)

$$\frac{x+8}{2x+13} = \frac{3}{5}$$
$$5(x+8) = 3(2x+13)$$
$$5x+40 = 6x+39$$
$$40-39 = 6x-5x$$
$$1 = x$$

$$* \quad \frac{x}{2x+5} = \frac{1}{2(1)+5} = \frac{1}{2+5} \Rightarrow \frac{1}{7}$$

Respondent 12

Process Skills (Perform the indicated problem, and evaluate using the given value)

$$5x+25 = 3(21+2x)$$
$$5x+25 = 63+6x$$
$$x = 63-25$$
$$x = 38$$

Respondent 24

(4.2)
Age Problem

Process Skills (Perform the indicated problem, and evaluate using the given value)

$$x+y = 32$$
$$x+12 = 32$$
$$x = 32-12$$
$$x = 20$$

$$2y-8+y-4$$
$$3y-12 = 24$$
$$3y = 24+12$$
$$\frac{3y}{3} = \frac{36}{3}$$
$$y = 12$$

Respondent 32

Process Skills (Perform the indicated problem, and evaluate using the given value)

~~$2J+J-4=24$, $3J=28$~~

$$2J-8+J-4 = 24$$
$$3J = 36$$
$$J = 12$$

John = 12
Mary = 20

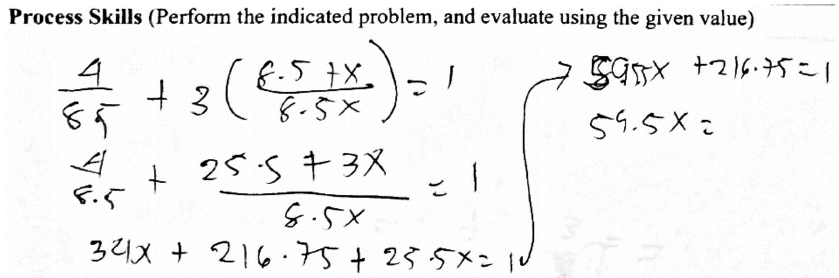
Respondent 5

(4.3)
Work Problem

Process Skills (Perform the indicated problem, and evaluate using the given value)

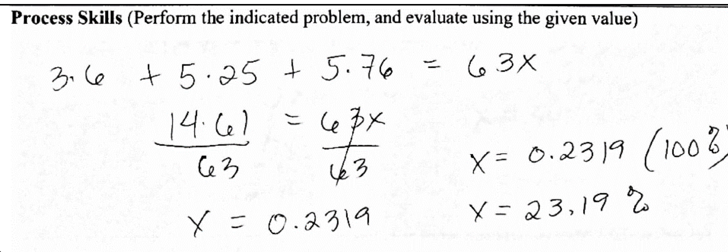
$$4\left(\frac{1}{\frac{17}{2}}\right) + 3\left(\frac{1}{\frac{17}{2}} + \frac{1}{x}\right) = 1$$
$$4\left(\frac{2}{17}\right) + 3\left(\frac{2}{17} + \frac{1}{x}\right) = 1$$
$$\frac{8}{17} + \frac{6}{17} + \frac{3}{x} = 1$$
$$17x\left[\frac{14}{17} + \frac{3}{x} = 1\right]$$
$$14x + 51 = 17x$$
$$51 = 17x - 14x$$
$$\frac{51}{3} = \frac{3x}{3}$$
$$17 = x$$

Respondent 17

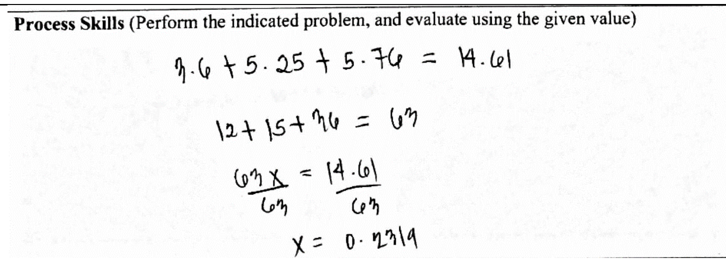


Respondent 25

(4.4)
Mixture Problem

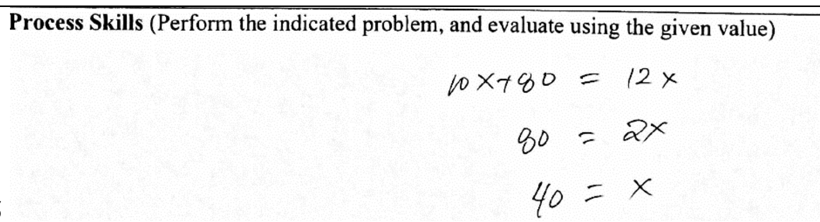


Respondent 22



Respondent 24

(4.5)
Rate Problem



Respondent 5

Figure 4: Process skills errors in solving verbal problems – sample of correct and incorrect answers

Encoding errors

The respondents found age problem easier than the rest of the problems. It should be noted that five (or 13%) had incorrectly provided a conclusion or failed to write any answer, and more than half (25 or 66%) of them had either provided incorrect conclusions or did not answer in the posted work problem. Common encoding errors committed by the respondents were incorrect grammar and failure to convert the answer which lead them to incorrect answers.

To be more specific, 26 (or 68%) of the respondents had committed no encoding errors in digit problem while one (or 3%) of them committed some, and 11 (29%) of them had either incorrectly provided a conclusion or did not answer. As shown in Figure 5.1, Respondent 5 had drawn a precise conclusion. The rest of the respondents did not.

In the given age problem, 29 (or 76%) did not commit any encoding error while four (or 11%) of them had committed some and another five (or 13%) had incorrectly provided a conclusion or failed to write any answer. As presented in Figure 5.2, Respondent 9 had precisely drawn a conclusion, hence was scored 2 whereas Respondent 6 arrived at a wrong answer hence, arrived at a wrong conclusion therefore was scored 0.

In the given work problem, 10 (or 26%) scored perfect or had not committed any encoding error; three (or 8%) committee some; whereas 25 (or 66%) of them had either provided incorrect conclusion or did not answer. As seen in the sample answer in Figure 5.3, Respondent 17 wrote down a precise conclusion while Respondent 35 had a wrong answer which is why the conclusion drawn was also incorrect.

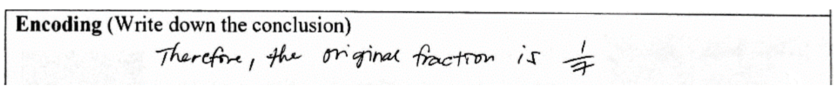
In the given mixture problem, 13 (or 34%) of them committed no encoding errors; 13 (or 34%) failed to do so; while 12 (or 32%) did not answer or had incorrectly written conclusion. As shown in the sample answer in Figure 5.4, Respondent 8 drew a precise conclusion whereas Respondent 14 had drawn otherwise.

Finally, in the given rate problem, nine (or 24%) provided a concise conclusion; six (or 15%) provided a less precise conclusion while 23 (or 61%) failed to provide a conclusion. As presented in Figure 5.5, Respondent 10 had a precise conclusion while Respondent 19 had a less precise conclusion. Said respondent had provided the correct answer but the grammar was slightly incorrect which is why a score of 1 was provided.

(5.1)

Digit Problem

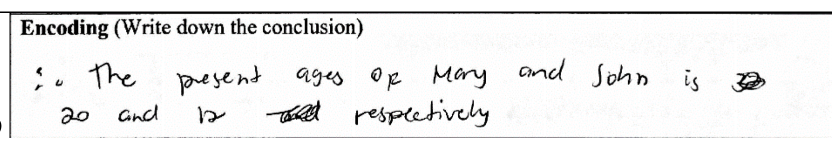
Respondent 5


Encoding (Write down the conclusion)
Therefore, the original fraction is 1/7

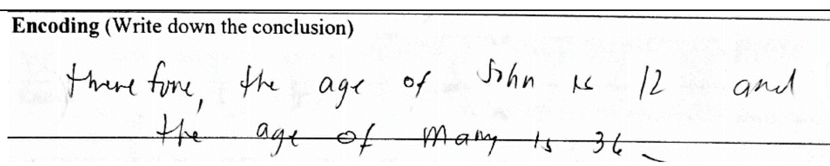
(5.2)

Age Problem

Respondent 9


Encoding (Write down the conclusion)
∴ The present ages of Mary and John is [illegible] 20 and 12 [illegible] respectively

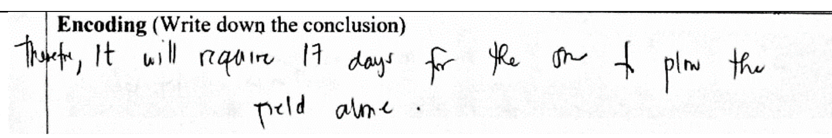
Respondent 6


Encoding (Write down the conclusion)
therefore, the age of John is 12 and ~~the age of Mary is 36~~.

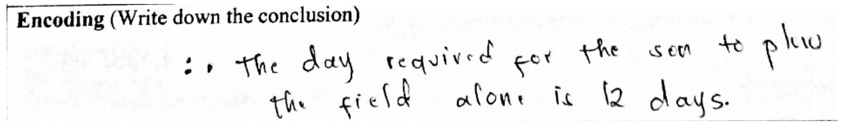
(5.3)

Work Problem

Respondent 17

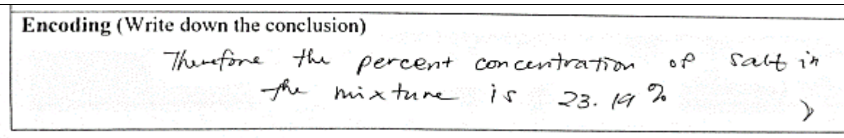

Encoding (Write down the conclusion)
Therefore, It will require 17 days for the one to plow the field alone.

Respondent 35

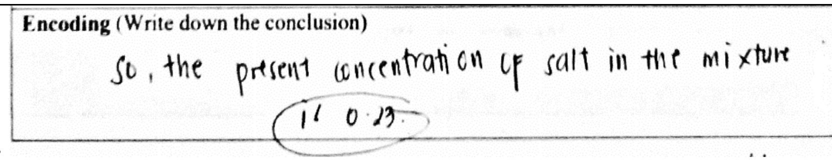

Encoding (Write down the conclusion)
∴ The day required for the son to plow the field alone is 12 days.

(5.4) *Mixture Problem*

Respondent 8

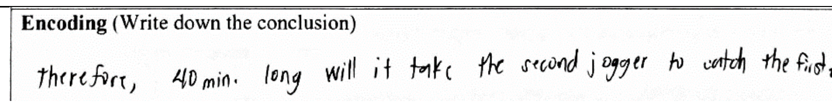

Encoding (Write down the conclusion)
Therefore the percent concentration of salt in the mixture is 23.19%

Respondent 14


Encoding (Write down the conclusion)
So, the present concentration of salt in the mixture is 0.23.

(5.5) *Rate Problem*

Respondent 10


Encoding (Write down the conclusion)
therefore, 40 min. long will it take the second jogger to catch the first.

Respondent 19

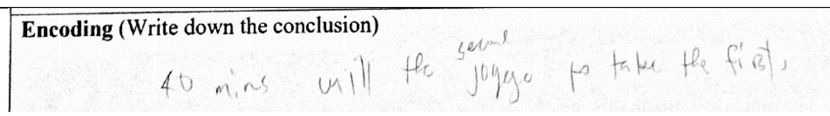

Encoding (Write down the conclusion)
40 mins will the second joggo to take the first.

Figure 5: Encoding errors in solving verbal problems – sample of correct and incorrect answers

Factors of error in solving verbal problems

Through the analysis of the answer sheet of the respondents, factors that resulted in the committed errors in solving verbal problems were revealed. First, incomplete answers and solutions were present in all stages, implying that it is a prominent factor that causes errors in verbal problem solving. According to Dofková and Surá (2021), complete and accurate solutions increase the likelihood of getting the right answers, but students are more likely to doubt their solutions and delete or struggle to finish. Thus, students' cognitive processes made them skeptical, which prevented them from addressing the problem.

Misspelling was a prominent error in the reading stage. Capone et al. (2021) found that several respondents miswrote and misspelled words and figures, causing errors in subsequent stages. Lopez (2004) found that students' mathematical problem-solving thoughts can alter their spelling. Due to erroneous values and words in the first stage, the remaining procedures will be inaccurate, independent of tool use.

Another cause of errors was poor grammar, especially while encoding. Considering that the given problems were in English language, the respondents can be deemed susceptible to error considering that the Philippines is slipping in its rank in terms of English proficiency, indicating a declining performance (Suelto, 2012). However, the education system in the country is implementing mother tongue-based multilingual education, especially in early education, because it is proven that a good mother tongue foundation can result to strong literacy abilities (Bernardo et al., 2012). Thus, the pursuit of enhancing literacy abilities among Filipino students have been implemented. In line with the findings, Guce (2017) found that while formulas and equations are crucial to mathematical work, ordinary grammatical norms also apply. Along with subject-verb agreement, mistaking a phrase for a sentence is a common grammar error. An inaccuracy in grammar might change meaning, causing errors.

In the transformation stage, incorrect construction of formulas or equations was most evident. Winarso and Toheri (2021) found that students can misidentify the right theorem or tool and be inconsistent with their equation. Thus, this evidence suggests that pre-service teachers build formulas or equations incorrectly, which causes errors, especially during the transformation stage, when these formulas are critical to solving the problem.

Similarly, there was also a problem with the incorrect process, especially in the process skills stage. According to Gurat (2018), student teachers can also be challenged in undertaking proper procedures for problem solving. Thus, this data may infer that pre-service teachers commit incorrect construction of formula or equation that leads to error, especially in the transformation stage where these formulas are crucial to be used accurately to arrive at the accurate answer.

Additionally, other errors involve the usage of wrong solution, but arriving at the correct answer. Such phenomenon has been explored by Tong and Loc (2017) that the solutions used by students for their chosen operations are incorrect, but they arrive at to correct answer, which they perceived to be more important. This data may infer that students have their own strategy, which focuses on getting the accurate answer rather than undertaking the right process or right solution. However, it can also be inferred that even though the given answers were correct, but the solution was incorrect, it can be suspected that logical thinking and conceptual understanding of students are still in need to be further developed (Kholid et al., 2021).

Meanwhile, process skills and encoding stage entail another prominent factor resulting in error, which was the failure to convert the answer. As cited in the study of Chizeck et al. (2009), students can commit mistakes in failing to convert their answers, especially on the measurement unit of their answers. This data may infer that arriving to the right answer also involves the proper conversion that is suitable to the context of the problem. Essentially, students may utilize the Polya four-step problem-solving process that demands the problem to be thoroughly understood, developing a plan through working with an equation, implementing the plan, and to look back with the problem to validate the answer (Airth & Boddie, 2019).

Finally, failing to answer or not answering was shown in all stages. In the study of Carson (2007), found that students get trapped in problem-solving stages, leading to hopelessness and failure. It also involves gaps between theory and practice, where learning mathematical theories is distinct from using them to solve problems. Since respondents failed to answer or provide a response, this element challenges their knowledge and skills.

CONCLUSIONS

This study focused on the error analysis of pre-service mathematics teachers' verbal problem solving using the Newman's Error Analysis, which included five stages. The data analysis demonstrated that the respondents made few errors during the first stage of problem solving, which was the reading stage. However, moderate level of errors, which were also shown in the latter stages, such as process skills and encoding stages. As a result, it can be concluded that respondents are more likely to make mistakes throughout the process of performing the appropriate operation for the problem and then interpreting the outcome of the computation. In line with this, a deeper understanding of the mathematical concepts and operations is necessary as well as the ability to have a grasp on the context of the problem in pursuit of an accurate interpretation of the result.

As a result, the researcher suggests that academe and instructors in tertiary level further strengthen their focus on lessons on the proper utility of operations, equations, and interpretation of results in solving verbal problems by thoughtful teaching strategies and active learning approaches. Because they are the final steps of the problem-solving process, it is crucial that they lessen errors because previous answer may have been ineffective. Furthermore, educators should encourage critical thinking, encourage active participation in the subject, and provide opportunities for application and practice. Encouraging students to ask questions, seek clarification, and explore topics in-depth can enhance their understanding and fill in the gaps in their knowledge.

Also, educators should develop and implement classroom activities that enable pre-service mathematics teachers to highly understand the construction and usage of the mathematical equations, especially the processes they entail. The use of various and relevant examples to demonstrate the practical applications of the concepts that help them for more complete understanding is also beneficial to the students. Additionally, pre-service teachers may work on their knowledge and skills in terms of mastering the operations, and equations and interpreting the results of the verbal problems they intend to solve.

Lastly, the researcher suggests conducting a refresher program among graduating pre-service mathematics teachers, which can serve as an assessment regarding their knowledge and skills, especially in verbal problem solving, and assistance in preparation for their board exam. This program can focus on pre-service mathematics teachers who took their course during the pandemic considering that there can be lapses in their understanding and lesson retention. These lapses can be caused by the fact that online learning during the pandemic was conducted in their respective houses, which is not a suitable learning environment. The problems and procedures in this study can be adapted for this program.

ACKNOWLEDGMENTS

The authors extend their heartfelt gratitude to the Department of Science and Technology - Science Education Institute – Capacity Building Program in Science and Mathematics Education (DOST-SEI-CBPSME) for the financial support in this study.

REFERENCES

[1] Abu Bakar, S., Salim, N. R., Ayub, A. F. M., & Gopal, K. (2021). Success Indicators of Mathematical Problem-Solving Performance Among Malaysian Matriculation Students. *International Journal of Learning, Teaching and Educational Research, 20(3),* 97–116. https://doi.org/10.26803/ijlter.20.3.7

[2] Agustina, E. N. S., Widahdah, S., & Nisa, P. A. (2021). Developing realistic mathematics problems based on Sidoarjo local wisdom. *Mathematics Teaching Research Journal, 13(4).* https://eric.ed.gov/?id=EJ1332336

[3] Airth, M., & Boddie, K. (2019). Polya's Four-Step Problem-Solving Process Video with Lesson Transcript. *Study.com*. https://study.com/academy/lesson/polyas-four-step-problem-solving-process.html

[4] Albay, E. M. (2019). Analyzing the effects of the problem solving approach to the performance and attitude of first year university students. *Social Sciences & Humanities Open, 1(1),* 100006. Sciencedirect. https://doi.org/10.1016/j.ssaho.2019.100006

[5] Aulia Adytia Putri, Nanang Priatna, & Kusnandi Kusnandi. (2023). Analysis of Student Errors in Solving Mathematics Problems Based on Newman Procedure and Providing Scaffolding. *Numerical:Jurnal Matematika dan Pendidikan Matematika* 7(2), 321–332. https://doi.org/10.25217/numerical.v7i2.3993

[6] Bernardo, E., Aggabao, N., & Tarun, J. (2012). Implementation of the Mother Tongue-Based Multilingual Education (MTB- MLE) Program: Reactions, Attitudes and Perceptions of Teachers. *The International Academic Forum.* https://papers.iafor.org/wp-content/uploads/papers/aceid2018/ACEID2018_39734.pdf

[7] Boonen, A. J. H., de Koning, B. B., Jolles, J., & van der Schoot, M. (2016). Word Problem Solving in Contemporary Math Education: A Plea for Reading Comprehension Skills Training. *Frontiers in Psychology, 7(191).* https://doi.org/10.3389/fpsyg.2016.00191

[8] Capone, R., Filiberti, F., & Lemmo, A. (2021). Analyzing Difficulties in Arithmetic Word Problem Solving: An Epistemological Case Study in Primary School. *Education Sciences, 11(10),* 596. https://doi.org/10.3390/educsci11100596

[9] Carson, J. (2007). A Problem With Problem Solving: Teaching Thinking Without Teaching Knowledge. *The Mathematics Educator, 17(2),* 7–14. https://files.eric.ed.gov/fulltext/EJ841561.pdf

[10] Chirimbana, M., Nghipandulwa, L. T., & Kamati, F. N. (2022). An Investigation of the Factors That Contribute to Poor Problem-Solving Skills in Grade 8 Mathematics Learners in Namibia. *Open Journal of Social Sciences, 10(12),* 614–628. https://doi.org/10.4236/jss.2022.1012042

[11] Chizeck, Butterworth, E., & Bassingthwaighte, J. B. (2009). Error detection and unit conversion. *IEEE Eng Med Biol Mag, 28(3),* 50–58. https://doi.org/10.1109/memb.2009.932477

[12] Cocjin, A. L. (2021). Error analysis in the written texts of pre-service teachers. *Asian Journal of Research in Education and Social Sciences, 3(4),* 17-27. https://repository.cpu.edu.ph/handle/20.500.12852/1804

[13] Dofková, R., & Surá, M. (2021). Nonstandard Math Word Problems And Analysis Of The Partial Stages Of Its Solution. *Problems of Education in the 21st Century, 79(5),* 716–727. https://doi.org/10.33225/pec/21.79.716

[14] Gholami, H. (2021). Improving the Performance of Mathematics Teachers through Preparing a Research Lesson on the Topic of Maximum and Minimum Values of a Trigonometric Function. *Mathematics Teaching Research Journal, 14(2).* https://eric.ed.gov/?id=EJ1350529

[15] Guce, I. K. (2017). Mathematical Writing Errors in Expository Writings of College Mathematics Students. *International Journal of Evaluation and Research in Education (IJERE), 6(3),* 233. https://doi.org/10.11591/ijere.v6i3.8549

[16] Gurat, M. (2018). Mathematical Problem-Solving Heuristics Among Student Teachers. *Journal on Efficiency and Responsibility in Education and Science, 11(3),* 53–64. https://doi.org/10.7160/eriesj.2018.110302

[17] Haryanti, M. D., Herman, T., & Prabawanto, S. (2019). Analysis of students' error in solving mathematical word problems in geometry. *Journal of Physics: Conference Series, 1157,* 042084. https://doi.org/10.1088/1742-6596/1157/4/042084

[18] Hijada, M. V., & Dela Cruz, M. L. (2022). The Gap between Comprehension Level and Problem-Solving Skills in Learning Mathematics. In *ERIC* (Vol. 1). https://eric.ed.gov/?id=ED621118

[19] Khalo, X., Bayaga, A., & Wadesango, N. (2015). Error Analysis: Case of Pre-service Teachers. *International Journal of Educational Sciences, 9(2),* 173–179. https://doi.org/10.1080/09751122.2015.11890307

[20] Khatimah, H., & Sugiman, S. (2019). The effect of problem solving approach to mathematics problem solving ability in fifth grade. *Journal of Physics: Conference Series, 1157,* 042104. https://doi.org/10.1088/1742-6596/1157/4/042104

[21] Kholid, M. N., Imawati, A., Swastika, A., Maharani, S., & Pradana, L. N. (2021). How are Students' Conceptual Understanding for Solving Mathematical Problem? *Journal of Physics: Conference Series*, *1776*(1), 012018. https://doi.org/10.1088/1742-6596/1776/1/012018

[22] Klang, N., Karlsson, N., Kilborn, W., Eriksson, P., & Karlberg, M. (2021). Mathematical Problem-Solving Through Cooperative Learning—The Importance of Peer Acceptance and Friendships. *Frontiers in Education,* 6. https://doi.org/10.3389/feduc.2021.710296

[23] Kurshumlia, R., & Vula, E. (2019). The Impact Of Reading Comprehension On Mathematics Word Problem Solving. *Education and New Developments*. https://doi.org/10.36315/2019v2end076

[24] Ling, E. Y. (2020). Assessing Students' Learning through the Newman Error Analysis Guidelines. *Mathematics Teaching in Singapore,* 53–67. https://doi.org/10.1142/9789811220159_0004

[25] Lopez, L. (2004). Helping at-risk students solve mathematical word problems through the use of direct instruction and problem solving strategies. https://stars.library.ucf.edu/cgi/viewcontent.cgi?article=4668&context=etd

[26] Mumu, J., Prahmana, R. C. I., Sabariah, V., Tanujaya, B., Bawole, R., & Monim, H. O. L. (2021). Students' ability to solve mathematical problems in the context of environmental issues. *Mathematics Teaching Research Journal, 13(4).* https://eric.ed.gov/?id=EJ1332360

[27] Nasir, S. N. C., Alias, Z., Jaffar, S. F., & Hidayat. (2022). Rural secondary teachers' readiness to teach online in the pandemic era: a case study. *Mathematics Teaching Research Journal, 14(3).* https://eric.ed.gov/?id=EJ1361575

[28] Ocampo, D. (2021). 21st Pedagogical Competence of Pre-Service Teachers in the New Normal Modalities. *Journal of Progressive Education, 11(1).* https://eric.ed.gov/?id=ED613644

[29] OECD. (2023). PISA 2022 Results (Volume I): The State of Learning and Equity in Education. https://www.oecd-ilibrary.org/sites/9149c2f5-en/index.html?itemId=/content/component/9149c2f5-en#biblio-d1e3995-92e1015b90

[30] Panthi, R. K., Acharya, B. R., Kshetree, M. P. Khanal, B., & Belbase, S. (2019). Mathematics Teachers' perspectives on emergent issues in teaching and learning mathematics in Nepal. *Mathematics Teaching Research Journal, 13(2).* https://eric.ed.gov/?id=EJ1383144

[31] Palinkas, L. A., Horwitz, S. M., Green, C. A., Wisdom, J. P., Duan, N., & Hoagwood, K. (2015). Purposeful sampling for qualitative data collection and analysis in mixed method implementation research. *Administration and policy in mental health and mental health services research, 42,* 533-544. https://doi.org/10.1007/s10488-013-0528-y

[32] Patac, A., & Patac, L. (2015). An Application of Student Self- Assessment and Newman Error Analysis in Solving Math Problems. *Recoletos Multidisciplinary Research Journal, 3(1),* 1–1. https://doi.org/10.32871/rmrj1503.01.17

[33] Ratnaningsih, N., & Hidayat, E. (2021). Error analysis and its causal factors in solving mathematical literacy problems in terms of habits of mind. *Journal of Physics: Conference Series, 1764(1),* 012104. https://doi.org/10.1088/1742-6596/1764/1/012104

[34] Rohmah, M., & Sutiarso, S. (2018). Analysis Problem Solving in Mathematical Using Theory Newman. *EURASIA Journal of Mathematics, Science and Technology Education, 14(2).* https://doi.org/10.12973/ejmste/80630

[35] Rushton, S. J. (2018). Teaching and Learning Mathematics through Error Analysis. *Fields Mathematics Education Journal, 3(1)*. https://doi.org/10.1186/s40928-018-0009-y

[36] Saleh, K., Yuwono, I., Rahman As'ari, A., & Sa'dijah, C. (2022). Errors analysis solving problems analogies by Newman procedure using analogical reasoning. *International Journal of Humanities and Social Sciences, 9(1),* 17–26. https://ijhss.net/index.php/ijhss/article/download/253/89

[37] Segura, C., & Ferrando, I. (2021). Classification and Analysis of Pre-Service Teachers' Errors in Solving Fermi Problems. *Education Sciences, 11(8),* 451. https://doi.org/10.3390/educsci11080451

[38] Seng, N. B. (2020). Newman Error Analysis For Errors In Mathematical Word Questions Among Year Three Students In Sekolah Kebangsaan Taman Kluang Barat. *International Journal of Novel Research in Education and Learning, 7(2),* 58-63. https://www.noveltyjournals.com/upload/paper/Newman%20Error%20Analysis-2300.pdf

[39] Setyaningsih, N., Juniati, D., & Suwarsono. (2018). Student's scheme in solving mathematics problems. *Journal of Physics: Conference Series, 974,* 012012. https://doi.org/10.1088/1742-6596/974/1/012012

[40] Sinaga, B., Sitorus, J., & Situmeang, T. (2023). The influence of students' problem-solving understanding and results of students' mathematics learning. *Frontiers in Education, 8*. https://doi.org/10.3389/feduc.2023.1088556

[41] Singh, P., Rahman, A. A., & Hoon, T. S. (2010). The Newman Procedure for Analyzing Primary Four Pupils Errors on Written Mathematical Tasks: A Malaysian Perspective. *Procedia - Social and Behavioral Sciences, 8,* 264–271. https://doi.org/10.1016/j.sbspro.2010.12.036

[42] Suelto, S. L. (2012, June 10). Pursuing English Language Proficiency Among Filipino Students. Exegesis. https://bsuexegesis.wordpress.com/authors/language-education/pursuing-english-language-proficiency-among-filipino-students/

[43] Tong, D. H., & Loc, N. P. (2017). Students' errors in solving mathematical word problems and their ability in identifying errors in wrong solutions. *European Journal of Education Studies, 3(6)*. https://doi.org/10.5281/zenodo.581482

[44] Triliana, T., & Asih, E. C. M. (2019). Analysis of students' errors in solving probability based on Newman's error analysis. *Journal of Physics: Conference Series, 1211*, 012061. https://doi.org/10.1088/1742-6596/1211/1/012061

[45] University at Buffalo. (2023). Teaching Methods. https://www.buffalo.edu/catt/develop/design/teaching-methods.html

[46] Vera, B. O. de. (2021, July 1). 80% of PH kids don't know what they should know – World Bank. *INQUIRER.net*. https://newsinfo.inquirer.net/1453814/wb-80-of-ph-kids-dont-know-what-they-should-know

[47] White, A. L. (2010). Numeracy, Literacy and Newman's Error Analysis. *Journal of Science and Mathematics Education in Southeast Asia, 33(2),* 129–148. https://eric.ed.gov/?id=EJ970194

[48] Winarso, W., & Toheri, T. (2021). An Analysis of Students' Error in Learning Mathematical Problem Solving; the Perspective of David Kolb's Theory. *Turkish Journal of Computer and Mathematics Education (TURCOMAT)*. https://doi.org/10.16949/turkbilmat.753899

[49] Xu, Z., & Qi, C. (2022). Middle school students' mathematical problem-solving ability and the influencing factors in mainland China. *Frontiers in Psychology, 13*. https://doi.org/10.3389/fpsyg.2022.1042315

[50] Zamzam, K. F., & Patricia, F. A. (2018, January 1). Error Analysis of Newman to Solve the Geometry Problem in Terms of Cognitive Style. Www.atlantis-Press.com; *Atlantis Press*. https://doi.org/10.2991/incomed-17.2018.5

[51] Zamzam, K., & Alfiana, F. (2017). Error Analysis of Newman to Solve the Geometry Problem in Terms of Cognitive Style Mathematics education IKIP Budi Utomo Malang INDONESIA 1. https://www.atlantis-press.com/article/25893788.pdf

[52] Zulyanty, M., & Mardia, A. (2022). Do students' errors still occur in mathematical word problem-solving?: A newman error analysis. Al-Jabar : *Jurnal Pendidikan Matematika, 13(2),* 343–353. https://doi.org/10.24042/ajpm.v13i2.13519

Problem Posing in Mathematics Teacher Training: Developing Proportional Reasoning

María Burgos[1], Jorhan Chaverri[2], José M. Muñoz-Escolano[3]

[1] University of Granada, Spain

[2] University of Costa Rica, Costa Rica

[3] University of Zaragoza, Zaragoza, Spain

mariaburgos@ugr.es, jorhan2009@hotmail.com, jmescola@unizar.es

Abstract: The aim of this paper is to describe and analyze how a group of prospective teachers create problems to develop proportional reasoning either freely or from a given situation across different contexts, and the difficulties they encounter. Additionally, it identifies their beliefs about what constitutes a good problem and assesses whether these beliefs are reflected in their problem creation. This is a descriptive-qualitative study that utilizes theoretical and methodological tools from the Onto-semiotic Approach in the content analysis of participants' responses. The results indicate that the prospective teachers' beliefs about what makes a good problem do not always manifest in their practice. The prospective teachers faced challenges in inventing problems that meet the established didactic-mathematical purpose, related to insufficient didactic-mathematical knowledge of proportional reasoning, achieving better outcomes in the arithmetic context and in free creation.

Keywords: didactic-mathematical knowledge problem posing, problem solving, proportional reasoning, teacher training.

INTRODUCTION

Problem posing has emerged as a significant topic in mathematics education, with extensive research over the past few decades (Baumanns & Rott, 2021; Cai & Huang, 2020; Lee et al., 2018) underscoring its critical role in teacher training programs (Grundmeier, 2015; Malaspina et al., 2019). This approach not only introduces trainee teachers to the nuances of teaching mathematics but also deepens their understanding of mathematical content and helps identify their teaching deficiencies (Tichá & Hošpesová, 2013).

Recent studies have leveraged problem posing as a tool to develop and enhance the competencies and knowledge of mathematics teachers, particularly in the realm of proportional reasoning (Ellerton, 2013; Mallart et al., 2018). This research focuses on the outcomes of a formative intervention aimed at prospective secondary mathematics teachers, emphasizing the creation of problems that involve proportionality.

Despite the recognized importance of this approach, challenges persist in the didactic-mathematical preparation necessary for teaching proportionality. Research highlights difficulties that both trainee and practicing teachers encounter, such as teaching proportional reasoning and related concepts effectively (Ben-Chaim et al., 2012; Burgos et al., 2018; Burgos & Castillo, 2022; Burgos & Chaverri, 2022, 2023a, 2023b; Hilton & Hilton, 2019; Izsák & Jacobson, 2017; Weiland et al., 2020). Common issues include an over-reliance on arithmetical methods like the rule of three, limited development of proportional skills, underestimation of the ratio concept in favor of fractions, and confusion between linear and non-linear relationships (Cuevas-Vallejo et al., 2023).

Moreover, there is an evident gap in research addressing how teachers in training create problems in contexts beyond the arithmetic, such as functional, geometric, and probabilistic contexts where understanding proportionality is crucial. Past findings reveal that prospective teachers often struggle to devise meaningful and contextually appropriate proportionality problems, leading to tasks that may not effectively contribute to students' learning (Bayazit & Kirnap-Donmez, 2017; Tichá & Hošpesová, 2013; Xie & Masingila, 2017). Such problems are often irrelevant, improperly leveled, or flawed, highlighting a need for focused research on problem posing within diverse mathematical contexts. On the other hand, works like that of Li et al. (2020) demonstrate that teachers' beliefs about problem posing can determine how they use this strategy in their teaching practice. However, despite the importance of problem posing in mathematics teaching and learning, and of the beliefs of both preservice and inservice teachers, the study of beliefs about problem posing has received little attention (Li et al., 2020).

In the formative action described in this article, prospective teachers must develop problems in a given context (arithmetic, functional, geometric, or probabilistic) freely or from a given situation. In the first case, the goal is for future teachers to identify the complexity of the proposed problem based on the mathematical elements involved in its resolution. In the second case, teachers in training need to analyze its potential to develop essential aspects, such as the proportional nature of percentages or the properties that characterize the relationship of proportionality. The following research objectives are proposed:

1. Identify the beliefs that future secondary teachers have about what constitutes a good mathematical problem and how these influence the creation of proportionality problems.
2. Study the difficulties that future teachers manifest in posing proportionality problems freely and in a semi-structured manner in various contexts.
3. Describe the knowledge about proportional reasoning that future teachers demonstrate in problem posing.

Theoretical Framework

Practices, objects, and processes in the analysis of mathematical activity

The Onto-semiotic Approach (OSA) to mathematical knowledge and instruction, as outlined by Godino et al. (2007), presents a pragmatic and anthropological view of mathematics, focusing on the meanings of mathematical objects within systems of practices. These practices, executed either individually to highlight personal meanings or collectively within an institution to create institutional meanings, involve a variety of entities classified as mathematical objects. These objects include problem-situations, languages, concepts, propositions, procedures, and arguments.

A mathematical process is considered to be any sequence of actions activated or developed over a certain period to achieve a goal, typically the response to a proposed task subject to mathematical or metamathematical rules such as posing or solving mathematical problems or communicating solutions. The objects emerge from the systems of practices through the respective processes of problematization, communication, definition, enunciation, algorithmization, and argumentation. This framework allows the creation of onto-semiotic configurations of practices, objects and processes, forming articulated networks where objects and processes play crucial roles within their originating practice systems (Godino et al. 2007).

This structure is critical for analyzing mathematical activities, enabling educators to discern and address potential learning difficulties, evaluate mathematical competencies, and ensure timely recall of key concepts during instruction. It provides tools for both epistemic or institutional and cognitive or personal interpretations, helping to define both institutional and personal mathematical knowledge (Godino et al., 2017).

Meanings of proportionality

The objects and processes involved in problem-solving practices that involve proportionality depend on the contexts of application, as demonstrated by numerous studies on the nature and development of proportional reasoning (Ben-Chaim et al., 2012). Therefore, specific meanings can be delineated for different fields of application of proportionality: arithmetic, algebraic-functional, geometric, probabilistic, etc.

The arithmetic approach, centered on the notion of ratio and proportion, has been predominant in curriculum developments and research proposals. A ratio is an ordered pair of quantities of magnitudes (homogeneous or heterogeneous) compared multiplicatively. Each of these quantities is expressed using a real number and a unit of measure. A ratio may appear as a fraction when the units of measure of the related quantities are disregarded; in this case, a proportion is the equality of two equivalent fractions.

This approach essentially distinguishes two categories of proportionality problems: comparison and missing value (Cramer & Post, 1993). In the arithmetic approach, various constructs associated with the rational number, particularly the percentage, are of special importance. Although the meanings of percentages are diverse, such as a number (which can be written as a fraction or decimal), as an intensive quantity, part-whole relationship, part-part relationship, or as an operator, it is fundamentally based on the need to compare two quantities not only in an absolute manner but also relatively. Percentage allows for the concise expression of proportionality relationships (Parker & Leinhardt, 1995). Knowledge about percentages involves much more than conversions, calculations, and applications—it implies seeing the percentage as a proportion (Dole, 2010).

In the algebraic approach, proportionality is recognized as a situation in which there is a constant multiplicative functional relationship between two covarying magnitudes. Problems in an (algebraic-)functional context are characterized by the application of the notion of linear function and resolution techniques based on the properties of such a function (additive, $f(x_1 + x_2) = f(x_1) + f(x_2)$, scalar-multiplicative $f(\lambda x) = \lambda f(x)$, functional-multiplicative $f(x) = kx$, for any real number x, x_1, x_2, λ; k being the constant of proportionality). A translation of a linear function $f(x) = mx$ can lead to the affine function $g(x) = mx + n$, or conversely, a linear function is a case of the affine function $g(x) = mx + n$ such that $g(0) = 0$. Linear and affine functions are the first examples of real functions of a real variable in Secondary Education.

The geometric approach is based on the notion of similarity of figures and scales in which the ratios and proportions are established between segments. Tasks related to the application of the Theorem of Thales, scales, enlargements, and reductions of figures while preserving shape, particularly reproducing a puzzle at a different scale, fall under the geometric approach (Aroza et al., 2016; Ben-Chaim et al., 2012).

Finally, works such as those by Bryant and Nunes (2012) show that proportional reasoning is a key factor in children's ability to understand and apply probabilistic concepts. Proportional reasoning is part of the analysis of the sample space, the quantification of probabilities, the study of the random variable and sampling, and the understanding and use of correlations (Bryant & Nunes, 2012), making it an essential element of probabilistic reasoning. Moreover, insufficient proportional reasoning and the close cognitive and intuitive connection between the notions of chance and proportion may underlie many of the conceptual and procedural errors in the field of probability (Bryant & Nunes, 2012), emphasizing the importance of providing opportunities for students to develop proportional reasoning in the probabilistic context (Begolli et al., 2021)

Didactic-Mathematical Knowledge and Competence Model

The Didactic-Mathematical Knowledge and Competence model (DMKC) for teachers, developed within the Onto-semiotic Approach (OSA) framework, articulates the categories of knowledge and competencies of mathematics teachers, through the facets and components of the processes of mathematical study considered in this framework (Godino et al., 2017). Thus, it is accepted that the teacher must possess mathematical knowledge per se, which is common, relative to the educational level where they teach (shared with their students and sufficient to solve problems and tasks proposed in the curriculum), and expanded, which allows them to link it with higher levels. Additionally, as any mathematical content comes into play, the teacher must have didactic-mathematical knowledge of the various facets that affect the educational process: epistemic, ecological, cognitive, affective, mediational, and interactional. These facets interact in any process aimed at teaching and learning mathematical content.

In addition to possessing these knowledge bases, the DMKC model proposes that the teacher should be competent to describe, explain, and judge what has happened in the study process and make improvement proposals (Godino et al., 2017). Specifically, the competencies in *analyzing global meanings*—identifying and describing operative and discursive practices in mathematical activity—and in *onto-semiotic analysis of practices*—recognizing the configurations of objects and processes emerging from mathematical practices—are fundamental for problem posing with didactic purposes. In turn, problem posing serves as a means to develop these competencies, as it requires: reflecting on the global structure of the problem to understand its objectives and assess if the provided information is sufficient; analyzing potential solutions and the interrelationships of mathematical objects and processes involved; and identifying potential difficulties students may encounter, along with strategies to address these in new problem formulations.

Problem Posing in Teacher Training

Mathematical problem posing is a fundamental skill for teachers, not just in solving problems but in selecting, modifying, and designing them with an educational goal (Malaspina et al., 2015). This skill enhances students' conceptual understanding and is a vital tool for assessing their knowledge and gaps (Kaur & Rosli, 2021; Kılıç, 2017; Lee et al., 2018; Kwek, 2015).

The literature describes various methodologies and categories for problem posing, as highlighted by Stoyanova and Ellerton (1996). They categorize problem creation into three types: free, semi-structured, and structured. In free situations, students create problems without restrictions, drawing from personal experiences. Semi-structured situations provide a partial framework that students complete using mathematical knowledge and experiences. Common tools in semi-structured problem posing include images, graphs, and tables (Silver & Cai, 2005). Akay and Boz (2010) distinguish various semi-structured situations based on the starting scenario. Finally, structured situations are based on reformulating existing problems, altering their conditions or questions to fit new contexts (Prabhu & Czarnocha, 2013; Van Harpen & Presmeg, 2013).

The Cruz model (2006) guides teachers through posing new problems in educational settings. This process starts with selecting a mathematical object based on didactic needs, followed by analyzing and possibly transforming this object to *posing* a new problem, emphasizing the relationship between mathematical concepts and their properties.

Grundmeier (2015) explores problem posing in teacher training from two angles: *reformulation* and *generation*. Reformulation allows for various modifications to a problem, such as swapping the given and required elements, altering the context while keeping the structure, adding data, changing wording, or expanding the problem to broader contexts. Structural reformulation demands greater creativity and a deeper grasp of the mathematical content, distinguishing it from more superficial techniques like merely adding information or modifying the wording of the problem.

Lee et al. (2018) differentiates between generation and reformulation by their purposes: generation fosters creativity by connecting mathematics with real-life situations, while reformulation encourages a deeper reflection on existing problems, enhancing the student's creative and analytical skills.

Malaspina (2013) defines the elements that characterize a mathematical problem: information, requirement, context, and mathematical environment. The context could be intra-mathematical or extra-mathematical, focusing on how the problem is framed within a specific mathematical domain, such as algebraic or geometric contexts. The process of creating a new problem can occur through variation—modifying elements of an existing problem—or elaboration—constructing a problem freely from a situation, focusing on the relationships between the information provided and the mathematical framework involved.

This detailed approach to problem posing is crucial for teachers, as it not only helps in understanding how to construct educational and challenging problems but also aids in evaluating the didactic-mathematical knowledge required for effective teaching.

METHOD

The study is framed within a descriptive research approach that is essentially qualitative, as it aims to describe and interpret how prospective teachers create problems to develop proportional reasoning and the difficulties, they face in creating problems, as well as their beliefs about what constitutes a good problem and how to create one. Content analysis (Cohen et al., 2018) is used to examine the response protocols of the prospective teachers who participated in the formative intervention.

Research Context

The formative action that frames the research was conducted with 16 students training to be teachers (5 women and 11 men) in the context of a mathematics specialization course of the Master's in Secondary Education Teaching at a Spanish university (March 2023). The prior degrees of the participants entering the Master's program include Bachelor's in Mathematics (11) and Bachelor's in Physics (5). As part of their training in the master's program, the prospective teachers (PTs henceforth) had also received training in another course on rich tasks to be developed in the secondary education classroom (Arce et al., 2019), as well as aspects of teaching and learning proportionality based on reading various research articles such as the historical phenomenology, types of problems and different solving strategies, students' learning difficulties, and the common treatment of proportionality in textbooks (Cramer & Post, 1993; Fernández & Llinares, 2010; Martínez-Juste et al., 2014; Oller-Marcén & Gairín, 2015; Steinthorsdottir, 2006 ; Tinoco et al., 2021).

Once informed of the research purposes, the PTs signed a consent form and individually solved the proposed assessment tasks over two hours. In addition to individual work reports, evaluations made by the PTs about some of the assigned tasks and the first impressions made by some of the participants upon submission are also available. Finally, they were encouraged to attend the next class sessions, after the practicum period, to share insights on the questionnaire responses and complete the module by presenting research findings and the didactic proposal on proportionality included in Martínez-Juste (2022).

Data Collection Instrument

The questions and tasks proposed as part of the data collection instrument were designed by the researchers, taking into account the results obtained in previous studies (Burgos et al., 2018; Burgos & Chaverri, 2022; Mallart et al., 2016). These tasks were applied to a pilot group of preservice primary education teachers. To understand the participants' beliefs about problem creation, they were initially asked to respond to the following questions:

- *What characteristics do you think a good mathematical problem should have?*
- *What factors do you believe determine the complexity of a mathematical problem?*
- *What knowledge and skills do you think a teacher needs to create a good mathematical problem?*

Subsequently, two tasks were proposed. In the first task (Figure 1) students were required to freely create a problem (without a starting situation) involving proportional reasoning in three different contexts: arithmetic, geometric, and probabilistic.

TASK 1: Create a problem that is solved by applying proportional reasoning for each of the following three contexts: arithmetic, geometric, and probabilistic. Then, solve it.

In each problem:

- Indicate how proportional reasoning is used.
- Identify the objects (concepts, procedures, properties, and arguments) and the mathematical processes (enunciation, signification, algorithmization, argumentation, representation, particularization, generalization, etc.) that are involved.
- Justifiably identify the degree of complexity involved in solving each one.

Figure 1: Free creation of proportionality problem

After creating each problem, the participants must justify how proportional reasoning is involved, identifying the network of objects and processes, so as to verify the relevance of the problems they have devised in meeting the established purpose. Additionally, this process is expected to help them recognize the level of complexity involved in their problems.

In the second task (Figure 2), three scenarios were proposed: an image about percentages and discounts in stores (Situation A), a graph of two functions (linear and affine) (Situation B), and a puzzle with given measurements of the pieces (Situation C).

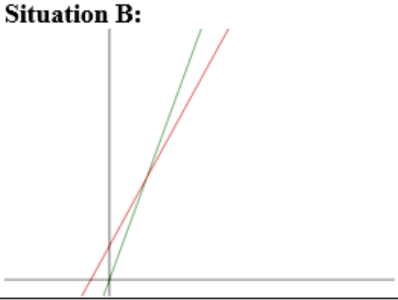
TASK 2: Below, some situations described through images are proposed. For each situation, you must create a problem that involves proportional reasoning in the given context and meets the established requirement.

Situation A:

50% off on the 2nd unit — *3 for 2 on over 1500 products* — *2nd unit at 70% off*

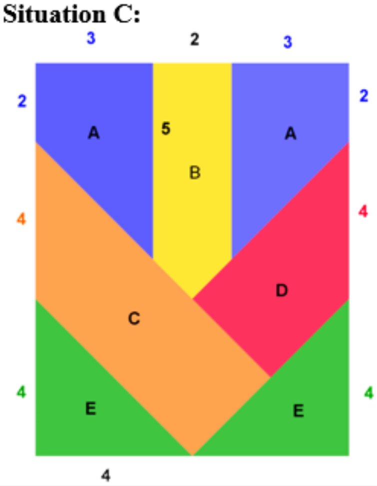
Context: Arithmetic.
Requirement: The problem should motivate the student to understand the proportional nature of percentages.
Solve the problem, highlighting how these properties are used.

Situation B:



Context: Functional.
Requirement: The problem should encourage the student to use the properties of the direct proportionality relationship (additive, multiplicative: scalar relationship and functional relationship).
Solve the problem, highlighting how these properties are used.

Situation C:



Context: Geometric.
Requirement: The problem should encourage the student to use the properties of the direct proportionality relationship (additive, multiplicative: scalar relationship and functional relationship).
Solve the problem, highlighting how these properties are used.

Figure 2: Creation of proportionality problems from situations

Students were asked to create and solve a problem for each of the associated contexts (arithmetic, functional, and geometric, respectively) with a given purpose related to the proportional nature of percentages or the properties of the direct proportionality relationship.

Analysis Guidelines

The responses from the PTs were analyzed by two authors based on pre-established criteria by the research team.

To analyze the responses regarding beliefs about what constitutes a good problem or what determines its complexity, content analysis was applied to the participants' written responses, identifying and isolating descriptions that could be associated with characteristics related to the problem statement itself, the activity it develops, or the potential solvers.

The analysis assessed the significance and pertinence of the posed problems. A problem is *significant* if it clearly establishes a mathematical challenge that is solvable, the solution is not implicit, the wording is clear and unambiguous, and its elements (context, information, requirement, environment) are clearly identified.

A problem may be significant but not *pertinent*. For example, in Task 1 (Figure 1), non-pertinence occurs if the problem lacks proportional reasoning or fails to match the requested context. In Task 2 (Figure 2), a problem is non-pertinent if it deviates substantially from the given situation or overlooks the intended context or the didactic-mathematical requirement. Conversely, a pertinent problem accurately reflects the situation and addresses the requirements.

Task 1 demands a precise description of how the problem involves proportional reasoning. We aim to understand the factors PTs consider when evaluating the complexity of their problems, using Stein et al.'s (1996) model to categorize them as memorization, procedures without connection, procedures with connection, or doing mathematics. Task 2 analyzing how PTs recognize the proportional nature of percentages in Situation A and apply proportionality properties in Situations B and C.

RESULTS

Beliefs

The reflections of the PTs about the characteristics that a good mathematical problem should have can be organized around three dimensions: the statement of the problem, the mathematical activity it motivates, and the demands it places on students.

According to Table 1, for the PTs, a good mathematical problem is characterized by a clear and unambiguous statement, a motivating context, as well as a requirement that stimulates reasoning and allows reaching its solution in an accessible way and by different strategies. These ideas coincide with and expand the results obtained by Mallart et al. (2016).

Characteristic	Frequency
The statement	
Clear, unambiguous, includes necessary information	7
Appropriate, enriching, motivating context	5
Coherent with the mathematical object being taught	2
The mathematical activity promoted	
Allows different resolution strategies	7
Questions encourage reasoning, checking, reflection	5
Mobilizes competencies (non-algorithmic)	3
Motivates the need for mathematical objects, fosters questioning and the extension of its solution to other settings	3
The students	
Accessible to all students, not difficult to pose and begin to solve	4

Table 1: Characteristics of a Good Problem

When referring to the factors that determine the complexity of a problem (second question, Table 2), they again referred to conditions about the formulation of the problem, highlighting how data and questions appear ordered and related, as well as the "scaffolding" (PT3) of these, about the mathematical activity they involve, specifying that the level of abstraction or the deployment of "happy ideas" (PT16), that is creativity and ingenuity, raises the complexity of a problem. Finally, they considered the knowledge and skills that students need to face them as an aspect that influences their complexity.

Factors	Frequency
About the problem statement	
Information. Amount of data, order and relation between questions	4
Requirement. Difficulty in interpreting the statement, what it requires, selecting what's relevant	6
Context. Connection with other fields	2
Environment. Complexity of the concepts and processes involved	5
The mathematical activity promoted	
Existence of more than one way to solve	3
Abstraction and creativity	3
About the Students	
Required knowledge and competencias	7

Table 2: Factors determining the complexity of a problem

PTs found it more complex to specify the knowledge and competencies that a teacher requires to create good problems (Table 3), which may be justified by their lack of training and experience in this area, as they themselves acknowledged. In fact, of the 16 participants, two did not respond to this question.

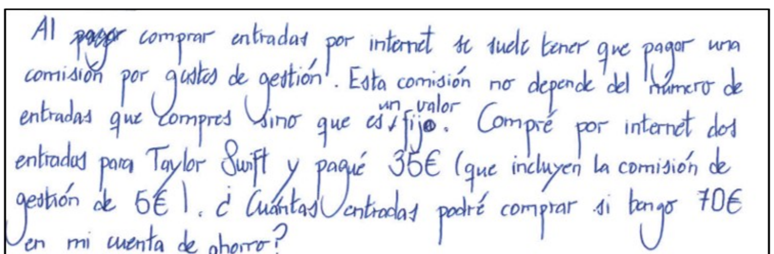
Knowledge and Skills	Frequency
Related to Content	
Knowledge of the content and problem field	5
Knowledge of the goals pursued and how to achieve them	4
Creativity	6
Related to Students	
About students' prior knowledge	3
About potential difficulties, frequent errors	4
About students' tastes and interests	5

Table 3: Knowledge and skills required for a teacher to create problems

For the PTs, a "deep knowledge of mathematics" (PT9) is necessary, including understanding of the mathematical objects and how they are used to solve problems. They also considered it important to be creative in proposing attractive problems in useful and real contexts, taking into account the goal being pursued ("knowing what you want to achieve with the problem and how to generate new knowledge from it", PT16). Only some participants recognized as important having experience with problem creation (PT6, PT16) as well as "writing skills" (PT8).

Free Creation of Proportionality Problems

Out of the 16 participants, two did not create problems in the arithmetic context, and two others proposed non-significant problems, as they did not establish the regularity needed to apply a proportionality relation. The remaining PTs created relevant problems, mostly (nine) involving missing values (Figure 3), although they also formulated problems on comparing ratios, proportional distribution, or compound proportionality.


Al ~~pagar~~ comprar entradas por internet te suele tener que pagar una comisión por gastos de gestión. Esta comisión no depende del número de entradas que compres sino que es un valor fijo. Compré por internet dos entradas para Taylor Swift y pagué 35€ (que incluyen la comisión de gestión de 5€). ¿Cuántas entradas podré comprar si tengo 70€ en mi cuenta de ahorro?

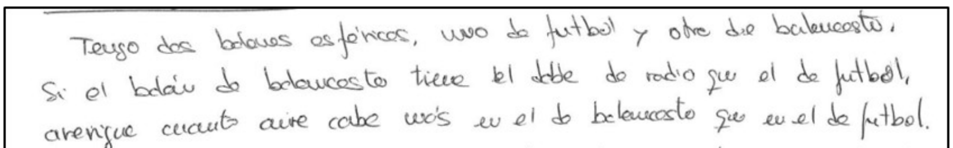
When purchasing tickets online, a fixed service fee is typically charged. This fee does not depend on the number of tickets purchased. I bought two tickets for Taylor Swift online and paid 35 euros (which includes a 5-euro service fee). How many tickets can I buy if I have 70 euros in my savings account?

Figure 3: Proportionality problem in arithmetic context (PT6)

For instance, in the missing value problem included in Figure 3, a direct proportionality relationship is established between the number of tickets for the concert and the price paid for them (excluding the service fee).

Participants encountered significant challenges when posing proportionality problems in the geometric context. Four PTs did not propose any problems, indicating uncertainty about applying proportionality in geometry compared to arithmetic, and a general aversion to the subject. Additionally, two submitted non-significant problems (ambiguous, lack of regularity conditions), and another provided a significant problem that, nonetheless, failed to incorporate proportional reasoning. Successful submissions typically involved the similarity of geometric figures, primarily triangles and, in one case, squares, aiming to calculate lengths (in three instances) or areas (in another three).

They also proposed two statements on scales (determining length from scale, or the relationship between scales) and one statement that demanded the relationship between the volumes of two spheres (Figure 4).


Tengo dos balones esféricos, uno de futbol y otro de baloncesto. Si el balón de baloncesto tiene el doble de radio que el de futbol, averigue cuanto aire cabe más en el de baloncesto que en el de futbol.

I have two spherical balls, one soccer ball and one basketball. If the basketball has twice the radius of the soccer ball, find out how much more air the basketball can hold than the soccer ball.

Figure 4: Proportionality problem in geometric context relationship between volumes of spheres (PT1)

Posing problems that involve proportional reasoning in the probabilistic context also proved complex. Four PTs did not respond to this task (three of whom also did not do so in the geometric context and two also not in the arithmetic), another four proposed non-significant problems, due to the information provided not allowing the problem to be solved, and one proposed a significant problem, but not pertinent as it did not involve proportional reasoning ("In the lottery the probability of winning is 1/1000. If you play every week, does it increase?", PT2).

In this context, the pertinent problems primarily required calculating and comparing simple probabilities, with exceptions including a fair play scenario, determining the composition of an urn based on a given probability, and a compound experiment. For example, PT3's problem asked, "If a basketball player makes 3 out of every 10 shots, how many shots must he average to make at least 17?" The ratio was interpreted as the probability of success, fitting a missing value problem of direct proportionality, where "on average" suggests uncertainty. In this case, PT3 considered that the difficulty was due to "the number of shots will not come out whole, hence the word at least in the statement, because they will have to add 1 to the integer part of that number."

Participants needed to solve the problems they created and identify the involved mathematical objects and processes. Out of 14 arithmetic context problems, seven were correctly solved, identifying proportional reasoning through the direct proportionality relationship between the magnitudes. PT6, for instance, identified in relation to the problem included in Figure 3, "Proportional reasoning appears in the relationship of Dir. Prop [direct proportionality] between Entries and money", indicating later as objects the "ratio €/entry, direct proportionality between € and entries" and as mathematical processes, "signification", which links to interpreting the commission, "algorithmization", which he understands as applying the unit reduction technique to solve the problem and "argumentation". However, the PTs generally did not identify objects beyond basic concepts of ratio, fraction, and direct proportionality, nor did they recognize processes beyond basic arithmetic operations and rule of three (the routine procedure to find the fourth term of a proportion when the other three are given, based on the equality of the cross product of the means and extremes of the proportion) (Arican et al, 2023), except PT3 who mentioned "generalization" relating to regularity.

The PTs assessed problem complexity based on the difficulty level for students. The 10 participants who made some reflection on this point highlight aspects that affect the greater or lesser difficulty of one task compared to another. Most found their problems to be of low difficulty, citing "clear language" and straightforward steps, although PT6 and PT11 noted medium-high difficulty due to the need to distinguish between different types of relationships from the additive (the service fee) and handle non-whole ratios when the magnitudes were discrete ("fraction of an egg in a cooking recipe", "contradictions may arise").

From an expert viewpoint, all the proposed problems corresponded to a level of cognitive demand for memorization or procedures without connection according to Stein et al. (1996), except for a couple of problems in the level of procedures with connection where it was necessary to compare ratios in compound proportionality (PT16) or obtain the relationship between flows of two hoses knowing the relationship between the volume (PT1). In both cases, the PTs indicated that the problems were not complex.

In the case of problems in the geometric context, only five PTs solved their problem and correctly identified proportional reasoning. Two others did not solve it but did indicate how proportional reasoning was involved: through the use of scale or the similarity relationship between figures. In this case, only four PTs identified some concepts and properties (right triangle, similar triangles, proportionality, area, similarity of triangles) and only one PT also mentioned the processes of modeling, representation, and argumentation. Regarding the complexity of the problems posed in this context, only nine of the participants made any mention of it. It is observed that, although they mostly still think that their problems are not difficult, they consider them more complex than those posed in the arithmetic context, referring to the context itself ("relate the proportion to the scale characteristic", PT10; "use concepts of geometry and be less explicit the method of solution", PT8) or the use of new magnitudes. For example, PT3 considers that his/her problem "is difficult due to the paradigm shift that occurs when changing from direct to quadratic proportionality" and PT1 considers that in his/her problem (Figure 4).

> *"Students must find proportional relationships among 3D figures. They need to understand that they are asked to find the relationship between the volumes of the spheres, disguised as elements common to their environment. That is, finding that magnitude may pose an additional problem".*

In this case, the cognitive demand of the proposed problems is higher than those posed in the arithmetic context (only one of them is memorization and there are five that involve procedures with connection, for example, the one included in Figure 4).

Only three PTs solved the proposed probability problem and another two, although did not solve it, specified that proportional reasoning was involved in the calculation and comparison of probabilities or in the use of Laplace's Rule. The objects indicated by these five PTs included the concepts of probability, favorable cases, possible, simple probability, conditioned probability, frequentist probability and only PT3 indicates as a process the generalization (linked to the assumption of equiprobability).

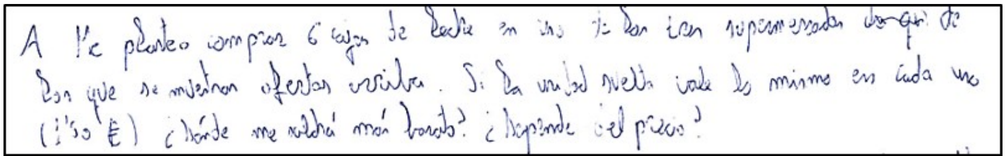
Of the eight PTs who reflected on the degree of complexity of their problem, three indicated that it was low, considering that only "basic concepts" are used (PT14), that the statement is clear (direct) and few steps are needed (PT13), while the others considered that the complexity is greater due to the context itself and the implications it has on the relationships between the numbers ("thinking if a fraction that expresses a probability makes sense", PT2; "understanding the frequentist meaning of probability", PT11). In this respect, except for one problem at the level of memorization and another of procedure without connection, all the problems created in the context of probability responded to the level of procedure with connection.

Creation of Proportionality Problems from Situations

In Task 2 (Figure 2) the PTs had to create proportionality problems from situations described by images, in different contexts and with the intention that their solution involve certain fundamental knowledge of proportional reasoning.

In situation A, which describes three common commercial offer options (50% discount on the second item, buy 2 and get 3 products, and 70% discount on the second item), the given context is arithmetic and the problem posed should motivate reflection on the proportional nature of percentages. In this case, two PTs did not propose any problem; another three formulated non-significant problems (lack of requirement or incomplete information) and another participant posed a problem that although it was significant substantially changed the information (offer 3x2 and second unit 60%).

The rest (ten PTs) created pertinent problems, that is, significant, elaborated from the situation and involving the proportional nature of the percentage. Most of these problems (seven) lead to deciding which is the best offer in the purchase of a given number of units assuming that the price of the unit is always the same, known (in three cases) or unknown (in the rest) (see, for example, Figure 6). In two other cases, it must be decided in which offer less is paid per unit ("which offer implies a lower cost per product", PT5; understanding that two units or three units are bought, depending on the offer).

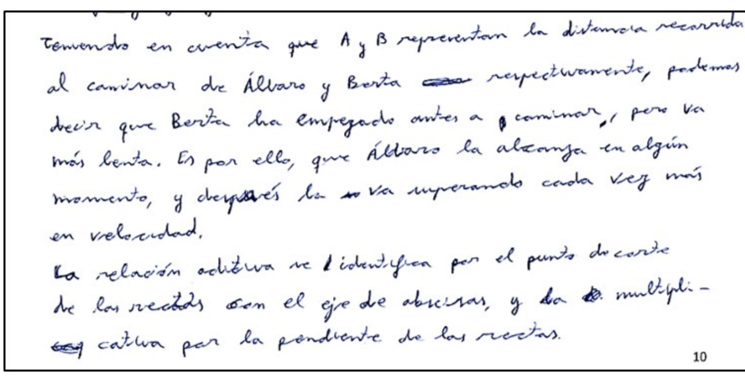

A Me planteo comprar 6 cajas de leche en uno de los tres supermercados donde se muestran ofertas arriba. Si la unidad suelta vale lo mismo en cada uno (1'50 €) ¿Dónde me saldrá más barato? ¿Depende del precio?

I am considering buying 6 boxes of milk from one of the three supermarkets showing offers above. If the price per individual unit is the same at each one (1.50 euros), where will it be cheaper? Does it depend on the price?

Figure 5: Elaboration from situation A (PT11) pertinent problem

In three statements, as shown in Figure 5, the PTs asked how the price of an item influences which offer is superior. Although they solved the problems, they seldom discussed how proportional reasoning and percentages are involved, except for PT11 (problem posed in Figure 5) who associated proportionality with discounts, noting, "It does not depend on the price. Note that the proportionality of these discounts is used where it is maintained (in pairs or trios)," and PT4 who stated, "the ratio/relationship of what you pay per unit is used." Generally, the PTs used the percentage as an operator to calculate and determine the proportional part of the unit price for each offer to then compare them.

Posing problems from situation B, featuring a joint graph of a linear and an affine function, proved challenging for the PTs. It was asked that the problem inspire students to use proportionality properties. Five PTs failed to devise any problems, four presented non-significant problems (ambiguous or insufficient information), two offered significant but unrelated problems (involving shopping, comparing two affine functions), and one created a relevant but not proportionality-focused problem (calculating function intersections). In the pertinent responses, the PTs needed to align the functions to the graphical representation, interpreting slope and the "additive part" in the affine function, as shown in Figure 6.


Teniendo en cuenta que A y B representan la distancia recorrida al caminar de Álvaro y Berta respectivamente, podemos decir que Berta ha empezado antes a caminar, pero va más lenta. Es por ello, que Álvaro la alcanza en algún momento, y después la va superando cada vez más en velocidad.
La relación aditiva se identifica por el punto de corte de las rectas con el eje de abscisas, y la multiplicativa por la pendiente de las rectas.
10

Considering that A and B represent the distance walked by Alvaro and Berta respectively, we can say that Berta started walking earlier, but she is slower. That is why Alvaro catches up with her at some point, and then increasingly surpasses her in speed.

The additive relationship is identified by the point where the lines intersect the x-axis, and the multiplicative relationship is identified by the slope of the lines.

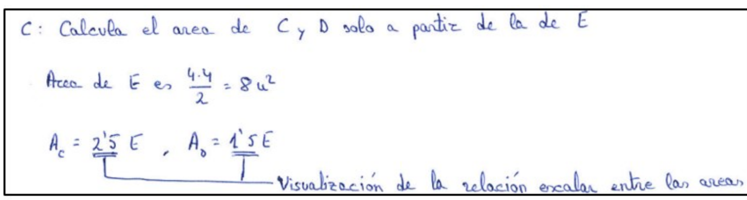
Figure 6: Problem created by elaboration from situation B and solution (PT14)

When solving the problem, on rare occasions did they mention how the didactic-mathematical purpose is contemplated. Nevertheless, it is possible to observe some confusion with the additive and multiplicative properties of the proportionality relation. Either they associated the multiplicative property with the slope or growth of the lines (Figure 6), or they consider that this is a possible relationship between the slopes ("the slopes are proportional"; PT11, PT16). As for the additive property, it seems to be interpreted (Figure 4) as the point of intersection with the abscissa axis (so that in a linear function that additive part would be 0) or also as the difference between the functions for x=0 ("original difference", PT11).

Finally, only six PTs posed a problem for situation C, openly stating their inability to create a proportionality problem from this situation (PT3) or showing their difficulty in specifying the ideas in a statement:

> *"I can't think of anything beyond asking them to relate quantities such as the areas of the polygons shown in the image. It would be interesting to ask them to explain how the area of one of them would vary if the numerical values shown in the figure were modified.*" (PT4)

Of the six problems posed, all significant non-pertinent, two did not take into account the starting situation and the other four, although they did incorporate the puzzle of C in their approach, did not consider the intended didactic-mathematical purpose or the context was arithmetic (Figure 7).



Calculate the area of C and D based solely on E.
Area of E is $\frac{4\cdot 4}{2} = 8u^2$
$A_C = 2.5\ E\ ; A_D = 1.5\ E$
Visualization of the scalar relationship between the areas.

Figure 7: Problem elaborated by PT4 from situation C

As shown in Figure 7, PT4 asked to determine the area of two pieces of the puzzle (C and D) using as a unit of measure the piece E, obtaining "how many times E fits" in C and D (two and a half times, as a decimal 2.5 in C and one and a half times, as a decimal, 1.5, in D). Regarding the properties of proportionality, PT4 mentioned the "scalar relationship between areas", however, no proportionality relation between magnitudes is established.

While in the previous task of free creation, the PTs utilized the concept of similarity of figures (relationship between lengths or areas of similar figures), the notion of scale, or the application of Thales' theorem, they did not resort to these concepts or know how to incorporate them into the established purpose when starting from Situation C. As shown in Figure 8, PT12 used Situation C to formulate the problem; however, it does not represent a geometric meaning of proportionality, as it proposes a relationship between the number of figures and the area they occupy.

situacion C: ¿es el numero de figuras de cada (A y D) tipo proporcional al área que ocupan en la figura?
Situation C: Is the number of figures of each type proportional to the area they occupy in the figure?

Figure 8: Proposal by PT12 of a problem from situation C

DISCUSSION AND CONCLUSIONS

Teachers, in addition to being competent in solving the problems they propose to their students, must also know how to choose, modify, and create them with an educational purpose (Malaspina et al., 2015). The creation of problems enhances students' conceptual understanding (Kaur & Rosli, 2021; Kılıç, 2017; Lee et al., 2018) and serves as a crucial tool for assessing what students know and do not know (Kwek, 2015). Furthermore, problem posing contributes to the development of mathematical knowledge during initial teacher training, prompting them to "rethink" the nature of mathematical objects before explicit instruction (Kılıç, 2017). Achieving this competency requires that teacher training includes the design and implementation of specific actions to ensure the necessary didactic-mathematical knowledge. When the goal is epistemic in nature, these knowledges involve aspects specific to the mathematical content, in this case, proportional reasoning. Since different application contexts of the notions of ratio and proportion involve the participation of specific objects and processes from those fields in the corresponding practices, ideal teaching of proportionality must consider the different meanings—arithmetic, algebraic-functional, geometric, probabilistic—in an articulated manner. The teacher must thus be able to pose proportionality problems in different contexts, motivating students to learn through their resolution the essential properties of the proportionality relationship.

In this article, we have reported the results of a formative experience with a group of prospective secondary teachers focused on the creation of proportionality problems with a didactic purpose. The findings described contribute to understanding what they value in a good problem, the difficulties they face in posing problems, and diagnosing their didactic-mathematical knowledge.

Regarding our first objective, the PTs' beliefs about what constitutes a good mathematical problem and what skills they need to develop them are not consistent with each other or with their proposed statements. For example, from an instructional perspective, PTs consider one of the characteristics of a good mathematical problem to be a clear and unambiguous statement (Table 1), yet they designed non-significant problems (ambiguous) in both tasks. Similarly, PTs believe a good problem allows for various solution strategies, however, there is no evidence that they verified the possibility of addressing the requirement in their problems in more than one way, or that the problems could indeed be solved. This coincides with findings in research such as that of Bayazit and Kirnap-Donmez (2017). From a cognitive-affective standpoint, although they consider students' prior knowledge as one of the essential aspects for determining the complexity of the problems (Table 2), they prioritize creativity or the ability to cater to their interests as necessary skills for creating good problems (Table 3).

Concerning the second and third objectives, the PTs' competency to create pertinent problems is adequate in the arithmetic context but very limited in the functional, geometric, and probabilistic contexts. This reveals that the knowledge required to create a mathematical problem in one context does not transfer to another and that the success in developing problems is determined by the field in which they are framed. It is significant, however, that the complexity of the problems created was higher in the probabilistic and geometric contexts than in the arithmetic one.

Like in previous research (Burgos et al., 2018; Burgos & Chaverri, 2022; Mallart et al., 2016), PTs (despite their higher mathematical education) had limitations in identifying the mathematical objects and processes emerging from the solution to the proposed problems. This deficiency could explain why some significant problems were not pertinent as they did not respond to the epistemic purpose established in the task. Despite this, the few PTs who reflected on the complexity of their proposed problems explicitly mentioned objects and processes (even though they had not referred to them before). This suggests a need in secondary PT training to "strengthen the study of task complexity based on the analysis of the mathematical objects and processes involved in their resolution" (Burgos & Chaverri, 2022).

Although previous studies like those of Şengül and Katranci (2015) or Bayazit and Kirnap-Donmez (2017) observed that teachers in training had less success in freely creating problems than when doing so in a semi-structured manner (in an arithmetic context), our research shows better performance in the free case than in the semi-structured. In this case, responding to the didactic purpose established in the guideline could have been the biggest obstacle. The PTs need to recognize from the information provided which magnitudes can be related proportionally, know and recognize in the situation the properties of the functional relationship, and establish the requirement in such a way that solving the problem requires using that relationship and its properties. As observed in previous research with primary school teachers in the arithmetic context (Burgos & Chaverri, 2022), behind this difficulty could be an insufficient knowledge of proportional reasoning in the probabilistic and geometric contexts (Batanero et al., 2015; Copur-Gencturk, et al., 2023).

Aware of the limitation that the sample size imposes for generalizing the results, it is necessary to continue researching and making didactic proposals for problem posing, paying attention to the influence of context, educational purpose, and the involved didactic-mathematical knowledge. Considering the results obtained, in new interventions it would be necessary to know the mathematical and didactic-mathematical knowledge of prospective teachers, since the training received in their undergraduate and master's degrees in teaching might not be as expected. Similarly, it would be advisable to reinforce their knowledge of proportional reasoning in the geometric context and even more so, the probabilistic one. Also, dealing with structured situations, which were not considered in this research, and incorporating inverse proportionality. In this last case, there might be a lower success rate, given the more extensive and cross-sectional presence of direct proportionality in the curriculum compared to the inverse, and the potential interferences between both structures caused by prior knowledge (Parameswari et al., 2024). It would be desirable to design tasks that focus on a single context, for example, functional, geometric, or probabilistic, since they were the least successful. It would also be advisable to address cognitive-type purposes, for example, creating problems with a certain complexity or cognitive demand. As suggested by Crespo and Harper (2020) "promoting students' reasoning and problem-solving depends on the teacher's ability to identify, construct, and pose mathematically rich problems from a cognitive perspective" (p. 1).

The flexibility of the methodology allows designing and implementing new interventions with practicing teachers from different educational stages, in which we would expect better results than the current ones, although research suggests that, despite their experience, they have difficulties in creating problems (Kaur & Rosli, 2021). The integration or expansion of spaces where PTs can practice and develop the competency of problem creation helps to bridge the gap between teachers' knowledge and their teaching practices (Lee et al., 2018).

Acknowledgments

Research conducted as part of the Grant PID2022-139748NB-I00 funded by MICIU/AEI/10.13039/501100011033 and "FEDER/EU", with support from the Research Groups FQM-126 (Junta de Andalucía, Spain) and S60_23R (Government of Aragon, Spain).

References

[1] Akay, H., & Boz, N. (2010). The Effect of Problem Posing Oriented Analyses-II Course on the Attitudes toward Mathematics and Mathematics Self-Efficacy of Elementary Prospective Mathematics Teachers. *Australian Journal of Teacher Education, 35*(1), 59–75. https://doi.org/10.14221/ajte.2010v35n1.6

[2] Arce, M., Conejo, L., & Muñoz-Escolano, J.M. (2019). *Aprendizaje y enseñanza de las matemáticas*. Síntesis.

[3] Arican, M., Verschaffel, L., & Van Dooren, W. (2023). Preservice middle school mathematics teachers' strategy repertoire in proportional problem solving. *Research in Mathematics Education,* 1-21. https://doi.org/10.1080/14794802.2023.2212260

[4] Aroza, C. J., Godino, J. D., & Beltrán-Pellicer, P. (2016). Iniciación a la innovación e investigación educativa mediante el análisis de la idoneidad didáctica de una experiencia de enseñanza sobre proporcionalidad. *Aires,* *6*(1), 1–29. https://enfoqueontosemiotico.ugr.es/documentos/Aroza_Godino_Beltran.pdf

[5] Baumanns, L., & Rott, B. (2021). Rethinking problem-posing situations: A review. *Investigations in Mathematics Leaning,* *13*(2), 59–76. https://doi.org/10.1080/19477503.2020.1841501

[6] Bayazit, I., & Kirnap-Donmez, S. M. (2017). Prospective teachers' proficiencies at problem posing in the context of proportional reasoning. *Turkish Journal of Computer and Mathematics Education, 8*(1), 130–160. https://doi.org/10.16949/turkbilmat.303759

[7] Batanero, C., Gómez, E., Contreras, J. M., & Díaz, C. (2015). Conocimiento matemático de profesores de primaria en formación para la enseñanza de la probabilidad: Un estudio exploratorio. *Práxis Educativa 10*(1), 11-34. https://doi.org/10.5212/PraxEduc.v.10i1.0001

[8] Begolli, K.N., Dai, T., McGinn, K.M., & Booth, J. L. (2021) Could probability be out of proportion? Self-explanation and example-based practice help students with lower proportional reasoning skills learn probability. *Instructional Science 49*, 441–473. https://doi.org/10.1007/s11251-021-09550-9

[9] Ben-Chaim, D., Keret, Y., & Ilany, B. (2012). *Ratio and proportion: Research and teaching in mathematics teachers' education*. Sense Publisher. https://doi.org/10.1007/978-94-6091-784-4

[10] Bryant, P., & Nunes, T. (2012). *Children's understanding of probability: A literature review* (full report). London: The Nuffield Foundation.

[11] Burgos, M., Beltrán-Pellicer, P., Giacomone, B., & Godino, J. D. (2018). Conocimientos y competencia de futuros profesores de matemáticas en tareas de proporcionalidad. *Educação e Pesquisa, 44*, 1–22. https://doi.org/10.1590/s1678-4634201844182013

[12] Burgos, M., & Castillo, M. J. (2022). Developing Reflective Competence in Preservice Teachers by Analysing Textbook Lessons: The Case of Proportionality. *Mathematics Teaching Research Journal, 14*(4), 165-191.

[13] Burgos, M., & Chaverri, J. (2022). Knowledge and Competencies of Prospective Teachers for the Creation of Proportionality Problems. *Acta Scientiae, 24*(6), 270–306. https://doi.org/10.17648/acta.scientiae.7061

[14] Burgos, M., & Chaverri, J. (2023a). Creación de problemas de proporcionalidad en la formación de docentes de primaria. *Uniciencia, 37*(1), 1–24. http://dx.doi.org/10.15359/ru.37-1.14

[15] Burgos, M., & Chaverri, J. (2023b). Explorando la percepción de futuros maestros de primaria sobre el pensamiento matemático de los alumnos en un problema de proporcionalidad. *Aula Abierta, 52*(1), 43–53. https://doi.org/10.17811/rifie.52.1.2023.43-52

[16] Cai, J., & Hwang, S. (2020). Learning to teach through mathematical problem posing: Theoretical considerations, methodology, and directions for future research. *International Journal of Educational Research, 102*, 101391. https://doi.org/10.1016/j.ijer.2019.01.001

[17] Cohen, L., Manion, L., & Morrison, K. (2018). *Research methods in education* (8va ed.). Routledge. https://doi.org/10.4324/9781315456539

[18] Copur-Gencturk, Y., Baek, C., & Doleck, T. A. (2023) Closer Look at Teachers' Proportional Reasoning. *Int J of Sci and Math Educ, 21*, 113–129. https://doi.org/10.1007/s10763-022-10249-7

[19] Cramer, K., & Post, T. (1993). Connecting Research to Teaching Proportional Reasoning. *Mathematics Teacher, 86*(5), 404–407.

[20] Crespo, S., & Harper, F. K. (2020). Learning to pose collaborative mathematics problems with secondary prospective teachers. *International Journal of Educational Research, 102*. https://doi.org/10.1016/j.ijer.2019.05.003

[21] Cruz, M. (2006). A mathematical problem–formulating strategy. *International Journal for Mathematics Teaching and Learning*, 79–90.

[22] Cuevas-Vallejo, A., Islas-Ortiz, E., & Orozco-Santiago, J. (2023). Promover el razonamiento proporcional mediante la tecnología digital. *Apertura, 15*(1), 84–101. http://doi.org/10.32870/Ap.v15n1.2344

[23] Dole, S. (2010). Promoting Percent as a Proportion in Eighth-Grade Mathematics. *School Science and Mathematics, 10*(7), 345–396. https://doi.org/10.1111/j.1949-8594.2000.tb18180.x

[24] Ellerton, N. F. (2013). Engaging pre-service middle-school teacher-education students in mathematical problem posing: development of an active learning framework. *Educational Studies in Mathematics*, *83*(1), 87–101. https://doi.org/10.1007/s10649-012-9449-z

[25] Fernández, C., & Llinares, S. (2010). Evolución de los perfiles de los estudiantes de primaria y secundaria cuando resuelven problemas lineales. In M. Moreno, J. Carrillo, & A. Estrada (Eds.), *Investigación en Educación Matemática XIV* (pp. 281-290). SEIEM.

[26] Font, V., Breda, A., & Seckel, M. J. (2017) Algunas implicaciones didácticas derivadas de la complejidad de los objetos matemáticos cuando estos se aplican a distintos contextos, *Revista Brasileira de Ensino de Ciência e Tecnologia, 10*(2), 1–23. https://doi.org/10.3895/rbect.v10n2.5981

[27] Font, V., Godino, J. D., & Gallardo, J. (2013). The emergence of objects from mathematical practices. *Educational Studies in Mathematics*, *82*, 97–124. https://doi.org/10.1007/s10649-012-9411-0

[28] Godino, J. D., Batanero, C., & Font, V. (2007). The onto-semiotic approach to research in mathematics education. *ZDM. The International Journal on Mathematics Education, 39*(1), 127–135. https://doi.org/10.1007/s11858-006-0004-1

[29] Godino, J. D., Giacomone, B., Batanero, C., & Font, V. (2017). Enfoque Ontosemiótico de los Conocimientos y Competencias del Profesor de Matemáticas. *Bolema, 31*(57), 90–113. https://doi.org/10.1590/1980-4415v31n57a05

[30] Grundmeier, T. (2015). Developing the Problem-Posing Abilities of Prospective Elementary and Middle School Teachers. In F.M. Singer et al. (eds.), *Mathematical Problem Posing, Research in Mathematics Education*. https://doi.org/10.1007/978-1-4614-6258-3_20

[31] Hilton, A., & Hilton, G. (2019). Primary school teachers implementing structured mathematics interventions to promote their mathematics knowledge for teaching proportional reasoning. *Journal of Mathematics Teacher Education, 22*, 545–574. https://doi.org/10.1007/s10857-018-9405-7

[32] Izsák, A., & Jacobson, E. (2017). Preservice teachers' reasoning about relationships that are and are not proportional: A knowledge-in-pieces account. *Journal for Research in Mathematics Education, 48*(3), 300–339. https://doi.org/10.5951/jresematheduc.48.3.0300

[33] Kaur, A., & Rosli, R. (2021). Problem Posing in Mathematics Education Research: A Systematic Review. *International Journal of Academic Research in Progressive Education and Development, 10*(1), 438–456. https://doi.org/10.6007/IJARPED/v10-i1/8641

[34] Kılıç, Ç. (2017). A new problem-posing approach based on problem-solving strategy: Analyzing pre-service primary school teachers' performance. *Educational Sciences: Theory & Practice, 17*, 771–789. http://dx.doi.org/10.12738/ estp.2017.3.0017

[35] Kwek, M. L. (2015). Using problem posing as a formative assessment tool. In F. Singer, N. Ellerton y J. Cai (Eds.), *Mathematical problem posing: from research to effective practice* (pp. 273–292). Springer. https://doi.org/10.1007/978-1-4614-6258-3_13

[36] Lee, Y., Capraro, R. M., & Capraro, M. M. (2018). Mathematics Teachers' Subject Matter Knowledge and Pedagogical Content Knowledge in Problem Posing. *International Electronic Journal of Mathematics Education, 13*(2), 75–90. https://doi.org/10.12973/iejme/2698

[37] Li, X, Song, N., Hwang, S., & Cai, J. (2020). Learning to teach mathematics through problem posing: teachers' beliefs and performance on problem posing. *Educational Studies in Mathematics, 105*, 325–347. https://doi.org/10.1007/s10649-020-09981-0

[38] Malaspina, U. (2013). La creación de problemas de matemáticas en la formación de profesores. En SEMUR, *Sociedad de Educación Matemática Uruguaya (Ed.), VII Congreso Iberoamericano de Educación Matemática* (pp. 129–140). SEMUR.

[39] Malaspina, U., Mallart, A., & Font, V. (2015). Development of teachers' mathematical and didactic competencies by means of problem posing. In K. Krainer y N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 2861–2866). Proceedings of the CERME 9. https://hal.archives-ouvertes.fr/hal-01289630

[40] Malaspina, U., Torres, C., & Rubio, N. (2019). How to stimulate in-service teachers' didactic analysis competence by means of problem posing. In P. Liljedahl, y L. Santos-Trigo (Eds.),

Mathematical Problem Solving (pp. 133–151). Suiza: Springer. https://doi.org/10.1007/978-3-030-10472-6_7

[41] Mallart, A., Font, V., & Malaspina, U. (2016). Reflexión sobre el significado de qué es un buen problema en la formación inicial de maestros. *Perfiles educativos, 38*(152), 14–30. https://doi.org/10.22201/iisue.24486167e.2016.152.57585

[42] Mallart, A., Font, V., & Diez, J. (2018). Case Study on Mathematics Pre-service Teachers' Difficulties in Problem Posing. *Eurasia Journal of Mathematics, Science and Technology Education, 14*(4), 1465–1481. https://doi.org/10.29333/ejmste/83682

[43] Martínez-Juste, S. (2022). *Diseño, implementación y análisis de una propuesta didáctica para la proporcionalidad en el primer ciclo de Secundaria*. [Doctoral dissertation, University of Valladolid]. https://uvadoc.uva.es/handle/10324/52863

[44] Martínez-Juste, S., Muñoz-Escolano, J. M., & Oller-Marcén, A. M. (2014). Tratamiento de la proporcionalidad compuesta en cuatro libros de texto españoles. In M. T. González, M. Codes, D. Arnau, & T. Ortega (Eds.), *Investigación en Educación Matemática XVIII* (pp. 435–444). SEIEM.

[45] Oller-Marcén, A. M., & Gairín, J. M. (2016). Proportionality problems in some mathematical texts prior to fourteenth century. In K. Krainer, & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education (CERME 9)* (pp. 1859–1865).

[46] Parameswari, P., Purwanto, Sudirman, & Susiswo (2024). Students' Proactive Interference in Solving Proportion Problems: How Was the Met-Before? *Mathematics Teaching Research Journal, 15*(6), 93-115.

[47] Parker, M., & Leinhardt, G. (1995) Percent: a privileged proportion. *Review of Educational Research, 65*(4), 421–481. https://doi.org/10.2307/1170703

[48] Prabhu, V. & Czarnocha, B. (2013). Problem posing, problem solving dynamics in the context of teaching- research and discovery method. *Mathematics Teaching-Research Journal, 6*(1&2), 100-122.

[49] Şengül, S., & Katranci, Y. (2015). The analysis of the problems posed by prospective mathematics teachers about 'ratio and proportion' subject. *Procedia. Social and Behavioral Sciences, 174*, 1364–1370. https://doi.org/10.1016/j.sbspro.2015.01.760

[50] Silver, E. A., & Cai, J. (2005). Assessing students' mathematical problem posing. *Teaching children mathematics, 12*(3), 129–135. https://doi.org/10.5951/TCM.12.3.0129

[51] Stein, M. K., Grover, B. W., & Henningen, M. (1996). Building student capacity for mathematical thinking and reasoning: an analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*, 455–488. https://doi.org/10.3102/00028312033002455

[52] Steinthorsdottir, O. B. (2006). Proportional reasoning: Variable influencing of the problems difficulty level and one's use of problem solving strategies. In J. Novotná, H. Moraová, K. M., & N. Stehliková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 169–176). PME.

[53] Stoyanova, E. & Ellerton, N.F. (1996). A framework for research into students' problem posing. In P. Clarkson (Ed.), *Technology in mathematics education* (pp. 518–525). Mathematics Education Research Group of Australasia.

[54] Tichá, M., & Hošpesová, A. (2013). Developing teachers' subject didactic competence through problem posing. *Educational Studies in Mathematics, 83*(1), 133–143. https://doi.org/10.1007/s10649-012-9455-1

[55] Tinoco, J. C., Albarracín, L., & Deulofeu, J. (2021). Estrategias de proporcionalidad simple en las aulas de Matemáticas y de Física. In P.D. Diago, D.F. Yáñez, M.T. González-Astudillo, & D. Carrillo (Eds.), *Investigación en Educación Matemática XXIV* (pp. 587 – 594). SEIEM.

[56] Van Harpen, X. Y., & Presmeg, N. C. (2013). An investigation of relationships between students' mathematical problem-posing abilities and their mathematical content knowledge. *Educational Studies in Mathematics, 83*, 117–132. https://doi.org/10.1007/s10649-012-9456-0

[57] Weiland, T., Orrill, C.H., Nagar, G.G., Brown, R., & Burke, J. (2020). Framing a robust understanding of proportional reasoning for teachers. *Journal of Mathematics Teacher Education, 24*(2), 179–202. https://doi.org/10.1007/s10857-019-09453-0

[58] Xie, J., & Masingila, J. (2017). Examining interactions between problem posing and problem solving with prospective primary teachers: A case of using fractions. *Educational Studies in Mathematics*, *96*(1), 101–118. https://doi.org/10.1007/s10649-017-9760-9

Harmony in Teaching: Unraveling the Interplay between Pre-Service Teachers' Mathematical Knowledge Fractions and Classroom Practices

Charles Kwabena Sie[1], Douglas Darko Agyei[2]

[1]Nsawkaw College of Education, Tain District, Ghana
[2] Department of Mathematics and ICT Education, University of Cape Coast, Ghana
Siecharles78@gmail.com, ddagyei@ucc.edu.gh

Abstract: This study delves into the intricate relationship between pre-service teachers' (PSTs') Mathematical Knowledge for Teaching Fractions (MKTF) and its influence on their teaching practices. Grounded in the premise that MKTF domains exhibit interconnectivity, shaping the constructs of teaching practices, the study employed the mathematical task framework and the framework for mathematical knowledge for teaching. Utilizing the Mathematical Knowledge for Teaching Fractions test and the Teaching Practices test, data were collected from 171 PSTs. Regression analyses uncovered significant effects of MKTF domains on five teaching practice components, underscoring the pivotal role of a teacher's mathematical knowledge in effective teaching. Notably, among the six MKTF domains, the KCFS domain emerged as the most fundamental, strongly predicting various MKTF domains and influencing teaching practice constructs. This study underscores the significance of the KCFS domain in shaping both MKTF domains and instructional practices. The findings bear implications for the education of PSTs in Ghana and other nations facing similar educational landscapes.

Keywords: Mathematical knowledge for teaching, Teaching practices, Mathematical task Framework, Lesson script

INTRODUCTION

Studies have shown that mathematical knowledge for teaching influences teaching practice (Hoover et al. 2016). These influences are however not clear since there is a complex connection between mathematical knowledge for teaching and teaching practices that lead to the quality of instruction. Notwithstanding this, Hill, Umland, Litke, and Kapitula (2012) study have reported that, weak mathematical knowledge for teaching predicts low quality teaching practices, and strong mathematical knowledge for teaching predicts high quality teaching practices. On the other hand, Hill et al. (2008) suggest that there are other factors such as: professional development, supplemental curriculum materials and teacher beliefs that potentially have influence on the quality of teaching practices, but these factors may cut both ways depending on the teachers' mathematical knowledge for teaching. In addition, efforts to clarify the conceptualization of mathematical knowledge for teaching continue to be concerned with the dynamic nature of mathematical knowledge for teaching, the usefulness of knowledge, and whether, when, and how it plays in teaching (Hoover et al, 2016; Kersting et al, 2012). Although mathematical knowledge for teaching (MKT) have been shown to have influence on teaching practices, the question of what kind of teaching tasks require which domain, still require further attention (Ball et al., 2008; Markworth, Goodwin, & Glisson, 2009). These discussions suggest that along teachers' mathematical knowledge, teachers' teaching practices are key in order to produce lessons in which learners will be exposed to high quality tasks that help them to learn concepts and procedures in mathematics with understanding (Addae & Agyei, 2018). This in turn produces in learners' self-confidence to engage in challenging mathematical tasks that are provided in a rich mathematics curriculum (NCTM, 2000).

From analysis of video-recorded classroom observations and teacher interviews, Cengiz, Kline and Grant (2011) provide detailed accounts of teaching and "demonstrate that MKT matters in the way teachers pursue student thinking" (Cengiz et al., 2011). Their analysis of data from one of the participating teachers "provide evidence that a lack of certain aspects of knowledge can negatively impact a teacher's pursuit of student thinking" (p. 372). Steele and Rogers (2012) argue that the more experienced and MKT-knowledgeable teacher not only enacts a stronger and more nuanced lesson on mathematical proof, but her students end up having more mathematical authority. A study by Tanase (2011) suggests that teachers' knowledge goes beyond their own mathematical understanding. Differences are observed in teachers' ability to make connections between fraction concepts and other mathematical concepts, how they set different objectives for students as well as the extent to which they challenge students in their mathematical work (Charalambous, 2008). Tanase also observed that teachers who have strong mathematical knowledge for teaching, are able to produce lessons with high quality of instruction. Johnson and Larsen's (2012) study posit that teachers need not only knowledge of students' misconceptions, but also knowledge of when and why students are likely to be confused and display misconceptions and of the consequences of such misconceptions when students engage in new activities. In his study of mathematics teacher knowledge and its impact on how teachers engage students with challenging tasks, Choppin (2011) noted that teacher's knowledge appears to influence teaching in the adaptation of tasks. Engaging students with challenging tasks is an important component of the work of teaching mathematics, and so is the selection and use of appropriate examples. Charalambous (2008) explored the relationships between pre-service teachers' MKT and their five teaching practices (selecting and using tasks, using representations; providing explanations, responding to students' requests for help and analysing student's work/contributions) required for quality teaching. Charalambous study did not show the impact of the domains of MKT on the five teaching practices. Additionally, his study considered only two mathematical knowledge for teaching domains (common content knowledge and special content knowledge).

Similar to Charalambous (2008) study, our previous study (Sie & Agyei, 2023) also identified significant relationships between pre-service teachers' MKTF domains and five teaching practices constructs. The present study, unlike the previous studies (Charalambous, 2008; Sie & Agyei, 2023) focused on examining the impact of the domains of MKT on the five teaching practices constructs. Applying the mathematical knowledge for teaching domains and the mathematical tasks framework we hypothesis that mathematical knowledge for teaching domains contribute to the ability to perform the five teaching practices.

Conceptual Framework for the Study

Ball et al. (2008) developed their conceptualization of MKT, which is based on the actual teaching practices of mathematics teachers, to include six knowledge domains: Common Content Knowledge (CCK), Specialized Content Knowledge (SCK), Horizon Content Knowledge (HCK), Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT), and Knowledge of Content and Curriculum (KCC). Researchers have recognized the following issues that still need more research on MKT as a common framework for understanding teachers' mathematical knowledge for teaching: (a) Which domain is required for what kinds of teaching tasks? (b) What connection exists between the MKT domains? (c) What are the MKT domains' actual definitions? (Ball et al., 2008; Markworth, Goodwin, & Glisson, 2009).

The design, presentation, and execution of tasks—the three phases of the instructional process—are analyzed using the Mathematical Task Framework (MTF) (Stein & Smith, 1998; Stein et al., 2000). According to Stein and Smith (1998), students engage in one of two forms of thinking depending on whether they are required to memorize methods in a systematic way (instrumental thinking) or think conceptually and draw connections during a task (relational thinking). This indicates that how tasks are selected and carried out during instruction has an impact on what students learn.

Based on the MTF, Charalambous (2008) suggested a few teaching practices under each of the three stages that instructional tasks go through. These teaching practices, which are thought to improve the quality of mathematics instruction, were used in the study. They included choosing and using tasks, using representations, giving explanations, responding to students' direct or indirect requests for help, and analyzing students' work and contributions. Teachers are expected to carry out specific duties (such as selecting instructional tasks, modifying/adapting instructional tasks, sequencing instructional tasks, and anticipating students' faults or difficulties) and create lesson plans during the planning phase. During the presentation phase, teachers are expected to present definitions, explain concepts, give examples and counterexamples, use analogies, use representations and manipulatives, establish connections between various concepts and representations, and simplify tasks to support student success. Additionally, during the enactment phase, teachers work alongside students on assigned activities or tasks while implementing specific teaching strategies (e.g., responding to students' help requests, monitoring and analyzing students' thinking, spotting mistakes, appreciating students' alternative approaches, posing probing questions, and facilitating the exchange of multiple ideas or solutions) (Charalambous, 2008; Fumador & Agyei, 2018). Despite being discussed and presented individually, these three phases have no distinct limits, according to Charalambous. Charalambous noted that though these three phases are discussed and presented separately, there are no clear boundaries between them. This study adopted a conceptual framework based on the earlier work of the authors (Sie & Agyei, 2023) that is depicted in Figure 1 by combining the theory of MKT with the mathematical task framework.

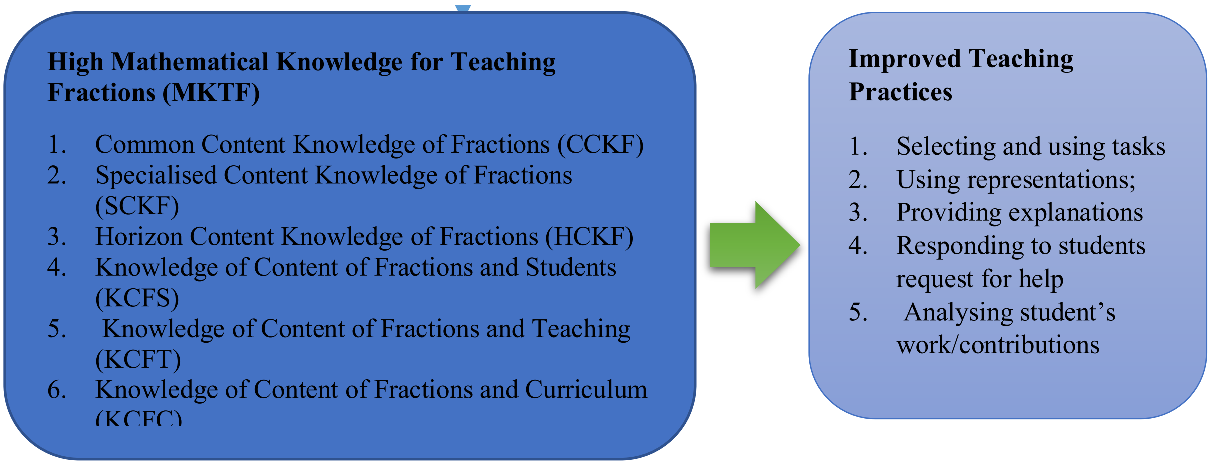


Figure 1: Conceptual framework of the study

From the figure, the study hypothesizes that mathematical knowledge for teaching domains have great impact on the five teaching practices constructs and that MKTF domains impacts positively to influence each other and influence the teaching practices constructs.

The MKTF domains are described in this study as follows:

- *Common Content Knowledge of Fractions (CCKF*) is the level of proficiency in fractional knowledge that comprises an understanding of the ideas, terms, definitions, rules, and symbols used in fractions that all workers who use fractions must possess.
- *Specialized Content Knowledge of Fractions (SCKF)* is the knowledge of: multiple solution strategies, generalizations, figuring out why an algorithm works or makes sense, explaining ideas by using appropriate examples and representations to visualize fractions, making connections between various representations, and figuring out actual definitions.
- *Horizon Content Knowledge of Fractions (HCKF)* refers to knowledge and awareness of how topics in the mathematics curriculum are related so that teachers can draw connections to topics while teaching fractions.
- *Knowledge of Content of Fractions and Students (KCFS*) refers to understanding of how students learn fractions, including understanding of frequent mistakes, common misconceptions, and challenges that students have in learning fractions.

- *Knowledge of Content of Fractions and Teaching (KCFT)* refers to understanding the fundamental concepts behind fractions as well as the various instructional approaches that can be utilized to teach them.

- *Knowledge of Content of Fractions and Curriculum (KCFC)* is the understanding of the contents as well as their organization that is required for teaching fractions at a particular level and at different levels.

The teaching practices constructs are also described as follows:

- *Selection and using instructional tasks* refer to the capacity to select, modify, and arrange instructional tasks in a way that challenges students' cognitive abilities to learn and establish connections for conceptual understanding.
- *Providing explanations* refers to the capacity of a teacher to give concise explanations that aid pupils in understanding the mathematics being taught. Here, a teacher creates and provides simple, student-understandable mathematical examples, counterexamples, and analogies.
- *Using representations* is the ability to enhance student learning by working with and around representational modes.
- *Analysing students' work and contributions* is the ability to evaluate student explanations and decipher what they say, to assess the validity of students' mathematical strategies and non-routine approaches to problem-solving, to assess what students know and their knowledge gaps based on their work and contributions or their errors
- *Responding to student's requests for help* refers to the ability of a teacher to respond to and attend to students' requests, either directly or indirectly.

Research Design and Questions

The study employed a correlational research design in which the researchers sought to identify the relationship between pre-service teachers MKTF and their teaching practices and identify how PSTs MKTF predict their teaching practices. The study addressed two main research questions: (i) what is the impact of each MKTF domains on the other domains? And (ii) what is the impact of each of the six MKTF domains (*CCKF, SCKF, HCKF, KCFT, KCFS, and KCFC*) on the five constructs of teaching practices?

METHOD

Respondents

The study's population was pre-service mathematics teachers at Ghana's 46 public colleges of education. To avoid the dangers of long-distance travel during the COVID outbreak, the available population of the study was chosen to include pre-service mathematics teachers from five colleges which were conveniently sampled. Out of 1445 prospective mathematics teachers from the five colleges of education, 171 were chosen at random to make up the study's sample using the stratified random sampling method.

Instruments

The MKTF domains and teaching practices components of the 171 PSTs were assessed using two tests, the Mathematical Knowledge for Teaching Fractions Test and the Teaching Practices Test, respectively. The sections below that follow discuss these instruments in more detail.

Mathematical Knowledge for Teaching Fractions Test

To assess PSTs' MKT in fractions, the researcher modified the online sample of the Learning Mathematics for Teaching (LMT) test items by Hill, Schilling, and Ball (2004). There are 64 test items on number, algebra, and operations in the online LMT sample test items. Several of the items on this instrument were found to be unrelated to the current study after analysis. As a result, 11 fraction-related items were chosen, altered, and used in the research. The online LMT test questions include questions that could assess knowledge in the CCK, SCK, KCT, and KCS knowledge domains of MKT. By reviewing earlier studies (Shulman, 1986; Ball, et al., 2008; Cole, 2012; Sugilar, 2016; Avcu, 2019) that highlighted the concepts and skills that instructors must possess in order to teach fractions properly, we were able to expand the LMT items by 33 to adequately cover all six knowledge domains of MKT (Ball, et al., 2008).

The Mathematical Knowledge for Teaching Fractions Test, which was composed of redesigned and modified LMT test items, had closed-ended questions. The responses of pre-service teachers to each question on the MKTF test were dichotomously evaluated on a 2-point scale: 0 for an incorrect response and 1 for an appropriate one. A total of 31 out of the test's 44 questions were scored and categorized into the six MKTF domains: CCKF (8 questions), SCKF (6 questions), HCKF (3 questions), KCFT (3 questions), KCFS (4 questions), and KCFC (7 questions). For ease of comparison, the total score for each MKTF domain was standardized to a maximum value of 8 points. A score of 4 was considered as the average score point value. A score of 4 or higher was regarded as a high MKTF score, while a score of lower than 4 was regarded as a low MKTF score. The MKTF domains' Kuder-Richardson reliabilities were higher than the acceptable cutoff value of 0.60, ranging from 0.64 to 0.82 (CCKF, =0.72; SCKF, =0.70; HCKF, =0.82; KCFT, =0.64; KCFS, =0.73; and KCFC, =0.79).

Teaching Practices Test

The five teaching practices of selecting and using tasks, using representations, providing explanations, responding to students' requests for assistance, and analyzing student work/contributions were all included in the measurement of teaching practices. For the study, the Teaching Practice Test was modified from Charalambous (2008) teaching practices interview guide. Charalambous used a 24-item interview guide to examine the effectiveness of 20 pre-service teachers in five different teaching practices. However, the adapted teaching practices test in this study included 27 test items, to which respondents were required to answer at various stages with information regarding what they had observed, how they had interpreted it, and how they would have carried out such activities. A lesson script containing the five teaching practices a teacher might employ in a lesson on fractional division was provided along with the test. The PSTs were required to read the lesson script and respond to test questions on the teaching practices they observed, how they perceived or assessed them, and how they would put those interpretations or evaluations into practice.

The test, referred to as the "teaching practices test," was made up of closed-ended questions, and responses from pre-service teachers were graded on a dichotomous 2-point scale of 0 and 1, where a score of zero (0) indicated an incorrect response and a score of one (1) indicated a correct response. Twenty of the test's twenty-seven items were scored, and they were categorized into groups according to the five teaching practices constructs: selecting and using tasks (6 items), using representations (4 items), providing explanations (4 items), responding to students' requests for help (3 items), and analyzing students' work and contributions (3 items). The overall score for each of the teaching practices constructs was standardized to the same scale maximum value of 6 points to make scoring easier to compare. Obtaining a score of 3 was regarded as the average point value. Thus, a high score in teaching practices was understood to mean having a score of 3 or higher, while a low score in teaching practices was understood to imply having a score lower than 3. Three of the teaching practices constructs had Kuder-Richardson reliabilities that ranged from 0.61 to 0.81, exceeding the acceptable threshold value of 0.60 (providing explanations, α =0.61; analyzing student work/contributions, α =0.81; and using representations, α =0.68), whiles the Kuder-Richardson reliabilities for the remaining two teaching practices constructs (selecting and using tasks, $\alpha = 0.54$; and responding to students' requests for help, $\alpha = 0.51$) which did not meet the acceptable threshold of 0.60; where later accepted by the researchers as having moderate reliabilities based on Hinton et al.'s (2014) guidance on appropriate cut-off points for reliability coefficients.

Data analysis

The positivist approach was used in this study to analyze numerical information about MKTF and teaching practices from a sample of 171 PSTs. The data was analyzed using both descriptive (mean and standard deviation) and inferential (regression) statistical methodologies.

RESULT

We conducted descriptive analyses to determine the mean and standard deviation of the mathematical knowledge for teaching fractions domains and the teaching practices constructs before determining the effects of pre-service teachers' MKT on their teaching practices. In tables 1 and 2, the outcomes of the descriptive analyses are displayed. The descriptive statistics for each of the six categories of mathematical knowledge for teaching fractions are shown in Table 1 below.

MKT Domain	Mean	Std. Deviation
Horizon Content Knowledge of Fractions (HCKF)	4.83	3.342
Knowledge of Content of Fractions and Curriculum (KCFC)	4.54	2.350
Knowledge of Content of Fractions and Teaching (KCFT)	3.58	2.020
Special Content Knowledge of Fractions (SCKF)	3.36	2.109
Common Content Knowledge of Fractions (CCKF)	3.22	2.381
Knowledge of Content of Fractions and Students (KCFS)	2.91	2.051

Table 1: The descriptive statistics of the six MKTF domains (N = 171)

The range of MKT domains' average scores was 0.291 to 4.83. The findings reveal a variability in the MKTF domains' ratings, which went from 2.020 to 3.342. In comparison to the average score point value of 4, the PSTs' mean scores (HCKF, M = 4.83; KCFC, M = 4.54) showed that PSTs generally performed well in these two MKTF domains. The PSTs mean scores in these four domains (KCFT, M = 3.58; SCKF, M = 3.36; CCKF, M = 3.22; and KCFS, M = 2.91) were, nevertheless, low when compared to the average score point value of 4. Additionally, PSTs scored the lowest on average in the KCFS.

For the pre-service teachers teaching practices, the descriptive statistics for the five constructs of teaching practices are shown in Table 2.

Teaching Practices	Mean	Standard Deviation
Tasks	2.17	1.371
Explanations	2.04	1.441
Analysing	2.03	1.708
Representations	1.48	1.699
Requests	1.26	1.320

Table 2: The descriptive statistics of the five teaching practices (N = 171)

Table 2 shows the average scores pre-service teachers obtained for the five components of teaching practices, which ranged from 1.26 to 2.17. The PST results revealed a spread in scores for the five teaching practices components, ranging from 1.320 to 1.708. The PSTs' mean scores (Tasks, M = 2.17; Explanations, M = 2.04; Analyzing, M = 2.03; Representations, M = 1.48; and Requests, M = 1.26) showed that, on average, PSTs performed poorly in all five teaching practices components when compared to the average score point value of 3. Responding to students' requests for help received the lowest average score among the PSTs' teaching practice constructs.

The researchers performed multiple linear regression analyses to explore the impact of each MKTF domain on the other domains (*CCKF, SCKF, HCKF, KCFT, KCFS, and KCFC*). Table 3 below shows the results of these analyses.

Independent Variables		Dependent Variable					
		CCKF	SCKF	HCKF	KCFT	KCFS	KCFC
CCKF	Beta		0.144	-0.028	-0.050	0.133	0.076
	(Sig)		(0.031)	(0.694)	(0.524)	(0.039)	(260)
SCKF	Beta	0.194		-0.054	0.150	0.381	0.255
	(Sig)	(0.031)		(0.505)	(0.097)	(0.000)	(0.001)
HCKF	Beta	-0.034	-0.050		0.320	0.224	0.343
	(Sig)	(0.694)	(0.505)		(0.000)	(0.002)	(0.000)
KCFT	Beta	-0.049	0.111	0.257		-0.228	0.018
	(Sig)	(0.524)	(0.097)	(0.000)		(0.000)	(791)
KCFS	Beta	0.192	0.410	0.263	-0.333		0.147
	(Sig)	(0.039)	(0.000)	(0.002)	(0.000)		(0.070
KCFC	Beta	0.101	0.251	0.368	0.024	0.134	
	(Sig)	(0.260)	(0.001)	(0.000)	(0.791)	(0.070)	

Table 3: Predicting each MKTF domain by other MKTF domains

From table 3, the results of the multiple linear regression analyses reveal significant positive impact of: SCKF (β = 0.194, P = 0.031); KCFS (β = 0.192, P = 0.039) in predicting PSTs *CCKF*. However, the results did not reveal any significant impacts of: HCKF (β = -0.034, P = 0.694); KCFT (β = -0.049, P = 0.524); and KCFC (β = 0.101, P = 0.260*)* in predicting the PSTs CCKF. The result therefore shows that two other MKTF domains of PSTs (SCKF and KCFS) impacted positively to predict their CCKF domain of MKT which appears to suggest that the PSTs in these two knowledge domains have direct influence on their development of the CCKF knowledge domain. Of the two MKTF domains (SCKF and KCFS) that significantly predicted PSTs CCKF domain of MKTF, the impact of SCKF was a little higher than the impact of KCFs. Thus, the study identified the PSTs' SCKF domain as having the greatest influence on their development of CCKF domain.

With regards to the SCKF domain, the results showed significant positive impacts of three other MKTF domains: CCKF (β = 0.144, P = 0.031); KCFS (β = 0.410, P = 0.000); and KCFC (β = 0.251, P = 0.001) in predicting the PSTs SCKF domain. On the other hand, the results for two other MKTF domains: HCKF (β = -0.050, P = 0.505); and KCFT (β = 0.111, P = 0.097) were not significant in predicting the PSTs SCKF domain. Of the three MKTF domains (CCKF, KCFS and KCFC) that significantly predicted PSTs SCKF knowledge domain, the impact of KCFS was the highest. Thus, identifying PSTs' KCFS as the best strongest predictor of their SCKF knowledge domain, and hence having a pronounced positive influence on the development of the PSTs SCKF knowledge domain.

With regard to PSTs other MKTF domains predicting their HCKF domain, the results have shown significant positive impacts of three MKTF domains: KCFT (β = 0.257, P = 0.000); KCFS (β = 0.263, P = 0.002); and KCFC (β = 0.368, P = 0.000*)*, at α=0.05. However, the impacts of: CCKF (β = -0.028, P = 0.694); and SCKF (β = -0.054, P = 0.505) did not show significant predictions of the PSTs HCKF knowledge domain. Of the MKTF domains that significantly predicted PSTs HCKF domain, the impact of KCFC seems to be very pronounced compared to the others, followed by KCFS and least by KCFT. Thus, the result has shown that the impact of PSTs KCFC was most pronounced on their HCKF domain, an indication that the PSTs KCFC knowledge domain has the greatest influence on the development of their HCKF knowledge domain.

With respect to PSTs other MKTF domains predicting KCFT domain, the results have shown a significant positive impact of: HCKF (β = 0.320, P = 0.000); and negative impact of KCFS (β = -0.333, P = 0.000) in predicting the PSTs' KCFT. However, the results did not show significant impacts of: CCKF (β = - 0.050, P = 0.524); SCKF (β = 0.150, P = 0.097); and KCFC (β = 0.024, P = 0.791) in predicting the PSTs KCFT. Of the two MKTF domains that significantly predicted the PSTs KCFT domain, the impact of KCFS was negative whiles the impact of HCKF was positive, suggesting that a PST KCFS has negative influence on the development of their KCFT whiles PSTs HCKF has positive influence on the development of their KCFT knowledge domain.

With regard to predicting the PSTs KCFS domain, the results have shown the impacts of three other MKTF domains: CCKF (β = 0.133, P = 0.039); SCKF (β = 0.381, P = 0.000); HCKF (β = 0.224, P = 0.002); as positive and significant and one other MKTF domain: KCFT (β = -0.228, P = 0.000) as negative and significant. However, the impact of KCFC (β = 0.134, P = 0.070) was not significant indicating that the PSTs knowledge in this domain did not significantly predict their KCFS knowledge domain. Of the four MKTF domains that significantly predicted the PSTs KCFT domain, the results showed negative influence by the PSTs KCFT domain and showed positive influence by the other three MKTF domains (CCKF, SCKF and HCKF). The results also revealed that, of the MKTF domains that significantly predicted PSTs KCFS, the impact of SCKF appears to be very pronounced compared to the others. The finding therefore show that the PSTs SCKF knowledge domain has the greatest influence on the development of their KCFS knowledge domain.

Regarding the PSTs other MKTF domains predicting the PSTs KCFC domain, the results have shown significant positive impacts of: SCKF (β = 0.133, P = 0.039); and HCKF (β = 0.224, P = 0.002) in predicting PSTs KCFC domain. However, the impact of: KCFC (β = 0.134, P = 0.070); KCFC (β = 0.134, P = 0.070); and KCFC (β = 0.134, P = 0.070) were not significant indicating that the PSTs knowledge in these domains did not significantly predict their KCFC knowledge domain. Of the MKTF domains that significantly predicted PSTs KCFC, the impact of HCKF appears to be very pronounced compared to the impact of SCKF. The finding therefore show that the PSTs HCKF knowledge domain has the greatest influence on the development of their KCFC knowledge domain.

To sum up, the results showed that each of the PSTs MKTF domains was significantly predicted by at least two other MKTF domains an indication that the PSTs MKTF domains influence each other. The results have shown the PSTs KCFS as a very influential knowledge domain as it significantly predicted most of the MKTF domains of the PSTs.

The researchers performed multiple linear regression analyses to explore the effect of each of the six MKTF domains (CCKF, SCKF, HCKF, KCFT, KCFS, and KCFC) in predicting each of the five constructs of teaching practices. Table 4 below provides the summary of the results.

MKTF Domain	Standardised Coefficients (Beta)				
	Tasks	Representations	Explanations	Requests	Analysing
CCKF	0.086	0.174	0.252	-0.020	0.251
(Sig)	(0.266)	(0.013)	(0.001)	(0.786)	(0.001)
SCKF	-0.080	0.129	0.084	0.059	0.162
(Sig)	(0.372)	(0.110)	(0.337)	(0.498)	(0.056)
HCKF	-0.031	0.078	0.164	0.051	0.159
(Sig)	(0.722)	(0.314)	(0.052)	(0.537)	(0.049)
KCFT	0.005	0.048	0.143	0.142	0.254
(Sig)	(0.945)	(0.484)	(0.059)	(0.054)	(0.001)
KCFS	0.349	0.314	-0.248	0.422	0.116
(Sig)	(0.000)	(0.000)	(0.007)	(0.000)	(0.187)
KCFC	0.138	0.095	0.189	0.033	-0.070
(Sig)	(0.122)	(0.237)	(0.031)	(0.700)	(0.403)

Table 4: The effect of each of the six MKTF domains in predicting each of the five constructs of teaching practices

From Table 4, the results have shown a significant positive impact of KCFS ($\beta = 0.349$, $P = 0.000$) in predicting PSTs selection and using tasks. The finding showed that there was a significant increase of 0.349 units in PSTs' practice of selection and using tasks for every unit change in PSTs' KCFS, when the variance explained by all other MKTF domains in the model is controlled for. However, the results did not show significant impacts of: CCKF ($\beta = 0.086$, $P = 0.266$); SCKF ($\beta = -0.080$, $P = 0.372$); HCKF ($\beta = -0.031$, $P = 0.722$); KCFT ($\beta = 0.005$, $P = 0.945$); and KCFC ($\beta = 0.138$, $P = 0.122$*)* in predicting their selection and using tasks. The result therefore shows that, of the six MKTF domains of PSTs, only KCFS significantly impacted positively to predict their practice of selection and using tasks which appears to suggest that a pre-service teacher with a very strong KCFS has the potential to perform well in the practice of selecting and using tasks.

With regards to the domain on representation, the results showed significant positive impacts of two of the MKTF domains: CCKF ($\beta = 0.174$, $P = 0.013$); and KCFS ($\beta = 0.314$, $P = 0.000$) in predicting the PSTs practice of using representations. The results: SCKF ($\beta = 0.129$, $P = 0.110$); HCKF ($\beta = 0.078$, $P = 0.314$); KCFT ($\beta = 0.048$, $P = 0.484$) and KCFC ($\beta = 0.095$, $P = 0.237$) however, were not significant for four of the domains in predicting the PSTs practice of using representations. Of the two MKTF domains (CCKF and KCFS) that significantly predicted PSTs use of representations, the impact of KCFS was higher compared to the impact of KCCF. Thus, identifying PSTs' KCFS as the best strongest predictor of their practice of using representations, suggesting that a pre-service teacher with a strong KCFS has the potential to perform well in the practice of using representations.

With regard to PSTs MKTF domains predicting their practice of providing explanations, the results have shown significant positive impacts of two domains: CCKF (β = 0.252, P = 0.001); KCFC (β = 0.189, P = 0.031), and significant negative impact of one domain: KCFS (β = -0.248, P = 0.007), at α=0.05. However, the impacts of: SCKF (β = 0.084, P = 0.337); HCKF (β = 0.164, P = 0.052); and KCFT (β = 0.143, P = 0.059) did not show significant predictions of the PSTs practice of providing explanations. Of the MKTF domains that significantly predicted PSTs practice of *providing explanations*, the impact of CCKF seems to be very pronounced compared to the others, followed by KCFS and least by KCFC. Thus, the result has shown that the impact of PSTs CCKF was most pronounced on their practice of providing explanations, an indication that a pre-service teacher with a strong CCKF has the potential to perform well in providing explanations as a teaching practice.

With respect to PSTs MKTF domains predicting their practice of responding to students' requests for help, the results have shown a significant positive impact of KCFS (β = 0.422, P = 0.000) in predicting the PSTs' practice of responding to students' requests for help. This showed that there was a significant increase of 0.422 units in PSTs' practice of responding to students' requests for help for every unit change in PSTs' KCFS, when the variance explained by all other MKTF domains is controlled for. However, the results did not show significant impacts of: CCKF (β = -0.020, P = 0.786); SCKF (β = 0.059, P = 0.498); HCKF (β = 0.051, P = 0.537); KCFT (β = 0.142, P = 0.054); and KCFC (β = 0.033, P = 0.700) in predicting the PSTs practice of responding to students' requests for help. The finding therefore, show that, of the six MKTF domains of PSTs, only KCFS significantly impacted positively to predict their practice of responding to students' requests for help, suggesting that a PST with a strong KCFS has the greatest potential to do well in the practice of responding to students' requests for help.

With regard to predicting the PSTs practice of analyzing students' works and contributions by their MKTF domains, the result has shown the impacts of: CCKF (β = 0.251, P = 0.001); HCKF (β = 0.159, P = 0.049); and KCFT (β = 0.254, P = 0.001) as positive and significant in predicting PSTs practice of analyzing students' works and contributions. However, the impacts of: SCKF (β = 0.162, P = 0.056); KCFS (β = 0.116, P = 0.187); and KCFC (β = - 0.070, P = 0.403), were not significant indicating that the PSTs knowledge in these three MKTF domains did not significantly predict their practice of analyzing students' works and contributions. Interestingly, the impact of SCKF (β = 0.162) was higher than the impact of HCKF (β = 0.159) in predicting the practice of analyzing students' works and contributions, yet the impact of HCKF was significant while the impact of SCKF was not. This might be due to the fact that the PSTs SCKF did not correlate with their KCFT as they were too far apart uncorrelated (see Table 4). Of the MKTF domains that significantly predicted PSTs practice of analyzing students' works and contributions, the impact of KCFT appears to be very pronounced compared to the others, followed by CCKF and least by HCKF. The finding therefore show that the impact of PSTs KCFT was most pronounced on their practice of analyzing students' works and contributions, an indication that a PST needs a strong knowledge in KCFT to be able to perform well in the practice of analyzing students' works and contributions.

To sum up, the results have shown significant predictive models of PSTs teaching practices constructs by their mathematical knowledge for teaching fraction domains an indication that MKTF domains have impact on teaching practices. The results have also revealed that the impact of PSTs KCFS was most pronounced in predicting three constructs of teaching practices (selection and using tasks, using representations, and responding to students' requests for help). Moreover, the results have shown that the PSTs CCKF was the strongest predictor of their practice *of* providing explanations. Again, the results have also shown that the impact of PSTs KCFT was most pronounced in predicting their practice of analyzing students' works and contributions. Interestingly, the results have shown the impact of SCKF higher than the impact of HCKF in predicting PSTs practice of analyzing students' works and contributions, yet the impact of HCKF was significant whiles the impact of SCKF was not. The results have shown from the perspectives of the pre-service teachers that KCFS knowledge domain has great impact on teaching practices as it significantly predicted majority of the pre-service teaching practice.

DISCUSSION

This research aimed to assess the influence of pre-service teachers' Mathematical Knowledge for Teaching (MKTF) domains on their instructional methods. The study employed the Mathematical Knowledge for Teaching framework and the Mathematical Task framework to identify six domains of teacher knowledge related to fractions (CCKF, SCKF, HCKF, KCFT, KCFS, and KCFC), along with five essential teaching practices: selecting and using tasks, utilizing representations, providing explanations, responding to students' requests for help, and analyzing students' work and contributions. Specifically, the investigation focused on understanding how pre-service teachers' mathematical knowledge for teaching fractions affects their actual teaching practices in the classroom.

The research revealed a significant correlation between each of the pre-service teachers' (PSTs) Mathematical Knowledge for Teaching (MKTF) domains, with each domain being notably predicted by at least two other MKTF domains. This finding suggests a mutual influence among the PSTs' MKTF domains, aligning with the perspective of Fennema and Franke (1992), who asserted that teacher knowledge domains have interrelated effects. Understanding these interactions is crucial, as it is challenging to comprehend the role of a single MKT domain in the broader context of teacher knowledge without insight into how these domains mutually affect each other. From this study, we identified the domain of KCFS as being a fundamental knowledge among the MKTF domains since it significantly predicted majority of the PSTs MKTF domains. This means that KCFS played a significant role in relation to the totality of the MKTF domains.

The study has found that the impact of PSTs KCFS was most pronounced in predicting three constructs of teaching practices (selection and using tasks, using representations, and responding to students' requests for help). This means KCFS was fundamental (foundational knowledge) to PSTs' ability to uniquely perform majority of mathematics teaching practices. This means that as a teacher, the ability to know the subject matter of mathematics and know your students are key to be able to teach and perform majority of mathematics teaching practices in order to bring about instructional quality. The result has shown that the PSTs CCKF was the best strongest predictor of their practice of providing explanations. This means that PST CCKF was a foundational knowledge needed to uniquely perform the task of providing explanations in the instructional process. The finding shows that the impact of PSTs KCFT was most pronounced in predicting their practice of analysing students' work and contributions. This contradicted Rosebery's (2005) assertion that a thorough understanding of the topic and students is necessary for analysing student errors or challenges, which in turn allows the instructor to plan ahead of time in order to maintain a cognitive level of tasks and support students so that they remain engaged. Though, as established in a previous study (Sie & Agyei, 2023), KCFT did not correlate with majority of PSTs' teaching practices, the results of the multiple regression performed in the present study revealed that it was still a significant foundational knowledge requirement of PSTs' to be able to effectively analyse students' errors in mathematics. This means that a PST needs to uniquely show sufficient knowledge of contents of mathematics and knowledge of teaching the contents in order to effectively analyse students work and contributions when teaching mathematics.

The results showed that KCFS was the most foundational knowledge as it significantly predicted majority of the PSTs MKTF domains and imparted on majority of the teaching practices constructs. This means that KCFS played a significant role in relation to the totality of the MKTF domains and teaching practices constructs. Interestingly, the result has shown the impact of SCKF higher than the impact of HCKF to the model that predicted PSTs practice of analysing students' work and contributions, yet the impact of HCKF was significant whiles the impact of SCKF was not. This was attributed to the fact that the PSTs SCKF did not correlate with their KCFT as they were too far apart uncorrelated (see Table 4). This is consistent with researchers (Pedhazur, 1997; Pedhazur & Schmelkin, 1991) who describe beta weight values as partly a function of the correlations between the predictors themselves. That means a certain predictor variable may have a high beta coefficient than another variable in a model but may fail to be significant to the model because of how weak it correlates with other predictors, whiles the other variable with less beta coefficient but correlate well with other predictors in the model would instead be significant. Thus, the results therefore failed to provide evidence to support that teachers SCK helps them to uniquely teach mathematics (Ball et al., 2008; Hill & Ball, 2004).

CONCLUSIONS

This study was not without limitations. Teaching practices in this study were explored using teaching approximations, thus identifying and interpreting teaching practices contained in lesson scripts. It would have been more appropriate to use observation data instead of using tests in which PSTs were made to read, identify and interpret the appropriateness of the teaching practices contained in lesson scripts. However, the use of a test enabled us to obtain data from PSTs about the same teaching practices for easy comparison that would have been difficult if observation data was used. Future research is therefore needed to use both observation data and lesson scripts with accompanying tests to explore the teaching practices of PSTs in order to compare whether PSTs' performance in the test is similar to their performance in the observation of teaching data. The study only explored MKT and teaching practices in fractions. It is not clear if the same results will be obtained using different topics in mathematics. This, to some extent limits the generalization of the results to other areas in mathematics. Future research is needed using other topics in mathematics to examine the relationship between teachers' MKT and their teaching practices.

Notwithstanding these limitations, the findings provide some insights into how pre-service teachers in Ghana and countries with similar contexts could conceptualise MKT and teaching practices. The findings show that the domains of MKTF have great influence on teaching practices constructs which suggest that teacher knowledge is key to the enhancement of quality mathematics teaching which consequently, could lead to improved students' performance. The findings here confirm the hypothesis that mathematical knowledge for teaching domains contribute to the ability to perform the five teaching practices. The findings implies that teachers need MKT domains to be able to teach to bring instructional quality that involves *selecting and using tasks; using representations; providing explanations; responding to students' direct or indirect requests for help; and analysing students' works and contributions*. These findings suggest that teacher training institutions should focus on training PSTs to explicitly acquire the six knowledge domains of MKT that will have great influence on their ability to perform mathematics teaching practices that bring about instructional quality and consequently lead to quality learning. The findings further implied that in the training of prospective teachers these three knowledge domains (CCK, KCT, and KCS) are foundational to the acquisition of their knowledge as they will help them to uniquely perform specific tasks of teaching mathematics.

References

[1] Addae, B. D., & Agyei, D. D. (2018). High school students' attitudes towards the study of mathematics and their perceived teachers' teaching practices. *European Journal of Educational and Development Psychology*, 6(2), 1-14.

[2] Avcu, R. (2019). Turkish pre-service middle level mathematics teachers' knowledge for teaching fractions. *RMLE Online*, *42*(9), 1-20. https://doi.org/10.1080/19404476.2019.1681624

[3] Ball, D. L., Thames, M. A., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389-407. https://doi.org/10.1177/0022487108324554

[4] Cengiz, N., Kline, K., & Grant, T. J. (2011). Extending students' mathematical thinking during whole-group discussions. *Journal of Mathematics Teacher Education*, *14*, 355-374

[5] Charalambous, C. Y. (2008). *Preservice Teachers' Mathematical Knowledge for Teaching and Their Performance in Selected Teaching Practices: Exploring a Complex Relationship* (Doctoral dissertation).

[6] Choppin, J. (2011). The role of local theories: Teacher knowledge and its impact on engaging students with challenging tasks. *Mathematics Education Research Journal*, *23*, 5-25. A

[7] Cole, Y. (2012). Assessing elemental validity: The transfer and use of mathematical knowledge for teaching measures in Ghana. *ZDM*, *44*(3), 415-426. https://doi.org/10.1007/s11858-012-0380-7

[8] Fennema, E., & Franke, L. M. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164). New York, NY: Macmillan.

[9] Fumador, E. S., & Agyei, D. D. (2018). Students' errors and misconceptions in algebra: Exploring the impact of remedy using diagnostic conflict and conventional teaching approaches.

[10] Hill, H. C., & Ball, D. L. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. *Journal for research in mathematics education*, *35*(5), 330-351.

[11] Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and instruction*, *26*(4), 430-511. https://doi.org/10.1080/07370000802177235

[12] Hill, H.C., Schilling, S.G., & Ball, D.L. (2004) Developing measures of teachers' mathematics knowledge for teaching. *Elementary School Journal, 105*, 11-30. https://doi.org/10.1086/428763

[13] Hill, H. C., Umland, K., Litke, E., & Kapitula, L. R. (2012). Teacher quality and quality teaching: Examining the relationship of a teacher assessment to practice. *American Journal of Education*, *118*(4), 489-519.

[14] Hinton, P. R., McMurray, I., & Brownlow, C. (2014). *SPSS explained*. Routledge.

[15] Hoover, M., Mosvold, R., Ball, D. L., & Lai, Y. (2016). Making progress on mathematical knowledge for teaching. *The Mathematics Enthusiast*, *13*(1), 3-34. https://doi.org/10.54870/1551-3440.1363

[16] Johnson, E. M., & Larsen, S. P. (2012). Teacher listening: The role of knowledge of content and students. *The Journal of Mathematical Behavior*, *31*(1), 117-129.

[17] Kersting, N. B., Givvin, K. B., Thompson, B. J., Santagata, R., & Stigler, J. W. (2012). Measuring usable knowledge: Teachers' analyses of mathematics classroom videos predict teaching quality and student learning. *American Educational Research Journal*, *49*(3), 568-589.

[18] Markworth, K., Goodwin, T., & Glisson, K. (2009). The development of mathematical knowledge for teaching in the student teaching practicum. *AMTE monograph*, *6*, 67-83.

[19] Principles, N. C. T. M. (2000). standards for school mathematics. Reston, VA: The National Council of Teachers of Mathematics..

[20] Pedhazur, E. J. (1997). *Multiple Regression in Behavioral Research (3rd ed.)*. Orlando, FL: Harcourt Brace.

[21] Pedhazur, E. J., & Schmelkin, L. P. (1991). *Measurement, design, and analysis: An integrated approach.* Hillsdale, NJ: Erlbaum.

[22] Rosebery, A. (2005). What are we going to do next? *A case study of lesson planning. In R. Nemirovsky, A. Rosebery, B. Warren, & J. Solomon (Eds.), Everyday matters in mathematics and science: Studies of complex classroom events*, 299-328.

[23] Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15 (2), 4-14. https://doi.org/10.3102/0013189X015002004

[24] Sie, C. K., & Agyei, D. D. (2023). Building network of relationships between teachers' mathematical knowledge for teaching fractions and teaching practices. *Eurasia Journal of Mathematics, Science and Technology Education*, *19*(4), em2251. https://doi.org/10.29333/ejmste/13087

[25] Steele, M. D., & Rogers, K. C. (2012). Relationships between mathematical knowledge for teaching and teaching practice: the case of proof. *Journal of Mathematics Teacher Education*, *15*, 159-180.

[26] Stein, K. M, & Smith, M. S. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle School*, 3 (4), 268275. https://doi.org/10.5951/MTMS.3.4.0268

[27] Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). *Implementing Standards-based mathematics instruction: A casebook for professional development*. New York: Teachers College Press.

[28] Sugilar, S. (2016). Factors of Students Participating in Online Examination. *Journal of Education and Learning*, *10*(2), 119-126. https://doi.org/10.11591/edulearn.v10i2.3256

[29] Tanase, M. (2011). Teaching Place Value Concepts to First Grade Romanian Students: Teacher Knowledge and its Influence on Student Learning. *International Journal for mathematics teaching and learning*.

Development of Prospective Teacher Student Worksheets Through Interactive Case-Based Learning Model Assisted by Cublend App to Improve Mathematical Literacy Skills

Ramdani Miftah, Lia Kurniawati*, Kamal Fikri Musa

Universitas Islam Negeri Syarif Hidayatullah Jakarta, Indonesia

ramdani.miftah@uinjkt.ac.id, lia.kurniawati@uinjkt.ac.id*, kamalfiqry@uinjkt.ac.id

Abstract: In this 21st century, learners are required to be able to adapt to new challenges so that they can solve problems in the real world. This poses challenges in learning, especially online learning, which will continue to grow. This research aims to produce worksheets for prospective teachers using the Interactive Case-Based Learning (ICBL) model assisted by the Cublend Application and oriented to quality mathematical literacy skills in terms of validity, practicality and effectiveness. The R&D development model consists of initial investigation, design, realization, validation & Revision, implementation, and evaluation. The instruments used were an expert validation questionnaire, a student response questionnaire and a mathematical literacy test. The results showed that: (1). Student worksheets using the ICBL model assisted by Cublend App (ICBL-Cublend App) are declared valid and very feasible to use based on expert assessment with a V coefficient of 0.82 and a percentage of 85.50%; (2). Student worksheets using the ICBL-Cublend App model are classified as practical based on the responses of prospective mathematics teacher students with a percentage of 79% (3). Applying the ICBL-Cublend App model effectively improves mathematical literacy skills and significantly influences prospective mathematics teachers' mathematical literacy skills. This study's results imply that students based on the ICBL-Cublend App model can be used as an alternative learning model in university teacher professional development programs.

Keywords: cublend app, interactive case-based learning model, mathematical complex problem solving, mathematical literacy skills

INTRODUCTION

In the 21st century, learners are required to adapt to new challenges, and skills are one of the solutions to adapting to the new environment. The importance of literacy has been widely expressed by experts such as Akyüz (2014) and Arslan & Yavuz, (2012), who see the importance of literacy as a solution to various problems in the country both in the economic, social, and cultural fields. Literacy is one of the abilities that must be integrated into the learning process, including mathematical literacy. Everyone needs to have mathematical literacy skills to deal with various problems related to work or tasks in real life (Kusumah, 2012; OECD (2013). Mathematical literacy is needed to adapt to innovation (Pugalee & Chamblee, 2000).

The results of the 2021 PISA survey stated that Indonesia ranked 70th with a score of 366 out of 81 countries that participated in this survey (Schleicher, 2022). Although Indonesia's ranking has increased compared to the previous PISA results, the score has decreased. The scores obtained have declined since Indonesia participated in the last three PISA surveys. From 2015, with a score of 386, it decreased in 2018 to 379 and decreased again in 2021. In addition to the PISA results, other studies conducted by Kurniawati et al. (2022), Amelia et al., (2021), Hardiani & Desmayanasari (2022), and Aisyah et al., (2018) show that the mathematical literacy skills of students in Indonesia are at low criteria. The following graph shows the results of Indonesia's PISA score since 2000 (OECD, 2022).

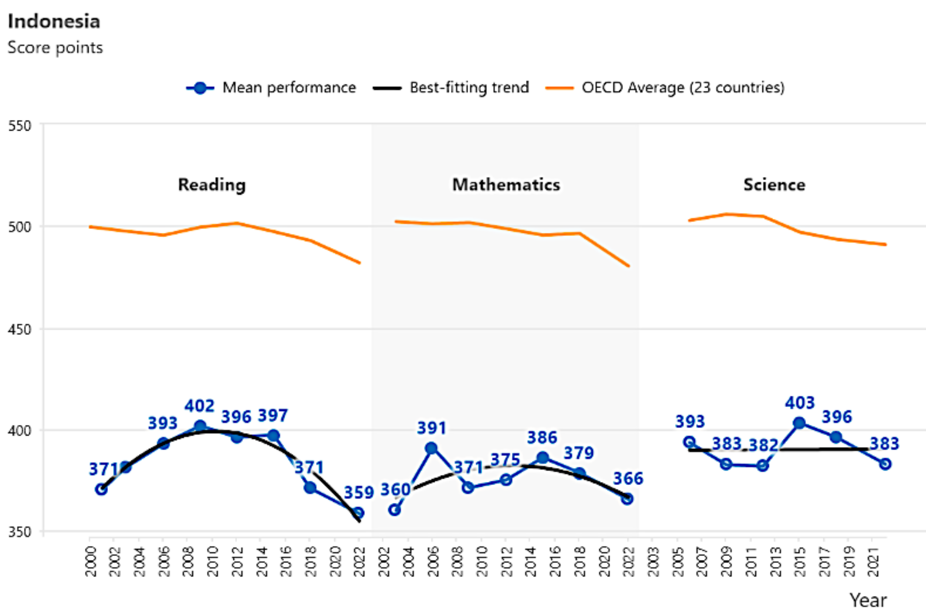


Figure 1: Indonesian Students' Scores in PISA

Educators have a crucial role in the learning process and significantly influence learning success in the classroom (Barlia, 2010). An alternative solution that educators can use to overcome the problem of students' low mathematical literacy skills is the use of the Interactive Case-Based Learning (ICBL) model. This model was developed by adapting the complex problem-solving process model according to Frensch & Funke (1995), Grünig & Kühn (2017), and Chevvalier (2016). Therefore, the ICBL model is defined as learning that is closely related to cases that have characteristics of complex, realistic, and relevant problems with the material being studied where students actively participate in finding various pertinent information to solve the case based on their knowledge and previous experience (Miftah et al., 2024). This model is designed to help students develop higher-order thinking skills, such as mathematical literacy.

In addition, online learning will continue to grow in the future, including in higher education. Online learning allows for developing learning environments that replicate the complexity of real-world problem-solving situations, thus expanding the boundaries of classroom-based learning (Schon, 1987). Furthermore, learners who utilize technology in learning show significantly better communication, complex problem-solving, and creativity (Lai & Hwang, 2014). However, many universities are not ready due to the limitations of existing applications in accommodating the use of constructivism-based learning models. Constructivism-Based Blended Learning (Cublend App) is an application designed to accommodate learning that uses constructivism-based learning models such as the ICBL model. Therefore, it is necessary to develop teaching materials that use the ICBL model assisted by Cublend App (ICBL-Cublend App Model) to improve the mathematical literacy skills of prospective teacher students.

Research Question

Based on the research background and review of relevant research in this study, we sought to answer the following research questions:

1. What is the validity of the learner worksheet using the ICBL-Cublend App model to improve the mathematical literacy skills of prospective teacher students?
2. What is the practicality of the learner worksheet using the ICBL-Cublend App model to improve the mathematical literacy skills of prospective teacher students?
3. What is the effectiveness of the learner worksheets developed using the ICBL-Cublend App model in improving the mathematical literacy skills of prospective teacher students?

Literature Review

Interactive Case-Based Learning Model

Literature related to instructional design around unstructured problem-solving is lacking, especially about how to support students and teachers in instructional design (Choi & Lee, 2009). Therefore, researchers took the initiative to develop a case-based learning model (CBL) by presenting complex cases. CBL is considered more efficient in content knowledge acquisition (Kirschner et al., 2006) and more effective in helping students solve unstructured problems (Williams, 2005). The selection of cases in CBL begins with the search for the issues around students that student may face in the future (Bridges & Hallinger, 1999). Barrows (1986) proposed a taxonomy that classifies problem-based learning (PBL) into two variables, namely self-directedness and problem-structuredness, which was later illustrated by Hung (2011). The following is presented in Figure 2, a modification of the PBL taxonomy according to Barrows (1986).

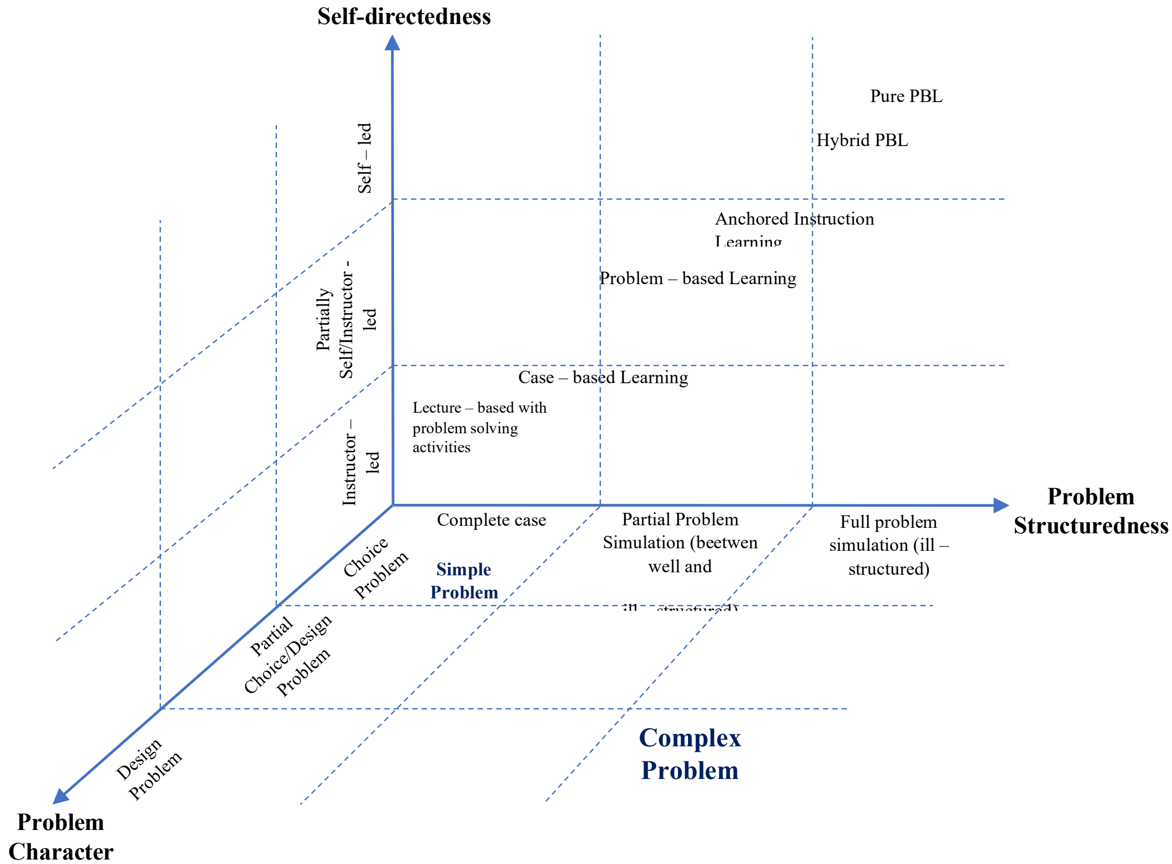


Figure 2: Modified PBL Taxonomy (Miftah et al., 2024)

In solving problems, a person will be influenced by the problem character, which consists of choice and design problems. Choice problems are problems whose solution options are known from the start, while design problems can only be solved by breaking them down into several sub-problems so that the new problem can be solved step by step. There are three possibilities for a problem to be a design problem, namely when the problem can only be solved by: (1). Series, when the problem is broken down into several sub-problems, and one of the sub-problems is solved first, followed by solving the next sub-problem; (2). Parallel, when the problem is broken down into several independent sub-problems, the solution results from coordinating the solutions of several sub-problems; (3). Combined Series-Parallel, when the problem is solved in both Series and parallel ways (Miftah et al., 2024).

The term "interactive" is used for a complex case, such as the one presented in the learning. Therefore, this is called the Interactive Case-Based Learning (ICBL) model. The ICBL model is learning closely related to cases with complex, realistic, and relevant problems with the material being studied, where students actively find pertinent information to solve the case based on their knowledge and previous experience. The learning steps of the ICBL model were developed from CBL learning according to Williams (2005) so that it becomes (1). Dividing learners heterogeneously into several groups; (2). Presenting an interactive (complex) case; (3). Determine objectives based on a review of the complex problem; (4). Diagnosing the issue; (5). Creating a solution.

Constructivism-Based Blended Learning Application (Cubelnd App)

The utilization of technology in learning has been proven to significantly improve better problem-solving skills (Lai & Hwang, 2014). Many LMS have been developed and applied in learning, such as Google Classroom, moodle, and Schoology. However, the LMS has not specifically accommodated the learning process based on constructivism, especially in monitoring and organizing activities at each learning stage (syntax). Therefore, an LMS that accommodates the implementation of constructivism-based learning has been developed, named "Cublend," which stands for constructivism-based blended learning.

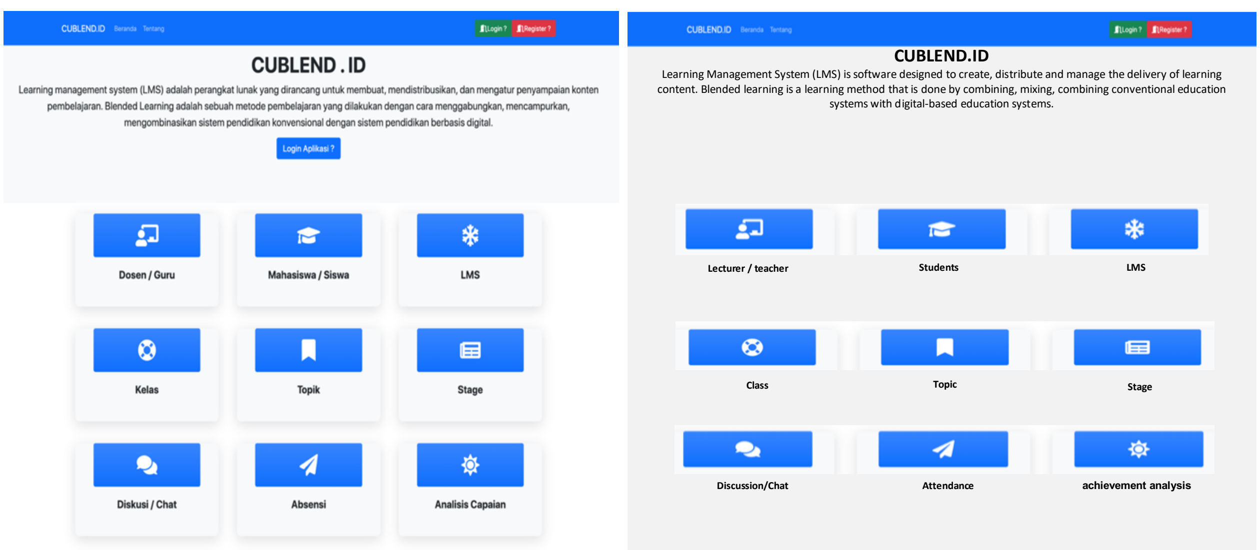


Figure 3: Cublend App Homepage

The learning theory underlying the development of the Cublend App is constructivism. Constructivism requires students to build their knowledge through a thinking process based on objects, experiences, environment, and student activeness. The Cublend App can be used in all lessons using a constructivism-based learning approach. In the Cublend App, educators are free to design learning based on the syntax of the learning approach and can monitor students' responses to each learning stage. Cublend App allows educators and students to discuss things to make learning livelier and more meaningful.

Mathematical Literacy Skills

Mathematical literacy is defined differently in different countries around the world. Some refer to it as quantitative or mathematical literacy, or it can be called numeracy (Rosa & Orey, 2015). Mathematical literacy is the power to use mathematical thinking in problem-solving, reasoning, communicating, and explaining in solving everyday problems to be better prepared for life's challenges (Stacey & Tuner, 2015). Mathematical literacy is an individual's capacity to formulate, use, and interpret mathematics in various contexts (OECD, 2013). Internationally, a person can be called mathematically literate if they can apply mathematics to practical problems that are part of their daily life routine and arrive at a solution (Botha & Putten, 2018).

The indicators of mathematical literacy skills developed in this study are: (1). Formulate identifies opportunities to use mathematics and develop mathematical structures in contextual problems; (2). Employers can apply mathematical concepts, facts, procedures, and mathematical reasoning to solve mathematical problems and obtain systematic conclusions (3). Interpret can reflect on mathematical solutions in the form of conclusions and interpret them in everyday life.

METHOD

This research is a Research and Development (R&D) study. According to Borg & Gall (2003), R&D is an industry-based development model where research results are used to design new products and procedures, which are systematically tested, evaluated, and refined until they meet the criteria of effectiveness and quality. The product of this research is teaching materials that use the ICBL model assisted by the Cublend application. This application is built using the PHP programming language and uses MySQL as a Data Base Management System.

Developing teaching materials using the ICBL model uses the following learning stages: (1). Dividing students heterogeneously into several groups; (2). Presenting an interactive (complex) case; (3). Determining objectives based on a review of the complex problem; (4). Diagnosing the problem; (5). Creating a solution. The instruments used in this research are an expert validation questionnaire, a student response questionnaire, and a mathematical literacy ability test. According to PISA (Thomson, 2013), the indicators of mathematical literacy skills used are formulate, employ, and interpret. The development model used is the model of Plomp (1997) combined with models from Joyce & Weil (2004), Nieveen (1999) and Dick & Carey (2005).

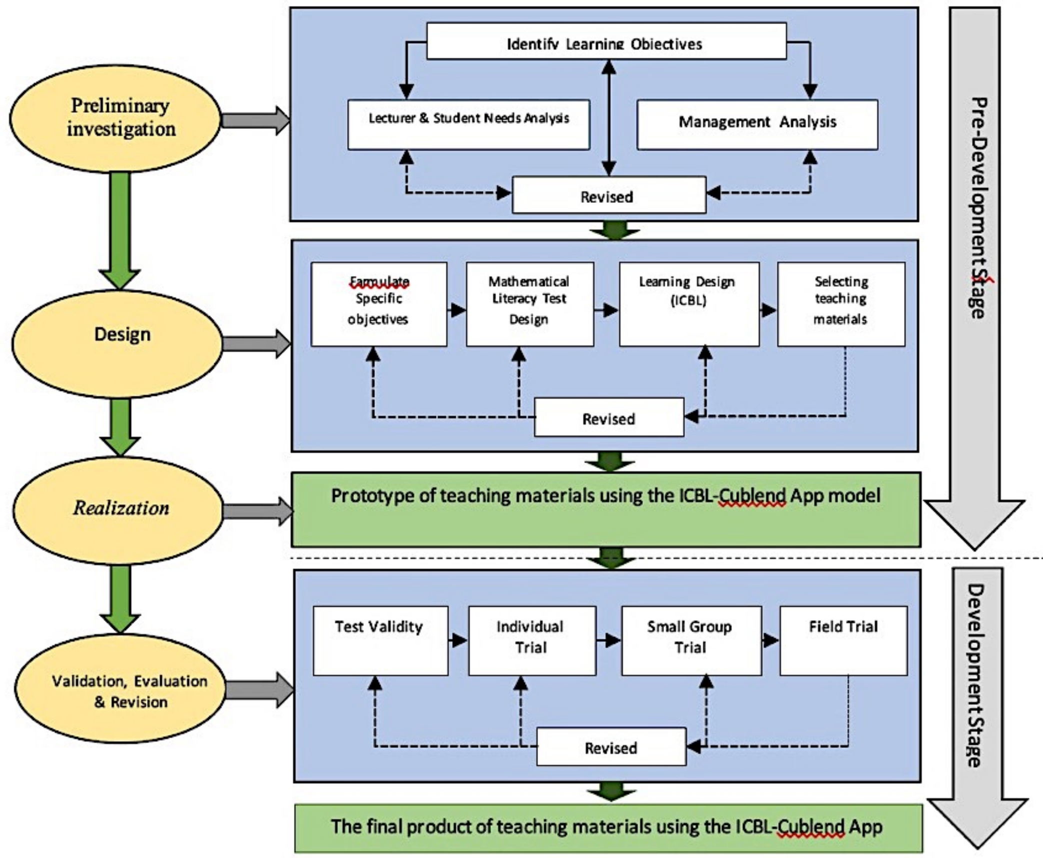


Figure 4: R&D development model

The development stages used in this study are:

(1) Preliminary investigation, which consists of analyzing student needs, the lecturer needs, and learning management;

(2) Design, consisting of formulating specific objectives and evaluation tools, determining learning strategies, and selecting learning materials;

(3) Realization, in the form of prototype realization of model guidelines, learning tools, and instruments;

(4) Validation & Revision. Validation by at least three experts, namely information technology/e-learning and mathematics learning design experts. The validity test is obtained from the average calculation and inter-rater test using the Cohen Kappa formula with the assessment criteria based on Murti (1997), namely if more than or equal to 0.75, then it is categorized as very good if between 0.4 to 0.75 is classified as good and if less than 0.4 is classified as poor. This application is valid if it obtains a minimum value of 3 and a minimum kappa value of 0.4. The practicality test was carried out by distributing Likert scale questionnaires converted to a ratio scale. Table 1 is the eligibility criteria used.

No	Percentage (%)	Criteria
1	81 – 100	Very Decent
2	61 – 80	Worth
3	41 – 60	Decent Enough
4	21 – 40	Not Feasible
5	0 – 20	Very inappropriate

Table 1: Eligibility criteria

(5) Implementation: this stage is carried out to see the effectiveness of applying the ICBL-Cublend App model in mathematics learning to improve the mathematical literacy skills of prospective teacher students. The mathematics learning was carried out on prospective teacher students at the mathematics education department of UIN Syarif Hidayatullah Jakarta in semester II of the 2023/2024 academic year.

(6) Evaluation. One-to-one evaluation where researchers provide opportunities for educators to use LKPD with the ICBL-Cublend App model in learning mathematics as a basis for seeing the effectiveness of this LKPD seen from student activities, student responses and the results of improving students' mathematical literacy skills obtained through pretests and post-tests.

RESULT

Preliminary Investigation Stage

At this stage, a content analysis was carried out as a literature review of previous studies on developing teaching materials, the ICBL model, and mathematical literacy skills. In addition, the curriculum and teaching materials used in the classroom are analyzed, and it was found that the curriculum uses the KKNI-based curriculum. The material analysis results showed that the material of rows and Series is taught in high school mathematics courses in the last material. The materials taught are arithmetic sequence, geometric sequence, and geometric sequence.

Design Stage

At this stage, the design of teaching materials in the form of Learner Worksheets (LKPD) using the ICBL model on the material of rows and Series consisting of five LKPDs according to the material that has been determined. Five LKPDs were developed, and two LKPDs whose learning is designed using the Cublend App, namely on the LKPD of arithmetic sequence material and the LKPD of geometric sequence material. The following displays the Cublend App when presenting complex cases on arithmetic sequence material.

Realization Stage

At this stage, the teaching materials developed are in the form of prototypes complete with learning tools and test instruments for mathematical literacy skills of prospective teacher students according to the indicators of mathematical literacy skills, namely formulate, employ, and interpret according to the material of rows and Series.

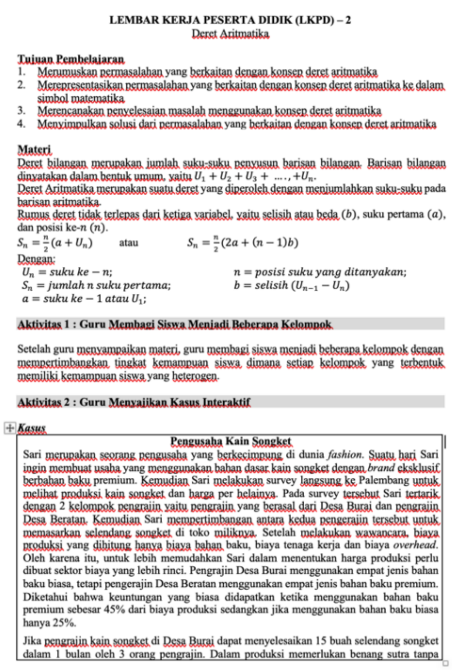
Validation and Revision Stage

At this stage, the teaching materials that have been developed are validated by experts, namely material experts, media experts, and practitioners (lecturers). Table 2 is the recapitulation of LKPD assessment results by experts using the ICBl-Cublend App model.

No	Aspect	Percentage	V	Conclusion
1	Supporting Theory	84	0,81	Very Appropriate, Valid
2	Anatomy and Purpose	84	0,80	Very Appropriate, Valid
3	Semester Learning Plan	84	0,80	Very Appropriate, Valid
4	Format and Display	89	0,86	Very Appropriate, Valid
5	Contents	87	0,83	Very Appropriate, Valid
6	Language	85	0,82	Very Appropriate, Valid
	Total	85,50	0,82	Very Appropriate, Valid

Table 2: Assessment results of prototype teaching materials using the icbl model - cublend

The assessment of the mathematical literacy test instrument was prepared to determine the suitability of each item with the indicators of mathematical literacy skills used. Based on 6 items tested using the Aiken method, 5 items were valid, and 1 item needed to be corrected. After the experts assessed the LKPD and mathematical literacy test instruments, several suggestions and inputs were obtained as improvement materials to improve the product before it was implemented in the test subjects. Some improvements to the LKPD are as follows.



LEMBAR KERJA PESERTA DIDIK (LKPD) – 2
Deret Aritmatika

Tujuan Pembelajaran
1. Merumuskan permasalahan yang berkaitan dengan konsep deret aritmatika
2. Merepresentasikan permasalahan yang berkaitan dengan konsep deret aritmatika ke dalam simbol matematika
3. Merencanakan penyelesaian masalah menggunakan konsep deret aritmatika
4. Menyimpulkan solusi dari permasalahan yang berkaitan dengan konsep deret aritmatika

Materi
Deret bilangan merupakan jumlah suku-suku penyusun barisan bilangan. Barisan bilangan dinyatakan dalam bentuk umum, yaitu $U_1 + U_2 + U_3 + \ldots, + U_n$.
Deret Aritmatika merupakan suatu deret yang diperoleh dengan menjumlahkan suku-suku pada barisan aritmatika.
Rumus deret tidak terlepas dari ketiga variabel, yaitu selisih atau beda (b), suku pertama (a), dan posisi ke-n (n).
$S_n = \frac{n}{2}(a + U_n)$ atau $S_n = \frac{n}{2}(2a + (n-1)b)$
Dengan:
$U_n = suku\ ke - n;$ $n = posisi\ suku\ yang\ ditanyakan;$
$S_n = jumlah\ n\ suku\ pertama;$ $b = selisih\ (U_{n-1} - U_n)$
$a = suku\ ke - 1\ atau\ U_1;$

Aktivitas 1 : Guru Membagi Siswa Menjadi Beberapa Kelompok

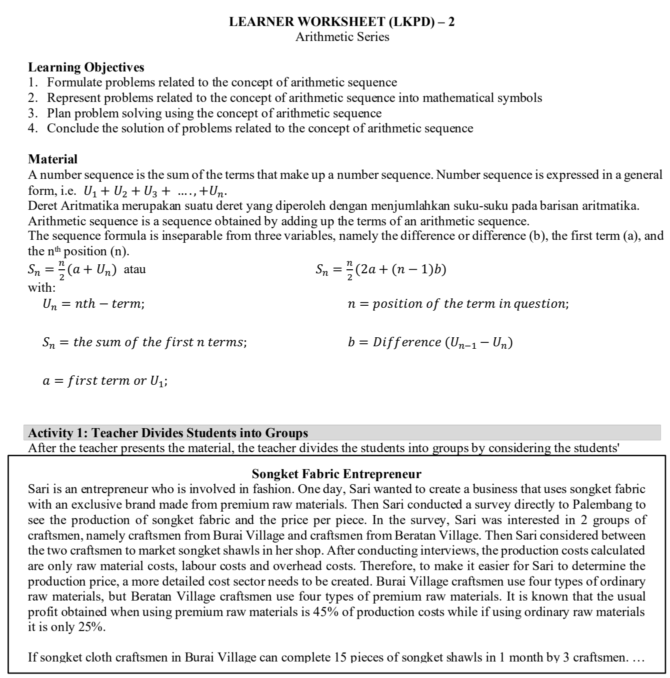
Setelah guru menyampaikan materi, guru membagi siswa menjadi beberapa kelompok dengan mempertimbangkan tingkat kemampuan siswa dimana setiap kelompok yang terbentuk memiliki kemampuan siswa yang heterogen.

Aktivitas 2 : Guru Menyajikan Kasus Interaktif

Kasus

Pengusaha Kain Songket
Sari merupakan seorang pengusaha yang berkecimpung di dunia *fashion*. Suatu hari Sari ingin membuat usaha yang menggunakan bahan dasar kain songket dengan *brand* eksklusif berbahan baku premium. Kemudian Sari melakukan survey langsung ke Palembang untuk melihat produksi kain songket dan harga per helainya. Pada survey tersebut Sari tertarik dengan 2 kelompok pengrajin yaitu pengrajin yang berasal dari Desa Burai dan pengrajin Desa Beratan. Kemudian Sari mempertimbangan antara kedua pengerajin tersebut untuk memasarkan selendang songket di toko miliknya. Setelah melakukan wawancara, biaya produksi yang dihitung hanya biaya bahan baku, biaya tenaga kerja dan biaya *overhead*. Oleh karena itu, untuk lebih memudahkan Sari dalam menentukan harga produksi perlu dibuat sektor biaya yang lebih rinci. Pengrajin Desa Burai menggunakan empat jenis bahan baku biasa, tetapi pengerajin Desa Beratan menggunakan empat jenis bahan baku premium. Diketahui bahwa keuntungan yang biasa didapatkan ketika menggunakan bahan baku premium sebesar 45% dari biaya produksi sedangkan jika menggunakan bahan baku biasa hanya 25%.

Jika pengrajin kain songket di Desa Burai dapat menyelesaikan 15 buah selendang songket dalam 1 bulan oleh 3 orang pengrajin. Dalam produksi memerlukan benang sutra tanpa



LEARNER WORKSHEET (LKPD) – 2
Arithmetic Series

Learning Objectives
1. Formulate problems related to the concept of arithmetic sequence
2. Represent problems related to the concept of arithmetic sequence into mathematical symbols
3. Plan problem solving using the concept of arithmetic sequence
4. Conclude the solution of problems related to the concept of arithmetic sequence

Material
A number sequence is the sum of the terms that make up a number sequence. Number sequence is expressed in a general form, i.e. $U_1 + U_2 + U_3 + \ldots, + U_n$.
Deret Aritmatika merupakan suatu deret yang diperoleh dengan menjumlahkan suku-suku pada barisan aritmatika.
Arithmetic sequence is a sequence obtained by adding up the terms of an arithmetic sequence.
The sequence formula is inseparable from three variables, namely the difference or difference (b), the first term (a), and the n^{th} position (n).
$S_n = \frac{n}{2}(a + U_n)$ atau $S_n = \frac{n}{2}(2a + (n-1)b)$
with:
$U_n = nth - term;$ $n = position\ of\ the\ term\ in\ question;$
$S_n = the\ sum\ of\ the\ first\ n\ terms;$ $b = Difference\ (U_{n-1} - U_n)$
$a = first\ term\ or\ U_1;$

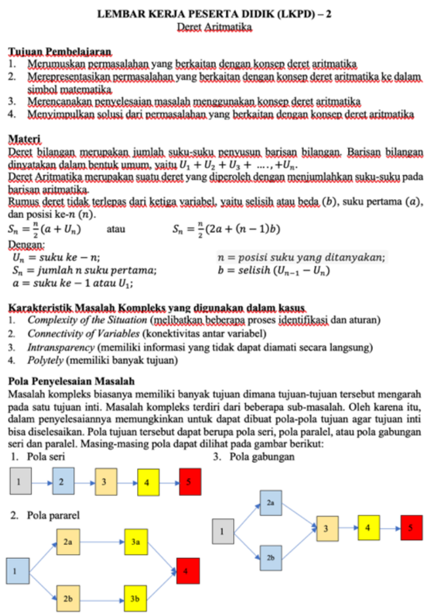
Activity 1: Teacher Divides Students into Groups
After the teacher presents the material, the teacher divides the students into groups by considering the students'

Songket Fabric Entrepreneur
Sari is an entrepreneur who is involved in fashion. One day, Sari wanted to create a business that uses songket fabric with an exclusive brand made from premium raw materials. Then Sari conducted a survey directly to Palembang to see the production of songket fabric and the price per piece. In the survey, Sari was interested in 2 groups of craftsmen, namely craftsmen from Burai Village and craftsmen from Beratan Village. Then Sari considered between the two craftsmen to market songket shawls in her shop. After conducting interviews, the production costs calculated are only raw material costs, labour costs and overhead costs. Therefore, to make it easier for Sari to determine the production price, a more detailed cost sector needs to be created. Burai Village craftsmen use four types of ordinary raw materials, but Beratan Village craftsmen use four types of premium raw materials. It is known that the usual profit obtained when using premium raw materials is 45% of production costs while if using ordinary raw materials it is only 25%.

If songket cloth craftsmen in Burai Village can complete 15 pieces of songket shawls in 1 month by 3 craftsmen. …

(a)

MATHEMATICS TEACHING RESEARCH JOURNAL
Golden Fall 2024
Vol 16 no 4



LEMBAR KERJA PESERTA DIDIK (LKPD) – 2
Deret Aritmatika

Tujuan Pembelajaran

1. Merumuskan permasalahan yang berkaitan dengan konsep deret aritmatika
2. Merepresentasikan permasalahan yang berkaitan dengan konsep deret aritmatika ke dalam simbol matematika
3. Merencanakan penyelesaian masalah menggunakan konsep deret aritmatika
4. Menyimpulkan solusi dari permasalahan yang berkaitan dengan konsep deret aritmatika

Materi

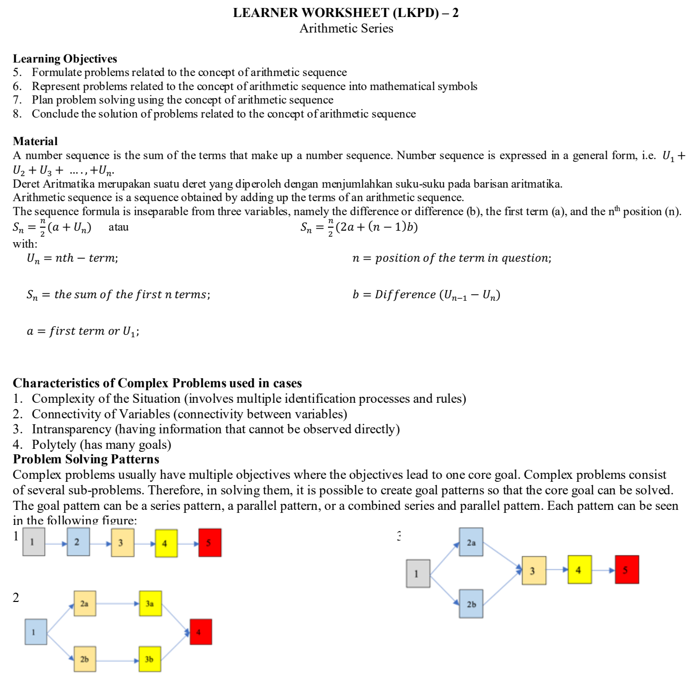
Deret bilangan merupakan jumlah suku-suku penyusun barisan bilangan. Barisan bilangan dinyatakan dalam bentuk umum, yaitu $U_1 + U_2 + U_3 + \dots, + U_n$.
Deret Aritmatika merupakan suatu deret yang diperoleh dengan menjumlahkan suku-suku pada barisan aritmatika.
Rumus deret tidak terlepas dari ketiga variabel, yaitu selisih atau beda (b), suku pertama (a), dan posisi ke-n (n).
$S_n = \frac{n}{2}(a + U_n)$ atau $S_n = \frac{n}{2}(2a + (n-1)b)$
Dengan:
$U_n = suku\ ke - n;$ $n = posisi\ suku\ yang\ ditanyakan;$
$S_n = jumlah\ n\ suku\ pertama;$ $b = selisih\ (U_{n-1} - U_n)$
$a = suku\ ke - 1\ atau\ U_1;$

Karakteristik Masalah Kompleks yang digunakan dalam kasus

1. *Complexity of the Situation* (melibatkan beberapa proses identifikasi dan aturan)
2. *Connectivity of Variables* (konektivitas antar variabel)
3. *Intransparency* (memiliki informasi yang tidak dapat diamati secara langsung)
4. *Polytely* (memiliki banyak tujuan)

Pola Penyelesaian Masalah

Masalah kompleks biasanya memiliki banyak tujuan dimana tujuan-tujuan tersebut mengarah pada satu tujuan inti. Masalah kompleks terdiri dari beberapa sub-masalah. Oleh karena itu, dalam penyelesaiannya memungkinkan untuk dapat dibuat pola-pola tujuan agar tujuan inti bisa diselesaikan. Pola tujuan tersebut dapat berupa pola seri, pola paralel, atau pola gabungan seri dan paralel. Masing-masing pola dapat dilihat pada gambar berikut:



LEARNER WORKSHEET (LKPD) – 2
Arithmetic Series

Learning Objectives

5. Formulate problems related to the concept of arithmetic sequence
6. Represent problems related to the concept of arithmetic sequence into mathematical symbols
7. Plan problem solving using the concept of arithmetic sequence
8. Conclude the solution of problems related to the concept of arithmetic sequence

Material

A number sequence is the sum of the terms that make up a number sequence. Number sequence is expressed in a general form, i.e. $U_1 + U_2 + U_3 + \dots, + U_n$.
Deret Aritmatika merupakan suatu deret yang diperoleh dengan menjumlahkan suku-suku pada barisan aritmatika.
Arithmetic sequence is a sequence obtained by adding up the terms of an arithmetic sequence.
The sequence formula is inseparable from three variables, namely the difference or difference (b), the first term (a), and the n[th] position (n).
$S_n = \frac{n}{2}(a + U_n)$ atau $S_n = \frac{n}{2}(2a + (n-1)b)$
with:
$U_n = nth - term;$ $n = position\ of\ the\ term\ in\ question;$
$S_n = the\ sum\ of\ the\ first\ n\ terms;$ $b = Difference\ (U_{n-1} - U_n)$
$a = first\ term\ or\ U_1;$

Characteristics of Complex Problems used in cases

1. Complexity of the Situation (involves multiple identification processes and rules)
2. Connectivity of Variables (connectivity between variables)
3. Intransparency (having information that cannot be observed directly)
4. Polytely (has many goals)

Problem Solving Patterns

Complex problems usually have multiple objectives where the objectives lead to one core goal. Complex problems consist of several sub-problems. Therefore, in solving them, it is possible to create goal patterns so that the core goal can be solved. The goal pattern can be a series pattern, a parallel pattern, or a combined series and parallel pattern. Each pattern can be seen in the following figure:

(b)

Figure 5: Before (a) and After Revision (b) of LKPD-2

In Figure 5, the results of expert validation of LKPD-2 on arithmetic sequence material can be seen. Figure 5(a) is the initial design for LKPD-2. Figure 5(b) is the result of revisions according to expert input related to the characteristics of complex problems that must be presented in LKPD and the presentation of possible problem-solving patterns (cases) so that students do not experience difficulties in learning arithmetic sequence material using the ICBL learning model. Some suggestions related to the use of the Cublend application include a less attractive screen display, monotonous writing, discussion tools that cannot be deleted/edited if you make a comment error, less soft colours, not too striking and dominant red and blue colors, writing settings that are sometimes wrong in beheading words, not normative like when typing in Microsoft Word.

Implementation Stage

At this stage, mathematics learning is carried out on the material of rows and Series using teaching materials made with the ICBL-Cublend App model. After the implementation stage was completed, the results of students' work on the mathematical literacy test instrument were analyzed to determine the effectiveness of teaching materials. The average score of students during the pretest was 20, while during the post-test was 48. To see a significant increase in students' mathematical communication skills, a comparison test was conducted with a paired sample T-test. Based on the statistical test, the p-value <0.05 is 0.000, so that $H0$ is rejected. Therefore, it is concluded that a significant difference exists between students' average mathematical literacy skills before and after applying mathematics learning using the ICBL-Cublend App model. The effect size value is 0.696, more than 0.25, so the effect of teaching materials using the ICBL-

Cublend App model on improving students' mathematical literacy skills is 69.6% and is classified as significant.

Evaluation Stage

After the implementation stage was completed, the results of students' work on the mathematical literacy test instrument were analyzed to determine the effectiveness of the LKPD. LKPD is said to be effective if there is an increase in the average value of students' mathematical literacy skills during the pretest and post-test. Table 3 shows increased students' mathematical literacy skills, so using LKPD is declared effective. The average score of students during the pretest was 20, while during the post-test was 48. Descriptive statistics of students' mathematical literacy scores during the pretest and post-test are presented in Table 3 below.

Statistics	Pretest	Post-test
Lowest score	4	13
Highest score	50	88
Average	20	48
Median	17	50
Mode	17	71

Table 3: Descriptive statistics of mathematical literacy ability of prospective teacher students

Meanwhile, a paired sample T-test was conducted to see a significant increase in students' mathematical literacy skills. The results are presented in Table 4 below.

Paired Differences										
		Mean	Std. Deviation	Std. Error Mean	95% Confidence Interval of The Difference		t	df	Sig. (2-tailed)	Effect Size
					Lower	Upper				
Pair 1	Post-Test – Pretest	28,33	18,97	3,25	21,70	34,95	8,70	33	0,000	0,696

Table 4: Paired sample t-test result

Based on Table 4, the p-value <0.05 is 0.000, so that H_0 is rejected and H_1 is accepted. Therefore, it is concluded that there is a significant difference in the average student learning outcomes before and after using the LKPD. The effect size value is 0.696> 0.25, and the effect of LKPD on improving students' mathematical literacy skills is 69% and is classified as significant.

Discussion

The research and development process begins by conducting content analysis and prospective teacher students, which includes curriculum analysis, teaching materials, materials, and students' mathematical literacy skills. Case-based learning is a learning approach in mathematics education that focuses on delivering the curriculum and not on what is taught (Talmage & Hart, 1977).

Therefore, this research was conducted to produce LKPD products that use the ICBL-Cublend App model and test instruments for students' mathematical literacy skills. The material used is row and sequence material. The product design process starts with designing the prototype of LKPD, mathematical literacy test instruments, and product assessment instruments for validators and trial subjects. After the product was created, it was validated and improved according to the suggestions of the validator. The product was tested and responded to by students who became test subjects. In determining the quality of the LKPD with the ICBL-Cubelnd App model, the criteria according to Nieveen (1999) are used. Namely, the development product is said to be of good quality if it meets the valid, effective, and practical requirements.

Validity Analysis

The validity of the LKPD is based on the expert's assessment. Based on Table 2, a percentage of 85.50 and a V coefficient of 0.82 were obtained. This means that LKPD with the ICBL-Cublend App model is feasible and valid regarding content. This is in accordance with the research of Fauzia et al. (2021), which states that case-based learning teaching materials on statistics material are suitable for support for learning mathematics at school. Likewise, research by Kamaruddin et al., (2021) using the 4D development model resulted in the case-based learning e-learning prototype being developed into a complete mathematics learning media. As for the validity of the mathematical literacy test instrument, 5 valid items were obtained, and 1 item needed to be revised. The question that needs to be revised contains an indicator of explaining why mathematical results or conclusions make sense or do not make sense, related to the context of the problem given. Furthermore, improvements were made to the test instrument in accordance with the suggestions and comments from the expert.

Practicality Analysis

The practicality of teaching materials is based on the prospective teacher-student response questionnaire results to the LKPD with the ICBL-Cublend App model. It was obtained that students responded to the LKPD with the ICBL-Cublend App model, with a total percentage of 79%. Students responded well by saying that the developed LKPD could be easily understood, which helped them solve the problem of rows and Series. Thus, it is concluded that the LKPD with the ICBL-Cublend App model is considered practical and can be used by lecturers and prospective mathematics teacher students, especially on the material of rows and Series. The following is one of the results of student work in the learning process using LKPD on arithmetic series material, especially when determining the solution relationship between objectives. In Figure 6, it can be seen that students can evaluate various objectives related to the main solution of the learning case given.

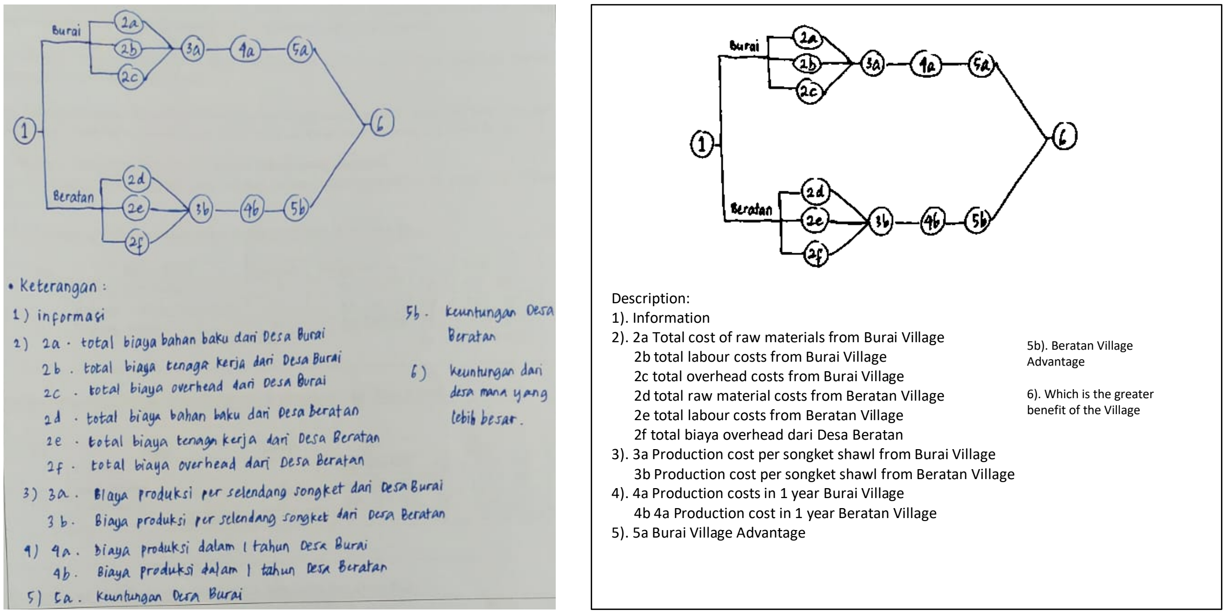
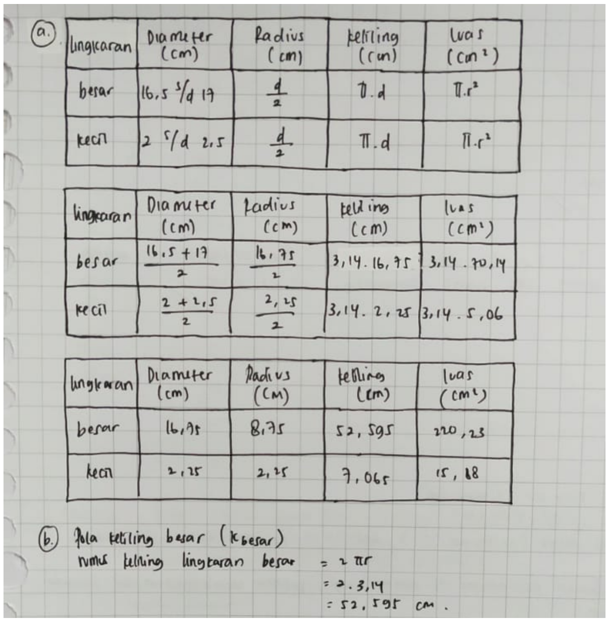


Figure 6: The results of student entries in making case settlement patterns

Effectiveness Analysis

The effectiveness of LKPD using the ICBL-Cublend App model is based on the difference in the average pretest and post-test results of prospective teacher students' mathematical literacy ability instrument. The increase in the average score of students before and after using teaching materials with the ICBL-Cublend App model was 28. The increase was proven significant based on the results of the paired sample t-test. In addition, based on the effect size coefficient of 69.6%, it demonstrates that the developed LKPD has a significant effect on improving students' mathematical literacy skills. Thus, it is concluded that LKPD with the ICBL-Cublend App model effectively improves mathematical literacy skills. The role of technology in learning shows high efficiency in improving students' mathematical literacy (Smirnov et al., 2021). The following presents the results of student work during the post-test related to students' ability to represent problem situations mathematically using tables as one of the indicators of the formulating ability of mathematical literacy.



Circle	Diameter (cm)	Radius (cm)	Circumference (cm)	Area (cm^2)
Big	16,5 to 17	$\frac{d}{2}$	$\pi.d$	$\pi.r^2$
Small	2 to 2,5		$\pi.d$	$\pi.r^2$

Circle	Diameter (cm)	Radius (cm)	Circumference (cm)	Area (cm^2)
Big	$\frac{16,5\ +\ 17}{2}$	$\frac{16,75}{2}$	$3,14\ \times 16,75$	$3,14\ \times 70,14$
Small	$\frac{2\ +2,5}{2}$	$\frac{2,25}{2}$	$3,14\ \times 2,25$	$3,14\ \times 5,06$

Circle	Diameter (cm)	Radius (cm)	Circumference (cm)	Area (cm^2)
Big	16,75	8,75	52,595	220,23
Small	2,25	2,25	7,065	15,88

Figure 7: One of the results of student answers on the formulate indicator

In Figure 7, it can be seen that students can understand the problem presented by calculating in detail the circumference and area of the large circle and the circumference and area of the small circle of the Talempong musical instrument. This did not happen when students worked on pretest questions. At the time of the pretest, students could not accurately represent the problem given. This illustrates that student teachers, after learning with LKPD using the ICBL-Cublend App, students' mathematical literacy skills increase.

This study has several limitations, including: (1). Variations in implementing the ICBL-Cubpend App between different teachers and institutions may lead to inconsistencies in the results; (2). Limited access to technology skills among learners may be an obstacle to optimizing the ICBL-Cubpend App model to improve mathematical literacy skills; (3). The study's relatively short duration may not be sufficient to capture the long-term impact of using this model on the mathematical literacy skills of student teachers.

CONCLUSION

The results showed that teaching materials using the ICBL-Cublend App model were declared valid and feasible for mathematics learning. Learning mathematics using the ICBL-Cublend App model effectively improves mathematical literacy skills and significantly influences mathematical literacy skills. The use of teaching materials using the ICBL-Cublend App model is classified as practical based on the responses of prospective mathematics teacher students. Students of prospective mathematics teachers generally stated that learning mathematics using the ICBL-Cublend App model was interesting, that the material taught was easy to understand, and could help improve mathematical literacy skills.

Using the Cublend application to learn mathematics has been proven effective in improving mathematical literacy skills. Therefore, this research contributes to the development of teaching materials in the form of student worksheets that are more interactive and contextual. The worksheets developed using the ICBL-Cublend App model can attract prospective teacher students' interest in mathematics. This study showed a significant increase in the mathematical literacy skills of prospective mathematics teachers. The results of this study imply that LKS based on the ICBL-Cublend App model can be used as an alternative learning model in higher education. In addition, the results of this study can also be used as material in teacher professional development programs.

Acknowledgements

We would like to thank the research and publishing centre of UIN Syarif Hidayatullah Jakarta for providing financial support for this research.

References

[1] Aisyah, P. N., Khasanah, S.U.N., Yuliani, A., & Rohaeti, E. E. (2018). AnalisisKemampuan Pemecahan Masalah Matematis Siswa SMP pada Materi Segiempat danSegitiga. JPMI – Jurnal Pembelajaran Matematika Inovatif, 1 (5), 1025-1036. https://doi.org/10.22460/jpmi.v1i5.p1025-1036

[2] Akyüz, G. (2014). PISA 2003 Sonuçlarına göre Öğrenci ve Sınıf Özelliklerinin Matematik Okuryazarlığına ve Problem Çözme Becerilerine Etkisi. *PISA 2003 Sonuçlarına Göre Öğrenci ve Sınıf Özelliklerinin Matematik Okuryazarlığına ve Problem Çözme Becerilerine Etkisi*, *9*(2), 668–678. https://doi.org/10.17051/io.74461

[3] Amelia, Kiki Nia Sania Effendi, dan Karunia Eka Lestari (2021). Analisis Kemampuan Literasi Matematis Siswa Kelas X SMA Dalam Menyelesaikan Soal Pisa. Majamath: Jurnal Matematika dan Pendidikan Matematika, Vol. 4, No. 2, 2021, h. 136–145. https://doi.org/10.36815/majamath.v4i2.1270

[4] Arslan, C., & Yavuz, G. (2012). A Study on Mathematical Literacy Self-Efficacy Beliefs of Prospective Teachers. *Procedia - Social and Behavioral Sciences*, *46*, 5622–5625. https://doi.org/10.1016/j.sbspro.2012.06.484

[5] Barlia .(2010). Elementary School Teachers' Personality In Students' Learning Motivation To Understand Concept Of Science. *Cakrawala Pendidikan*, Februari 2010, Th. XXIX, No. 1. https://doi.org/10.21831/cp.v1i1.215

[6] Barrows, H. S. (1986). A taxonomy of problem-based learning methods. *Medical Education*, *20*(6), 481–486. https://doi.org/10.1111/j.1365-2923.1986.tb01386.x

[7] Botha. H & Putten, S. V. (2018). How Mathematical Literacy Teachers Facilitate Mathematization in Modelling Situations. *African Journal of Research in Mathematics, Science and Technology Education, doi: http://10.1080/18117295.2018.1437337*

[8] Borg, W.R and Gall, M.D. (2003). Educational Research: An Introduction 4th Edition. London: Longman Inc. https://doi.org/10.2307/3121583

[9] Bridges, E. M., & Hallinger, P. (1999). The Use Of Cases In Problem-Based Learning. *The Journal Cases Educational Leadership*, *2*(2), 144–152. https://doi.org/https://doi.org/10.1177/155545899900200201

[10] Choi, I., Lee, S. J., & Kang, J. (2009). Implementing a case-based e-learning environment in a lecture-oriented anaesthesiology class: Do learning styles matter in complex problem solving over time? *British Journal of Educational Technology*, *40*(5), 933–947. https://doi.org/10.1111/j.1467-8535.2008.00884.x

[11] Chevallier, A. (2016). Strategic Thinking In Complex Problem Solving. In *Oxford University Press*. https://doi.org/10.1093/acprof:oso/9780190463908.001.0001

[12] Daar, G., Supartini, N., Sulasmini, N., Ekasani, K., Lestari, D., & Kesumayathi, I. (2023). Students' Perception of the Use of Learning Management System in Learning English for Specific Purpose During the Pandemic: Evidence From Rural Area in Indonesia. Journal of Language Teaching and Research. https://doi.org/10.17507/jltr.1402.16.

[13] Dick, W and L. Carey, J. O. Carey. (2005). The systematic Design of Instruction. New York : Logman. https://doi.org/10.1007/s11423-006-9606-0

[14] Fauzia, D. N., Sobiruddin, D., & Khairunnisa, K. (2021). Development Of Teaching Materials Based On Case-Based Learning On Statistics. *Algoritma: Journal of Mathematics Education*, *3*(1), 27-40. https://doi.org/10.15408/ajme.v3i1.20613

[15] Frensch, P.A. & Funke, J. (1995). *Complex Problem Solving: The European Perspective*. Routledge: Abingdon, UK. https://www.researchgate.net/publication/200134353_Complex_Problem_Solving-The_European_Perspective

[16] Gamit, A. (2023). Embracing Digital Technologies into Mathematics Education. Journal of Curriculum and Teaching. https://doi.org/10.5430/jct.v12n1p283.

[17] Grünig. R & Kühn. R. (2017). Solving Complex Decision Problems: A Heuristic Process. *Springer International Publishing AG.* https://doi.org/10.1007/978-3-662-53814-2

[18] Hendriana, Heris, dkk. 2017. Hard Skills dan Soft Skills Matematik Siswa. Cetakan Ke-2. Bandung: PT Refika Aditama.

[19] Hung, W. (2011). Theory to reality: A few issues in implementing problem-based learning. *Educational Technology Research and Development*, *59*(4), 529–552. https://doi.org/10.1007/s11423-011-9198-1

[20] Joyce, B., Weil, M., & Calhoun, E. (2004). Models of Teaching (7th ed.). Boston: Allyn and Bacon. https://www.scirp.org/reference/ReferencesPapers?ReferenceID=1656383

[21] Kamaruddin, E., Wijaya, Y., Avianti, R., Rahardjo, I., & Sunawar, A. (2021). Development of prototype e-learning mathematics learning tools using Moodle. IOP Conference Series: Materials Science and Engineering, 1098. https://doi.org/10.1088/1757-899X/1098/2/022095

[22] Kusumah, Y. S. (2012). *Literasi Matematis*. *Disajikan pada Seminar Nasional Matematika*, *Universitas Bandar Lampung*.

[23] Kirschner, P. A., Sweller, J., & Clark, R. E. (2006). Why minimal guidance during instruction does not work: An analysis of the failure of constructivist, discovery, problem-based, experiential, and inquiry-based teaching. *Educational Psychologist*, *41*(2), 75–86. https://doi.org/10.1207/s15326985ep4102_1

[24] Kurniawati. L, Miftah. R, Kadir. Muin. A. (2021). Student Mathematical Literacy Skill of Madrasah in Indonesia with Islamic Context. Tarbiya: Journal of Education in Muslim Society, Vol. 8, No. 1, 2021, h. 108–118. https://doi.org/10.15408/tjems.v8i1.3184

[25] Lai, C., & Hwang, G.-J. (2014). Effects of mobile learning time on students' conception of collaboration, communication , awareness and creativity. *Int. J. Mobile Learning and Organisation*, *8*, 276–291. https://doi.org/10.1504/IJMLO.2014.067029

[26] Miftah, R. Dahlan, JA. Kurniawati, L. Herman, T and Lutfiana, L (2024). How does interactive case-based learning improve students' complex mathematical problem-solving abilities?. Journal of Honai Math, Vol 7, No 2 (2024), h. 307-326. https://doi.org/10.30862/jhm.v7i2.622

[27] Murti, B. (1997) Prinsip dan Metode Riset Epidemiologi. Yogyakarta: Gajahmada

[28] Nazhifah, N., & Fathurohman, A. (2023). Teachers' Perspectives on the Learning Management System (LMS) in Physics Subject: A Preliminary Study. Jurnal Ilmiah Pendidikan Fisika. https://doi.org/10.20527/jipf.v7i1.7224.

[29] Nieveen, N. (1999). Prototyping to Reach Product Quality.London: Kluwer Academic Publisher. https://doi.org/10.12691/education-7-12-1

[30] OECD. (2013). PISA 2012 Assessment and Analytical Framework: Mathematics, Reading, Science,Problem Solving and Financial Literacy. In *OECD Publishing*.

[31] OECD. (2022). PISA 2022 Results: Factsheets – Indonesia. www.oecd.org/pisa

[32] Pugalee, D. K., & Chamblee, G. (2000). *Mathematical and Technological Literacy:*

Developing an Integrated 21st Century Model David. https://files.eric.ed.gov/fulltext/ED445908.pdf

[33] Plomp, T., & Nieveen, N. (2007). An Introduction to Educational Design Research. *Proceedings of the seminar conducted at the East China Normal University* (p. 77). Shanghai (PR China), November 23-26. https://research.utwente.nl/en/publications/an-introduction-to-educational-design-research-proceedings-of-the

[34] Rosa, M., & Orey, D. (2015). Social-Critical Dimension Of Mathematical Modelling. *In G. A. Stillman, W. Blum, & M. Biembengut (Eds.), Mathematical modelling in education research and practice: Cultural, social and cognitive influences (pp. 385–395). Cham: Springer.* https://doi.org/10.1007/978-3-319-18272-8_32

[35] Schon, D. A. (1987). *Educating the Reflective Practitioner : Toward a New Design for Teaching and Learning in the Professions*. Jossey-Bass Inc. https://doi.org/10.1002/chp.4750090207

[36] Schleicher. A. (2022). PISA 2022: Insights and Interpretations were. https://www.oecd.org/publication/pisa-2022-results/

[37] Smirnov, E., Tikhomirov, S., & Abaturova, V. (2021). The phenomenon of complex knowledge in teaching mathematics as a factor of mathematical literacy forming of school students. Perspectives of Science and Education. https://doi.org/10.32744/pse.2021.6.19.

[38] Stacey, K dan Tuner, R. (2015). Assessing Mathematical Literacy: The PISA Experience, Australia. Springer. https://www.amazon.com/Assessing-Mathematical-Literacy-PISA-Experience/dp/331910120X

[39] Talmage, H., & Hart, A. (1977). Investigative Teaching of Mathematics and Its Effect on the Classroom Learning Environment.. Journal for Research in Mathematics Education, 8, 345. https://doi.org/10.2307/748406

[40] Thomson, S. (2013). *A Teacher's Guide to PISA Mathematical Literacy*. https://www.researchgate.net/publication/263553588_Programme_for_International_Student_Assessment_A_teacher's_guide_to_PISA_mathematical_literacy

[41] Williams, B. (2005). Case based learning - A review of the literature: Is there scope for this educational paradigm in prehospital education? *Emergency Medicine Journal*, *22*(8), 577–581. https://doi.org/10.1136/emj.2004.022707

Figural and Non-Figural Linear Pattern: Case of Primary Mathematical Gifted Students' Functional Thinking

M. Syawahid[1], Nasrun[2], Rully Charitas Indra Prahmana[3]

[1]Universitas Islam Negeri Mataram, Indonesia,

[2] Universitas Muhammadiyah Makasar, Indonesia

[3]Universitas Ahmad Dahlan, Indonesia

[1]syawahid@uinmataram.com, [2]nasrun@unismuh.ac.id, [3]rully.indra@mpmat.uad.ac.id

Abstract: Mathematically gifted students have a potential for understanding and connecting mathematics concept. Pattern generalization as a part of functional thinking becomes one of the benchmarks for gifted students in mathematics. The mathematics curriculum in Indonesia that has not accommodated the functional thinking ability of elementary school students is the basis for this study. It focuses on describing mathematically gifted students functional thinking in solving figural and non-figural linear pattern task. Functional thinking abilities in this study consist of thinking process in near generalization, far generalization, formal generalization and determine inverse. Case study of qualitative approach used in describing mathematically gifted students thinking. Data were collected from 5^th^-grade of gifted student's problem solving in figural and non-figural linear patterns task. the finding showed that gifted students are able in functional thinking in different ways. They represented the relationship of two quantities symbolically. In solving figural linear pattern task, gifted students perform FT consist of: near generalization by counting, multiplicative approach, and contextual strategy; far generalization by contextual strategy; formal generalization by multiple difference and proportional strategy; and determine inverse by using general rule. In solving non-figural linear pattern task, gifted students perform FT consist of near generalization, far generalization, and formal generalization by multiple difference strategy; and determine inverse by using general rule.

Keywords: functional thinking; gifted students; linear pattern.

INTRODUCTION

Mathematically gifted students (MGS) have become an interesting topic in recent years. MGS refers to students who have mathematics abilities that manifest in the form of successful performance and creativity in mathematics tasks (Krutetskii, 1976). When compared to the top 10% of peers their own age, gifted students are those who have outstanding potential in one or more human ability domains, such as intelligence, creativity, social skills, or mindset (Gagné, 1985). MGS also have a unique strategy for reasoning and problem solving (Pitta-Pantazi, Christou, Kontoyianni, and Kattou, 2011). Krutetskii (1976) described mathematics giftedness as formal perception abilities, logical thought abilities, mathematical symbol abilities, generalization abilities, mathematical reasoning and structured abilities, flexibility of mental process in mathematical activities, striving for clarity, simplicity, economy, and rationality of solutions; mathematical memory; mathematical cast of minds expressed in striving to interpret the environment mathematically. Students with good problem-solving skills, metacognitive skills, creative mathematical thinking, and high ability or performance in mathematical problem solving are typically considered to be gifted in mathematics (Leikin, 2018, 2021).

There has been a lot of research on mathematically gifted students. Pitta-Pantazi et al. (2011) constructed a structured Model indicating that mathematical abilities contribute more than mathematical creativity. The structured Model also confirmed that the nature of cognitive abilities (fluid intelligence and working memory) predicts mathematical giftedness. Gutierrez et al. (2018) found that MGS are much faster in learning mathematics, they used different strategies in generalization linear patterns. It also showed that students made all necessary cognitive effort, as much as was possible due to his limited knowledge of algebra. Paz-Baruch et al. (2022) revealed five main cognitive factors: visual-serial processing (VSP); arithmetic abilities (AA); pattern recognition (PR); auditory working memory (AWM); visual-spatial working memory (VSWM); and Structural equation Modeling (SEM) based on the factor analysis revealed clear differences in the role of cognitive abilities as predictors of EM, G, and MG.

Other research characterized MGS as generalization abilities in mathematics structure and pattern (Assmus & Fritzlar, 2022; Erdogan & Gul, 2023; Gutierrez et al., 2018; Krutetskii, 1976; Paz-Baruch et al., 2022; Pitta-Pantazi et al., 2011). Assmus & Fritzlar (2022) found that mathematical giftedness and mathematical creativity have a high correspondence in concerning the invention of figural patterns. MGS used figural reasoning in generalizing linear patterns and numerical reasoning in generalizing non-linear patterns (Girit Yildiz & Durmaz, 2021). They also used different strategies in solving linear and nonlinear pattern (recursive, chinking, contextual, and functional) (Erdogan & Gul, 2023).

Pattern generalization involved the ability to relate and represent two quantities as words, tables, graphics, or symbols. This ability is termed functional thinking (FT). FT is a fundamental part of algebraic reasoning (Blanton & Kaput, 2005; Smith, 2008). It's key for algebraic thinking because it involves generalizations of how quantities are related (Tanıslı, 2011). Functional thinking is a representational thinking that focuses on the relationship between two (or more) quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances (Smith, 2008). Blanton et al. (2011) define functional thinking as generalizing relationships between co-varying quantities, expressing those relationships in words, symbols, tables, or graphs, and reasoning with these various representations to analyze function behavior. According to Smith (2008) and Blanton et al. (2011), functional thinking consists of the generalization and representation of relationships between two variables.

There are two types of pattern generalization tasks: figural and non-figural pattern generalization tasks (Chua & Hoyles, 2014; F. Rivera & Becker, 2003). Figural pattern generalization task included a task with the pattern listed as a sequence of pictorial context. Non-figural pattern generalization is often called numerical pattern generalization when the pattern is listed as a sequence of numbers.

Generalization is part of three activities: identifying similarities in a case, expanding one's reasoning beyond the range of origin, and obtaining broader results from certain cases (Kaput, 1999). Chua & Hoyles (2014) stated that generalization is a process involving at least one of the following activities: (i) to examine a few particular cases to identify a commonality; (ii) to extend one's reasoning beyond those particular cases; and (iii) to establish a broader result for those particular cases. Generalization starts with a sense of pattern, using patterns, making conjectures, and testing the results of generalizations. Thus, it can be said that the process of generalization is related to the understanding of patterns and conjectures (Mason, Stacey, and Burton, 2010).

Concerning linear patterns, Stacey (1989) distinguishes between ‘‘near generalization’’ tasks, which include finding the next pattern or elements that can be reached by counting, drawing, or forming a table, and ‘‘far generalization’’ tasks, in which finding a pattern requires an understanding of the general rule. Amit & Neria (2008) add the “formal generalization” term as an explicit requirement for representing a generalization in a formal mode, striving toward algebra.

Several studies have revealed about students' functional thinking. Warren et al. (2006) found that elementary students are capable not only of developing functional thinking but also of communicating their thinking both verbally and symbolically. Blanton & Kaput (2004, 2005) found that students were able to think functionally at the kindergarten level co-variationally and were able to think functionally as a correspondent in the 1st grade. Tanıslı (2011) found that five-grader students thought on co-variation while working with the linear function tables. Wilkie & Clarke (2016) found four types of visual structure in functional thinking, and Stephens et al. (2017) found three types and ten levels of *student sophistication* in functional thinking.

There are few studies about functional thinking for elementary students in Indonesia. Rusdiana et al. (2018, 2017) reported that there are two aspects of elementary students pattern generalization in Indonesia: focus on the number of patterns and focus on the figure of patterns. Syawahid et al. (2020) revealed that elementary school students in Indonesia are capable of functional thinking by starting recursively to find a corresponding formula. Functional thinking of elementary students in Indonesia can also be categorized as recursive-verbal, correspondence-verbal, and recursive-to-correspondence-symbolic (Syawahid, 2022).

The limited number of studies on functional thinking of elementary school students in Indonesia may be due to the applied mathematics curriculum. Based on the National Council Teachers of Mathematics (NCTM) standard, pattern generalization as a core of FT was thought of from grade 3 to grade 5 of elementary school (NCTM, 2000). However, in the Indonesia curriculum, pattern generalization was thought of in secondary school (MoEC, 2016). Despite the difference, it’s suspected that some Indonesian students in elementary school are able to generalize patterns, or FT (Rusdiana et al., 2017, 2018; Syawahid et al., 2020; Syawahid, 2022).

Previous studies of functional thinking students of elementary students in Indonesia just revealed students thinking in solving figural pattern tasks (Rusdiana et al., 2017, 2018; Syawahid et al., 2020; Syawahid, 2022). There are no studies that reveal students functional thinking in solving non-figural pattern tasks. This allows for an in-depth study of elementary students' functional thinking in solving figural and non-figural patterns.

This study aims to describe MGS FT in solving figural and non-figural linear pattern tasks. FT in this study consists of near generalization, far generalization, formal generalization, and determining the inverse. Near, far, and formal generalization are part of generalization and representation relationship, and determining the invers is part of analyze function behavior.

METHOD

Research Design

This study tries to describe gifted elementary students functional thinking in solving figural and non-figural linear patterns. It also identifies the gifted students' strategies in performing functional thinking. This study used a case study of a qualitative research approach. It involve a detailed study of one or a few individuals (Fraenkel, Wallen, and Hyun, 2012). The detailed study in this research refers to a case of a few gifted students' functional thinking. A case study allows searching for a selected subject in detail (Cohen, Manion, and Morrison, 2000) and exploring problems to find an in-depth understanding (Creswell, 2012). The types of cases in this study include the intrinsic cases of gifted elementary students performing functional thinking (Creswell, 2012; Fraenkel et al., 2012). A typical sample of purposive sampling is used to select research subjects. A typical sample is considered to be representative of that which is being studied (Fraenkel et al., 2012).

Participant and Instrument

This study involved 62 *Athirah* elementary school at Makasar, Indonesia. They were 13-year-olds of fifth grade. We gave a linear pattern task (Wilkie & Clarke, 2016) to 62 students and found that there were two students who had a correct answer. Two students who had a correct answer have the initial AA and AG. Based on interviews with mathematics teachers, AA and AG are classified as students with high achievement and often represent the school in mathematic competitions (e.g., mathematics Olympiads).

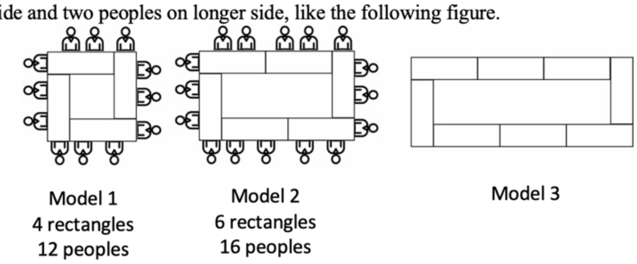
Data in this study involved qualitative data consisting of student's answers and interviews in-depth. It was carried out by giving functional thinking tests and interview protocols. Functional thinking tests consist of figural linear pattern problems and non-figural linear pattern problems. Figural linear pattern problems were adopted from Lepak, Wernet, and Ayieko (2018). These problems describe the arranged tables provided by an accompany for business meeting. There is a role for this arrangement in that the table is a rectangle and can be occupied by one person on the shorter side and two people on the longer side. A picture of arranged tables is given to be a hint in this problem. Second, non-pictorial linear pattern problem was produced by research. This problem illustrates plant height growth observed by researchers on the first, second, and third days. Students are asked to determine plant height on the day after and determine general rules for growth in plant height.

Aspect	Indicator
Near generalization	Determine the nearest unknown quantity of the dependent variable from a given pattern
Far generalization	Determine a certain unknown quantity of the dependent variable from a given pattern
Formal generalization	Determine a relationship between dependent and independent variable (e.g., by word, table, graph, or symbolic)
Determine inverse	Determine the quantity of the independent variable for the known dependent variable

Table 1: Functional thinking aspect

Arranging Tables

A company provides tables for business meetings. For each table, there is one people can seat on shorter side and two peoples on longer side, like the following figure.



1. How many people can sit on model 3? Explain how you got it!
2. How many people can sit on model 13? Explain how you got it!
3. Write an equation for P people number who can sit at S model? Explain how you got it!
4. Mr. Yogi wants a meeting with 75 peoples. Which model will he use?

Plant Growth

A researcher was observing the growth of a plant. On first day (H1), the plant height was 4 cm, on second day (H2), the plant height was 6 cm and on the third day (H3) the plant height was 8 cm.

1. Determine the plant height on 4th days (H4) and 5th days (H5)!
2. Determine the plant height on 7th days (H7) and 10th days (H10)!
3. What can you notice about the structure of the plant's growth every day? Write down how you got the height of the plant (T) on a certain day (H).
4. On which day the plant height was 86 cm? Explain how you got your answer!

Figure 1: Figural and non-figural functional thinking task

Data Collection

This study was conducted using a task-based interviews. Task-based interview present figural and non-figural linear pattern problems and participants were required to explain their solution. One of the purposes of conducting task-based interviews is to identify patterns of subject behavior when working on tasks, such as success in completing tasks, strategies used, expressions of curiosity, and certain actions associated with student success and failure in completing tasks.(Mejía-Ramos & Weber, 2020).

Data Analysis

The qualitative data from student's answer and transcript interviews was analyzed by comparative analysis between each category, and new categories emerged (Creswell, 2012; Leedy & Ormrod, 2019). Interview data and transcripts were reduced to fragments involving explanations of student's main ideas. The data were coded, sorted, and read repeatedly to answer research questions. In addition, the pre-established categories (Table 2) were considered to interpret the student's functional thinking emerging from their answer.

Category	Description
Counting	Draw the next figures and count their element
Recursive	Continue the sequence using the numerical difference between consecutive terms or explicit the recursive relation between consecutive terms
Multiple difference	Use the difference between consecutive terms as a multiplicative factor (adjusting or not the result) to obtain distant terms or the general term.
Proportional	Use multiplicative strategies, starting from one known term of the sequence to find distant terms or the general term
Visual	Express a relation between the two varying quantities for a distant term or in the general term, based on the characteristics of the pictorial representation
Numerical	Express a relation between the two varying quantities for a distant term or in the general term, based on the numerical sequence.
Contextual	Constructing a rule based on the information provided in the situation; relating the rule to a counting technique
Guest and check	Guessing a rule without regard to why this rule might work. Usually, this involves experimenting with various operations and numbers provided in the problem situation

Table 2.:Strategies in generalize function relation (Lannin, 2005; Oliveira, Polo-Blanco, and Henriques, 2021)

In order to conceptualize qualitative investigations, research must be trustworthy when collecting, analyzing, and interpreting the results (Merriam, 2015). The study employs both the triangulation method and triangulation theories to ensure its validity. The triangulation method is used by comparing the data collection method and triangulation theories by comparing the data with the relevance theories (Merriam, 2015). In this study, we applied the triangulation method by analyzing written documents from students, which included their solutions and interview outcomes. We applied the triangulation theory by evaluating the study's findings against the relevance of multiple journal-published studies.

RESULTS

This section presents gifted students functional thinking (AA and AG), which consists of near generalization, far generalization, formal generalization, and determining the inverse in solving figural and non-figural linear pattern tasks. AA and AG's functional thinking data is described in the form of strategies used in generalizing and representing figural and non-figural linear pattern tasks.

Figural Linear Task

In a figural linear task, gifted students are asked to complete a task that contains a table setting picture pattern. Gifted students are asked to perform near generalizations, far generalizations, formal generalizations, and determine inverses.

AA's functional thinking in solving a figural linear pattern task involved AA's abilities in generalizing two quantities (people number and Model) and representing them by words, table, graphic, or symbolic. Generalization abilities in this study consist of near generalization, far generalization, and formal generalization.

At near generalization, AA tries to determine people number at Model 3. Firstly, AA draws the Model 3 figure and writes a number based on people number seated (figure 2). Furthermore, AA associates the number with a multiplicative approach (from $2+2+2+2+2+2+2+2+1+1+1+1$ to $(8\times 2)+(4\times 1)$). In this stage, AA uses a counting strategy by drawing the next figure and counting its element (Lannin, 2005; Oliveira et al., 2021).

R : how did you get people number at Model 3?

AA : first, I draw Model 3 and write a number representing the people sitting on the table, i.e., 1 and 2. Second, I count the number by associating it with a similar number to get $(8\times 2)+(4\times 1)=20$.

R : how did you write a number representing the people sitting on the table? Can you explain?

AA : as we can see in Model 1 and Model 2 that there are two people sitting at a table with long sides and one person sitting at a table with a wide side. So, I wrote a number based on that.

Before AA performs a far generalization, AA reflects on Model 3 that there are two people sitting in pairs at the length of three tables above and two people sitting in pairs at the length of three tables below. While there are 8 other people, consisting of 1 person at the width of the tables above, 1 person at width of the tables below, 3 people at the table on the left, and 3 people at the table on the right. This reflection was used by AA to get the people number at Model 13 (far generalization). AA imagines that at Model 13 there are two people sitting in pairs at the 13 tables above and two people sitting in pairs at the 13 tables below $(2 \times 13 + 2 \times 13 = 4 \times 13)$. While there are 8 other people at Model 13, consisting of 1 person at the table above, 1 person at the table below, 3 people at table on the left and 3 people at table on the right $(1 + 1 + (2 + 1) + (2 + 1) = 8)$. In this stage, AA used a contextual strategy by constructing a rule based on information provided in the situation; relating the rule to a counting technique (Lannin, 2005).

R : did you get people number at Model 13 by previous way?

AA : no, I can't draw the table at Model 13.

R : so, how did you get people number at Model 13? Can you explain?

AA : I observe the people number at Model 3 that there are three table above, three table below and two table in side. At three table above and below there are $3 \times 2 \times 2$ *people. At two table in side, there are 8 people. It means that for Model 13 there are 13 table above, 13 table below and 2 table in side. At 13 table above and below there* $13 \times 2 \times 2$ *people and at two table in side there are* 8 *people.*

In formal generalization, AA tries to get a general rule of the people numbers. AA observes that the people numbers in Model 1, Model 2, and Model 3 construct a sequence with the same difference. The sequence has a difference of 4. AA realizes that the people number at Model 2 consists of the people number at Model 1 and the difference $(16 = 12 + 4 \times 1)$, while the people number at Model 3 consists of the people number at Model 2 and the difference $(20 = 16 + 4 = 12 + 4 \times 2)$. In this stage, AA used a multiple difference strategy by using the difference between consecutive terms as a multiplicative factor (adjusting or not the result) to obtain distant terms or the general term (Oliveira et al., 2021).

R : how did you get the relationship between people number (P) and Model S?

AA : I noticed the people numbers at 1st, 2nd, and 3rd, and got that the people numbers have the same difference for each term. For example, people numbers at 2nd equal to people numbers at 1st and difference $(16 = 12 + 4)$*, while the people number at 3rd equal to the people number at 2nd and difference* $(20 = 16 + 4 = 12 + 2 \times 4)$*. I decide that for Model S, the people number equal to the people number at 1st and* $(S - 1)$ *multiply by difference,* $12 + (S - 1) \times 4$*.*

To determine the inverse, AA tries to determine which Model to use for 75 people. AA used a general rule obtained at formal generalization by substituting P by $75 + 1$ and performing an algebraic operation to gen $S = 17$. AA realizes that to get S, he must provide an even of the people number. Therefore, he substitutes P by $75 + 1$.

R : how did you get which Model will be used for 75 people?

AA : I used the previous formula $P = 12 + (S - 1) \times 4$

R : how did you use the formula? Can you explain?

AA : firstly, I substituted P by 75 to get S, but the coefficient of S and a Constanta was an even number, so I decide to add up 75 by 1 and then I performed the mathematics' operation to get $S = 17$. .

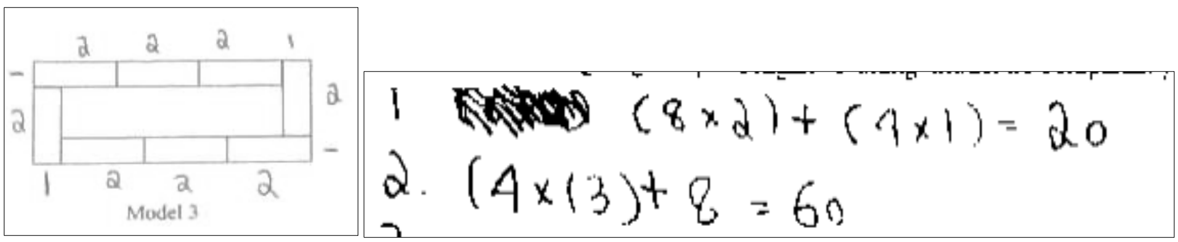


Figure 2: AA's counting and contextual strategies in near and far generalization

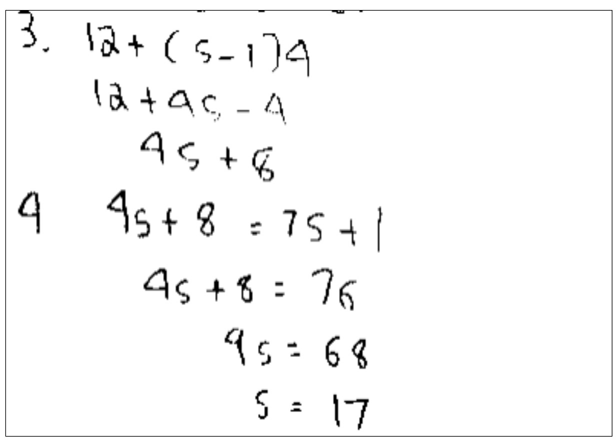


Figure 3: AA's multiple difference strategies in formal generalization and determine inverse.

In performing generalization, AA was capable of representing the relation between two quantities by symbol. He writes general rule of the relationship between people number quantities and Model quantities as $P = 4S + 8$. It shown that AA was capable of thinking functionally by correspondence, which means the emphasis is on the relation between corresponding pairs of variables (Confrey & Smith, 1991; Smith, 2008).

AG functional thinking in solving figural linear pattern tasks involved AG's abilities in generalizing two quantities (people number and Model) and representing them by words, table, graphic or symbolic. Generalization abilities in this study consist of near generalization, far generalization, and formal generalization.

At near generalization, AG tries to determine the people number at Model 3. Firstly, AG noticed the people number at Model 1, consisting of three people sitting above, below, left side, and right side of the table, and it can be written mathematically by "3×4". Secondly, AG noticed people number at Model 2 consisting of people number at Model 1 and two people sitting on the table above and two people sitting on the table below. It can be written by "$(3 \times 4) + (2 \times 2)$".

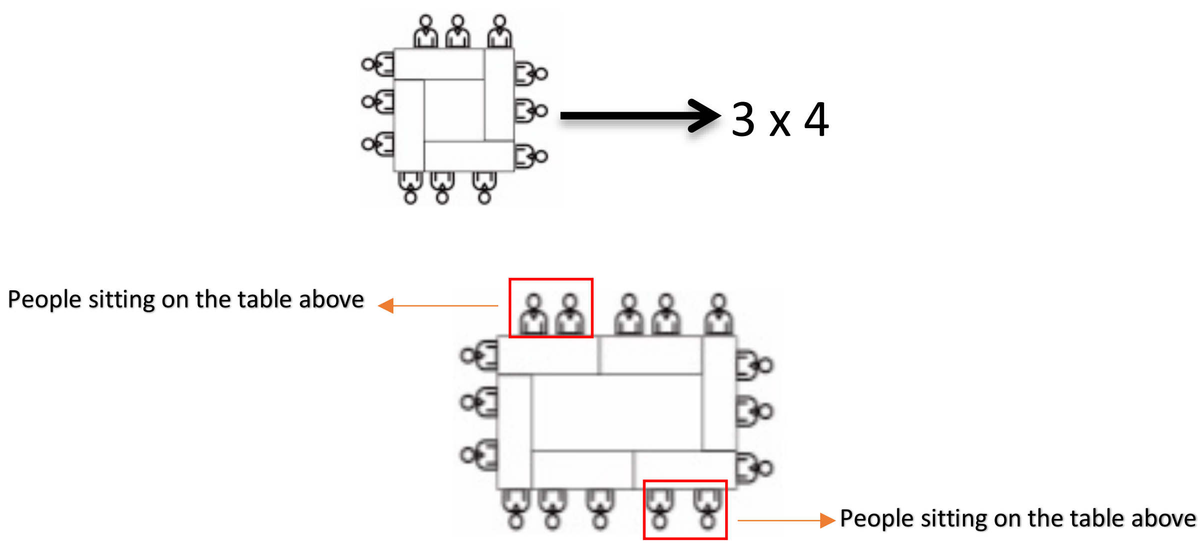


Figure 4: AG noticing of people number at Model 1 and Model 2

Based on these understandings, AG determined the people number at Model 3 by expressing that people number consists of people number at Model 1 and four people sitting on the table above and four people sitting on the table below. It can be written as "$(3 \times 4) + (4 \times 2)$". In this stage, AG used a counting strategy by drawing the next figures and counting their elements (Oliveira et al., 2021). Moreover, AG used contextual strategy by constructing a rule based on the information provided in the situation (Lannin, 2005).

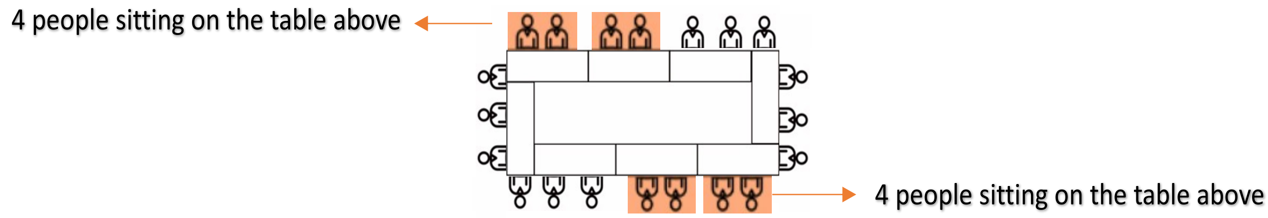


Figure 5: AG express the people number at Model 3.

In far generalization, AG tries to determine the people number at Model 13. Firstly, AG tries to construct a general form of people number by determining people number at Model 4. She used the same strategy previously to get people number at Model 4. She realized that people number at Model 4 consist of people number at Model 1, six people sitting on the table above, and six people sitting on the table below. It can be written by "$(3 \times 4) + (6 \times 2)$". Secondly, AG arranges the people number at Model 1, Model 2, Model 3, and Model 4 consecutively. In this stage, AG constructed a general form of people number at Model s consisting of people number at Model 1 and multiplication between “2” and $2n$ subtracted by 2.

$$
\begin{gathered}
P(1) = (3 \times 4) \\
P(2) = (3 \times 4) + (2 \times 2) \\
P(3) = (3 \times 4) + (4 \times 2) \\
P(4) = (3 \times 4) + (6 \times 2) \\
\vdots \\
P(n) = (3 \times 4) + ((2n - 2) \times 2)
\end{gathered}
$$

Finally, AG determined the people number at Model 13 using a general form by substituting s with 13 and performing a mathematical operation to get people number equal to 60. In this stage, AG performs a formal first for far generalization. She used a numerical correspondence strategy by expressing a relation between quantity of people number and Model s quantity for a distance term in general term.

At formal generalization, AG constructed a final general form of the relationship between people number quantity and Model s quantity using basic algebraic operations to get $P = 4s + 8$, where P refers to people number quantity and s refers to Model s quantity. In addition, AG used the final general form to determine inverse. She substituted P with 76 and performed a mathematical operation to get $s = 17$. It was concluded by AG that there was Model 17 for 75 people.

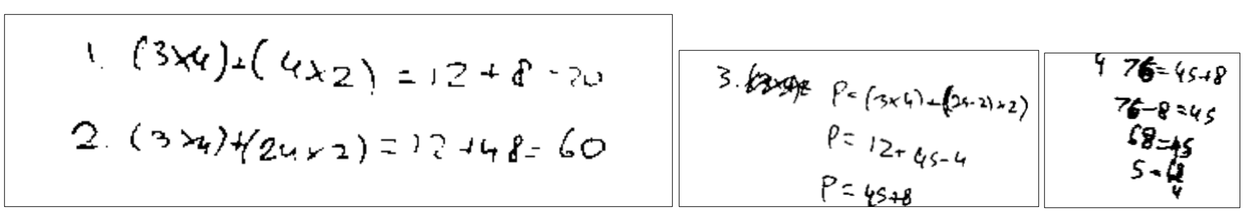


Figure 6: AG answer in solving figural linear pattern task.

Non-Figural Linear Patterns Task

Non-figural linear pattern task involves the generalization of plant height and day quantity. It consists of questions of near generalization, far generalization, formal generalization and determining the inverse.

In this task, AA tries to generalize the relationship between two quantities consisting of days (H) and plant height (T). Here, the information provided consists of plant's height at the 1st, 2nd, and 3rd days.

At near generalization, AA tries to determine plant height at 4th days. Firstly, AA processes an information of plant height at 1st, 2nd, and 3rd days. He realizes that the plant height has the same difference for each day. Furthermore, AA constructed the plant height using the multiple difference strategy, for example: plant height at the 2nd day equals plant height at the 1st day and the difference ($H(2) = H(1) + 2$) while plant height at 3rd day is equal to plant height at 2nd and difference ($H(3) = H(2) + 2 = H(1) + 2 \times 2$). With this strategy, AA generalizes that the plant height at 4th days is equal to plant height at 1st day and the difference multiply by 3 ($H(4) = 4 + (3 \times 2)$). "3" refers to the day number (4th) subtracted by 1 ($3 = 4 - 1$).

AA uses the same strategy to get plant height at 5th days, which is equal to plant height at 1st day and the difference multiplied by 4. "4" refers to the day number (5th) subtracted by 1 ($4 = 5 - 1$). At this stage, AA performs a formal generalization. AA not only uses the multiple of difference strategy, but also develops to numerical correspondence strategy by expressing a relation between the two varying quantities for a distant term or in the general term, based on the numerical sequence (Oliveira et al., 2021). In this case, AA generalizes the general form of relationship between day quantity and plant height quantity as $T = 4 + 2(H - 1)$, where T refers to plant height, 4 refers to plant height at 1st day, 2 refers to difference, and H refers to day number.

At far generalization, AA used a general form of relationship between day quantity and plant height quantity previously obtained ($T = 4 + 2(H - 1)$). AA gets the plant height at 7th days by adding up the plant height at 1st day and multiplying the difference by 6 (obtained from $7 - 1$). Likewise, AA gets the plant height at 10th day by adding up the plant height at 1st day and multiplying the difference by 9 (obtained from $10 - 1$).

In determining the inverse, AA tries to determine on what day the plant has 86 cm of height. AA used the general form of relationship between day quantity and plant height quantity "$T = 4 + 2(H - 1)$", substituted T with 86, and performed a mathematical operation to get $H = 42$. In this stage, AA has an understanding of algebraic operation by substituting and operating mathematical symbol.

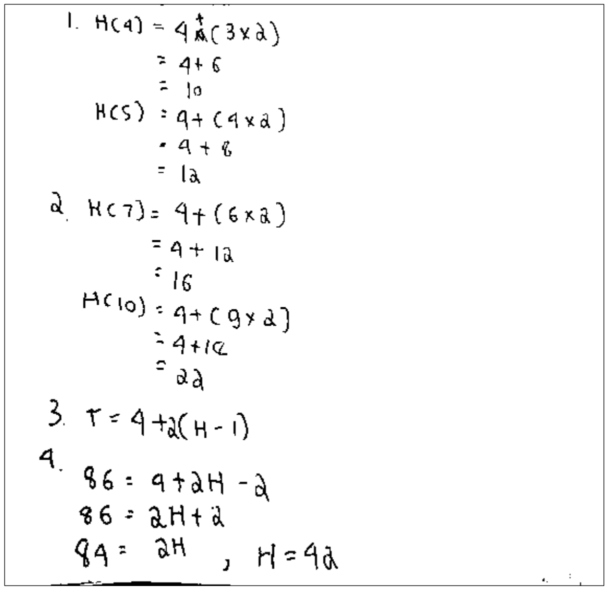


Figure 7: AA answer in solving non-figural linear pattern task

At near generalization, AG observed the plant height on the 1st, 2nd, and 3rd days. She found that the plant height had the same difference for each term. After realizing that there is a constant difference for each term, AG conjectured the plant height at 2nd day as plant height at 1st and the difference, likewise, the plant height at 3rd was equal to the plant height at 2nd and the difference.

$$H1: T = 4$$

$$H2: T = 4 + 2 = 6$$

$$H3: T = 4 + 2 \times 2 = 8$$

AG determined the plant height at 4th day by expressing an equation involving plant height at 1st and multiplying the difference by 3. 3 is obtained by 4 subtracted by 1, where 4 refers to 4th term. its' written with "$H4: T = 4 + (3 \times 2)$". She also expressed plant height at 5th day as plant height at 1st day and multiplied the difference by 4, where 4 is obtained by subtracted 5th term by 1, "$H5: T = 4 + 4 \times 2$". In this stage, AG used a multiple of difference strategy by use the difference between consecutive terms as a multiplicative factor to obtain the distant term (Oliveira et al., 2021).

4 + 2 + 2 + 2 = 4 + 3 x 2

↓ 1st day ↓ 2th day ↓ 3th day ↓ 3th day

Figure 8: AG strategies in determining plant height at 4th days

At far generalization, AG determined plant height at 7th and 10th day. She used the same strategy previously by constructing plant height as a sum between plant height at 1st day and multiplicative factor. For example, AG expressed plant height at 7th day as a sum between plant height at 1st and the difference multiplied by 6, where 6 is obtained by subtracted 7th term by 1 ($H7: T = 4 + 6 \times 2$). Likewise, AG expressed plant height at 10th day as a sum between plant height at 1st and the difference multiplied by 9, where 9 is obtained by subtracted 10th term by 1 ($H10: T = 4 + 9 \times 2$).

Plant Height at 7th days	$4 + (7 - 1) \times 2 = 4 + 6 \times 2$
Plant Height at 10th days	$4 + (10 - 1) \times 2 = 4 + 9 \times 2$
	↓ 1st day ↓ difference

Figure 9: AG strategies in determining plant height at 7th and 10th days

At formal generalization, AG expressed the relationship between plant height quantity and day quantity symbolically. She conjectured a general term for these relationships from activities at near generalization and far generalization. The general term of the relationship between plant height and the day quantity was expressed as $T = 4 + (H - 1) \times 2$, where T refers to plant height, 4 refers to plant height at 1st day, H refers to the day's quantity, and 2 refers to the difference.

$$H1: T = 4$$

$$H2: T = 4 + 1 \times 2$$

$$H3: T = 4 + 2 \times 2$$

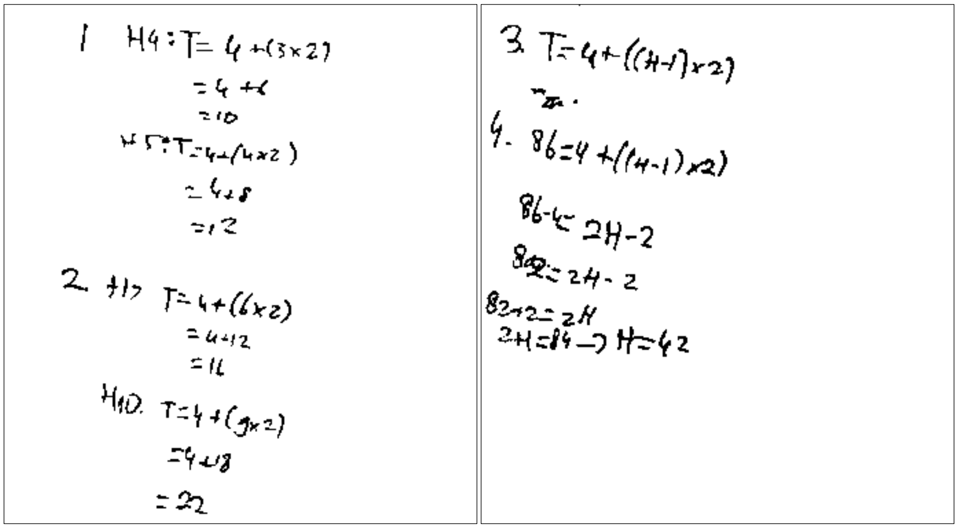
$$H4: T = 4 + 3 \times 2$$

$$H5: T = 4 + 4 \times 2$$

$$\vdots$$

$$Hn: T = 4 + (n - 1) \times 2$$

To determine the inverse, AG tries to determine on what day the plant height was 75. She used the general form previously by substituting T with 86 and performing mathematical operations to get $H = 42$.



(a) (b)

Figure 10: (a) AG Near and Far Generalization; (b) AG Formal Generalization and Determining Inverse in Solving Non-Figural Linear Pattern Task

	Figural Linear Pattern	**Non-figural Linear pattern**
AA	Near generalization - Use counting strategy by drawing next figure and count people number by multiplicative approach $[(8 \times 2) + (4 \times 1)]$ Far generalization - Use contextual strategy by constructing a rule based on information providing about people number and Model size, then relating the rule to a counting technique $[(4 \times 13) + 8]$. Formal generalization - Use a multiple difference strategy by using the difference between consecutive terms of people number as a multiplicative factor (adjusting or not the result) to	Near generalization - Use multiple of difference strategy by using the difference between consecutive terms of people number as a multiplicative factor (adjusting or not the result) to obtain the distance term $[H(4) = 4 + (3 \times 2)\ dan\ H(5) = 4 + (4 \times 2)]$ Far generalization - Use multiple of difference strategy by using the difference between consecutive terms of people number as a multiplicative factor (adjusting or not the result) to obtain the distance term $[H(7) = 4 + (6 \times 2)\ dan\ H(10) = 4 + (9 \times 2)]$.

	Figural Linear Pattern	Non-figural Linear pattern
	obtain the general term ($P = 12 + (S-1)4$) Determine inverse - Use a general rule in formal generalization ($P = 4S + 8$) and substitute people number known then operate by mathematical operation to get final result.	Formal generalization - Use a multiple difference strategy by using the difference between consecutive terms of people number as a multiplicative factor (adjusting or not the result) to obtain the general term [$T = 4 + 2(H-1)$. Determine inverse - Use a general rule in formal generalization ($T = 4 + 2(H-1)$ and substitute people number known then operate by mathematical operation to get final result.
AG	Near generalization - Use counting strategy by drawing next figure and count people number by multiplicative approach. - Use contextual strategy by constructing a rule based on information provided in the situation [$(3 \times 4) + (4 \times 2)$]. Far generalization - Use contextual strategy by constructing a rule based on information providing about people number and Model size, then relating the rule to a counting technique [$(3 \times 4) + (24 \times 2)$]. Formal generalization - Use a multiplicative reasoning by proportional strategy by use multiplicative strategies, starting from one known term of the	Near generalization - Use multiple of difference strategy by using the difference between consecutive terms of people number as a multiplicative factor (adjusting or not the result) to obtain the distance term [$H_4 = 4 + (3 \times 2)\ dan\ H_5 = 4 + (4 \times 2)$] Far generalization - Use multiple of difference strategy by using the difference between consecutive terms of people number as a multiplicative factor (adjusting or not the result) to obtain the distance term [$H_7 = 4 + (6 \times 2)\ dan\ H_{10} = 4 + (9 \times 2)$]. Formal generalization - Use a multiple difference strategy by using the difference between consecutive terms of people number as a multiplicative factor

Figural Linear Pattern	Non-figural Linear pattern
sequence to find distant terms or the general term $P = 12 + (2S - 2)2$	(adjusting or not the result) to obtain the general term $[T = 4 + (H - 1) \times 2$.
Determine inverse	Determine inverse
Use a general rule in formal generalization $(P = 4S + 8)$ and substitute people number known then operate by mathematical operation to get final result.	Use a general rule in formal generalization $(T = 4 + (H - 1) \times 2$ and substitute people number known then operate by mathematical operation to get final result.

Table 3: Differences in functional thinking strategies of AA and AG

The finding showed that gifted students of primary school are able to perform functional thinking even though they have not obtained number pattern material in their school. Students not only perform functional thinking recursively, but they are also able to conjecture a general form of the relationship between two variables symbolically. It's in line with previous studies (Blanton & Kaput, 2004; Blanton & Kaput, 2005; Tanıslı, 2011; Warren et al., 2006; Warren & Cooper, 2005) which revealed that elementary students are able to generalize and represen the relationship by correspondence. Blanton & Kaput (2004, 2005) found that students were able to think functionally at the kindergarten level co-variationally and were able to think functionally as a correspondent in the 1st grade. Warren et al. (2006) found that elementary students are capable not only of developing functional thinking but also of communicating their thinking both verbally and symbolically. Tanıslı (2011) found that fifth-grader students thought on co-variation while working with the linear function tables.

This study revealed that gifted primary students perform functional thinking by generalizing the relationship between two variables in different ways. They used some strategies in performing the generalization, such as counting, multiple difference, and contextual strategy (Lannin, 2005; Oliveira et al., 2021). it's in line with Erdogan & Gul (2023) finding that gifted students used some strategies in generalizing linear pattern tasks, such as functional, chunking, and contextual strategies. This finding also promotes the study by Gutierrez et al. (2018) which found that mathematically gifted students are much faster than average students in learning mathematics content. It showed that gifted students are able to generalize geometric patterns in different ways, from recursive type to functional type.

Gifted students in this study used a multiplicative difference strategy to get a general term of relationship between two quantities. It was shown that gifted students are able to develop deconstructive generalization (DG), which refers to direct or closed polynomial formula that students construct form known stage as a result of figure (Rivera & Becker, 2011). In this study, gifted students are able to developing DG in solving figural and non-figural linear pattern task. They constructed a polynomial formula as a general rule using multiplicative difference strategy.

Another aspect of gifted students' mental flexibility was found in switching from one solution method to another. In solving figural linear patterns, AA switched from counting strategy to multiple difference strategy and AG from counting strategy to multiplicative reasoning by proportional strategy. Previous studies support this finding (Amit & Neria, 2008; Assmus & Fritzlar, 2022; Gutierrez et al., 2018). Amit & Neria (2008) found that students who began solving problems using the recursive method, usually showed flexibility in trying an alternative approach. Assmus & Fritzlar (2022) suggested that gifted students show flexibility in mathematical mental process. Gutierrez et al. (2018) declared that gifted students quickly move from one strategy to another, which they think is more useful and beneficial.

In solving a figural linear pattern, gifted students performed a reflection by observing the pattern, grasping its central attribute, and performing far generalizations. This finding highlights the inseparable connection between generalization and reflection (Amit & Neria, 2008). Ellis (2007) introduced reflection generalization, which refers to the final statement of a verval or written generalization. In this study, gifted students performed reflection generalization by writing a general rule of relationship between two quantities symbolically in solving figural and non-figural linear pattern tasks.

In the near and far generalization of a figural linear pattern, gifted students observed people sitting on the table figure and determined the near and far terms using counting strategy. Gutierrez et al. (2018) stated that most geometric patterns show a procedure to split the figures into parts that can be considered like independent patterns, making it easy to find a general procedure to calculate the values of the terms in the sequence. This procedure is termed by functional figural decomposition of the pattern with a cognitive demand in the procedures with connections level (Gutierrez et al., 2018).

There are different strategies used by gifted students in determining general rule of relationship between two quantities of a figural linear pattern. AA used a multiple difference strategy and found the general rule as $T = 12 + 4(S - 1)$, while AG used multiplicative approach to find general rule as $T = 12 + 2(2S - 2)$. It showed that gifted students have multiplicative constructive nonstandard of algebraic generalization type which refers to seeing figural pattern as consisting of nonoverlapping part (Rivera, 2010). Student's multiplicative reasoning plays a role in functional thinking development (Askew, 2018).

In solving figural linear pattern, gifted students inclined in using non-explicit counting strategy. They were able to developing recursive strategy to multiple differences. Recursive rules involve recognizing and using the change from term-to-term in the dependent variable (Lannin, Barker, and Townsend, 2006). Students understand that there is a same difference from term-to-term and they add the difference that they find recursively. However, recursive reasoning can limit the depth of functional thinking that students attain (Tanıslı, 2011).

In solving non-figural linear pattern, gifted students performed the generalization by correspondence relationship. It's based on identifying correlations between variables (M. Blanton, 2008). Gifted students are able to build a conjecture between the day and plant height variable. Drawing the finding, following the previous study (Blanton & Kaput, 2004; Blanton & Kaput, 2005; Tanıslı, 2011; E. A. Warren et al., 2006; E. Warren & Cooper, 2005) that elementary students are able to generalize and represent the relationship by correspondence. Blanton & Kaput (2004, 2005) found that students were able to think functionally at the kindergarten level co-variationally and were able to think functionally as a correspondent in the 1st grade. Warren et al. (2006) found that elementary students are capable not only of developing functional thinking but also of communicating their thinking both verbally and symbolically. Tanıslı (2011) found that five-grader students thought on co-variation while working with the linear function tables.

CONCLUSION

This study found that mathematically gifted students are able to use functional thinking in solving a figural and non-figural linear pattern task. In solving a figural linear pattern task, gifted students perform FT consisting of: (a) near generalization by counting, multiplicative approach, and contextual strategy; (b) far generalization by contextual strategy; (c) formal generalization by multiple difference and proportional strategy; and (d) determining inverse by using general rule. In solving a non-figural linear pattern task, gifted students perform FT consist of: (a) near generalization, far generalization, and formal generalization by multiple difference strategy; and (b) determine inverse by using general rule.

The findings of this study suggest that gifted students of elementary school in Indonesia have a potential in developing functional thinking. They are able in performing functional thinking in different ways. They also used different strategies in solving a figural and non-figural linear pattern.

References

[1] Amit, M., & Neria, D. (2008). Rising to the challenge: using generalization in pattern problems to unearth the algebraic skills of talented pre-algebra students. *ZDM Mathematics Education*, *40*(2), 111–129. https://doi.org/10.1007/s11858-007-0069-5

[2] Askew, M. (2018). Multiplicative reasoning: teaching primary pupils in ways that focus on functional relations. *Curriculum Journal*, *29*(3), 406–423. https://doi.org/10.1080/09585176.2018.1433545

[3] Assmus, D., & Fritzlar, T. (2022). Mathematical creativity and mathematical giftedness in the primary school age range: an interview study on creating figural patterns. *ZDM - Mathematics Education*, *54*(1), 113–131. https://doi.org/10.1007/s11858-022-01328-8

[4] Blanton, M. (2008). *Algebra and the elementary classroom. Transforming thinking, transforming practice*. Portsmounth,NH: Heinemann.

[5] Blanton, M., & Kaput, J. J. (2004). Elementary Grades Students ' Capacity for Functional

Thinking. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, *2*, 135–142.

[6] Blanton, M. L., & Kaput, J. J. (2005). Characterizing a Classroom Practice That Promotes Algebraic Reasoning. *Journal for Research in Mathematics Education*, *36*(5), 412–446. https://doi.org/https://doi.org/10.2307/30034944

[7] Blanton, M., Levi, L., Crites, T., & Dougherty, B. J. (2011). *Developing essential understandings of algebraic thinking, Grades 3-5*. Reston, VA: The National Council of Teachers of Mathematics.

[8] Chua, B. L., & Hoyles, C. (2014). Generalisation of Linear Figural Patterns in Secondary School Mathematics. *The Mathematics Educator*, *15*(2), 1–30. Retrieved from http://math.nie.edu.sg/ame/matheduc/journal/v15_2/n1.aspx

[9] Cohen, L., Manion, L., & Morrison, K. (2000). *Research methods in education* (5th editio). London: Routledge.

[10] Confrey, J., & Smith, E. (1991). A framework for functions: Prototypes, multiple representations, and transformations. In R. Underhill & C. Brown (Eds.), *Proceedings of the thirteenth annual meeting of the north American chapter of the international group for the psychology of mathematics education* (pp. 57–63). Blacksburg, VA: Virginia Polytechnic Institute & State University.

[11] Creswell, J. W. (2012). *Educational Research: Planning, Conducting, and Evaluating Quantitative and Qualitative Research* (Fourth). Boston: Pearson Education.

[12] Ellis, A. B. (2007). A Taxonomy for Categorizing Generalizations : Generalizing Actions and Reflection Generalizations. *The Journal of The Learning Sciences*, *16*(2), 221–262. https://doi.org/10.1080/10508400701193705

[13] Erdogan, F., & Gul, N. (2023). Reflections from the generalization strategies used by gifted students in the growing geometric pattern task. *Journal of Gifted Education and Creativity*, *9*(4), 369–385. Retrieved from https://dergipark.org.tr/en/pub/jgedc/issue/74010/1223156

[14] Fraenkel, J. R., Wallen, N. E., & Hyun, H. H. (2012). *How to Design and Evaluate Research in Education*. New York: McGraw-Hill.

[15] Gagné, F. (1985). Giftedness and Talent: Reexamining a Reexamination of the Definitions. *Gifted Child Quarterly*, *29*(3), 103–112. https://doi.org/10.1177/001698628502900302

[16] Girit Yildiz, D., & Durmaz, B. (2021). A Gifted High School Student's Generalization Strategies of Linear and Nonlinear Patterns via Gauss's Approach. *Journal for the Education of the Gifted*, *44*(1), 56–80. https://doi.org/10.1177/0162353220978295

[17] Gutierrez, A., Benedicto, C., Jaime, A., & Arbona, E. (2018). The Cognitive Demand of a Gifted Student's Answers to Geometric Pattern Problems. In F. M. Singer (Ed.), *Mathematical Creativity and Mathematical Giftedness: Enhancing Creative Capacities in Mathematically Promising Students* (ICME-13 Mo, pp. 169–198). Cham: Springer International Publishing. https://doi.org/10.1007/978-3-319-73156-8_7

[18] Kaput, J. (1999). Teaching and learning a new algebra with understanding. In E. Fennema & T. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 133–155). Mahwah: Erlbaum.

[19] Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren* (Vol.

8). Chicago: University of Chicago Press.

[20] Lannin, J. K. (2005). Generalization and Justification : The Challenge of Introducing Algebraic Reasoning Through Patterning Activities. *Mathematical Thinking and Learning*, *7*(3), 231–258. https://doi.org/DOI: 10.1207/s15327833mtl0703_3

[21] Lannin, J. K., Barker, D. D., & Townsend, B. E. (2006). Recursive and explicit rules: How can we build student algebraic understanding? *Journal of Mathematical Behavior*, *25*(4), 299–317. https://doi.org/10.1016/j.jmathb.2006.11.004

[22] Leedy, P. D., & Ormrod, J. E. (2019). *Practical Research: Planning and Design* (12th Editi). United State: Pearson Education.

[23] Leikin, R. (2018). Giftedness and High Ability in Mathematics. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 1–11). Cham: Springer International Publishing. https://doi.org/10.1007/978-3-319-77487-9_65-4

[24] Leikin, R. (2021). When practice needs more research: the nature and nurture of mathematical giftedness. *ZDM - Mathematics Education*, *53*(7), 1579–1589. https://doi.org/10.1007/s11858-021-01276-9

[25] Lepak, J. R., Wernet, J. L. W., & Ayieko, R. A. (2018). Capturing and characterizing students' strategic algebraic reasoning through cognitively demanding tasks with focus on representations. *Journal of Mathematical Behavior*, *50*(January), 57–73. https://doi.org/10.1016/j.jmathb.2018.01.003

[26] Mason, J., Stacey, K., & Burton, L. (2010). *Thinking Mathematically*. Edinburgh: Pearson.

[27] Mejía-Ramos, J. P., & Weber, K. (2020). Using task-based interviews to generate hypotheses about mathematical practice: mathematics education research on mathematicians' use of examples in proof-related activities. *ZDM - Mathematics Education*, *52*(6), 1099–1112. https://doi.org/10.1007/s11858-020-01170-w

[28] Merriam, S. B. (2015). *Qualitative research: A guide to design and implementation* (4th ed.). San Fransisco: Jossey-Bass A Wiley Brand.

[29] Ministary of Educations and Culture (MoEC). (2016). *Peraturan Menteri Pendidikan dan Kebudayaan Republik Indonesia Nomor 21 Tahun 2016* (No. 21). Jakarta, Indonesia: Kementerian Pendidikan dan Kebudayaan RI.

[30] NCTM. (2000). *Principless and Standards for School Mathematics*. Reston: NCTM.

[31] Oliveira, H., Polo-Blanco, I., & Henriques, A. (2021). Exploring prospective elementary mathematics teachers' knowledge: A focus on functional thinking. *Journal on Mathematics Education*, *12*(2), 257–278. https://doi.org/10.22342/jme.12.2.13745.257-278

[32] Paz-Baruch, N., Leikin, M., & Leikin, R. (2022). Not any gifted is an expert in mathematics and not any expert in mathematics is gifted. *Gifted and Talented International*, *37*(1), 25–41. https://doi.org/10.1080/15332276.2021.2010244

[33] Pitta-Pantazi, D., Christou, C., Kontoyianni, K., & Kattou, M. (2011). A Model of mathematical giftedness: Integrating natural, creative, and mathematical abilities. *Canadian Journal of Science, Mathematics and Technology Education*, *11*(1), 39–54. https://doi.org/10.1080/14926156.2011.548900

[34] Rivera, F., & Becker, J. R. (2003). the Effects of Numerical and Figural Cues on the Induction Processes of Preservice Elementary Teachers. *Proceedings of the 27th Conference*

of the International Group for the Psychology of \nMathematics Education Held Jointly with the 25th Conference of PME-NA, *4*, 63–70.

[35] Rivera, F. D. (2010). Visual templates in pattern generalization activity. *Educational Studies in Mathematics*, *73*(3), 297–328. https://doi.org/10.1007/s10649-009-9222-0

[36] Rivera, F. D., & Becker, J. R. (2011). Formation of Pattern Generalization Involving Linear Figural Patterns Among Middle School Students: Results of a Three-Year Study. In J. Cai & E. Knuth (Eds.), *Early Algebraization, Advances in Mathematics Education* (pp. 323–366). Berlin: Springer-Verlag Berlin Heidelberg. https://doi.org/10.1007/978-3-642-17735-4_18

[37] Rusdiana, M., Suriaty, M., Sutawidjaja, A., & Irawan, E. B. (2017). *Pattern Generalization by Elementary Students*. *100*, 379–381. https://doi.org/10.2991/seadric-17.2017.82

[38] Rusdiana, Sutawidjaja, A., Irawan, E. B., & Sudirman. (2018). Students perception on a problem of pattern generalization. *Journal of Physics: Conference Series*, *1116*(2). https://doi.org/10.1088/1742-6596/1116/2/022040

[39] Smith, E. (2008). Representational thinking as a framework for introducing functions in the elementary curriculum. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 95–132). London & Newyork: Lawrence Erlbaum/Taylor & Francis Group & NCTM.

[40] Stacey, K. (1989). Finding and Using Patterns in Liniar generalising Problems. *Educational Studies in Mathematics*, *20*(2), 147–164. https://doi.org/10.1007/bf00579460

[41] Stephens, A. C., Fonger, N., Strachota, S., Isler, I., Blanton, M., Knuth, E., & Gardiner, A. M. (2017). A Learning Progression for Elementary Students ' Functional Thinking A Learning Progression for Elementary Students ' Functional. *Mathematical Thinking and Learning*, *19*(3), 143–166. https://doi.org/10.1080/10986065.2017.1328636

[42] Syawahid, M., Purwanto, Sukoriyanto, & Sulandra, I. M. (2020). Elementary students' functional thinking: From recursive to correspondence. *Journal for the Education of Gifted Young Scientists*, *8*(3), 1031–1043. https://doi.org/10.17478/JEGYS.765395

[43] Syawahid, Muhammad. (2022). Elementary students' functional thinking in solving context-based linear pattern problems. *Beta: Jurnal Tadris Matematika*, *15*(1), 37–52. https://doi.org/10.20414/betajtm.v15i1.497

[44] Tanıslı, D. (2011). Functional thinking ways in relation to linear function tables of elementary school students. *The Journal of Mathematical Behavior*, *30*, 206–223. https://doi.org/10.1016/j.jmathb.2011.08.001

[45] Warren, E. A., Cooper, T. J., & Lamb, J. T. (2006). Investigating functional thinking in the elementary classroom : Foundations of early algebraic reasoning. *Journal of Mathematical Behavior*, *25*, 208–223. https://doi.org/10.1016/j.jmathb.2006.09.006

[46] Warren, E., & Cooper, T. O. M. (2005). Introducing Functional Thinking in Year 2 : a case study of early algebra teaching. *Contemporary Issues in Early Childhood*, *6*(2), 150–162.

[47] Wilkie, K. J., & Clarke, D. M. (2016). Developing students ' functional thinking in algebra through different visualisations of a growing pattern ' s structure. *Mathematics Education Research Journal*, *28*(2), 223–243. https://doi.org/10.1007/s13394-015-0146-y

Thoughts of Prospective Mathematics Teachers on Educating Mathematically Gifted Students

Gönül Yazgan-Sağ
Gazi University, Ankara, Turkey
gonulyazgan@gazi.edu.tr

Abstract: The aim of this qualitative study was to explore the thoughts of primary and secondary prospective mathematics teachers about educating mathematically gifted students. For this purpose, this research was conducted with 40 prospective mathematics teachers, 17 of whom were secondary mathematics prospective teachers, and 23 were primary mathematics prospective teachers. The data was collected through (i) written explanations of all prospective teachers, (ii) one classroom discussion, and (iii) three focus group interviews. The data was analyzed by using content analysis. The findings indicated that prospective mathematics teachers mostly associated education of gifted students within the school context, such as what can be done in the classroom or out of the classroom for these students. It can be interpreted that the participants will tend to focus on out-of-class activities rather than in-class activities for the mathematically gifted students in their future classrooms.

Keywords: Mathematically gifted students; prospective mathematics teachers; teacher education; gifted education; mixed classroom

INTRODUCTION

The improvement of a society is undoubtedly related to the talents and creative potentials of its individuals. These highly potential individuals take their education in either homogenous or heterogeneous classroom settings (Davis, Rimm & Siegle, 2014; Dimitriadis, 2016b; Leikin, 2010), namely different countries have different education policies in organizing environments for these gifted students (Davis et al., 2014; Gómez-Arizaga, Conejeros-Solar & Martin, 2016; Leikin, 2011b). Consequently, equity and differentiation in education, challenging situations with gifted students in the classrooms, and teachers' awareness of gifted students' needs are some of the issues examined in the literature (Shayshon et al., 2014; Leikin, 2011b). With the awareness of the educational needs of gifted students in societies, the research has increasingly emphasized the role of several components, such as the education of gifted students, since the beginning of the 1980s (Dimitriadis, 2016a). How to educate gifted students, such as grouping the gifted students either homogeneously according to their abilities or heterogeneously with all other students, is still open to discussion (Davis et al., 2014). There is no consensus for the right educational practices in the gifted literature (Shayshon et al., 2014). While some of these practices like grouping and compacting curriculum have been criticized because of not considering individual differences and motivation factors for gifted students, other practices have suggested modifications in all school programmes, such as differentiating curriculum for every student including gifted students (Dimitriadis, 2012). All these practices are being employed in the educational systems in line with the needs of gifted students and schools.

Educational systems mainly rely on classroom teachers to differentiate curriculum and support gifted students (Diezmann & Watters, 2000). In their first years of careers, mathematics teachers have the vital role of identifying students' strengths and adapting lessons accordingly. Suppose teachers choose tasks and activities for the average level in mixed classes. In that case, then mathematically gifted students are not able to notice their potential and get bored, or are usually labeled failed, even though teachers realize their talents (Applebaum, Freiman & Leikin, 2011; Fraser-Seeto, 2013; Leikin, 2011b; Levenson & Gal, 2013; Levenson, Tirosh & Tsamir, 2009). Teachers may find it challenging to work with gifted students due to the unique needs of them (Karp, 2010). Although studies devoted to the characteristics of mathematically gifted students are vast, there is a need for a clear study of the teaching methods in mixed classes (Karp, 2010; Karsenty, 2014; Leikin, 2011a; Levenson & Gal, 2013, Reed, 2004; Yazgan-Sağ, 2022). In this respect, for mathematically gifted students to reach their full potential, teachers' knowledge and beliefs on how to educate them are essential (Even, Karsenty & Friedlander, 2009). This study might be seen as an initial step in cultivating the awareness of prospective teachers on the challenges associated with educating these students. Therefore, this study aims to reveal the prospective mathematics teachers' thoughts on educating mathematically gifted students, especially in mixed classrooms.

Education of mathematically gifted students

In this study, "mathematically gifted students" refers to students who have potential in mathematics and display significant mathematical abilities in society. One of the common myths is that due to giftedness, a gifted student does not need any special education or support in order to nurture their talents (Copper, 2009; Moon, 2009). Unlike what is believed, gifted students need to be motivated and guided purposely (Clark, 2013; Levenson & Gal, 2013; Yazgan-Sağ, 2019). Establishing of an environment for learning opportunities is crucial to making gifted students realize their potential (Dimitriadis, 2016a, 2016b; Leikin, 2010). The seminal work of Krutetskii (1976) revealed that having experiences in challenging tasks and using targeted teaching methods can lead to the development of mathematical ability. Leikin (2010) also argued that challenging environments can support mathematically gifted students in nurturing their potential. "When challenge is coupled with student choice and interest, the outcome is an intellectually stimulating learning environment that is enjoyable and meaningful" (Davis et al., 2014). However, there is no agreement in the literature on how to enhance gifted students' potential (Shayshon et al., 2014). The issue of designing an education program that can be merged into a heterogeneous classroom or a separate program from the regular one is still open to discussion (Davis et al., 2014; Dimitriadis, 2016b). There are several educational approaches, such as acceleration, grouping, compacting, and differentiation. The curriculum aims to organize such a challenging context for gifted students (Davis et al., 2014; Leikin, 2010, 2011a; Sriraman & Haavold, 2017; Renzulli & Reis, 1985; VanTassel-Baska & Little, 2003). These approaches mostly describe the environments that might be the most suitable for gifted students and their needs. The curricular modifications also aim to construct an environment for critical, creative, and high-level thinking and to work as independent researchers in the mathematics discipline for gifted students (Little, 2018; Sriraman & Haavold, 2017). Namely, in-school or out-of-school adjustments should be made in teaching and learning methods according to the students' differentiated abilities and thinking processes (Freeman, 1999; Little, 2018; Shore & Kanevsky, 1993; Tomlinson, 1999). For example, teachers can provide extra tasks to students instead or after the main task given to the class, and they can add extra projects in line with their interests. A school can select gifted students for subject-skipping or grade-skipping and organize cluster grouping for all gifted students at each grade level. A student can take a university course, a part-time special gifted class, or go to a special school for gifted students (Davis et al., 2014).

Mathematics educators mostly highlight mathematics content and teaching action rather than the program itself for the education of mathematically gifted students (Koshy, Ernest & Casey, 2009; Sheffield, 1999). For example, Sheffield (1999) proposed the tasks should be complex, original, and open-ended which opens students' minds to think creatively. In fact, the literature offers to trigger creativity in all types of classes (Levenson, 2013, Prabhu & Czarnocha, 2014). Researchers also stated that teaching mathematically gifted students involves a higher level of challenge and requires a higher level of creativity and critical thinking from students (Casey, 1999; Koshy, 2001). In this manner, Leikin (2010) clarified how mathematical tasks (e.g., proof, inquiry, and multiple solutions) can challenge gifted students in the classrooms. Opportunities such as math contests and Olympiads, mathematical clubs, and conferences can also be offered to all students, especially for gifted students as in-school or out-of-school activities (Leikin, 2010; Sriraman & Haavold, 2017). It is clear that not all of these practices can be implemented in the heterogeneous classroom context. However, modifying ways of teaching in all of these contexts is one of the essential requirements for educating mathematically gifted students (Karp & Busev, 2015). For the mixed classroom, mathematics teachers can meet all students' needs with differentiated education (Mellroth, 2020; Tomlinson, 2016). In order not to make the students get bored and frustrated in the school context, the curriculum that will be used in the education of mathematically gifted students should be designed as accelerated, deeper, and abstract (Diezmann & Watters, 2000; Dimitriadis, 2016a; Lubinski & Benbow, 2006).

METHOD

The aim of this study was to reflect the prospective mathematics teachers' thoughts about educating mathematically gifted students. The participants of the study were 40 prospective mathematics teachers (17 prospective secondary mathematics teachers and 23 prospective primary mathematics teachers) in the Mathematics Education teaching program in Turkey. They had never taken any course related to giftedness during their education program. The researcher of the current study was also the lecturer of the course titled "Teaching Methods on Mathematics Education" for both primary and secondary mathematics teaching programs. Prospective primary teachers took this course in their fifth semester, and prospective secondary teachers took the course in the seventh semester of their teaching program.

In this study, the data was collected through (i) written explanations, (ii) classroom discussions, and (iii) focus group interviews with prospective mathematics teachers. Firstly, open-ended questions were asked which are related to (mathematical) giftedness, mathematically gifted students, education and teachers of these students, such as "What do you think of the education of (mathematically) gifted students?". Writing down their opinions about these questions took about one course hour (45-50 minutes). Secondly, an audio-recorded classroom discussion was organized with 23 prospective primary mathematics teachers after their written explanations were taken. The discussion lasted approximately 35 minutes. Then, two groups (primary and secondary) were asked if they would like to participate in the focus group interviews. In these focus group interviews, seven prospective secondary mathematics teachers and 11 prospective primary mathematics teachers volunteered to participate. Three focus group interviews were made with these participants. While seven prospective secondary mathematics teachers were the first focus group interview participants, the second and third interviews were conveyed with 11 prospective primary mathematics teachers by dividing them into groups of 5 and 6 participants. Each of the three focus group interviews, which were also video-typed, took approximately 100 minutes. Both classroom discussions and focus group interviews enabled the participants to reveal their thoughts more specifically. Thus, they had an opportunity to consider different aspects of giftedness, think deeply, and reflect on the other participants' views. Questions for focus group interviews were prepared by considering each participant's written statements. Yet, in order to keep the participants focused on the subject, prompts during the focus group interviews were used.

The data of this qualitative study, which aims to reveal the prospective mathematics teachers' thoughts, were analyzed through content analysis (Patton, 2002). First of all, the row data were transcribed in order to prepare them for analysis. Prospective secondary and primary mathematics teachers' written explanations were read several times and the whole classroom discussion was listened, and three focus group interviews were watched. The patterns that indicate the issues related to the education of mathematically gifted students were clarified. Literature served as a guidance while determining these patterns as categories in line with the aim of the study (Karp, 2010; Karsenty, 2014; Leikin, 2011a, 2011b; Leikin & Stanger, 2011; Levenson & Gal, 2013; Tomlinson, 1999; Reed, 2004). For instance, when the statements of the participants such as "Out-of-the class, I can ask challenging and advanced problems such as intelligence questions that enable the gifted students to reflect on them." were observed in the data sources, they were coded as "educating mathematically gifted students out the classroom" category. A number was given to each paper which included prospective teachers' written explanations. Then, these prospective teachers' names were coded as PT-X if the paper of written explanations had the number X.

FINDINGS

In this section, the prospective mathematics teachers' thoughts about how to educate mathematically gifted students will be presented. The sources of the data presented in this section are (i) written explanations, (ii) classroom discussions, or (iii) focus group interviews with prospective mathematics teachers. The prospective mathematics teachers' thoughts were coded into three categories. Firstly, the participants' general comments on these students' education will be given. Then, the prospective teachers' thoughts about how to educate these students in the school context, which were divided into two subsections as "in the classroom" and "out the classroom" will be presented.

General comments on educating mathematically gifted students

The thoughts not directly relevant to the mathematics classroom or the relation between teachers and gifted students were coded as "general comments on education mathematically gifted students". Although there were some voices among the participants that the gifted students should be educated with all the other students, the majority of the participants stated that it would be better *to educate all gifted students together in the same school* context. The participants specified several reasons for educating gifted students in the specialized schools:

> PT-20: We could educate them as scientists or successful people in several disciplines […]. If we educate them with other students, we are likely to lose those students because this education would not be sufficient for them. Therefore, I think they should be educated in separate schools both for their own future and for the future of the country.

PT-20 considered not only the individualized education of gifted students but also the education policy of their country. She emphasized educating gifted students for the sake of national interests. PT-1 also thought that improving the abilities of these students and satisfying these students' curiosity could be more feasible in such schools. In a similar manner, PT-34 highlighted that those students can get bored easily and may not be able to realize their abilities in regular schools. Prospective teachers also suggested the use of a differentiated curriculum for gifted students. PT-19's statement is as follows: "Regular curriculum can distract gifted students' attentions, and it would not satisfy them." On the other hand, some prospective mathematics teachers *hesitated to raise gifted students in special or regular schools.* Here is PT-16's explanation:

> PT-16: I am between a rock and a hard place; I could not decide on this situation. If we isolate the gifted students, they will have problems socializing with normal people. However, being in a normal class is a waste of time for those mathematically gifted students. If there are many gifted students and the teacher makes the other students feel this situation, it can cause problems.

PT-16 especially underlined the socialization issue of gifted students in their daily lives. She also declared that although they would have mathematical knowledge, being with only gifted students could cause them to lose their self-confidence and lower their academic success. A number of participants also thought that *gifted students should come together out of their regular schools or classrooms:*

> PT-25: There are schools called Science and Art Centers. I heard that gifted students go to these schools two days a week. They can go to their regular schools the other three days of the week.

PT-25 also stated that it is essential to come together with gifted students to foster their own abilities. Besides, PT-18 added that regular schools are substantial for gifted students to adapt themselves to society, knowing their differences from others. Some of the participants also offered that there could be some special classes or inclusive education for gifted students in regular schools. For example, PT-35 said that "Perhaps a special education can be provided for these students both in mainstreaming and out of school education." Alternatively, PT-26 and PT-7 believed that having a teacher specializing in these students could be helpful in regular schools.

Prospective mathematics teachers have different views on *whether teachers of gifted students are supposed to be gifted or not.* Most participants believed that *a teacher does not have to be gifted* for a gifted student, but it is important to get education on mathematically gifted students. Here are the statements of PT-10 and PT-24:

> PT-10: The teachers don't have to be gifted. They just have to know how to use approaches and methods for gifted students.

> PT-24: The teachers do not need to be gifted, but need to get a special education. Since the teacher's communication with the student is crucial here, the teachers cannot understand what the student is thinking, and if they cannot analyze student's answers, they may misunderstand their students.

The other participants, on the other hand, stated that *it might be an advantage for the students to have a gifted teacher, but it is not necessary to* be a gifted teacher for a gifted student. For instance, PT-7 explains as "It could be an advantage if the teacher should be trained on this subject. At the very least, teachers should be able to recognize the gifted students and refer them to the experts." Some of the participants said that *gifted students' teachers must be gifted persons:*

> PT-25: Gifted teachers can better understand what gifted students can understand and how they can think. Therefore, these students don't have any troubles.

> PT-15: Because the gifted teacher knows best what they encountered and what they questioned […] In other words, that teacher is the person who has passed those roads and can best understand those students. Therefore, these students don't have any troubles.

These prospective teachers claim that gifted teachers can be more insightful and helpful for gifted students because of sharing similar problems in society and the classroom context. PT-33 also highlighted that gifted teacher can promote students' giftedness; otherwise, gifted students can lose their abilities.

Educating mathematically gifted students in the classroom

The participants' explanations were coded based on teachers' actions in the classroom context as "educating mathematically gifted students in the classroom". Firstly, most of the prospective teachers considered *being equal to all students.* They stated that it would be a problem, especially not for gifted students but for all the other students in the classroom.

> PT-20: We must act following the principle of equality in education. If not, the other students will feel bad, and I lose them.

As seen from above, PT-20 perceived equity principle as giving attention to all students in the same way and not behaving differently to the gifted students in the classrooms. The participants mainly care about other students' thoughts regarding their teaching actions, regardless of whether they consider gifted students' needs. A couple of prospective teachers offered to *share gifted students' needs explicitly with the whole classroom*. Here is PT-2's explanation:

> PT-2: If we work with older students, we can share their needs with others. While that gifted student is absent, we can say: "Everyone has different abilities; he/she is talented in this field. I will work with him/her separately, if there are others who are interested, we can also work together". […] I think it is very important to give feedback here. The teacher should make it feel: "You don't miss anything; he/she is just a little further."

PT-2 emphasized that studying differently with gifted students does not mean the teacher is less interested in the other students. However, the other participants do not favor this statement. They suggested that students of that age might not be able to handle this speech because of the students' own perceptions.

The prospective mathematics teachers also stated that their thoughts are related to gifted students' possible questions during the lesson. They have proposed various *ways to answer the gifted students' questions in the classroom.* Most of the participants agreed that it is important *to answer these questions at the moment the gifted students* asked. For instance, PT-23 argued about answering gifted students' questions as much as possible when they asked during the lesson. He also stated that the teacher should not restrict the students' thinking by saying, "It is enough for you." PT-22 added that the teacher would examine why the student was asking the mentioned question:

> PT-22: While solving equations, the student may ask "Who found the equals sign?" I would ask firstly, "Why did you wonder?" Because that question is irrelevant in that moment, then I would talk about Euler.

With this statement of PT-22, PT-19 offered to discuss the equality concept at the beginning of the lesson. In this way, the teacher can answer the possible questions before the gifted students ask them. PT-18 also contributed to PT-22's explanation by saying, "With a catchy short story, we can say that Euler found the equals sign." PT-23 proposed that the teacher can guide the students to investigate these questions even if the teacher knows the answer. He thought the teacher should not answer to give students more permanent knowledge. Similar to PT-23, some of the other prospective teachers stated that the teacher can *answer the question partially* during the lesson:

> PT-2: We can give a little information to satisfy student's curiosity and say, "Let's talk about this during the break". I think necessary information should also be given there.

> PT-6: What if I don't know the answer? If I don't know, I can't answer at that moment, nor can I lie. I can say, "Let's not interrupt the lesson for now; you write it down, come and discuss it in free time during lunch break."

While PT-2 stated that she would give a part of knowledge in order to satisfy the gifted student's curiosity at that moment and then answer the question during break time; PT-6 said that she could not explain if she did not know the answer. PT-4 also preferred to give little information about the question; otherwise, she thought that it would be confusing for the other students if they did not have enough knowledge to understand both the question and its answer.

A number of the participants expressed that a teacher can *differentiate lesson activities according to mathematically gifted students* in the classroom. They offered to adjust the task, paying attention to mathematically gifted students. However, the other participants found this teaching action challenging in the classroom context. Here are some explanations:

> PT-22: We can make plan B for the tasks. We can give them to gifted students by saying, "Now you can work on this problem, let's solve it" I think we can make different arrangements in different ways without changing our lesson plan.

> PT-21: When these students finish earlier, we can say, "You can look at this and think about this" without separating them from the class. This tactic also seems to be useful. We can use the existing task and add something extra.

These participants especially emphasized using the same activities during the lesson by differentiating the related tasks for gifted students and giving extra tasks while they finished them. In the same line, PT-26 stressed that mathematics teachers should *give advanced knowledge to gifted students aside from the curriculum.* PT-2 also offered to *relate the subject with the history of mathematics*. For example, a teacher can share unsolved problems like Fermat's problems with all the students but also can give these problems only to gifted students to raise their awareness and think on them during the lessons.

Some of the participants *offered to give different activities* for mathematically gifted students. For instance, PT-15 said that advanced mathematical problems and puzzles or something that is improving can be given to the gifted students. At the same time, the teacher continues the lesson plan with the other students. Similarly, PT-5 proposed to share advanced-level mathematics books related to algebra, calculus can be given to the gifted students during the lesson. PT-25 also added that if the teacher uses the same activities for the gifted students, they can easily get bored and disturb the other students in the classroom.

Some of the prospective mathematics teachers proposed to *use activities not only for mathematically gifted students but also for other students.* PT-16's statement is as follows: "For instance, after explaining negative numbers, we can ask questions like "If you defined a new negative number, how would you define it?" She mentioned *asking concept related questions* to all the students in the classroom context. However, PT-15 claimed that these questions could potentially confuse the other students' minds. Here is PT-16's thought against PT-15's statement:

> PT-16: Let them be confused too. […] Let everyone think and define something on their own. I don't expect a correct answer from everyone. For instance, we can ask why the zero power of 2 is 1. Instead of giving the proof, we can just let them think about what they would do.

Similar to PT-16's explanation, PT-26 also emphasized on *asking open-ended tasks* related to problem solving and posing activities to all students in the classroom. She said that after lecturing a subject, a teacher could give a problem-posing task to all students; in this way, every student could produce a problem according to their own abilities, and gifted students would not be bored during the lesson.

Educating mathematically gifted students out of the classroom

Prospective mathematics teachers declared their thoughts on what can be done with gifted students out of the classroom context as well. The participants proposed that a teacher should study with mathematically gifted students individually aside from the mathematics lessons. For instance, PT-15 confirmed this statement with the following explanation: "Out of the class, I can ask challenging and advanced problems such as intelligence questions that enable the gifted students to reflect on them". In a similar manner, PT-26 said that a teacher can *ask gifted students advanced problems* to prove some formulas out of the classroom:

> PT-26: For example, we can think of dividing by 7. Instead of just accepting such rules, we can ask: "How do you think we could have reached this formula? or "Why is the zero power of any number 1?" Then we can say: "Let's think about this at home."

PT-16 confirmed PT-26's statement by saying, "We can ask them to explain why exponential numbers are needed; we can make them question and investigate". PT-6 suggested that *giving an advanced level of books* can be helpful for gifted students out of the classroom context. PT-6 also stated that it is important to investigate *gifted students' questions together with them out of the lesson hours* so of not disturbing the lesson flow for the other students. PT-3 confirmed PT-6 with his explanation: "If we constantly try to answer that students' questions in the classroom, the other students will say "the teacher is fully interested in them".

The participants also emphasized *guiding the mathematically gifted students to create a product.* They proposed that a teacher should direct them to share their products with the other students. PT-16's explanation is as follows:

> PT-16: We can refer special students to math clubs. For example, the child enters the lesson I teach, and after the lesson we refer him/her to the club. They can publish journals about mathematics in the club. For example, one of our professors told us that a teacher guided a student by giving tasks related to Gauss's work, because he/she was curious about Gauss's life, and then the student stopped disturbing the lesson. Similarly, the history of mathematics can be studied with gifted students. It can be helpful to create a new puzzle type suck as Sudoku. We can ask them: "What kind of puzzles do you create?" or "Let's design puzzles, what would you do if you adapted them to mathematics?" Such studies can be done with the club. It can also be shared on school boards and in the magazines that students produce.

PT-16 stated that mathematically gifted students can study in school clubs related to mathematics. By this way, they might investigate the lives of famous mathematicians, produce puzzles in these clubs, and then share their works in some platforms such as magazines or school panels. In a similar way, PT-19 proposed that *gifted students can be guided and prepared to participate in mathematics Olympiads, projects or contests* by their mathematics teachers.

DISCUSSION AND CONCLUSION

In this study, 40 prospective mathematics teachers' thoughts were investigated on educating mathematically gifted students, and the data were gathered with their written explanations, one classroom discussion, and three focus group interviews. Although the number of participants is relatively small, this study provides an overall picture of possible teaching acts of prospective mathematics teachers who do not have any formal university education or course on giftedness.

Prospective mathematics teachers proposed a variety of arguments about how to educate mathematically gifted students. As it is known, there are different points of views which make educating mathematically gifted students very complicated (Leikin, 2011b). Similarly, the findings revealed that the prospective mathematics teachers mentioned different options for educating gifted students. Most participants thought it would be better to educate gifted students in specialized schools. The reasons behind their thoughts were meeting their needs more easily, making them aware of their own abilities, preventing them from boredom, and training them as scientists for national interests. However, some of the other participants cited negative consequences in special schools, such as not being able to socialize appropriately in their daily lives, losing their self-confidence and getting their education in mainstream classrooms and coming together with other gifted students in schools like Science and Art Centers or special classes only for gifted students in regular schools. In brief, the participants were quite sensitive to mathematically gifted students in terms of both academic and affective needs. Although they had no formal education on giftedness, the findings revealed that most of the participants made discussions about gifted education similar to the literature. The majority of the prospective mathematics teachers considered that a teacher of gifted students was not necessarily a gifted person, but he must be an expert or educated about mathematically gifted students as stated in the literature (Leikin, 2011a; Rosemarin, 2014). They stated that knowing how to communicate and teach gifted students is crucial in order to understand the ways of their thinking process. Besides, some participants said that gifted students would benefit from having a gifted teacher who also knows what being a gifted person is and that gifted teachers could effectively understand, recognize, and guide their gifted students.

The participants also stated their thoughts about what can be done inside and outside the classroom regarding the education of mathematically gifted students. Firstly, the prospective teachers prioritized behaving all students in the same way. These thoughts may be due to their concerns about not being equal to all the students in their classrooms. In fact, the participants misinterpreted the equity principle which does not mean to give exactly the same instructions to all students. Instead, the equity principle "demands that reasonable and appropriate accommodations be made to promote access and attainment for all students" (National Council of Teachers of Mathematics [NCTM], 2000, p. 12). They mostly prioritized other students' situations in a heterogeneous classroom and preferred not to reveal that they knew the giftedness of relevant students.

The study showed that most participants offered to answer gifted students' questions as soon as they asked during the lesson. However, they also stated that it is important to make both gifted and the other students think about the related concepts of the questions. Some of the prospective students mentioned that they just gave part of the answers because they did not know the whole answer, did not confuse the other students' minds, and guided gifted students to research independently. The prospective teachers also suggested differentiating the lesson plan to nurture the needs of gifted students, such as making plan B's, giving extra tasks, solving advanced problems and books, and asking concept-related questions to gifted students. Some of the participants offered to ask concept-related questions and open-ended questions to all students in the classroom. For example, PT-16 offered to ask how to define a negative number while introducing negative numbers. This approach can be helpful to improve both gifted students' and other students' abilities to use the language of mathematics and mathematical notations to express mathematical ideas clearly (NCTM, 2000). Similarly, PT-16 and PT-26 gave a conceptual example of exponential numbers: "how can we define the zero power of any number?" Rather than directly giving proof to the students, using this kind of concept-related questions in learning environments can trigger especially gifted students' mathematical reasoning in a creative way (Singer, 2018). Mathematically gifted students can reflect on both mathematical concepts and the relations between these concepts. PT-26 offered to differentiate problems after introducing the related subject. This approach can be associated with making plan B's for the class. She exemplified such differentiation with problem-posing activities. Students most probably pose the problems according to their cognitive levels. In this sense, mathematically gifted students may pose relatively difficult problems. It can be used in mixed classrooms and also improve mathematically gifted students' talents. Mathematics teachers can meet gifted students' needs in this way (Singer et al., 2016). These strategies are beneficial ideas that can be implemented in classrooms where mathematically gifted students are also involved. However, prospective mathematics teachers also predicted that the organization of the differentiation would be challenging for the teachers (Sisk, 2009; VanTassel-Baska & Stambaugh, 2005). They emphasized taking advantage of the history of mathematics and transcending the curriculum with gifted students. They thought studying with gifted students outside of the classroom would be better. This finding is parallel with the literature which stresses that teachers less prefer to adjust their lesson plans in relation to gifted students in regular classes (Leikin, 2011b; Leikin & Stanger, 2011; Levenson et al., 2009; VanTassel-Baska & Stambaugh, 2005). The participants proposed to guide mathematically gifted students out of the classroom by asking advanced problems and books, answering their questions during the break time, or giving them research work. The participants also mentioned that they guided gifted students to create mathematics magazines and puzzles, and investigate the famous mathematicians' lives and works. They also suggested guiding gifted students to participate in mathematics-related contests, which play a crucial role in the improvement of their motivation and knowledge (Renzulli, 1994). It is clear that the prospective mathematics teachers suggested a wide variety of creative ideas that could be applied while educating gifted students inside and outside the classroom. This may be the result of their taking the teaching methods course. When they become mathematics teachers, the next step might be to observe and investigate if these participants will use their ideas with mathematically gifted students in their classrooms.

Despite the limited time the researcher spends with participants and the limited data collection tools, it can be assumed that these prospective mathematics teachers will pay attention to potentially gifted students in their future classes with out-of-class activities instead of differentiating the lesson plan. This may be because they did not have any professional experience with mathematical giftedness during their teacher education. However, "it is worth noting that teacher education by no means ends in college." (Karp, 2010). If they take formal gifted education courses or interact with gifted students in their professional careers, their thoughts may change in other ways. Teacher education programs should also enhance prospective mathematics teachers' knowledge and awareness about the diversity of classrooms including gifted students. As Yazgan-Sağ (2020) stated, discussing possible mathematics classroom environment scenarios related to gifted students can also broaden the prospective teachers' ideas about what to do in the classrooms in teaching education programs.

References

[1] Applebaum, M., Freiman, V., & Leikin, R. (2011). Prospective conceptions about teaching mathematically talented students: Comparative examples from Canada and Israel. *The Montana Mathematics Enthusiast, 8*(1-2), 255–290. https://doi.org/10.54870/1551-3440.1216

[2] Casey, R. (1999). A key concepts model for teaching and learning mathematics. Mathematics in School, 28(3), 13–14.

[3] Clark, B. (2013). *Growing up gifted* (8th ed.). Boston, MA: Pearson.

[4] Cooper, C. R. (2009). Myth 18: It is fair to teach all children the same way.Gifted Child Quarterly, 53(4), 283–285

[5] Davis, G.A., Rimm, S.B., & Siegle, D. (2014). *Education of the gifted and talented* (6th ed.). Essex, UK: Pearson Education Limited.

[6] Diezmann, C. M., & Watters, J. J. (2000). Catering for mathematically gifted elementary students: Learning from challenging tasks. *Gifted Child Today, 23*(4), 14-52. https://doi.org/10.4219/gct-2000-737

[7] Dimitriadis, C. (2012). Provision for mathematically gifted children in primary schools: An investigation of four different methods of organisational provision. *Educational Review, 64*(2), 241-260. https://doi.org/10.1080/00131911.2011.598920

[8] Dimitriadis, C. (2016a). Gifted programs cannot be successful without gifted research and theory: Evidence from practice with gifted students of mathematics. *Journal for the Education of the Gifted*, *39*(3), 221-236. https://doi.org/10.1177/0162353216657185

[9] Dimitriadis, C. (2016b). Nurturing mathematical promise in a regular elementary classroom: Exploring the role of the teacher and classroom environment. *Roeper Review, 38*(2), 107-122. https://doi.org/10.1080/02783193.2016.1150375

[10] Even, R., Karsenty, R., & Friedlander, A. (2009). Mathematical creativity and giftedness in teacher professional development In R. Leikin, A. Berman & B. Koichu (eds.), *Creativity in Mathematics and the Education of Gifted Students* (pp. 309–324). Rotterdam, The Netherlands: Sense Publishers.

[11] Freiman, V., & Sriraman, B. (2007). Does mathematics gifted education need a working philosophy of creativity? *Mediterranean Journal for Research in Mathematics Education, 6*(1-2), 23–46.

[12] Freeman, J. (1999). Teaching gifted pupils, *Journal of Biological Education, (33)*4, 185-190. https://doi.org/10.1080/00219266.1999.9655663

[13] Gómez-Arizaga, M. P., Conejeros-Solar, M. L. & Martin, A. (2016). How good is good enough? A community-based assessment of teacher competencies for gifted students. *SAGE Open, 6*, 1-14. https://doi.org/10.1177/2158244016680687

[14] Fraser-Seeto, K. (2013). Pre-service teacher training in gifted and talented education: An Australian perspective. *Journal of Student Engagement: Education Matters, 3*(1), 29-38. https://ro.uow.edu.au/jseem/vol3/iss1/5/

[15] Karp, A. (2010). Teachers of the mathematically gifted tell about themselves and their profession. Roeper *Review, 32*(4), 272–280. https://doi.org/10.1080/02783193.2010.485306

[16] Karp, A. & Busev, V (2015). Teachers of the mathematically gifted: Two case studies. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the Ninth Conference of the European Society for Research in Mathematics Education* (pp. 1010-1015). Prague, Czech Republic: Charles University and ERME. https://hal.science/hal-01287303/

[17] Karsenty, R. (2014). Who can teach the mathematically gifted? Characterizing and preparing mathematics teachers of highly able students at the secondary level. *Gifted and Talented International, 29*(1-2), 161–174. https://doi.org/10.1080/15332276.2014.11678438

[18] [17] Koshy, V. (2001). *Teaching mathematics to able children*. London, England: David Fulton. https://doi.org/10.4324/9780203065198

[19] Koshy, V., Ernest, P., & Casey, R. (2009). Mathematically gifted and talented learners: Theory and practice. *International Journal of Mathematical Education in Science and Technology*, *40*, 213–228. https://doi.org/10.1080/00207390802566907

[20] Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren*. Chicago: University of Chicago Press.

[21] Leikin, R. (2010). Teaching the mathematically gifted. *Gifted Education International, 27,* 161–175. https://doi.org/10.1177/026142941002700206

[22] Leikin, R. (2011a). Teaching the mathematically gifted: Featuring a teacher. *Canadian Journal of Science, Mathematics and Technology Education, 11*(1), 78–89. https://doi.org/10.1080/14926156.2011.548902

[23] Leikin, R. (2011b). The education of mathematically gifted students: On some complexities and questions. *Montana Mathematical Enthusiast Journal 8*(2), 167-188. https://doi.org/10.54870/1551-3440.1211

[24] Leikin, R., & Stanger, O. (2011). Teachers' images of gifted students and the role assigned to them in heterogeneous mathematics classes. In B. Sriraman & K. W. Lee (Eds.), *The elements of creativity and giftedness in mathematics* (pp. 103–118). Rotterdam: Sense Publishers.

[25] Levenson, E. (2013). Tasks that may occasion mathematical creativity: Teachers' choices. *Journal of Mathematics* Teacher *Education*, *16*(4), 269–291. https://doi.org/10.1007/s10857-012-9229-9

[26] Levenson, E., Tirosh, D. & Tsamir, P. (2009). Students' perceived sociomathematical norms: The missing paradigm. *The Journal of Mathematical Behavior, 28*(2–3), 83–95. https://doi.org/10.1016/j.jmathb.2009.09.001

[27] Levenson, E., & Gal, H. (2013). Insights from a teacher professional development course: Rona's changing perspectives regarding mathematically-talented students. *International*

Journal of Science and Mathematics Education, 11(5), 1087-1114. https://doi.org/10.1007/s10763-012-9368-6

[28] Little, C. A. (2018). Teaching strategies to support the education of gifted learners. In S. I. Pfeiffer, E. Shaunessy-Dedrick, & M. Foley-Nicpon (Eds.), *APA handbook of giftedness and talent* (p. 371–385). Washington, DC: American Psychological Association. https://doi.org/10.1037/0000038-024

[29] Lubinski, D., & Benbow, C. P. (2006). Study of mathematically precocious youth after 35 years: Uncovering antecedents for the development of math-science expertise. *Perspectives on Psychological Science*, *1*, 316–345. https://doi.org/10.1111/j.1745-6916.2006.00019.x

[30] Mellroth E. (2021). Teachers' views on teaching highly able pupils in a heterogeneous mathematic sclassroom, *Scandinavian Journal of Educational Research*,65(3), 481–499

[31] Moon, S. M. (2009). Myth 15: High-ability students don't face problems and challenges. *Gifted Child* Quarterly*, 53*(4), 274–276. https://doi.org/10.1177/0016986209346943

[32] National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: NCTM.

[33] Patton, M. Q. (2002). *Qualitative research and evaluation methods.* Newbury Park: Sage Publication.

[34] Prabhu, V. & Czarnocha, B. (2014). Democratizing mathematical creativity through Koestler's bisociation theory. *Mathematics Teaching-Research Journal Online, 6*(4), 33-46. https://eric.ed.gov/?id=ED600005

[35] Reed, C. F. (2004). Mathematically gifted in the heterogeneously grouped mathematics classroom: What is a teacher to do?. *The Journal of Secondary Gifted Education, 3*, 89– 95. https://doi.org/10.4219/jsge-2004-453

[36] Renzulli, J. S. (1994). *Schools for talent development: A practical plan for total school.* Mansfield Center, CT: Creative Learning Press.

[37] Renzulli, J. S., & Reis, S. M. (1985). *The schoolwide enrichment model: A comprehensive plan for* educational *excellence*. Mansfield Center, CT: Creative Learning Press.

[38] Rosemarin, S., (2014). Should the teacher of the gifted be gifted? *Gifted Education International. 30*(3), 263–270. https://doi.org/10.1177/0261429413486577

[39] Shayshon, B., Gal, H., Tesler, B., & Ko, E. (2014). Teaching mathematically talented students: A cross-cultural study about their teachers' views. *Educational Studies in Mathematics, 87*(3), 409–438. https://doi.org/10.1007/s10649-014-9568-9

[40] Sheffield, L. J. (1999). Serving the needs of the mathematically promising. In L. J. Sheffield (Ed.), Developing *mathematically promising students* (pp. 43–55). Reston, VA: National Council of Teachers of Mathematics.

[41] Shore, B. M., & Kanevsky, L. (1993). Thinking processes: Being and becoming gifted. In K. A. Heller, F. J. Moenks, & A. H. Passow (Eds.), *International handbook of research and development of giftedness and talent* (pp. 133−147). Oxford, UK: Pergamon. https://www.weizmann.ac.il/st/blonder/sites/st.blonder/files/uploads/shore-and-kanevsky-1993_thinking.pdf

[42] Singer, F. M., Sheffield, L., Freiman, V., & Brandl, M. (2016). *Research on and activities for mathematically gifted students.* New York: Springer Nature. https://library.oapen.org/bitstream/handle/20.500.12657/27730/1/1002275.pdf

[43] Singer, M. F. (2018). Enhancing creative capacities in mathematically-promising students. Challenges and limits. In M. F. Singer (Ed.), *Mathematical creativity and mathematical giftedness. Enhancing creative capacities in mathematically promising students* (pp. 1–23). New York: Springer. https://doi.org/10.1007/978-3-319-73156-8_1

[44] Sisk, D. (2009). Myth 13: The regular classroom teacher can "go it alone". *Gifted Child Quarterly, 53*(4), 269-271. https://doi.org/10.1177/0016986209346939

[45] Sriraman, B., & Haavold, P. (2017). Creativity and giftedness in mathematics education: A pragmatic view. In J. Cai (Ed.), *First compendium for research in mathematics education.* Reston: National Council of Teachers of Mathematics.

[46] Tomlinson, C. A. (2014). *The differentiated classroom: Responding to the needs of all learners*. Alexandria, VA: Association for Supervision and Curriculum Development.

[47] Tomlinson, C. A. (2016). *The differentiated classroom: Responding to the needs of all learners.* Alexandria, VA: Pearson education.

[48] VanTassel-Baska, J., & Little, C. A. (Eds.). (2003). *Content based curriculum for gifted learners.* Waco, TX: Prufrock Press.

[49] VanTassel-Baska, J., & Stambaugh, T. (2005). Challenges and possibilities for serving gifted learners in the regular classroom. *Theory into Practice, 44*(3), 211-217. https://doi.org/10.1207/s15430421tip4403_5

[50] Yazgan-Sağ, G. (2019). A theoretical view to mathematical giftedness. *Milli Eğitim Dergisi, 48*(221), 159–174.

[51] Yazgan-Sağ, G. (2020). Possible interactions with mathematically gifted students: Views of prospective teachers. *Research in Pedagogy, 10*(2), 121–132.

[52] Yazgan-Sağ, G. (2022). Views on mathematical giftedness and characteristics of mathematically gifted students: The case of prospective primary mathematics teachers. *Mathematics Teaching-Research Journal Online, 14*(5), 128-140. https://eric.ed.gov/?id=EJ1382367

Teaching Algebra to a Grade 7 Student: Action Research Intervention

Gembo Tshering

Paro College of Education, Royal University of Bhutan, Bhutan
gembotshering.pce@rub.edu.bt

Abstract: Algebra is critical in shaping future mathematics success and is integral to the K-12 curriculum. Despite its inclusion, a common challenge arises as students' progress to higher grades without a solid foundation, resulting in challenging learning experiences. This action research study focuses on the algebraic learning experience of a Grade 7 student. The research explores various teaching and learning approaches, including consideration of cognitive development milestones, real-world applications, interactive learning, differentiated instruction, personalized learning plans, technology integration, and innovative assessment methods. Findings suggest that aligning instructional methods with cognitive development, incorporating real-world applications, facilitating interactive learning, offering differentiated instruction, personalizing learning plans, integrating technology, and utilizing innovative assessment approaches enhance the engagement and comprehension of algebraic concepts.

Keywords: Grade 7 algebra, mathematics, curriculum, foundational, action research, interactive learning.

INTRODUCTION

In the dynamic landscape of modern education, the traditional boundaries of when and how education systems introduce complex mathematical concepts are evolving. The spotlight is now turning to the early years of a child's education, where the seeds of mathematical understanding are sown. Among these foundational concepts, algebra emerges as a critical player. It is traditionally associated with later academic pursuits but is increasingly recognized as a potent catalyst for cognitive development in primary students (Star et al., 2015).

At the heart of this paradigm shift is the acknowledgment that primary students are not just capable of grappling with abstract ideas; they thrive when presented with challenges that stimulate their growing minds. Algebra, emphasizing symbols, relationships, and problem-solving, stands as a bridge between the tangible world of arithmetic and the abstract realm of higher mathematics (Oppenzato & Ginsburg, 2018; Star et al., 2015). Education systems aim to tap into young minds' natural curiosity and flexibility by introducing algebra at this juncture, fostering a solid foundation for future mathematical prowess. Nevertheless, I encounter a challenge concerning the feasibility of achieving this objective concerning my 11-year-old son, referred to as the learner in this paper, who is in Grade 7.

The learner would put in an hour-long self-study on math every day, excluding the time he would spend on math homework. He would manage to complete the homework and additional questions from the math textbook, albeit with my interventions. However, his scores on unit tests would hover around the average score, indicating partial comprehension of the finished topics. As he moves ahead with unit after unit in tandem with the routines of his school, what still needs to be mastered in the completed topics accumulates, and the cumulative effect from such a process result in variegated learning that undermines his ability to cope with new issues. Being a former math teacher in schools, I might have missed this revelation amidst the busy schedules of instructional activities. However, as a parent and faculty member at a teacher education college, I have been helping the learner with his math. As the early introduction of algebra is not merely an adjustment to curriculum sequencing but a deliberate and strategic investment in the intellectual growth of our youngest learners, I will share how I assisted the learner in learning algebra and how the assistance enabled the learner to comprehend algebraic problems.

As I assisted the learner in learning algebra, the central question was, how can I teach algebra to a Grade 7 student?

I sought to address the central question by exploring the theoretical underpinnings of introducing algebra to primary students and the practical strategies and approaches to make this venture feasible and enriching. By examining the cognitive milestones of early development, the relevance of real-world applications, and the integration of interactive and technological tools, we embarked on a journey to learn algebra.

LITERATURE

Cognitive Development and Algebra

The integration of algebra into the primary curriculum is not merely a pedagogical experiment; it aligns with the natural trajectory of cognitive development in young minds. In their formative years, primary students undergo a phase marked by significant cognitive growth and an expanding capacity for abstract thinking.

During the concrete operational period, typically ages 7 to 11, children develop the cognitive ability to think logically about concrete situations (Babakr et al., 2019; Sherrell, 2023; Virtual Lab School, 2023). This foundational phase sets the stage for educators to introduce algebraic concepts strategically, capitalizing on the newfound logical reasoning skills. Through tangible examples and real-world scenarios, educators use manipulatives, visual aids, and relatable problems to guide students through the transition from concrete operations to abstract algebraic thinking (Sa'adah et al., 2023). This intentional approach equips students to understand and apply abstract mathematical concepts in the concrete operational period, establishing a robust foundation for future learning.

Algebra is a natural progression from concrete arithmetic to abstract reasoning (Breiteig & Grevholm, 2006; Driscoll, 1999; Staff, 2019; Star et al., 2015). During this cognitive shift, students move from manipulating tangible objects to manipulating symbols, representing relationships between quantities (Sa'adah et al., 2023). This developmentally appropriate introduction encourages the cultivation of problem-solving skills and logical reasoning, aligning seamlessly with the broader goals of primary education (Breiteig & Grevholm, 2006; Star et al., 2015). Early exposure to algebra establishes a sturdy foundation for future mathematical endeavors, enabling students to comfortably navigate advanced concepts in later grades (Demme, 2018; Gojak, 2013).

Beyond its cognitive benefits, integrating algebra into the primary curriculum promises to enhance motivation and engagement (Centres for Excellence in Maths, 2020). Naturally curious and eager to explore, young learners discover a captivating outlet in algebra's puzzle-like nature and real-world applications. This intentional integration taps into their innate curiosity, transforming mathematical exploration into a dynamic and enjoyable experience. Consequently, this approach not only aligns with the cognitive development of primary students but also sets the stage for a lifelong appreciation and mastery of mathematical concepts.

Real-World Applications

One of the most compelling reasons to introduce algebra to primary students is its immediate and tangible relevance to their world. Far from an abstract set of rules and symbols, algebra finds its roots in real-world scenarios, making the learning experience meaningful and applicable to children's daily challenges.

Recognizing that algebra has immediate and tangible relevance to the world inhabited by young learners, exploring diverse real-world applications that bring algebra to life in the primary classroom can enhance the learners' curiosity. From routine trips to the grocery store (Trethewy, 2022) to the intricate patterns found in nature (Lassiter, 2023), the design of dream bedrooms (Hello Learning, n.d.), the captivating arena of sports statistics (West, 2022), and even the playful experimentation in a sandbox (Driscoll, 1999), each facet contributes to the overarching goal of making algebra not only meaningful but also an integral tool for understanding and navigating the environment of primary students.

Incorporating real-world applications into teaching algebra to primary students transforms abstract concepts into practical tools for understanding and navigating their environment. By embedding algebra in everyday experiences, educators can bridge the gap between theoretical knowledge and practical application, fostering a deep and lasting appreciation for algebra's relevance in the lives of young learners.

Interactive Learning Strategies for Algebra

Interactive learning, centered on collaboration and group activities, turns algebra into a shared adventure (Jackson et al., 1998). Games, from board games to apps, make algebra fun and competitive (Sun-Lin & Chiou, 2019; Timotheou et al., 2023). Tactile tools, like manipulatives, make abstract algebra tangible and memorable (Hodge-Zickerman et al., 2020). Through apps and online platforms, technology offers a dynamic virtual playground for personalized learning (Attard et al., 2015).

Interactive learning includes real-time feedback and reflection, refining understanding, and promoting metacognitive skills. Technology is crucial in teaching algebra to primary students, offering dynamic and interactive platforms. Apps and software gamify learning, making algebra accessible and enjoyable. Virtual manipulatives bring abstract concepts to life in a digital space (Freina & Ott, 2015). Online platforms foster collaborative learning beyond the classroom, breaking down geographical barriers.

Technology enables adaptive learning systems tailoring instruction to individual needs. Multimedia presentations and interactive lessons replace traditional lectures, enhancing comprehension. Augmented and virtual reality provide immersive experiences, opening new frontiers for teaching algebra.

Teaching Strategies for Algebra

In primary education, algebra teaching introduces challenges characterized by variations in cognitive development and diverse learning paces (Arcavi et al., 2016). This examination delves into the strategies to address these challenges while nurturing an inclusive and supportive learning environment for primary students.

Recognition of the unique characteristics inherent in each learner within the primary classroom prompts educators to adopt differentiated instruction (Tomlinson, 2017). Through tailoring teaching methods, materials, and assessments to accommodate individual learning styles and paces, this approach aims to ensure that algebraic concepts resonate with each student, averting feelings of overwhelm and fostering a culture of inclusivity. As the exploration extends into the intricacies of primary algebra education, the concept of personalized learning plans emerges as a pivotal tool designed to cater to the diverse needs of primary students (Shemshack & Spector, 2020). By identifying individual strengths, weaknesses, and preferences, educators create targeted frameworks for interventions or extensions, ensuring that algebra instruction is accessible and appropriately challenging (Zheng et al., 2022). Proactive measures, encompassing the early identification and rectification of potential misconceptions, contribute to a classroom culture that values questions, encourages critical thinking, and utilizes formative assessments. This personalized and proactive approach extends to scaffolded learning (Grotherus et al., 2019), the prioritization of conceptual understanding (Joffrion, 2007), fostering a growth mindset (Breckenridge, 2014), encouraging peer collaboration (Alegre et al., 2019), and involving parents in the learning process (Muir, 2012).

These strategies establish a supportive and empowering space for primary students to thrive in their algebraic learning journey.

Assessment Strategies for Algebra

Assessing primary students in algebra requires a nuanced approach considering developmental stages, diverse learning styles, and the dynamic nature of understanding (Oguguo et al., 2024). Formative assessments, like quizzes and interactive activities, offer real-time insights for tailored instruction. Performance-based assessments evaluate application skills through real-world scenarios (Oguguo et al., 2024; Tejeda & Gallardo, 2017). Portfolios showcase progression over time, capturing diverse skills and encouraging reflection (Torres et al., 2016). Observational (Schoenfeld et al., 2018) and interview assessments (Ardiansari et al., 2023) during activities and discussions provide qualitative data on problem-solving and collaboration. Self-assessment promotes metacognition, involving students in the evaluation process (Babic et al., 2019). Technology facilitates adaptive assessments, providing immediate feedback and personalized experiences (Wu et al., 2023). Clear rubrics standardize evaluation, offering criteria and constructive feedback (Poh et al., 2015).

METHOD

I employed participatory action research because of its attributes of pragmatism, baseline data, use of intervention, post-intervention data, evaluation of interventions, and reflexivity (Kemmis et al., 2014; McNiff, 2013; MicNiff & Whitehead, 2006; Mills, 2014; Stringer, 2021).

I had only one student participant, my son studying in Grade 7. In the study, the participant was referred to as a learner. The teaching and learning sessions occurred at home after school hours and on weekends, and they were one to two hours daily.

Baseline Data Collection

My baseline data was sourced from the Grade 7 math textbook, the learner’s notebook, and conversations with the learner. Grade 7 math textbook has units, and the units have chapters, and the chapters have topics, examples, and exercise questions (refer to Table 1).

Chapter	Topic
Chapter 1	Patterns and Relationships; Using variables to describe pattern rules; Creating and evaluating expressions; and Simplifying expressions
Chapter 2	Solving Equations; Solving equations using models; Solving equations using guess and test; Solving equations using inverse operations; and Solving equations using reasoning
Chapter 3	Graphical Representations; Graphing a relationship; Examining a straight-line graph; Describing change on a graph; and Relationship graphs straight lines

Table 1: Grade 7 algebra unit

A series of instructional activities were organized to collect baseline data. First, I would teach the topics to the learner using the chalk and blackboard approach, a typical variant of the traditional teaching approach (Bamne & Bamne, 2016). Second, I would give the learner tasks on the topics taught. The learner would complete the tasks during the teaching sessions, while I would observe the process the learner would follow to complete the tasks. The process followed by the learner when doing the tasks would be recorded with anecdotal records (Obano& Enowoghommwenma, 2021). Third, I would assign the learner homework; homework in this study was independent tasks the learner would do after the teaching sessions. I would assess the homework and give symbolic feedback through ticks or cross marks (Chappuis, 2012; Hattie & Timperley, 2007). I would use analytic rubrics to evaluate the homework (Brookhart, 2013). This would continue for the whole unit on algebra. I would instruct the learner to redo the incorrect work, but the learner would complete the job with the same errors, and I would reteach the relevant topics again in the same way. This cycle would continue until my focus would shift to the next topic simply because there would be more topics to teach—a phenomenon that compelled D Carlo (2009) to sloganeer, “Too much content, not enough thinking, and too little FUN.” The number of anecdotal records and analytic rubrics would increase over time without much attention from me as I moved ahead with topic after topic, assuming that the learner would have acquired the intended knowledge and skills from the taught topics until the unit tests were conducted. The unit test results would indicate what the learner knows and what he can do with what he knows. A set of sample questions from the algebra unit test is depicted below:

Q1 Simplify $(3n+2) + (5n+6)$.

Q2 List the variables, coefficients, and the constants in $-2m-4$.

Q3 Evaluate the expression $3h-4+7h+8$, when $h=4$.

Q4 Kinley earns Nu. 7000 each month. He also receives an allowance of Nu. 500 each month.
(a) Write an algebraic expression that represents how much Kinley receives in m months.
(b) Evaluate the expression to determine his annual income.

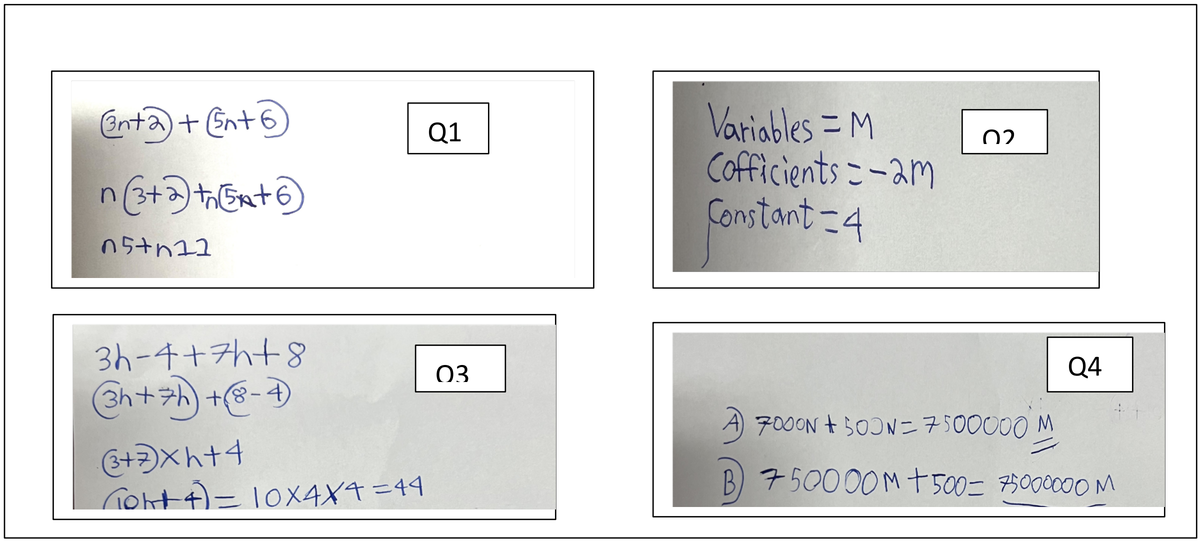


Figure 1: The learner's responses

Figure 1 shows a sample test result of the learner on the algebra unit. The learner's total score on a 100-mark test was 40 marks.

Analysis of the Learner's Responses to the Algebra Unit Test

In scrutinizing the learner's responses, I adopted a systematic approach outlined in Tables 2 and 3, extending the analysis to the teaching sessions and homework materials. The comprehensive assessment of during-the-session tasks and homework unveiled challenges encompassing generalization, function, structures, variables, constants, the concept of division in algebra, the principles of addition, subtraction, multiplication, and simplifying algebraic fractions.

No.	Learner's Answer	My Interpretation
1	$n(3+2)+n(5+6)$ $n5+n11$	The mistake here is a failure to distribute the terms within the parentheses correctly. Specifically, the student didn't combine the 'n' terms separately from the constant terms. It's a common error known as not applying the distributive property properly when simplifying expressions.
2	Coefficient=2 Variable=m Constant=4	The mistake could be a misunderstanding or oversight regarding the inclusion of the negative sign in both the coefficient and the constant. This is a common error when learning about terms with negative coefficients and constants. It's an opportunity for the student to refine their understanding of signed numbers in algebraic expressions. Also, the student might be confusing the coefficient with the absolute value of the coefficient.

No.	Learner's Answer	My Interpretation
3	(3*h*+7*h*) + (8-4) (3+7) x *h* +4 10x4x4=44	So, the student's error is in the calculation, not in the substitution of the value. The correct calculation yields 44, but the student wrote 1044=44, which is an incorrect statement and a calculation error. The error could be attributed to a miscalculation or misunderstanding of the steps involved in evaluating the expression.
4	(a) 7000 x *m* (b) 9000	The key knowledge and skills that seem to be missing relate to the translation of verbal descriptions into algebraic expressions and equations. In part (a), it involves understanding that m × (7000+500) represents the total income for m months. In part (b), it involves recognizing the need to multiply by the number of months in a year (12) to find the annual income. Improving in this area would involve practicing translating verbal statements into algebraic expressions and equations, understanding the structure of mathematical expressions, and recognizing the operations needed to represent real-world scenarios algebraically.

Table 2: Analysis of the responses

I analyzed the results of the during-the-session tasks, homework, and algebra unit tests in Table 3. Table 3 delineates the frequency and distribution of errors across these domains, shedding light on notable patterns. The elevated percentage of mistakes in variables and constants is noteworthy, underscoring a concentrated struggle in this realm. Generalization and division concepts in algebra persist as formidable challenges, as evidenced by consistently high mistake counts. This detailed breakdown affords a holistic understanding of the learner's hurdles, guiding targeted interventions to fortify comprehension in specific algebraic facets.

Area of Difficulty	Mistake Count (% of Total)	Tasks			
		Classwork	Homework	Test	Total
Generalization	10 (47.6%)	10	5	6	21
Function	3 (15.8%)	10	4	5	19
Variables and constants	7 (58.3%)	8	3	1	12
Concepts of division in algebra	10 (35.7%)	15	5	3	28
Principle of addition, subtraction, and multiplication	6 (30.0%)	10	6	4	20
Simplifying algebraic fractions	8 (38.1%)	12	5	4	21

Table 3: Areas of difficulty and mistake counts

Interventions

The intervention strategies employed in this study were meticulously designed to address specific areas of difficulty identified in the learner's comprehension of algebraic concepts. Table 4 provides a comprehensive overview of these difficulties and the interventions implemented during the study.

Area of Difficulty	Conceptual Underpinnings of Intervention	Instructional Activities
Generalization	Emphasize that the distributive property works for both addition and subtraction. Provided examples of distributing a number across a subtraction operation to illustrate the general applicability of the rule. Reinforce the idea that the same principle applies to different scenarios involving parentheses and operations. Generalization in mathematics refers to the process of identifying and describing patterns, trends, or relationships that hold true across a range of situations or numbers. In algebraic terms, generalization often involves finding a formula or expression that represents a pattern or relationship for a broader set of values. It's about creating a rule that can be applied to a variety of specific cases.	Differentiated Instruction: Offered various examples that cater to different learning styles. Some students may benefit from visual representations, while others may prefer symbolic or verbal explanations. Personalized Learning Plans: Identified individual students' preferred learning styles and tailor examples accordingly. Provided additional practice problems with varying contexts to reinforce the generalization concept.
Function	Emphasize the need to distribute 3 to both terms inside the parentheses. Provide step-by-step guidance on simplifying the expression correctly. A function is a mathematical relationship between two sets of elements, where each element in the first set (domain) is related to exactly	Differentiated Instruction: Used multiple representations such as graphs, tables, and verbal descriptions to cater to diverse learning preferences. Offered different levels of complexity for functions to challenge advanced learners.

Area of Difficulty	Conceptual Underpinnings of Intervention	Instructional Activities
	one element in the second set (codomain). In algebra, a function is usually represented by an equation or expression that relates an input variable (usually denoted as 'x') to an output variable (usually denoted as 'y'). Each input value corresponds to exactly one output value. For example, y=2x+3 represents a linear function where for every x, there's a unique y.	Personalized Learning Plans: Assessed each student's proficiency in understanding functions and provide additional resources or challenges based on their current level of mastery.
Variables and constants	Review the definition of variables and constants in a typical equation like: y=mx+b. Reinforce that *m* and *b* are constants with known values, while x is the variable.	Differentiated Instruction: Used real-world examples and analogies to explain the difference between variables and constants (provided concrete instances where constants remain fixed while variables changed). Provide concrete instances where constants remain fixed while variables change. Personalized Learning Plans: Assessed each student's grasp of variables and constants and tailor additional examples or activities to reinforce the understanding of these foundational concepts.
Concepts of Division in Algebra	Clarify that a/b means *a* divided by *b*, not the other way around. Provide examples to illustrate the correct interpretation.	Differentiated Instruction: Offer multiple approaches to explaining division in algebra, including visual models and real-world scenarios. Use concrete examples to illustrate the concept. Personalized Learning Plans: Identify individual misconceptions about algebraic division and

Area of Difficulty	Conceptual Underpinnings of Intervention	Instructional Activities
		provide targeted interventions. Offer additional practice problems that focus on the specific challenges each student faces.
Principle of Addition, Subtraction, and Multiplication	Emphasize that addition and subtraction are not commutative. Use specific examples to illustrate the inverse relationship between these operations.	Differentiated Instruction: Utilize manipulatives, visuals, and real-world examples to illustrate the principles of addition, subtraction, and multiplication. Offer a variety of problems with different levels of complexity. Personalized Learning Plans: Assess each student's understanding of these principles and provide targeted reinforcement based on their specific needs. Adjust the difficulty of problems to match individual proficiency levels.
Simplifying Algebraic Fractions	Guide the learner through the correct process of simplifying algebraic fractions by canceling out common factors. Emphasize simplifying both the coefficient and the variable term.	Differentiated Instruction: Break down the steps of simplifying algebraic fractions using visual aids and interactive examples. Offer alternative methods for students who may struggle with the traditional approach. Personalized Learning Plans: Identify specific errors or misconceptions related to simplifying algebraic fractions. Provide individualized practice problems and additional resources to address these challenges.

Table 4: Areas of difficulties and interventions

Furthermore, I incorporated scaffolded learning and nurtured a growth mindset. I also facilitated peer collaboration by asking the learner to take some tasks to the school and solve them with his peers. The instructional approach involved a gradual elevation in problem complexity, offering support and guidance in tandem with students' progression. Commencing with more straightforward examples, I systematically introduced increasingly challenging problems. I underscored the significance of a growth mindset, highlighting that mistakes are invaluable learning opportunities. The emphasis was on fostering a positive attitude towards challenges, cultivating resilience, and instilling a belief in the inherent capacity for improvement. Collaborative activities were implemented to promote an interactive and supportive learning atmosphere, encouraging the learner to work with his school peers to solve problems. This peer collaboration not only provided varied perspectives but also contributed to the creation of a nurturing and conducive learning environment. In addition to these focused interventions, I incorporated real-world problem-solving scenarios in the tasks to provide a practical context for algebraic concepts. Activities such as grocery shopping, studying patterns in plant leaves, measuring vegetable growth, solving car-related problems, using scoreboards, and plotting electricity bills were presented to the learner. Integrating interactive websites, including ChatGPT (Open AI, 2023), further enriched the intervention process. ChatGPT has many helpful features.

ChatGPT offers tailored examples and explanations for individual learning styles. The learner can interact with ChatGPT to seek clarifications or explore alternative approaches to generalization. It generates graphs, tables, or verbal descriptions based on students' preferences, fostering dynamic interaction to deepen understanding. ChatGPT reinforces the distinction between variables and constants by engaging the learner in conversations about real-world examples, providing personalized analogies, and answering specific queries. Through interactive discussions, ChatGPT provides visual models and real-world scenarios, ensuring a comprehensive understanding of algebraic division. It assists in creating manipulative scenarios and real-world examples, offering personalized reinforcement, and addressing specific challenges related to addition, subtraction, and multiplication principles. ChatGPT breaks down the steps of simplifying algebraic fractions using visual aids and interactive examples, providing alternative methods, and engaging in conversations to address the learner's specific challenges. The conversations enabled me to understand the learner's deep-seated challenges, which were not normally accessible through word problems or written tasks (Yildirim-Elbasali et al., 2023). Therefore, the conversations were pointers for the zone of proximal development, a gap between what is known and what is not known (Vygotsky, 1978). For instance, the conversations about constants and variables helped identify misconceptions arising from proactive interferences of arithmetic rules when solving algebraic problems. The conversations also helped me understand how the learner construed the words' meanings when teaching, facilitating clarifications later.

These interventions were administered over three weeks, creating a structured and immersive learning experience for the learner. The study aimed to enhance the learner's holistic grasp of algebraic principles by combining targeted academic interventions with real-world applications and interactive technologies.

Post-Intervention Data Collection

The post-intervention phase involved carefully evaluating the learner's proficiency through a 100-mark test strategically designed with randomly selected questions from the algebra unit of the math textbook. This assessment aimed to gauge the learner's grasp on diverse algebraic concepts, covering tasks such as identifying variables, coefficients, and constants in expressions, devising pattern rules for geometric figures, evaluating complex algebraic expressions, constructing word problems based on given expressions, simplifying algebraic expressions, and solving equations. The learner's commendable total score of 70 marks on the test serves as an initial indicator of the efficacy of the interventions implemented to address the identified difficulties. Some sample questions from the test are provided below:

(1) List the variable, the coefficient, and the constant in -3*h*+2*f*-5.

(2) (a) Complete the table below.

(b) Write a pattern rule that could be used to find the number of squares if you know the figure number.

Figure number	Figure	Number of squares
1	□ □ □ □ □	6
2	□ □ □ □ □ □ □ □ □ □ □	11
3	□ □ □ □ □ □ □ □ □ □ □ □ □ □ □ □	16
4		
5		

(3) Evaluate $\frac{(3n+2)(n-4)}{2n}, when\ n = 5$.

(4) Write a word problem that could be solved by using the expression 2*x*+60.

(5) Simplify $(4x - 3) + (2x + 6) - (-3x - 4)$.

(6) Solve $10 + 5m = 55$.

Analysis of the Post-Intervention Data

Figure 2 visually represents the learner's sample responses, offering a qualitative glimpse into the learner's approach and understanding of the varied algebraic tasks presented in the test. As shown in Figure 2, the learner responded correctly to all sample questions, but there were mistakes in other questions of the 100-mark test.

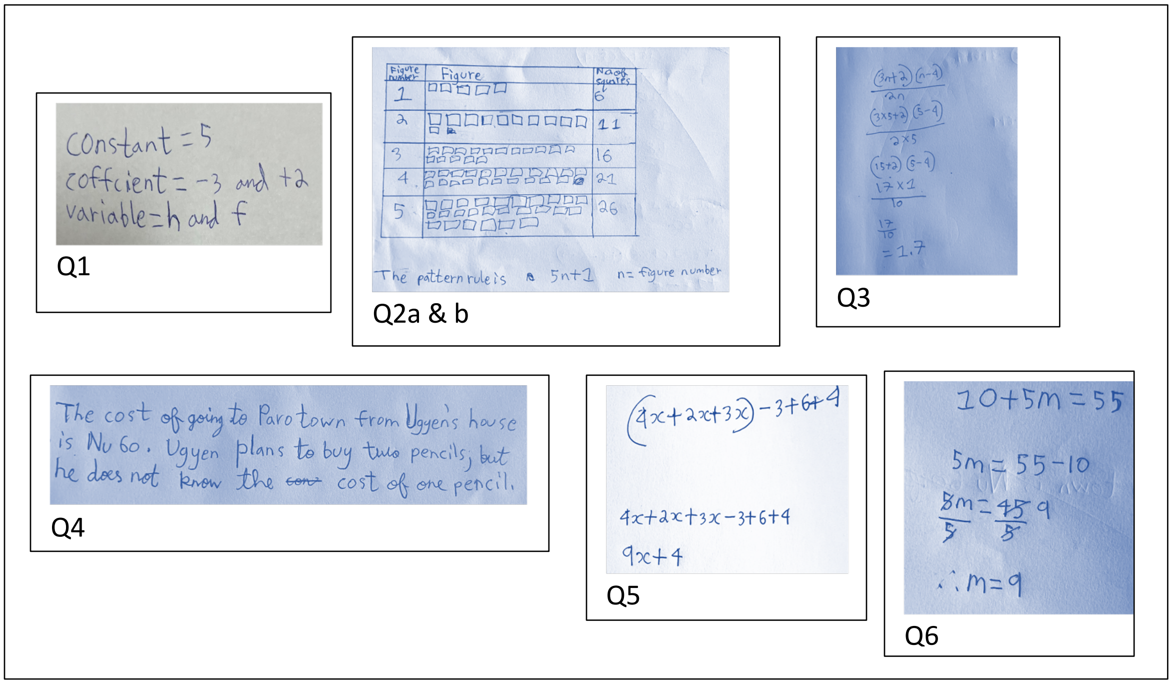


Figure 2: The learner's responses to the sample questions

The detailed examination of individual responses allows for a nuanced understanding of the learner's application of mathematical principles and identifies specific areas where further improvement or intervention may be warranted. The subsequent analysis, as illustrated in Table 5, delves into the learner's performance across distinct areas of difficulty, providing a comprehensive breakdown of mistake counts in classwork, homework, and the test.

Area of Difficulty	Tasks				
	Mistake Count (% of Total)	**Classwork**	**Homework**	**Test**	**Total**
Generalization	3 (20%)	6	3	6	15
Function	2 (12.5%)	7	4	5	16
Variables and constants	0	3	2	1	6
Concepts of division in algebra	2 (11.1%)	12	3	3	18
Principle of addition, subtraction, and multiplication	0	8	4	4	16
Simplifying algebraic fractions	2 (11.1%)	9	5	4	18

Table 5: Area of difficulty and mistake counts

Notably, there are observable reductions in mistake percentages across various algebraic concepts, signifying positive progress from the targeted interventions. These findings affirm the effectiveness of the intervention strategies in addressing the learner's challenges and enhancing their comprehension and proficiency in specific algebraic domains.

RESULTS

Comparing Baseline and Post-Intervention Data

I conducted a comparative analysis between the baseline data and the post-intervention results, scrutiny encapsulated in Table 6. This table is a comprehensive snapshot, capturing the landscape of mistake percentages before and after implementing targeted interventions across various algebraic domains. The comparison unravels a compelling narrative of improvement, signaling a positive trajectory in the learner's grasp of algebraic concepts.

One notable transformation emerges in generalization, where the learner initially grappled with a 47.6% mistake rate. This percentage significantly plummeted to a commendable 20% through the strategic interventions, showcasing a substantial leap in understanding this intricate algebraic facet. Similarly, the domain of variables and constants witnessed a remarkable turnaround. Initially burdened with a daunting 58.3% mistake count, interventions wrought a complete elimination of errors, portraying a resounding success in addressing challenges in this area.

Areas of Difficulty	Mistake Counts before Interventions	Mistake Counts after Interventions
Generalization	47.6%	20%
Function	15.8%	12.5%
Variables and constants	58.3%	0
Concepts of division in algebra	35.7%	11.1%
Principle of addition, subtraction, and multiplication	30.0%	0
Simplifying algebraic fractions	38.1%	11.1%

Table 6: Comparing the percentages of mistake counts before and after interventions

The overarching theme resonating from Table 6 is one of progress and improvement. The reduction in mistake percentages across the spectrum—in functions, concepts of division in algebra, or simplifying algebraic fractions—paints a picture of successful intervention strategies. This nuanced analysis not only underscores the positive impact of interventions but also provides a roadmap for refining future strategies to fortify the learner's grasp of algebraic principles.

Overall, Table 6 suggests that interventions had a positive impact, reducing mistake percentages across various algebraic concepts, with some areas showing significant improvement.

DISCUSSION

The choice of action research as the methodology for this study aligns with the pragmatic nature of addressing real-world challenges in the classroom setting. Action research, as described by McNiff (2013), McNiff and Whitehead (2006), Stringer (2021), Mills (2014), and Kemmis et al. (2014), provided a systematic approach to understanding and improving the teaching and learning process.

The baseline data collection comprehensively examined the Grade 7 math curriculum, utilizing textbooks and learners' notebooks. This approach allowed for a detailed breakdown of the topics covered in the algebra unit. The instructional activities employed for baseline data collection provided insights into the learner's understanding of algebraic concepts. Using classwork, homework, assessments, and detailed feedback mechanisms ensured a holistic view of the learner's progress.

Drawing upon the baseline data, the analysis of the learner's responses to algebraic problems revealed specific areas of difficulty, such as generalization, function, variables and constants, concepts of division in algebra, and principles of addition, subtraction, and multiplication. This detailed breakdown illuminated the nuances of the learner's struggles, guiding the subsequent interventions (Shemshack & Spector, 2020; Zheng et al., 2022).

My interventions targeted each identified difficulty, demonstrating a tailored approach to addressing the learner's challenges. From emphasizing the distributive property to providing real-world applications of algebraic concepts, the interventions aimed to enhance the learner's understanding and application of mathematical principles (Lassiter, 2023; West, 2022). Also, the incorporation of real-world problems and interactive tools like ChatGPT in the intervention process showcased a creative and practical dimension to teaching algebra. These approaches sought to bridge the gap between theoretical knowledge and its real-world applicability, fostering a more engaging and immersive learning experience. The conversations with the learner about his tasks effectively brought forth the deep-seated challenges in the learner, which were not accessible with written responses. Mostly, the written responses showed what the learner knew and did not know. However, understanding the gap between the two is crucial for the learner to further his learning, and it is where the learner’s deep-seated challenges are concentrated (Vygotsky, 1978). Conversations helped me recognize the challenges ((Yildirim-Elbasali et al., 2023)

The post-intervention data collection involved thoroughly examining the learner's performance on a follow-up test. Randomly selecting questions from the algebra unit ensured a comprehensive assessment of the learner's progress. Comparing baseline and post-intervention data revealed a positive impact, significantly reducing mistake percentages across various algebraic concepts.

This study opens new paths in math education, suggesting a long-term perspective by tracking learners for insights into sustained intervention impact. Future research may explore adaptive tech, like AI tutoring and optimizing algebra learning. Tailoring interventions to diverse learning profiles is crucial. Collaborative interventions across disciplines, drawing on cognitive science and psychology, offer novel methods for mathematical comprehension. Exploring culturally responsive pedagogy in algebraic instruction is promising. Future directions may involve targeted teacher professional development for a responsive educational environment. Extending interventions to diverse settings informs broader educational practices. Investigating metacognitive dimensions in algebraic learning is a future research frontier, exploring how teaching metacognitive strategies enhances autonomy in problem-solving. Furthermore, future research may use interactive, conversation-based assessments with students as additional data sources to gain greater insight into their challenges.

CONCLUSIONS

In reflection, this study has not only shed light on the intricacies of teaching algebra but has also catalyzed profound changes in my approach to mathematics education. Through the lens of action research, I delved deep into the challenges faced by learners and crafted interventions that went beyond mere numerical improvements, aiming to foster a conceptual understanding of algebraic principles.

One of the most transformative aspects of this study was the realization that traditional teaching methods must be transcended and embrace a more dynamic and engaging approach. By using personalized learning plans, differentiated instructions, scaffolding, and incorporating real-world applications and interactive tools, I made my teaching immersive and relatable for the learner. Rather than focusing solely on numerical results, I prioritized conceptual understanding, encouraging the learner to explore algebra's practical implications in everyday life, which motivated him to study mathematics. ChatGPT made the learner self-regulating as he interacted with it for procedural doubts and validation of his solutions. Furthermore, conversations with the learner about the lessons and assessment tasks were essential to understanding his deep-seated, recurring challenges. The study showed that verbal tasks highlighted what the learner knew and did not know, not the deep-seated difficulties between the two—a space commonly known as the zone of proximal development, a gold mine for teachers. Teaching algebra requires a holistic approach.

Moreover, the collaborative nature of this study, drawing insights from diverse disciplines such as cognitive science and psychology, underscored the importance of interdisciplinary approaches in mathematics education. I integrated principles from these fields into my teaching practice, recognizing the value of holistic and multidisciplinary perspectives in enhancing mathematical comprehension. I was able to comprehend the learners' challenges from different perspectives.

Furthermore, this study highlighted the significance of cultural responsiveness in algebra instruction. Acknowledging and embracing my learner's experiences, I created a supportive learning environment for him. Through culturally relevant pedagogy, I sought to empower my learner to see himself reflected in the mathematics he learned, fostering a more profound sense of belonging and engagement.

In essence, this study provided valuable insights into the teaching and learning of algebra and ignited a passion for transformative pedagogy within me. Based on this insight, I am committed to embracing innovation, collaboration, and cultural responsiveness in my teaching practice to empower the learners to become critical thinkers and lifelong learners in mathematics and beyond.

Reference

[1] Alegre, F., Moliner, L., Maroto, A., & Lorenzo-Valentin, G. (2019). Peer tutoring in algebra: A study in middle school. *The Journal of Educational Research*, *112*(6), 693-699. https://doi.org/10.1080/00220671.2019.1693947

[2] Ardiansari, L., Suryadi, D., & Dasari, D. (2023). Is fruit salad algebra still a favorite menu in introducing algebra in schools? *Mathematics Teaching Research Journal, 15*(3), 73-88. https://files.eric.ed.gov/fulltext/EJ1408197.pdf

[3] Arcavi, A., Drijvers, P., & Stacey, K. (2016). *The learning and teaching of algebra: Ideas, insights and activities*. Routledge.

[4] Attard, C., Calder, N., Holmes, K., Larkin, K., & Trenholm, S. (2020). Teaching and learning mathematics with digital technologies. *Research in Mathematics Education in Australasia 2016–2019*, 319-347. https://doi.org/10.1007/978-981-15-4269-5_13

[5] Babakr, Zana H.; Mohamedamin, Pakstan; Kakamad, Karwan. (2019). Piaget's Cognitive Developmental Theory: Critical Review. *Education Quarterly Reviews*, *2*(3), 517-524. https://files.eric.ed.gov/fulltext/EJ1274368.pdf

[6] Babić, T., Lacković, A., & Matejić, M. (2019, May). Critical thinking and creative thinking-The self-assessment of Algebra University College students. In *2019 42nd International Convention on Information and Communication Technology, Electronics and Microelectronics (MIPRO)* (pp. 843-848). IEEE. https://10.23919/MIPRO.2019.8757107

[7] Bamne, S.N. & Bamne, A.S. (2016). Comparative study of chalkboard teaching over PowerPoint teaching as a teaching tool in undergraduate medical teaching. *International Journal of Medical Science and Public Health, 5*(12). https://www.bibliomed.org/mnsfulltext/67/67-1467436103.pdf?1698550936

[8] Breckenridge, C. T. (2014). Becoming the gardeners of their own educational growth: evaluating an intervention designed to cultivate achievement and a growth mindset with seventh grade pre-algebra students (Doctoral dissertation, California State University, Sacramento).

[9] Breiteig, T. & Grevholm, B. (2006). The transition from arithmetic to algebra: To reason, explain, argue, generalize and justify. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of*

Mathematics Education, 2. 225-232. https://www.diva-portal.org/smash/get/diva2:1005151/FULLTEXT01.pdf

[10] Brookhart, S.M. (2013). How to create and use rubrics for formative assessment and grading. ASCD.

[11] Centres for Excellence for Maths. (2020*). Principles and practice: Motivating and engaging students in further education maths*. Pearson

[12] Chappuis, J. (2012). Leadership for Learning: How am I doing? *Educational Leadership, 70*(1), 36-41. https://www.ascd.org/el/articles/how-am-i-doing

[13] DCarlo, S.E. (2009). Too much content, not enough thinking, and too little FUN. *Advances in Physiology Education, 33*(4), 257-264. https://journals.physiology.org/doi/epdf/10.1152/advan.00075.2009

[14] Demme, I. (2018). *Six reasons why we learn algebra*. Demme Learning. https://demmelearning.com/blog/why-we-learn-algebra/

[15] Driscoll, M. (1999). *Fostering algebraic thinking: A guide for teachers grade 6-10*. Heinemann.

[16] Egner, T. (2023). Principles of cognitive control over task focus and task switching. *Nat Rev Psychol*. https://doi.org/10.1038/s44159-023-00234-4

[17] Freina, L., & Ott, M. (2015, April). A literature review on immersive virtual reality in education: state of the art and perspectives. In *The International Scientific Conference eLearning and Software for Education,1*(133),10-1007.

[18] Gojak, L.M. (2013). Algebra: Not 'If', but 'When'. National Council of Teachers of Mathematics. https://www.nctm.org/News-and-Calendar/Messages-from-the-President/Archive/Linda-M_-Gojak/Algebra_-Not-_If_-but-_When_/

[19] Hattie, J., & Timperley, H. (2007). The power of feedback. *Review of Educational Research,* 77(1), 81-112. https://doi.org/10.3102/003465430298487

[20] Hello Learning. (n.d.). *Design a dream bedroom math project-based learning-area and perimeter project.* https://www.teacherspayteachers.com/Product/Design-a-Dream-Bedroom-Math-Project-Based-Learning-Area-and-Perimeter-Project-4058860

[21] Hodge-Zickerman, A., Stade, E., York, C. S., & Rech, J. (2020). TACTivities: fostering creativity through tactile learning activities. *Journal of Humanistic Mathematics*, *10*(2), 377-390.

[22] Jackson, S. L., Krajcik, J., & Soloway, E. (1998). The design of guided learner-adaptable scaffolding in interactive learning environments. In *Proceedings of the SIGCHI conference on Human factors in computing systems,* 187-194.

[23] Joffrion, H. K. (2007). *Conceptual and procedural understanding of algebra concepts in the middle grades* (Doctoral dissertation, Texas A&M University).

[24] Obano, E. J., & Enowoghomwenma, D.-E. (2021). Teachers' knowledge andapplication of anecdotal record in assessing basic schools learners' progress in OredoLocal Government Area, Edo State, Nigeria. Benin Journal of Educational Studies, 27(1), 160–169.https://beninjes.com/index.php/bjes/article/view/81

[25] Kemmis, S., McTaggart, R., & Nixon, R. (2014). *The action research planner: Doing critical participatory action research (1st ed)*. Springer

[26] Lassiter, N. (2022). What is a pattern in nature? https://study.com/learn/lesson/pattern-nature-repeating-mathematical-animal.html

[27] McNiff, J. (2013). *Action research: Principles and practice*. London: Routledge.

[28] MicNiff, J. & Whitehead, J. (2006). *All you need to know about action research: An introduction*. SAGE.

[29] Mills, E. G. (2014). *Action research: A guide for the teacher researcher* (*5th ed*.). Pearson.

[30] Muir, T. (2012). Numeracy at home: Involving parents in mathematics education. International *Journal for Mathematics Teaching and Learning*, *25*, 1-13.

[31] Obano, E. J., & Enowoghomwenma, D.E. (2021). Teachers' knowledge and application of anecdotal record in assessing basic schools learners' progress in Oredo Local Government Area, Edo State, Nigeria . *Benin Journal of Educational Studies, 27*(1), 160–169. https://beninjes.com/index.php/bjes/article/view/81

[32] Oguguo, B.C.E, Oche, E.S., Ezeanya, C.O., et al. (2024). Relative effectiveness of formative assessment techniques on students' academic achievement in mathematics classroom teaching and learning. *Mathematics Teaching Research, 15*(6), 191-193. https://orcid.or/0000-0002-0723-2248

[33] OpenAI. (2023). ChatGPT. https://chat.openai.com

[34] Oppenzato, C. & Ginsburg, H.P. (2018). Teaching algebraic thinking to young children. Department and Research in Early Math Education. https://prek-math-te.stanford.edu/patterns-algebra/teaching-algebraic-thinking-young-children

[35] Poh, B. L. G., Muthoosamy, K., Lai, C. C., & Hoe, G. B. (2015). A marking scheme rubric: to assess students' mathematical knowledge for applied algebra test. *Asian Social Science*, *11*(24), 18.

[36] Sa'adah, N., Faizah, S., Sa'dijah, C., Khabibah, S., & Kurniati, D. (2023). Students'mathematical thinking process in algebraic verification based on crystalline concept. *Mathematics Teaching Research Journal, 15*(1), 90-107.

[37] Schoenfeld, A. H., Floden, R., El Chidiac, F., Gillingham, D., Fink, H., Hu, S., ... & Zarkh, A. (2018). On classroom observations. *Journal for STEM Education Research*, *1*, 34-59.

[38] Shemshack, A., & Spector, J. M. (2020). A systematic literature review of personalized learning terms. *Smart Learning Environments*, *7*(1), 1-20.

[39] Sherrell, Z. (2023). What are Piaget's stages of development, and what are examples of each? *Medical News Today*. https://www.medicalnewstoday.com/articles/325030#piagets-stages

[40] Staff, S. (2019). *How to prepare elementary school students for algebra?* Haughton Mifflin Harcourt. https://www.hmhco.com/blog/how-to-prepare-elementary-school-students-for-algebra

[41] Star, J. R., Caronongan, P., Foegen, A., Furgeson, J., Keating, B., Larson, M. R., Lyskawa, J., McCallum, W. G., Porath, J., & Zbiek, R. M. (2015). *Teaching strategies for improving algebra knowledge in middle and high school students* (NCEE 2014-4333). National Center for Education Evaluation and Regional Assistance (NCEE), Institute of Education Sciences, U.S. Department of Education. https://ies.ed.gov/ncee/wwc/docs/practiceguide/wwc_algebra_040715.pdf

[42] Sun-Lin, H. Z., & Chiou, G. F. (2019). Effects of gamified comparison on sixth graders' algebra word problem solving and learning attitude. *Journal of Educational Technology & Society*, *22*(1), 120-130.

[43] Grothérus, A., Jeppsson, F., & Samuelsson, J. (2019). Formative Scaffolding: how to alter the level and strength of self-efficacy and foster self-regulation in a mathematics test situation. *Educational Action Research*, *27*(5), 667–690. https://doi.org/10.1080/09650792.2018.1538893

[44] Tejeda, S., & Gallardo, K. (2017). Performance assessment on high school advanced algebra. International *Electronic Journal of Mathematics Education*, *12*(3), 777-798.

[45] Timotheou, S., Miliou, O., Dimitriadis, Y., Sobrino, S. V., Giannoutsou, N., Cachia, R., ... & Ioannou, A. (2023). Impacts of digital technologies on education and factors influencing schools' digital capacity and transformation: A literature review. *Education and Information Technologies*, *28*(6), 6695-6726.

[46] Tomlinson, C. A. (2017). Differentiated instruction. In *Fundamentals of gifted education,* 279-292. Routledge.

[47] Torres, J. T., García-Planas, M. I., & Domínguez-García, S. (2016). The Use of E-Portfolio in a Linear Algebra Course. Journal of Technology and Science Education, 6(1), 52-61.

[48] Trethewy, K. (2022). 55 math activities for middle school: Algebra, fractions, exponents, and more! *Teaching Expertise*. https://www.teachingexpertise.com/classroom-ideas/math-activities-for-middle-school/

[49] Uddin, L.Q. (2021). Cognitive and behavioural flexibility: Neural mechanisms and clinical considerations. *Nature Reviews Neuroscience, 22*. 167-179. https://doi.org/10.1038/s41583-021-00428-w

[50] Virtual Lab School. (2023). Cognitive development: School-age. https://www.virtuallabschool.org/school-age/cognitive-development/lesson-2#:~:text=They%20begin%20to%20develop%20a,perform%20simple%20addition%20and%20subtraction.

[51] Vygotsky, L.S. (1978). *Mind in society: The development of higher psychological processes.* Harvard University Press.

[52] West, M. (2022). *Did you know how much math is there in sports?* https://www.mathnasium.com/madisonwest/news/did-you-know-how-much-math-there-sports-mw

[53] Wu, T. T., Lee, H. Y., Li, P. H., Huang, C. N., & Huang, Y. M. (2023). Promoting self-regulation progress and knowledge construction in blended learning via ChatGPT-based learning aid. *Journal of Educational Computing Research*, 07356331231191125.

[54] Yildirim-Erbasli, S. N., Bulut, O., Demmans Epp, C., & Cui, Y. (2023). Conversation-based assessments in education: Design, implementation, and cognitive walkthroughs for usability testing. *Journal of Educational Technology Systems, 52*(1), 27-51. https://doi.org/10.1177/00472395231178943

[55] Zheng, L., Long, M., Zhong, L., & Gyasi, J. F. (2022). The effectiveness of technology-facilitated personalized learning on learning achievements and learning perceptions: a meta-analysis. Education *and Information Technologies*, *27*(8), 11807-11830.

Encouraging Undergraduate Students to Explore Multiple Proofs of the Multinomial Theorem

Christian Farkash, Michael Storm, Thomas Palmeri, Chunhui Yu

Mathematics Department, Farmingdale State College, NY USA

chunhui.yu@farmingdale.edu

Abstract: *Several studies indicate that exploring mathematical ideas by using more than one approach to prove the same statement is an important matter in mathematics education. In this work, we have collected a few different methods of proving the multinomial theorem. The goal is to help further the understanding of this theorem for those who may not be familiar with it. These proofs can also be used by undergraduate college instructors in a calculus, a discrete mathematics or a probability course.*

Keywords: Multinomial distribution; Differential calculus; Probability; Combinatorics

INTRODUCTION

The multinomial theorem is used to expand any sum to an integer power and is an extension of the binomial theorem. The binomial theorem only deals with the addition of two variables to an integer power, whereas the multinomial theorem deals with more than two variables. The binomial and multinomial theorems are important results in elementary mathematics, and aside from the straightforward application of expanding polynomials of high degree, they also have applications in probability, combinatorics, number theory, and several other fields of mathematics. The multinomial theorem is written as follows:

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{\sum_{i=1}^{m} k_i = n} \frac{n!}{k_1!\, k_2! \cdots k_m!} {x_1}^{k_1} {x_2}^{k_2} \cdots {x_m}^{k_m}$$

Here, $k_1, k_2, \cdots, k_m \geq 0$ and the multinomial coefficient $\frac{n!}{k_1!k_2!\cdots k_m!} = \binom{n}{k_1, k_2, \cdots, k_m}$ is the number of possible ways to put n balls into m boxes.

When introducing the binomial theorem, most instructors often employ various methods to engage students and deepen their understanding (Flusser & Francia, 2000). Two typical approaches are:

1. Algebraic derivation: The instructor may start by introducing the binomial theorem and its formula, then proceed to prove it algebraically using mathematical induction or combinatorial arguments.
2. Pascal's triangle: The instructor may start by introducing Pascal's triangle and its connection to binomial coefficients, then show how each row corresponds to the coefficients in the binomial expansion.

The Multinomial theorem serves as a generalization of the binomial theorem, extending its principles from binomials to multinomials. We will present several approaches to prove the multinomial theorem in the following section.

Mathematics educators agree that exploring mathematical ideas by using more than one approach to solving the same problem (e.g., proving the same statement) is an essential element in the development of mathematical reasoning (NCTM, 2000; Polya, 2004; Schoenfeld, 2014; Dreyfus, Nardi & Leikin, 2012; Stupel & Ben-Chaim, 2013; Stupel & Ben-Chaim, 2017). Dreyfus, Nardi & Leikin (2012) discusses the pedagogical importance of multiple proof tasks and of taking into account the mathematical, pedagogical, and cognitive structures related to the effective teaching of proof and proving. Leikin (2009) indicates that the differences between the proofs are based on using: (1) different representations of a mathematical concept; (2) different properties (definitions or theorems) of mathematical concepts from a particular mathematical topic; (3) different mathematics tools and theorems from different branches of mathematics; or (4) different tools and theorems from different subjects (not necessarily mathematics). In our case, we apply the third type of differences between the proofs; we shall present various proofs using the tools and theorems of combinatorics, induction, probability, and differential calculus.

Proofs of the Multinomial Theorem

Combinatorial proof and induction proof are two classical methods which can be easily found in a standard textbook or with an online search. For readers' convenience, we state them here first.

Combinatorial Proof

Given variables $x_1, x_2, \cdots x_m$, we look to expand

$$(x_1 + x_2 + \cdots + x_m)^n \tag{1.1}$$

By definition, we know that this can be expressed as

$$\underbrace{(x_1 + x_2 + \cdots + x_m)(x_1 + x_2 + \cdots + x_m)\cdots(x_1 + x_2 + \cdots + x_m)}_{\text{n times}} \tag{1.2}$$

Each term of this expression when expanded will be of the form

$$Cx_1^{k_1}x_2^{k_2}\cdots x_m^{k_m} \tag{1.3}$$

where $\sum_{i=1}^{m} k_i = n$ and C is a numerical coefficient. To determine the value of C, consider the following method. Suppose we choose x_1 from k_1 sets of parentheses; there are $\binom{n}{k_1}$ ways this can be done because there is no more than one of x_1 in each set. Now when we choose x_2 from k_2 sets of parentheses, we cannot choose the same set that we have already chosen x_1 from. This means that we are left with $n-k_1$ sets from which we can choose x_2. So the number of ways to choose x_1 from k_1 sets and x_2 from k_2 sets is as follows:

$$\binom{n}{k_1}\binom{n-k_1}{k_2} \tag{1.4}$$

Following the same approach for the remaining x values, we can see that the coefficient for each term can be represented as such:

$$\binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3}\cdots\binom{n-k_1-k_2-\cdots-k_{m-1}}{k_m} \tag{1.5}$$

By definition,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{1.6}$$

When we expand our coefficient by the definition above, many of the terms will cancel, leaving the following value for determining the coefficient:

$$\frac{n!}{k_1!k_2!k_3!\cdots k_m!} \tag{1.7}$$

Finally, to obtain every term from the expansion of $(x_1 + x_2 + \cdots + x_m)^n$, we add together every possible combination of $k_1 + k_2 + \cdots + k_m = n$.

$$\sum_{\sum_{i=1}^{m} k_i = n} \frac{n!}{k_1!k_2!\cdots k_m!} x_1^{k_1}x_2^{k_2}\cdots x_m^{k_m} \tag{1.8}$$

Induction Proof

Proof. We will prove this with induction on m. To start, we show that this holds for $m = 1$.

$$(x_1)^n = \sum_{\sum_{i=1}^{1} k_i = n} \binom{n}{k_1} x_1^{k_1} = x_1^n \tag{2.1}$$

Next, suppose the multinomial theorem holds for m. Then

$$(x_1 + x_2 + \cdots + (x_m + x_{m+1}))^n = \sum_{\sum_{i=1}^{m-1} k_i + K = n} \binom{n}{k_1 k_2 \cdots k_{m-1} K} x_1^{k_1}x_2^{k_2}\cdots x_{m-1}^{k_{m-1}}(x_m + x_{m+1})^K$$

(2.2)

Applying the binomial theorem to the right-hand side gives us

$$(x_1 + x_2 + \cdots + (x_m + x_{m+1}))^n$$
$$= \sum_{\sum_{i=1}^{m-1} k_i + K = n} \left[\binom{n}{k_1 k_2 \cdots k_{m-1} K} x_1^{k_1} x_2^{k_2} \cdots x_{m-1}^{k_{m-1}} \sum_{k_m + k_{m+1} = K} \binom{K}{k_m k_{m+1}} x_m^{k_m} x_{m+1}^{k_{m+1}} \right]$$

(2.3)

Then since

$$\binom{n}{k_1 k_2 \cdots k_{m-1} K} \binom{K}{k_m k_{m+1}} = \frac{n!}{k_1! k_2! \cdots k_{m-1}! K!} \cdot \frac{K!}{k_m! k_{m+1}!} = \frac{n!}{k_1! k_2! \cdots k_m! k_{m+1}!} \quad (2.4)$$

it follows that

$$(x_1 + x_2 + \cdots + (x_m + x_{m+1}))^n = \sum_{\sum_{i=1}^{m} k_i = n} \binom{n}{k_1 k_2 \cdots k_{m+1}} x_1^{k_1} x_2^{k_2} \cdots x_{m+1}^{k_{m+1}}$$

(2.5)

Since we now have shown that $m \Rightarrow m + 1$, we can conclude by the principle of induction that this statement holds for all integers m greater than or equal to 1.

Probability Proof

The following is a proof in Kataria (2016), which is an extension of the proof in Rosalsky (2007). Consider an experiment with n independent trials. The outcome of each trial results in the occurrence of one of the m mutually exclusive and exhaustive events $E_1, E_2, \cdots, E_m$. For each $i = 1, 2, \cdots, m$, let p_i be the constant probability of the occurrence of the event E_i and X_i be the random variable that denotes the number of times event E_i has occurred. Then, the joint probability mass function of the random variables $X_1, X_2, \cdots, X_m$ is

$$P\{X_1 = k_1, X_2 = k_2, \cdots, X_m = k_m\} = n! \prod_{j=1}^{m} \frac{p_j^{k_j}}{k_j!} \quad (3.1)$$

where $\sum_{i=1}^{m} k_i = n$. Also, since (3.1) is a valid statistical distribution, we have

$$1 = \sum_{\sum_{i=1}^{m} k_i = n} n! \prod_{j=1}^{m} \frac{p_j^{k_j}}{k_j!} \quad (3.2)$$

By using the distributive property in (1.2), it follows that for all real number x_i's,

$$(x_1+x_2+\cdots+x_m)^n=\sum_{\sum_{i=1}^m k_i=n} C(n,k_1,k_2,\cdots,k_m){x_1}^{k_1}{x_2}^{k_2}\cdots{x_m}^{k_m} \tag{3.3}$$

where $C(n,k_1,k_2,\cdots,k_m)$ are positive integers and k_i's are nonnegative integers which satisfy $\sum_{i=1}^m k_i=n$. Now we need to show that

$$C(n,k_1,k_2,\cdots,k_m)=\frac{n!}{k_1!k_2!\cdots k_m!} \tag{3.4}$$

Assume $x_i > 0$ for all $i = 1,2,\cdots,m$ and define

$$p_i=\frac{x_i}{x_1+x_2+\cdots+x_m} \tag{3.5}$$

It follows that $0 < p_i < 1$ and $\sum_{i=1}^m p_i = 1$. Substituting (3.5) into (3.2), we obtain for positive reals

$$(x_1+x_2+\cdots+x_m)^n=\sum_{\sum_{i=1}^m k_i=n}\frac{n!}{k_1!k_2!\cdots k_m!}x_1^{k_1}x_2^{k_2}\cdots x_m^{k_m} \tag{3.6}$$

Finally, subtracting (3.6) from (3.3),

$$\sum_{\sum_{i=1}^m k_i=n}\left(C(n,k_1,k_2,\cdots,k_m)-\frac{n!}{k_1!\,k_2!\cdots k_m!}\right){x_1}^{k_1}{x_2}^{k_2}\cdots{x_m}^{k_m}=0, x_i>0 \tag{3.7}$$

Since (3.7) shows the left-hand side equals zero when subtracting (3.6) from (3.3), it follows that (3.4) is true.

Proof with Differential Calculus

The following proof extends Hwang's proof of the binomial theorem in Hwang (2009) using differential calculus into the multinomial theorem.

Again, it follows that upon distribution that for any integer n

$$(x_1+x_2+\cdots+x_m)^n=\sum_{\sum_{i=1}^m k_i=n} C(n,k_1,k_2,\cdots,k_m)\prod_{i=1}^m {x_i}^{k_i} \tag{4.1}$$

where the k_i's are nonnegative integers and $C(n,k_1,k_2,\cdots,k_m)$ are positive integers.
Given any set of nonnegative integers $c_1,c_2,\cdots,c_m$ such that $\sum_{i=1}^m c_i=n$, we calculate the partial derivatives of both sides of (4.1) with respect to each x_i c_i times for $i=1,2,\cdots,m$.

For the left side of (4.1), since $\sum_{i=1}^{m} c_i = n$,

$$\frac{\partial^n}{\partial x_1^{c_1} \partial x_2^{c_2} \cdots \partial x_m^{c_m}} (x_1 + x_2 + \cdots + x_m)^n = n! \tag{4.2}$$

For the right side of (4.1), if and only if $c_i = k_i$ for all $i = 1,2,\cdots,m$, then

$$\frac{\partial^n}{\partial x_1^{c_1} \partial x_2^{c_2} \cdots \partial x_m^{c_m}} \prod_{i=1}^{m} {x_i}^{k_i} = \frac{\partial^n}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_m^{k_m}} \prod_{i=1}^{m} {x_i}^{k_i} = k_1!\, k_2! \cdots k_m! \tag{4.3}$$

Otherwise,

$$\frac{\partial^n}{\partial x_1^{c_1} \partial x_2^{c_2} \cdots \partial x_m^{c_m}} \prod_{i=1}^{m} {x_i}^{k_i} = 0 \tag{4.4}$$

Therefore,

$$\frac{\partial^n}{\partial x_1^{c_1} \partial x_2^{c_2} \cdots \partial x_m^{c_m}} \left[\sum_{\sum_{i=1}^{m} k_i = n} C(n, k_1, k_2, \cdots, k_m) \prod_{i=1}^{m} {x_i}^{k_i} \right] = C(n, k_1, k_2, \cdots, k_m) k_1!\, k_2! \cdots k_m! \tag{4.5}$$

Then from (4.1), (4.2), and (4.5), we have for nonnegative integers $k_i$satisfying$\sum_{i=1}^{m} k_i = n$,

$$C(n, k_1, k_2, \cdots, k_m) k_1!\, k_2! \cdots k_m! = n!$$

Which means that

$$C(n, k_1, k_2, \cdots, k_m) = \frac{n!}{k_1!\, k_2! \cdots k_m!}$$

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{\sum_{i=1}^{m} k_i = n} \frac{n!}{k_1!\, k_2! \cdots k_m!} {x_1}^{k_1} {x_2}^{k_2} \cdots {x_m}^{k_m}$$

Thus, we have collected multiple methods of proving the multinomial theorem.

1. A combinatorial proof.
2. A proof by induction.
3. A probability proof.
4. A proof with differential calculus.

METHOD

A case study was conducted in an undergraduate math major junior and senior level topic class at Farmingdale State College, with a total of 12 students. The aim of this survey includes:

1. Test the ability of junior and senior level college math major students to prove the multinomial theorem;
2. Examine math major undergraduate students' attitudes toward presenting multiple proof approaches for the multinomial theorem;

Step 1: The following question was asked in class.

Do you think it is valuable to present multiple proof approaches for a same mathematics statement? Can you give one or more examples that can be proved in different methods?

Step 2: The multinomial theorem was presented in class.

Step 3: The combinatory and induction methods were presented in class.

Step 4: The probability and differential calculus method were presented in class.

Step 5: The following question was asked in class:

Assuming you are the instructor, will you present multiple proofs of the multinomial theorem? Will you require your students to know all of them?

RESULTS

In step 1, all 12 students agreed that presenting multiple proof approaches for a same mathematics statement is important and valuable. However, only 6 students could provide meaningful examples. With instructor's hints, they recalled the proofs of Pythagorean theorem and some geometry and combinatory properties.

In step 2, although 8 students claimed familiarity of the multinomial theorem, initially none of the students felt confident in proving it completely.

In step 3, all 12 students followed the combinatory and induction proofs comfortably, with some remembering their use in proving the binomial theorem.

In step 4, none of the students had learned the probability method or the differential calculus method before. After a brief review of the same proof methods for the binomial theorem, they all gained better understanding of the same proof methods applied to the multinomial theorem and appeared to be impressed with these two additional proofs, especially the probability one.

In step 5, students were allowed to discuss this question. Their consensus was with a suggestion that at least present the combinatorial and induction methods. If class time permits, consider introducing the probability proof as well. The differential calculus method can be left as optional homework for interested students. This approach allows flexibility and caters to varying levels of interest and readiness. Some students did worry that presenting more than two methods simultaneously might overwhelm and confuse the class.

DISCUSSION

While some students struggled initially, exposure to various methods surely enhanced their understanding. It can develop students' divergent reasoning (Kwon et al. 2006), as well as their mental flexibility and fluency (Dreyfus, Nardi & Leikin, 2012; Leikin, 2009; Silver, 1997; Sriraman, 2003). As an instructor, presenting multiple proofs can enrich students' mathematical experience and foster deeper comprehension of theorems. From an educational viewpoint, such a comparison provides teachers and students with interesting connections between different viewpoints. Of course, the perspective presented requires a good level of epistemological skill on the part of teachers (Bagni, 2008).

CONCLUSION

For the multinomial theorem, the classroom study indicates that students find the induction approach most rigorous. Connecting it to the simpler binomial theorem, which they are already familiar with, makes it more accessible. Additionally, the combinatorial proof by counting also provides a concrete interpretation and students who enjoying combinatorial structure tend to find this approach appealing. Some students also like the probability proof, especially after gaining a clear understanding of the ideas presented in Rosalsky (2007). However, some students are not accustomed to the differential calculus method, as it can feel quite abstract. Only students with a strong background in multivariable calculus tend to follow it through well.

In general, comparison of different proofs can be an appropriate method to make the nature of proof visible to the students (Pfeiffer, 2010). Educators should consider using alternative methods for proof, which can provide students with alternative strategies to approach complex problems and enhance their understanding of underlying concepts (Mowahed & Mayar, 2023). Exposure to diverse proofs also hones students' problem-solving skills. They learn to adapt, generalize, and apply techniques across different scenarios (Stylianides & Ball, 2008). In an undergraduate-level mathematics course, when introducing the multinomial theorem, instructors can cover the induction and combinatorial methods with students during classroom lectures. For the probability and differential calculus approaches, instructors can provide students with materials related to proofs for binomial theorem, such as those found in reference Rosalsky (2007) and Hwang (2009), and encourage them to extend these concepts to the multinomial theorem case. This approach allows students to explore this topic from different angles and deepen their understanding. Further interested instructors and advanced students can even refer to Noble (2022) for a detailed historical background review.

References

[1] Dreyfus, T., Nardi, E., & Leikin, R. (2012). Forms of proof and proving in the classroom. Proof and proving in mathematics education: The 19th ICMI study, 191-213. https://doi.org/10.1007/978-94-007-2129-6_20

[2] Polya, G. (2004). How to solve it: A new aspect of mathematical method (No. 246). Princeton university press.

[3] Schoenfeld, A. H. (2014). Mathematical problem solving. Elsevier.

[4] Bagni, G. T. (2008). Theorem and Its Different Proofs: History, Mathematics Education, and the Semiotic-Cultural Perspective. Canadian Journal of Science, Mathematics and Technology Education, 8(3), 217-232. https://doi.org/10.1080/14926150802169297

[5] Stupel, M., & Ben-Chaim, D. (2017). Using multiple solutions to mathematical problems to develop pedagogical and mathematical thinking: A case study in a teacher education program. Investigations in Mathematics Learning, 9(2), 86–108. https://doi.org/10.1080/19477503.2017.1283179

[6] Kataria, K. K. (2016). A probabilistic proof of the multinomial theorem. The American Mathematical Monthly, 123(1), 94-96.

[7] Rosalsky, A. (2007). A simple and probabilistic proof of the binomial theorem. The American Statistician, 61(2), 161-162. https://doi.org/10.1198/000313007X188397

[8] Hwang, L. C. (2009). A simple proof of the binomial theorem using differential calculus. The American Statistician, 63(1), 43-44. https://doi.org/10.1198/tast.2009.0009

[9] Noble, E. (2022). A History of the Binomial and Multinomial Theorems. In The Rise and Fall of the German Combinatorial Analysis (pp. 17-66). Cham: Springer International Publishing. https://doi.org/10.1007/978-3-030-93820-8_2

[10] Pfeiffer, K. (2010). How do students evaluate and compare mathematical proofs? Research in Mathematics Education, 12(2), 161–162. https://doi.org/10.1080/14794802.2010.496986

[11] Mowahed, A. K., & Mayar, J. A. (2023). Problematic and Supportive Aspects of Indirect Proof in Afghan Undergraduate Students' Proofs of the Irrationality of $\sqrt{3}$ and $\sqrt{5/8}$. Mathematics Teaching Research Journal, 15(4), 124-135. https://eric.ed.gov/?id=EJ1409364

[12] Stylianides, A. J., & Ball, D. L. (2008). Understanding and describing mathematical knowledge for teaching: Knowledge about proof for engaging students in the activity of proving.

Journal of mathematics teacher education, 11, 307-332. https://doi.org/10.1007/s10857-008-9077-9

[13] Flusser, P., & Francia, G. A. (2000). Derivation and visualization of the binomial theorem. International Journal of Computers for Mathematical Learning, 5, 3-24. https://doi.org/10.1023/A:1009873212702

[14] Leikin, R. (2009). Multiple proof tasks: Teacher practice and teacher education. ICME Study, 19(2), 31-36.

[15] Stupel, M., & Ben-Chaim, D. (2013). One problem, multiple solutions: How multiple proofs can connect several areas of mathematics. Far East Journal of Mathematical Education, 11(2), 129.

[16] National Council of Teachers of Mathematics (2000), Principles and Standards for School Mathematics, NCTM, Reston, VA, 2000.

[17] Kwon, O. N., Park, J. S., & Park, J. H. (2006). Cultivating divergent thinking in mathematics through an open-ended approach. Asia Pacific Education Review, 7, 51–61. https://doi.org/10.1007/BF03036784

[18] Leikin, R. (2009b). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A. Berman, & B. Koichu (Eds.), Creativity in mathematics and the education of gifted students (pp. 129–145). Rotterdam: Sense Publishers.

[19] Silver, E. A. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. ZDM – The International Journal on Mathematics Education, 3, 75–80. https://10.1007/s11858-997-0003-x

[20] Sriraman, B. (2003). Mathematical giftedness, problem solving, and the ability to formulate generalizations. Journal of Secondary Gifted Education, 14, 151–165. https://doi.org/10.4219/jsge-2003-425

Trigonometric Ratios in High School Students: From Instrumental Understanding to Relational Understanding through their Application in Motion Vectors

Ivonne Alejandra Toledo-Nieto, José Antonio Juárez-López

Faculty of Physical and Mathematical Sciences, Meritorious Autonomous University of Puebla, México

ivonne.toledo@correo.buap.mx, jajul@fcfm.buap.mx

Abstract: The present educational intervention study focused on addressing some areas of improvement in the teaching and learning of trigonometric ratios in high school students. A diagnostic evaluation was performed that revealed an instrumental understanding of trigonometric concepts by the students. Subsequently, a didactic sequence was implemented that incorporated practical applications of trigonometric ratios in the context of displacement vectors. The results of the study indicated significant improvements in students' understanding after the intervention, marking a shift toward relational understanding. This proposal is presented as an effective strategy and suggests promising ways to improve pedagogy in this field.

Keywords: relational understanding, trigonometric ratios, rectangular components, displacement vector.

INTRODUCTION

Trigonometry, a fundamental branch of mathematics, has been praised for its invaluable contribution to the development of natural sciences and technology (Aray et al., 2020). Researchers who have explored students' trigonometric learning (Dunghana et al., 2023; Moore, 2013, 2014; Thompson et al., 2007; Weber, 2005; Yigit, 2014) state that it is a very difficult area for both students as for teachers.

In this regard, authors such as González et al. (2017) highlight the need to overcome the static and rigid teaching of trigonometry and advocate a more dynamic and meaningful approach. Cantoral et al. (2015) emphasize the importance of using a variety of resources and practices, while Solanilla (2015) highlights the need for effective teaching strategies. Aray et al. (2020) support the idea of giving meaning to teaching, while Suárez et al. (2017) highlight the importance of comprehensive mathematics education to prepare students for the university world.

To address the concern about the teaching of trigonometry in high school, an educational intervention investigation was carried out in the month of September 2023. Two groups of pre-university students from the city of Puebla, Mexico who had passed the trigonometry subject in previous semester of high school were selected. A pretest was administered to measure their level of understanding in trigonometry and, after the intervention, a posttest was administered to evaluate the results, so the present study is quantitative.

The diagnostic test consisted of three items on trigonometric ratios. Three levels of cognitive performance were evaluated: item 1, knowledge, item 2, understanding, and item 3, application. Although the test had multiple choice answers, students were asked to justify their answers by expressing their ideas, procedures, formulas, or thoughts related to the solution of each exercise, see Appendix 1.

For the intervention, six 50-minute sessions of a workshop were designed that focused its speech on the application of trigonometric ratios in the context of uniform rectilinear motion (MRU) and displacement vectors. One of the concepts that requires a solid understanding of trigonometric ratios is that of displacement vectors, which play a crucial role in physics and in a wide variety of practical applications, from navigation to computer animation. However, to effectively analyze displacement vectors, it is essential to understand trigonometric ratios and their relationship to the decomposition of vectors into their Cartesian components.

A displacement vector provides precise information about how much an object moves and in what direction it moves in space. Trigonometric ratios, such as cosine and sine, are essential to accurately describe the horizontal and vertical components of the magnitude and direction of said movement, while the tangent allows calculating the direction, that is, the angle between the displacement vector and the abscissa axis (Serway & Vuille, 2012). Although there are various graphic methods to determine the characteristics of the displacement vectors, the analytical method provides greater precision as it eliminates instrumental errors.

Through this intervention, it is intended to take students from a superficial instrumental understanding to a relational understanding (Herheim, 2023) in which they can appreciate the real and significant application of trigonometric ratios in motion vectors. The findings of a literature review study conducted by Hamzah et al. (2021) revealed that, in the last 10 years, most of the research in the field of trigonometry teaching has focused on the identification of misconceptions and errors made by students and very few provide concrete solutions to minimize them.

On the other hand, the results of Aray et al. (2020) have highlighted an additional concern related to the teaching of trigonometry, specifically about the superficiality in which this discipline is often approached, which not only affects the understanding of trigonometric ratios themselves, but also creates a void in students' holistic mathematical knowledge. This, in turn, makes it difficult to teach and learn later subjects such as calculus, linear algebra, physics, and other areas that require a solid foundation in trigonometry.

In this regard, De Villiers and Jugmohan (2012) state that students often lack a consolidated understanding of trigonometric principles, which leads them to rely excessively on memorizing procedures and rules. This practice, while it may lead to success at a procedural level, masks underlying conceptual deficiencies.

In this context, the present research acquires significance by addressing the development and implementation of effective pedagogical strategies to improve students' understanding and application of trigonometric ratios. The research aims to transform the teaching of trigonometry, moving students from a superficial instrumental understanding to a relational and applied understanding of these ratios, particularly in the context of motion vectors. We assume that this approach will not only improve the quality of high school mathematics instruction but will also prepare students more effectively to meet the academic and university challenges that await them in the future, thus addressing a critical need in the field of mathematics education.

Literature Review

Richard Skemp (2006) establishes a distinction between two types of mathematical understanding: instrumental understanding and relational understanding. Instrumental understanding refers to the ability to apply a mathematical procedure without needing to understand its meaning or relationship with other mathematical concepts, that is, use the formula indicating its variables and operate arithmetically. On the other hand, relational understanding involves the ability to understand how mathematical concepts are interconnected and how they relate to each other. Skemp (2006) argues that relational understanding is essential for a deep and lasting consolidation of knowledge, while instrumental understanding can be useful in specific situations, but is not sufficient for a complete and meaningful understanding of mathematical concepts. This latter approach goes beyond simply “knowing what to do” by training students to explain “why to do it” (Skemp, 2006).

In this regard, Hiebert and Lefevre (2013) describe instrumental understanding as that which focuses on the isolated use of concepts and rules to complete mathematical tasks, while relational understanding focuses on establishing connections and relationships between these concepts, thus promoting deeper and more meaningful knowledge. Herheim (2023) provides another important difference between instrumental understanding and relational understanding, which is the way in which mathematical discourse is approached.

Instrumental understanding accepts the explanations given without questioning them, while relational understanding critically examines the logic of the explanations through dialogic discourse. Furthermore, instrumental understanding expresses explanations in a rigid way, relying on visual prototypes, while relational understanding expresses explanations in a flexible way, allowing students to adapt their understanding to different contexts.

On the other hand, Weber (2005) highlights that students' difficulties in trigonometry often arise from a poor initial understanding of fundamental concepts, such as angles, angle measurement and right triangles.

In this regard, Maknun, Rosjanuardi and Jupri (2022) identified some epistemological obstacles that students face when learning trigonometry, among which are: difficulty in understanding the relationship between trigonometric ratios and angles, difficulty in applying trigonometric ratios in practical situations, confusion between different trigonometric functions and their properties, etc. His article suggests that addressing these obstacles is essential to improving mathematics learning in general.Hamzah et al. (2021) identified, in their review, studies that have focused on correcting misconceptions in trigonometry. These interventions include the use of manipulative materials (Ulyani & Qohar, 2021), gamification through digital applications (Prabowo et al., 2018, 2019), the use of software such as Geogebra (Ibrahim & Ilyas, 2016), the learning approach game-based (Jorda & Santos, 2015), the 5E learning model (Tuna & Kacar, 2013) and discovery learning (Ngu & Phan, 2020; Hadi & Faradillah, 2020). All of these approaches aim to address conceptual deficiencies and improve understanding of trigonometry among students.

Taking into account this theoretical basis and the need to address conceptual difficulties in trigonometry, an intervention strategy is proposed that uses the concept of displacement vectors. By linking trigonometric ratios with the analysis of uniform rectilinear motion, we seek to show their practical applicability in real-world situations. Additionally, by providing a concrete context for understanding trigonometric ratios, students are expected to be able to articulate "why" these are critical in motion vector analysis, thus promoting relational and meaningful understanding.

METHOD

A sample of two groups of 32 members each of high school students from the center of the city of Puebla, Mexico, was selected as study subjects of comparison of static groups (experimental-control). The items extracted for the pretest are part of the diagnostic exam developed by the Evaluation Commission of the Meritorious Autonomous University of Puebla (BUAP by its acronym in Spanish) High School Physics Academy, of which the first author is a member. This exam was created based on the mathematical knowledge necessary to successfully undertake the Physics I course. However, it is important to highlight that of the 15 items that make it up, only 3 are related to trigonometry. The piloting process of this exam began in 2020, and the items were reviewed and adjusted, retaining, modifying, or discarding questions based on the confidence intervals. This process continued throughout 2021 and 2022, until reaching its final version.

In the month of August 2023, the test was administered again, using a questionnaire prepared in Microsoft Forms to a population of 5,433 fifth semester students.

Taking into account the recommendations of González et al. (2017), Cantoral et al. (2015), Solanilla (2015), Aray et al. (2020) and Suárez et al. (2017), Ulyani and Qohar (2021), Prabowo et al. (2018, 2019), Ibrahim & Ilyas (2016), Jorda & Santos (2015) and Ngu & Phan (2020) and

Hadi & Faradillah (2020), the didactic sequence showing in Table 1 was designed, which was applied five weeks later to the sample groups.

SPECIFIC CONTENT:	Vectors	NUMBER OF SESSIONS:	6	SESSION TIME UNITS:	50 minutes
EXPECTED LEARNING					
Explain the relationship of Physics with other areas of knowledge by applying vector magnitudes to solve problems.					

MOMENT	SESSION	ACTIVITY	TEACHING STRATEGIES	LEARNIGN STRATEGIES	RESOURCES
Opening **50 minutes**	*1*	Introduction	The teacher provides a game where the character visually demonstrates that the displacement is formed from the distance traveled along the horizontal and vertical components. Ask students to recount moments in daily life that involve traveling on straight roads.	It uses the interactive Cartesian plane game in which you are asked to reach a specific coordinate, using the cursor and marking the location. Perform active observation. Participate in the Brainstorm.	• Cellphones or computers • Wi-Fi access • Game: https://www.cokitos.com/juego-coordenadas-cartesianas-matematicas/
Develop-ment **150 minutes**	*2*	Exploring the difference between the concepts of distance and displacement.	Presents the simulator for the student to explore and requests the preparation of a comparative table on the differences between distance and displacement. Submit the reading for the next session about vector addition.	Explore the simulator and generate the comparison table.	• Cellphones or computers • Wi-Fi access • Simulator: https://www.educaplus.org/game/distancia-y-desplazamiento • Reading material

	3	Knowing the graphic methods of vector addition.	The teacher conducts questioning to generate brainstorming about reading about vector addition (Inverted Classroom), then consolidates the information and shows examples of solutions to exercises that involve the graphic method, finally providing the exercises that must be done. solve the students.	Through reading the parallelogram method and the polygon method, students work on determining the resulting vector on a scale, with ruler and protractor measurements.	• Board and markers • Ruler and protractor • Exercises • Notebook
	4	Knowing the analytical method of vector addition.	In a plenary session, a debate is generated on the conclusions of the previous session, where emphasis will be placed on the use and relevance of trigonometric ratios. The teacher shows on the blackboard how addition with vectors is done analytically using geometry and trigonometry.	Students solve the same exercises they did in session 2, but now using the mathematical approach. They compare both results and develop conclusions.	• Board • Markers • Exercises • Notebook
Closing **100 minutes**	*5*	Applying the concepts acquired through a virtual laboratory.	Indicate the didactic situation. Organize teams of 5 members. Give students the instruction manual to follow to carry out the	Use Google Maps to plot trajectories from various starting points to a meeting point in the center of the city of Puebla (whose streets are perfect grids). Represent these trajectories as	• Cellphones or computers • Wi-Fi access • GeoGebra • https://www.geogebra.org/calculator • Google Maps https://www.g

			practice. See **Appendix 2** Provide links and explain the use of the ruler tool in Google maps.	displacement vectors, measure their magnitude and determine the direction using Cartesian coordinates. Subsequently, they entered GeoGebra to represent the displacement vectors both graphically and analytically and compared the results with the length of the trajectories provided by Google Maps. Preparation of virtual laboratory report.	oogle.com.mx/maps/search/zocalo/@19.0429833,-98.2005476,17z?entry=ttu
	6	Self-assessment.	Provide link to self-assessable exercises. Reinforce and consolidate relational understanding of concepts.	Perform interactive self-assessment exercises, test their understanding of concepts and evaluate their progress.	• Cellphones or computers • Wi-Fi access • Suggestion for self-assessment exercises: https://es.khanacademy.org/math/linear-algebra/vectors-and-spaces/vectors/e/adding_vectors

EVALUATION OF LEARNING (Technique, instrument, type, and performance descriptors)	
Diagnostic	Interrogation through written test
Formative	Comparative table Evaluation technique: Performance analysis Evaluation instrument: Checklist. Type of evaluation: Hetero-assessment
Summative	Exercises Technique evaluation: Performance analysis. Evaluation instrument: Rubric. Type of evaluation: Self-assessment

Table 1: Didactic sequence

In the initial session, students begin to become familiar with basic concepts related to location and movement. While this session does not focus directly on trigonometric ratios, it lays the foundation for understanding the importance of measuring distances and displacements, which are key aspects in the context of trigonometry.

In the third session the stage fosters an understanding of how trigonometric ratios, such as sine and cosine, are applied in solving vector problems.

In the third session is common for students to mention that to save time it is better to use the Pythagorean Theorem because the legs of a right triangle are the horizontal and vertical components of a vector. Questions also arise as to whether the hypotenuse provides the magnitude of the vector, how to determine the direction or angle of inclination of the vector? The most prepared students would indicate that using trigonometric ratios, hence the teacher could take advantage of the situation to develop the topic.

At the end of the six sessions of the didactic sequence, a post-test was applied with the same items from the diagnostic test, on trigonometric ratios. As this is a quantitative study in which the aim is to measure the impact of a pedagogical intervention strategy on student learning, the T-Student test for related samples was carried out in order to compare the differences between the pretest scores and the posttest This allowed us to determine if there was a statistically significant change in learning after the intervention. Data analysis was carried out through the process of data reduction and visualization to draw conclusions.

RESULTS

Figures 1, 2 and 3 show the responses to the items evaluated in the diagnostic test. he asterisk symbol (*) indicates the correct answer.

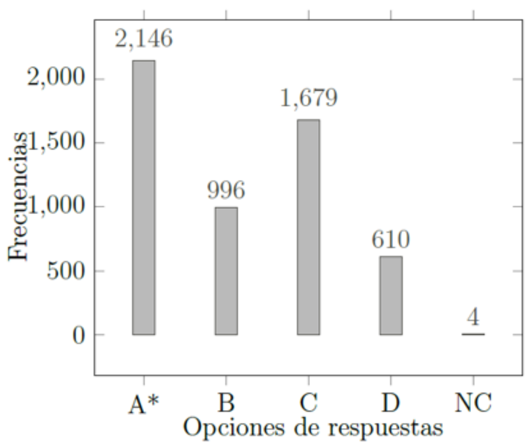


Figure 1: Answers to item 1

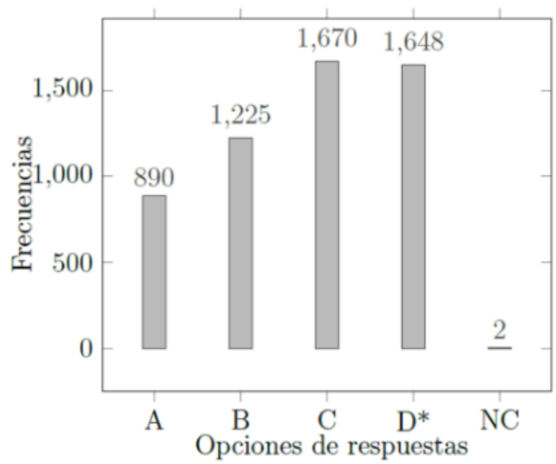


Figure 2: Answers to item 2

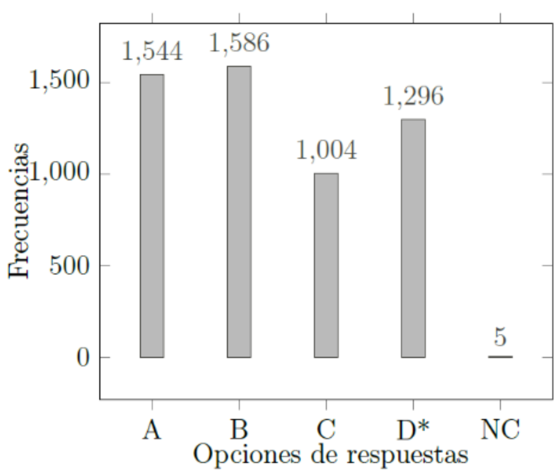


Figure 3: Answers to item 3

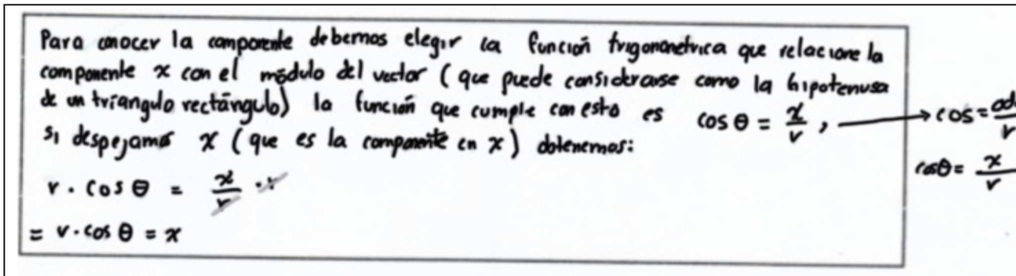
Table 2 presents the classification of academic performance in trigonometry. In this classification, performance is considered "Low" if students answered 1 or no test items correctly. It is classified as "Fair" if they answered 2 items correctly, "Good" if they answered 3 items correctly, and "Excellent" if they answered correctly to all 4 items of the test.

	Supporters	Percentage
Low	3339	61.44
Regular	1460	26.86
Well	518	9.53
Excellent	118	2.17

Table 2: Academic performance in trigonometry

In order to go deeper, Table 3 shows examples of answers provided in the posttest that give indications of relational understanding, the relevance of which is discussed in the conclusions.

Item 1



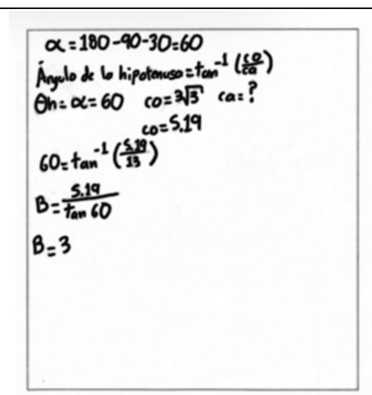
To determine the component, we must choose the trigonometric ratio that relates the x-component with the magnitude of the vector (which can be considered as the hypotenuse of a right triangle). The function that fulfills this is $Cos\ \theta = \frac{x}{r}$,

$Cos\ = \frac{ady}{r}$

If we solve for \(x\) (which is the x-component), we get: $r * Cos\ \theta = \frac{x}{r} r$

$= r * Cos\ \theta = x$

Item 2



$\alpha = 180 - 90 - 30 = 60$

$The\ angle\ of\ the\ hypotenuse = Tan^{-1} \frac{Co}{Ca}$

	$\theta h = \alpha = 60$ $\quad Co = 3\sqrt{3}$ $\quad Ca = ?$ $Co = 5.19$ $B = \dfrac{5.19}{tan60}$ $B = 3$
Item 3	

el área del triángulo es $a = \frac{bh}{2}$
ó, visto desde otra perspectiva
$a = \frac{Cx \cdot Cy}{2}$
Para obtener los componentes:
$Sen\alpha = \frac{O}{r}$ $\quad Cos\alpha = \frac{a}{r}$
$O = Sen\alpha \cdot r$ $\quad \alpha = Cos\alpha \cdot r$
$y = Sen(225) 4\sqrt{2}$ $\quad x = cos(225) r$
$y = -\frac{\sqrt{2}}{2} \cdot 4\sqrt{2}$ $\quad x = -\frac{\sqrt{2}}{2} \cdot 4\sqrt{2}$
$y = -\frac{(\sqrt{2})^2 \cdot 4}{2}$ $\quad x = -\frac{(\sqrt{2})^2 \cdot 4}{2}$
$y = -\frac{8}{2}$ $\quad x = -\frac{8}{2}$
$y = -4$ $\quad x = -4$

The area of the triangle is $a = \frac{bh}{2}$ or, from another perspective,

$a = \frac{Cx*Cy}{2}$

To obtain the components.

$$Sin \propto = \frac{O}{r} \qquad Cos\alpha = \frac{A}{r}$$

$$O = Sin\alpha r \qquad \alpha = Cos\,(225)r$$

$$y = Sin(225)4\sqrt{2} \qquad x = Cos(225)r$$

$$y = \frac{-\sqrt{2} * 4\sqrt{2}}{2} \qquad x = \frac{-\sqrt{2} * 4\sqrt{2}}{2}$$

$$y = \frac{-8}{2} \qquad x = \frac{-8}{2}$$

$$y = -4 \qquad x = -4$$

Table 3: Examples of responses in the posttest

Seven months after the implementation of the didactic sequence, a post-test is applied that includes 10 items of which 7 correspond to topics related to vectors and the same 3 trigonometry items applied in the diagnostic evaluation are interspersed.

The data are analyzed in the SPSS statistical software and it is obtained that the samples are non-parametric as they do not fit the normal distribution, so they are analyzed with the Wilcoxon rank sum to compare two related or paired samples (pre and post of the group control, apart, pre and post of the experimental group). This test is used to compare the performance of a group of students, at different times, before and after receiving an educational intervention and evaluate its effectiveness.

Table 4 presents the results of the Wilcoxon signed rank test, the interpretation of which is explained in the discussion and conclusions section.

	PreCon - PostCon	PreExp - PostExp
Z	-0.26	-4.45
asymptotic sig.	0.79	0.00

Table 4: Wilcoxon Signed Rank Test

Table 5 describes the positive, negative differences and ties between the test results in both groups.

	Experimental	Control
Negative differences	1	6
Positive differences	26	7
Ties	5	19

Table 5: Range test

Table 6 shows the average whose maximum value is 3.

	$\bar{x}$
Pretest Experimental	0.93
Postest Experimental	2.24
Pretest Control	0.81
Postest Control	0.87

Table 6: Arithmetic average

DISCUSSION

Pretest

In the diagnostic test, we identified a partial instrumental understanding in the response justifications of item 1, where phrases such as: "At this moment I do not remember how tangent, sine and cosine work. I do not remember how to obtain sides with angles, nor the trigonometric laws. I use the Pythagorean theorem $c = \sqrt{a^2 + b^2}$, oh no! it has an angle... forget it, I don't remember the process. I have the idea that the sine is used and I play with the data. They were reasons trigonometric, but I don't remember the procedure. I tried to find it with the angles, but I don't remember the formula." Surprisingly, even though 41% of the students selected the correct answer, 90% of them claimed to have chosen the option at random.

In this case, instrumental understanding is revealed, since it is possible that students were exposed to the repetition of mathematical tasks, which potentially led them to obtain knowledge without a deep understanding of the concepts involved.

Although in item 1 presence of the mnemonic "SOH–CAH–TOA" was observed, as evidenced in the study by Yigit (2016). This acronym is used to identify the acquisition of knowledge through the ratio method between the proportions of pairs of sides in a right triangle, and no indicators of the use of the unit circle method were detected in any of the questionnaires. In items 2 and 3, students stated that they did not have adequate basic knowledge or had difficulties reasoning about the task.

Postest

In item 1, 76% of correct answers are presented, very detailed justifications are collected, in which students generally describe that the figure is a right triangle, the modulus vector r is the hypotenuse and the x component is the adjacent leg; and that if the angle is known, the cosine and a clearance can be used, given that this ratio relates to said magnitudes. In addition, two tests were found that use the unit circle as a representation.

It is observed that they refer to the ray r as a vector and the x coordinate as its horizontal component, showing indicators of relational understanding, since, although they are presented with a geometry reagent, they are able to transfer the situation to the vector context.

Item 2 stands out as the most challenging, observing an increase in correct answers from 24% to 44%. However, it supports the idea that there is a solid understanding of trigonometric ratios, which involves overcoming the underlying conceptual deficiencies mentioned by De Villiers and Jugmohan (2012).

In the third item, once again evidence of relational understanding is observed, which implies the ability to understand the interconnection of mathematical concepts and their relationships with diverse situations. This is reflected in the fact that 61% of the students were able to correctly identify the trigonometric expressions necessary to solve an area calculation task that was unrelated to the class topic, focused on displacement vectors. However, they approached the problem of finding the legs as if they were the horizontal and vertical components of a vector.

CONCLUSIONS

Relational understanding, according to Skemp's (2006) model, implies that students not only understand mathematical concepts in isolation, but that they are able to relate and connect these concepts with each other.

The implementation of the didactic sequence and its effectiveness was evaluated through the Wilcoxon statistics. This finding provides strong statistical evidence to conclude that there is a significant difference between the Pretest and Posttest scores only in the experimental group. A value of Z = -4-45, in a sample of 32, provides strong evidence to reject the null hypothesis, thus concluding that there is a significant difference between before and after the intervention in the experimental group.

When comparing the results of the pretest and posttest for the control group, a value of Z=0.26 is obtained, which in the context of statistical hypothesis testing means that the difference observed between is null or insignificant. In other words, it suggests that the two samples are statistically equal and that the difference between the sample means is very small.

In the didactic sequence, students not only learned to calculate trigonometric ratios, but also understood how these ratios are related to the addition of vectors and how they are applied in practical situations. Although the teaching sequence was not designed for a trigonometry class, it facilitated relational understanding by showing how mathematical concepts are interconnected and applied in the real world.

Acknowledgments

We would like to thank CONAHCYT for the financial support provided during the realization of this Project, to the Evaluation Commission directed by Master Rogelio Paredes Jaramillo and to the General Academy of Physics of the BUAP High Schools, for their invaluable support in the application of the diagnostic instrument.

REFERENCES

[1] Araya, A., Monge, A., & Morales, C. (2007). Comprensión de las razones trigonométricas: Niveles de comprensión, indicadores y tareas para su análisis. *Actualidades Investigativas en Educación,* **7**(2), 123-133.

[2] Aray, C., Guerrero, Y., Montenegro, L. & Navarrete, S. (2020). La superficialidad en la enseñanza de la trigonometría en el bachillerato y su incidencia en el aprendizaje del cálculo en el nivel universitario. *ReHuSo, 5*(2), 62-69.

[3] Cantoral, R., Montiel, G., & Reyes, D. (2015). Análisis del discurso matemático escolar en los libros de texto, una mirada desde la Teoría Socioepistemológica. *Avances de Investigación en Educación Matemática,* **5**(8), 9-28. https://dialnet.unirioja.es/servlet/articulo?codigo=5672145

[4] De Villiers, M. & Jugmohan, J. (2012). Learners' conceptualization of the sine function during an introductory activity using sketchpad at grade 10 level. *Revista Educação Matemática Pesquisa*., São Paulo, **14**(1), 9-30. https://revistas.pucsp.br/emp/article/view/8750

[5] Dhungana, S., Pant, B. P., & Dahal, N. (2023) Students' experience in learning trigonometry in high school mathematics: A phenomenological study. *Mathematics Teaching Research Journal,* **15**(4), 184-201. https://eric.ed.gov/?id=EJ1409234

[6] González, M., Matilla, J., & Rosales, F. (2017). Potencialidades del software Geogebra en la enseñanza de la matemática: estudio de caso de su aplicación en la trigonometría. *Roca: Revista Científico Educacional de la provincia Granma,* **13**(4), 401-415.

[7] Hadi, W. & Faradillah, A. (2020). Application of discovery learning method in mathematical proof of students in trigonometry. *Desimal: Jurnal Matematika, 3*(1), 73–82. doi:10.24042/djm.v3i1.5713. http://dx.doi.org/10.24042/djm.v3i1.5713

[8] Hamzah, N., Maat, S. M., & Ikhsan, Z. (2021). A systematic review on pupils' misconceptions and errors in trigonometry. *Pegem Journal of Education and Instruction*, **11**(4), 209– 218. https://doi.org/10.47750/pegegog.11.04.20

[9] Herheim, R. (2023). On the origin, characteristics, and usefulness of instrumental and relational understanding. *Educational Studies in Mathematics*, **113**, 389–404. https://doi.org/10.1007/s10649-023-10225-0

[10] Hiebert, J. & Lefevre, P. (2013). Conceptual and procedural knowledge in mathematics: an introductory analysis. In: Hiebert, J. (Ed.) *Conceptual and procedural knowledge: the case of mathematics.* Hillsdale (NJ): Lawrence Erlbaum Associates, (pp. 1–27).

[11] Ibrahim, K. & llyas, Y. (2016). Teaching a concept with GeoGebra: Periodicity of trigonometric functions. *Educational Research and Reviews,* **11**(8), 573–581. https://eric.ed.gov/?id=EJ1098194

[12] Jorda, E. & De los Santos, O. (2015). Effect of computer game-based learning on the performance in trigonometry of the ESEP high school students. *Technological University of the Philippines*, (pp. 1–9)

[13] Kurniati, N., Suhendra, S., Priatna, N., & Prabawanto, S. (2022). An exploration student's errors in solving trigonometric ratio problems with its scaffolding. *AKSIOMA Jurnal Program Studi Pendidikan Matematika,* **11**(3), 2235-2247. http://dx.doi.org/10.24127/ajpm.v11i3.523

[14] Maknun, C. L., Rosjanuardi, R., & Jupri, A. (2022). Epistemological obstacle in learning trigonometry, *Mathematics Teaching Research Journal*, **14**(2), 5-25. https://eric.ed.gov/?id=EJ1350528

[15] Moore, K. (2013). Making sense by measuring arcs: a teaching experiment in angle measure. *Educational Studies in Mathematics*, **83**(2), 225–245. https://doi.org/10.1007/s10649-012-9450-6

[16] Moore, K. (2014). Quantitative reasoning and the sine function: the case of Zac. *Journal for Research in Mathematics Education,* **45**(1), 102–138. https://doi.org/10.5951/jresematheduc.45.1.0102

[17] Ngu, B. H. & Phan, H. P. (2020). Learning to solve trigonometry problems that involve algebraic transformation skills via learning by analogy and learning by comparison. *Frontiers in Psychology,* **11**(1), 1–11.

[18] Ogbonnaya, U., & Mogari, D. (2014). The relationship between grade 11 students' achievement in trigonometric functions and their teachers' content knowledge. *Mediterranean Journal of Social Sciences,* **5**(4), 443-451. https://pdfs.semanticscholar.org/5847/04830f61bb22417e23f2ca70c4315142ae7f.pdf

[19] Prabowo, A., Anggoro, R. P., Adiyanto, R. & Rahmawati, U. (2018). Interactive multimedia-based teaching material for trigonometry. *Journal of Physics: Conference Series,* **1097** 012138. https://10.1088/1742-6596/1097/1/012138

[20] Prabowo, A., Usodo, B. & Pambudi, I. (2019). Field-independence versus field-dependence: A serious game on trigonometry learning. *Journal of Physics: Conference Series,* **1188** 012100. https://10.1088/1742-6596/1188/1/012100

[21] Serway R. & Vuille C. (2012) Fundamentos de Física. *Cengage Learning*, 56-61.

[22] Solanilla, O. (2015). Implementación de herramientas didácticas y tecnológicas para mejorar el nivel de aprendizaje de la trigonometría. Tesis de Maestría. Universidad del Tolima, Ibagué, Colombia.

[23] Suárez, M., Chaves, A., & Fernández, E. (2017). De las fórmulas fundamentales en la trigonometría esférica a las fórmulas fundamentales de la trigonometría hiperbólica. *Sigma,* **13**(2), 1-15.

[24] Skemp, R. (2006). Relational understanding and instrumental understanding. *International Journal of Mathematical Education in Science and Technology,* **12**(20), 88–95. https://doi.org/10.5951/MTMS.12.2.0088

[25] Thompson P., Carlson M. & Silverman J. (2007). The design of tasks in support of teachers' development of coherent mathematical meanings. *Journal of Mathematics Teacher Education*, **10**(4–6), 415–432. https://doi.org/10.1007/s10857-007-9054-8

[26] Tuna, A. & Kacar, A. (2013). The effect of 5E learning cycle model in teaching trigonometry on students' academic achievement and the permanence of their knowledge. *International Journal on New Trends in Education and Their Implications,* **4**(1), 73–87.

[27] Ulyani, O. & Qohar, A. (2021). Development of manipulative media to improve students' motivation and learning outcomes on the trigonometry topic. *AIP Conference Proceedings,* **2330**. https://doi.org/10.1063/5.0043142

[28] Weber, K. (2005). Students' understanding of trigonometric functions. *Mathematics Education Research Journal*, **17**(3), 91–112. https://doi.org/10.1007/BF03217423

[29] Yigit, M. (2014). Pre-service secondary mathematics teachers' conceptions on angles. *The Mathematics Enthusiast*, **11**(3), 707–736.

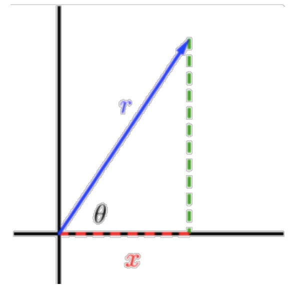
[30] Yigit, M. (2016). Mathematics education graduate students' understanding of trigonometric ratios. *International Journal of Mathematical Education in Science and Technology,* 47, 1028-1047. https://doi.org/10.1080/0020739X.2016.1155774

APPENDIX 1

Prestest-Posttest Items

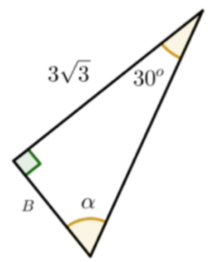
1. The figure shows a ray with an angle in the first quadrant. Find the x coordinate. Justify your answer.

a) $\boldsymbol{r\,Cos\,(\theta\,)}$
b) $\boldsymbol{r\,Sen\,(\theta\,)}$
c) $\boldsymbol{r\,Tan\,(\theta\,)}$
d) $\boldsymbol{r\,Csc\,(\theta\,)}$

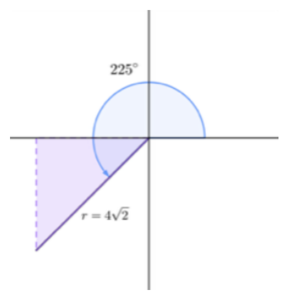


2. Find the length of vector B. Justify your answer.

a) 1
b) 2
c) $2\sqrt{2}$
d) 3



3. Find the area of the right triangle. Justify your answer.
   a) 8
   b) $8\sqrt{2}$
   c) $6\sqrt{2}$
   d) 16



APPENDIX 2

Instructions for virtual laboratory practice

1. Enter Google Maps and represent the following situation: 5 tourists have agreed to meet at the access door to the Cathedral of Puebla located on the corner of September 16th and 3 east streets, they are located at
    A. Rosario Chapel – East 4th and May 5th
    B. Amparo Museum – 2 south and 7 east
    C. El Parian craft market – 6 north and 4 east
    D. El Mural de los Poblanos Restaurant - September 16th and 7 east
    E. Automobile Museum -3 south and 15 west
2. Each member must choose a starting point and trace a trajectory to the meeting point using displacement vectors.
3. Use the ruler tool to measure the magnitude of the vectors, to determine the direction properly define the coordinate axes (x,y)
4. Enter GeoGebra by scanning the QR code, each member will represent the displacement vectors corresponding to their trajectory, remember that they can represent them to scale.
5. Add the displacement vectors of each trajectory using the graphical method and take screenshots.
6. Represent the displacement vectors of your trajectory in their Cartesian components and perform the same sums from the previous step using the analytical method.
7. Compare the vector sum of the displacement vectors with the length of the path you chose.
8. Draw conclusions, prepare your report and send it for review.

Checklist for Virtual Laboratory Report

Criterion	Accomplished	In progress	Unachieved
Identify the variables involved in the observed phenomenon			
Distinguish the type of physical magnitude that corresponds to each variable			
Correctly relate the phenomenon to the mathematical model (equations).			
Correctly perform the required calculations			
Correctly record data			
Interpret data correctly			
Make appropriate graphic representations of the information			
Answer the questions posed with scientific arguments consistent with the topic of analysis			
Includes a conclusion			

The Problem Corner

Ivan Retamoso, PhD, *The Problem Corner* Editor

Borough of Manhattan Community College

iretamoso@bmcc.cuny.edu

The Purpose of ***The Problem Corner*** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem and its solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Hello, problem solvers!

I'm delighted to announce that we have received both precise and thoughtful solutions for Problem 28 and Problem 29 in The Problem Corner. These submissions excel in both accuracy and strategic problem-solving. Our aim is to highlight exemplary solutions that foster inspiration and advance mathematical insights on a global scale.

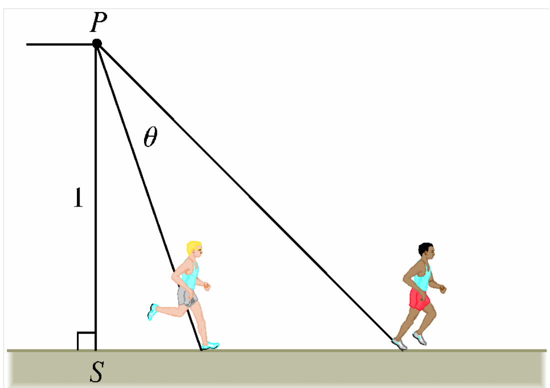
Solutions to **Problems** from the Previous Issue.

"Running a race" problem.

Problem 28

Proposed by Ivan Retamoso, BMCC, USA.

An observer is positioned at point P, one unit away from a track. Two runners begin at point S, which is illustrated in the diagram, and move along the track. One of the runners runs at a speed three times faster than the other. Determine the maximum angle θ that the observer's line of sight forms between the two runners.

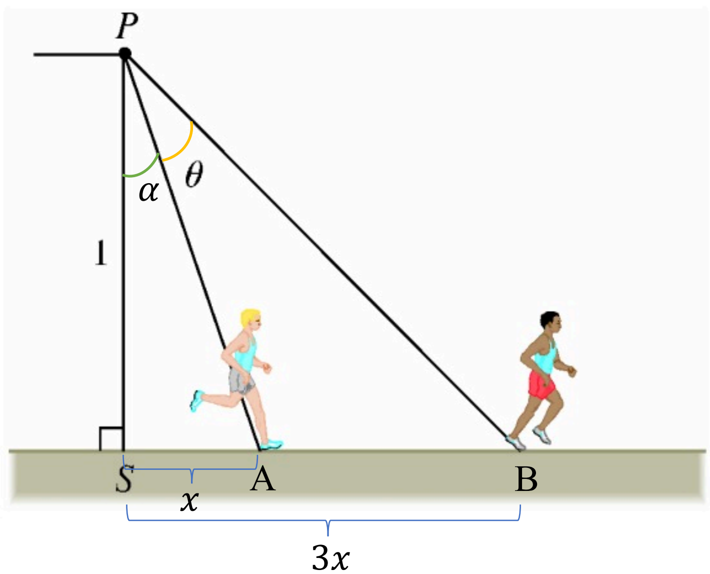


Solution to problem 28

By Dr. Abdullah Kurudirek, Mathematics Ed. Department, Tishk International University.

After carefully labeling the angles and distances, our solver uses the trigonometric expansion for the tangent of the sum of two angles to express the tangent of an angle in terms of a single variable. Finally, by leveraging the fact that for acute angles, the tangent function and the angle increase together, the function is maximized to find the solution.

We can denote points A, B, angle α and distance covered by runners in terms of x on the figure as shown below.



By using the tan $(a + b)$ sum formula on the right-angled triangle PSB:

$\tan(\alpha + \theta) = 3x$

$\frac{\tan\alpha + tan\theta}{1 - tan\alpha . tan\theta} = 3x$

Knowing that $tan\alpha = x$ in the right-angled triangle PSA, we substitute $tan\alpha = x$ and get:

$\frac{x + tan\theta}{1 - x.tan\theta} = 3x$

After necessary algebraic operations, we get:

$tan\theta = \frac{2x}{1+3x^2}$

Now we need the differentiate this function to get the maximum value of θ.

Notice that for θ an acute angle, $\tan\theta$ is strictly increasing so that θ is maximized when $\tan\theta$ is maximized.

$\left(\frac{2x}{1+3x^2}\right)' = 0$

By using the quotient rule, we get:

$\frac{2-6x^2}{(1+3x^2)^2} = 0$

Then,

$$x = \frac{\sqrt{3}}{3}$$

To be sure that $tan\theta$ gets maximized when $x = \frac{\sqrt{3}}{3}$, you may use a behavior table or the second derivative test.

Now, if we substitute $x = \frac{\sqrt{3}}{3}$,

$$tan\theta = \frac{2\frac{\sqrt{3}}{3}}{1+3\left(\frac{\sqrt{3}}{3}\right)^2} = \frac{\sqrt{3}}{3} \Rightarrow \theta = 30^0 = \frac{\pi}{6}$$

"STOP sign" probability problem.

Problem 29

Proposed by Ivan Retamoso, BMCC, USA.

A regular octagon $ABCDEFGH$ has sides that are 2 units in length. The points W, X, Y, and Z are the midpoints of the sides $\overline{AB}$, $\overline{CD}$, $\overline{EF}$, and $\overline{GH}$, respectively. Find the probability that a point chosen uniformly at random from inside the octagon $ABCDEFGH$ will be located inside the quadrilateral $WXYZ$. Give your answer in exact form.

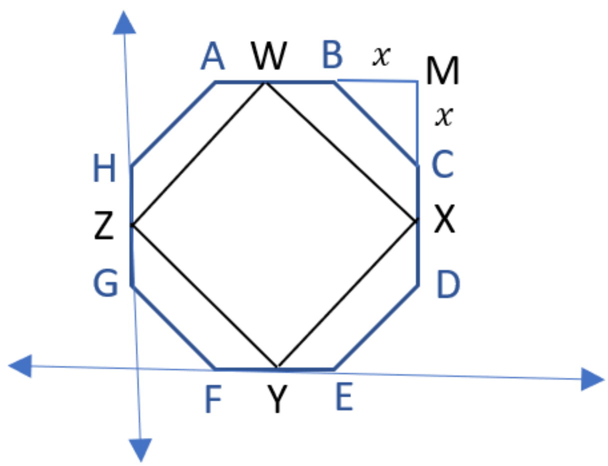
First solution to problem 29

By Dr. Hosseinali Gholami, University Putra Malaysia, Serdang, Malaysia

Our solver presents two solutions: one based on Cartesian geometry, utilizing the distance and slope formulas, and the other based on the Pythagorean theorem and the properties of isosceles triangles. Both solutions are explained in detail for your understanding and enjoyment.

Solution 1:

The following shape shows the regular octagon $ABCDEFGH$ on the coordinates axes that its sides are 2 units in length.



The coordinates of octagon vertexes are clear, because based on the equation $x^2 + x^2 = 4$ the value of x is $\sqrt{2}$. Also, the coordinates of quadrilateral vertexes are determinable easily, since the points W, X, Y, and Z are the midpoints of the sides AB, CD, EF, and GH, respectively. As respect to the coordinates of points $X(2\sqrt{2}+2, 1+\sqrt{2})$, $Y(1+\sqrt{2}, 0)$ and $Z(0, 1+\sqrt{2})$, the slope of sides XY and ZY are calculated as follows:

$$a_{XY} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\left(1+\sqrt{2}\right) - 0}{\left(2\sqrt{2}+2\right) - \left(1+\sqrt{2}\right)} = 1$$

$$a_{ZY} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\left(1+\sqrt{2}\right) - 0}{0 - \left(1+\sqrt{2}\right)} = -1$$

It shows $XY \perp ZY$ $(a_{XY}a_{ZY} = 1(-1) = -1)$ therefore, $WXYZ$ is a square. The length of square side is $ZY = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = \sqrt{(1+\sqrt{2}-0)^2 + (0-1-\sqrt{2})^2} = \sqrt{4\sqrt{2}+6}$. The surface of this square is $S_{WXYZ} = ZY^2 = (\sqrt{4\sqrt{2}+6})^2 = 4\sqrt{2}+6$. The surface of regular octagon is determined as the following:

$S_{ABCDEFGH} = (2+2\sqrt{2})^2 - 4\left(\frac{\sqrt{2}\times\sqrt{2}}{2}\right) = 8 + 8\sqrt{2}.$

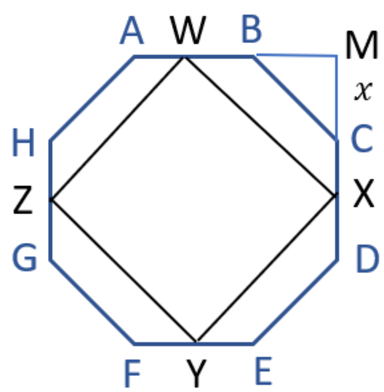
Therefore, the probability that a point chosen uniformly at random from inside the octagon $ABCDEFGH$ will be located inside the quadrilateral $WXYZ$ is calculated using the probability formula as below:

$$P = \frac{S_{WXYZ}}{S_{ABCDEFGH}} = \frac{4\sqrt{2}+6}{8+8\sqrt{2}} = \frac{\sqrt{2}+1}{4}.$$

Solution 2:

Based on the following figure, the surface of regular octagon is determined as the following:

$$S_{ABCDEFGH} = (2+2\sqrt{2})^2 - 4\left(\frac{\sqrt{2}\times\sqrt{2}}{2}\right) = 8+8\sqrt{2}.$$



In an isosceles triangle WMX, angle MWX is 45 degrees. In a similar way, it can be shown that the angle AWZ is 45 degrees. As a result, angle ZWX is 90 degrees, which shows that the quadrilateral $WXYZ$ is a square. The length of the side of this square is calculated as follows.

$$WX^2 = WM^2 + MX^2 = (1+\sqrt{2})^2 + (1+\sqrt{2})^2 = 6+4\sqrt{2}.$$

Therefore, the surface of this square is $S_{WXYZ} = WX^2 = 6+4\sqrt{2}$. The desired probability is obtained as follows.

$$P = \frac{S_{WXYZ}}{S_{ABCDEFGH}} = \frac{4\sqrt{2}+6}{8+8\sqrt{2}} = \frac{\sqrt{2}+1}{4}.$$

Second Solution to problem 29

By Dr. Abdullah Kurudirek, Mathematics Ed. Department, Tishk International University.

Our solver provides a concise solution, relying on familiar formulas for regular polygons, specifically the octagon and square. By combining side lengths, areas, and symmetry, the ratio of areas is calculated, which leads to determining the desired probability. A graph is included to further clarify the explanation.

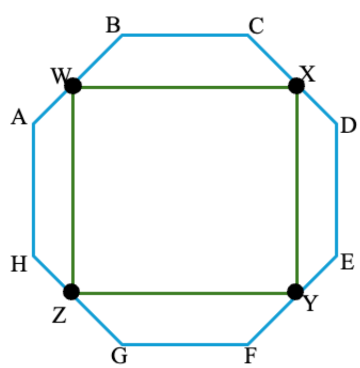
Solution 29

As it is known, the area of a regular octagon is $2a^2(1+\sqrt{2})$ where a is the side length.

For $a = 2$, therefore

$$A(ABCDEFGH) = 8(1+\sqrt{2})$$

According to the given information, when the midpoints of alternating sides of a regular octagon are joined, we get a square WXYZ. If one side of the regular octagon is 2, then one side of the square will be $2+\sqrt{2}$.



Let P be the probability that a point chosen uniformly at random from inside the octagon ABCDEFGH will be located inside the quadrilateral WXYZ then

$$P = \frac{Area(WXYZ)}{Area(ABCDEFGH)} = \frac{(2+\sqrt{2})^2}{8(1+\sqrt{2})} = \frac{1+\sqrt{2}}{4}$$

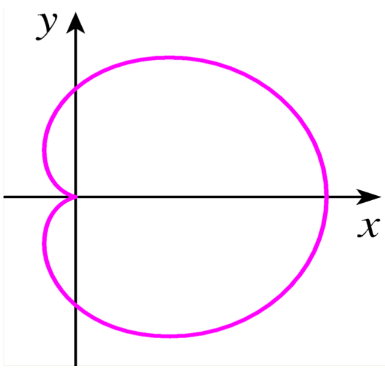
Dear fellow problem solvers,

I'm glad you enjoyed tackling problems 28 and 29 and that you've expanded your approach to mathematics. Let's jump into the next set of problems to keep advancing your skills.

Problem 30

Proposed by Ivan Retamoso, BMCC, USA.

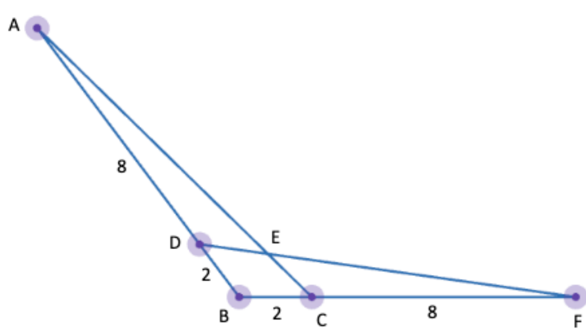
Find the coordinates of the intersection point of the tangent lines at the highest and lowest y-intercepts of the cardioid described by $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$.



Problem 31

Proposed by Ivan Retamoso, BMCC, USA.

In the figure below, $AD = CF = 8\ cm$, $DB = BC = 2\ cm$, and the area of triangle ABC is $7.2\ \ cm^2$. Find the area of the quadrilateral $DECB$.



How to Motivate Students Using Augmented Reality in The Mathematics Classroom? An Experimental Study

Wanda Nugroho Yanuarto[1], Elfis Suanto[2], Ira Hapsari[1], Aulia Nisa Khusnia[3]

[1]Universitas Muhammadiyah Purwokerto, Indonesia,

[2]Universitas Riau, Indonesia,

[3]Universitas Perwira Purbalingga, Indonesia

wandanugrohoyanuarto@ump.ac.id

Abstract: An Augmented Reality (AR) approach to creating a maths education app is the focus of this research. To turn the 2D floor plan into 3D objects, students will use their mobile devices to scan the cards. Students can have a better understanding of the 3D shape from various angles by interacting with virtual 3D items. Students can gain a deeper understanding of the method of volume computation by utilising the decomposition or combined functions of the Assembler Edu app. This study compares the digital learning results of students who excel in maths with those of students who struggle, and it does so by analysing the students' motivation both before and after using an AR Assembler Edu app. The study analyses the learning impacts and experiences of the AR Assembler Edu App through the use of achievement assessments, questionnaires, and interviews with teachers and students. The results show that students are receptive to the institute's AR maths learning app and that it might pique their interest in studying. Compared to the pre- and post-study results, the learning effects are much more pronounced. Teacher interviews revealed that students' motivation and three-dimensional composition were significantly improved after using this digital learning resource and interactive experience model.

Keywords: Augmented reality, experimental study, geometry, students' motivation

INTRODUCTION

Mathematics is viewed as a foundation for the development of science, and technology; it is also the source of our geometric and spatial abilities, which allow us to recognize the sizes, shapes, and placements of objects in our environment. Using actions like combining, pushing and stacking, and measuring, Geometry and Spatial Competence Training can be used to construct virtual spaces, which can increase 2D and 3D spatial cognition. According to Ahmad and Junaini (2020), who also found that visual imagery and spatial skills were critical to mathematical thinking and learning, there was a degree of relationship between spatial talents and Math learning.

Important parts of learning mathematics include following traditional lesson plans, studying logical reasoning, and building formulas and definitions using very abstract symbols (Voronina et al., 2019). The practice of spoon-feeding mathematics to students usually results in a lack of interest in the subject. Research shows that when students feel good about themselves, they do better in maths (Cabero-Almenara et al., 2021; Özçakır & Özdemir, 2022). Despite its obvious importance, math is a daunting subject for many lower-secondary school students (Su et al., 2022). Strong mathematical foundations can aid students' future scientific or ideological pursuits, thus it's important to inspire their natural curiosity and love for the topic (Schutera et al., 2021).

In 2018, the Ministry of Education in Indonesia implemented a 9-year Integrated Curriculum that covered four branches of mathematics: Number and Quantity, Graphics and Space, Statistics and Probability, and Algebra. "Graphics & Space" has an ethereal quality to it, and it's not an easy course to finish (Chani & Susilowati, 2022). Other challenging courses, like "Number & Quantity", use mathematics (Lauren, 2021). Students have a more difficult time mentally converting from two-dimensional to three-dimensional images, and they also have more difficulty learning in three dimensions through direct use of instructional materials or indirect learning through floor plans. Different modules receive different amounts of time from teachers (Elsayed & Al-Najrani, 2021). To make sure that students fully understand volume, teachers should employ a variety of supplemental materials.

Making better use of Information and Communication Technology (ICT) to enhance education is one of the most distinctive features of modern civilization. Digital learning encompasses the public's and businesses' massive demands for online education (Nasrudin et al., 2021). With the proliferation of digital technologies, interactive media has become a popular subject across many sectors. Educational materials, including apps that use game-based learning, are now accessible on the move because to the growth of mobile devices. With the advent of Augmented Reality (AR) technology, it is no longer necessary to recreate the imaginative experience since the virtual world can be seamlessly integrated with the human sensory nerve and attached to the actual world in a matter of seconds (Wahyudi & Arwansyah, 2019). The usage of augmented reality has been widespread in several industries, including gaming, publishing, and navigation. Math, spatial transformation, and understanding course material are among of the many areas where augmented reality is finding increasing use in the classroom (Pujiastuti et al., 2020). Teachers can engage their students more deeply in the subject matter and help them grasp it at a deeper level by utilising the potential of digital technology to create extra teaching aids.

In order enable seventh graders to understand the benefits of incorporating digital technology into the 'Geometry' unit of study, the study has created an augmented reality maths learning app. The main objective of developing these educational resources is to motivate students to work on their own to gain a deeper understanding of *"3D shapes"*. The interactive operation method helps students better understand the composition and structure of cubes. The objectives of this research are threefold: first, to examine the relative merits of traditional and digital mathematic education; second, to gauge the motivation of the students to these changes; and third, to monitor the results of incorporating technology into the classroom.

LITERATURE REVIEW

Incorporating interactive behaviours like voices and gestures into the study in recent years has created the illusion of a more natural way of functioning. One area where information technology has the potential to advance is in the classroom, where students can use a variety of devices to engage with virtual learning resources made possible by virtual reality technology (Hamzah et al., 2021). AR, a branch of Virtual Reality (VR), integrates both digital and physical components to build a mixed-reality system that improves the quality of instruction. Using Human-Computer Interaction (HCI) in mixed reality contexts increased student engagement and retention (Dinayusadewi et al., 2020).

The three main characteristics of augmented reality apps are the ability to quickly interact with virtual objects and animals in three dimensions, to combine the real and virtual worlds, and to integrate them. Khanchandani et al. (2021) stated that AR has numerous uses in the classroom, including the following: facilitating unobtrusive interactions between the virtual and physical worlds; simulating a touchable interface for object operations; and allowing for seamless transitions between the two. Abdullah et al. (2022) expressed that students' awareness of their actual surroundings can be enhanced through the use of AR, which differentiates from earlier computer interaction technologies by allowing users to fully immerse themselves in virtual worlds. As a result, AR has a lot of potential as a technology, can help students become more engaged and motivated to learn, and can supplement classroom instruction. According to Cai et al. (2019), this data is derived from the rapid evolution of products such as mobile devices, VR, and AR has led to changes in the way users interact with technological goods. Teaching using technology is another important topic that needs discussing. As the future of human-computer interaction evolves away from the traditional mouse and keyboard and towards an image-based interface and sensory control, experts and researchers from many walks of life are more concerned with user experience issues. Avila-Garzon et al. (2021), the designer of the system, service, or product is obligated to improve the user's experience by modifying the content of the interface in response to feedback received from the user at various points on the satisfaction spectrum (Korkmaz & Morali, 2022). In particular, new technology is being produced and utilised in the education sector due to the growing importance of understanding the science and technology behind the system as well as its practical applications through first-hand experience.

3D is cutting-edge software that allows users to observe and interact with three-dimensional objects in a visual context. On top of that, you can add the virtual items to a control trigger event to make the interactive effects happen instantly (Suwayid & Rezqallah, 2022). Users may find that AR provides a more compelling experience than the more traditional real-world interface. AR has a lot of promise as a teaching tool because of this and many other reasons. AR integrates digital and physical components to provide a more lifelike and engaging experience for consumers. Several fields rely heavily on AR technologies, such as education, ecology, engineering, medical, and aviation (Fendi et al., 2021; Widyasari & Mastura, 2020). With the use of AR, which combines the features of mobile devices (such as phones) with those of real-world objects (such as buildings, signs, or landscapes), users are able to immerse themselves in a scenario where they are actively involved in real-life activities.

In a wide variety of subject areas, including engineering (Munir et al., 2022) history (Studies), design (Design and Mathematics Education), mathematics (Education), and natural science (Guntur et al., 2019), AR has proven to significantly enhance learning outcomes in K-12 and higher education settings. There is hope that incorporating AR into the design classroom can help students become more self-reliant, creative, and critical thinkers. Nugraha (2023) found that when used in conjunction with traditional maths lessons in high school, AR can make pupils more engaged and successful in the subject. Putrie and Syah (2023) demonstrated that AR technology could be a useful tool for capturing the interest of junior high school history students. One significant advantage of AR technology is virtual reality's capacity to contextualise abstract concepts. Pramuditya et al. (2022) employed AR in their research to simulate the three-dimensional operation of real electrical engineering instruments, enabling students to engage in independent learning and improve their ability to connect theoretical concepts with practical applications. Permatasari and Andayani (2021) showed that students were more motivated and learned more when they used AR to simulate electrical experiments.

AR environments have numerous scientific and artistic applications. Riza et al. (2023) developed an AR system that is library-based. Researchers discovered that it not only improved students' motivation and interest in learning, but also their academic achievement. Miundy et al. (2019) investigated the impact of the AR system on the motivation to learn of Visual Arts course participants. The use of AR enhanced the motivation to learn among lower secondary school students, according to the study. Furthermore, AR has a myriad of applications in fields such as situational investigation and related learning research. The research conducted by Kazanidis and Pellas (2019) utilized an AR smartphone navigation system to assist college students in exploring cultural places. Using AR in conjunction with a mobile learning system, Khanchandani et al. (2021) helped pre-service teachers with teaching scientific and technology subjects. Researchers found that employing AR picture books and e-book reading to encourage parent-child connection and sharing activities was far more effective than using traditional methods of training (Avila-

Garzon et al., 2021). Further exploration of the subject matter and the implementation of effective strategies for reshaping the classroom were also helpful.

Earlier studies have offered taxonomies as a means of directing the development of AR applications. These taxonomies highlight potential design aspects that could improve learning results while increasing the complexity of AR applications. For instance, in their taxonomy, Kazanidis and Pellas (2019) include dimensions like reference system, which refers to how firmly the AR content is tied to the real world, and visual connection, which describes how firmly the AR visualisations are physically attached to objects or how they are connected to them through other mechanisms, like virtual lines. While AR apps that use spatially anchored virtual content to make things more understandable for users can be useful in the classroom (Munir et al., 2022), they do necessitate more involved software development for object tracking (Suwayid & Rezqallah, 2022). Multiple studies have shown that learners benefit from experiences with more visualisations (Cabero-Almenara et al., 2021; Özçakır & Özdemir, 2022). Nevertheless, as mentioned earlier, the effects of these particular design decisions on pupils are hardly investigated in experimental studies on augmented reality teaching. To build upon this previous work, we compare two augmented reality applications and look at how the design differences affect student learning and inquiry. One application has multiple anchored, dynamically changing 3D visualisations, while the other uses a smaller number of 2D representations that don't move with the real objects.

In the field of AR for education, studies have demonstrated that AR visual representations, which are also known as AR visualisations, can direct student curiosity, promote active cooperation, and enhance problem-solving strategies among classmates. As an example, Pramuditya et al. (2022) discovered that while participating in a group AR mathematics activity, participants make good use of AR visualisations as grounding representations, which facilitates questioning and explanations amongst participants. For instance, when students work together in AR-enhanced investigations of *"3D shapes"* (Korkmaz & Morali, 2022) the presence of shared representations in AR can help equalise contributions and reduce leadership dominance. The use of augmented reality visualisations improved group learning and attitudes towards problem-based learning activities, according to research by Avila-Garzon et al. (2021), when contrasted with more conventional methods of instruction that did not incorporate AR. According to research by Putrie and Syah (2023), students in a group mathematics class were more likely to actively learn through experimentation when given access to augmented reality representations, which facilitated better problem-solving and learning overall. Based on these findings, it seems that AR visualisations, especially when used in student-to-student interactions, can help students think more critically and actively learn by facilitating experimentation, quick feedback, and better communication. The authors may anticipate comparable outcomes from AR tutoring via distant connections, and we can quantify the effect of AR visualisations through qualitative analysis of student learning.

METHOD

Development of Mathematics Teaching Materials

According to a study, lower secondary school students had difficulty grasping the *"Graphics and Space"* subfield of mathematics (Korkmaz & Morali 2022). The *"3D shapes"* units of lower secondary school mathematics provide the basis of the study's material. With a combined ten years of classroom experience, two senior teachers co-created the mathematical curriculum. This AR math app was built using the Assembler Edu. The Assembler Edu app allows users to create games that run on a variety of platforms, including iOS, Android, Windows, Mac OS X, and Linux. When it comes to creating AR/VR applications, Assembler Edu 3D is the way to go. It has an excellent module for digital sceneries, 3D objects, and a logic technique for programming.

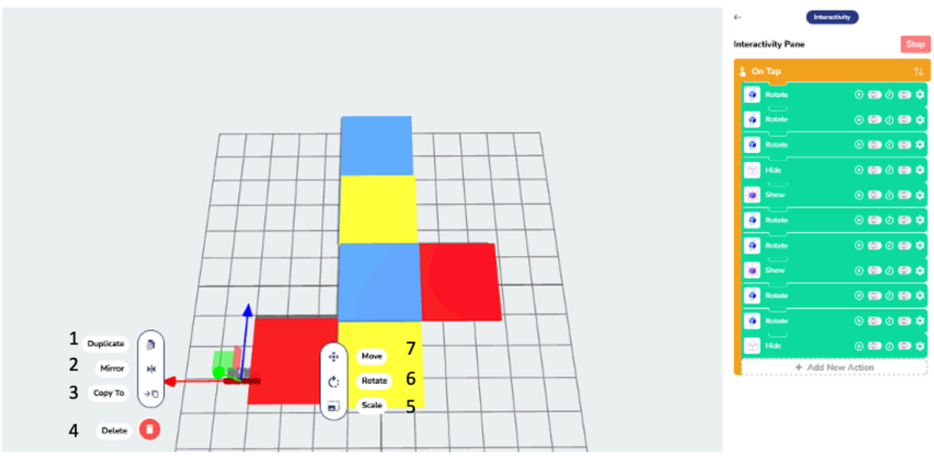


Figure 1: Assembler edu apps interaction

A mathematics software with augmented reality that is both directed and participatory has been developed by the research group. The app can be accessed and installed by students using their mobile devices. The software's built-in interactive operating instructions and user-friendly design make math easier to remember and increase motivation to learn. Figure 1 and Table 1 describe the App's interactive functions.

No.	Descriptions Menu
1	Alternative duplicate
2	Mirror
3	Copy
4	Delete
5	Scale figure
6	Rotate figure
7	Move figure

Table 1: Assembler edu menu descriptions

This study's overarching goal is to recommend digital learning tools that seventh graders can utilize to better understand the mathematical concept of the *"3D shapes"* unit. Students are guided in their understanding of 3D composition, and geometry through the use of information technology and an interactive learning paradigm. The story goes on to describe an AR math learning tool and how it can improve students' grades, motivation, and attitude towards science and technology.

Participants

A total of ninety graders from two different classrooms took part in the study; forty-six of them were assigned in the experimental and forty-four control groups students (see Table 2). Both maths classes were taught by the same teacher. The effectiveness of the method was tested in a four-week teaching trial utilizing the Maths *"3D shapes"* course material.

Profile	Experimental Group	Control Group	Total
Male	28	24	52
Female	18	20	38
Total	46	44	90

Table 2: The participants profile

Research Questions

Based on the study's methodology, we developed the following research questions to help shed light on it: 1. What is the difference between the experimental and control groups regarding students' learning achievement? 2. How does the experimental group compare in terms of students' motivation? 3. What is the difference in terms of the experimental group's and the control group's technological acceptance value?

Experimental Process

To ensure that the two groups of students using the Assembler Edu app for mathematics classrooms had the same prior knowledge, they were asked to complete a pre-test and a pre-study survey before the course began.

The students on the experimental team attended a tutorial before the activity started to make sure they were familiar with the augmented reality programme. Thereafter, a wide range of educational activities were carried out by two sets of students. The subjects in the experiment learned the information through an AR Assembler Edu app, while the subjects in the control group learned it through the use of computers and video data. Figure 2 shows the experimental flowchart after both groups completed the post-test and post-study surveys and interview. In order to measure the effect, we administered a questionnaire to students before and after the experiment to gauge their level of motivation.

Along with the study questionnaire, there is a survey of students' attitudes towards technology. Subsequently, to determine how well the AR technologies worked, both the experimental and control groups were given an achievement test after the experiment. Students who volunteered to be part of the experimental group also had to endure interviews. We asked the students for their thoughts on the execution plan.

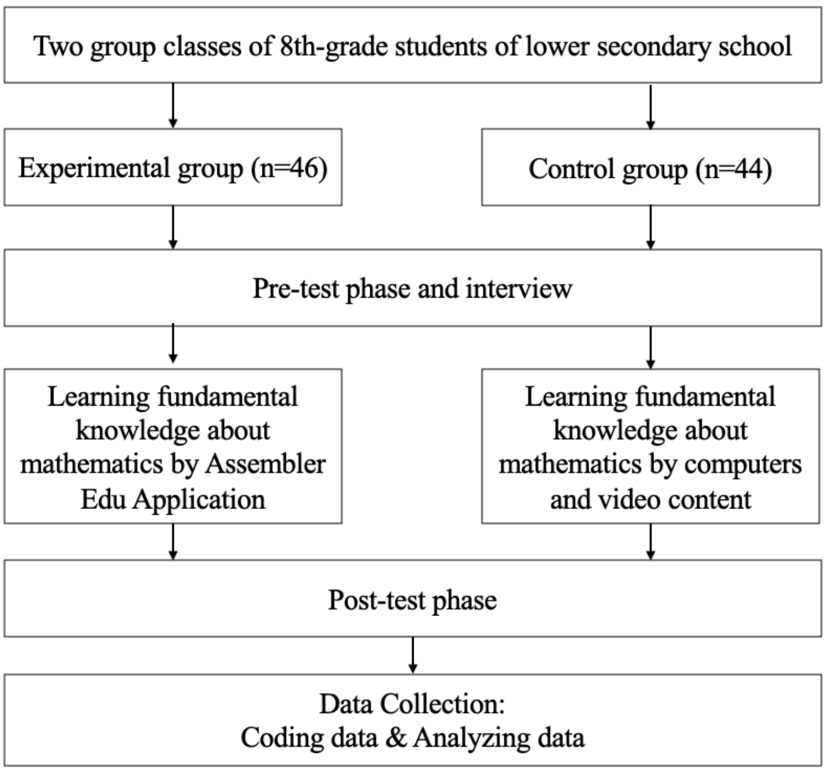


Figure 2: The experimental phase

Meanwhile, figure 3 shows a math 3D shape unit that incorporates multiple interactive operations into its learning activities.

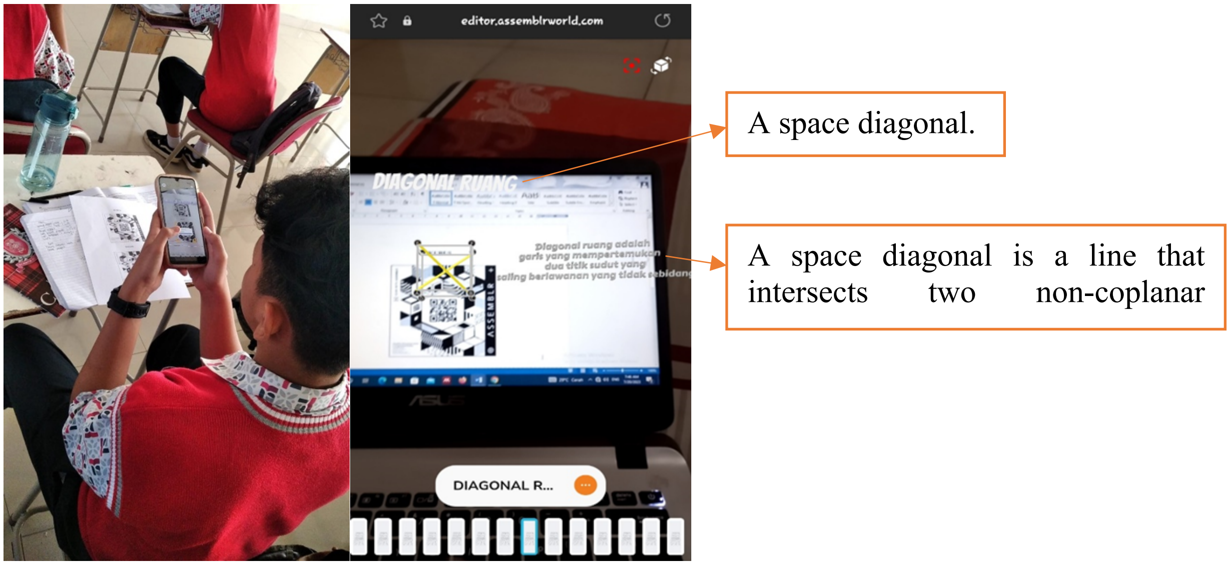


Figure 3: Assembler edu apps of students' activity in experimental class

Research Tools

The research instruments utilized in this project include a learning accomplishment exam and a questionnaire. A questionnaire was used to gauge the student's openness to technology and their motivation for learning. The content of the accomplishment test was developed by two teachers with a combined ten years of classroom experience. Preliminary testing focuses on students' 3D common sense. The test content consisted of ten questions for a total of one hundred points. Students' prior knowledge of 3D shapes could be gauged through pre-testing. The post-test had a total of 100 points and consisted of 10 questions. Using a knowledge quiz and their grasp of the graphic concepts of 3D shapes, and 3D composition, the issue assessed whether students could effectively distinguish between the three after engaging in digital learning.

Following the trial, participants filled out science and technology acceptance assessment forms. After some revisions to meet the requirements of the study, the technology acceptance rating measures developed by Cai et al. (2019) were used. Two parts of the survey, "Cognitive Usefulness" and "Cognitive Easy-to-use," comprised thirteen items administered using Likert's 5-point scale. To what extent did users' acquisition of learning aids improve their learning? That was the "cognitive usefulness" focus. Cognitive Easy-to-use was employed to ascertain if learners could utilize the learning aids without difficulty. The computations required to prove the test's validity and reliability were all carried out. Using Cronbach's Alpha, we found that the test has a reliability of 0.910. Scale dependability is vital when Cronbach's alpha is between 0.6 and 0.80.

The study's qualitative section made use of a self-designed semi-structured interview guide. After the study, the survey's main goal was to get the experimental group students' thoughts on how AR was used. The questionnaire was double-checked for validity and reliability by interviewing two additional experts alongside the researchers. The experts' advice was carefully followed, and the required changes were made without a hitch.

Data Analysis

The quantitative data collected for the study's main objective was examined using the statistical analysis program SPSS 28.0. Before running any statistical tests on the data, it was crucial to determine whether the data adhered to a normal distribution and how similar the variance was. The researchers employed the Levene test to investigate the homogeneity of conflicts in the dataset and the Kolmogorov-Smirnov test to evaluate the normalcy assumption. The homogeneity of group variances and the data set's normal distribution ($p > .05$) impacted the parametric tests selected for the study.

Analysis was conducted using the independent sample t-test on data obtained from both the experimental and control groups both before and after the test was administered. After the experiment, a descriptive analysis was done on the qualitative data collected from student interviews. The students' comments were examined using a method called content analysis. Consequently, content analysis identifies the core concepts and relationships that provide the most satisfactory explanation for the collected data (Robinson, 2017). For each subject, the content analysis used actual quotes from students to establish the credibility of the findings. The authorities have not been altered in any way; they were extracted straight from the student interviews. To keep track of which kids were the most vocal, we used the abbreviations S1, S2, S3, etc. For more reliable results, the researchers and two experts from the outside coded the qualitative data independently. Having impartial experts present significantly bolstered the analysis's credibility. The data analysis's dependability was assessed using the article's reliability formula (Ishtiaq, 2019). The computation yielded a 94% confidence level. The study's findings supported this particular conclusion. The qualitative data was evaluated, and the interview was scheduled using NVIVO 12.

RESULTS

Learning Achievement Analysis – RQ1

To evaluate the math-based abilities of the two teams prior to the experiment and determine whether the students' prior knowledge was similar before the learning activity, we utilised an independent samples t test on the pre-test findings. The control group averaged 67.41 points on the pre-test Maths, while the experimental group averaged 67.21 points, according to the statistical analysis. The experimental group and the control group did not differ substantially, according to the data ($t = 0.439$, $p > 0.1$). The findings showed that the students in the experimental and control groups had the same level of pre-experiment mathematical proficiency.

We examined the post-test results of the experimental and control groups using analysis of covariance (ANCOVA) to see whether there was a statistically significant difference in their learning outcomes. To see if our theories regarding the covariance analysis held true, we looked at two sets of data.

The pre- and post-test scores are related variables, as can be shown from the fact that the regression coefficients were the same ($F=0.526$, $p =.470>.05$). We know that changes in the treatment level of the independent variable will not have an impact on the regression coefficient since covariance analysis is predicated on the idea that the regression coefficient is homogeneous in the covariate set.

Table 3 shows that there was a significant difference between two test scores after the Math test ($F = 4.348$, $p =.039 <.05$) after controlling for the influence of covariance (pre-test score) on the dependent test items (post-test score). The experimental squad's average score was 75.86, with a 14.83 standard deviation. With a standard deviation of 20.32, the control group's average score was 69.75. Stated differently, traditional information coupled with math learning was outperformed by interactive AR app math learning to the statistical significance level.

Criteria Group	**N**	**M**	**SD**	**Adjusted Mean**	**S.E**	**F**
Experimental group	46	78.31	15.32	79.45	2.53	4.525*
Control group	44	69.42	18.46	69.73	2.01	

*p < .05

Table 3: The ANCOVA value of the pst-test study

Math Students' Motivation Analysis – RQ2

Researchers looked at two teams' pre- and post-test results on the Math learning motivation evaluation scale to see how different teaching philosophies affected students' motivation to study. This gave them the chance to talk about whether interactive augmented reality learning could improve the experimental team's motivation to learn maths.

An independent sample t-test was performed on the total score obtained from the Pickett-Like 5-point assessment of learning motivation scale prior to the instructional activity. The control group did remarkably well, with an average score of 4.27 points and a standard deviation of 1.06. The average score for the entire experimental group was 4.28, with a standard deviation of 1.17. The t-test results showed that there was no significant difference between the two teams' pre-lesson motivation. "p" =.957>.05. and "t" = 0.054>.05. After the trial, a motivational questionnaire was given to each of the two groups, and one-factor covariates (ANCOVA) were used to assess the data.

Firstly, we tested the homogeneity of regression coefficients to confirm the covariates' (pretest scores) relationship with the dependent variables. The results showed that the coefficients were homogeneous (F = 2.921, p =.09 >.05). the outcomes of the following evaluation It was not different because the treatment levels of the independent variables were different, assuming that the covariate's regression coefficients are homogeneous.

The outcomes of the ensuing covariate analysis are displayed in Table 4. After adjusting for the impact of the independent variable (pre-motivation questionnaire) on the dependent variable (post-motivation questionnaire), there were significant differences between the post-Math motivation levels of the two teams. There is statistical evidence to corroborate this (F = 8.80, p =.004 <.01). That is to say, the statistical significance of learning through interactive augmented reality apps was significantly higher than that of learning through more traditional techniques that blended maths with conventional knowledge.

Criteria Group	**N**	**M**	**SD**	**Adjusted Mean**	**S.E**	**F**
Experimental group	46	4.83	0.93	4.85	0.105	8.74**
Control group	44	4.15	1.26	4.17	0.103	

** p < .01

Table 4: The ANCOVA value of students' motivation in post-test study

Subsequently, figure 4 shows that the majority of students had results on the Students' Motivation test that fell into the top two categories. As an example, 84.3% of the participants (76 out of 90) thought AR applications were very attractive. In addition, the top-2-box score shows that 74.3% of participants agreed that the given authoring choices were useful for building AR applications. Furthermore, it is worth mentioning that a considerable majority of the participants, specifically 82.6% (74 out of 90), thought that augmented reality technologies can be useful while teaching mathematics. In a similar vein, a sizeable majority of respondents (78.3 percent, or 70 out of 90) agreed that augmented reality tools are usually appropriate for developing educational interventions. Finally, 73.3% of people think it's easy to combine AR with a traditional textbook.

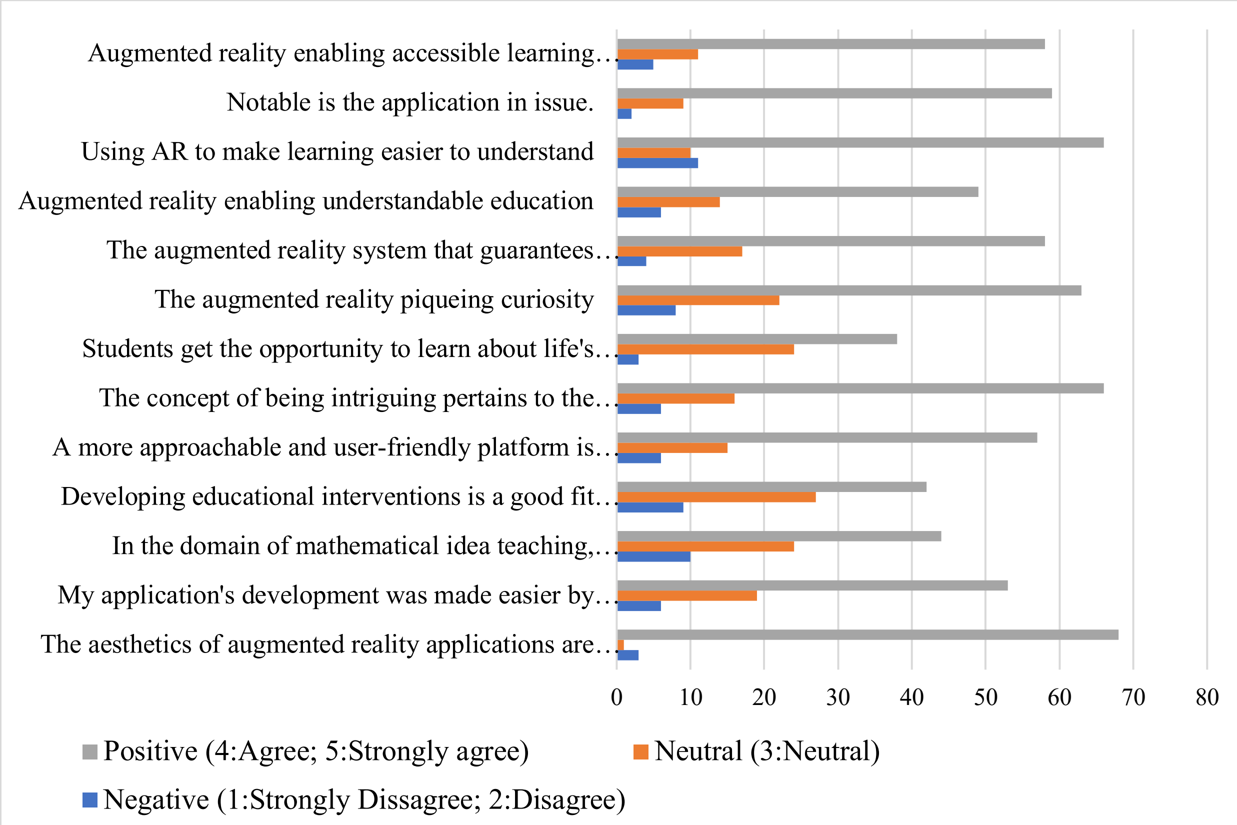


Figure 4: Vertical graph on students' experience on motivation

Numerous participants have reported having excellent experiences using AR. We chose "Both AR tools are easily learnable without any prior knowledge of computer programming or domain-specific expertise" because it best describes the scenario. A different students commented, "It is noteworthy that, three weeks prior, I possessed no foresight regarding the opportunity that would arise for me to engage in the development of AR app." With the use of supplemental materials, I was able to become proficient in fractional mathematics, and for that I am grateful.

Extra information regarding the use of AR technology in mathematics instruction was gleaned from short discussions with focus group session participants. The majority of participants expressed a strong preference for AR and expressed their intention to incorporate it into several fields. The use of an AR writing tool has been suggested by students as a potential solution to the problems that exist in cooperation. Using AR in the classroom is something that the majority of kids are interested in doing. During a discussion on augmented reality authoring tools, three students voiced a need for more feature-rich tools for AR reality applications, such a hybrid of Unity 3D and Vuforia. The majority of students in this class thought the AR exercise was the best part of the class. However, participants reported that making instructional videos was the most time-consuming and challenging task. They also came to the conclusion that mathematical themes were properly covered by an existing classroom content development method. More than that, a handful of students said they had problems with the technical aspects of their software programmes. With the help of their classmates, students were able to fix the problems in the majority of cases. On rare occasions, pupils may choose to approach the instructor for individual attention.

After conducting an experiment, we gathered student feedback on AR activities. In this particular context, the students were mostly asked about the motivational signs of using AR in the classroom. As shown in Table 5, we analysed and modelled the responses of the participants.

Providing visually engaging surroundings and effective learning tools.
Providing the opportunity to utilise technological benefits
Developing abilities in research and inquiry
Providing a plethora of options to gain access to important resources
Visual intelligence stimulation and application
Opening up possibilities for observation
Giving students the opportunity to learn about the nuts and bolts of life
Providing accessible, thorough, and efficient educational opportunities
Providing thorough instructional guidance
The development of creative thinking
Notable is the application in issue.
Providing visually engaging surroundings and effective learning tools
Providing the opportunity to utilise technological benefits

Table 5: The impact of assembler edu apps on students' motivation

Student responses describing the positive effects of AR applications in the classroom are included in Table 5. One of the possible uses for augmented reality is to help people learn in visually rich environments; another is to let them "become" another object or idea. Someone once claimed, "I learned better because it was visual," and they couldn't have been more right. My learning and comprehension were both enhanced by this. Part 5. "It was absolutely lifelike; I felt like I was smack dab in the middle of the action," said another pupil. The level of detail in the planet photos was breathtaking. In contrast to the first, who showed no interest, the second one said, "I was very fascinated by that" (S1). This programme has the potential to be useful in a range of educational contexts (S9). According to another student, the incorporation of augmented reality software into the class enhances the retention of information. Using the applications (S12) piqued my interest in the lesson and helped me understand the content better. "The application of AR was interesting for the topic," another student commented. Learning is also made more interesting and entertaining by the application's graphically rich environment. An easy learning curve is the end outcome" (S6).

Technology Acceptance Analysis – RQ3

To learn more about the degrees of technology adoption in the control group and the experimental group after the introduction of digital learning for maths. The researchers evaluated the results of two teams on science technology acceptance scales to determine how various learning activity strategies affected the acceptability of science technology. Twelve questions each on the topics of "Cognitive Usefulness" and "Cognitive Easy-to-use" were included in the test. The Likert 5-point scale was used to score the questions. The science technology acceptability scores were examined using an independent sample t test. The experimental group used an augmented reality app to learn, whereas the control group used more traditional digital approaches. Table 6's results demonstrate a substantial difference (p =.00 <.001) between the experimental and control teams' embrace of science and technology. This indicates that pupils were able to give normal digital learning a lower technological acceptance rating than AR App Learning.

Criteria Group	N	M	SD	F
Experimental group	46	4.47	0.72	4.29***
Control group	44	3.39	0.59	

Table 6: The t-test value for technology acceptance of this study

Meanwhile, an additional component of the research was the inclusion of students' viewpoints regarding the limitations of AR applications and their recommendations for enhancements. Table 7 shows how the students feel about it.

Category	Recommendations
Cognitive usefulness	The student takes on the position of an observer and listener.
	Very confusing user text.
	Some students might not be able to use it.
	Applying a conceptual framework that functions inside a more complex and detailed world can shed light on the phenomenon.
	It is recommended to take suggestions into account and put them into action at each level before moving forward.
	The ability to produce audible sounds is possessed by auditory stimuli.
Cognitive easy-to-use	It is necessary to create graphics that are more accurate.
	Perhaps a more interesting way to convey the text might make it more effective.
	Possible savings might materialise.
	All fields of study must make heavy use of this resource.
	It is possible to increase the level of usefulness.
	The object's three-dimensional projection needs to be shown in an appropriate spot.
	It is recommended that students cultivate an attitude of living in the here and now.
	Making sure that AR App are used to conduct tests.

Table 7: The students' viewpoints of limitations and recommendations

Table 7 highlights several problems, including excessive expense, instructors falling behind, students not actively participating, application challenges due to a large student body, insufficient availability, and time spent. The students' responses to these problems included suggestions to make the app more interesting to use, reduce expenses, and provide more realistic images and sounds. "It is not good that the connection and the technical problems we had in class take much time" (S2) is one possible student complaint. "It is a very nice app, but it is not good because we do not have money to buy it, and it is expensive" (S6) is another student's take on the downsides of AR apps. "It sometimes confused our class," another student stated, adding to the class's confusion. "What if I wanted to reach out and touch the sun?" In Section 3. Think about the consequences if we experienced it Part fourteen "For this course, augmented reality is a waste of time and energy. The student examines the digital representation. No one can really benefit from it (S11). "I think it would be good for my brother if this were used in all courses" (S7). The students who participated in the study expressed their discontent with the use of AR and provided suggestions for how to fix it (Table 7). According to a student's suggestion, "The application was interesting and exciting, but more realistic sounds and images could be used during the application" (S12), the programme may have been enhanced. A different student brought up the fact that these apps are expensive and said, "Not all students can afford to use them." as an argument against their widespread use. Making software that doesn't break the bank is essential (S8).

DISCUSSION

According to the experimental findings, AR can be a helpful tool for motivating students, making it easier to handle objects using a tactile interface metaphor, and enabling a smooth transition between the physical and virtual worlds. These advantages are in line with the need for mathematics education, particularly in mathematical modelling where students demand a relationship between mathematical concepts and real-world circumstances. This software allows students to match 3D objects with their appropriate real-world situations, in line with Cabero-Almenara et al. (2021) idea of mathematical modelling as a sub-competence, which aids students in understanding real-world difficulties.

According to Salim et al. (2020), students can create mathematical models based on real-world examples with the use of AR app in students' motivation. This is a part of the stage of students' motivation in classroom instruction. The motivation of each assignment supports these two stages of instruction. Additionally, Elsayed and Al-Najrani (2021), when stuck, students can apply previously learned mathematical ideas to particular mathematical situations by using the app's hint feature.

Here, the recommendations feature refers to the idea of students' motivation scaffolding (Özçakır & Özdemir, 2022). In light of recent technology improvements, digital tools have become an effective tool for teaching students motivation (Hamzah et al., 2021; Nasrudin et al., 2021). This study 3D shape display offers clues that resemble the actual object that needs to be found in addition to the criteria.

Dinayusadewi et al. (2020) claim that this visual assistance facilitates students' ability to draw meaningful links between pertinent mathematical settings and real-world situations. According to Kazanidis and Pellas (2019), students can convert problems from the actual world into more formal mathematics. With this function, students are guided to the next level of problem solving in mathematical models. Subsequently, they conduct diverse modelling procedures as required and convert mathematical inferences into practical models, scenarios, and testing approaches. Fendi et al. (2021) noted that shapes are now easier to view and interact with AR.

Meanwhile, the findings demonstrate how AR seamlessly connects the actual and virtual worlds. By presenting geometric concepts in various ways and from many angles, AR might help students comprehend them better and lessen the possibility of misconceptions brought on by students' motivation difficulties. This is supported by the findings of Cai et al. (2019), who discovered that AR holds a lot of potential for application in the classroom, especially in subjects that require a lot of motivation aids. The positive feedback students left about using the AR mobile app also showed how excited and interested they were in learning. This finding supports the idea that more people use AR to create creative classroom learning environments (Korkmaz & Morali, 2022; Suwayid & Rezqallah, 2022). Students can employ AR, which bridges the gap between virtual and real-world scenarios, to tackle Assembler Edu tasks using mathematical modeling cycles. The study's findings can help Assembler Edu achieve its development goals, which include improving students' motivation (Fendi et al., 2021). In this educational programme, AR has brought a virtual concept into the real environment about technology (Dinayusadewi et al., 2020).

Based on the findings, the AR Assembler Edu App has a favorable impact on student's motivation for learning tasks. This training curriculum improves the ability to model mathematically. AR technology can benefit students in terms of student motivation, especially when learning to understand real-world events, build physical models, and then use those models to generate student motivation. Furthermore, field measurements verified the presence of a relationship between the students' motivation created during instrumentation and the instrumented approaches used in the AR app.

LIMITATIONS

Several limitations are associated with the present study. Because the study was designed with only two conditions, we don't know how the specific changes in complexity between them affected the students' learning and curiosity. The conditions differed in terms of the quantity of AR representations, the sorts of AR representations, and how they were anchored. To better understand how AR application designs vary along these particular dimensions, future research is required to examine more controlled variation. Both the findings and the discussion are very conjectural and point to potential avenues for further investigation due to the small sample size, which diminished the statistical ability to identify conditional differences. Furthermore, there were not many question prompts in the instructional exercise, which may have contributed to a low question volume. A bigger sample size participating in an open-ended tutoring activity for an extended length of time should allow for the detection of more statistical effects.

In addition, the data used for qualitative analyses of students' mentions of AR visuals is limited to questions that directly addressed the visuals; however, students did ask numerous questions that did not directly address the visuals, and this data could potentially provide more accurate comparisons between the groups. When applied to bigger datasets, the statistical implications and repeatability of the descriptive findings from this study should be explored in future studies. Furthermore, there was a significant gender imbalance in our sample; in each condition, women made up about two-thirds of the participants. The current study did not do a gender analysis, thus even while the gender disparity seems to be equal across situations, it is unclear if this influenced the outcomes. Whether or not the current findings are applicable to larger populations may be confirmed by future research. The perception of students' silences also has its limitations. Both outwardly visible behaviours (such as asking questions or requesting activities) and internally visible behaviors (such as students thinking critically about the subject) can constitute active learning. While the Full-AR group did show more overt indicators of active learning in our data, it doesn't imply that all students weren't involved. Students in the Basic-AR group did appear to be paying attention in class; they just weren't expressing themselves as clearly. Research in the future can make use of other tools for gauging students' internal processes, such as cognitive load measurements and more targeted pre- and post-test inquiries. In conclusion, we recognise that the activity has an effect on the sorts of inquiry that students do; students may display distinct patterns of inquiry with varied learning content and with tutoring activities that are less direction driven.

CONCLUSION

Students may need help understanding more abstract concepts taught in the classroom using traditional approaches. However, developments in AR apps can provide several levels of interaction and information presentation, improving students' motivation for learning. In recent years, there has been a steady increase in research studies and projects addressing this topic. Implementing AR apps in educational settings yields numerous advantages, such as creating a captivating learning atmosphere, raising student motivation, and enabling more profound comprehension of the subject matter. However, utilizing technology to support instruction does not guarantee that students will learn more motivated. Effective teaching practices must be implemented in addition to recently released technology so that students can benefit from it. The study results indicate that employing an AR maths app to learn the content of the "3D shape" unit can improve students' learning outcomes and motivation through interactive operations and modifications to 3D space. Students who use AR apps for learning are also more likely than the control group to accept technology, which aligns with another research.

References

[1] Abdullah, N., Baskaran, V. L., Mustafa, Z., Ali, S. R., & Zaini, S. H. (2022). Augmented Reality: The Effect in Students' Achievement, Satisfaction and Interest in Science Education. *International Journal of Learning, Teaching and Educational Research*, *21*(5), 326–350. https://doi.org/10.26803/ijlter.21.5.17

[2] Ahmad, N. I. N., & Junaini, S. N. (2020). Augmented Reality for Learning Mathematics: A Systematic Literature Review. *International Journal of Emerging Technologies in Learning*, *15*(16), 106–122. https://doi.org/10.3991/ijet.v15i16.14961

[3] Avila-Garzon, C., Bacca-Acosta, J., Kinshuk, , Duarte, J., & Betancourt, J. (2021). Augmented Reality in Education: An Overview of Twenty-Five Years of Research. *Contemporary Educational Technology*, *13*(3), 1–29. https://doi.org/10.30935/cedtech/10865

[4] Cabero-Almenara, J., Barroso-Osuna, J., & Martinez-Roig, R. (2021). Mixed, augmented and virtual, reality applied to the teaching of mathematics for architects. *Applied Sciences*, *11*(15), 1–16. https://doi.org/10.3390/app11157125

[5] Cai, S., Liu, E., Yang, Y., & Liang, J. C. (2019). Tablet-based AR technology: Impacts on students' conceptions and approaches to learning mathematics according to their self-efficacy. *British Journal of Educational Technology*, *50*(1), 248–263. https://doi.org/10.1111/bjet.12718

[6] Chani Saputri, D. S., & Susilowati, D. (2022). Augmented Reality in Indonesia's Primary School: Systematic Mapping Study. *International Journal of Engineering and Computer Science Applications (IJECSA)*, *1*(1), 43–50. https://doi.org/10.30812/ijecsa.v1i1.1817

[7] Dinayusadewi, N. P., Ngurah, G., & Agustika, S. (2020). Development Of Augmented Reality Application As A Mathematics Learning Media In Elementary School Geometry Materials. *Journal of Education Technology*, *1*(2), 204–210.

[8] Elsayed, S. A., & Al-Najrani, H. I. (2021). Effectiveness of the Augmented Reality on Improving the Visual Thinking in Mathematics and Academic Motivation for Middle School Students. *Eurasia Journal of Mathematics, Science and Technology Education*, *17*(8), 1–16. https://doi.org/10.29333/ejmste/11069

[9] Fendi, R. D., Suyatna, A., & Abdurrahman, A. (2021). Augmented Reality-Based Student Worksheet to Stimulate Students' Critical Thinking Skills. *Indonesian Journal of Science and Mathematics Education*, *4*(2), 118–133. https://doi.org/10.24042/ijsme.v4i2.9017

[10] Guntur, M. I. S., Setyaningrum, W., Retnawati, H., Marsigit, M., Saragih, N. A., & Noordin, M. K. bin. (2019). Developing augmented reality in mathematics learning: The challenges and strategies. *Jurnal Riset Pendidikan Matematika*, *6*(2), 211–221. https://doi.org/10.21831/jrpm.v6i2.28454

[11] Hamzah, M. L., Ambiyar, Rizal, F., Simatupang, W., Irfan, D., & Refdinal. (2021). Development of Augmented Reality Application for Learning Computer Network Device. *International Journal of Interactive Mobile Technologies*, *15*(12), 47–64. https://doi.org/10.3991/ijim.v15i12.21993

[12] Ishtiaq, M. (2019). Book Review Creswell, J. W. (2014). Research Design: Qualitative, Quantitative and Mixed Methods Approaches (4th ed.). Thousand Oaks, CA: Sage. *English Language Teaching*, *12*(5), 40–51. https://doi.org/10.5539/elt.v12n5p40

[13] Kazanidis, I., & Pellas, N. (2019). Developing and Assessing Augmented Reality Applications for Mathematics with Trainee Instructional Media Designers: An Exploratory Study on User Experience. *Journal of Universal Computer Science*, *25*(5), 489–514. https://www.blippar.com/

[14] Khanchandani, K., Shah, M., Shah, K., Panchal, V., & Professor, A. (2021). A Review on Augmented Reality and Virtual Reality in Education. *International Research Journal of Engineering and Technology*, *8*(2), 961–968. www.irjet.net

[15] Korkmaz, E., & Morali, H. S. (2022). A meta-synthesis of studies on the use of augmented reality in mathematics education. *International Electronic Journal of Mathematics Education*, *17*(4), 1–21. https://doi.org/10.29333/iejme/12269

[16] Lauren, B. (2021). Strengthening Digital Learning across Indonesia: A Study Brief. *Journal of Educational Learning*, *4*(2), 44–58. https://blogs.worldbank.org/eastasiapacific/COVID-19-and-learning-inequities-indonesia-four-ways-bridge-gap

[17] Miundy, K., Badioze Zaman, H., Nosrdin, A., & Hui Ng, K. (2019). Evaluation Of Visual Based Augmented Reality (AR) Learning Application (V-ARA-Dculia) For Dyscalculia Learners. *International Journal on Informatics Visualization*, *3*(4), 343–354.

[18] Munir, N. P., Anas, A., Suryani, L., & Munir, F. S. (2022). The Practicality of Geometry Learning Media based on Augmented Reality. *Al-Khwarizmi: Jurnal Pendidikan Matematika Dan Ilmu Pengetahuan Alam*, *10*(1), 75–84. https://doi.org/10.24256/jpmipa.v10i1.2241

[19] Nasrudin, N. H., Azfar, M. H., Hatta, M., Azmi, A. Z., & Junid, R. A. (2021). Mobile Application: Learning Basic Mathematics Operation using Augmented Reality. *Mathematical Sciences and Informatics Journal*, *2*(1), 9–20. http://www.mijuitmjournal.com

[20] Nugraha, I. (2023). Application development of augmented reality for elementary English teaching. *Indonesian EFL Journal (IEFLJ)*, *9*(1), 19–24. https://doi.org/10.25134/ieflj.v9i1.6423

[21] Özçakır, B., & Özdemir, D. (2022). Reliability and Validity Study of an Augmented Reality Supported Mathematics Education Attitude Scale. *International Journal of Human-Computer Interaction*, *38*(17), 1638–1650. https://doi.org/10.1080/10447318.2022.2092955

[22] Permatasari, G. I., & Andayani, S. (2021). Teachers' Challenges in Teaching Geometry Using Augmented Reality Learning Media. *AKSIOMA: Jurnal Program Studi Pendidikan Matematika*, *10*(4), 2159. https://doi.org/10.24127/ajpm.v10i4.3889

[23] Pramuditya, S. A., Pitriayana, S., Subroto, T., & Wafiqoh, R. (2022). Implementation of augmented reality-assisted learning media on three-dimensional shapes. *Jurnal Elemen*, *8*(2), 480–493. https://doi.org/10.29408/jel.v8i2.5238

[24] Pujiastuti, H., Haryadi, R., & Arifin, M. (2020). The development of Augmented Reality-based learning media to improve students' ability to understand mathematics concept. *Unnes Journal of Mathematics Education*, *9*(2), 92–101. https://doi.org/10.15294/ujme.v9i2.39340

[25] Putrie, S. N., & Syah, M. N. S. (2023). Development of 3D Math AR Applications as Mathematics Learning Media Augmented Reality Based. *Hipotenusa : Journal of Mathematical Society*, *5*(1), 72–81. https://doi.org/10.18326/hipotenusa.v5i1.6401

[26] Riza, A., Atika, Z., & Maira, N. (2023). Application of Augmented Reality in Geometry Learning in Increasing Student Learning Motivation. *Journal of Curriculum and Pedagogic Studies (JCPS)*, *2*(1), 40–50. https://e-journal.lp2m.uinjambi.ac.id/ojp/index.php/jcps

[27] Robinson, M. A. (2017). Using multi-item psychometric scales for research and practice in human resource management. *Hum Resour Manage*, 1–12. https://doi.org/10.1002/hrm.21852

[28] Salim, S., Fitrah, D., & Jainuddin. (2020). Augmented Reality-based Mathematics Worksheet for Online Learning During Covid-19 Pandemic. *Indonesian Journal of Educational Studies (IJES)*, *24*(1), 81–90.

[29] Schutera, S., Schnierle, M., Wu, M., Pertzel, T., Seybold, J., Bauer, P., Teutscher, D., Raedle, M., Heß-Mohr, N., Röck, S., & Krause, M. J. (2021). On the potential of augmented reality for mathematics teaching with the application cleARmaths. *Education Sciences*, *11*(8), 1–18. https://doi.org/10.3390/educsci11080368

[30] Su, Y.-S., Cheng, H.-W., & Lai, C.-F. (2022). Study of Virtual Reality Immersive Technology Enhanced Mathematics Geometry Learning. *Frontiers in Psychology*, *13*(February), 1–8. https://doi.org/10.3389/fpsyg.2022.760418

[31] Suwayid Alqarni, A., & Rezqallah Alzahrani, R. (2022). The Impact of Augmented Reality on Developing Students' Mathematical Thinking Skills. *IJCSNS International Journal of Computer Science and Network Security*, *22*(3), 553. https://doi.org/10.22937/IJCSNS.2022.22.3.71

[32] Voronina, M. V., Tretyakova, Z. O., Krivonozhkina, E. G., Buslaev, S. I., & Sidorenko, G. G. (2019). Augmented reality in teaching descriptive geometry, engineering and computer graphics-systematic review and results of the russian teachers' experience. *Eurasia Journal of Mathematics, Science and Technology Education*, *15*(12), 1–17. https://doi.org/10.29333/ejmste/113503

[33] Wahyudi, U. M. W., & Arwansyah, Y. B. (2019). Developing Augmented Reality-based Learning Media to Improve Student Visual Spatial Intelligence. *Indonesian Journal of Curriculum and Educational Technology Studies*, *7*(2), 89–95. https://doi.org/10.15294/ijcets.v7i2.36039

[34] Widyasari, N., & Mastura, L. I. (2020). Improving Geometry Thinking Ability through Augmented Reality Based Learning Media. *Eduma: Mathematics Education Learning and Teaching*, *9*(1), 80–85. https://doi.org/10.24235/eduma.v8i2.5304

MATHEMATICS TEACHING RESEARCH JOURNAL
Golden Fall 2024
Vol 16 no 4

Math anxiety in the virtual classroom during covid-19 pandemic and its relationship to academic achievement

Georgios Polydoros

University of Thessaly, Greece

gpolydoros@uth.gr

Abstract: The plethora of research studies on the impact of the COVID-19 pandemic on education gives us a first picture of the difficulties and challenges encountered at the various educational levels, especially at the elementary level, by students aged from 6 to 12 years old. The aim was to examine the existence of math anxiety in primary education and how math anxiety differentiates performance according to gender during the COVID-19 confinement. A quantitative empirical research carried out, using a fifteen-question questionnaire, with a 5-point Likert scale, which employed the Mathematics Anxiety Rating Scale (MARS), the Mathematics Anxiety Scale (MAS), the Abbreviated Math Anxiety Scale (AMAS) and a math test. The sample consisted of 173 Greek sixth grade students. The questionnaire and the math test were provided to the students by the teachers at the beginning of the school year in September 2021. Sixth graders, especially girl students, get quite anxious when involved with math learning activities. Math anxiety affected more girl than boy students' grades in the math test. Math anxiety affected primary 11 to 12-year-old students, while girl students exhibited higher math anxiety than their counterparts. In addition, math anxiety affected girls' performance in the math test.

Keywords: Elementary school; Online Learning; Gender; Math Anxiety; Achievement

INTRODUCTION

One issue we need to clarify from the start is the differences between distance education and emergency digital education implemented during the pandemic. Distance education is an alternative and flexible learning option for learners, an interdisciplinary field that has evolved over time and is characterized by the spatial distance that separates instructors from learners, the use of specialized digital resources and tools designed to respond to particular learning needs and guidance through open educational practices (Bozkurt et al., 2019; Zawacki-Richter et al., 2020). In contrast, emergency online digital education was formed without long-term planning in a short period of time. It was a necessity for educational systems to respond to the special conditions of the total ban on mobility that prevailed around the world and the general shift to teleworking and social distancing. It was therefore an obligation, a temporary educational solution that tried to respond to existing problems that forced educational communities to temporarily adopt different strategies and set different priorities in the educational process (Bozkurt & Sharma, 2020).

The rich literature that has developed at a rapid pace around the question of the effects of the crisis on education gives us a first picture of the difficulties and challenges that the various educational systems encountered as well as of the ways and innovations that they developed to adapt to the new conditions (Bao, 2020; Flores & Gago, 2020; Moorhouse, 2020).

Greece, like most countries, announced emergency remote teaching at all levels of education in March 2020 (UNESCO, 2020). In a short period of time, the educational units were called on to adapt to the urgent and unpredictable condition of the spread of the pandemic due to the COVID-19 disease by taking the necessary measures to limit movement from place to place, gatherings and communication. In order to continue in safe conditions, the educational units were asked to interrupt the learning process requiring physical presence in the classrooms and to adapt to the emergency distance education model (Crompton et al., 2021).

The literature review clearly shows that the consequences of the pandemic were more severe on children (Fore, 2020; Lee, 2020; Liu et al., 2020). School closures had serious psychosocial effects on children (Spinelli et al., 2020). Children, showed high levels of anxiety and insecurity, as they had great difficulty understanding what was happening (Shapiro et al., 2020).

The long period of online learning and the substantial interruption of all those factors present in live teaching raises concerns about the effects on the cognitive development, the motivation to learn and the learning progress of children. It was imperative that the school developed a comprehensive plan to address the potential learning and mental health needs of children (Ghosh et al., 2020; Phelps & Sperry, 2020).

ICT integration in the modern classroom has already been advocated by the OECD (2001) and used in different situations (Crompton et al., 2021). However, until the onset of the COVID-19 pandemic, the majority of European educational systems had not altered their primary method of instruction—that is, teaching in classrooms (Schleicher, 2020).

While the coronavirus pandemic brought digital learning into educational institutions, the vast majority of schools and teachers was caught unprepared and struggled to adapt to the new teaching framework (Hodges et al., 2020), 'victimizing' certain groups of students, such as females.

According to a flurry of new studies (e.g., Patrinos et al., 2022; Werner & Woessmann, 2023), the COVID-19 pandemic was an obstacle that hindered students' learning. For example, Di Pietro's (2023) meta-analysis**,** with a view to investigating the impact of COVID-19 on student achievement—239 estimations from 39 studies spanning 19 nations were included in the collection—found that the learning results were generally negatively impacted by the COVID-19 pandemic, i.e., students have fallen behind in math and science compared to other subjects, while students are still recuperating from the initial learning losses at least one year following COVID-

19; thus, new studies are needed because findings from new studies will show the level of the impact of school closure during the pandemic on students' learning and on education in general.

According to the World Economic Forum (2022), the full effects of the COVID-19 pandemic are just now beginning to become apparent, thus future effects on students' competencies, abilities, knowledge (Hanushek & Woessmann, 2020), and the psychological aspects of learning should be taken into consideration.

Determining whether and to what extent the interruption of face-to-face instruction resulted in student learning deficits is crucial for this reason. In order to help students recover from the learning deficit brought on by the closure of schools during the COVID-19 pandemic, educators and decision-makers in the field of education must first identify the categories of students who may require additional support (Di Pietro, 2023).

Since a part of empirical studies points out gender differences in e-learning (Devine et al., 2012; Drabowicz, 2014; Wongwatkit et al., 2020; Khasawneh et al., 2021; Bertoletti et al., 2023) it is censorious to investigate, among other factors, MA gender differences and performance in digital environments, because one of the main driving factors interpreting this gender gap is the students' anxiety during the COVID-19 school closure (Bertoletti et al., 2023). Meanwhile, to date few studies have explored specific gender differences in online-learning math activities (Aguillon et al., 2020), e.g. math anxiety. Therefore, the overall objective of the research is to determine the level of math anxiety and in what way it affected sixth graders' math performance during the school closure in the COVID-19 pandemic.

The three specific pointed objectives are:

- To confirm the existence of math anxiety in sixth graders
- To determine whether math anxiety differs in gender
- To assess performance in relation to math anxiety during the COVID-19 lockdown period.

Finally, the main research question addressed by the research is how math anxiety affects the performance of sixth-grade students in mathematics during the COVID-19 pandemic, followed by a second one exploring the existence of math anxiety among sixth graders.

LITERATURE

Math anxiety

While some difficulties with mathematics can be linked to a lack of understanding of the material, other difficulties can be brought on by emotional issues. The term MA refers to the adverse emotional response that is triggered when someone meets a mathematical task, according to Cipora et al. (2015). MA, which is defined as a negative emotional reaction to mathematics, may affect a person's ability to accomplish mathematical assignments. Furthermore, MA makes students feel less confident about their mathematical skills and keeps them from generally relishing their journey into the "mathland."

To help understand when the state of MA is triggered, the 'Growth Zone Model' of Johnston-Wilder et al. (2013) was deployed. As shown in Fig. 1, there are three zones when the student tackles a mathematical situation: the Comfort zone, where the students feels confident; the Growth zone, where the students experiences learning; the Anxiety zone, where the students begins to expose themselves to math anxiety.

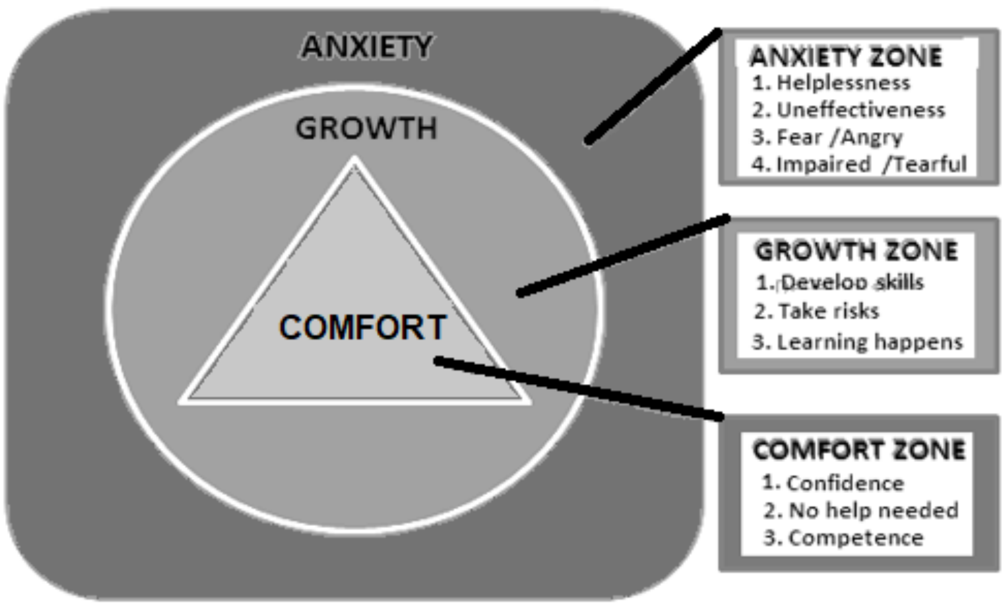


Figure 1: Growth Zone Model

MA is a condition which can cause several symptoms, including emotional -for instance, feelings of apprehension, aversion, tension, distress, irritation or fear-, physical -for instance, butterflies, tachycardia, struggling to catch your breath-, or behavioral -for instance, not acting properly in class, avoiding math assignments and studying (Hembree, 1990). Young school children are reported to exhibit math anxiety from approximately the age of six (Krinzinger et al., 2009; Beilock et al., 2010; Vukovic et al., 2013). As Dowker et al. (2016) mentioned, negative attitudes towards mathematics usually increase at the age of secondary school and remain during post-secondary education and throughout adulthood. It is hard to determine how often math anxiety occurs, because measurements of math anxiety are continuous and there is no clear limit regarding whether an individual is math-anxious or not (Devine et al., 2018).

Math anxiety has an impact on individual wellbeing – e.g., some students will face their math lessons with fear or skip their math homework due to a dislike of experiencing negative emotions (Dowker et al., 2016). Additionally, intellectual factors may be associated with MA. Although there is increasing evidence that children demonstrate MA as early as first grade (Ramirez et al., 2013), little is known about the change of MA across childhood. Few studies showed that the level of MA increases with age (Dowker et al., 2012; Krinzinger et al., 2009).

Children with developmental dyscalculia (a specific deficit in the acquisition of mathematics skills) and other learning disabilities in mathematics are more likely to have MA (Rubinsten & Tannock 2010; Passolunghi, 2011). Furthermore, MA may be influenced by an individual's other personal characteristics. For instance:

- Gender - girls are more likely to experience anxiety about math (Bieg et al., 2015);
- Self-esteem - a lower self-esteem leads to higher levels of math anxiety (Abbasi et al., 2013);
- Instruction and learning style – students who had more negative experiences with instructional methods had higher levels of MA compared to those who had fewer negative experiences (O'Leary et al., 2017), and
- Attitude towards math – those who generally like math usually appear to have lower math anxiety levels than those who dislike it (Kargar et al., 2010).

Math anxiety and Gender differences

Studies regarding adult populations have revealed females to have higher MA than males (Ferguson et al., 2015; Van Mier et al., 2019).

Girls showed greater anxiety than boys in the majority of the PISA participating nations (OECD, 2013). In addition, experts are increasingly of the opinion that MA develops in childhood (Vukovic et al., 2013). Studies examining gender-related MA in elementary education during the COVID-19 pandemic are few, nevertheless. Although some studies found no gender differences, other research has found that there exists a MA gender difference among students in elementary schools (Griggs et al., 2013). However, recent research by Puteh and Khalin (2016) found no differences in MA levels between male and female students. Yet, it is ambiguous if boy or girl primary school students experience different levels of MA (Hill et al., 2016).

Collectively, although previous reports on MA differences between genders have been based upon results obtained from secondary school samples, MA differences are likely to develop during primary school years or even earlier (Hill et al., 2016). For example, in the study of Mitchell and George (2022), the mathematics anxiety score for males was lower than that of female sixth grade students.

Ayuso et al. (2020) mentioned more negative feelings for girls than boys during primary education years in math classes, which was later confirmed by the research of Arnal-Palacián et al. (2022). Thus, according to Ayuso et al. (2020) during the teaching of mathematics in primary education, girls experience less positive emotions, which is in the same line as the conclusions of Hembree (1990). On the other hand, Sorvo et al. (2017) found no gender differences in second to fifth-grade students in Finland at the beginning of the school year and stated that math anxiety appears later in the school year in some situations during the lesson, as in question-and-answer exercises, where the researchers found that it was higher among girls than boys.

Effective learning of mathematics at in primary education is often a challenging situation for most school-aged children. The research of Mata et al. (2022) suggested that primary education students' emotions are complex and it is important to understand them to improve math learning outcomes.

Math anxiety and performance

Around seventy years ago research by Dreger and Aiken (1957) revealed mathematics anxiety and since then it still seems to majorly affect students' performance (Siaw et al., 2021). By common confession, MA is described as a predominantly negative emotion, which, in the literature, has been linked to a multitude of reactions. Cemen (1987) reports that students with math anxiety were so overwhelmed by negative emotions that they could not assimilate mathematical concepts, no matter how well someone explained them.

Since then, a plethora of studies have pointed out math anxiety as a significant criterion of learning mathematics and of low students' test-scores, for both standard and custom assessments, with an impact on school grades (e.g. Núñez-Peña et al., 2013; Foley et al., 2017; Zhang et al., 2019; Siaw et al., 2021) and performance (Gunderson et al., 2018; Yuan et al., 2023). For example, in their research on the effects of math anxiety on student achievement, Nunez-Pena et al. (2013) concluded that math anxiety is the most critical factor explaining low test scores of students at university.

Math anxiety is an important cause of math difficulties, as students who are highly apprehensive about math tend to fail math-related activities more often than students who do not experience such anxiety (Caviola et al., 2017). In addition, negative thoughts are evoked in the person, as they believe they may fail, and they are essentially confused when dealing with the negative thoughts and solving a math problem at the same time (Ashcraft & Kirk, 2001). Although moderate levels of stress can help to enhance motivation, excessive stress can become detrimental (Wang et al., 2015).

Moreover, a recent meta-analysis of 49 studies (Zhang et al., 2019) suggested a powerful negative link between math anxiety and performance, especially among older high school students. In addition, they found a more solid correlation between math anxiety and performance amid students from Asia than amid students from Europe or America. A cross-national study of Yuan et al. (2023) used a sample group of 17,284 fifteen-year-old students from the PISA-2012 database, regarding mathematics performance and MA, which firmly established negative effects of math anxiety in math performance in either lower or higher math achievement countries.

However, in reference to primary school children, the relationship between MA and mathematics performance remains unclear. Findings from previous studies are contradictory, with some studies suggesting that primary school students' MA is unrelated to achievement (e.g. Dowker et al., 2012; Haase et al., 2012). In particular, in their research, Dowker et al. (2012) studied the relationship between mathematics achievement, mathematics anxiety, unhappiness due to low achievement, willingness to engage in mathematics and self-assessment concerning primary school students studying in the third grade (ages 7-8) and in the fifth grade (ages 9-10). The students were asked to solve exercises that required numeracy skills, as well as to complete a questionnaire on attitude and anxiety towards mathematics. The results of the research showed that MA was not related to performance, in contrast to self-assessment, which showed a significant correlation. Meanwhile, other scholars suggested the existence of a negative correlation with mathematical performance at this age (e.g. Harari et al., 2013; Jameson, 2013; Vukovic et al., 2013; Wu et al., 2014; Villavicencio & Bernardo, 2016; Gabriel, 2022).

Similarly, Wu et al. (2014) investigated the relationship between early math ability, math anxiety, and internalizing-externalizing behaviors in a group of 366 second and third grade students. The research showed that math achievement is negatively correlated with MA. In regard to the gender differences in math performance, the UN published news in concerning the global fight for gender equality and opportunity, specifically in relation to gender differences in math performance, showing that girls are now performing as well as boys in the classroom when it comes to mathematics - despite the fact they are hampered by numerous hurdles (United Nations, 2022).

METHOD

Participants

The number of primary school students participating in the research was N=173. Four out of seven elementary public schools responded positively and finally participated in the research, after getting the necessary ethical approvals. Schools were selected randomly. Also, all students were attending sixth grade at public schools in Attica, Greece; in the research sample, 69 were male students and 104 were female students.

Procedure

The research was empirical and used the quantitative approach using as a tool a questionnaire consisting of 15 statements/items regarding MA and a test in mathematics containing exercises on fractions. In the MA questionnaire, a five-point Likert scale indicated whether the students “strongly agree” (weighted 5 points), “agree” (weighted 4 points), “maybe” (weighted 3 points), “disagree” (weighted 2 points) or “strongly disagree” (weighted 1 point). Then, the following formula was used to find the mean value of each statement for each student and for their gender add all the weights, then divide by 8, number of positive feelings statements, or by 7, number of negative feeling statements. In order to calculate the mean value of all negative feeling or positive feeling statements for the two genders, the mean values of all boys or girls were summed up and divided by 69 (number of boys in the sample) or by 104 (number of girls in the sample). At the end, the total scores for each feeling question were converted into a percentage, and rounded off to the nearest integer. In addition, the mean value of positive feelings and negative feelings was calculated for each gender.

Next the mean value of the 15 statements/items of MA questionnaire as well as the mean value of mathematics test for each student was calculated. For the interpretation of MA questionnaire's mean value, a higher mean value indicates higher math anxiety. In addition, the math test was formulated based on the teacher's textbook using similar items.

Instruments

The data was collected via a structured questionnaire related to the student experience of MA, and was administered in schools. The questionnaire consisted of two parts. Part A requested information**,** about gender and age, while the fifteen questions in Part B collected information about the MA level of the students; 7 questions on negative feelings and 8 questions on positive feelings. The questionnaire consisted of the following statements (1, 4, 5, 6, 8, 11, 13, 15 on positive feelings about mathematics and 2, 3, 7, 9, 10, 12, and 14 on negative feelings about math):

1. I have usually been at ease in math classes.
2. I see math as a subject I will rarely use.
3. I'm not good at math.
4. Generally, I have felt secure about doing math.
5. I'll need mathematics for my future work.
6. I usually get good grades in mathematics.
7. I don't think that I could do advanced math.
8. It wouldn't bother me at all to take more math classes.
9. Even though I study, math seems unusually hard for me.
10. I am unable to think clearly when working in mathematics.
11. Knowing mathematics will help earn a living.
12. Math has been my worst subject.
13. I think I could handle more difficult mathematics.
14. I'm not the type to do well in mathematics.
15. Math doesn't scare me at all.

The questionnaire was designed to measure student math anxiety and to examine math anxiety gender differences. Response options followed a 5-point Likert scale; Not at all – 1, 2, 3, 4, 5-Very much.

The next measurement scales were used as references: The Mathematics Anxiety Rating Scale (MARS), the Mathematics Anxiety Scale (MAS), and the Abbreviated Math Anxiety Scale (AMAS). For example number 6 “I usually get good grades in mathematics” derive from the statement “Waiting to get a mathematics test returned in which you expected to do well” of MARS survey. MARS and MAS are the most widely cited measurements used to explore math anxiety (Luttenberger et al., 2018) and the AMAS is acknowledged as suitable for testing math anxiety for students between 11 and 16 years old (Devine et al., 2012), while it can also work competently in contexts with cultural and linguistic diversity (Cipora et al., 2015).

In addition, the math test examined students’ both mathematical knowledge of fractions and how well they can use that knowledge to solve problems. The problem given to the students was “*five friends ordered two same size pizzas*” as seen in the Figure 2.

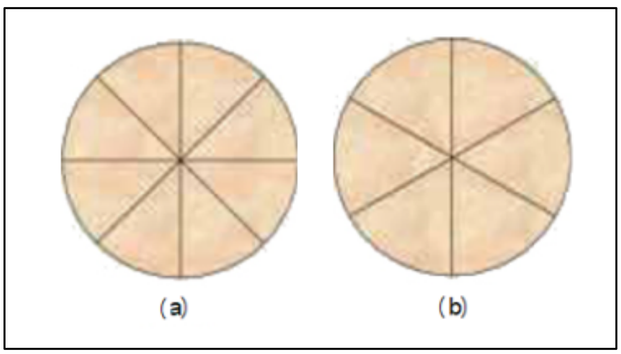


Figure 2: The shape of the two pizzas

A. *Vasilis ate 5 pieces of pizza (a), Georgios ate 3 pieces of pizza (b), and Margarita ate one piece of pizza (a).*

   1. Write the fraction of the pizza they ate separately.

      Vassilis ate…….of pizza (a), Georgios ate…….. of pizza (b), and Margarita ate …… of pizza (a).

   2. Write each fraction of the pizza they ate in ascending order using the less-than sign.

      Answer: ……………………………………………………………………

B. *Georgios ate $\frac{3}{8}$ of pizza (a) and Marcos ate $\frac{3}{6}$ pizza (b). Who ate more pizza?*

   Check the correct answer: I) Georgios II) Marcos

C. *Georgios ate* $\frac{3}{6}$ *of pizza (a) and Maria ate* $\frac{4}{8}$ *of pizza (b). Use number lines to compare the fractions.*

Check the correct answer: i. Georgios ate more

ii. They ate the same amount of pizza

iii. Maria ate more

D. *Vasilis ate* $\frac{2}{8}$ *of pizza (a) and* $\frac{3}{6}$ *of the same pizza.*

i. Total part of pizza eaten by Vasilis:

Check the correct answer: A. $\frac{5}{14}$ B. $\frac{3}{4}$ C. $\frac{6}{48}$

ii. Maria said "Only a slice of pizza (a) is left".

I. Agree II. Disagree

Data analysis

Statistical tests were conducted, one non-parametric test and two parametric tests. The non-parametric Kruskal-Wallis test was applied to assess significant differences in the dependent variable "math test score" by the independent variable "math anxiety". Two independent t-tests were conducted for the dependent variables "Mean scores of Negative feeling" and "Mean scores of Positive feeling" across the independent variable "Students' Gender".

The SPSS 21 statistical software was the major statistical tool processing the data while the Excel application was used for the other calculations; the percentages and the mean values.

Reliability and Validity

It was determined to establish the validity and reliability of the research instrument because it had been altered to fit the research. The criteria set for the evaluation survey questionnaire put out by Good and Scates (1954) followed in determining the validity of the research instrument. Good and Scates instrument had eight criteria (this research used an instrument of ten criteria, see Appendix 3) for validation of research instruments by experts: 1) Is the question on the subject? 2) Is the question perfectly clear and unambiguous? 3) Does the question get at something stable? Something relatively deep-seated, well-considered, non-superficial, and not ephemeral, but something which is typical of the individual or the situation? 4) Does the question pull? 5) Do the responses show a reasonable range of variation? 6) Is the information obtained consistent? 7) Is the item sufficiently inclusive? 8) Is there a possibility of using an external criterion to evaluate the questionnaire?

The validity of the research tool was evaluated by three experts, yielding a validity rating of 4.55; mean value of the eleven number-responses. This validity index shows that the redesigned research instrument had excellent levels of validity. According to Tavakol and Dennick (2011), the alpha range should be between 0.7 and 0.9. Cronbach alpha computed and determined reliability alpha overall coefficient to be 0.87. Cronbach alpha for negative feelings was 0.88 and for positive feelings Cronbach alpha was 0.86. These reliability coefficients demonstrated a high level of instrument reliability.

RESULTS

Results for MA

Primarily, the questionnaire was designed to provide information about the MA at the primary educational level. Secondarily, but not of lesser importance, the questionnaire was used to gather data regarding the gender differences with regard to MA. The following Figure 3 demonstrates that girls have lower confidence than boys concerning their math skills.

The results concerning positive feelings about mathematics statements 1, 4, 5, 6, 8, 11, 13, 15), show that girls have a lower score than boys. However, in statement 4, "Generally I have felt secure about trying math", as well as in statement 6, "I usually get good grades in mathematics" the girls' percentage is quite high, demonstrating that girls are competitive. On the other hand, it is worth noting that girls scored very low compared to boys in all positive-feeling statements, hence, reporting greater emotional stress towards mathematics.

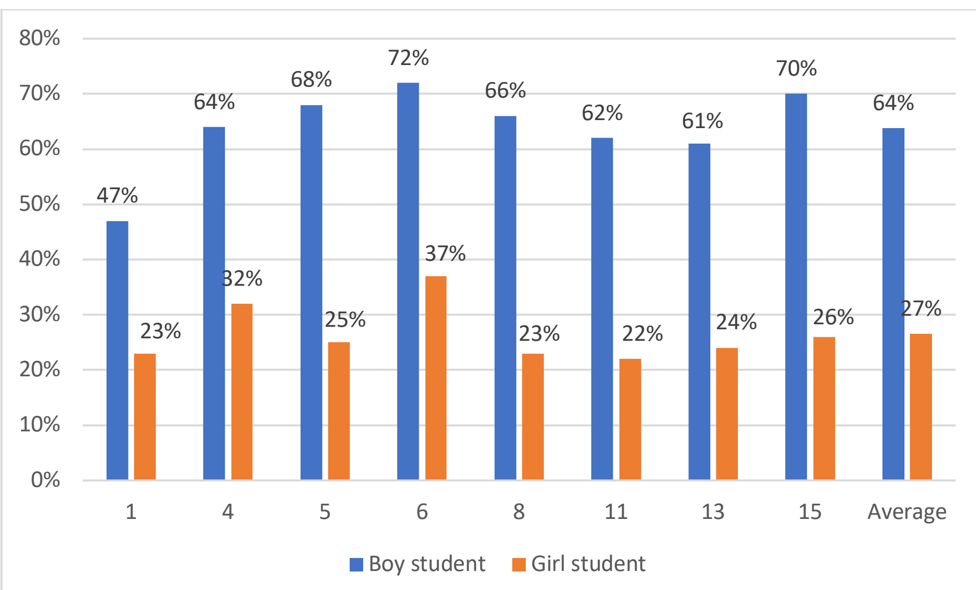


Figure 3: Gender percentages for the positive-feeling statements

In coherence with the previous results, the bar graph in Fig. 4 revealed a high percentage of negative feelings for the girls suggesting a 'handicap' in relation to math skills. It can be seen from the results of the negative-feeling scores about math statements 2, 3, 7, 9, 10, 12, 14) in Fig. 4. Girls have a much higher score than boys. Therefore, the results depicted in Figure 4 support Figure 3 findings associated with the undoubtedly negative influence math anxiety has on girl students. Furthermore, this suggests a strong negative emotional reaction towards math on behalf of girls. It can be seen that the percentages of statement 10, "I am unable to think clearly when working in mathematics" are of great interest; the corresponding percentage of girls was 65% and that of boys was 32%. It may well be argued that these high percentages revealed that both genders had experienced difficulties when they were dealt with math assignments.

Moreover, the girls' high-percentage score (74%) in statement 9, "Even though I study, math seems unusually hard for me", highlights issues relevant to the instruction of mathematics in sixth grade. Therefore, issues related to instructional problems are a challenge for teachers to consider and deal with. Although the results showed a higher rate of mathematical anxiety for girls than boys, even boys were exposed to a moderate level of math anxiety.

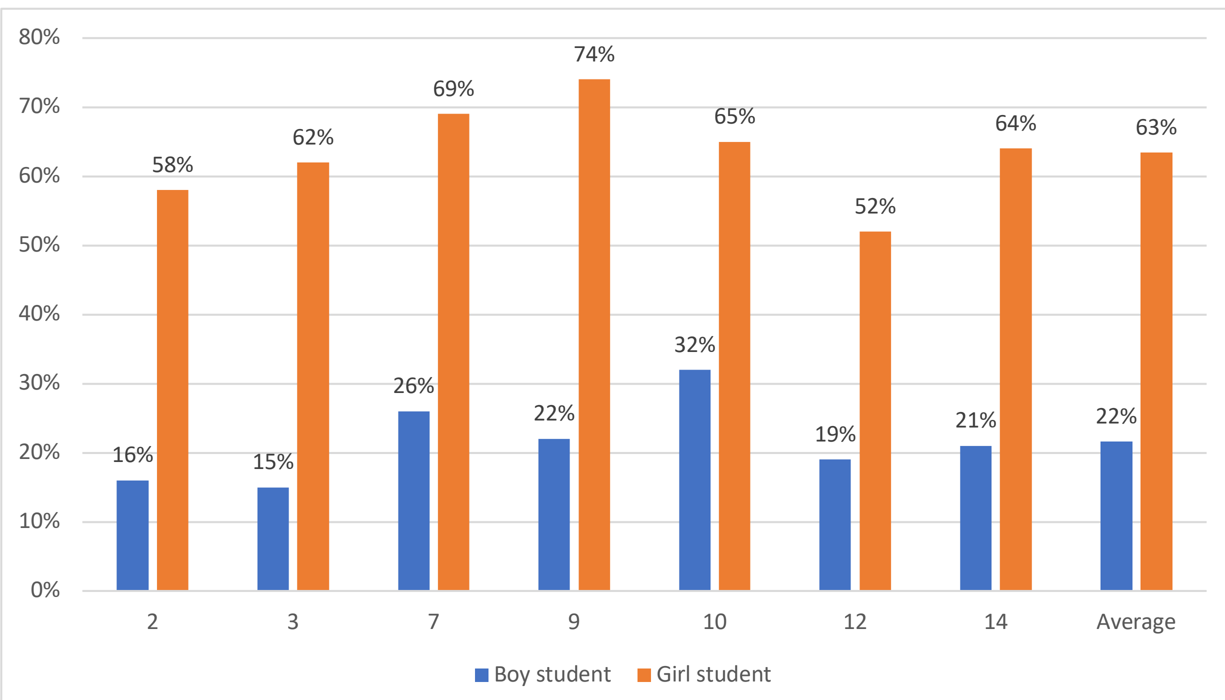


Figure 4: Gender percentages for the negative-feeling statements

Next, Table 1 summarizes the results of all the previous data from Figure 3 and Figure 4 concerning the percentages for each statement of positive and negative feelings about mathematics for both genders.

Statement	Emotion	Boy	Girl
1. I have usually been at ease in math classes.	positive	47%	23%
2. I see math as a subject I will rarely use.	negative	16%	58%
3. I'm not good at math.	negative	15%	62%
4. Generally, I have felt secure about doing math.	positive	64%	32%
5. I'll need mathematics for my future work.	positive	68%	25%
6. I usually get good grades in mathematics.	positive	72%	37%
7. I don't think that I could do advanced math.	negative	26%	69%
8. It wouldn't bother me at all to take more math classes.	positive	66%	23%
9. Even though I study, math seems unusually hard for me.	negative	22%	74%
10. I am unable to think clearly when working in mathematics.	negative	32%	65%
11. Knowing mathematics will help earn a living.	positive	22%	62%
12. Math has been my worst subject.	negative	19%	52%
13. I think I could handle more difficult mathematics.	positive	61%	24%
14. I'm not the type to do well in mathematics.	negative	21%	64%
15. Math doesn't scare me at all.	positive	70%	26%

Table 1: Display (%) of gender emotions about Mathematics

Results of Gender math anxiety test

For a better understanding of the results, the percentages of the mean value of positive and negative responses were calculated for each gender. By looking at Figure 5, it is obvious that math anxiety exists among sixth grade students. Meanwhile, as it appears in Figure 5, positive feelings about math were a lot higher for boys (55% boys - 27% girls) in contrast to negative feelings (63% girl students - 22% boy students). These results determined that there is a positive relationship between math anxiety and gender, female students exhibiting higher negative feelings than their male student counterparts.

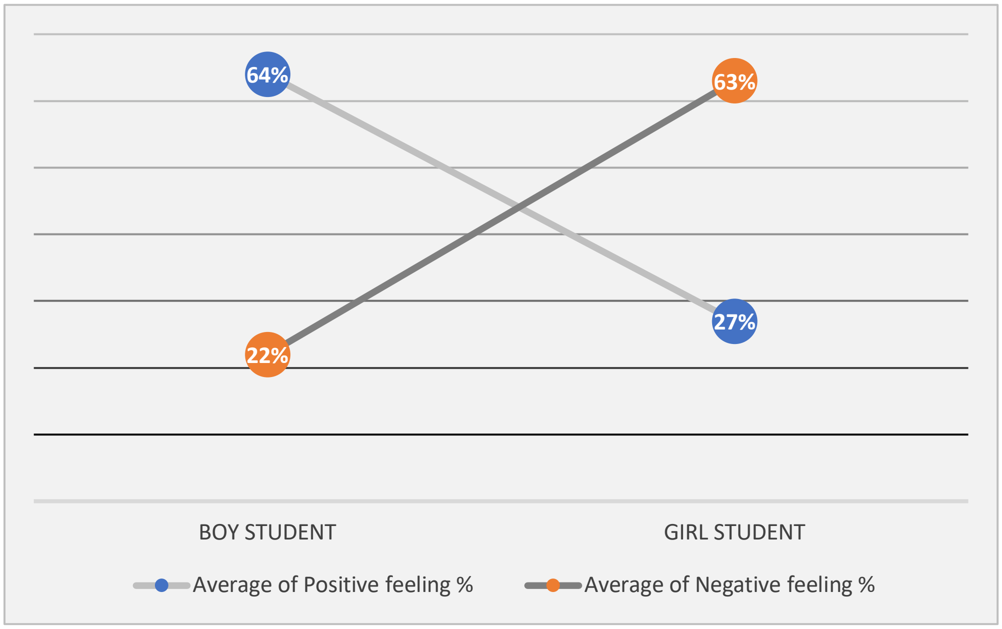


Figure 5: Gender positive and negative feeling average (%) values

For a statistical assessment of the differences between the two genders, two independent t-tests were conducted, as the data followed the normal distribution (Table 2). To check the normality for the dependent variables "Mean Score of Negative feeling" and "Mean Score of Positive feeling" across the two levels of the independent variable "Students' Gender", boy students and girl students, the Shapiro-Wilk test was applied to the sample.

	Students' Gender	Shapiro-Wilk
Mean Score of Positive feeling	Girl Students	0.087
	Boy Students	0.095
Mean Score of Negative feeling	Girl Students	0.075
	Boy Students	0.064

Table 2: Normality tests for both Negative and Positive feelings

Moreover, there were no outliers (Figure 6).

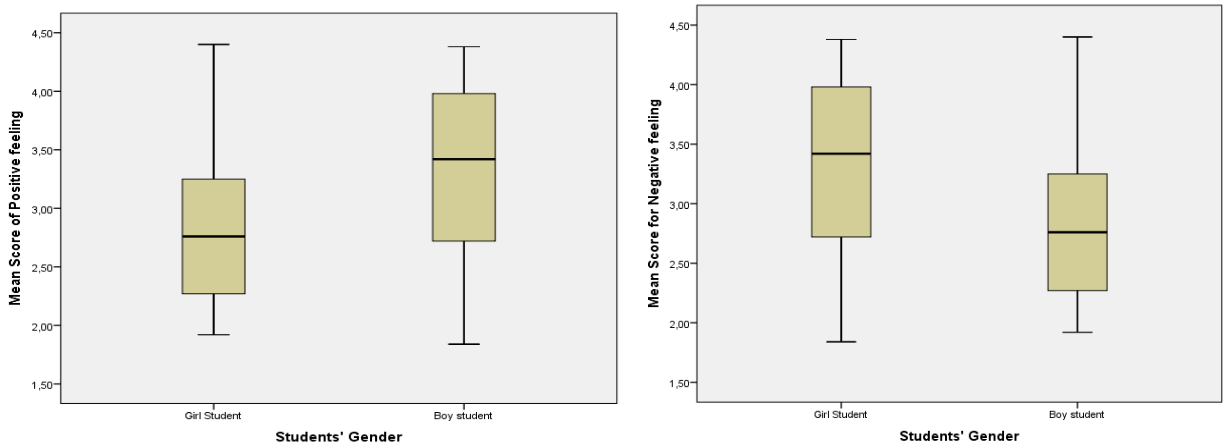


Figure 6: Boxplots for both "Mean Score" variables

Levene's test assumption for equality of variances was found to be violated for the present analysis; for the variable "Mean scores of Negative feeling" the significance of Levene's test was 0.012< 0.05 and for the variable "Mean scores of Positive feeling", 0.014< 0.05. Due to this violation, independent t-tests not assuming homogeneity of variance were computed.

The results of the independent t-test samples showed there was a significant difference in the Mean scores of Negative feeling for Girl Students (M=3.32, SD=0.77) and Boy students (M=2.91, SD=0.67). This test was statistically significant, t (171) = 3.84, $p < .05$; d=0.56. The effect size for this analysis (d = 0.56) was found to be Cohen's (1988) convention for a medium effect (d=.05). These results indicate that boy students experienced less negative feeling than did girls did.

Next, the t-test results of the independent samples showed there was a significant difference in the Mean scores of Positive feeling for Girl Students (M=2.89, SD=0.65) and Boy students (M=3.38, SD=0.75). This test was statistically significant, t(171) = - 4.6, p < .05; d=0.71. The effect size for this analysis (d = 0.71) was found to be Cohen's (1988) convention for a large effect (d=.08). These results indicate that girl students experienced less positive feeling than boys did.

Also, because the 95% CI (Confidence Interval) of the difference does not include zero; lower limit at 0.207 and upper limit at 0.648- in the "Mean scores of Negative feeling", it can be derived as a conclusion that the difference in the negative feeling between the two genders does exist in the population. Moreover, the test for the variable "Mean scores of Positive feeling" returned the same result, as the lower limit of CI was at -0.710 and the upper limit was at -0.284.

The following Table 3 describes the results of the two independent t-tests of the variables "Mean scores of Negative feeling" and "Mean scores of Positive feeling" across the variable "Students' Gender".

	Students'bGender	*M*	*SD*	*p*	*Cohen's d*
Mean Score of Negative feeling	Girl Students	3.32	0.77	0.00	0.56
	Boy Students	2,91	0.67		
Mean Score of Positive feeling	Girl Students	2.89	0.65	0.00	0.71
	Boy Students	3.38	0.75		

Table 3: The results of the two independent t-tests

Results of Math test score and MA score

The reliability of all 15 statements was good giving a Cronbach alpha of 0.865.

The Shapiro-Wilk normality test determined for both continuous variables Math test score and MA score, that the p-values were more than 0.05 (Table 4). Thus, the distributions of math test scores for these five "New Negative Feeling" groups are not normally distributed. For that reason, a non-parametric statistical test was suitable to analyze the data.

Variable	*p*
MA score	.088
Math test score	.344

Table 4: The Shapiro-Wilk normality test

Additionally, the scatterplot suggested that there was a linear relationship between the continuous variables, Math test score and MA score, and that the assumption of homoscedasticity was not violated. A Pearson's correlation analysis indicated that there was a very strong, negative, and significant correlation between Math test score and MA score, r (171) = - 0.956, $p < 0.01$ (Table 5).

Variable	M	SD	1	2
1. MA score	2.89	0.89	--	
2. Math test score	7.29	1.30	-0.956	--

Table 5: Pearson's correlation analysis

Next an ANCOVA test was conducted giving a significant difference in mean Math test score, with F (1,170) = 4.608, p=0.006, between the two genders, whilst adjusting for MA score. The partial Eta Squared value was 0.81 indicating a large effect size that is gender explains approximately 81% of the variance in the dependent variable Math test score.

On the whole, these findings imply that gender has a significant effect on the dependent variable Math test score when considering the effects of MA score. This underscores the significant impact of gender on the variable Math test score when accounting for the influence of the variable MA score.

DISCUSSION

Mathematics is a discipline studied almost all over the world. However, research shows that some students don't like to study the subject because they believe it is difficult. The research complements pieces of the puzzle called MA, looking at this phenomenon from the Primary Education's point of view, where there is a gap in literature during the COVID-19 pandemic.

According to the study, the results strongly indicated that MA is an existing phenomenon in children who attend Primary School (Beilock et al., 2010; Vukovic et al., 2013; Wu et al., 2014; Hill et al., 2016), specifically eleven to twelve-year-old children. This emotional state appeared when students crossed the line of the Growth Zone and entered the Anxiety Zone in the *Growth Zone Model* (Cipora et al., 2015). Thus, the importance of teaching Mathematics in a more organized manner is highlighted.

In addition, the research findings are in agreement with the results of various studies which determined significant differences in the levels of MA according to gender, with the girl students being more vulnerable to the effects of MA (e.g. Griggs, et al., 2013; Bieg et al., 2015; Ferguson et al., 2015; OECD, 2015; Foley, et al. 2017; Van Mier et al., 2019). This result is in line with the conclusions of Coronado-Hijón (2017), Ayuso et al. (2020) and Arnal-Palacián et al. (2022), MA being a significant factor**,** negatively affecting overall girl students' feelings. However, the research findings contradict the findings of Puteh and Khalin (2016), who found no significant differences between the genders.

Furthermore, this study revealed unfavorable effects of math anxiety in math performance, which are consistent with the findings of Yuan et al. (2023) regarding mathematics performance and MA. According to numerous studies (e.g. Nunez-Pena et al., 2013; Foley et al., 2017; Gunderson et al., 2018; Zhang et al., 2019; Siaw et al., 2021; Yuan et al., 2023), MA is a major cause of issues relevant to learning mathematical concepts. This is because it inhibits students' performance. Moreover, according to the study, boy students performed higher than girl students in the math test, which is contrary to assertions made by the United Nations (United Nations, 2022). From the previous discussion the research question was effectively addressed by the study, as it investigated the impact of MA on the academic performance of sixth-grade students in primary education during the COVID-19 pandemic.

Summarizing the results, the study confirmed the presence of MA among eleven to twelve-year-old children, highlighting the emotional distress experienced when students transitioned from the Growth Zone to the Anxiety Zone in the Growth Zone Model. Furthermore, it was found that there were considerable differences in math anxiety between genders; with girls being more susceptible to it. The study also showed that there is a negative relationship between MA and math performance, which means that any presence of math anxiety has an inhibitory effect on learning mathematics. These results demonstrate the significance of structured and supportive mathematics teaching in elementary schools and the importance of specific approaches for helping children overcome math anxiety.

The research made significant progress towards addressing the research question by providing empirical evidence of the impact of math anxiety on academic performance in sixth-grade students. In addition, it expands upon the existing body of literature by validating the presence of MA among sixth graders and its negative impact on math performance, notably among female students. Additionally, the study illuminated the interplay between gender, math anxiety, and academic performance, underscoring the significance of targeted interventions in order to mitigate MA's negative effects.

A limitation to the study was certain school Principals' denial to participate in the research, the reason for the denial of access to these schools being, as suspected, the concern about a health threat to the school population. In addition, the study examined one of the three sub-categories (Luttenberger et al., 2018) of MA that nearly all researchers agree upon. So, future studies could focus on all three MA sub-categories or on each one separately.

Due to the fact that MA negatively impacts math learning, even in online environments, the education systems must develop effective strategies to strengthen the self-esteem of students. In addition, since it seems that female students have higher MA than male students, appropriate approaches should be deployed focusing on the characteristics of the students and their learning needs. Moreover, studies should be focused on interventions to minimize MA. Finally, additional research should be undertaken on the nature of MA in order to find out the origins of and its effects on other aspects.

CONCLUSION

The research results show that MA can significantly affect math performance of sixth graders; specifically among girls. It is important to put in place teaching methods that take care of different learning styles and solve specific problems for students affected by MA; however, by understanding MA' s nature and consequences educators can include interventions fostering primary school learning milieu for success in mathematics.

Acknowledgment
This research did not receive support from any funding agency

Disclosure statement
No potential conflict of interest was reported by the author.

Data availability statement (DAS)
Upon request, the corresponding author will supply the data supporting the study's conclusions upon reasonable inquiry

References

[1] Abbasi, M., Samadzadeh, M., & Shahbazzadegan, B. (2013). Study of Mathematics Anxiety in High School Students and it's Relationship with Self-esteem and Teachers' Personality Characteristics. *Procedia - Social and Behavioral Sciences*, *83*, 672-677, https://doi.org/10.1016/j.sbspro.2013.06.127

[2] Aguillon, S., Siegmund, G-F., Petipas, R., Drake, A., Cotner, S., & Ballen, C. (2020). Gender Differences in Student Participation in an Active-Learning Classroom. *CBE—Life Sciences Education*, *19*, Article 2. https://doi.org/10.1187/cbe.19-03-0048

[3] Arnal-Palacián, M., Arnal, A., & Blanco, C. (2022). Math Anxiety in Primary Education during Covid-19 Confinement: Influence on Age and Gender. *Acta Scientiae*, *24*(1), 145-170. https://doi.org/10.17648/acta.scientiae.6745

[4] Ashcraft, M. H., & Kirk, E. P. (2001). The relationships among working memory, math anxiety, and performance. *Journal of Experimental Psychology: General, 130,* 224–237. https://doi.org/10.1037/0096-3445.130.2.224

[5] Ayuso, N., Fillola, E., Masiá, B., Murillo, A. C., Trillo-Lado, R., Baldassarri, S., Cerezo, E., Ruberte, L., Mariscal, D.M., & Villarroya-Gaudó, M. (2020). Gender Gap in STEM: A Cross-Sectional Study of Primary School Students' Self-Perception and Test Anxiety in Mathematics. *IEEE Transactions on Education*, *64*(1), 40-49. https://doi.org/10.1109/TE.2020.3004075

[6] Bao, W. (2020). COVID-19 and Online Teaching in Higher Education: A Case Study of Peking University Human. *Behavior and Emerging Technologies*, *2*, 113–115. https://doi.org/10.1002/hbe2.191

[7] Beilock, S.L., Gunderson, E.A., Ramirez, G., & Levine, S.C. (2010). Female teachers' math anxiety affects girls' math achievement. *Proceedings of the National Academy of Sciences of the United States of America*, *107* (5), 1860-1863. https://doi.org/10.1073/pnas.0910967107

[8] Bertoletti, A., Biagi, F., Di Pietro, G., & Karpiński, Z. (2023).The effect of the COVID-19 disruption on the gender gap in students' performance: a cross-country analysis. *Large-scale Assessments in Education*, *11*, Article 6. https://doi.org/10.1186/s40536-023-00154-y

[9] Bieg, M., Goetz, T., Wolter, I., & Hall, N. (2015). Gender stereotype endorsement differentially predicts girls' and boys' trait-state discrepancy in math anxiety. *Frontiers in Psychology*, *6*, Article 1404. https://doi.org/10.3389/fpsyg.2015.01404

[10] Bozkurt, A., Koseoglu, S., & Singh, L. (2019). An analysis of peer reviewed publications on openness in education in half a century: Trends and patterns in the open hemisphere. *Australasian Journal of Educational Technology*, *35*(4), 68-97. https://doi.org/10.14742/ajet.4252

[11] Bozkurt, A., & Sharma, RC. (2020). Education in normal, new normal, and next normal: Observations from the past, insights from the present and projections for the future. *Asian Journal of Distance Education*, *15*(2), 1–5. https://doi.org/10.5281/zenodo.4362664

[12] Caviola, S., Primi, C., Chiesi, F., & Mammarella, I. C. (2017). Psychometric properties of the Abbreviated Math Anxiety Scale (AMAS) in Italian primary school children. Learning and Individual Differences, 55, 174– 182. doi: https://doi.org/10.1016/j.lindif.2017.03.006

[13] Cemen, P. B. (1987). *The nature of mathematics anxiety*. Oklahoma State University. https://eric.ed.gov/?id=ED287729

[14] Cipora, K., Szczygiel, M., Willmes, K., & Nuerk, H.C. (2015). Math anxiety assessment with the abbreviated math anxiety scale: Applicability and usefulness: Insights from the Polish adaptation. *Frontiers in Psychology*, *6*, Article 1833. https://doi.org/10.3389/fpsyg.2015.01833

[15] Cohen, J. (1988). *Statistical power and analysis for the behavioral sciences* (2nd ed.). Hillsdale, NJ: Lawrence Erlbaum Associates.

[16] Coronado-Hijón, A. (2017). The Mathematics Anxiety: a transcultural perspective. *Procedia - Social and Behavioral Sciences*, 237, 1061-1065. https://doi.org/10.1016/j.sbspro.2017.02.155

[17] Crompton, H., Burke, D., Jordan, K., & Wilson, S. W. G. (2021). Learning with technology during emergencies: A systematic review of K-12 education. *British Journal of Educational Technology*, *52*, 1554–1575. https://doi.org/10.1111/bjet.13114

[18] Devine, A., Fawcett, K., Szucs, D., & Dowker, A. (2012). Gender differences in mathematics anxiety and the relation to mathematics performance while controlling for test anxiety. *Behavioral and Brain Functions*, *8*(33), 1–9. https://doi.org/10.1186/1744-9081-8-33

[19] Devine, A., Hill, F., Carey, E., & Szűcs, D. (2018). Cognitive and emotional math problems largely dissociate: Prevalence of developmental dyscalculia and mathematics anxiety. *Journal of Educational Psychology*, *110*(3), 431-444. http://dx.doi.org/10.1037/edu0000222

[20] Di Pietro, G. (2023). The impact of Covid-19 on student achievement: Evidence from a recent meta-analysis. *Educational Research Review*, *39*, Article 100530. https://doi.org/10.1016/j.edurev.2023.100530

[21] Dowker, A., Ashcraft, M., & Krinzinger, H. (Eds.). (2012). The Development of Attitudes and Emotions Related to Mathematics [Special issue]. *Child Development Research,* Article 124939. https://doi.org/10.1155/2012/238435

[22] Dowker, A., Sarkar, A., & Looi, C. Y. (2016). Mathematics Anxiety: What Have We Learned in 60 Years? *Frontiers in Psychology*, *7*, Article 508. https://doi.org/10.3389/fpsyg.2016.00508

[23] Drabowicz, T. (2014). Gender and digital usage inequality among adolescents: A comparative study of 39 countries. *Computers & Education, 74*, 98-111. https://doi.org/10.1016/j.compedu.2014.01.016

[24] Dreger, R. M., & Aiken, L. R., Jr. (1957). The identification of number anxiety in a college population. *Journal of Educational Psychology*, *48*(6), 344–351. https://doi.org/10.1037/h0045894

[25] Ferguson, A.M., Maloney, E.A., Fugelsang, J., & Risko, E.F. (2015). On the relation between math and spatial ability: The case of math anxiety. *Learning and Individual Differences*, *39*, 1–12. https://doi.org/10.1016/j.lindif.2015.02.007

[26] Flores, M. A., & Gago, M. (2020). Teacher Education in Times of COVID-19 Pandemic in Portugal: National, Institutional and Pedagogical Responses. *Journal of Education for Teaching*, *46*(4), 507-516. https://doi.org/10.1080/02607476.2020.1799709

[27] Foley, A., Herts, J., Borgonovi, F., Guerriero, S., Levine, S., & Beilock, S. (2017). The Math Anxiety-Performance Link: A Global Phenomenon. *Current Directions in Psychological Science*, *26* (1), 52–58. https://doi.org/10.1177/09637214166724

[28] Fore, H. H. (2020). A Wake-Up Call: COVID-19 and its Impact on Children's Health and Wellbeing. *Lancet Global Health*, *8*(7), 861 – 862. https://doi.org/10.1016/S2214-109X(20)30238-2

[29] Gabriel, F. (2022). Maths Anxiety – And How to Overcome it. *Significance*, *19*(1), 34–35. https://doi.org/10.1111/1740-9713.01612

[30] Ghosh, R., Dubey, M. J., Chatterjee, S., & Dubey, S. (2020). Impact of COVID-19 on children: special focus on the psychosocial aspect. *Minerva Pediatrica*, *72*(3), 226-235. https://doi.org/10.23736/S0026-4946.20.05887-9

[31] Good, C., & Scates, D. (1954). *Methods of Research: Educational, Psychological and Sociological*. New York: Appleton-Century-Crofts.

[32] Griggs, M.S., Rimm-Kaufman, S.E., Merritt, E.G., & Patton, C.L. (2013). The Responsive Classroom approach and fifth grade students' math and science anxiety and self-efficacy. *School Psychology Quarterly*, *28*(4), 360–373. https://doi.org/10.1037/spq0000026

[33] Gunderson, E. A., Park, D., Maloney, E. A., Beilock, S. L., & Levine, S. C. (2018). Reciprocal relations among motivational frameworks, math anxiety, and math achievement in early elementary school. *Journal of Cognition and Development, 19*(1), 21–46. https://doi.org/10.1080/15248372.2017.1421538

[34] Haase, V. G., Júlio-Costa, A., Pinheiro-Chagas, P., Oliveira, L. F. S., Micheli, L. R., & Wood, G. (Eds.) (2012). Math self-assessement, but not negative feelings, predicts mathematics performance of elementary school children [Special issue]. *Child Development Research*, *10*(1). https://doi.org/10.1155/2012/982672

[35] Hanushek, E. & Woessmann, L. (2020, September). *The economic impacts of learning losses* (OECD Education Working Papers No. 225). https://doi.org/10.1787/21908d74-en

[36] Harari R. R., Vukovic R. K., Bailey S. P. (2013). Mathematics anxiety in young children: an exploratory study. *The Journal of Experimental Education*, *81*(4), 538-555. https://doi.org/10.1080/00220973.2012.727888

[37] Hembree, R. (1990). The Nature, Effects, and Relief of Mathematics Anxiety. *Journal for Research in Mathematics Education*, *21*(1), 33-46. https://doi.org/10.2307/749455

[38] Hill, F., Mammarella, I. C., Devine, A., Caviola, S., Passolunghi, M. C., & Szûcs D. (2016). Maths anxiety in primary and secondary school students: gender differences, developmental changes and anxiety specificity. *Learning and Individual Differences*, *48*, 45–53. https://doi.org/10.1016/j.lindif.2016.02.006

[39] Hodges, C., Moore, S., Lockee, B., Trust, T., & Bond, A. (2020). The Difference between Emergency Remote Teaching and Online Learning. *EDUCAUSE Review*. https://er.educause.edu/articles/2020/3/the-difference-between-emergency-remote-teaching-and-online-learning

[40] Jameson, M. M. (2013). The Development and Validation of the Children's Anxiety in Math Scale. *Journal of Psychoeducational Assessment*, *31*(4), 391–395. https://doi.org/10.1177/0734282912470131

[41] Johnston-Wilder, S., Lee, C., Garton, L., Goodlad, S., & Brindley, J. (2013, November 18-20). *Developing Coaches for Mathematical Resilience*. ICERI 2013: 6th International Conference on Education, Research and Innovation, Seville, Spain. http://oro.open.ac.uk/38989/.

[42] Kargar, M., Tarmizi, R.A., & Bayat, S. (2010). Relationship between Mathematical Thinking, Mathematics Anxiety and Mathematics Attitudes among University Students. Procedia Social and Behavioral Sciences *8*(3), 537-542. https://doi.org/10.1016/j.sbspro.2010.12.074.

[43] Khasawneh, E., Gosling, C., & Williams, B. (2021). What impact does maths anxiety have on university students? *BMC Psychology*, *9*, Article 37. https://doi.org/10.1186/s40359-021-00537-2

[44] Krinzinger, H., Kaufmann, L., & Willmes, K. (2009). Math Anxiety and Math Ability in Early Primary School Years. *Journal of Psychoeducational Assessment*, 27(3), 206–225. https://doi.org/10.1177/0734282908330583

[45] Lee, J. (2020). Mental health effects of school closures during COVID-19. *The Lancet Child and Adolescent Health*, *4*(6). https://doi.org/10.1016/S2352-4642(20)30109-7

[46] Liu, J. J., Bao, Y., Huang, X., Shi, J., & Lu, L. (2020). Mental health considerations for children quarantined because of COVID-19. *The Lancet Child & Adolescent Health*, *4*(5), 347 – 349. https://doi.org/10.1016/S2352-4642(20)30096-1

[47] Luttenberger, S., Wimmer, S., & Paechter, M. (2018). Spotlight on math anxiety. *Psychology research and behavior management*, *11*, 311–322. https://doi.org/10.2147/PRBM.S141421

[48] Mata, L., Monteiro, V., Peixoto, F., Santos, N. N., Sanches C., & Gomes, M. (2022). Emotional profiles regarding maths among primary school children – A two-year longitudinal study. *European Journal of Psychology of Education, 37*(1), 391–415. https://doi.org/10.1007/s10212-020-00527-9

[49] Mitchell, L., & George, L. (2022). Exploring mathematics anxiety among primary school students: Prevalence, mathematics performance and gender. *International Electronic Journal of Mathematics Education*, 17(3), Article em0692. https://doi.org/10.29333/iejme/12073

[50] Moorhouse, B. L. (2020). Adaptations to a face-to-face initial teacher education course 'forced' online due to the COVID-19 pandemic. *Journal of Education for Teaching*, *46*(4), 609–611. https://doi.org/10.1080/02607476.2020.1755205

[51] Núñez-Peña, M.I., Suárez-Pellicioni, M., & Bono, R. (2013). Effects of Math Anxiety on Student Success in Higher Education. *International Journal of Educational Research, 58*, 36-43. https://doi.org/10.1016/j.ijer.2012.12.004

[52] OECD (2001). Learning to change: ICT in schools. Paris: Organisation forEconomic Co-operation and Development. https://www.oecd.org/digital/learningtochangeictinschools.htm

[53] OECD (2013*). PISA 2012 Results: Ready to Learn: Students' Engagement, Drive and Self-Beliefs*, (Volume III). Paris: OECD Publishing. https://doi.org/10.1787/19963777

[54] OECD (2015). *The ABC of Gender Equality in Education: Aptitude, Behaviour, Confidence*. Paris: OECD Publishing.

[55] O'Leary, D., Fitzpatrick, C., & Hallett, D. (2017). Math Anxiety Is Related to Some, but Not All, Experiences with Math. *Frontiers in Psychology*, *8*, Article 2067. https://doi.org/10.3389/fpsyg.2017.02067

[56] Passolunghi, M.C. (2011). Cognitive and Emotional Factors in Children with Mathematical Learning Disabilities. *International Journal of Disability Development and Education 58*(1), 61–73. https://doi.org/10.1080/1034912X.2011.547351

[57] Patrinos, H.A. (2022). Learning loss and learning recovery. *Decision, 49*(2), 183–188. https://doi.org/10.1007/s40622-022-00317-w

[58] Puteh, M., & Khalin, S.Z. (2016). Mathematics Anxiety and Its Relationship with the Achievement of Secondary Students in Malaysia. International Journal of Social Science and Humanity *6* (2), 119-122. https://doi.org/10.7763/IJSSH.2016.V6.630

[59] Phelps, C., & Sperry, L. L. (2020). Children and the COVID-19 pandemic. *Psychological Trauma: Theory, Research, Practice, and Policy*, *12*(S1), 73–75. 5 ISSN: 1942-9681 http://dx.doi.org/10.1037/tra0000861

[60] Ramirez, G. Gunderson, E. A., Levine, S. C., & Beilock, S. L. (2013). Math Anxiety, Working Memory, and Math Achievement in Early Elementary School. *Journal of Cognition and Development*, *14*(2), 187-202. https://doi.org/10.1080/15248372.2012.664593

[61] Rubinsten, O., & Tannock, R. (2010). Mathematics anxiety in children with developmental dyscalculia. *Behavioral and Brain Functions*, *6*(1), 46-58. https://doi.org/10.1186/1744-9081-6-46.

[62] Shapiro, M. O., Gros, D. F., & McCabe, R. E. (2020). Intolerance of Uncertainty andSocial Anxiety while Utilizing a Hybrid Approach to Symptom Assessment. *International Journal of Cognitive Therapy*, *13*(1), 189-202. https://doi.org/10.1007/s41811-020-00068-5

[63] Schleicher, A. (2020). *The Impact of COVID-19 on Education: Insights from "Education at a Glance 2020"*. OECD Publishing. https://eric.ed.gov/?id=ED616315

[64] Siaw, E.S., Shim, G.T.G., Azizan, F.L., & Shaipullah, N.M. (2021). Understanding the relationship between students' mathematics anxiety levels and mathematics performances at the foundation level. *Journal of Education and Learning*, *10*(1), 47–54. https://doi.org/10.5539/jel.v10n1p47

[65] Sorvo, R., Koponen, T., Viholainen, H., Aro, T., Räikkönen, E., Peura, P., Dowker, A., & Aro, M. (2017). Math anxiety and its relationship with basic arithmetic skills among primary school children. *The British Journal of Educational Psychology*, *87*(3), 309-327. https://doi.org/10.1111/bjep.12151

[66] Spinelli, M., Lionetti, F., Pastore, M., & Fasolo, M. (2020). Parents' stress and children's psychological problems in families facing the COVID-19 outbreak in Italy. *Frontiers in Psychology*, *11*. https://doi.org/10.3389/fpsyg.2020.01713

[67] Tavakol, M., & Dennick, R. (2011). Making sense of Cronbach's alpha. *International journal of medical education*, *2*, 53–55. https://doi.org/10.5116/ijme.4dfb.8dfd

[68] Villavicencio, F. T., & Bernardo, A. B. I. (2016). Beyond math anxiety: Positive emotions predict mathematics achievement, self-regulation, and self-efficacy. *Asia Pacific Education Researcher*, *25*(3), 415-422. https://doi.org/10.1007/s40299-015-0251-4

[69] Vukovic, R. K., Kieffer, M. J., Bailey, S. P., & Harari, R. R. (2013). Mathematics anxiety in young children: Concurrent and longitudinal associations with mathematical performance. *Contemporary Educational Psychology*, *38*(1), 1-10. https://doi.org/10.1016/j.cedpsych.2012.09.001

[70]UNESCO (2020). Education: From disruption to recovery. https://en.unesco.org/covid19/educationresponse

[71] United Nations (2022, April 27). *Girls' performance in maths 'starting to add up to boys', says UNESCO*. UN News. https://news.un.org/en/story/2022/04/1117082

[72] Van Mier, H.I., Schleepen, T.M.J., Van den Berg, F.C.G. (2019). Gender Differences Regarding the Impact of Math Anxiety on Arithmetic Performance in Second and Fourth Graders. *Frontiers in Psychology*, *9*, Article 2690. https://doi.org/10.3389/fpsyg.2018.02690

[73] Wang, Z., Lukowski, S. L., Hart, S. A., Lyons, I. M., Thompson, L. A., Kovas, Y., Mazzocco, M. M. M., Plomin, R., & Petrill, S. A. (2015). Is Math Anxiety Always Bad for Math Learning? The Role of Math Motivation. *Psychological Science*, *26*(12), 1863–1876. https://doi.org/10.1177/0956797615602471

[74] Werner, K., & Woessmann, L. (2023). The Legacy of Covid-19 in Education. *Economic Policy*, eiad016. https://doi.org/10.1093/epolic/eiad016

[75] Wongwatkit, C., Panjaburee, P., Srisawasdi, N., *&* Seprum, P. (2020). Moderating effects of gender differences on the relationships between perceived learning support, intention to use, and learning performance in a personalized e-learning. *Journal of Computers in Education, 7*, 229–255. https://doi.org/10.1007/s40692-020-00154-9

[76] World Economic Forum (2022, November 14). *Here's how COVID-19 affected education- and how we can get children's learning back* Broom, https://www.weforum.org/agenda/2022/11/covid19-education-impact-legacy/

[77] Wu, S. S., Willcutt, E. G., Escovar, E., & Menon, V. (2014). Mathematics achievement and anxiety and their relation to internalizing and externalizing behaviors. *Journal of Learning Disabilities, 47*(6), 503–514. https://doi.org/10.1177/0022219412473154

[78] Yuan, Z., Tan, J., & Ye, R. (2023). A Cross-national Study of Mathematics Anxiety. *The Asia-Pacific Education Researcher*, Article 32, 295–306. https://doi.org/10.1007/s40299-022-00652-7

[79] Zawacki-Richter, O., Conrad, D., Bozkurt, A., Aydin, C. H., Bedenlier, S., Jung, I., Stöter, J., Veletsianos, G., Blaschke, L. M., Bond, M., Broens, A., Bruhn, E., Dolch, C., Kalz, M., Kondakci, Y., Marin, V., Mayrberger, K., Müskens, W., Naidu, S., … Xiao, J. (2020). Elements of Open Education: An Invitation to Future Research. *The International Review of Research in Open and Distributed Learning*, *21*(3), 319-334. https://doi.org/10.19173/irrodl.v21i3.4659

[80] Zhang, J., Zhao, N., & Kong, Q. P. (2019). The Relationship between Math Anxiety and Math Performance: A Meta-Analytic Investigation. *Frontiers in psychology*, *10*, Article 1613. https://doi.org/10.3389/fpsyg.2019.01613

APPENDIX

1. MA Test

Instruction: Check the box below that represents your most appropriate response to each statement -Not at all – 1, 2, 3, 4, 5-Very much-using check mark, an X, to the following statements:

e.g.

Statement	1 Not at all	2	3	4	5 Very much
…………		X			

STATEMENTS

Statement	1	2	3	4	5
1.I have usually been at ease in math classes					
2. I see math as a subject I will rarely use					
3. I'm not good at math					
4. Generally, I have felt secure about doing math					
5. I'll need mathematics for my future work					
6. I usually get good grades in mathematics					
7. I don't think that I could do advanced math					
8. It wouldn't bother me at all to take more math classes					
9. Even though I study, math seems unusually hard for me					
10. I am unable to think clearly when working in mathematics					
11. Knowing mathematics will help earn a living					
12. Math has been my worst subject					
13. I think I could handle more difficult mathematics					
14. I'm not the type to do well in mathematics					
15. Math doesn't scare me at all					

2. Validity: Extended Evaluation survey questionnaire

Survey instrument validation rating scale

Instruction: Please specify your level of agreement or disagreement on the next agree-disagree scale for the therteen statements.

Circle the number that describes how much you agree with each statement

1 - Strongly Disagree 2 - Disagree 3 - Undecided 4 - Agree 5 - Strongly Agree

Criteria	Level of Agreement or Disagreement				
The items in the instrument are relevant to answer the objectives of the study.	1	2	3	4	5
The items in the instrument can obtain depth to constructs being measured.	1	2	3	4	5
The instrument has an appropriate sample of items for the construct being measured.	1	2	3	4	5
The items and their alternatives are neither too narrow nor limited in its content.	1	2	3	4	5
The items in the instrument are stated clearly.	1	2	3	4	5
The items on the instrument can elicit responses which are stable, definite, consistent and not conflicting.	1	2	3	4	5
The layout or format of the instrument is technically sound.	1	2	3	4	5
The responses on the scale show a reasonable range of variation.	1	2	3	4	5
The instrument is not too short or long enough that the participants will be able to answer it within a given time.	1	2	3	4	5
The instrument is interesting such that participants will be induced to respond to it and accomplish it fully.	1	2	3	4	5
The instrument as a whole could answer the basic purpose for which it is designed.	1	2	3	4	5

Comments and Suggestions:

. . . .
. . . .
. . . .
. . . .

Signature over Printed Name

3. **Correlations**

		MA mean score	Math test mean score
MA mean score	Pearson Correlation	1	-,956**
	Sig. (2-tailed)		,000
	N	173	173
Math test mean score	Pearson Correlation	-,956**	1
	Sig. (2-tailed)	,000	
	N	173	173

**. Correlation is significant at the 0.01 level (2-tailed).

4. ANCOVA TEST

Tests of Between-Subjects Effects

Dependent Variable: Math test mean score

Source	Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared
Corrected Model	267,390[a]	2	133,695	924,612	,000	,916
Intercept	1910,889	1	1910,889	13215,362	,000	,987
MA_score	265,881	1	265,881	1838,785	,000	,915
Gender	,666	1	,666	4,608	,006	,806
Error	24,581	170	,145			
Total	9493,630	173				
Corrected Total	291,972	172				

a. R Squared = ,916 (Adjusted R Squared = ,915)

5. **Descriptive Statistics**

	N	Mean	Std. Deviation	Skewness		Kurtosis	
	Statistic	Statistic	Statistic	Statistic	Std. Error	Statistic	Std. Error
MA mean score	173	2,8904	,88839	,164	,185	,347	,367
Math test mean score	173	7,293	1,3029	,079	,185	-,531	,367
Valid N (listwise)	173						