Trigonometric Ratios in High School Students: From Instrumental Understanding to Relational Understanding through their Application in Motion Vectors

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Abstract: The present educational intervention study focused on addressing some areas of improvement in the teaching and learning of trigonometric ratios in high school students. A diagnostic evaluation was performed that revealed an instrumental understanding of trigonometric concepts by the students. Subsequently, a didactic sequence was implemented that incorporated practical applications of trigonometric ratios in the context of displacement vectors. The results of the study indicated significant improvements in students' understanding after the intervention, marking a shift toward relational understanding. This proposal is presented as an effective strategy and suggests promising ways to improve pedagogy in this field.

Keywords: relational understanding, trigonometric ratios, rectangular components, displacement vector.

INTRODUCTION

Trigonometry, a fundamental branch of mathematics, has been praised for its invaluable contribution to the development of natural sciences and technology (Aray et al., 2020). Researchers who have explored students' trigonometric learning (Dunghana et al., 2023; Moore, 2013, 2014; Thompson et al., 2007; Weber, 2005; Yigit, 2014) state that it is a very difficult area for both students as for teachers.

In this regard, authors such as González et al. (2017) highlight the need to overcome the static and rigid teaching of trigonometry and advocate a more dynamic and meaningful approach. Cantoral et al. (2015) emphasize the importance of using a variety of resources and practices, while Solanilla (2015) highlights the need for effective teaching strategies. Aray et al. (2020) support the idea of giving meaning to teaching, while Suárez et al. (2017) highlight the importance of comprehensive mathematics education to prepare students for the university world.

To address the concern about the teaching of trigonometry in high school, an educational intervention investigation was carried out in the month of September 2023. Two groups of pre-university students from the city of Puebla, Mexico who had passed the trigonometry subject in previous semester of high school were selected. A pretest was administered to measure their level of understanding in trigonometry and, after the intervention, a posttest was administered to evaluate the results, so the present study is quantitative.

The diagnostic test consisted of three items on trigonometric ratios. Three levels of cognitive performance were evaluated: item 1, knowledge, item 2, understanding, and item 3, application. Although the test had multiple choice answers, students were asked to justify their answers by expressing their ideas, procedures, formulas, or thoughts related to the solution of each exercise, see Appendix 1.

For the intervention, six 50-minute sessions of a workshop were designed that focused its speech on the application of trigonometric ratios in the context of uniform rectilinear motion (MRU) and displacement vectors. One of the concepts that requires a solid understanding of trigonometric ratios is that of displacement vectors, which play a crucial role in physics and in a wide variety of practical applications, from navigation to computer animation. However, to effectively analyze displacement vectors, it is essential to understand trigonometric ratios and their relationship to the decomposition of vectors into their Cartesian components.

A displacement vector provides precise information about how much an object moves and in what direction it moves in space. Trigonometric ratios, such as cosine and sine, are essential to accurately describe the horizontal and vertical components of the magnitude and direction of said movement, while the tangent allows calculating the direction, that is, the angle between the displacement vector and the abscissa axis (Serway & Vuille, 2012). Although there are various graphic methods to determine the characteristics of the displacement vectors, the analytical method provides greater precision as it eliminates instrumental errors.

Through this intervention, it is intended to take students from a superficial instrumental understanding to a relational understanding (Herheim, 2023) in which they can appreciate the real and significant application of trigonometric ratios in motion vectors. The findings of a literature review study conducted by Hamzah et al. (2021) revealed that, in the last 10 years, most of the research in the field of trigonometry teaching has focused on the identification of misconceptions and errors made by students and very few provide concrete solutions to minimize them.

On the other hand, the results of Aray et al. (2020) have highlighted an additional concern related to the teaching of trigonometry, specifically about the superficiality in which this discipline is often approached, which not only affects the understanding of trigonometric ratios themselves, but also creates a void in students' holistic mathematical knowledge. This, in turn, makes it difficult to teach and learn later subjects such as calculus, linear algebra, physics, and other areas that require a solid foundation in trigonometry.

In this regard, De Villiers and Jugmohan (2012) state that students often lack a consolidated understanding of trigonometric principles, which leads them to rely excessively on memorizing procedures and rules. This practice, while it may lead to success at a procedural level, masks underlying conceptual deficiencies.

In this context, the present research acquires significance by addressing the development and implementation of effective pedagogical strategies to improve students' understanding and application of trigonometric ratios. The research aims to transform the teaching of trigonometry, moving students from a superficial instrumental understanding to a relational and applied understanding of these ratios, particularly in the context of motion vectors. We assume that this approach will not only improve the quality of high school mathematics instruction but will also prepare students more effectively to meet the academic and university challenges that await them in the future, thus addressing a critical need in the field of mathematics education.

Literature Review

Richard Skemp (2006) establishes a distinction between two types of mathematical understanding: instrumental understanding and relational understanding. Instrumental understanding refers to the ability to apply a mathematical procedure without needing to understand its meaning or relationship with other mathematical concepts, that is, use the formula indicating its variables and operate arithmetically. On the other hand, relational understanding involves the ability to understand how mathematical concepts are interconnected and how they relate to each other. Skemp (2006) argues that relational understanding is essential for a deep and lasting consolidation of knowledge, while instrumental understanding can be useful in specific situations, but is not sufficient for a complete and meaningful understanding of mathematical concepts. This latter approach goes beyond simply “knowing what to do” by training students to explain “why to do it” (Skemp, 2006).

In this regard, Hiebert and Lefevre (2013) describe instrumental understanding as that which focuses on the isolated use of concepts and rules to complete mathematical tasks, while relational understanding focuses on establishing connections and relationships between these concepts, thus promoting deeper and more meaningful knowledge.

Herheim (2023) provides another important difference between instrumental understanding and relational understanding, which is the way in which mathematical discourse is approached.

Instrumental understanding accepts the explanations given without questioning them, while relational understanding critically examines the logic of the explanations through dialogic discourse. Furthermore, instrumental understanding expresses explanations in a rigid way, relying on visual prototypes, while relational understanding expresses explanations in a flexible way, allowing students to adapt their understanding to different contexts.

On the other hand, Weber (2005) highlights that students' difficulties in trigonometry often arise from a poor initial understanding of fundamental concepts, such as angles, angle measurement and right triangles.

In this regard, Maknun, Rosjanuardi and Jupri (2022) identified some epistemological obstacles that students face when learning trigonometry, among which are: difficulty in understanding the relationship between trigonometric ratios and angles, difficulty in applying trigonometric ratios in practical situations, confusion between different trigonometric functions and their properties, etc. His article suggests that addressing these obstacles is essential to improving mathematics learning in general. Hamzah et al. (2021) identified, in their review, studies that have focused on correcting misconceptions in trigonometry. These interventions include the use of manipulative materials (Ulyani & Qohar, 2021), gamification through digital applications (Prabowo et al., 2018, 2019), the use of software such as Geogebra (Ibrahim & Ilyas, 2016), the learning approach game-based (Jorda & Santos, 2015), the 5E learning model (Tuna & Kacar, 2013) and discovery learning (Ngu & Phan, 2020; Hadi & Faradillah, 2020). All of these approaches aim to address conceptual deficiencies and improve understanding of trigonometry among students.

Taking into account this theoretical basis and the need to address conceptual difficulties in trigonometry, an intervention strategy is proposed that uses the concept of displacement vectors. By linking trigonometric ratios with the analysis of uniform rectilinear motion, we seek to show their practical applicability in real-world situations. Additionally, by providing a concrete context for understanding trigonometric ratios, students are expected to be able to articulate "why" these are critical in motion vector analysis, thus promoting relational and meaningful understanding.

METHOD

A sample of two groups of 32 members each of high school students from the center of the city of Puebla, Mexico, was selected as study subjects of comparison of static groups (experimental-control).

The items extracted for the pretest are part of the diagnostic exam developed by the Evaluation Commission of the Meritorious Autonomous University of Puebla (BUAP by its acronym in Spanish) High School Physics Academy, of which the first author is a member. This exam was created based on the mathematical knowledge necessary to successfully undertake the Physics I course. However, it is important to highlight that of the 15 items that make it up, only 3 are related to trigonometry. The piloting process of this exam began in 2020, and the items were reviewed and adjusted, retaining, modifying, or discarding questions based on the confidence intervals. This process continued throughout 2021 and 2022, until reaching its final version.

In the month of August 2023, the test was administered again, using a questionnaire prepared in Microsoft Forms to a population of 5,433 fifth semester students.

Taking into account the recommendations of González et al. (2017), Cantoral et al. (2015), Solanilla (2015), Aray et al. (2020) and Suárez et al. (2017), Ulyani and Qohar (2021), Prabowo et al. (2018, 2019), Ibrahim & Ilyas (2016), Jorda & Santos (2015) and Ngu & Phan (2020) and Hadi & Faradillah (2020), the didactic sequence showing in Table 1 was designed, which was applied five weeks later to the sample groups.

SPECIFIC CONTENT:	Vectors	**NUMBER OF SESSIONS:**	**6**	**SESSION TIME UNITS:**	**50 minutes**

EXPECTED LEARNING					
Explain the relationship of Physics with other areas of knowledge by applying vector magnitudes to solve problems.					
MOMENT	**SESSION**	**ACTIVITY**	**TEACHING STRATEGIES**	**LEARNIGN STRATEGIES**	**RESOURCES**
Opening **50 minutes**	*1*	Introduction	The teacher provides a game where the character visually demonstrates that the displacement is formed from the distance traveled along the horizontal and vertical components. Ask students to recount moments in daily life that involve traveling on straight roads.	It uses the interactive Cartesian plane game in which you are asked to reach a specific coordinate, using the cursor and marking the location. Perform active observation. Participate in the Brainstorm.	• Cellphones or computers • Wi-Fi access • Game: https://www.cokitos.com/juego-coordenadas-cartesianas-matematicas/
Develop-ment **150 minutes**	*2*	Exploring the difference between the concepts of distance and displacement.	Presents the simulator for the student to explore and requests the preparation of a comparative table on the differences between distance and displacement. Submit the reading for the next session about vector addition.	Explore the simulator and generate the comparison table.	• Cellphones or computers • Wi-Fi access • Simulator: https://www.educaplus.org/game/distancia-y-desplazamiento • Reading material
	3	Knowing the graphic methods of	The teacher conducts questioning to generate brainstorming about reading about vector addition (Inverted	Through reading the parallelogram method and the polygon method, students work on	• Board and markers • Ruler and protractor

		vector addition.	Classroom), then consolidates the information and shows examples of solutions to exercises that involve the graphic method, finally providing the exercises that must be done. solve the students.	determining the resulting vector on a scale, with ruler and protractor measurements.	• Exercises • Notebook
	4	Knowing the analytical method of vector addition.	In a plenary session, a debate is generated on the conclusions of the previous session, where emphasis will be placed on the use and relevance of trigonometric ratios. The teacher shows on the blackboard how addition with vectors is done analytically using geometry and trigonometry.	Students solve the same exercises they did in session 2, but now using the mathematical approach. They compare both results and develop conclusions.	• Board • Markers • Exercises • Notebook
Closing **100 minutes**	*5*	Applying the concepts acquired through a virtual laboratory.	Indicate the didactic situation. Organize teams of 5 members. Give students the instruction manual to follow to carry out the practice. See **Appendix 2** Provide links and explain the use of the ruler tool in Google maps.	Use Google Maps to plot trajectories from various starting points to a meeting point in the center of the city of Puebla (whose streets are perfect grids). Represent these trajectories as displacement vectors, measure their magnitude and determine the direction using Cartesian	• Cellphones or computers • Wi-Fi access • GeoGebra • https://www.geogebra.org/calculator • Google Maps https://www.google.com.mx/maps/search/zocalo/@19.0429833,-

				coordinates. Subsequently, they entered GeoGebra to represent the displacement vectors both graphically and analytically and compared the results with the length of the trajectories provided by Google Maps. Preparation of virtual laboratory report.	98.2005476,17z?entry=ttu
	6	Self-assessment.	Provide link to self-assessable exercises. Reinforce and consolidate relational understanding of concepts.	Perform interactive self-assessment exercises, test their understanding of concepts and evaluate their progress.	• Cellphones or computers • Wi-Fi access • Suggestion for self-assessment exercises: https://es.khanacademy.org/math/linear-algebra/vectors-and-spaces/vectors/e/adding_vectors

EVALUATION OF LEARNING (Technique, instrument, type, and performance descriptors)	
Diagnostic	Interrogation through written test
Formative	Comparative table Evaluation technique: Performance analysis

	Evaluation instrument: Checklist. Type of evaluation: Hetero-assessment
Summative	Exercises Technique evaluation: Performance analysis. Evaluation instrument: Rubric. Type of evaluation: Self-assessment

Table 1: Didactic sequence

In the initial session, students begin to become familiar with basic concepts related to location and movement. While this session does not focus directly on trigonometric ratios, it lays the foundation for understanding the importance of measuring distances and displacements, which are key aspects in the context of trigonometry.

In the third session the stage fosters an understanding of how trigonometric ratios, such as sine and cosine, are applied in solving vector problems.

In the third session is common for students to mention that to save time it is better to use the Pythagorean Theorem because the legs of a right triangle are the horizontal and vertical components of a vector. Questions also arise as to whether the hypotenuse provides the magnitude of the vector, how to determine the direction or angle of inclination of the vector? The most prepared students would indicate that using trigonometric ratios, hence the teacher could take advantage of the situation to develop the topic.

At the end of the six sessions of the didactic sequence, a post-test was applied with the same items from the diagnostic test, on trigonometric ratios. As this is a quantitative study in which the aim is to measure the impact of a pedagogical intervention strategy on student learning, the T-Student test for related samples was carried out in order to compare the differences between the pretest scores and the posttest This allowed us to determine if there was a statistically significant change in learning after the intervention. Data analysis was carried out through the process of data reduction and visualization to draw conclusions.

RESULTS

Figures 1, 2 and 3 show the responses to the items evaluated in the diagnostic test. he asterisk symbol (*) indicates the correct answer.

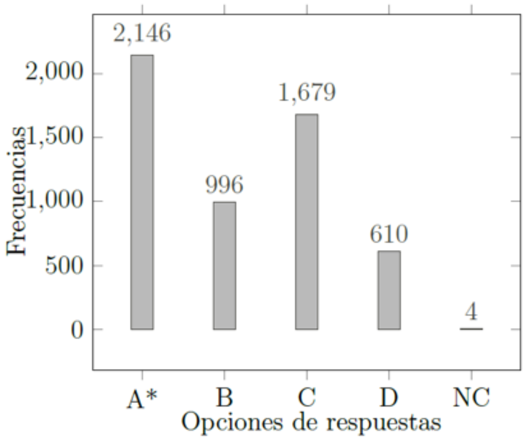


Figure 1: Answers to item 1

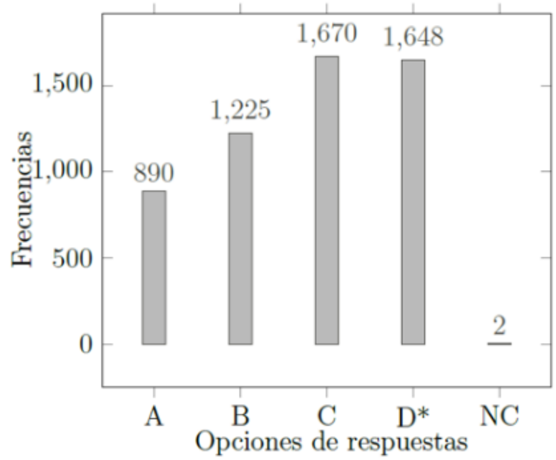


Figure 2: Answers to item 2

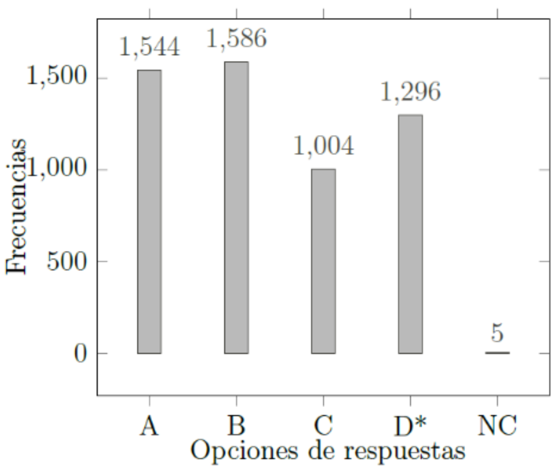


Figure 3: Answers to item 3

Table 2 presents the classification of academic performance in trigonometry. In this classification, performance is considered "Low" if students answered 1 or no test items correctly. It is classified as "Fair" if they answered 2 items correctly, "Good" if they answered 3 items correctly, and "Excellent" if they answered correctly to all 4 items of the test.

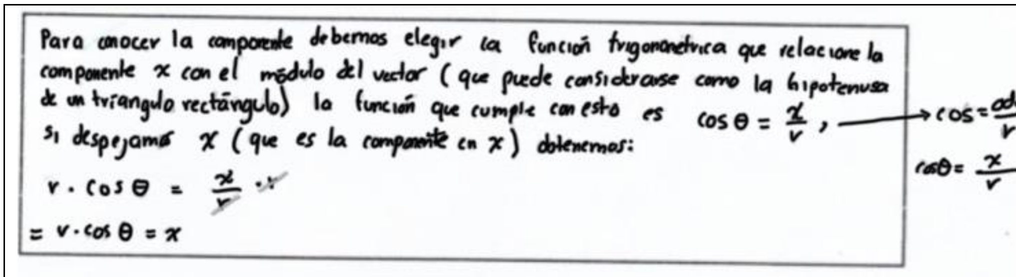
	Supporters	Percentage
Low	3339	61.44
Regular	1460	26.86
Well	518	9.53

Excellent	118	2.17

Table 2: Academic performance in trigonometry

In order to go deeper, Table 3 shows examples of answers provided in the posttest that give indications of relational understanding, the relevance of which is discussed in the conclusions.

Item 1


Para conocer la componente debemos elegir la función trigonométrica que relacione la componente x con el módulo del vector (que puede considerarse como la hipotenusa de un triángulo rectángulo) la función que cumple con esto es $\cos\theta = \frac{x}{v}$, → $\cos = \frac{ady}{v}$

$\cos\theta = \frac{x}{v}$

Si despejamos x (que es la componente en x) obtenemos:

$v \cdot \cos\theta = \frac{x}{v} \cdot v$

$= v \cdot \cos\theta = x$

To determine the component, we must choose the trigonometric ratio that relates the x-component with the magnitude of the vector (which can be considered as the hypotenuse of a right triangle). The function that fulfills this is $Cos\ \theta = \frac{x}{r}$,

$Cos\ = \frac{ady}{r}$

If we solve for \(x\) (which is the x-component), we get: $r * Cos\ \theta = \frac{x}{r} r$

$= r * Cos\ \theta = x$

Item 2


$\alpha = 180 - 90 - 30 = 60$

Ángulo de la hipotenusa $= \tan^{-1}\left(\frac{co}{ca}\right)$

$\theta h = \alpha = 60$ $co = 3\sqrt{3}$ $ca = ?$

$co = 5.19$

$60 = \tan^{-1}\left(\frac{5.19}{B}\right)$

$B = \frac{5.19}{\tan 60}$

$B = 3$

$\alpha = 180 - 90 - 30 = 60$

$$The\ angle\ of\ the\ hypotenuse = Tan^{-1}\frac{Co}{Ca}$$

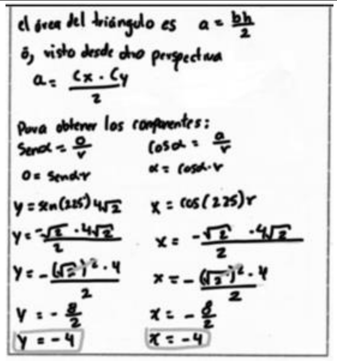
$\theta h = \alpha = 60$ $Co = 3\sqrt{3}$ $Ca = ?$

$Co = 5.19$

$$B = \frac{5.19}{tan60}$$

$B = 3$

Item 3


el área del triángulo es $a = \frac{bh}{2}$
ó, visto desde otra perspectiva
$a = \frac{Cx \cdot Cy}{2}$
Para obtener los componentes:
$Sen\alpha = \frac{O}{r}$ $Cos\alpha = \frac{A}{r}$
$O = Sen\alpha r$ $\alpha = Cos\alpha \cdot r$
$y = sen(225)4\sqrt{2}$ $x = cos(225)r$
$y = \frac{-\sqrt{2} \cdot 4\sqrt{2}}{2}$ $x = \frac{-\sqrt{2} \cdot 4\sqrt{2}}{2}$
$y = \frac{-(\sqrt{2})^2 \cdot 4}{2}$ $x = \frac{-(\sqrt{2})^2 \cdot 4}{2}$
$y = -\frac{8}{2}$ $x = -\frac{8}{2}$
$\boxed{y = -4}$ $\boxed{x = -4}$

The area of the triangle is $a = \frac{bh}{2}$ or, from another perspective,
$a = \frac{Cx*Cy}{2}$

To obtain the components.

$$Sin \propto = \frac{O}{r} \qquad Cos\alpha = \frac{A}{r}$$

$$O = Sin\alpha r \qquad \alpha = Cos\,(225)r$$

$$y = Sin(225)4\sqrt{2} \qquad x = Cos(225)r$$

$$y = \frac{-\sqrt{2} * 4\sqrt{2}}{2} \qquad x = \frac{-\sqrt{2} * 4\sqrt{2}}{2}$$

$$y = \frac{-8}{2} \qquad x = \frac{-8}{2}$$

$$y = -4 \qquad x = -4$$

Table 3: Examples of responses in the posttest

Seven months after the implementation of the didactic sequence, a post-test is applied that includes 10 items of which 7 correspond to topics related to vectors and the same 3 trigonometry items applied in the diagnostic evaluation are interspersed.

The data are analyzed in the SPSS statistical software and it is obtained that the samples are non-parametric as they do not fit the normal distribution, so they are analyzed with the Wilcoxon rank sum to compare two related or paired samples (pre and post of the group control, apart, pre and post of the experimental group). This test is used to compare the performance of a group of students, at different times, before and after receiving an educational intervention and evaluate its effectiveness.

Table 4 presents the results of the Wilcoxon signed rank test, the interpretation of which is explained in the discussion and conclusions section.

	PreCon - PostCon	PreExp - PostExp
Z	-0.26	-4.45
asymptotic sig.	0.79	0.00

Table 4: Wilcoxon Signed Rank Test

Table 5 describes the positive, negative differences and ties between the test results in both groups.

	Experimental	Control
Negative differences	1	6
Positive differences	26	7
Ties	5	19

Table 5: Range test

Table 6 shows the average whose maximum value is 3.

	$\bar{x}$
Pretest Experimental	0.93
Postest Experimental	2.24
Pretest Control	0.81
Postest Control	0.87

Table 6: Arithmetic average

DISCUSSION

Pretest

In the diagnostic test, we identified a partial instrumental understanding in the response justifications of item 1, where phrases such as: "At this moment I do not remember how tangent, sine and cosine work. I do not remember how to obtain sides with angles, nor the trigonometric laws. I use the Pythagorean theorem $c = \sqrt{a^2 + b^2}$, oh no! it has an angle... forget it, I don't remember the process. I have the idea that the sine is used and I play with the data. They were reasons trigonometric, but I don't remember the procedure. I tried to find it with the angles, but I don't remember the formula." Surprisingly, even though 41% of the students selected the correct answer, 90% of them claimed to have chosen the option at random.

In this case, instrumental understanding is revealed, since it is possible that students were exposed to the repetition of mathematical tasks, which potentially led them to obtain knowledge without a deep understanding of the concepts involved.

Although in item 1 presence of the mnemonic "SOH–CAH–TOA" was observed, as evidenced in the study by Yigit (2016). This acronym is used to identify the acquisition of knowledge through the ratio method between the proportions of pairs of sides in a right triangle, and no indicators of the use of the unit circle method were detected in any of the questionnaires. In items 2 and 3, students stated that they did not have adequate basic knowledge or had difficulties reasoning about the task.

Postest

In item 1, 76% of correct answers are presented, very detailed justifications are collected, in which students generally describe that the figure is a right triangle, the modulus vector r is the hypotenuse and the x component is the adjacent leg; and that if the angle is known, the cosine and a clearance can be used, given that this ratio relates to said magnitudes. In addition, two tests were found that use the unit circle as a representation.

It is observed that they refer to the ray r as a vector and the x coordinate as its horizontal component, showing indicators of relational understanding, since, although they are presented with a geometry reagent, they are able to transfer the situation to the vector context.

Item 2 stands out as the most challenging, observing an increase in correct answers from 24% to 44%. However, it supports the idea that there is a solid understanding of trigonometric ratios, which involves overcoming the underlying conceptual deficiencies mentioned by De Villiers and Jugmohan (2012).

In the third item, once again evidence of relational understanding is observed, which implies the ability to understand the interconnection of mathematical concepts and their relationships with diverse situations. This is reflected in the fact that 61% of the students were able to correctly identify the trigonometric expressions necessary to solve an area calculation task that was unrelated to the class topic, focused on displacement vectors. However, they approached the problem of finding the legs as if they were the horizontal and vertical components of a vector.

CONCLUSIONS

Relational understanding, according to Skemp's (2006) model, implies that students not only understand mathematical concepts in isolation, but that they are able to relate and connect these concepts with each other.

The implementation of the didactic sequence and its effectiveness was evaluated through the Wilcoxon statistics. This finding provides strong statistical evidence to conclude that there is a significant difference between the Pretest and Posttest scores only in the experimental group. A value of Z = -4-45, in a sample of 32, provides strong evidence to reject the null hypothesis, thus concluding that there is a significant difference between before and after the intervention in the experimental group.

When comparing the results of the pretest and posttest for the control group, a value of Z=0.26 is obtained, which in the context of statistical hypothesis testing means that the difference observed between is null or insignificant. In other words, it suggests that the two samples are statistically equal and that the difference between the sample means is very small.

In the didactic sequence, students not only learned to calculate trigonometric ratios, but also understood how these ratios are related to the addition of vectors and how they are applied in practical situations. Although the teaching sequence was not designed for a trigonometry class, it facilitated relational understanding by showing how mathematical concepts are interconnected and applied in the real world.

Acknowledgments

We would like to thank CONAHCYT for the financial support provided during the realization of this Project, to the Evaluation Commission directed by Master Rogelio Paredes Jaramillo and to the General Academy of Physics of the BUAP High Schools, for their invaluable support in the application of the diagnostic instrument.

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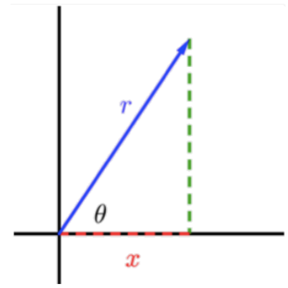
APPENDIX 1

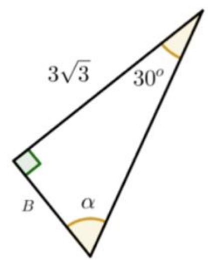
Prestest-Posttest Items

1. The figure shows a ray with an angle in the first quadrant. Find the x coordinate. Justify your answer.



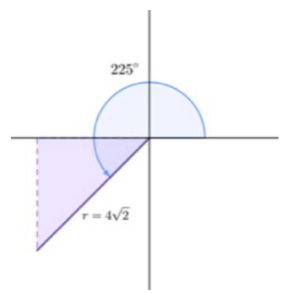
a) $\boldsymbol{r\ Cos\ (\theta\ )}$
b) $\boldsymbol{r\ Sen\ (\theta\ )}$
c) $\boldsymbol{r\ Tan\ (\theta\ )}$
d) $\boldsymbol{r\ Csc\ (\theta\ )}$

2. Find the length of vector B. Justify your answer.



a) 1
b) 2
c) $2\sqrt{2}$
d) 3

3. Find the area of the right triangle. Justify your answer.



   a) 8
   b) $8\sqrt{2}$
   c) $6\sqrt{2}$
   d) 16

APPENDIX 2

Instructions for virtual laboratory practice

1. Enter Google Maps and represent the following situation: 5 tourists have agreed to meet at the access door to the Cathedral of Puebla located on the corner of September 16th and 3 east streets, they are located at
   A. Rosario Chapel – East 4th and May 5th
   B. Amparo Museum – 2 south and 7 east
   C. El Parian craft market – 6 north and 4 east
   D. El Mural de los Poblanos Restaurant - September 16th and 7 east
   E. Automobile Museum -3 south and 15 west
2. Each member must choose a starting point and trace a trajectory to the meeting point using displacement vectors.
3. Use the ruler tool to measure the magnitude of the vectors, to determine the direction properly define the coordinate axes (x,y)
4. Enter GeoGebra by scanning the QR code, each member will represent the displacement vectors corresponding to their trajectory, remember that they can represent them to scale.
5. Add the displacement vectors of each trajectory using the graphical method and take screenshots.
6. Represent the displacement vectors of your trajectory in their Cartesian components and perform the same sums from the previous step using the analytical method.
7. Compare the vector sum of the displacement vectors with the length of the path you chose.
8. Draw conclusions, prepare your report and send it for review.

Checklist for Virtual Laboratory Report

Criterion	**Accomplished**	**In progress**	**Unachieved**
Identify the variables involved in the observed phenomenon			
Distinguish the type of physical magnitude that corresponds to each variable			
Correctly relate the phenomenon to the mathematical model (equations).			
Correctly perform the required calculations			
Correctly record data			
Interpret data correctly			
Make appropriate graphic representations of the information			
Answer the questions posed with scientific arguments consistent with the topic of analysis			
Includes a conclusion			