Figural and Non-Figural Linear Pattern: Case of Primary Mathematical Gifted Students' Functional Thinking

M. Syawahid[1], Nasrun[2], Rully Charitas Indra Prahmana[3]

[1]Universitas Islam Negeri Mataram, Indonesia,

[2] Universitas Muhammadiyah Makasar, Indonesia

[3]Universitas Ahmad Dahlan, Indonesia

[1]syawahid@uinmataram.com, [2]nasrun@unismuh.ac.id, [3]rully.indra@mpmat.uad.ac.id

Abstract: Mathematically gifted students have a potential for understanding and connecting mathematics concept. Pattern generalization as a part of functional thinking becomes one of the benchmarks for gifted students in mathematics. The mathematics curriculum in Indonesia that has not accommodated the functional thinking ability of elementary school students is the basis for this study. It focuses on describing mathematically gifted students functional thinking in solving figural and non-figural linear pattern task. Functional thinking abilities in this study consist of thinking process in near generalization, far generalization, formal generalization and determine inverse. Case study of qualitative approach used in describing mathematically gifted students thinking. Data were collected from 5th-grade of gifted student's problem solving in figural and non-figural linear patterns task. the finding showed that gifted students are able in functional thinking in different ways. They represented the relationship of two quantities symbolically. In solving figural linear pattern task, gifted students perform FT consist of: near generalization by counting, multiplicative approach, and contextual strategy; far generalization by contextual strategy; formal generalization by multiple difference and proportional strategy; and determine inverse by using general rule. In solving non-figural linear pattern task, gifted students perform FT consist of near generalization, far generalization, and formal generalization by multiple difference strategy; and determine inverse by using general rule.

Keywords: functional thinking; gifted students; linear pattern.

INTRODUCTION

Mathematically gifted students (MGS) have become an interesting topic in recent years. MGS refers to students who have mathematics abilities that manifest in the form of successful performance and creativity in mathematics tasks (Krutetskii, 1976). When compared to the top 10% of peers their own age, gifted students are those who have outstanding potential in one or more human ability domains, such as intelligence, creativity, social skills, or mindset (Gagné, 1985). MGS also have a unique strategy for reasoning and problem solving (Pitta-Pantazi, Christou, Kontoyianni, and Kattou, 2011). Krutetskii (1976) described mathematics giftedness as formal perception abilities, logical thought abilities, mathematical symbol abilities, generalization abilities, mathematical reasoning and structured abilities, flexibility of mental process in mathematical activities, striving for clarity, simplicity, economy, and rationality of solutions; mathematical memory; mathematical cast of minds expressed in striving to interpret the environment mathematically. Students with good problem-solving skills, metacognitive skills, creative mathematical thinking, and high ability or performance in mathematical problem solving are typically considered to be gifted in mathematics (Leikin, 2018, 2021).

There has been a lot of research on mathematically gifted students. Pitta-Pantazi et al. (2011) constructed a structured Model indicating that mathematical abilities contribute more than mathematical creativity. The structured Model also confirmed that the nature of cognitive abilities (fluid intelligence and working memory) predicts mathematical giftedness. Gutierrez et al. (2018) found that MGS are much faster in learning mathematics, they used different strategies in generalization linear patterns. It also showed that students made all necessary cognitive effort, as much as was possible due to his limited knowledge of algebra. Paz-Baruch et al. (2022) revealed five main cognitive factors: visual-serial processing (VSP); arithmetic abilities (AA); pattern recognition (PR); auditory working memory (AWM); visual-spatial working memory (VSWM); and Structural equation Modeling (SEM) based on the factor analysis revealed clear differences in the role of cognitive abilities as predictors of EM, G, and MG.

Other research characterized MGS as generalization abilities in mathematics structure and pattern (Assmus & Fritzlar, 2022; Erdogan & Gul, 2023; Gutierrez et al., 2018; Krutetskii, 1976; Paz-Baruch et al., 2022; Pitta-Pantazi et al., 2011). Assmus & Fritzlar (2022) found that mathematical giftedness and mathematical creativity have a high correspondence in concerning the invention of figural patterns. MGS used figural reasoning in generalizing linear patterns and numerical reasoning in generalizing non-linear patterns (Girit Yildiz & Durmaz, 2021). They also used different strategies in solving linear and nonlinear pattern (recursive, chinking, contextual, and functional) (Erdogan & Gul, 2023).

Pattern generalization involved the ability to relate and represent two quantities as words, tables, graphics, or symbols. This ability is termed functional thinking (FT). FT is a fundamental part of algebraic reasoning (Blanton & Kaput, 2005; Smith, 2008). It's key for algebraic thinking because it involves generalizations of how quantities are related (Tanıslı, 2011). Functional thinking is a representational thinking that focuses on the relationship between two (or more) quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances (Smith, 2008). Blanton et al. (2011) define functional thinking as generalizing relationships between co-varying quantities, expressing those relationships in words, symbols, tables, or graphs, and reasoning with these various representations to analyze function behavior. According to Smith (2008) and Blanton et al. (2011), functional thinking consists of the generalization and representation of relationships between two variables.

There are two types of pattern generalization tasks: figural and non-figural pattern generalization tasks (Chua & Hoyles, 2014; F. Rivera & Becker, 2003). Figural pattern generalization task included a task with the pattern listed as a sequence of pictorial context. Non-figural pattern generalization is often called numerical pattern generalization when the pattern is listed as a sequence of numbers.

Generalization is part of three activities: identifying similarities in a case, expanding one's reasoning beyond the range of origin, and obtaining broader results from certain cases (Kaput, 1999). Chua & Hoyles (2014) stated that generalization is a process involving at least one of the following activities: (i) to examine a few particular cases to identify a commonality; (ii) to extend one’s reasoning beyond those particular cases; and (iii) to establish a broader result for those particular cases. Generalization starts with a sense of pattern, using patterns, making conjectures, and testing the results of generalizations. Thus, it can be said that the process of generalization is related to the understanding of patterns and conjectures (Mason, Stacey, and Burton, 2010).

Concerning linear patterns, Stacey (1989) distinguishes between ‘‘near generalization’’ tasks, which include finding the next pattern or elements that can be reached by counting, drawing, or forming a table, and ‘‘far generalization’’ tasks, in which finding a pattern requires an understanding of the general rule. Amit & Neria (2008) add the “formal generalization” term as an explicit requirement for representing a generalization in a formal mode, striving toward algebra.

Several studies have revealed about students' functional thinking. Warren et al. (2006) found that elementary students are capable not only of developing functional thinking but also of communicating their thinking both verbally and symbolically. Blanton & Kaput (2004, 2005) found that students were able to think functionally at the kindergarten level co-variationally and were able to think functionally as a correspondent in the 1st grade. Tanıslı (2011) found that five-grader students thought on co-variation while working with the linear function tables. Wilkie & Clarke (2016) found four types of visual structure in functional thinking, and Stephens et al. (2017) found three types and ten levels of *student sophistication* in functional thinking.

There are few studies about functional thinking for elementary students in Indonesia. Rusdiana et al. (2018, 2017) reported that there are two aspects of elementary students pattern generalization in Indonesia: focus on the number of patterns and focus on the figure of patterns. Syawahid et al. (2020) revealed that elementary school students in Indonesia are capable of functional thinking by starting recursively to find a corresponding formula. Functional thinking of elementary students in Indonesia can also be categorized as recursive-verbal, correspondence-verbal, and recursive-to-correspondence-symbolic (Syawahid, 2022).

The limited number of studies on functional thinking of elementary school students in Indonesia may be due to the applied mathematics curriculum. Based on the National Council Teachers of Mathematics (NCTM) standard, pattern generalization as a core of FT was thought of from grade 3 to grade 5 of elementary school (NCTM, 2000). However, in the Indonesia curriculum, pattern generalization was thought of in secondary school (MoEC, 2016). Despite the difference, it’s suspected that some Indonesian students in elementary school are able to generalize patterns, or FT (Rusdiana et al., 2017, 2018; Syawahid et al., 2020; Syawahid, 2022).

Previous studies of functional thinking students of elementary students in Indonesia just revealed students thinking in solving figural pattern tasks (Rusdiana et al., 2017, 2018; Syawahid et al., 2020; Syawahid, 2022). There are no studies that reveal students functional thinking in solving non-figural pattern tasks. This allows for an in-depth study of elementary students' functional thinking in solving figural and non-figural patterns.

This study aims to describe MGS FT in solving figural and non-figural linear pattern tasks. FT in this study consists of near generalization, far generalization, formal generalization, and determining the inverse. Near, far, and formal generalization are part of generalization and representation relationship, and determining the invers is part of analyze function behavior.

METHOD

Research Design

This study tries to describe gifted elementary students functional thinking in solving figural and non-figural linear patterns. It also identifies the gifted students' strategies in performing functional thinking. This study used a case study of a qualitative research approach. It involve a detailed study of one or a few individuals (Fraenkel, Wallen, and Hyun, 2012). The detailed study in this research refers to a case of a few gifted students' functional thinking. A case study allows searching for a selected subject in detail (Cohen, Manion, and Morrison, 2000) and exploring problems to find an in-depth understanding (Creswell, 2012). The types of cases in this study include the intrinsic cases of gifted elementary students performing functional thinking (Creswell, 2012; Fraenkel et al., 2012). A typical sample of purposive sampling is used to select research subjects. A typical sample is considered to be representative of that which is being studied (Fraenkel et al., 2012).

Participant and Instrument

This study involved 62 *Athirah* elementary school at Makasar, Indonesia. They were 13-year-olds of fifth grade. We gave a linear pattern task (Wilkie & Clarke, 2016) to 62 students and found that there were two students who had a correct answer. Two students who had a correct answer have the initial AA and AG. Based on interviews with mathematics teachers, AA and AG are classified as students with high achievement and often represent the school in mathematic competitions (e.g., mathematics Olympiads).

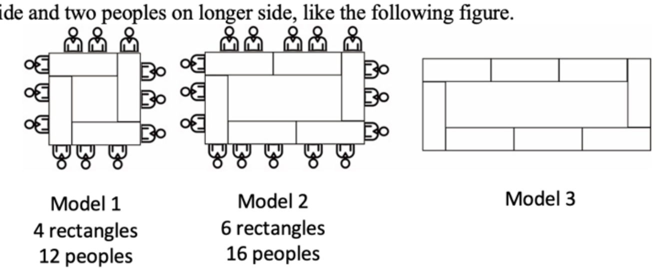
Data in this study involved qualitative data consisting of student's answers and interviews in-depth. It was carried out by giving functional thinking tests and interview protocols. Functional thinking tests consist of figural linear pattern problems and non-figural linear pattern problems. Figural linear pattern problems were adopted from Lepak, Wernet, and Ayieko (2018). These problems describe the arranged tables provided by an accompany for business meeting. There is a role for this arrangement in that the table is a rectangle and can be occupied by one person on the shorter side and two people on the longer side. A picture of arranged tables is given to be a hint in this problem. Second, non-pictorial linear pattern problem was produced by research. This problem illustrates plant height growth observed by researchers on the first, second, and third days. Students are asked to determine plant height on the day after and determine general rules for growth in plant height.

Aspect	Indicator
Near generalization	Determine the nearest unknown quantity of the dependent variable from a given pattern
Far generalization	Determine a certain unknown quantity of the dependent variable from a given pattern
Formal generalization	Determine a relationship between dependent and independent variable (e.g., by word, table, graph, or symbolic)
Determine inverse	Determine the quantity of the independent variable for the known dependent variable

Table 1: Functional thinking aspect

Arranging Tables

A company provides tables for business meetings. For each table, there is one people can seat on shorter side and two peoples on longer side, like the following figure.



1. How many people can sit on model 3? Explain how you got it!
2. How many people can sit on model 13? Explain how you got it!
3. Write an equation for P people number who can sit at S model? Explain how you got it!
4. Mr. Yogi wants a meeting with 75 peoples. Which model will he use?

Plant Growth

A researcher was observing the growth of a plant. On first day (H1), the plant height was 4 cm, on second day (H2), the plant height was 6 cm and on the third day (H3) the plant height was 8 cm.

1. Determine the plant height on 4th days (H4) and 5th days (H5)!
2. Determine the plant height on 7th days (H7) and 10th days (H10)!
3. What can you notice about the structure of the plant's growth every day? Write down how you got the height of the plant (T) on a certain day (H).
4. On which day the plant height was 86 cm? Explain how you got your answer!

Figure 1: Figural and non-figural functional thinking task

Data Collection

This study was conducted using a task-based interviews. Task-based interview present figural and non-figural linear pattern problems and participants were required to explain their solution. One of the purposes of conducting task-based interviews is to identify patterns of subject behavior when working on tasks, such as success in completing tasks, strategies used, expressions of curiosity, and certain actions associated with student success and failure in completing tasks.(Mejía-Ramos & Weber, 2020).

Data Analysis

The qualitative data from student's answer and transcript interviews was analyzed by comparative analysis between each category, and new categories emerged (Creswell, 2012; Leedy & Ormrod, 2019). Interview data and transcripts were reduced to fragments involving explanations of student's main ideas. The data were coded, sorted, and read repeatedly to answer research questions. In addition, the pre-established categories (Table 2) were considered to interpret the student's functional thinking emerging from their answer.

Category	Description
Counting	Draw the next figures and count their element
Recursive	Continue the sequence using the numerical difference between consecutive terms or explicit the recursive relation between consecutive terms
Multiple difference	Use the difference between consecutive terms as a multiplicative factor (adjusting or not the result) to obtain distant terms or the general term.
Proportional	Use multiplicative strategies, starting from one known term of the sequence to find distant terms or the general term
Visual	Express a relation between the two varying quantities for a distant term or in the general term, based on the characteristics of the pictorial representation
Numerical	Express a relation between the two varying quantities for a distant term or in the general term, based on the numerical sequence.
Contextual	Constructing a rule based on the information provided in the situation; relating the rule to a counting technique
Guest and check	Guessing a rule without regard to why this rule might work. Usually, this involves experimenting with various operations and numbers provided in the problem situation

Table 2.:Strategies in generalize function relation (Lannin, 2005; Oliveira, Polo-Blanco, and Henriques, 2021)

In order to conceptualize qualitative investigations, research must be trustworthy when collecting, analyzing, and interpreting the results (Merriam, 2015). The study employs both the triangulation method and triangulation theories to ensure its validity. The triangulation method is used by comparing the data collection method and triangulation theories by comparing the data with the relevance theories (Merriam, 2015). In this study, we applied the triangulation method by analyzing written documents from students, which included their solutions and interview outcomes. We applied the triangulation theory by evaluating the study's findings against the relevance of multiple journal-published studies.

RESULTS

This section presents gifted students functional thinking (AA and AG), which consists of near generalization, far generalization, formal generalization, and determining the inverse in solving figural and non-figural linear pattern tasks. AA and AG's functional thinking data is described in the form of strategies used in generalizing and representing figural and non-figural linear pattern tasks.

Figural Linear Task

In a figural linear task, gifted students are asked to complete a task that contains a table setting picture pattern. Gifted students are asked to perform near generalizations, far generalizations, formal generalizations, and determine inverses.

AA's functional thinking in solving a figural linear pattern task involved AA's abilities in generalizing two quantities (people number and Model) and representing them by words, table, graphic, or symbolic. Generalization abilities in this study consist of near generalization, far generalization, and formal generalization.

At near generalization, AA tries to determine people number at Model 3. Firstly, AA draws the Model 3 figure and writes a number based on people number seated (figure 2). Furthermore, AA associates the number with a multiplicative approach (from $2+2+2+2+2+2+2+2+1+1+1+1$ to $(8\times 2)+(4\times 1)$). In this stage, AA uses a counting strategy by drawing the next figure and counting its element (Lannin, 2005; Oliveira et al., 2021).

R : how did you get people number at Model 3?

AA : first, I draw Model 3 and write a number representing the people sitting on the table, i.e., 1 and 2. Second, I count the number by associating it with a similar number to get $(8\times 2)+(4\times 1)=20$.

R : how did you write a number representing the people sitting on the table? Can you explain?

AA : as we can see in Model 1 and Model 2 that there are two people sitting at a table with long sides and one person sitting at a table with a wide side. So, I wrote a number based on that.

Before AA performs a far generalization, AA reflects on Model 3 that there are two people sitting in pairs at the length of three tables above and two people sitting in pairs at the length of three tables below. While there are 8 other people, consisting of 1 person at the width of the tables above, 1 person at width of the tables below, 3 people at the table on the left, and 3 people at the table on the right. This reflection was used by AA to get the people number at Model 13 (far generalization). AA imagines that at Model 13 there are two people sitting in pairs at the 13 tables above and two people sitting in pairs at the 13 tables below $(2\times 13+2\times 13=4\times 13)$. While there are 8 other people at Model 13, consisting of 1 person at the table above, 1 person at the table below, 3 people at table on the left and 3 people at table on the right $(1+1+(2+1)+(2+1)=8)$. In this stage, AA used a contextual strategy by constructing a rule based on information provided in the situation; relating the rule to a counting technique (Lannin, 2005).

R : did you get people number at Model 13 by previous way?

AA : no, I can't draw the table at Model 13.

R : so, how did you get people number at Model 13? Can you explain?

AA : I observe the people number at Model 3 that there are three table above, three table below and two table in side. At three table above and below there are $3 \times 2 \times 2$ *people. At two table in side, there are 8 people. It means that for Model 13 there are 13 table above, 13 table below and 2 table in side. At 13 table above and below there* $13 \times 2 \times 2$ *people and at two table in side there are* 8 *people.*

In formal generalization, AA tries to get a general rule of the people numbers. AA observes that the people numbers in Model 1, Model 2, and Model 3 construct a sequence with the same difference. The sequence has a difference of 4. AA realizes that the people number at Model 2 consists of the people number at Model 1 and the difference $(16 = 12 + 4 \times 1)$, while the people number at Model 3 consists of the people number at Model 2 and the difference $(20 = 16 + 4 = 12 + 4 \times 2)$. In this stage, AA used a multiple difference strategy by using the difference between consecutive terms as a multiplicative factor (adjusting or not the result) to obtain distant terms or the general term (Oliveira et al., 2021).

R : how did you get the relationship between people number (P) and Model S?

AA : I noticed the people numbers at 1st, 2nd, and 3rd, and got that the people numbers have the same difference for each term. For example, people numbers at 2nd equal to people numbers at 1st and difference $(16 = 12 + 4)$, *while the people number at 3rd equal to the people number at 2nd and difference* $(20 = 16 + 4 = 12 + 2 \times 4)$. *I decide that for Model S, the people number equal to the people number at 1st and* $(S-1)$ *multiply by difference,* $12 + (S-1) \times 4$.

To determine the inverse, AA tries to determine which Model to use for 75 people. AA used a general rule obtained at formal generalization by substituting P by $75 + 1$ and performing an algebraic operation to gen $S = 17$. AA realizes that to get S, he must provide an even of the people number. Therefore, he substitutes P by $75 + 1$.

R : how did you get which Model will be used for 75 people?

AA : I used the previous formula $P = 12 + (S-1) \times 4$

R : how did you use the formula? Can you explain?

AA : firstly, I substituted P by 75 to get S, but the coefficient of S and a Constanta was an even number, so I decide to add up 75 by 1 and then I performed the mathematics' operation to get $S = 17$. .

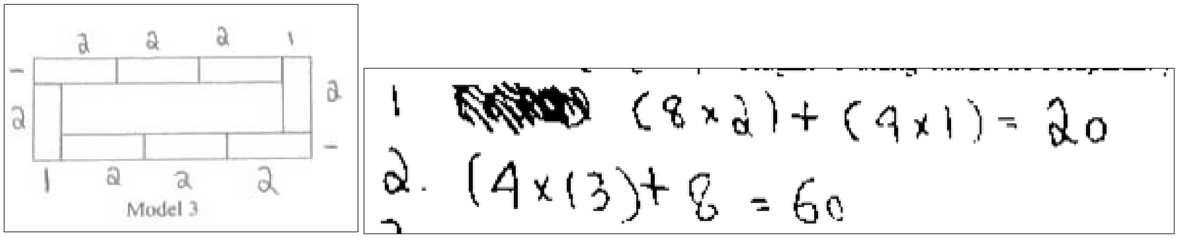


Figure 2: AA's counting and contextual strategies in near and far generalization

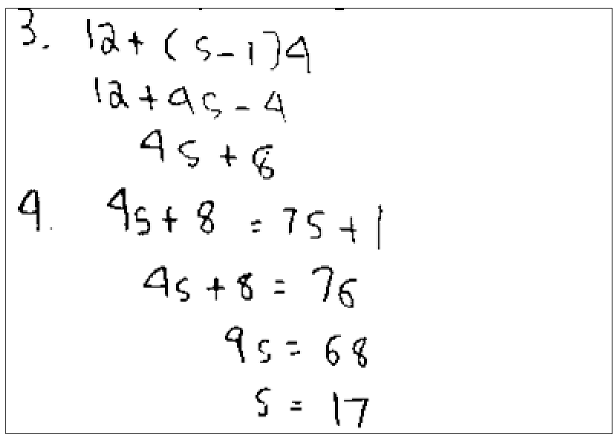


Figure 3: AA's multiple difference strategies in formal generalization and determine inverse.

In performing generalization, AA was capable of representing the relation between two quantities by symbol. He writes general rule of the relationship between people number quantities and Model quantities as $P = 4S + 8$. It shown that AA was capable of thinking functionally by correspondence, which means the emphasis is on the relation between corresponding pairs of variables (Confrey & Smith, 1991; Smith, 2008).

AG functional thinking in solving figural linear pattern tasks involved AG's abilities in generalizing two quantities (people number and Model) and representing them by words, table, graphic or symbolic. Generalization abilities in this study consist of near generalization, far generalization, and formal generalization.

At near generalization, AG tries to determine the people number at Model 3. Firstly, AG noticed the people number at Model 1, consisting of three people sitting above, below, left side, and right side of the table, and it can be written mathematically by "3×4". Secondly, AG noticed people number at Model 2 consisting of people number at Model 1 and two people sitting on the table above and two people sitting on the table below. It can be written by "$(3 \times 4) + (2 \times 2)$".

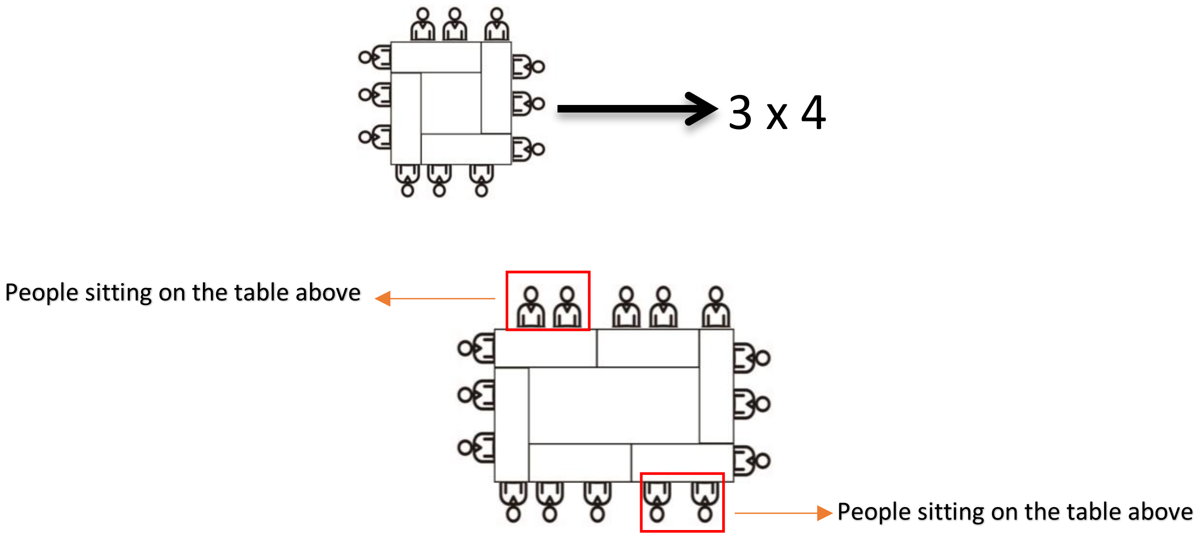


Figure 4: AG noticing of people number at Model 1 and Model 2

Based on these understandings, AG determined the people number at Model 3 by expressing that people number consists of people number at Model 1 and four people sitting on the table above and four people sitting on the table below. It can be written as "$(3 \times 4) + (4 \times 2)$". In this stage, AG used a counting strategy by drawing the next figures and counting their elements (Oliveira et al., 2021). Moreover, AG used contextual strategy by constructing a rule based on the information provided in the situation (Lannin, 2005).

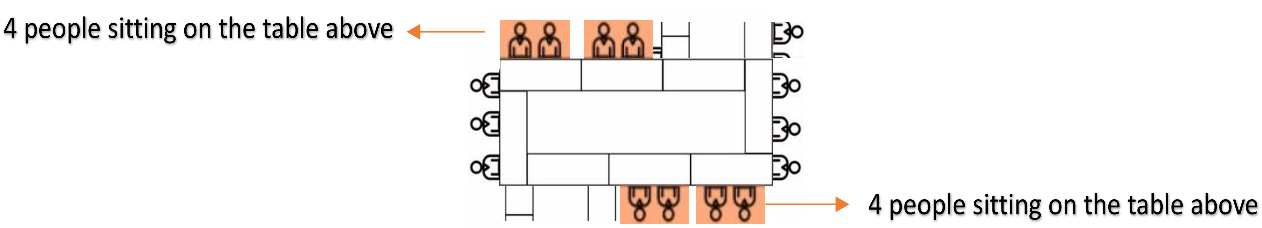


Figure 5: AG express the people number at Model 3.

In far generalization, AG tries to determine the people number at Model 13. Firstly, AG tries to construct a general form of people number by determining people number at Model 4. She used the same strategy previously to get people number at Model 4. She realized that people number at Model 4 consist of people number at Model 1, six people sitting on the table above, and six people sitting on the table below. It can be written by "$(3 \times 4) + (6 \times 2)$". Secondly, AG arranges the people number at Model 1, Model 2, Model 3, and Model 4 consecutively. In this stage, AG constructed a general form of people number at Model s consisting of people number at Model 1 and multiplication between "2" and $2n$ subtracted by 2.

$$P(1) = (3 \times 4)$$
$$P(2) = (3 \times 4) + (2 \times 2)$$
$$P(3) = (3 \times 4) + (4 \times 2)$$
$$P(4) = (3 \times 4) + (6 \times 2)$$
$$\vdots$$
$$P(n) = (3 \times 4) + ((2n - 2) \times 2)$$

Finally, AG determined the people number at Model 13 using a general form by substituting s with 13 and performing a mathematical operation to get people number equal to 60. In this stage, AG performs a formal first for far generalization. She used a numerical correspondence strategy by expressing a relation between quantity of people number and Model s quantity for a distance term in general term.

At formal generalization, AG constructed a final general form of the relationship between people number quantity and Model s quantity using basic algebraic operations to get $P = 4s + 8$, where P refers to people number quantity and s refers to Model s quantity. In addition, AG used the final general form to determine inverse. She substituted P with 76 and performed a mathematical operation to get $s = 17$. It was concluded by AG that there was Model 17 for 75 people.

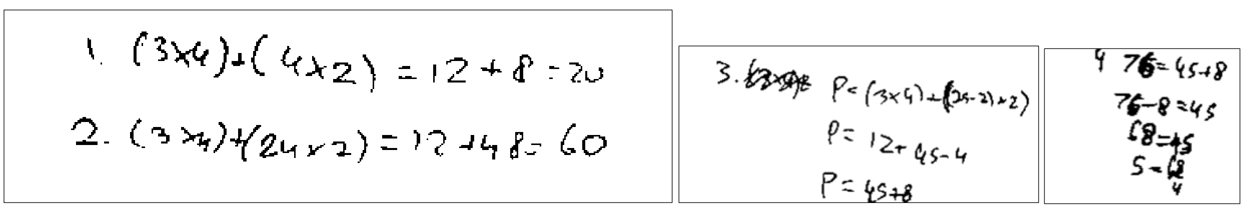


Figure 6: AG answer in solving figural linear pattern task.

Non-Figural Linear Patterns Task

Non-figural linear pattern task involves the generalization of plant height and day quantity. It consists of questions of near generalization, far generalization, formal generalization and determining the inverse.

In this task, AA tries to generalize the relationship between two quantities consisting of days (H) and plant height (T). Here, the information provided consists of plant's height at the 1st, 2nd, and 3rd days.

At near generalization, AA tries to determine plant height at 4th days. Firstly, AA processes an information of plant height at 1st, 2nd, and 3rd days. He realizes that the plant height has the same difference for each day. Furthermore, AA constructed the plant height using the multiple difference strategy, for example: plant height at the 2nd day equals plant height at the 1st day and the difference $(H(2) = H(1) + 2)$ while plant height at 3rd day is equal to plant height at 2nd and difference

$(H(3) = H(2) + 2 = H(1) + 2 \times 2)$. With this strategy, AA generalizes that the plant height at 4th days is equal to plant height at 1st day and the difference multiply by 3 $(H(4) = 4 + (3 \times 2))$. "3" refers to the day number (4th) subtracted by 1 $(3 = 4 - 1)$.

AA uses the same strategy to get plant height at 5th days, which is equal to plant height at 1st day and the difference multiplied by 4. "4" refers to the day number (5th) subtracted by 1 $(4 = 5 - 1)$. At this stage, AA performs a formal generalization. AA not only uses the multiple of difference strategy, but also develops to numerical correspondence strategy by expressing a relation between the two varying quantities for a distant term or in the general term, based on the numerical sequence (Oliveira et al., 2021). In this case, AA generalizes the general form of relationship between day quantity and plant height quantity as $T = 4 + 2(H - 1)$, where T refers to plant height, 4 refers to plant height at 1st day, 2 refers to difference, and H refers to day number.

At far generalization, AA used a general form of relationship between day quantity and plant height quantity previously obtained $(T = 4 + 2(H - 1))$. AA gets the plant height at 7th days by adding up the plant height at 1st day and multiplying the difference by 6 (obtained from $7 - 1$). Likewise, AA gets the plant height at 10th day by adding up the plant height at 1st day and multiplying the difference by 9 (obtained from $10 - 1$).

In determining the inverse, AA tries to determine on what day the plant has 86 cm of height. AA used the general form of relationship between day quantity and plant height quantity "$T = 4 + 2(H - 1)$", substituted T with 86, and performed a mathematical operation to get $H = 42$. In this stage, AA has an understanding of algebraic operation by substituting and operating mathematical symbol.

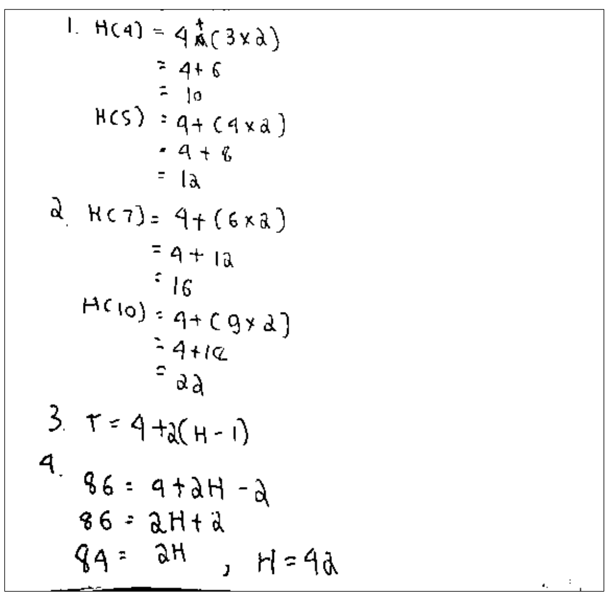

1. H(4) = 4+(3x2)
= 4+6
= 10
H(5) = 4+(4x2)
= 4+8
= 12
2. H(7) = 4+(6x2)
= 4+12
= 16
H(10) = 4+(9x2)
= 4+18
= 22
3. T = 4+2(H-1)
4. 86 = 4+2H-2
86 = 2H+2
84 = 2H , H = 42

Figure 7: AA answer in solving non-figural linear pattern task

At near generalization, AG observed the plant height on the 1st, 2nd, and 3rd days. She found that the plant height had the same difference for each term. After realizing that there is a constant difference for each term, AG conjectured the plant height at 2nd day as plant height at 1st and the difference, likewise, the plant height at 3rd was equal to the plant height at 2nd and the difference.

$$H1: T = 4$$

$$H2: T = 4 + 2 = 6$$

$$H3: T = 4 + 2 \times 2 = 8$$

AG determined the plant height at 4th day by expressing an equation involving plant height at 1st and multiplying the difference by 3. 3 is obtained by 4 subtracted by 1, where 4 refers to 4th term. its' written with "$H4: T = 4 + (3 \times 2)$". She also expressed plant height at 5th day as plant height at 1st day and multiplied the difference by 4, where 4 is obtained by subtracted 5th term by 1, "$H5: T = 4 + 4 \times 2$". In this stage, AG used a multiple of difference strategy by use the difference between consecutive terms as a multiplicative factor to obtain the distant term (Oliveira et al., 2021).

4 + 2 + 2 + 2 = 4 + 3 x 2

↓ ↓ ↓ ↓

1st day 2th day 3th day 3th day

Figure 8: AG strategies in determining plant height at 4th days

At far generalization, AG determined plant height at 7th and 10th day. She used the same strategy previously by constructing plant height as a sum between plant height at 1st day and multiplicative factor. For example, AG expressed plant height at 7th day as a sum between plant height at 1st and the difference multiplied by 6, where 6 is obtained by subtracted 7th term by 1 ($H7: T = 4 + 6 \times 2$). Likewise, AG expressed plant height at 10th day as a sum between plant height at 1st and the difference multiplied by 9, where 9 is obtained by subtracted 10th term by 1 ($H10: T = 4 + 9 \times 2$).

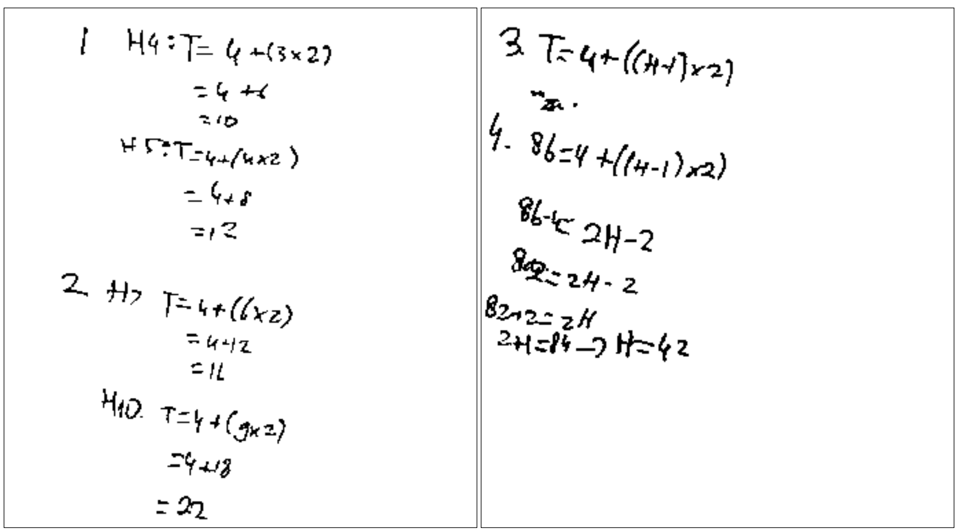
Plant Height at 7th days	$4 + (7 - 1) \times 2 = 4 + 6 \times 2$
Plant Height at 10th days	$4 + (10 - 1) \times 2 = 4 + 9 \times 2$
	1st day difference

Figure 9: AG strategies in determining plant height at 7th and 10th days

At formal generalization, AG expressed the relationship between plant height quantity and day quantity symbolically. She conjectured a general term for these relationships from activities at near generalization and far generalization. The general term of the relationship between plant height and the day quantity was expressed as $T = 4 + (H - 1) \times 2$, where T refers to plant height, 4 refers to plant height at 1st day, H refers to the day's quantity, and 2 refers to the difference.

$$H1{:}\, T = 4$$
$$H2{:}\, T = 4 + 1 \times 2$$
$$H3{:}\, T = 4 + 2 \times 2$$
$$H4{:}\, T = 4 + 3 \times 2$$
$$H5{:}\, T = 4 + 4 \times 2$$
$$\vdots$$
$$Hn{:}\, T = 4 + (n - 1) \times 2$$

To determine the inverse, AG tries to determine on what day the plant height was 75. She used the general form previously by substituting T with 86 and performing mathematical operations to get $H = 42$.



(a) (b)

Figure 10: (a) AG Near and Far Generalization; (b) AG Formal Generalization and Determining Inverse in Solving Non-Figural Linear Pattern Task

	Figural Linear Pattern	Non-figural Linear pattern
AA	Near generalization	Near generalization
	- Use counting strategy by drawing next figure and count people number by multiplicative approach $[(8 \times 2) + (4 \times 1)]$	- Use multiple of difference strategy by using the difference between consecutive terms of people number as a multiplicative factor

	Figural Linear Pattern	**Non-figural Linear pattern**
	Far generalization - Use contextual strategy by constructing a rule based on information providing about people number and Model size, then relating the rule to a counting technique $[(4 \times 13) + 8]$. Formal generalization - Use a multiple difference strategy by using the difference between consecutive terms of people number as a multiplicative factor (adjusting or not the result) to obtain the general term $(P = 12 + (S - 1)4)$ Determine inverse - Use a general rule in formal generalization $(P = 4S + 8)$ and substitute people number known then operate by mathematical operation to get final result.	(adjusting or not the result) to obtain the distance term $[H(4) = 4 + (3 \times 2)\ dan\ H(5) = 4 + (4 \times 2)]$ Far generalization - Use multiple of difference strategy by using the difference between consecutive terms of people number as a multiplicative factor (adjusting or not the result) to obtain the distance term $[H(7) = 4 + (6 \times 2)\ dan\ H(10) = 4 + (9 \times 2)]$. Formal generalization - Use a multiple difference strategy by using the difference between consecutive terms of people number as a multiplicative factor (adjusting or not the result) to obtain the general term $[T = 4 + 2(H - 1)$. Determine inverse - Use a general rule in formal generalization $(T = 4 + 2(H - 1)$ and substitute people number known then operate by mathematical operation to get final result.
AG	Near generalization - Use counting strategy by drawing next figure and count people number by multiplicative approach. - Use contextual strategy by constructing a rule based on information provided in the situation $[(3 \times 4) + (4 \times 2)]$.	Near generalization - Use multiple of difference strategy by using the difference between consecutive terms of people number as a multiplicative factor (adjusting or not the result) to obtain the distance term $[H_4 = 4 + (3 \times 2)\ dan\ H_5 = 4 + (4 \times 2)]$ Far generalization

Figural Linear Pattern	Non-figural Linear pattern
Far generalization - Use contextual strategy by constructing a rule based on information providing about people number and Model size, then relating the rule to a counting technique $[(3 \times 4) + (24 \times 2)]$.	- Use multiple of difference strategy by using the difference between consecutive terms of people number as a multiplicative factor (adjusting or not the result) to obtain the distance term $[H_7 = 4 + (6 \times 2)\ dan\ H_{10} = 4 + (9 \times 2)]$.
Formal generalization - Use a multiplicative reasoning by proportional strategy by use multiplicative strategies, starting from one known term of the sequence to find distant terms or the general term $P = 12 + (2S - 2)2$	Formal generalization - Use a multiple difference strategy by using the difference between consecutive terms of people number as a multiplicative factor (adjusting or not the result) to obtain the general term $[T = 4 + (H - 1) \times 2$.
Determine inverse Use a general rule in formal generalization $(P = 4S + 8)$ and substitute people number known then operate by mathematical operation to get final result.	Determine inverse Use a general rule in formal generalization $(T = 4 + (H - 1) \times 2$ and substitute people number known then operate by mathematical operation to get final result.

Table 3: Differences in functional thinking strategies of AA and AG

The finding showed that gifted students of primary school are able to perform functional thinking even though they have not obtained number pattern material in their school. Students not only perform functional thinking recursively, but they are also able to conjecture a general form of the relationship between two variables symbolically. It's in line with previous studies (Blanton & Kaput, 2004; Blanton & Kaput, 2005; Tanıslı, 2011; Warren et al., 2006; Warren & Cooper, 2005) which revealed that elementary students are able to generalize and represen the relationship by correspondence. Blanton & Kaput (2004, 2005) found that students were able to think functionally at the kindergarten level co-variationally and were able to think functionally as a correspondent in the 1st grade. Warren et al. (2006) found that elementary students are capable not only of developing functional thinking but also of communicating their thinking both verbally and symbolically. Tanıslı (2011) found that fifth-grader students thought on co-variation while working with the linear function tables.

This study revealed that gifted primary students perform functional thinking by generalizing the relationship between two variables in different ways. They used some strategies in performing the generalization, such as counting, multiple difference, and contextual strategy (Lannin, 2005;

Oliveira et al., 2021). it's in line with Erdogan & Gul (2023) finding that gifted students used some strategies in generalizing linear pattern tasks, such as functional, chunking, and contextual strategies. This finding also promotes the study by Gutierrez et al. (2018) which found that mathematically gifted students are much faster than average students in learning mathematics content. It showed that gifted students are able to generalize geometric patterns in different ways, from recursive type to functional type.

Gifted students in this study used a multiplicative difference strategy to get a general term of relationship between two quantities. It was shown that gifted students are able to develop deconstructive generalization (DG), which refers to direct or closed polynomial formula that students construct form known stage as a result of figure (Rivera & Becker, 2011). In this study, gifted students are able to developing DG in solving figural and non-figural linear pattern task. They constructed a polynomial formula as a general rule using multiplicative difference strategy.

Another aspect of gifted students' mental flexibility was found in switching from one solution method to another. In solving figural linear patterns, AA switched from counting strategy to multiple difference strategy and AG from counting strategy to multiplicative reasoning by proportional strategy. Previous studies support this finding (Amit & Neria, 2008; Assmus & Fritzlar, 2022; Gutierrez et al., 2018). Amit & Neria (2008) found that students who began solving problems using the recursive method, usually showed flexibility in trying an alternative approach. Assmus & Fritzlar (2022) suggested that gifted students show flexibility in mathematical mental process. Gutierrez et al. (2018) declared that gifted students quickly move from one strategy to another, which they think is more useful and beneficial.

In solving a figural linear pattern, gifted students performed a reflection by observing the pattern, grasping its central attribute, and performing far generalizations. This finding highlights the inseparable connection between generalization and reflection (Amit & Neria, 2008). Ellis (2007) introduced reflection generalization, which refers to the final statement of a verval or written generalization. In this study, gifted students performed reflection generalization by writing a general rule of relationship between two quantities symbolically in solving figural and non-figural linear pattern tasks.

In the near and far generalization of a figural linear pattern, gifted students observed people sitting on the table figure and determined the near and far terms using counting strategy. Gutierrez et al. (2018) stated that most geometric patterns show a procedure to split the figures into parts that can be considered like independent patterns, making it easy to find a general procedure to calculate the values of the terms in the sequence. This procedure is termed by functional figural decomposition of the pattern with a cognitive demand in the procedures with connections level (Gutierrez et al., 2018).

There are different strategies used by gifted students in determining general rule of relationship between two quantities of a figural linear pattern. AA used a multiple difference strategy and found the general rule as $T = 12 + 4(S - 1)$, while AG used multiplicative approach to find general rule as $T = 12 + 2(2S - 2)$. It showed that gifted students have multiplicative constructive nonstandard of algebraic generalization type which refers to seeing figural pattern as consisting of nonoverlapping part (Rivera, 2010). Student's multiplicative reasoning plays a role in functional thinking development (Askew, 2018).

In solving figural linear pattern, gifted students inclined in using non-explicit counting strategy. They were able to developing recursive strategy to multiple differences. Recursive rules involve recognizing and using the change from term-to-term in the dependent variable (Lannin, Barker, and Townsend, 2006). Students understand that there is a same difference from term-to-term and they add the difference that they find recursively. However, recursive reasoning can limit the depth of functional thinking that students attain (Tanıslı, 2011).

In solving non-figural linear pattern, gifted students performed the generalization by correspondence relationship. It's based on identifying correlations between variables (M. Blanton, 2008). Gifted students are able to build a conjecture between the day and plant height variable. Drawing the finding, following the previous study (Blanton & Kaput, 2004; Blanton & Kaput, 2005; Tanıslı, 2011; E. A. Warren et al., 2006; E. Warren & Cooper, 2005) that elementary students are able to generalize and represent the relationship by correspondence. Blanton & Kaput (2004, 2005) found that students were able to think functionally at the kindergarten level co-variationally and were able to think functionally as a correspondent in the 1st grade. Warren et al. (2006) found that elementary students are capable not only of developing functional thinking but also of communicating their thinking both verbally and symbolically. Tanıslı (2011) found that five-grader students thought on co-variation while working with the linear function tables.

CONCLUSION

This study found that mathematically gifted students are able to use functional thinking in solving a figural and non-figural linear pattern task. In solving a figural linear pattern task, gifted students perform FT consisting of: (a) near generalization by counting, multiplicative approach, and contextual strategy; (b) far generalization by contextual strategy; (c) formal generalization by multiple difference and proportional strategy; and (d) determining inverse by using general rule. In solving a non-figural linear pattern task, gifted students perform FT consist of: (a) near generalization, far generalization, and formal generalization by multiple difference strategy; and (b) determine inverse by using general rule.

The findings of this study suggest that gifted students of elementary school in Indonesia have a potential in developing functional thinking. They are able in performing functional thinking in different ways. They also used different strategies in solving a figural and non-figural linear pattern.

References

[1] Amit, M., & Neria, D. (2008). Rising to the challenge: using generalization in pattern problems to unearth the algebraic skills of talented pre-algebra students. *ZDM Mathematics Education*, *40*(2), 111–129. https://doi.org/10.1007/s11858-007-0069-5

[2] Askew, M. (2018). Multiplicative reasoning: teaching primary pupils in ways that focus on

functional relations. *Curriculum Journal*, *29*(3), 406–423. https://doi.org/10.1080/09585176.2018.1433545

[3] Assmus, D., & Fritzlar, T. (2022). Mathematical creativity and mathematical giftedness in the primary school age range: an interview study on creating figural patterns. *ZDM - Mathematics Education*, *54*(1), 113–131. https://doi.org/10.1007/s11858-022-01328-8

[4] Blanton, M. (2008). *Algebra and the elementary classroom. Transforming thinking, transforming practice*. Portsmounth,NH: Heinemann.

[5] Blanton, M., & Kaput, J. J. (2004). Elementary Grades Students ' Capacity for Functional Thinking. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, *2*, 135–142.

[6] Blanton, M. L., & Kaput, J. J. (2005). Characterizing a Classroom Practice That Promotes Algebraic Reasoning. *Journal for Research in Mathematics Education*, *36*(5), 412–446. https://doi.org/https://doi.org/10.2307/30034944

[7] Blanton, M., Levi, L., Crites, T., & Dougherty, B. J. (2011). *Developing essential understandings of algebraic thinking, Grades 3-5*. Reston, VA: The National Council of Teachers of Mathematics.

[8] Chua, B. L., & Hoyles, C. (2014). Generalisation of Linear Figural Patterns in Secondary School Mathematics. *The Mathematics Educator*, *15*(2), 1–30. Retrieved from http://math.nie.edu.sg/ame/matheduc/journal/v15_2/n1.aspx

[9] Cohen, L., Manion, L., & Morrison, K. (2000). *Research methods in education* (5th editio). London: Routledge.

[10] Confrey, J., & Smith, E. (1991). A framework for functions: Prototypes, multiple representations, and transformations. In R. Underhill & C. Brown (Eds.), *Proceedings of the thirteenth annual meeting of the north American chapter of the international group for the psychology of mathematics education* (pp. 57–63). Blacksburg, VA: Virginia Polytechnic Institute & State University.

[11] Creswell, J. W. (2012). *Educational Research: Planning, Conducting, and Evaluating Quantitative and Qualitative Research* (Fourth). Boston: Pearson Education.

[12] Ellis, A. B. (2007). A Taxonomy for Categorizing Generalizations : Generalizing Actions and Reflection Generalizations. *The Journal of The Learning Sciences*, *16*(2), 221–262. https://doi.org/10.1080/10508400701193705

[13] Erdogan, F., & Gul, N. (2023). Reflections from the generalization strategies used by gifted students in the growing geometric pattern task. *Journal of Gifted Education and Creativity*, *9*(4), 369–385. Retrieved from https://dergipark.org.tr/en/pub/jgedc/issue/74010/1223156

[14] Fraenkel, J. R., Wallen, N. E., & Hyun, H. H. (2012). *How to Design and Evaluate Research in Education*. New York: McGraw-Hill.

[15] Gagné, F. (1985). Giftedness and Talent: Reexamining a Reexamination of the Definitions. *Gifted Child Quarterly*, *29*(3), 103–112. https://doi.org/10.1177/001698628502900302

[16] Girit Yildiz, D., & Durmaz, B. (2021). A Gifted High School Student's Generalization Strategies of Linear and Nonlinear Patterns via Gauss's Approach. *Journal for the Education of the Gifted*, *44*(1), 56–80. https://doi.org/10.1177/0162353220978295

[17] Gutierrez, A., Benedicto, C., Jaime, A., & Arbona, E. (2018). The Cognitive Demand of a Gifted Student's Answers to Geometric Pattern Problems. In F. M. Singer (Ed.),

Mathematical Creativity and Mathematical Giftedness: Enhancing Creative Capacities in Mathematically Promising Students (ICME-13 Mo, pp. 169–198). Cham: Springer International Publishing. https://doi.org/10.1007/978-3-319-73156-8_7

[18] Kaput, J. (1999). Teaching and learning a new algebra with understanding. In E. Fennema & T. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 133–155). Mahwah: Erlbaum.

[19] Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren* (Vol. 8). Chicago: University of Chicago Press.

[20] Lannin, J. K. (2005). Generalization and Justification : The Challenge of Introducing Algebraic Reasoning Through Patterning Activities. *Mathematical Thinking and Learning*, *7*(3), 231–258. https://doi.org/DOI: 10.1207/s15327833mtl0703_3

[21] Lannin, J. K., Barker, D. D., & Townsend, B. E. (2006). Recursive and explicit rules: How can we build student algebraic understanding? *Journal of Mathematical Behavior*, *25*(4), 299–317. https://doi.org/10.1016/j.jmathb.2006.11.004

[22] Leedy, P. D., & Ormrod, J. E. (2019). *Practical Research: Planning and Design* (12th Editi). United State: Pearson Education.

[23] Leikin, R. (2018). Giftedness and High Ability in Mathematics. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 1–11). Cham: Springer International Publishing. https://doi.org/10.1007/978-3-319-77487-9_65-4

[24] Leikin, R. (2021). When practice needs more research: the nature and nurture of mathematical giftedness. *ZDM - Mathematics Education*, *53*(7), 1579–1589. https://doi.org/10.1007/s11858-021-01276-9

[25] Lepak, J. R., Wernet, J. L. W., & Ayieko, R. A. (2018). Capturing and characterizing students' strategic algebraic reasoning through cognitively demanding tasks with focus on representations. *Journal of Mathematical Behavior*, *50*(January), 57–73. https://doi.org/10.1016/j.jmathb.2018.01.003

[26] Mason, J., Stacey, K., & Burton, L. (2010). *Thinking Mathematically*. Edinburgh: Pearson.

[27] Mejía-Ramos, J. P., & Weber, K. (2020). Using task-based interviews to generate hypotheses about mathematical practice: mathematics education research on mathematicians' use of examples in proof-related activities. *ZDM - Mathematics Education*, *52*(6), 1099–1112. https://doi.org/10.1007/s11858-020-01170-w

[28] Merriam, S. B. (2015). *Qualitative research: A guide to design and implementation* (4th ed.). San Fransisco: Jossey-Bass A Wiley Brand.

[29] Ministary of Educations and Culture (MoEC). (2016). *Peraturan Menteri Pendidikan dan Kebudayaan Republik Indonesia Nomor 21 Tahun 2016* (No. 21). Jakarta, Indonesia: Kementerian Pendidikan dan Kebudayaan RI.

[30] NCTM. (2000). *Principless and Standards for School Mathematics*. Reston: NCTM.

[31] Oliveira, H., Polo-Blanco, I., & Henriques, A. (2021). Exploring prospective elementary mathematics teachers' knowledge: A focus on functional thinking. *Journal on Mathematics Education*, *12*(2), 257–278. https://doi.org/10.22342/jme.12.2.13745.257-278

[32] Paz-Baruch, N., Leikin, M., & Leikin, R. (2022). Not any gifted is an expert in mathematics and not any expert in mathematics is gifted. *Gifted and Talented International*, *37*(1), 25–41. https://doi.org/10.1080/15332276.2021.2010244

[33] Pitta-Pantazi, D., Christou, C., Kontoyianni, K., & Kattou, M. (2011). A Model of mathematical giftedness: Integrating natural, creative, and mathematical abilities. *Canadian Journal of Science, Mathematics and Technology Education*, *11*(1), 39–54. https://doi.org/10.1080/14926156.2011.548900

[34] Rivera, F., & Becker, J. R. (2003). the Effects of Numerical and Figural Cues on the Induction Processes of Preservice Elementary Teachers. *Proceedings of the 27th Conference of the International Group for the Psychology of \nMathematics Education Held Jointly with the 25th Conference of PME-NA*, *4*, 63–70.

[35] Rivera, F. D. (2010). Visual templates in pattern generalization activity. *Educational Studies in Mathematics*, *73*(3), 297–328. https://doi.org/10.1007/s10649-009-9222-0

[36] Rivera, F. D., & Becker, J. R. (2011). Formation of Pattern Generalization Involving Linear Figural Patterns Among Middle School Students: Results of a Three-Year Study. In J. Cai & E. Knuth (Eds.), *Early Algebraization, Advances in Mathematics Education* (pp. 323–366). Berlin: Springer-Verlag Berlin Heidelberg. https://doi.org/10.1007/978-3-642-17735-4_18

[37] Rusdiana, M., Suriaty, M., Sutawidjaja, A., & Irawan, E. B. (2017). *Pattern Generalization by Elementary Students*. *100*, 379–381. https://doi.org/10.2991/seadric-17.2017.82

[38] Rusdiana, Sutawidjaja, A., Irawan, E. B., & Sudirman. (2018). Students perception on a problem of pattern generalization. *Journal of Physics: Conference Series*, *1116*(2). https://doi.org/10.1088/1742-6596/1116/2/022040

[39] Smith, E. (2008). Representational thinking as a framework for introducing functions in the elementary curriculum. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 95–132). London & Newyork: Lawrence Erlbaum/Taylor & Francis Group & NCTM.

[40] Stacey, K. (1989). Finding and Using Patterns in Liniar generalising Problems. *Educational Studies in Mathematics*, *20*(2), 147–164. https://doi.org/10.1007/bf00579460

[41] Stephens, A. C., Fonger, N., Strachota, S., Isler, I., Blanton, M., Knuth, E., & Gardiner, A. M. (2017). A Learning Progression for Elementary Students ' Functional Thinking A Learning Progression for Elementary Students ' Functional. *Mathematical Thinking and Learning*, *19*(3), 143–166. https://doi.org/10.1080/10986065.2017.1328636

[42] Syawahid, M., Purwanto, Sukoriyanto, & Sulandra, I. M. (2020). Elementary students' functional thinking: From recursive to correspondence. *Journal for the Education of Gifted Young Scientists*, *8*(3), 1031–1043. https://doi.org/10.17478/JEGYS.765395

[43] Syawahid, Muhammad. (2022). Elementary students' functional thinking in solving context-based linear pattern problems. *Beta: Jurnal Tadris Matematika*, *15*(1), 37–52. https://doi.org/10.20414/betajtm.v15i1.497

[44] Tanıslı, D. (2011). Functional thinking ways in relation to linear function tables of elementary school students. *The Journal of Mathematical Behavior*, *30*, 206–223. https://doi.org/10.1016/j.jmathb.2011.08.001

[45] Warren, E. A., Cooper, T. J., & Lamb, J. T. (2006). Investigating functional thinking in the elementary classroom : Foundations of early algebraic reasoning. *Journal of Mathematical Behavior*, *25*, 208–223. https://doi.org/10.1016/j.jmathb.2006.09.006

[46] Warren, E., & Cooper, T. O. M. (2005). Introducing Functional Thinking in Year 2 : a case study of early algebra teaching. *Contemporary Issues in Early Childhood*, *6*(2), 150–162.

[47] Wilkie, K. J., & Clarke, D. M. (2016). Developing students ' functional thinking in algebra through different visualisations of a growing pattern ' s structure. *Mathematics Education Research Journal*, *28*(2), 223–243. https://doi.org/10.1007/s13394-015-0146-y