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The Problem Corner

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The Purpose of ***The Problem Corner*** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem and its solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Hello, problem solvers!

I'm pleased to report that we've received correct and insightful solutions for Problem 26 and Problem 27 in *The Problem Corner*. These submissions have not only met our accuracy requirements but also demonstrated thoughtful and effective problem-solving strategies. Our goal is to present exemplary solutions that inspire and advance mathematical knowledge worldwide.

Solutions to **Problems** from the Previous Issue.

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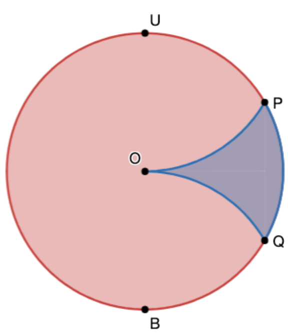
“Curvy” slice of a circle problem.

Problem 26

Proposed by Ivan Retamoso, BMCC, USA.

The diagram illustrates a circle with a radius of 6 inches and center O, with UB as its diameter. Points P and Q are positioned on the circle so that OP and OQ are arcs of circles with a radius of 6 inches and centers at U and B, respectively. Determine, in exact form, the area of the "blue" region OPQ.



First solution to problem 26

By Dr. Hosseinali Gholami, University Putra Malaysia, Serdang, Malaysia

Our solver delivered two equivalent solutions: one using only geometric formulas and symmetry, and the other using integral calculus. Each solution is explained in detail, with accompanying graphs that enhance understanding.

Solution 1:

As respect to the following shape, the surface of areas 1, 2, 3, 4 and 5 are equal, because each of these three triangles is an equilateral triangle. The surface of area 3 is calculated as below.

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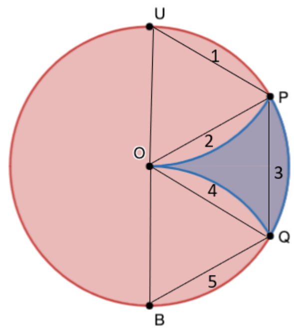
$$S = \frac{1}{6}\pi r^2 - \frac{r^2\sqrt{3}}{4} = \frac{1}{6}\pi 6^2 - \frac{6^2\sqrt{3}}{4} = 6\pi - 9\sqrt{3}.$$



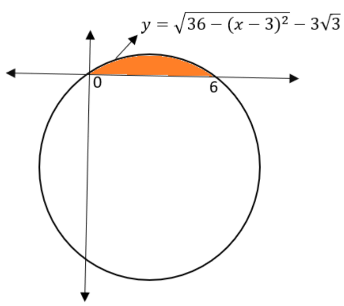
Therefore, the area of the "blue" region OPQ is determined as follows.

$$S_{OPQ} = \frac{1}{6}\pi r^2 - 2S = \frac{1}{6}\pi 6^2 - 2\left(6\pi - 9\sqrt{3}\right) = 18\sqrt{3} - 6\pi.$$

Solution 2:

Based on the figure above, we transfer the area 3 on the coordinates axes as shown in the following figure.

The equation of this circle with the center $(3, -3\sqrt{3})$ is $(x-3)^2 + (y - 3\sqrt{3})^2 = 36$.



This area is calculated as $S = \int_0^6 (\sqrt{36-(x-3)^2} - 3\sqrt{3})\, dx$. The value of S is determined according to the formula

$$\int \sqrt{36-(x-3)^2}\, dx = \frac{1}{2}(x-3)\sqrt{36-(x-3)^2} + 18\sin^{-1}(\frac{x-3}{6}) - 3\sqrt{3}x + c$$

as below.

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$$S = \left(\frac{3}{2} \times 3\sqrt{3} + 18 \times \frac{\pi}{6} - 3\sqrt{3} \times 6\right) - \left(\frac{-3}{2} \times 3\sqrt{3} - 18 \times \frac{\pi}{6}\right) = 6\pi - 9\sqrt{3}.$$

Therefore, the answer of this problem is obtained as follows.

$$S_{OPQ} = \frac{1}{6}\pi r^2 - 2S = \frac{1}{6}\pi 6^2 - 2\left(6\pi - 9\sqrt{3}\right) = 18\sqrt{3} - 6\pi.$$

Second solution to problem 26

By Dr. Aradhana Kumari, Borough of Manhattan Community College, USA.

This solution relies solely on Integral Calculus and symmetry. Our solver cleverly uses two curves to compute the area of the requested region. The diagrams and labeling make the solution appealing and easy to follow.

Solution: Consider the figure 1. Below

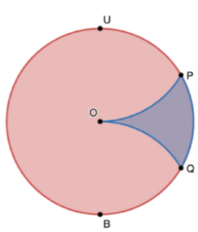

Figure 1.

Let's put the Figure 1. given in the problem to the rectangular coordinate system.

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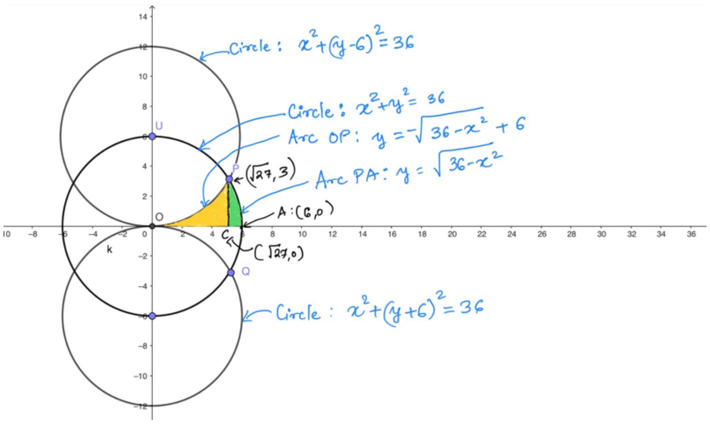

Figure 2.

Point P is the intersection of the circle given by equation $x^2 + y^2 = 36$ and the circle $x^2 + (y - 6)^2 = 36$. The coordinates of the point P can be obtained by solving

$x^2 + y^2 = x^2 + (y - 6)^2$,

i.e $y^2 = (y - 6)^2$

Hence $-12y = -36$

$y = 3$

Substituting the value of $y = 3$ in the equation $x^2 + y^2 = 36$ we get $x = \sqrt{27}$

Hence the coordinates of the point P $(\sqrt{27}, 3)$.

The required area is 2 times the area of yellow shaded region plus the area of green shaded region.

The equation of circle with center O $(0,0)$ and radius 6 is $x^2 + y^2 = 36$...................(1)

The equation of circle with center U $(0,6)$ and radius 6 is $x^2 + (y - 6)^2 = 36$ ....(2)

The equation of circle with center B $(0, -6)$ and radius 6 is $x^2 + (y + 6)^2 = 36$.......(3)

To find the area of yellow shaded region, we need to know the equation of curve passing through O to P in the first quadrant. This curve lies on the circle with center U and radius 6. Using equation given by (2) we get $y = -\sqrt{36 - x^2} + 6$

The area of the yellow shaded region is $\int_0^{\sqrt{27}} \{-\sqrt{36 - x^2} + 6\}\, dx$

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$= \int_0^{\sqrt{27}} \{-\sqrt{36-x^2}\}\, dx + \int_0^{\sqrt{27}} 6\ dx = \frac{-9\sqrt{3}}{2} - 6\pi + 6\sqrt{27}$

To find the area of the green shaded region, first we need to find the equation of the curve PC in the first quadrant. This curve lies on the circle with center O and radius 6. Using equation given by (1) we get $y = \sqrt{36-x^2}$

The area of the green shaded region is

$\int_0^{\sqrt{27}} \{\sqrt{36-x^2}\} dx = \int_{\sqrt{27}}^{6} \{\sqrt{36-x^2}\}\, dx = 3\pi - \frac{9\sqrt{3}}{2} - 6\pi$

The required area of the blue region $OPQ = 2\left(\frac{-9\sqrt{3}}{2} - 6\pi + 6\sqrt{27} + 3\pi - \frac{9\sqrt{3}}{2}\right)$

$= 2\left(-9\sqrt{3} - 3\pi + 18\sqrt{3}\right) = 2\left(9\sqrt{3} - 3\pi\right) = 18\sqrt{3} - 6\pi$

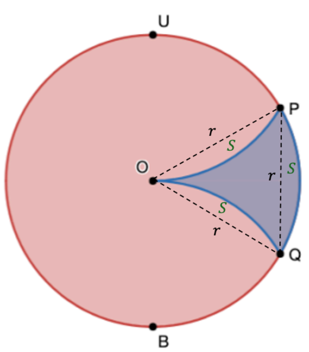
Third solution to problem 26

By Dr. Abdullah Kurudirek, Mathematics Ed. Department, Tishk International University.

By combining the proportions of circle sectors derived from central angles with the area formula for an equilateral triangle, our solver produces a solution that is both concise and elegant.

Solution:

We can join the points O, P, and Q by auxiliary lines (each has the same length $r = 6$ inches) and then we get an equilateral triangle ΔOPQ and its area equals $\frac{\sqrt{3}}{4}r^2$



Area of segment $S = \frac{\frac{\pi}{3}}{2\pi}\pi r^2 - \frac{\sqrt{3}}{4}r^2 = \frac{1}{6}\pi 6^2 - \frac{\sqrt{3}}{4}6^2 = 6\pi - 9\sqrt{3}$

And the area of the blue region OPQ will be as follows:

$$OPQ = Area\ of\ Sector - 2S = 6\pi - 2\left(6\pi - 9\sqrt{3}\right) = 18\sqrt{3} - 6\pi$$

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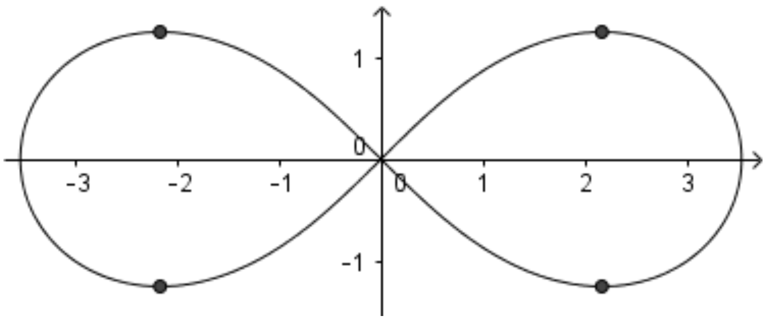
The "bow tie" problem

Problem 27

Proposed by Ivan Retamoso, BMCC, USA.

Consider the lemniscate curve $2(x^2 + y^2)^2 = 25(x^2 - y^2)$.



a. Find the slope of the tangent line to the lemniscate in terms of the variables x and y.

b. The four points on the lemniscate where the tangent line is horizontal are all on the intersection of the lemniscate with circle $x^2 + y^2 = k$, find the value of k.

First solution to problem 27

By Dr. Abdullah Kurudirek, Mathematics Ed. Department, Tishk International University.

The approach employed by our solver makes excellent use of implicit differentiation and partial derivatives. The final DESMOS graph plays a crucial role in visualizing and comprehending the results.

a) If we consider the lemniscate curve $2(x^2 + y^2)^2 = 25(x^2 - y^2)$ as $2x^4 + 4x^2y^2 + 2y^4 - 25x^2 + 25y^2 = 0$ then $\frac{dy}{dx} = -\frac{diff.\ w.r.t\ x}{diff.\ w.r.t\ y} = -\frac{x(4x^2 + 4y^2 - 25)}{y(4x^2 + 4y^2 + 25)}$ by using implicit differentiation, and this is the slope of the tangent line to the lemniscate in terms of the variables x and y.

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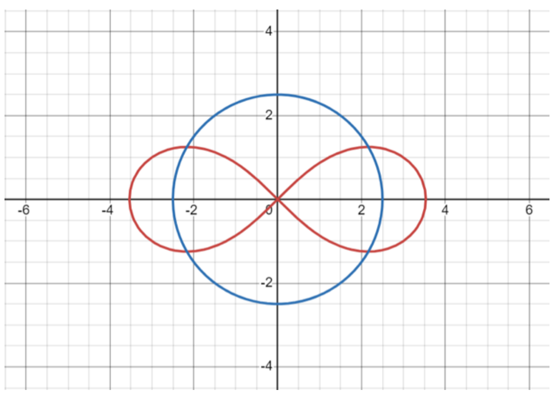
b) The four points on the lemniscate where the tangent line is horizontal means $\frac{dy}{dx} = -\frac{x(4x^2+4y^2-25)}{y(4x^2+4y^2+25)} = 0$ so $4x^2+4y^2-25=0$ and $x \neq 0$ (not one of the four points)

$4x^2+4y^2-25=0$ then $x^2+y^2=\frac{25}{4}$ so $k=\frac{25}{4}$, and by the way, if we substitute $x^2+y^2=\frac{25}{4}$ in the original equation, we find $x=\pm\frac{5\sqrt{3}}{4}$ and $y=\pm\frac{5}{4}$.

On the other hand, by using Desmos graphing calculator we can see the intersection of the lemniscate with the circle as follows.



Second solution to problem 27

By Dr. Aradhana Kumari, Borough of Manhattan Community College, USA.

By first applying implicit differentiation to compute dydx\frac{dy}{dx}dxdy, our solver then uses the property that horizontal lines have a slope of 0 to smartly determine the value of kkk. The step-by-step solution is clear and engaging to follow.

Solution a). Consider the lemniscate curve given by the equation below

$2\,(x^2+y^2)^2 = 25\,(x^2-y^2)$............(1)

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To find the slope of the tangent line to the lemniscate curve in terms of x and y we differentiate the equation given by (1) we get

$$\frac{d}{dx}2\,(x^2+y^2)^2 \;=\frac{d}{dx}25\,(x^2-y^2)$$

$4(x^2+y^2)\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}25\,(x^2-y^2)$

$4(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right)= \;25\left(2x-2y\frac{dy}{dx}\right)$

$4(x^2+y^2)\left(x+y\frac{dy}{dx}\right)= \;25\left(x-y\frac{dy}{dx}\right)$

$4(x^2+y^2)x+\;4\mathrm{y}(x^2+y^2)\frac{dy}{dx}\;=\;25x-25y\frac{dy}{dx}$

$4\mathrm{y}(x^2+y^2)\frac{dy}{dx}+25y\frac{dy}{dx}=25x-4(x^2+y^2)x$

$[4\mathrm{y}(x^2+y^2)+25y]\frac{dy}{dx}=x[(25-4(x^2+y^2)]$

$$\frac{dy}{dx}=\frac{x[25-4(x^2+y^2)]}{\mathrm{y}[25+4(x^2+y^2)]}$$

The slope of the tangent line at the point (x,y) is given by $\frac{x[25-4(x^2+y^2)]}{\mathrm{y}[25+4(x^2+y^2)]}$.

Solution b). If the tangent line is horizontal, then $\frac{dy}{dx}=0$ that means

$$\frac{dy}{dx}=\frac{x[25-4(x^2+y^2)]}{\mathrm{y}[25+4(x^2+y^2)]}=0$$

Hence $\quad x[(25-4(x^2+y^2)]=0$

Either $\quad x=0$ or $[25-4(x^2+y^2)]=0$

Consider $\;[25-4(x^2+y^2)]=0,$

$25=4(x^2+y^2)$

Hence $x^2+y^2=\frac{25}{4},\;$ therefore the required value of k is $\frac{25}{4}$.

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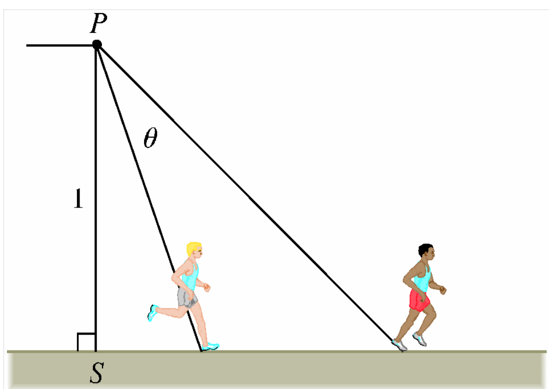
Dear fellow problem solvers,

I'm thrilled you enjoyed working through problems 26 and 27 and that you've expanded your mathematical strategies. Let's dive into the next set of problems to further develop your skills.

Problem 28

Proposed by Ivan Retamoso, BMCC, USA.

An observer is positioned at point P, one unit away from a track. Two runners begin at point S, which is illustrated in the diagram, and move along the track. One of the runners runs at a speed three times faster than the other. Determine the maximum angle θ that the observer's line of sight forms between the two runners.



Problem 29

Proposed by Ivan Retamoso, BMCC, USA.

A regular octagon $ABCDEFGH$ has sides that are 2 units in length. The points W, X, Y, and Z are the midpoints of the sides $\overline{AB}$, $\overline{CD}$, $\overline{EF}$, and $\overline{GH}$, respectively. Find the probability that a point chosen uniformly at random from inside the octagon $ABCDEFGH$ will be located inside the quadrilateral $WXYZ$. Give your answer in exact form.