

## Editorial from Bronisław Czarnocho, the Chief Editor of MTRJ



We would like to introduce this issue of MTRJ through two papers on learning and teaching trigonometry, the subject of mathematics whose pedagogy has not yet been deeply investigated. It could be due to the trigonometry's position between algebra and geometry. Such a position of a mathematical subject between two different areas of mathematics, while possibly difficult to address yet it hides within itself very creative possibilities, whose one of the primary examples is algebraic geometry. One can therefore surmise that pedagogy of trigonometry might hide some new and interesting ideas.

The first paper by colleagues from Indonesia addresses the connection between epistemological obstacles manifested by students and their misconceptions. Their work focuses on two such obstacles: the relationship between degrees and radians as measures of angles and related to it the meaning of  $\pi$ :  $22/7$  or  $180$ .

The second paper by the colleague from Malaysia addresses more limited issue of the misconception related to finding maxima and minima of trigonometric functions. At the same time, the author provides a clearly designed Japanese Lesson Study in relevant

trigonometry as an example of the helpful professional development of teachers; the author finds out that the teachers' main obstacle is the successful analysis of the function  $y = a \sin x \pm b \cos x$  and  $y = a \sin kx \pm b \cos kx + c$ . Interestingly enough teachers could deal with such functions if  $a$ ,  $b$ ,  $k$  and  $c$  are given numerically. Consequently, the problem may lie in the generalization process (quite involved as included examples show).

The third paper in this series by Edo and Tasik relates to student misconceptions in algebra, which occur in the context of solving Pisa-like problems of high difficulty. These misconceptions undermine student competence in algebraic modelling, and at the same time they show the serious absence of conceptual understanding as it relates to the role of algebraic symbols.

The two following papers have also an internal relationship. While one team of colleagues from Indonesia (Wijaya et al) investigates the development and role of analytic questions asked by students during the process of learning, another colleague, Aloisius Son from the same country investigates the synthetical abilities of students in terms of their ability to connect different concepts in mathematics as well as in the relations of mathematics with real World and other domains of science. Vijaya et al find that analytical questions, within a collaborative framework are generated by weaker students towards stronger students in a group aimed mostly at cognitive understanding. Their work is analyzed with the help of the revised Bloom's taxonomy. They have

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chosen five stages of the learning taxonomy: observing, asking, experimenting, associating, and communicating. They found out that the first stage of observing is the dominant stage when analytical questions are formulated and the last stage of communicating is where the least number of such questions appear.

On the other hand, Son has found out that the principles characterized by the CORE RME model, that is connecting, organizing, reflecting, and extending used to solve real world problems result in the highest degree of the development of student connecting abilities, as compared with other models.

Comparing work conducted in both papers we see that the lower stages of the Bloom taxonomy encourage analytical questions while the higher stages of the taxonomy promote synthetic abilities of students.

The next two papers come from Eastern Europe which is at present engaged in the tragic in my opinion war. We welcome both papers as the expression of commitment to reason and peace. Both papers are concerned with the efficiency in our work; the first one with the efficiency of teachers' assessment in courses with large (circa 1000) students using computer generated test questions, the second one with the efficiency of understanding mathematics by medical students pointing to the special role of visual representation. Both papers refer to courses in advanced mathematics. Colleagues from Russia bring forward a very nice metaphor for the composition of functions: the nested matryoshka dolls.

Following our regular by now Problem Corner, we present an unusual paper submitted by the 19-year-old student of the Indian Institute of Science Education and Research in Bhopal, Samir Sharma. Samir was inspired by the challenge posted by Professor James Tanton's Youtube channel (<https://www.youtube.com/watch?v=ImEcUQFw6DU>) titled Math Mystery for Young Mathematicians and proved an extension of properties shown there. for the relationship between loops, intersections and enclosed spaces in a two dimensional space. It might of interest for the reader to look up that video and then to look what a nice theory of loops, intersections and spaces has been created by Sameer Sharma.

The teaching-research report by colleagues from Germany, Caspari – Sadeghi et al, addresses an interesting theme of the role of student self-assessment with the help of a relatively new instrument called Certainty-Based Marking (CBM). The goal of CBM instrument is to cause student reflection upon their own answers and thus to increase student involvement in STEM learning. Student self-assessment is a subtle approach, it can play both positive and negative role in student motivation.

The next paper by colleagues from Indonesia by Ary Woro Kurniasih et al, addresses equally interesting and subtle work, facilitation of creative thinking by teachers in mathematics. The authors use the traditional by now approach grounded in Guilford/Torrance formulation of fluency, flexibility and originality as the critical markers.

The presentation by colleagues from Spain, Gavillán Isquierdo et al introduce here at MTRJ a new approach to learning formulated in the last decade and called called commognition, a purely socio-

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cultural approach based upon the analysis of discourse in mathematics classes. The authors focus their attention on the commognitive conflicts, which in that approach are the sources of learning, and in particular of learning through creativity. The authors have found object level and metalevel conflicts, each type related to learning at different cognitive levels.

We complete the issue with the review of a mathematics book introducing Gödel incompleteness theorem by Powell and Trimmer from US through a nicely narrated story by Hiroshi Yuko who specializes in writing books about mathematics from the student point of view. Nice and helpful reading about one of the deepest results in mathematics.

One cannot help but to admire the intensity of Indonesian work in Mathematics Education: 50% of the presented papers in this issue come from Indonesia.

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## Epistemological Obstacle in Learning Trigonometry

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**Abstract:** *Epistemological obstacle is emphasized in mathematics education. Students often have limited students' context of knowledge in understanding trigonometry. Knowing the epistemology obstacle can help a teacher understand the student's misconception. Therefore, this study aimed to identify students' epistemological obstacles of trigonometry and trigonometric function. This research used two subjects of 11th grade high learners of high school. The research data were collected in audio, photograph, and trigonometry test. Data were analyzed by using in and between conditions students interview. This study suggests that students tend to associate the trigonometric value into a particular angle, such as having difficulty using angle in radian and could not recognize the value of  $\pi$ . They also tend to follow the procedural steps in converting angle from radian to degree and vice versa without knowing how the formula is constructed. They have difficulty figuring out the trigonometric function's value, especially for angle in the quadrant and the coordinate of a point trigonometry graph, primarily related to the radian unit.*

## INTRODUCTION

Trigonometry, a mathematical concept, especially within the framework of a trigonometric function, has become a part of mathematics curricula in high school. As of the essential concepts of mathematics, trigonometry is both a unifier and a hypernym for many mathematical subjects such as geometry, function, and calculus (Weber, 2008). Trigonometric functions, formed by the definition of trigonometry as ratios of right-triangle side in a unit circle, are a part of calculus courses. Several studies have common conclusions that the traditional methods in teaching trigonometry are inadequate to introduce the students into a trigonometric function's concept (Kamber & Takaci, 2017; Orhun, 2001; Weber, 2005). The traditional methods mentioned here involve procedurally teaching the courses within the framework of definitions, theorems, proofs, and problem-solving of trigonometry as a ratio of right-triangle only.

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Understanding mathematical concept is quite challenging for students. However, if a student is not asking herself questions and solving problems, she is not doing mathematics. While doing a mathematics task, a student may manifest error. Errors are not only the effect of ignorance, of uncertainty, of chance but also the effect of a previous piece of knowledge that was interesting and successful but now is revealed as false or irrelevant. Errors of this type are not erratic and unexpected. They constitute obstacles (Brousseau, 1989). An error can be identified by seeing the result of students' work. For instance, in the research of Kamber & Takaci (2017) research, a student implements one of the properties  $\sin(mx) = m \sin(x)$  in determining the value of  $\sin 270^\circ$ . The student gets the result that:

$$\sin 270^\circ = \sin(3 \cdot 90^\circ) = 3 \cdot \sin 90^\circ \quad (1)$$

The student knows that the property is correct in algebra, so the student applies the equation to determine the other sine values of angles. While analyzing this error, it raises questions (Do the students not understand how to find out trigonometric values? Do the students use the wrong concepts? Do the students mix up the understanding of algebra and trigonometry? Do the students understand the meaning of sine? Do the errors occur because of prior knowledge of the students or occur during the learning process). For sure, each student has reasons or arguments that support the answer. The student uses a concept in a particular context and applies them to another context (Brousseau, 2002). The students might not realize that what they are doing is incorrect, for it makes sense to them. The error made by students indicates learning obstacles experienced by students.

The obstacles arose as a result of learning where situations experienced by students are insufficient in facilitating students to obtain correct and complete knowledge (Begg et al, 2003). Analysis of students' difficulty in learning trigonometry was studied by several researchers (Kamber & Takaci, 2017; Orhun, 2001; Weber, 2005). However, there was no analysis of students' understanding of trigonometry based on the epistemological obstacle. Classification of the students' difficulty through the classification of students' learning obstacles will facilitate the teacher, educators in solving the misconception and error. Hence, this article is concerned with identifying epistemological obstacles. This study aims to examine the epistemological obstacle of trigonometry and trigonometric functions and addresses the following research questions:

- What is the kind of epistemological obstacles of high school students in mathematics who have learned trigonometry within the context of the Theory of Didactical Situation?

This article is organized as follows: The following section presents the theoretical framework, the methodology adopted by this research, followed by a section that presents the results and provides the discussion of the results; and the last section concludes the paper with a summary of the discussion.

## THEORETICAL FRAMEWORK

### Obstacle in Theory of Didactical Situation

The theory of didactical situation, a learning theory in mathematics education, works in a framework that the knowledge gained by students comes from students' adaptation to the didactic situation given to the students (Brousseau, 2002). Thus, there will be situations that shape the students' knowledge. An appropriate learning situation can give correct concepts and then construct them into new knowledge. However, if a student experiences a learning situation that could not provide both the concept and knowledge construction, then the learning situation is insufficient to facilitate students to obtain the correct and complete knowledge (Artigue et al, 2014). In this case, there is a possibility that students' knowledge becomes an obstacle to further learning (Skordoulis et al, 2009).

Brousseau (2002) revealed that learning obstacles could be seen or indicated by students' mathematics errors. Errors can be explicitly identified in the students' work. However, errors indicated as an obstacle are not accidentally happened (e.g., careless in calculating), but those are resistant to changes and will be repeated by students. These errors occur consciously and are re-used and challenging to disappear. We illustrate obtaining new knowledge in the theory frame of a didactical situation as Figure 1.

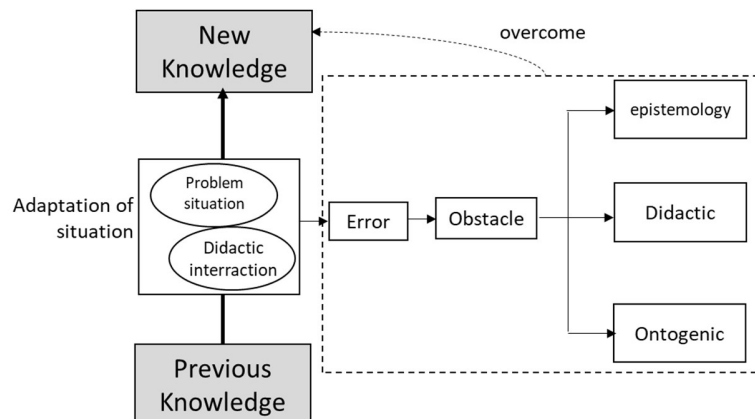


Figure 1 Scheme of the Obtaining New Knowledge

An obstacle is an integral part of the learning process. A reflection on obstacles becomes essential in changing learning models as important content in the learning process. It offers a thought-provoking opportunity to develop mathematical abilities because it can explicitly determine the direction of how the learning process takes place (Sztajn et al, 2012).

Brousseau (2002) revealed that the emergence of learning obstacles did not merely occur because of a single system of interaction. He determines that the learning obstacles can be derived from

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three origins (learning obstacles originating from ontogenic, didactical, and epistemological sources). The Ontogenic obstacles are developmental related to the stages of a child's mental growth. The Didactical obstacles arise as a result of teaching decisions. These obstacles can be avoided by developing alternative instructional approaches. Brousseau defines epistemological obstacles as forms of knowledge that have been relevant and successful in specific contexts, including school contexts, but that has become false or simply inadequate at some point, and whose traces can be found in the domain's historical development (Artigue et al., 2014). It emerges regardless of the instructional strategy since the concept itself is the source of the problem. Thus, identifying and characterizing an obstacle is essential to analyzing and constructing didactical situations.

Understanding and overcoming epistemological obstacles are essentially the same thing in many circumstances. Epistemological obstacles look backward, focusing on what was wrong, inadequate in the ways of knowing while understanding anticipates new ways of knowing (Sierpinska, 1990). Epistemological and cognitive considerations are, of course, not independent: The aim is to identify the conditions for a planned process of learning through which students construct and use those features of trigonometry that the epistemological analysis has identified as constituting the concept. This result of the study functions as an instrument to guide theoretically informed local decisions about teaching in trigonometry for teachers (Brousseau, 1989).

### Understanding the concept of trigonometry and trigonometric function

Of the many sentences that can be formulated about trigonometry and trigonometric functions, let us choose this one:

- Trigonometry: ratios of sides of a right-angled triangle.
- Trigonometric functions: Real Functions defined for the domain of real numbers which relate an angle and for the range of real numbers

The logical sentence of this sentence can be written as Figure 2:

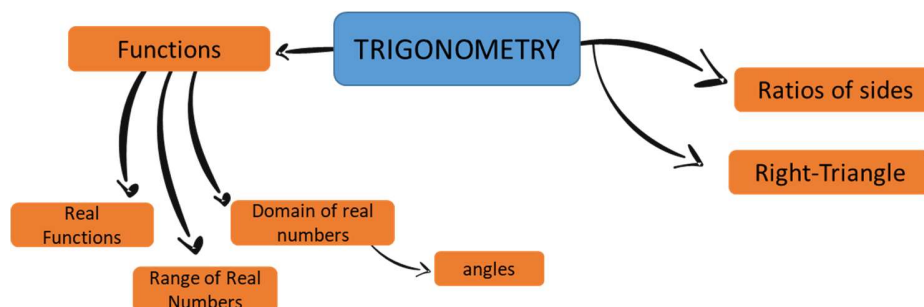


Figure 2 Logical structure of the trigonometry and trigonometric definition

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This structure defines the sense of the definition. It shows the difference between trigonometry and trigonometric functions. In trigonometry, the objects point out the ratios of two ideas: side and right triangle. It can be inferred that the object of trigonometry refers to the angle in the right triangle. In the case of trigonometric functions, an image from  $\mathbb{R}$  to  $\mathbb{R}$  generally points to an angle. It can be inferred that the object of trigonometric functions refers to any angle. Understanding both definitions will lead to the perception that there is no difference between trigonometry and trigonometric functions. However, in saying this, it defines a type of relation.

The trigonometric function definition is developed from the trigonometry concept in the right triangle. As a historical approach to trigonometry, the trigonometric function definition is related to the trigonometry in the right triangle (Moore, 2012). To explore this relation, let a right triangle  $ABC$  has right angle  $\theta$ . The  $\theta$  is in standart position, so that  $B'(x, y)$  is in unit circle and point  $C'(x, 0)$  is in x-axis (see Figure 3 ). The triangle  $ABC$  is congruent with triangle  $A'B'C'$ . Thus, these properties are followed:

$$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{1} = y \quad (2)$$

$$\frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{1} = x \quad (3)$$

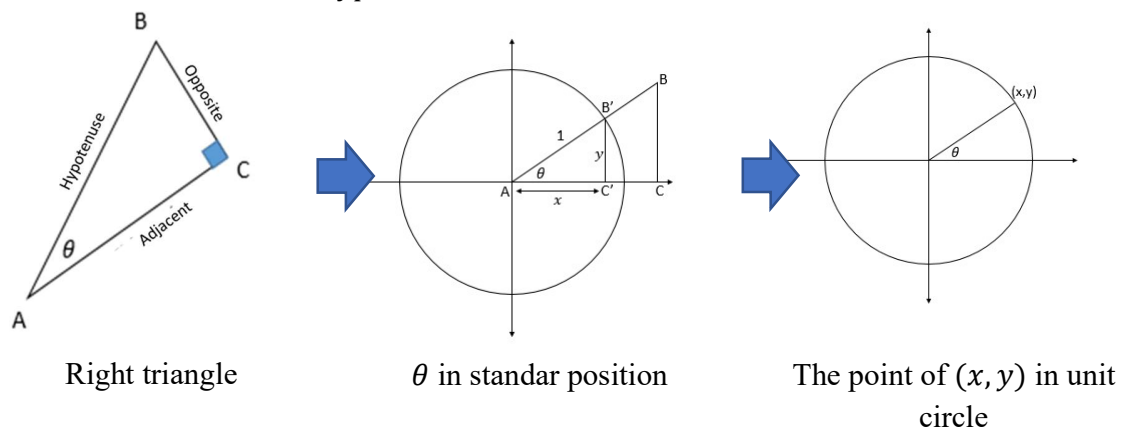


Figure 3. Consistency in Trigonometric Definition

The equation (2) and (3) are known as the definition of sine and cosine. Thus the value of  $\sin(\theta)$  is  $y$  and the value of  $\cos(\theta)$  is  $x$ . Thus, the coordinate of  $(x, y) = (\cos x, \sin x)$ . The process of drawing trigonometric function can be drawn from a unit circle (see: Figure 4)

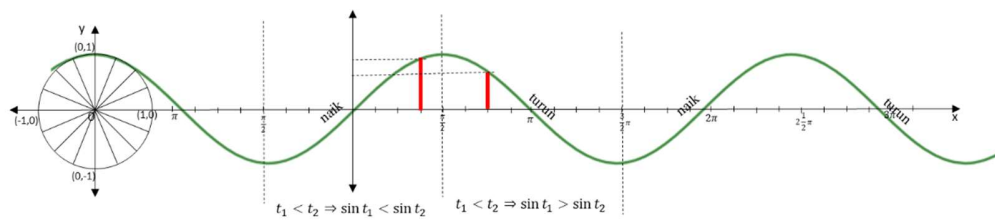


Figure 4 Sine Function Graph from A Unit Circle

## METHOD

This study is qualitative which was conducted in a case-study pattern. According to Yin (2003), a case study is an empirical research method used in cases where more than one source of proof or data is present. This study focuses on high school students' obstacles in trigonometric functions. For this purpose, the Theory of Didactical Situations was used, and then a coding table was formed. Based on this table, students' mental structures and mechanisms were coded within the context of the theory of the didactical situation. In line with this coding, the students were interviewed to reveal their trigonometric functions' mental structures and thinking mechanisms.

## Participant

This study was conducted in Indonesia with two high school students as participants. A purposive sampling method revealed the learning obstacle regarding their conceptual understanding of trigonometric function. The participants were selected according to the following criteria:

- (1) Participants are expected to have completed the trigonometry course in the curriculum. Trigonometry courses include the topics of ratios of right-triangle sides, radian, trigonometric graph, the value of trigonometry, period, maximum, and minimum of a trigonometric function. The reason behind this criterion is that the genetic decomposition steps for understanding the trigonometric function concept are among the learning outcomes of these courses. In line with the instructors' views teaching these courses, it is necessary for the students to freely express their ideas about a problem and provide a basis for them.
- (2) Based on the students' performance in mathematics. The categories were created for the students who have high performance in mathematics based on information from the teacher.

## Tools for data collection

In this study, the data were retrieved from two sources. The first one is the written responses that

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the students gave to the Trigonometric Function-Understanding test, and the second one is the tape recording of clinical interviews concerning these questions. A measurement tool was developed to analyze the learning obstacle of the trigonometric functions. The questions were taken from Kamber & Takaci (2017) and Weber (2005). Based on expert opinion, the first question regarding the mathematical argumentation was added from Maknun et al. (2018), and the second question regarding converting angle from radian to degree and vice versa were added. Further, the sixth question regarding the graphical representation of trigonometric function was adapted from Brousseau (2002). The previously constructed schemas, which the students are presumed to know to understand the trigonometric function, are as follows:

- (1) The intuitive notion of ratios of trigonometric such as formulas of sine, cosine, and tangent in right-triangle,
- (2) The Cartesian plane (that is, the concept of points as objects; functions, curves, and areas as processes resulting from the generalization of the action of representing the components of points),
- (3) Real numbers (that is, the concept of number as an object; arithmetic and algebraic transformations as processes),
- (4) Sets,
- (5) Real functions with real values (that is, the function as a process, operations with functions, and the coordination of the algebraic and geometric representations of functions)

The questions corresponding to the trigonometric functions used in this study to figure out students learning obstacles are presented in Table 1.

Questions	Questions corresponding to trigonometric understanding
1	The action of invoke the concept image of equation of trigonometric function and validate the statement
2	2a The action of converting angle from degree to radian 2b The action of converting angle from radian to degree
3	3a The action of sorting the sine value through estimating their value from the drawing angles 3b The action of sorting the cosine value through estimating their value from the drawing angles
4	The action of analysing about maximum and minimum values of sine function,
5	The action of determining the coordinate of particular point in graphical representation of trigonometric functions. The action of determining the domain and range of the trigonometric function through graphical representation.

Table 1 Questions corresponding to trigonometry test

## Desired Responses

### Question 1

It is not true that  $\sin x = \frac{1}{2}$  is  $30^\circ$ , because the domain of the  $x$  is  $90^\circ \leq x \leq 360^\circ$ . Thus the  $30^\circ$  is not the single answer. There is another solution, i.e.,  $150^\circ$ .

### Question 2

In this question, the students were not expected only on the question's correctness, but they could also explain how the formula is constructed. The construction of radian and degree equality is figure

$$\text{Full angle ( in radian)} = \frac{\text{circumference of circle}}{\text{radius}} \quad (4)$$

$$360^\circ = \frac{2\pi r}{r} \quad (5)$$

$$360^\circ = 2\pi \text{ radian} \quad (6)$$

$$180^\circ = \pi \text{ radian} \quad (7)$$

$$\frac{180^\circ}{\pi} = 57,3^\circ = 1 \text{ rad} \quad (8)$$

$$1^\circ = \frac{\pi}{180} = 0,0174 \quad (9)$$

### Question 3

Knowledge about the definition of trigonometric functions is needed to answer this question. In the cartesian plane, the angle coordinate represents the value of sine and cosine. Thus, if  $P(x, y)$  is a point in the unit circle, the sine value is  $y$ , and the cosine value is  $x$ . Hence, to arrange the value of sine and cosine to the correct order, the students should compare each coordinate. Since the coordinates were not available in this question, students needed to estimate each angle. The correct order is  $\cos \beta < \cos \alpha < \cos \theta$  and  $\sin \theta < \sin \alpha < \sin \beta$ .

### Question 4

In answering this question, the students could use any trigonometry context. One of the contexts they can use is the unit circle. The expected answer from students is that they can explain from the unit circle that there is no possible value of function  $y = \sin x$  such that smaller than  $-1$ . Otherwise, they can illustrate through trigonometry graph.

### Question 5

This question has a connection with the coordinate system that they have ever learned before. The cartesian plane was used to pair numbers from  $\mathbb{R}$  to  $\mathbb{R}$ . We used this question to see if the students used radian or degree to determine the coordinate. Using radians in answering this question is the response we expect to appear. Thus the correct answer to this question is  $(\pi, -1)$ .

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### Clinical interview

The students were interviewed to analyze their understanding and views of the mathematical concepts in the questions and obtain in-depth data by revealing their mental mathematical structures. The interviews were tape-recorded with consent from the students. There were no time limits during the application or the interview. Following the test analysis, the students have interviewed the questions coded as 0 and 1, according to Table 2. During the interviews, discussions were held concerning the answers given by the students to the questions chosen from the test. In the clinical interviews, the students were asked questions such as “What did you think in this question?” and “Can you explain what you have done in this question?” Furthermore, the students were asked some follow-up questions such as “How would you think if I give you these angles?”

### Data Analysis

The content analysis method was used to analyze the data obtained in the study. The answers given by the students were analyzed under the themes of learning obstacles (epistemological, didactical and ontological obstacle) proposed by (Artigue et al., 2014; Brousseau, 2002). In this study, we are re-examining the interpretation of students’ errors and how they are produced; we found recurrent errors, and showed that they are grouped around conceptions.

The written documents about the questions in the application were examined, and the answers given by the students were coded as 0, 1, and 2 to determine the incorrect or incomplete answers. The coding was performed according to Table 3. Then, the mathematical reasons behind appearing the epistemological obstacle for the questions coded as 0 or 1 were investigated through clinical interviews with the students. The interview recordings were transferred to a computer and transcribed. For example, the transcribed documents obtained from the test of S-1 and s-2 were examined; Questions 3 was coded as “0,” and S-1 and S-2 were interviewed about these questions. The interview showed that both S-1 and S-2 had difficulty in the geometric representation of the trigonometric value, particularly in a unit circle, although both the students had approached it through the positive and negative values in each quadrant. This situation may be attributed to the fact that the student could not find trigonometric value as a coordinate of points in a unit circle.

0 (incorrect)	1 (partially correct)	2 (correct)
An incorrect answer is given to the question, or the question is not answered (lack of schema that should have previously been known to understand the concept of trigonometry).	Some steps required to solve the question are performed, but the explanation about why perform those answers is not well explained.	A correct answer is given to the question and well explained.

Table 2 Code and explanation

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The questions given to the students in the interview strengthened the finding in students' work. Throughout the interview session, students could provide arguments for their answers related to trigonometry on why a mathematical statement is true or why the answers to math problems are obtained (Sriraman & Umland, 2014). The forms of arguments given by students can vary, for example, informal evidence, explanation, and working steps. Thus, the interview can determine students' understanding of trigonometry and students' mistakes when working on a given problem. The theory of learning obstacles proposed by Brousseau (2002) informed this study's theoretical backgrounds and goals. In the analysis process, the researcher identified common themes found in students' work, such as error and misunderstanding in trigonometric function, to discover the learning obstacles. After identifying these themes, the researcher searched the data for additional instances that could support and contradict these themes.

## RESULTS

In this section, the findings obtained from the data analysis are discussed. The findings include some of the answers given by the students to each question. The data was obtained from the interview analysis. The students learning obstacle of trigonometric was evaluated within the theory of the didactical situation. In the clinical interview dialogues, the researcher was denoted with the letter "R" whereas the students were denoted with the letter "S." The answers given by the students during the interview were represented as S-1 and S-2, as shown in Table 3.

	1	2a	2b	3a	3b	4	5
S-1	0	1	1	0	1	2	0
S-2	0	1	1	0	1	1	2

Table 3 Codings of the answers given to the questions by the students

### Question 1

In this question, the aim is to invoke the concept image of the equation of trigonometric function and validate the statement. When Table 3 is examined, it is seen that both the students answered the first question incorrectly. As indicated in Table 2, both students could not invoke the concept image of the equation of trigonometric function and validate the statement. The excerpts from the clinical interview are given below.

R: Can you explain why you think this statement is true?

S-1: "I remember it from the [trigonometric] table that  $\sin 30^\circ = \frac{1}{2}$ ,"

S-2: "Because  $\sin 30^\circ = \frac{1}{2}$ ,"

Both students expressed the concept of trigonometric values correctly. However, they ignored the parts where the value of  $x$  not only  $30^\circ$  since the domain of  $x$  is  $90^\circ \leq x \leq 360^\circ$ . As a result, it can be suggested that their intuitive notion of space schemas is incomplete. In this case, the

students tend to associate the sine values with the table, which lists the values of sines, cosines, and tangents at a particular angle given by the teacher. So the sine and cosine values that students know are  $0, \frac{1}{2}, \frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{3}$  and 1. It can also be seen in students' answers to other questions, limiting the sine value to  $0, \frac{1}{2}, \frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{3}$  and 1. Limiting sine values on that values will lead to obstacles. The sine value is a real number in the interval  $[-1, 1]$ .

## Question 2

In this question, the students were expected to convert an angle from degree to radian and vice versa. Both the students had the same accurate answers and procedure in answering question 2 as presented in Figure 1.

$$135^\circ = \frac{135}{180} \text{ rad}$$

$$\frac{\pi}{4} \text{ rad} = \frac{\pi}{4} \times 180^\circ = 45^\circ$$

$$\begin{array}{r} 4 \overline{)180} \\ \underline{160} \phantom{0} \\ 20 \end{array}$$

S-1 solution in question 2a and 2b

$$a. \quad 135^\circ = \dots \text{ rad}$$

$$\frac{135}{180} = \frac{3}{4} \text{ rad}$$

$$b. \quad \frac{\pi}{4} \text{ rad} = \dots^\circ$$

$$\frac{\pi}{4} \times 180 = 45^\circ$$

S-2 solution in question 2a and 2b

Figure 4 Students solution in question 2

Figure 4 demonstrates that S-1 and S-2 correctly answered Question 2b. S-1 gave the unfinished answer, while S-2 correctly answered the same question. Thus, it may be concluded that S-1 was unable to convert degree to radian. Below are the student's explanation and an excerpt from the interview.

R: Can you explain Why did in question 2a you divided the angle with 180, while in question 2b, you multiplied it with 180?

S-2: From the formula I learned from the teacher this formula, the degree to radian is divided by degree while radian to degree multiplied by 180

R: So, did you know how the formula was constructed? I mean, why does the formula provide that way?

S-1: I do not know

S-2: Perhaps there is a reason, but I do not know why. I did not learn that.

As can be seen, both the student could answer the question in a procedural action through a fixed formula. When asked why they should divide or multiply them by 180, they did not know the reasons and followed what had been taught. In converting angles from degrees to radians and vice

versa, there was a relationship between  $\pi$  and  $180^\circ$ . To further understand the students' understanding of  $\pi$ , we asked them to determine the value of  $\sin\left(\frac{\pi}{4}\right)$ , students answered. "Change it to degree first".

R: What is the value of  $\pi$ ?

S-1: In a circle, as far as I know  $\pi = \frac{22}{7}$ , or 3.14. However, the value of  $\pi$  can be different. It is not always  $\frac{22}{7}$ , especially if there is a word "sin" in front of it. In that case,  $\pi$  is 180"

S-2: It is 180

R: What about if we find out the circle area, what is the value of  $\pi$ ?

S-2: "... Hmm, it means  $\pi$  is not  $\frac{22}{7}$ , isn't it? (doubt). Yes, perhaps it is a provision. In the circle is  $\frac{22}{7}$ , but in degree becomes 180?"

The interview above found that the relationship between  $\pi$  and  $180^\circ$  influenced students to see the value of  $\pi$ . It can be seen that S-2 doubts why there are two different values of  $\pi$ . It can be seen that S-2 doubts why there are two different values of  $\pi$ , namely  $\frac{22}{7}$  and  $180^\circ$ . Whereas S-1 believes that the value of  $\pi$  will be different if the domain area is trigonometric.

### Question 3

In this question, the students were asked to sort the value of the sine of the angle from the smallest to the largest through the representation of the given angle without knowing how big the angle was in each picture. An excerpt from the interview with both students is presented below:

R: What is your answer?

S-1 : The order is  $\sin \alpha$ ,  $\sin \beta$ ,  $\sin \theta$

S-2 : The first order is  $\sin \theta$ , because it is negative, while  $\sin \alpha$  and  $\sin \beta$  are positive because they are in quadrant I and II. However, I do not know which one between  $\sin \alpha$  and  $\sin \beta$  is smaller since they are positive.

R: Can you explain the answer?

S-1: I predict the value. I assume there is an angle such as  $60^\circ$ . Then the one in quadrant II is likely  $120^\circ$  because it is more than  $90^\circ$  and this one is more that  $270^\circ$ . Let us say it is  $300^\circ$ , it is approximately about  $\frac{1}{2}\sqrt{3}$ . So, I predict the angle which is not too far from a special angle. So, th order is  $\sin \alpha$ ,  $\sin \beta$ ,  $\sin \theta$ . For cosine is the opposite [of sine] and will be like this:  $\cos \theta$ ,  $\cos \beta$ ,  $\cos \alpha$ . So evaluating from this [table of trigonometry], the cosine value is the opposite of sine



R: Why does a particular quadrant have positive and negative values in trigonometry?

S-1: I do not know the reason. That is what I learned

R: Can you explain your answer?

S-2: The  $\sin \theta$  is the smallest because it is negative [value], while  $\sin \alpha$  and  $\sin \beta$  are positive because [they are] in quadrants I and II. I do not know which the smallest one [between]  $\sin \alpha$  and  $\sin \beta$  because they are both positive. Quadrant I positive for sine, cosine, secan, cosec, tan, cotan; sine and cosec [positive in] quadrant I; tangent and cotangen [positive in] quadrant III; cosine equal secan [positive in] quadrant IV”

A further question was proposed for S-2 because he said that he could know exactly the order if he knew what the degree was.

R: What about these? Can you sort these from the smallest to the greatest order ( $\sin 10^\circ$ ,  $\sin 110^\circ$ ,  $\sin 250^\circ$ , and  $\sin 335^\circ$ .)

S-2: “[the order is]  $\sin 250^\circ$ ,  $\sin 335^\circ$ ,  $\sin 10^\circ$ ,  $\sin 110^\circ$  because this angle [pointed out to the angle  $250^\circ$  and  $335^\circ$ ] are negative, then to determine the smallest of them. Look at  $\sin 30^\circ$  and  $\sin 60^\circ$ .  $\sin 30^\circ$  [is]  $\frac{1}{2}$  and  $\sin 60^\circ$  is  $\frac{1}{2}\sqrt{3}$ , so if you have a bigger number [read: angles], then the sine value is also bigger. So this  $\sin 335^\circ$  is the smallest because it is negative.”

#### Question 4

In this question, the students were asked to analyze maximum and minimum values of a sine function, in contrast to Kamber and Takaci (2017), where most students can answer correctly and explain well the reasons why it is an impossible  $\sin x = 2$ . An excerpt from the interview is presented below.

R: What is your answer? Could you explain your answer?

S-1: “It is impossible because no matter how big the angles are, for example, more than 360, you will find the remaining angles and return to a special angle again”,

S-2: It is impossible, because the maximum sine value is 1.

R: Why cannot it be more than 1?

S-2: I do not know the reason.

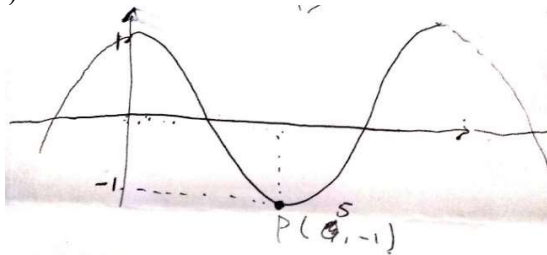
Although S-2 gave the correct answer, he could not explain why the sine value could not be more than one. In comparison, S-2 provided an argument by estimating for angles greater than  $360^\circ$  by using the concept of correlated angles. By analyzing that, any angle can undoubtedly be related to the angle in the first quadrant, wherein the first quadrant was found a particular angle with no value of more than two. S-1 concluded that the statement  $\sin x = 2$  is impossible.

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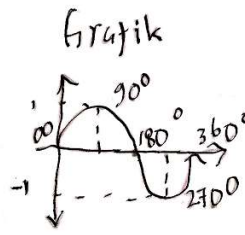


### Question 5

This question asked students to determine the coordinate of a particular point in a graphical representation of a trigonometric function. When Figure 5 is examined, it is seen that student S-1 answered the sixth question incorrectly, while student S-2 answered correctly. Student S-2 had no difficulty deciding the P coordinate with the cosine function graph he already knew. However, student S-1 provided a different perspective in finding the coordinate of P that her answer was (5, -1)



S-1 solution in question 5



S-2 solution on 5

Figure 5 Students' answer in question 5

The excerpts from the clinical interview are given below

R: Can you explain your answer?

S-1: This one [pointing at y-axis] is -1. Because the  $\cos 0^\circ = 1$  [pointing the graph], the cosine starts from here (pointing the peak of cosine graph) and about the x, it is 5 perhaps [thinking again for a moment] as I remembered, this one (pointing at x-axis) is (in) degree. That is the problem, I am not sure about it.

S-2: I draw the trigonometric function like this (Figure 5)

From question 5, the epistemological obstacle in deciding the point coordinate in a trigonometric graph can be known. S-1 could not recognize the accurate coordinate of a point. Instead of answering  $(\pi, -1)$ , she answered  $(5, -1)$  in which 5 represented her assumption of the length from the point to central coordinate in the x-axis. These epistemological obstacles may result from false intuition in determining prior knowledge to answer the questions or make less straightforward generalizations (Subroto & Suryadi, 2018). The students need to get more concrete activities in trigonometry (Maknun et al, 2020)

### DISCUSSION AND SUGGESTION

Students' errors in understanding trigonometry and trigonometric functions reveal students' obstacles related to these concepts. In this study, the students were asked five questions prepared to figure out students' errors of these concepts within the Theory of Didactical Situations framework.

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In this study, no students could invoke the concept image of the equation of trigonometric function and validate the statement. The students were unable to invoke the concept image of the equation of trigonometric function is they did not have the intuitive notions of trigonometric function in all angles. For example, S-1 could not assign the all value of the trigonometry equation because she only views the value for angle  $0^\circ \leq x \leq 90^\circ$ . In general, it was observed that the students who could not analyze trigonometric values in all angles were unable to evaluate a particular angle larger than  $90^\circ$ . Thus, it can be said that errors made regarding trigonometry values are consistent and lead to learning obstacles. Further, the students also had difficulty understanding the concept of radian measurement. Below we explain the learning obstacle related to the epistemological origin.

### **Obstacle 1 : Finding the value of trigonometry in the term of trigonometric functions**

Understanding trigonometric functions usually result in understanding trigonometry in the right triangle. However, this development is not possible without a concept shift of attention. In particular, focusing on the form of the angles or on the rule for generating terms of a function. Demir & Heck (2013) notice this matter and design the bridging concept through a unit circle. Below we shall make some comments on these obstacles:

#### *1. Domain of trigonometric functions in solving trigonometric equations*

This obstacle focuses on a trigonometric value on the triangle domain. However, it had been clear from the question that the domain of an angle is  $0^\circ \leq x \leq 360^\circ$ , both the students could not recognize the possibility of another value of  $x$ . This obstacle is not because they did not know that  $\sin(150^\circ) = \frac{1}{2}$ , but because it is unclear enough for students the term of  $0^\circ \leq x \leq 360^\circ$  is for, and because they focus only on the question itself. So that they directly answered that the value of  $x$  must be  $30^\circ$  only. Overcoming this obstacle amounts to defining trigonometry as a function, in a similar way to how the students learn functions.

#### *2. The bigger the angle the bigger the value of sine function*

The above conception quickly develops if the angles are introduced through excessive practice and stressing of certain angles ( $30^\circ, 45^\circ, 60^\circ, 90^\circ$ ). The student S-2 had difficulty in answering this problem, where she assumed that the bigger the angle was, the greater the trigonometric value would be. She analyzed from the sine values at special angles that she had memorized. In the particular angle that she memorized, from the selected angle, i.e.,  $0^\circ, 30^\circ, 45^\circ, 60^\circ$ , and  $90^\circ$ , indicating an increasingly more prominent sine value. She must assume the sequence of angles (according to the smallest to the biggest trigonometric value) on unclear angles on a unit circle. This case may tend to make unjustified inductive jumps and believe that if they observe on the sine value is getting more prominent as the angles bigger, and this means that the sequence of the trigonometric value depends on how big the angles are.

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### 3. The smaller the angle the smaller the value of cosine function

Like the finding in the sine case above, S-2 was asked to sort cosine values at the same angle. She chose to reverse the order of sine values immediately. It was widely known that the trigonometric value at that angle was that the sine value was getting greater along with the enhancement of the angle. On the contrary, the cosine value was getting smaller with the angle enhancement. The result above suggests that the students sort the sine and cosine values through the trigonometric value at a particular angle, most probably having in mind the table of trigonometric values.

Maknun et al. (2018) found that the particular angle studied by the students limited them to understand the trigonometric value, especially when the students were asked to memorize the table of trigonometric values in a particular angle.

Let us now consider the obstacle:

#### **Obstacle 2: Distinguish the value of $\pi$**

##### *1. $\pi$ has two different value (3,14... or 180)*

$\pi$  is the ratio of the circumference of any circle to the diameter of that circle.  $\pi$  has a decimal value of approximately 3,14. However, it is an irrational number, which cannot be expressed as the quotient of two integers. The students in this study were placed in a situation where they must answer the value of  $\pi$  and identify it in circle and trigonometry. The students were interviewed after they finished the task (converting radian to degree and vice versa). This situation raises their question concerning its value (why does it have a different value for the same mathematics symbol?). Before this interview, the students did not see any contradicts in the value of  $\pi$ . The feeling of paradox only when the students were remembered the  $\pi$  in a circle. No one can deliberately choose which value of  $\pi$  but agree that 3,14 and 180 are the values. The students said that both values are correct because the use of  $\pi$  depends on the context (is it trigonometry context or circle context?).

##### *2. Constructing The formula in converting degree to radian and vice versa*

Although, the students were able to convert angels from radians to degrees and vice versa well with the formula of multiplying and dividing by 180. However, when asked how the formula was originally from, the students found difficulty. This process revealed that procedural skill was the obstacle for students in understanding the whole mathematics concept. This difficulty is also found in the research of Kansanen and Meri (1999). This finding is similar to what Prihandhika et al. (2020) found that students still focus on procedural understanding in answering mathematical questions. Unlike the previous epistemological obstacles, the students generalized an outcome in a particular context when talking about the value of  $\pi$ . Students assume different values of  $\pi$  in circle and trigonometry context. This understanding was inseparable from the procedure by students in converting angels in radians to degrees and vice versa. Perform

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### Some Remarks on Obstacles in Trigonometry

It can be seen that students had understood the concept of trigonometric values, specifically at a particular angle. Students were also right in analyzing the trigonometric values at that particular angle. The sine value got more prominent when the angle got bigger and vice versa for the cosine. However, this fact did not apply to all circumstances. When discussing angles in general, a different perspective was needed. This perspective was a prerequisite for the emergence of an epistemological obstacle. In other words, students' knowledge of trigonometric values at certain angles becomes an epistemological obstacle in understanding trigonometric values at all angles (Brousseau, 2002). It means the students' knowledge was limited only to a particular context. This condition results in difficulties when students are given problems in a different context. This may occur due to the teachers of the students. First, if it occurred due to the teacher, the teacher tended to provide a single way or knowledge to the students when teaching a concept. To anticipate this matter, the teacher may collaborate with other teachers to enhance their creativity in designing the concept (Asari et al. 2018; Fauziyah et al., 2021). Second, if it occurred due to students' problems, students were incapable of keeping their pace with the teachers' explanation (Cesaria & Herman, 2019).

### Didactical handling of obstacle

The angle measurement forms an obstacle to the conception of trigonometric functions. For obvious reasons of the proximity of the depth analyses, this obstacle is more difficult to overcome than the one they pose to the conception of trigonometry as ratios of right-triangle. The learning obstacles are primarily about the conceptual problem, especially in trigonometry. This result supports the study of (Kurniasih & Rochmad, 2020), which stated that students with high mathematical beliefs experience conceptual problems in integrating their abilities. Thus, the lesson should be designed to fulfill the students' obstacles as follows:

- (1) Students tend to associate the trigonometric value into a particular angle
- (2) The students could not distinguish the value of  $\pi$
- (3) The students tend to follow the procedural steps in converting angle from radian to degree and vice versa without knowing how the formula constructed
- (4) The students challenging to figure out the value of a trigonometric function, especially for all angles
- (5) The students have difficulty figuring out the coordinate of a point in a trigonometric graph, primarily related to the radian unit.

Let us take the example of errors in point (4). To solve a problem, students intuitively guess required trigonometric values by finding the coordinate of sine and cosine in the unit circle. They fail when the angle is not available if placed in the quadrant with the same sign (positive and negative) trigonometric value, confusion with an angle bigger than  $90^\circ$ , or inability to understand the problem. Further, having previously succeeded in converting the degree to radian and vice versa does not entirely prevent the phenomenon from occurring obstacle in constructing the formulae.

Lastly, S-1 and S-2 are students with a high GPA among others; however. It was observed that their errors indicate the obstacles which might lead to the misconception. Thus, future studies may investigate a relationship between the learning obstacle of trigonometry and trigonometric functions and GPA. The prerequisite when choosing students was that they had already taken trigonometry courses and had completed them. When the learning objectives of these courses are examined, it is expected that the students who completed this course will be able to comprehend trigonometry and trigonometric functions. However, the results obtained from this study concerning the learning obstacle regarding trigonometry and trigonometric functions contradict these learning objectives. Thus, the sufficiency of curricula of trigonometry courses might be questioned. We can conclude that the traditional approach to trigonometry education is not adequate for conceptual understanding. This study suggests that the teaching of trigonometry courses should be evaluated and organized within the Theory of Didactical Situations framework to ensure that students can better understand the concepts in the trigonometry courses.

## CONCLUSION

The study revealed that epistemological obstacles affected students' understanding of trigonometry and trigonometric functions. It was also found that students' understanding of the trigonometry and trigonometric functions was related to the procedural skill in how to solve the questions. Drawing on the results of the study, it can be said that the errors appearing in students' answers come from the epistemology obstacle such as understanding the angle, understanding of the value of  $\pi$ . In this case, the lack of understanding of the concept of radians becomes an epistemological obstacle in understanding the value of  $\pi$ .

Obstacles can also cause more fundamental educational issues. Many didactical practices justified by the simply additive classical model must be reviewed and perhaps rejected. But this model affects both internal (within and between classes) and external (between teachers and society) negotiations with the educational system, in terms of the teaching curriculum. However didactical issue is not only the diagnosis of errors, their explanation, and the description that follows changed, but also the teacher's and students' roles and obligations have been reassigned. Teachers'

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epistemologies must be changed in order to integrate this new paradigm of didactical communication.

These findings contribute in several ways to our understanding of didactics obstacles encountered by the students in trigonometry and provide a basis for knowing what material should mostly lead to the students' misconception. Some practical recommendations for teachers in order to follow up these findings are:

- Teacher should not neglect the appearance of obstacle by students since the fact that various items of knowledge, even incorrect ones, may be required to enable the establishment of definitive knowledge (Brousseau, 2002).
- Design the learning sequences from what the students have known
- Unit circle can fill the gap between the definition of trigonometry as ratios of right-triangle and trigonometric functions
- Radian measurement should be stressed to make better understanding on trigonometric function especially on trigonometric graph.

#### DATA AVAILABILITY STATEMENT

The authors confirm that the data supporting the findings of this study are available within the article [and/or] its supplementary materials.

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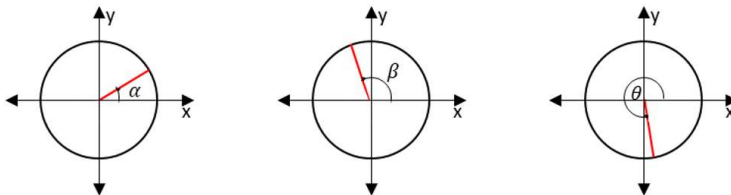
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#### APPENDIX: TRIGONOMETRY TEST

1 For  $90^\circ \leq x \leq 360^\circ$ , Is it true that the value of  $x$  in  $\sin x = \frac{1}{2}$  is  $30^\circ$ ?

- 2 a)  $135^\circ = \dots$  radian  
b)  $\frac{\pi}{4}$  radian =  $\dots$ (degree)

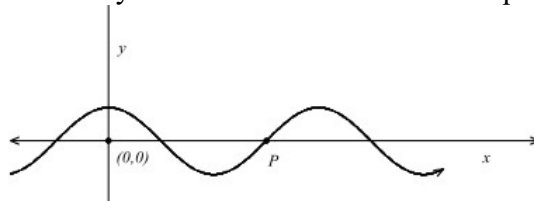
3



- a) Sort the sine value from the smallest to the largest value  
b) Sort the cosine value from the smallest to the largest value \*) problem adapted from (Weber, 2005)

4 Is there any value of  $x$  which fulfill the equation of  $\sin x = -2$ ?

5



What is the coordinate of the point P?

\*) problem adapted from (Brousseau, 2002)

## Improving the Performance of Mathematics Teachers through Preparing a Research Lesson on the Topic of Maximum and Minimum Values of a Trigonometric Function

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**Abstract:** *In each educational curriculum, trigonometry is an important subject at upper secondary schools and pre university level that apply in many other subjects such as algebra, calculus, geometry and physics. Many of students have serious problem in learning the trigonometric materials because usually mathematics educators transfer the trigonometric concepts to students through traditional methods that encourage students to memorize the trigonometric concepts. The purpose of this qualitative case study is to introduce the Lesson Study as a new teaching method based on problem solving approach in order to increase the performance of teachers in teaching trigonometry. In this study, a group of three mathematics teachers from an international pre-university centre in Malaysia and the researcher contributed in preparing a research lesson on the topic of the maximum and minimum values of a trigonometric function. Also, this research lesson improved in an existing class containing 10 students. Data collected through observations of discussion meetings and analyzed descriptively. In this research lesson, the researcher discussed teaching the maximum and minimum values of a trigonometric function through variety of solutions and likely misconceptions among students. Maybe this article helps mathematics educators to have better performance in teaching trigonometry through Lesson Study based on problem solving approach.*

**Keywords:** *Lesson Study, Misconception, Problem solving, Trigonometry*

### INTRODUCTION

Trigonometry is a complex part of mathematics that plays an important role in our daily life. Trigonometric concepts are difficult to understand by learners and usually this subject is challenging for teaching (Martin-Fernandez et al., 2019). For example, a study by Gholami et al. (2021) shows that none of the mathematics lecturers ( $n = 8$ ) chose teaching trigonometry as their favourite subject because of the complexity of this subject. They preferred to teach courses related to the calculus, algebra, statistics and probability. In teaching trigonometry, the challenge resides in the fact that many of traditional methods of teaching, primarily emphasizes superficial skills

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and such methods do not allow learners to understand the trigonometric topics conceptually (Altman & Kidron, 2016). Therefore, students face difficulties in learning trigonometry through problem solving due to misconceptions about trigonometric contents (Weber, 2005). In learning trigonometry, students experience numerous obstacles due to misconceptions, for instance, students' misunderstandings with the concepts angle and angle measure at the starting point for learning trigonometry are the most basic problem among students in depth understanding of trigonometric concepts (J. Nabie et al., 2018). Discussing different methods of solving trigonometric problems and common students' misunderstandings about them will help improve teachers' performance in the classroom. Based on Xenofontos and Andrews (2014) a mathematics task or a goal-directed activity is considered as a problem for students if this task is new and challenging to them. They further added that a mathematical exercise is not a problem because learners solve mathematics exercises by following steps they have learned. Teaching mathematics materials straight from textbooks is common in our educational institutions including schools and universities (Dhakal et al., 2020). Therefore, most of mathematics teachers still prefer to teach mathematical concepts through traditional methods by emphasizes on solving routine exercises in teaching (Voskoglou, 2019). It seems new methods of teaching such as Lesson Study requires a high level of mathematical knowledge and pedagogy to prepare suitable mathematical materials based on the ability of students. Considering appropriate activities and mathematics problems in the prepared lessons help learners to have better performance in the classes and enhance their abilities in problem solving (Gholami, Ayub, & Yunus, 2021).

The Japanese Lesson study approach, not only focuses on a team-oriented educational design and shared responsibility for the educational processes and outcomes but also clearly focuses on students' experience of learning process and not simply on the methods of teaching (Elliott, 2019; Hanfstingl et al., 2019). For example, familiarity with students' misunderstandings about different mathematical concepts provides a good opportunity for the Lesson Study team to provide appropriate research lessons. In this educational method, Lesson Study group members prepare lessons that are called research lessons in a participatory manner and after teaching in real classes, they constantly improve them (Coenders & Verhoef, 2019; Lewis et al., 2006). Therefore, Lesson Study as a kind of professional development programs improves the teaching knowledge of educators especially their pedagogical content knowledge through discussions among them regarding the students' learning (Coenders & Verhoef, 2019). This educational approach helps mathematics teachers to overcome difficulties facing students such as their misconceptions about mathematical concepts and to improve student learning (Leavy & Hourigan, 2018). Japanese Lesson Study has various models and is now spreading to educational systems of other countries in order to increases students' learning through supports teachers in improving their skills and teaching practices (Grimsaeth & Hallas, 2015). Research lesson is the most important part of Lesson Study and the procedure of preparing a research lesson is as follows (Lewis, 2002).

- 1) The Lesson Study group members set the goals for students' learning based on their abilities and skills
- 2) The members of the Lesson Study group collaborate to improve a plan for a teaching session to provide better learning situation for students

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- 3) One of the Lesson Study group members teaches the research lesson, while the others collect data by observation.
- 4) In a post-lesson discussion, the members of Lesson Study group analyze their observations in order to improve the quality of research lesson.
- 5) If necessary, the members of Lesson Study group plan to improve their teaching practice for a new research lesson.

Discussion about the misunderstandings of students about mathematical concepts is very beneficial for educators to have effective teaching. In other words, knowing the nature of misconceptions and misunderstandings and their sources regarding various contents of mathematics that are common among students of all educational levels, helps educators to plan suitable instructions for students' learning. Understanding mathematics concepts depend on linking from the prior knowledge and new topics, which may help or hinder the process of learning. Incorrect prior knowledge regarding mathematical concepts are called misconceptions, that cause a disability to learn the new contents (Alkhateeb, 2020). Mathematics educators can change the misconceptions after brief instructions and provide an appropriate situation for conceptually learning (Durkin & Rittle-Johnson, 2015). In fact, Suitable teaching methods eliminate the mistakes and misconceptions that students have about mathematical concepts (Yilmaz et al., 2018). In teaching trigonometry, mostly misconceptions arise from the teaching method. For instance, as identified by Tuna (2013) about 90% of the novice mathematics teachers had misconceptions regarding the definition of the trigonometric concept of radian. They explained incorrect definitions for this concept such as, "the expression of degree in terms of  $\pi$ ", "the unit of length of degree", and "I just know the formula of  $\frac{D}{180} = \frac{R}{\pi}$ " and "I do not know what radian is". Therefore, mathematics educators, beside the appropriate subject matter knowledge, require knowing the common misunderstandings and misconceptions that students face in a specific topic. Regarding this issue, knowing the variety of creatively solutions for trigonometric problems and students' misconceptions about them provide a good opportunity for educators to improve their teaching knowledge (Dundar, 2015). The purpose of this study is to prepare an effective research lesson for teaching a given trigonometry problem and investigate the misconceptions emerging in students learning regarding this problem in order to provide better teaching and learning situation.

## METHODOLOGY

### *Research Design and Sample*

This study was conducted during the academic year 2020. In this study, a group of three mathematics teachers (Two males and a female) from an international school in Malaysia and the researcher collaboratively planned, discussed and designed a research lesson on the topic of the maximum and minimum values of a trigonometric function. Meanwhile, all the members of the Lesson Study group were experienced teachers with at least 15 years experiences in teaching mathematics. Furthermore, 10 students (4 male and 6 female) from an existing class were participating in this study.

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### *Data Collection*

Finding the maximum and minimum values of a trigonometric function is an important part of trigonometry subject that apply in many trigonometric concepts such as the range of functions, drawing the graph of functions and optimization the real world problems. For example, the following problem shows an application of the maximum and minimum values of a trigonometric function in the real world.

Problem: In a four season country, the length of each day of a year calculated based on the following trigonometric function

$$L(t) = 12 + 2.4 \sin \frac{2\pi}{365}(t - 1)$$

where,  $t$  represents the order of days in the year (for example,  $t = 4$  means the fourth day of the year) and  $L(t)$  is the length of the day in hours. Determine the length of the longest and shortest day of the year.

Based on the importance of maximum and minimum values of the trigonometric functions, the researcher studied regarding the topic “Maximum and minimum values of the function  $f(x) = a \sin x \pm b \cos x$ , where  $a, b \in R$ ”. Therefore, in this research lesson, the Lesson Study group members suggested three solution methods for this problem and discussed the likely misconceptions of students about these solutions.

Two weeks before starting this study, the researcher introduced the topic of this research lesson to the members of Lesson Study group and asked them to prepare suitable material for a rich research lesson. In a meeting, they planned, discussed and designed a research lesson and a teacher of Lesson Study group taught this research lesson in a class and the others observed and collected data. In a post-discussion meeting, they tried to improve the quality of this research lesson.

### *Analyzing the Data*

All members of the Lesson Study group were familiar with the Lesson Study approach because they participated in an in-service program related to the Lesson Study a few months before starting this study. The researcher introduced the topic of this research lesson to teachers and asked them to share their knowledge and experience to produce a research lesson on the topic entitled “Maximum and minimum values of the function  $f(x) = a \sin x \pm b \cos x$ , where  $a, b \in R$ ”. During three sessions, they planned, discussed and designed a research lesson for pre-university level students. The materials in this research lesson gathered through observations of discussion meetings and teaching this research lesson for students in a real class by a member of Lesson Study group. The researcher analyzed the methods and suggestions of the Lesson Study group members descriptively to prepare a research lesson.

## FINDINGS ON THE PREPARED RESEARCH LESSON

The members of the Lesson Study group prepared a research lesson entitled “Maximum and minimum values of the function  $f(x) = a \sin x \pm b \cos x$ , where  $a, b \in R$ ” and they suggested the following solutions collaboratively. Furthermore, they referred to some common misconceptions among students about this topic.

### *First solution*

We know the formula

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y.$$

In this formula, for  $\sin(x \pm y)$ , there are two terms  $\sin x \cos y$  and  $\cos x \sin y$  with the following property:

The sum of squares of factors one from each of two terms is equal to one,  $\sin^2 x + \cos^2 x = 1$ . Similarly, the sum of squares of factors two from both terms is equal to one,  $\cos^2 y + \sin^2 y = 1$ . This property can be regarded as the condition for an expression of the trigonometric function  $y = a \sin x \pm b \cos y$  to be converted into an expression consisting of only sine function. The method and process for converting this trigonometric function into an expression consisting of only cosine function is similar.

In the function  $y = a \sin x \pm b \cos x$ , we have  $\sin^2 x + \cos^2 x = 1$  but the sum of squares of the first factors of terms  $a \sin x$  and  $b \cos x$  is  $a^2 + b^2$ . Since we do not know whether  $a^2 + b^2$  is equal to one or not, we multiply  $a$  and  $b$  by a number  $m$  such that  $(ma)^2 + (mb)^2 = 1$ .

$$\begin{aligned} (ma)^2 + (mb)^2 &= 1 \\ \Rightarrow m^2 a^2 + m^2 b^2 &= 1 \\ \Rightarrow m^2 (a^2 + b^2) &= 1 \\ \Rightarrow m^2 &= \frac{1}{a^2 + b^2} \Rightarrow m = \frac{1}{\sqrt{a^2 + b^2}} \end{aligned}$$

Therefore, we have

$$(ma)^2 + (mb)^2 = 1 \Rightarrow \left(\frac{a}{\sqrt{a^2 + b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2 + b^2}}\right)^2 = 1.$$

Now, we can write

$$\begin{aligned} y &= a \sin x \pm b \cos x \\ \Rightarrow y &= \frac{ma \sin x}{m} \pm \frac{mb \cos x}{m} \end{aligned}$$

$$\begin{aligned}\Rightarrow y &= \frac{1}{m}(ma \sin x \pm mb \cos x) \\ \Rightarrow y &= \frac{1}{\frac{1}{\sqrt{a^2 + b^2}}} \left( \frac{a}{\sqrt{a^2 + b^2}} \sin x \pm \frac{b}{\sqrt{a^2 + b^2}} \cos x \right) \\ \Rightarrow y &= \sqrt{a^2 + b^2} \left( \frac{a}{\sqrt{a^2 + b^2}} \sin x \pm \frac{b}{\sqrt{a^2 + b^2}} \cos x \right)\end{aligned}$$

Assume that  $\frac{a}{\sqrt{a^2 + b^2}} = \cos \theta$ , then,  $\sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \frac{a^2}{a^2 + b^2}} = \frac{b}{\sqrt{a^2 + b^2}}$ .

Based on the above calculations we obtain the following trigonometric function

$$\begin{aligned}y &= \sqrt{a^2 + b^2}(\cos \theta \sin x \pm \sin \theta \cos x) \\ \Rightarrow y &= \sqrt{a^2 + b^2} \sin(x \pm \theta).\end{aligned}$$

Since,  $-1 \leq \sin(x \pm \theta) \leq 1$ , then

$$-\sqrt{a^2 + b^2} \leq \sqrt{a^2 + b^2} \sin(x \pm \theta) \leq \sqrt{a^2 + b^2}.$$

It means that

$$-\sqrt{a^2 + b^2} \leq a \sin x \pm b \cos x \leq \sqrt{a^2 + b^2}.$$

*Second Solution:*

We convert this function into an expression consisting of only sine function as follows

$$\begin{aligned}a \sin x + b \cos x &= c \sin(k + x) \\ \Rightarrow a \sin x + b \cos x &= c \sin k \cos x + c \cos k \sin x.\end{aligned}$$

In the above equality, for any value of  $x$ , the coefficients of  $\sin x$  and  $\cos x$  should be equal on the left and right sides. Therefore,

$$a = c \cos k$$

$$b = c \sin k.$$

In this system of simultaneous equations we have,

$$a = c \cos k \Rightarrow \cos k = \frac{a}{c}$$

$$b = c \sin k \Rightarrow \sin k = \frac{b}{c}$$

$$\sin^2 k + \cos^2 k = 1 \Rightarrow \left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 = 1 \Rightarrow c = \pm\sqrt{a^2 + b^2}.$$

Now, according to the relations  $a = c \cos k$  and  $b = c \sin k$  we obtain

$$\frac{c \sin k}{c \cos k} = \frac{b}{a} \Rightarrow \tan k = \frac{b}{a} \Rightarrow k = \tan^{-1} \frac{b}{a}.$$

Therefore, we have

$$a \sin x + b \cos x = \pm\sqrt{a^2 + b^2} \sin\left(x + \tan^{-1} \frac{b}{a}\right).$$

If we limit the arctan to be within

$$-\frac{\pi}{2} < \tan^{-1} \frac{b}{a} < \frac{\pi}{2}.$$

Then we obtain the following relation

$$a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin\left(x + \tan^{-1} \frac{b}{a}\right).$$

Since  $-1 \leq \sin\left(x + \tan^{-1} \frac{b}{a}\right) \leq 1$ , we see that

$$\begin{aligned} -\sqrt{a^2 + b^2} &\leq \sqrt{a^2 + b^2} \sin\left(x + \tan^{-1} \frac{b}{a}\right) \leq \sqrt{a^2 + b^2} \\ \Rightarrow -\sqrt{a^2 + b^2} &\leq f(x) = a \sin x + b \cos x \leq \sqrt{a^2 + b^2}. \end{aligned}$$

Similarly, according to this method we can find the maximum and minimum values of the trigonometric function  $y = a \sin x - b \cos x$ .

*Third solution:*

Definition 1:

The dot product of two vectors  $\vec{u} = (a, b)$  and  $\vec{v} = (x, y)$ , written  $\vec{u} \cdot \vec{v}$  is given by the definition

$$\vec{u} \cdot \vec{v} = (a, b) \cdot (x, y) = ax + by.$$

Definition 2:

Assume that the angle between two vectors  $\vec{u} = (a, b)$  and  $\vec{v} = (x, y)$  is  $\theta$  then the dot product of these vectors is defined as

$$\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta$$

where  $|\vec{u}| = \sqrt{a^2 + b^2}$  and  $|\vec{v}| = \sqrt{x^2 + y^2}$ .



Now for solution of this given problem, we consider two vectors  $\vec{u}_1 = (a, b)$  and  $\vec{u}_2 = (\sin x, \cos x)$  and find the dot product of them through two different methods based on the definitions 1 and 2.

$$\vec{u}_1 \cdot \vec{u}_2 = (a, b) \cdot (\sin x, \cos x) = a \sin x + b \cos x$$

$$\vec{u}_1 \cdot \vec{u}_2 = |\vec{u}_1| |\vec{u}_2| \cos \theta = \sqrt{a^2 + b^2} \sqrt{\sin^2 \theta + \cos^2 \theta} \cos \theta = \sqrt{a^2 + b^2} \cos \theta$$

Since  $-1 \leq \cos \theta \leq 1$ ,

$$\begin{aligned} -\sqrt{a^2 + b^2} &\leq \sqrt{a^2 + b^2} \cos \theta \leq \sqrt{a^2 + b^2} \\ \Rightarrow -\sqrt{a^2 + b^2} &\leq \vec{u}_1 \cdot \vec{u}_2 \leq \sqrt{a^2 + b^2} \\ \Rightarrow -\sqrt{a^2 + b^2} &\leq a \sin x + b \cos x \leq \sqrt{a^2 + b^2}. \end{aligned}$$

Therefore, for the trigonometric function  $y = a \sin x + b \cos x$  we have

$$-\sqrt{a^2 + b^2} \leq y \leq \sqrt{a^2 + b^2}.$$

Through similar process, we can show that  $-\sqrt{a^2 + b^2} \leq a \sin x - b \cos x \leq \sqrt{a^2 + b^2}$ .

Example 1:

Find the range of the function  $f(x) = 2 + 3 \sin x - 4 \cos x$ .

Solution:

We know that

$$-\sqrt{3^2 + 4^2} \leq 3 \sin x - 4 \cos x \leq \sqrt{3^2 + 4^2} \Rightarrow -5 \leq 3 \sin x - 4 \cos x \leq 5.$$

Therefore,  $-3 \leq 2 + 3 \sin x - 4 \cos x \leq 7$  and the range of this function is  $R_f = [-3, 7]$ .

Example 2:

Determine the maximum and minimum values of the following two variables function

$$f(x, y) = 3 \sin x + 4 \cos x - 3 \cos y + 4 \sin y - 4.$$

Solution:

We know,

$$\begin{aligned} -\sqrt{3^2 + 4^2} &\leq 3 \sin x + 4 \cos x \leq \sqrt{3^2 + 4^2} \\ -\sqrt{3^2 + 4^2} &\leq 4 \sin y - 3 \cos y \leq \sqrt{3^2 + 4^2}. \end{aligned}$$

Therefore,

$$-10 \leq 3 \sin x + 4 \cos x + 4 \sin y - 3 \cos y \leq 10.$$

Now, we have

$$-14 \leq 3 \sin x + 4 \cos x + 4 \sin y - 3 \cos y - 4 \leq 6.$$

It means  $-14 \leq f(x, y) \leq 6$ .

*Generalization of this Trigonometric Problem*

Prove that the range of the function  $g(x) = a \sin kx + b \cos kx + c$  is calculated based on the rule  $R_g = [c - \sqrt{a^2 + b^2}, c + \sqrt{a^2 + b^2}]$ .

Proof:

Firstly, we show that,  $\forall a, b, k, x \in R, -\sqrt{a^2 + b^2} \leq a \sin kx + b \cos kx \leq \sqrt{a^2 + b^2}$ . The maximum and minimum values of the function  $y = a \sin kx \pm b \cos kx$  discussed as follows:

Since,  $-1 \leq \frac{a}{\sqrt{a^2 + b^2}} \leq 1$ , we assume that  $\cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}$  therefore,  $\sin \alpha = \frac{\pm b}{\sqrt{a^2 + b^2}}$ .

$$\begin{aligned} y &= a \sin kx \pm b \cos kx \\ \Rightarrow \frac{y}{\sqrt{a^2 + b^2}} &= \frac{a}{\sqrt{a^2 + b^2}} \sin kx \pm \frac{b}{\sqrt{a^2 + b^2}} \cos kx \\ \Rightarrow \frac{y}{\sqrt{a^2 + b^2}} &= \cos \alpha \sin kx \pm \sin \alpha \cos kx = \sin(kx \pm \alpha). \end{aligned}$$

We know,  $-1 \leq \sin(kx \pm \alpha) \leq 1$ , thus

$$\begin{aligned} -1 \leq \frac{y}{\sqrt{a^2 + b^2}} \leq 1 &\Rightarrow -\sqrt{a^2 + b^2} \leq y \leq \sqrt{a^2 + b^2} \\ \Rightarrow -\sqrt{a^2 + b^2} &\leq a \sin kx \pm b \cos kx \leq \sqrt{a^2 + b^2}. \end{aligned}$$

Therefore,

$$c - \sqrt{a^2 + b^2} \leq a \sin kx \pm b \cos kx + c \leq c + \sqrt{a^2 + b^2}$$

It means

$$\begin{aligned} c - \sqrt{a^2 + b^2} &\leq g(x) \leq c + \sqrt{a^2 + b^2} \\ \Rightarrow R_g &= [c - \sqrt{a^2 + b^2}, c + \sqrt{a^2 + b^2}]. \end{aligned}$$

Example 3:

Find the range of the function  $h(x) = [2 \sin 4x - 3 \cos 4x]$  where  $[\ ]$  is the symbol of partial integer.

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Solution:

The function  $h$  is continuous on the real numbers. As respect to the above theorem we have

$$-\sqrt{13} \leq 2 \sin 4x - 3 \cos 4x \leq \sqrt{13}$$

Therefore, we obtain

$$R_h = \{-4, -3, -2, -1, 0, 1, 2, 3\}.$$

The following four problems are suitable to discuss in the classroom, because such problems improve the ability of students in problem solving.

Problem 1:

For the function  $y = 12 \sin x + 5 \cos x$ , write the linear combination of sine and cosine as consisting of only cosine function.

Problem 2:

Find the maximum and minimum values of the function  $f(x) = \frac{2 + \sqrt{3} \sin x - 4 \cos x}{1 + \sqrt{4} \sin x + 3 \cos x}$ .

Problem 3:

Prove that

$$(\sin x + a \cos x)(\sin x + b \cos x) \leq 1 + \left(\frac{a+b}{2}\right)^2.$$

Problem 4:

Prove that

$$(\sin 3x + a \cos 3x)(\sin 3x + b \cos 3x) \leq \frac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2 b^2 + 1}).$$

## DISCUSSION

Based on a research by Gholami et al. (2021), some common misconceptions that students encounter regarding the problem “find the maximum and minimum values of the trigonometric function  $h(x) = \sin x + \cos x$ ” are as follows:

- a) According to inequalities  $-1 \leq \sin x \leq 1$  and  $-1 \leq \cos x \leq 1$  we have  $-2 \leq \sin x + \cos x \leq 2$  therefore, the maximum and minimum values of this function are 2 and  $-2$  respectively. In this argument, the two functions  $y = \sin x$  and  $y = \cos x$  considered as two independent functions, whereas the values of  $\sin x$  and  $\cos x$  are dependent based on the formula  $\sin^2 x + \cos^2 x = 1$ . For instance, for the angles in the first quartile of the unit circle, the value of  $\sin x$  increases when the value of  $\cos x$  decreases and conversely. It means, we

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cannot consider  $\sin x = 1$  and  $\cos x = 1$  simultaneously. By using the formula  $\sin^2 x + \cos^2 x = 1$  it is clear that  $\cos x = 0$  when  $\sin x = 1$ . The above argument is correct to find the maximum and minimum values of  $A = \sin x + \cos y$  because the two values of  $\sin x$  and  $\cos y$  are independent.

- b) By squaring both sides of the function  $h(x) = \sin x + \cos x$  we have

$$h^2(x) = (\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2 \sin x \cos x = 1 + \sin 2x.$$

$$0 \leq 1 + \sin 2x \leq 2 \Rightarrow 0 \leq h^2(x) \leq 2 \Rightarrow 0 \leq h(x) \leq \sqrt{2}.$$

In the above argument, the misconception is related to the concept  $0 \leq h^2(x) \leq 2 \Rightarrow 0 \leq h(x) \leq \sqrt{2}$  because the inequality  $0 \leq h^2(x) \leq 2$  is equivalent to  $h^2(x) \leq 2$  and hence  $-\sqrt{2} \leq h(x) \leq \sqrt{2}$ .

- c)  $h(x) = \sin x + \cos x \Rightarrow h(x) - \cos x = \sin x \Rightarrow -1 \leq h(x) - \cos x \leq 1$   
 $\Rightarrow -1 + \cos x \leq h(x) \leq 1 + \cos x \Rightarrow -2 \leq h(x) \leq 2.$

In this argument, obtaining the inequality  $-1 \leq h(x) - \cos x \leq 1$  is a misconception because the values  $\sin x$  and  $\cos x$  are dependent.

A logical argument for this problem was shown as follows

$$h(x) = \sin x + \cos x = \sqrt{2} \left( \frac{\sqrt{2}}{2} \sin x + \frac{\sqrt{2}}{2} \cos x \right) = \sqrt{2} (\cos 45 \sin x + \sin 45 \cos x)$$

$$\Rightarrow h(x) = \sqrt{2} \sin(x + 45).$$

Since,  $-1 \leq \sin(x + 45) \leq 1$  we have  $-\sqrt{2} \leq \sin(x + 45) \leq \sqrt{2}$ , thus  $-\sqrt{2} \leq h(x) \leq \sqrt{2}$ .

In this study, three students considered  $\sin x = 1$  and  $\cos x = 1$  to find  $a(1) + b(1) = a + b$  as the maximum value of the function  $f(x) = a \sin x + b \cos x$ . Also, they argued that the minimum value of this function is  $a(-1) + b(-1) = -(a + b)$ . In this argument, teachers can refer to the two important misconceptions that students face. Firstly, students cannot consider  $\sin x = \pm 1$  and  $\cos x = \pm 1$  simultaneously and another one is related to the sign of the variables  $a$  and  $b$  because these variables can be negative. Therefore, based on this incorrect argument, the maximum and minimum values of the function  $f(x) = a \sin x + b \cos x$  should be  $|a| + |b|$  and  $-(|a| + |b|)$  respectively. Students understood that the above argument is correct for the two variables function  $f(x, y) = a \sin x + b \cos y$  because in this function we can consider  $\sin x = \pm 1$  and  $\cos x = \pm 1$  simultaneously. Therefore, the maximum and minimum values of the function  $f(x, y) = a \sin x + b \cos y$  is  $|a| + |b|$  and  $-(|a| + |b|)$  respectively due to independency of  $\sin x$  and  $\cos y$ .

Also, two of them rewrite the function  $f(x) = a \sin x + b \cos x$  as  $f(x) - a \sin x = b \cos x$ , since  $-|b| \leq b \cos x \leq |b|$ , then  $-|b| \leq f(x) - a \sin x \leq |b|$ . Therefore,

$$-|b| \leq f(x) - a \sin x \leq |b|$$

$$\Rightarrow -|b| + a \sin x \leq f(x) \leq |b| + a \sin x$$

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$$\Rightarrow -|b| - |a| \leq f(x) \leq |b| + |a|.$$

In the above method of solution, students considered  $\sin x$  and  $\cos x$ , as two independent variables wrongly, whereas these two variables are dependent. Teachers explained this misconception to students using a specific function such as  $g(x) = 3 \sin x + 4 \cos x$  and helped students to understand we cannot obtain 7 as the maximum value of this function because when we put  $\sin x = 1$ , the value of  $\cos x$  should be only zero.

The rest of the students in the class got the right answer by considering specific situations for this problem. For example, one of the students considered equal value for both coefficients  $a$  and  $b$  and he found the values  $|a|\sqrt{2}$  and  $-|a|\sqrt{2}$  as the maximum and minimum values of the function  $h(x) = a \sin x + a \cos x$  respectively.

In the Problem 1, “For the function  $y = 12 \sin x + 5 \cos x$ , write the linear combination of sine and cosine as consisting of only cosine function” some students by using the formula  $\sin x = \pm\sqrt{1 - \cos^2 x}$  simply changed it as  $y = \pm 12\sqrt{1 - \cos^2 x} + 5 \cos x$ . They didn’t understand the concept of linear combination of two variables.

For the problem 2, “Find the maximum and minimum values of the function  $f(x) = \frac{2 + \sqrt{3} \sin x - 4 \cos x}{1 + \sqrt{4} \sin x + 3 \cos x}$ .” some students found the maximum and minimum values of numerator  $2 + \sqrt{3} \sin x - 4 \cos x$  as follows

$$\begin{aligned} -\sqrt{3^2 + (-4)^2} &\leq 3 \sin x - 4 \cos x \leq \sqrt{3^2 + (-4)^2} \\ \Rightarrow -5 &\leq 3 \sin x - 4 \cos x \leq 5 \\ \Rightarrow 0 &\leq \sqrt{3 \sin x - 4 \cos x} \leq \sqrt{5} \\ \Rightarrow 2 &\leq 2 + \sqrt{3 \sin x - 4 \cos x} \leq 2 + \sqrt{5}. \end{aligned}$$

Through similar calculation, they found the numbers 1 and  $1 + \sqrt{5}$  as the maximum and minimum values of denominator  $1 + \sqrt{4} \sin x + 3 \cos x$  respectively. They argued in order to find the maximum value of the function  $f(x)$  we require to divide the maximum value of numerator to the minimum value of denominator. Therefore, the maximum value of the function  $f(x)$  is  $\frac{2 + \sqrt{5}}{1} = 2 + \sqrt{5}$ . Similarly, the minimum value of this function should be  $\frac{2}{1 + \sqrt{5}}$ . This argument is not correct because the two values  $2 + \sqrt{3 \sin x - 4 \cos x}$  and  $1 + \sqrt{4 \sin x + 3 \cos x}$  are dependent. But this argument is true for the problem “find the maximum and minimum values of  $C = \frac{2 + \sqrt{3 \sin x - 4 \cos x}}{1 + \sqrt{4 \sin x + 3 \cos x}}$ .” because of independency of nominator and denominator of the statement  $C$ .

There is a common misconception among students regarding this problem that is so important for teachers to know. We have the following inequalities

$$2 \leq 2 + \sqrt{3 \sin x - 4 \cos x} \leq 2 + \sqrt{5}$$

$$1 \leq 1 + \sqrt{4 \sin x + 3 \cos x} \leq 1 + \sqrt{5}.$$

By dividing all sides of the above inequalities we obtain

$$2 \leq \frac{2 + \sqrt{3 \sin x - 4 \cos x}}{1 + \sqrt{4 \sin y + 3 \cos y}} \leq \frac{2 + \sqrt{5}}{1 + \sqrt{5}}.$$

It is clear that this result is not acceptable because  $2 > \frac{2 + \sqrt{5}}{1 + \sqrt{5}}$ .

The logical argument for this problem must be as follows

$$2 \leq 2 + \sqrt{3 \sin x - 4 \cos x} \leq 2 + \sqrt{5}$$

$$1 \leq 1 + \sqrt{4 \sin x + 3 \cos x} \leq 1 + \sqrt{5} \Rightarrow \frac{1}{1 + \sqrt{5}} \leq \frac{1}{1 + \sqrt{4 \sin x + 3 \cos x}} \leq 1.$$

By multiplying all sides of these inequalities we can see

$$\frac{2}{1 + \sqrt{5}} \leq \frac{2 + \sqrt{3 \sin x - 4 \cos x}}{1 + \sqrt{4 \sin y + 3 \cos y}} \leq 2 + \sqrt{5}.$$

Therefore, the maximum and minimum values of  $C = \frac{2 + \sqrt{3 \sin x - 4 \cos x}}{1 + \sqrt{4 \sin y + 3 \cos y}}$  are  $2 + \sqrt{5}$  and  $\frac{2}{1 + \sqrt{5}}$  respectively.

Another misconception is about the problem 3, "Prove that  $(\sin x + a \cos x)(\sin x + b \cos x) \leq 1 + \left(\frac{a+b}{2}\right)^2$ ."

For this problem, the members of Lesson Study group members were faced with the following argument

$$(\sin x + a \cos x)(\sin x + b \cos x) \leq \sqrt{1 + a^2} \sqrt{1 + b^2}.$$

After that students tried to prove  $\sqrt{1 + a^2} \sqrt{1 + b^2} \leq 1 + \left(\frac{a+b}{2}\right)^2$  but this inequality is not correct since, by setting  $a = 1$  and  $b = -1$  in this inequality, we obtain  $\sqrt{1 + (1)^2} \sqrt{1 + (-1)^2} \leq 1 + \left(\frac{1+(-1)}{2}\right)^2 \Rightarrow 2 \leq 1$ . This misconception is related to the dependency of the statements  $\sin x + a \cos x$  and  $\sin x + b \cos x$ . In fact, for maximization of the statement  $(\sin x + a \cos x)(\sin x + b \cos x)$  we cannot consider the maximum values of the statements  $\sin x + a \cos x$  and  $\sin x + b \cos x$  simultaneously. A logical proof for this problem that suggested by a student is as follows

Proof: If  $\cos x = 0$ , this inequality reduces to  $\sin^2 x \leq 1 + \left(\frac{a+b}{2}\right)^2$ , which is obviously true. We assume that  $\cos x \neq 0$ , thus by dividing both sides of the given inequality by  $\cos^2 x$  gives

$$(\tan x + a)(\tan x + b) \leq \left[1 + \left(\frac{a+b}{2}\right)^2\right] \sec^2 x.$$

Now, we set  $\tan x = m$ , then  $\sec^2 x = 1 + m^2$ . Therefore, the above inequality changes to

$$\begin{aligned} m^2 + (a+b)m + ab &\leq \left(\frac{a+b}{2}\right)^2 m^2 + m^2 + \left(\frac{a+b}{2}\right)^2 + 1 \\ \Rightarrow \left(\frac{a+b}{2}\right)^2 m^2 + 1 - (a+b)m + \left(\frac{a+b}{2}\right)^2 - ab &\geq 0 \\ \Rightarrow \left[\left(\frac{a+b}{2}\right)^2 m^2 + 1 - (a+b)m\right] + \left[\frac{a^2 + b^2 - 2ab}{4}\right] &\geq 0 \\ \Rightarrow \left(\frac{(a+b)m}{2} - 1\right)^2 + \left(\frac{a-b}{2}\right)^2 &\geq 0. \end{aligned}$$

The proof is complete because all of the above statements are reversible.

The same misconception is discussable for problem 4, “prove that  $(\sin 3x + a \cos 3x)(\sin 3x + b \cos 3x) \leq \frac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2 b^2 + 1})$ .” In this problem, we can write

$$(\sin 3x + a \cos 3x)(\sin 3x + b \cos 3x) \leq \sqrt{1 + a^2} \sqrt{1 + b^2}.$$

Now, we should prove

$$\sqrt{1 + a^2} \sqrt{1 + b^2} \leq \frac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2 b^2 + 1}).$$

The above inequality is a misunderstanding, because by setting  $a = 2$  and  $b = -2$  we obtain

$$\sqrt{1 + (2)^2} \sqrt{1 + (-2)^2} \leq \frac{1}{2}(1 + (2)(-2) + \sqrt{(2)^2 + (-2)^2 + (2)^2(-2)^2 + 1}) \Rightarrow 5 \leq 1.$$

A member of the Lesson Study group suggested a logical solution for this problem as follows

$$\begin{aligned} &(\sin 3x + a \cos 3x)(\sin 3x + b \cos 3x) \\ &= \sin^2 3x + b \sin 3x \cos 3x + a \cos 3x \sin 3x + ab \cos^2 3x \\ &= 1 + (ab - 1) \cos^2 3x + \left(\frac{a+b}{2}\right) \sin 6x \\ &= 1 + (ab - 1) \left(\frac{1 + \cos 6x}{2}\right) + \left(\frac{a+b}{2}\right) \sin 6x \\ &= \frac{ab + 1}{2} + \left(\frac{a+b}{2}\right) \sin 6x + \left(\frac{ab - 1}{2}\right) \cos 6x \end{aligned}$$

$$\begin{aligned} &\leq \frac{ab+1}{2} + \sqrt{\left(\frac{a+b}{2}\right)^2 + \left(\frac{ab-1}{2}\right)^2} \\ &= \frac{1}{2}(1+ab + \sqrt{a^2+b^2+a^2b^2+1}). \end{aligned}$$

## CONCLUSIONS

Improving the mathematical knowledge of mathematics teachers affects the quality of lesson design, teaching methods and classroom atmosphere (Copur-Gencturk, 2015). Therefore, teachers require enhancing their mathematical knowledge continually. One of the best ways regarding this issue is sharing their knowledge and experiences through Lesson Study approach. In teaching mathematics concepts through problem solving, teachers need to understand mathematical ideas regarding a problem in a deep and connected way, and further they should be familiar with different methods of solution (O. Masingila et al., 2018). In this study, the Lesson Study group members suggested three different solution methods to find the maximum and minimum values of the trigonometric function  $y = a \sin x + b \cos x$  and through similar ways they found the maximum and minimum values of the function  $y = a \sin x - b \cos x$ . Discussion about the variety of solution methods for the maximum and minimum values of the function  $y = a \sin x \pm b \cos x$  helps learners to improve their abilities in problem solving. Also, teachers generalized this given problem to finding the maximum and minimum values of the function  $g(x) = a \sin kx + b \cos kx + c$  that improve the skills of educators and students in generalizing the trigonometric concepts. They enhanced the quality of this research lesson by considering some suitable problems related to this given trigonometric problem and discussing regarding the variety of students' misconceptions in order to improve the performance of educators in their teaching. In this study, the members of the Lesson Study group found that students had serious problems to solve a general problem such as "find the maximum and minimum values of the function  $y = a \sin x \pm b \cos x$ " and "find the maximum and minimum values of the function  $y = a \sin kx \pm b \cos kx + c$ " because the coefficients  $a, b, c$  and  $k$  is not clear for students. They can solve the problems with clear coefficients such as "find the maximum and minimum values of the function  $y = 3 \sin x + 3 \cos x$ " and "find the maximum and minimum values of the function  $y = 4 \sin 2x + 3 \cos 2x - 5$ " easily because of the clearance coefficients in these functions.

In this article, the researcher discussed about the variety of solutions for a given trigonometric problem and related misconceptions to improve the mathematical knowledge of educators. Although mathematical misunderstandings are common among students, some novice teachers are also involved. Therefore, this research lesson may help teachers to provide a better learning environment for students regarding this trigonometric problem. Meanwhile, experienced teachers can improve this research lesson based on their students' abilities to reduce the trigonometric misconceptions among students.



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## Investigation of Students' Algebraic Conceptual Understanding and the Ability to Solve PISA-Like Mathematics Problems in a Modeling Task

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**Abstract:** *Several studies related to mathematics understanding found that many undergraduate students lack some basic knowledge of algebra. They memorized only a few topics, formulas, and algorithms without understanding them conceptually, even though they could manipulate those limited number of points correctly or incorrectly. In comparison, most high-achieving students have incomplete solutions in Modeling Mathematics PISA-like tasks in levels 5 and 6, related to the content of change and relationship. In contrast, students with moderate achievement can solve the problem using instinct, trial and error, and logic. Therefore, this study aims to analyze students' algebraic conceptual understanding related to their modeling competence in solving a mathematical problem that is an adapted PISA task, using qualitative research as an appropriate method. It emphasizes a holistic description of the phenomena studied concerning how students work with algebraic conceptual problems and Mathematics task-like PISA problems. This study involved 244 new vocational college students in 5 study programs. Data collection used in this research is students' worksheets, video recording, and interviewing some students to obtain more profound information about their thinking processes. Furthermore, the data was analyzed by holistic description. Interpretation and conclusion using the definition of equation and algebra expression and indicator of modeling competence, mathematical literacy refers to the proficiency level of PISA question given as a guideline to interpret and make a conclusion. Some discovered strategies for solving the problem and implications regarding their mathematical literacy skills related to these tasks are discussed.*

### INTRODUCTION

Algebra is a vital field of learning that plays a significant role in mathematical thinking and language expressed with symbols, tables, words, and graphs (Stacey & MacGregor, 1999). An understanding of algebra can help students to recognize the importance of mathematics. They should comprehend symbols and their manipulations to interpret the letters employed in various algebraic situations, the structural aspects, and the solution (Kieran, 2007; Sukirwan et al., 2018). Algebra is regarded as a gatekeeper course since students are expected to pass before moving to

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the next level. Wu (2001) argued that qualified teachers are needed to teach algebra successfully. Additionally, Gram and Jacobson (2000) stated that high school mathematics is widely regarded as the “gatekeeper” to teaching algebra, a subject that students in US public schools mostly fail.

Algebra is now a required part of most curricula, including vocational college. Therefore, all the college students included in the agriculture field need an understanding of the concepts and skills in using algebraic operations. Mathematics is not the primary major in the vocational curriculum, but it is one of the important subjects that students in agriculture majors are expected to master. Furthermore, agriculture is related to measurements, estimates, and projections involving algebraic operations, such as modeling growth, food supply models, and fisheries market trends.

The demand for algebra at more levels of education is increasing. Wiki Answers, one of the world’s most important questions and answers websites, outlines some current algebra uses (Gunawardena, 2011). For example, companies use algebra to determine their annual budget, including their annual expenditure. Additionally, various stores use algebra to predict the demand for a particular product and subsequently place their orders. It has individual applications in calculating annual taxable income, bank interest, and installment loans.

Several types of research have proposed that student misconceptions or gaps in conceptual knowledge of Algebra lead to incorrect and clumsy procedures for solving problems (Booth & Koedinger, 2008; Jacobson, 1981; Jupri & Drijvers, 2016; Nathan, 2000; Van Lehn & Johnes, 1993). However, in the domain of algebraic problem solving, one type of prior knowledge that is key to learning is a conceptual understanding of features in the problem, including equals sign, variables, like terms, and negative signs (Jupri & Drijvers, 2016). Conceptual knowledge of these features enables the user to recognize the symbols or perform an operation, as well as comprehend the purpose of the equation and the effect of relocating the feature on the overall problem (Nathan, 2000; Van Lehn & Johnes, 1993). Students should have a firm grasp of the problem’s fundamental elements to comprehend the instructional information completely (Booth & Koedinger, 2008; Jacobson, 1981). They are unlikely to demonstrate significant advances in procedural knowledge without this in-depth, relevant knowledge of problem characteristics. Deep strategy construction relies on the inclusion of sufficient information about the problematic aspects that make them appropriate or inappropriate. Therefore, having a high conceptual understanding may be required to solve equations appropriately. However, for students with an insufficient conceptual understanding of the problem’s characteristics, superficial methods such as the one outlined above are likely to dominate (Booth & Koedinger, 2008).

Incorrect procedures are typical when learning Algebra (Sebrechts et al., 1996), which inhibits accurate solutions. Moreover, many university students in the US also lack some basic understanding of algebra (Booth & Koedinger, 2008; Gunawardena, 2011; Jacobson, 1981). As a result, they commit the same mistakes as their secondary school counterparts. They memorized only a few facts, formulas, and algorithms without understanding the concept. However (Rittle-

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Johnson & Star, 2014; Star & Seifert, 2020) argued that a critical learning outcome in problem-solving domains is the development of flexible knowledge, where multiple strategies are learned and applied adaptively to a range of situations. Booth and Koedinger (2008) stated that in algebraic problem solving, one type of prior knowledge is the conceptual understanding of features in the problem, including equal signs, variables, terms, and negative signs. For example, understanding the equals sign has previously been crucial for algebraic problem solutions (Knuth et al., 2006).

PISA is the acronym for the ‘Programme for International Student Assessment,’ the OECD’s international program assessing reading, scientific and mathematical literacy ([www.oecd.org/pisa](http://www.oecd.org/pisa)). Mathematical literacy is an individual’s capacity to formulate, employ, and interpret mathematics in various contexts. It includes mathematical reasoning and using the concepts, procedures, facts, and tools to describe, explain and predict phenomena. It enables individuals to grasp the significance of mathematics in the world and make the sound judgments and choices required of constructive, engaged, and thoughtful citizens (Framework PISA, 2012). This conception supports the importance of students developing a solid understanding of pure mathematics and the benefits of exploring the abstract world. The construct of mathematical literacy, as defined for PISA, emphasizes the need to develop within students the capacity to use the context, and they are expected to have rich experiences within the classrooms to accomplish this.

Mathematical literacy refers to an individual’s capacity to *formulate*, *employ*, and *interpret real-world* problems. Stacey (2011) stated the close relationship of this concept to mathematical modeling. However, Edo et al. (2013) found that most high-achieving students have incomplete solutions in Modeling Mathematics Tasks like PISA levels 5 and 6, related to change and relationship content which refer to algebra framework. They cannot solve the non-routine problem correctly since problems cannot be formulated mathematically. In contrast, students with moderate achievement can solve the problem using their “instinct,” trial and error,” and “logic. Gunawardena (2011) argued that college students pursued similar difficulty in word problems. This is because the frequency of occurrence is very high in all problems, and they should interpret and convert everyday language into algebra. Therefore, this study aimed to examine students’ conceptual understanding of solving routine algebraic problems and their ability to solve non-routine problems adapted from PISA tasks.

## LITERATURE REVIEW

An equation is a phrase that expresses the equality of two algebraic expressions. For example, in  $x + 3 = 9$ ,  $x + 3$  is the left side, or left member, and 9 is the right or right member. An Equation may be a true, false or an open sentence such as  $2 + 3 = 5$ ,  $7 - 2 = 4$ , or  $x + 5 = 9$ . The number that can replace the variable in an open sentence to make it true is called a root or a solution of the equation (Usiskin, 1999).

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Variables have many possible definitions, referents, and symbols (Usiskin, 1999). The first conception considers algebra as generalized arithmetic, and in this conception, a variable is considered as a pattern of generalizing. The second conception suggests that it is a study of procedures for solving certain kinds of problems, and in this conception, a generalization was obtained for a particular question before solving the unknown. Therefore, variables are either unknowns or constants. In the third conception, algebra is the study of relationships among quantities, and these variables tend to vary. The fourth conception accepts algebra as the study of structures, where the variable is little more than an arbitrary symbol. The variable will become an arbitrary object in a structure related to certain properties  $2x^2 + ax + 12a^2$ . The conception of a variable represented is not the same as any previously discussed notions, and it does not act as an unknown or argument.

The framework of PISA 2012 explains, formulates, employs, and interprets processes related to Mathematical capabilities. For example, formulating situation activities identify the underlying variables and structures in the real-world problem and makes assumptions. Employing mathematical concepts, facts, methods, and reasoning entail tasks such as conceptualizing the problem or interpreting the solution within the original context. Meanwhile, interpreting, applying, and evaluating outcomes include activities to understand the extent and limits of a solution that results from the model employed.

## METHOD

This research used the qualitative method to investigate the quality of relationships, activities, situations, or materials. It places a premium on holistic description, that is, on detailing everything that occurs during a particular activity or scenario, rather than comparing the effect of a particular treatment or describing people's attitudes or behaviors.

The five steps in qualitative research used according to Fraenkel, Wallen, and Hyun (2014). Identification of the phenomenon to be studied: The quality relationships between students' conceptual understanding about equation and algebra expression and the ability to solve PISA-like mathematics problems in the model task. Therefore, this step involves developing and identifying a valid and reliable mathematical assignment for analyzing the relationship's quality. Identification of the participants in the study: The participants were 244 first-year students from five study programs of a public vocational college, East Nusa Tenggara, Indonesia. Generation of hypotheses: Students with a good conceptual understanding of equation and algebra expression can solve PISA-like mathematics problems related to the model task. Data collection: Using students' worksheets, video recording and interviewing some students to obtain deeper information of their thinking process; data analyzed by holistic descriptive. Interpretation and conclusion: Using the definition of equation and algebra expression and indicator of modeling competence, mathematical literacy refer to the proficiency level of PISA question given as a guideline to interpret and make a conclusion.

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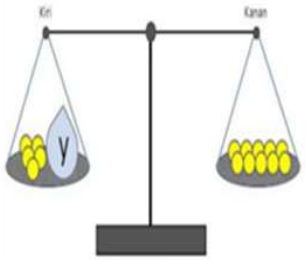

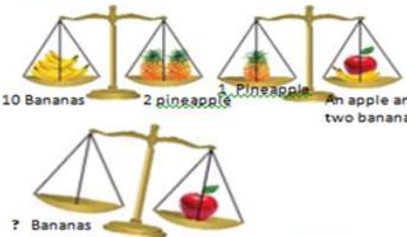


Students' algebraic conceptual understanding was investigated through their ability to solve three problems from junior high school textbooks containing the concept of the equation, variable, equal sign, and operation sign. Meanwhile, students' ability to Solve PISA-Like Mathematics Problems in Modeling Task were analyzed based on the sixth level of modeling Proficiency by (1) Applying given models, (2) recognizing, applying and interpreting basic given models, (3) using a different representational model, (4) Working with explicit models and related constraints with assumptions (5) developing and working with complex models that reflect on modeling processes and outcomes, (6) Conceptualizing and working with models of complex mathematical process and relationships, generalizing and explaining modeling outcomes. Mathematics tasks to investigate students' ability were taken from Pisa-like mathematics task content change and relationship in levels 2, 5, and 6.

## RESULTS

The validity, reliability of instruments, and difficulty level were analyzed using the quantitative method. The reliability test showed Cronbach's alpha 0.596, and they were reliable. Furthermore, the difficulty test showed that questions number 4 were easy, while 1,3 and 2 were moderate, and 5 and 6 were difficult.

The first type of problem consists of three mathematics tasks from junior and senior high school textbooks, as shown in Figure 1.

<p>1. Find the value of <math>y</math></p>  <p>Level 2</p>	<p>2. Find The Value of <math>x</math></p>  <p>Level 3</p>	<p>3. Find how many bananas that should be put in the left hand of third balance such that the <u>weight are</u> in balance</p>  <p>10 Bananas    2 pineapples    1 Pineapple    An apple and two bananas</p> <p>? Bananas    An apple and two bananas</p> <p>Level 4</p>
<p>Source: <a href="https://www.berpendidikan.com/2016/03/pengertian-dan-contoh-soal-persamaan-linear-satu-variabel-plsv.html">https://www.berpendidikan.com/2016/03/pengertian-dan-contoh-soal-persamaan-linear-satu-variabel-plsv.html</a></p>	<p>Source: Dris and Tasari (2010)</p>	<p>Source: As'ari et al. (2013)</p>

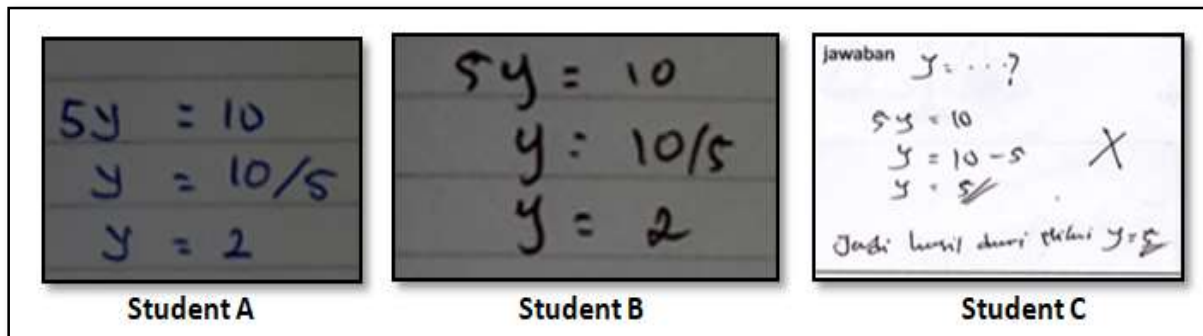
**Figure 1.** Linear equation problems to investigate students' conceptual understanding

The student came up with an unexpected outcome when answering question 1, and only 27.66% obtained the correct answers. Meanwhile, 10.64% of students could not construct a mathematical



model and gave the wrong answer by guessing. Subsequently, 61.70% made an error in the construct because of misconception or misreading and misinterpreted contextual language in mathematics.

Students' first misconception was assigned labels and arbitrary values, and the variables were misinterpreted as a "label" and as a "thing," as shown in Figure 2.



**Figure 2.** Students' answers in a one-to-one interview for question 1

In this context, students interpreted  $y$  as the label of things on the left side of the balance. For example, five oranges and an unknown substance are contained in the sack on the left side of the balance, which is balanced by the weight of ten. Some students translate the context to mathematics expression as  $5y = 10$ . Students A and B have similar answers, but they gave different conceptual understandings of the variable. The answers provided were recorded in the transcript as follow.

*Students A.*

*R: In your opinion, is this an easy, moderate, or difficult problem?*

*S: I felt that it is an easy question.*

*R: Please, can you elaborate on your response?*

*S: There is a balance, and the things on the left side are five oranges with a sack that contain unknown things. As  $y$  was used to signify the unknown, the mathematics equation for the items on the left side of balance was  $5y$ .*

*R: Are you confident,  $5y$ ? Would you mind checking your response against the context? Take care to consider the context.*

*S: While looking back to the context, there are five oranges and a sack; when the sack is denoted as  $y$ , the expression should be  $5y$ .*

*R: Ok. Let me know the meaning of  $5y$  in your opinion.*

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*S: Five oranges with unknown variables.*

*R: What is the answer to  $3 \times 4$ ?*

*S:  $3 \times 4 = 12$*

*R: Was  $3 \times 4$  calculated?*

*S: (did not respond)*

*R: Do you know that the concept of multiple comes from addition?*

*S: Yes, I know*

*R: how do you express  $4+4+4$  in multiple forms*

*S:  $3 \times 4$  or  $4 \times 3$*

*R:  $3 \times 4$  have a different meaning than  $4 \times 3$ ; which one is chosen?*

*S: (Students looked confused)*

*R: You add 4 three times. How is it converted to multiple concepts?*

*S: Four appeared three times, and the answer is  $3 \times 4$ .*

*R: What about  $5y$ ?*

*S: five and  $y$*

*R: Is  $5y = 5(y)$  and  $5 \times y$*

*S: All the variables that refer to the symbol of 5 times  $y$  are remembered.*

*R: The prior example  $3 \times 4 = 4+4+4$ , and  $5y = \dots\dots\dots$*

*S:  $y + y + y + y + y$*

*R: How many  $y$  is on the left side of the balance?*

*S: I did not know because the sack was closed.*

*R: In your opinion,  $y$  as a variable refers to an unknown thing. In this context,  $y$  refers to...*

*S: Oooo.... orange (not sure) ... a sack, maybe orange*

*R: Orange, sack, or others? Please make sure your answer is correct*

*S: sack because letter  $y$  is on a sack.*

*R: thank you for your time, and this will be discussed later when the algebra topics are considered.*

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*Student B*

*R: In your opinion, is this an easy, moderate, or difficult Problem?*

*S: It is one of the easy questions.*

*R: Can you please explain your answer, especially for the equation  $5y = 10$ ?*

*S: The balance context showed that the variables on both sides are the same since an equal sign was used. The left side of the balance has five oranges, and a sack contains an unknown number of oranges. Let orange be denoted as  $y$ ; then five oranges should be  $5y$ . Therefore, the equation;  $5y=10$  is formed. The fixed value of  $y$  is 2 or  $y =2$*

*R: Orange was denoted as  $y$ , then five oranges should be  $5y$ ” do you mean that  $y$  refers to orange?*

*S: yes, of course,  $y$  refers to orange for the question asked for the value of  $y$ .*

*R: what is the meaning of  $y =2$ ? Can you interpret the result?*

*S:  $y =2$  means 2 oranges in the sack.*

*R: Does it mean the total number of oranges on the left and right sides of the balance are 7 and 10?*

*S: No since the oranges in both sides have to be the same.*

*R: please look back to the balance! There are 10 oranges on the right side of the balance. The left side has five oranges and a sack containing an unknown number of oranges. How many oranges are in the sack equal 10 on the left side?*

*S: It is easy, add 5 oranges to make the numbers on the left side of the balance as many as on the right side.*

*R: What is the value of  $y$ ?*

*S:  $y$  is equal to five*

*R: How is the result interpreted?*

*S: The numbers of oranges in the sack are five.*

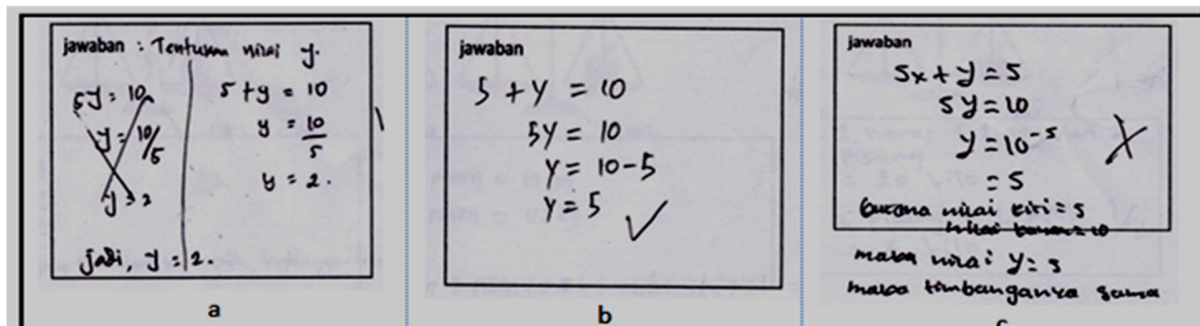
*R:  $y$  refers to orange, the number of oranges, and the number of oranges in the sack.*

*S:  $y$  refers to the number of oranges in the sack.*

*R: This will be discussed during the algebra topic.*

Students seem to lack understanding of the variable's concept and operation sign from the transcription. Instead, they bring prior knowledge and solve equations using routine algorithms without understanding. Meanwhile, there is no clear understanding of the function of the feature in the equation and how changing the location can affect the overall problem.

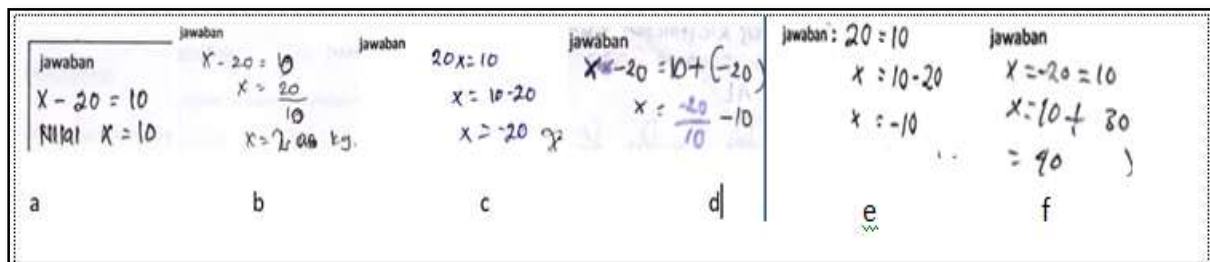
The second type of misconception was miscellaneous forms of an incorrect answer. Figure 3 illustrates students' responses to this type of inaccuracy.



**Figure 3.** Procedural errors students made for question1

There were different answers for the same question, significantly simplifying algebraic expressions, where incorrect rules were applied. Figures 3a and 3b showed that the problems were correctly formulated. However, they performed the wrong operation to simplify the equation. The student in Figure 3a separated the variable on the left side of the equal sign to move the constant to the right side. Then, it was divided on the right side of the equal sign to the constant on the left side. The algorithm's final steps are always reached by dividing the number on the right side by the left side. In contrast, Figure 3b does not have enough knowledge to operate algebra expressions and simplify the equation; hence, a double error was committed without understanding. The value of  $y$  was not substituted to the original equation to evaluate the equality of the expressions. Figures 3c showed that students found difficulties formulating the problem in the mathematics form. They had known that they should add five oranges on the left side of the balance to make it balance. However, they did not know how to communicate their thinking mathematically. Memorization algorithms or procedures were conducted to simplify algebra expressions to find the unknown value. They do not entirely understand the concept of equality and equation.

Students felt that question 2 was more accessible than 1, but some made the same error. For example, 65.95% of solved question 2 correctly, 21.28% cannot construct a mathematical model and gave the wrong answer by guessing, and 12.77% gave miscellaneous forms of incorrect answers. Students with problems performing algebra operations consistently made the same type of errors. For question 2, various incorrect solutions were provided, as shown in Figure 4.

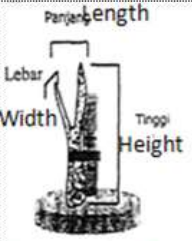


**Figure 4.** Students' miscellaneous forms of incorrect answers students made to question 2


Students have the same errors and misconceptions in solving questions 1 and 2. This fact is relevant to (Wu's 2001) statement that algebra is regarded as a gatekeeper course. Those who successfully pass through will be promoted to the next level. (Jacobson, 1981) also stated that High school mathematics is widely regarded as the "gatekeeper" to college. Students' answers in Figures 4e and 4f showed that they did not understand the concept of the equal sign.

In contrast, 91 students, or 39%, can correctly model problems in question 3. It was argued that this problem is like a two-linear equation system unless presented in the balancing context. Almost every student who fails to solve question 1 successfully solves question 3.

The second type of problem was PISA-Like Mathematical tasks taken from the contest of mathematics literacy questions in 2011 published by the Journal on Mathematics Education. There were 3 PISA-Like Mathematical tasks, to examine students' modeling proficiency, as shown in Figures 5 and 7



<http://www.google.co.id/imgres?q=TERUMBU+KARANG>



The growth of coral reef length can be predicted used the following function

$$y = 0,75x - 0,5.$$

y: the length of coral reef (mm)  
x: the height of coral reef (mm)

**Question 4**  
Find the length of coral reef if it has 40 mm in height.

Source: Contest Literacy of mathematics 2011 (Translate in English)

**Figure 5.** Question 4 (mathematics task like PISA level 2)

Question 4 was easy, and 72.34% solved this problem correctly. However, several students were put equal signs improperly. An example of errors and misconceptions in solving question 4 is shown in Figure 6.

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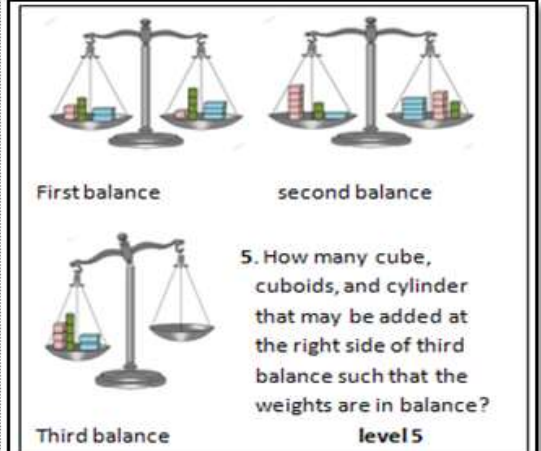


<p>Diket : <math>y = 0,75x - 0,5</math>          Dit : Panjang TK yang tinggi 4 mm          Jawab : <math>y = 0,75x - 5</math>  <math>y = 0,75 \times 40 - 0,5</math>  <math>y = 0,75 \times 40 = 30 - 0,5</math>  <math>= 29,5</math> mm          Jadi Panjang TK = dengan T 40 mm : 29,5"</p>	<p><b>Translate in english</b>          Known : <math>y = 0.75x - 0.5</math>          Unknown: The length of TK (acronym for <i>Terumbu Karang</i>) means coral reef which have 4 mm in height          Answer : <math>y = 0.75x - 5</math>  <math>y = 0.75 \times 40 - 0.5</math>  <math>y = 0.75 \times 40 = 30 - 0.5</math>  <math>= 29.5mm</math>          Thus, the length of TK = with 40 T mm = 29.5</p>
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**Figure 6.** Students lack understanding of the equal sign concept to question number 2

The student's answer showed in Figure 6, "... ,  $y = 0,75 \times 40 = 30 - 0,5, \dots$  ", where  $x$  was substituted with 40, and the result of 0.5 was subtracted from 0.75 (40). The lack of equal sign and operation sign concepts was inferred from these answers, and an equal sign was used to separate some calculation steps. In addition, the students did not understand the expressions  $0,75 \times 40 \neq 30 - 0,5$ .

The second and third mathematics tasks like PISA were in questions 5 and 6, as shown in Figure 7.

 <p>First balance      second balance</p> <p>Third balance</p> <p>5. How many cube, cuboids, and cylinder that may be added at the right side of third balance such that the weights are in balance? level 5</p>	<p>6. Ira and Ati have candies each other. Following are their conversations about the amounts of their candies.</p> <p>Ira: Ati, will you give me 5 candies of yours so the number of my candies as much as to your candies</p> <p>Ati: (Laughter) No Ira. Why do not you give me 5 candies so my candies as much twice of your candies.</p> <p><b>Question</b> How many candies are wholly owned by Ira and Ati</p>
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Source: Edo, Putri, and Hartono (2013)

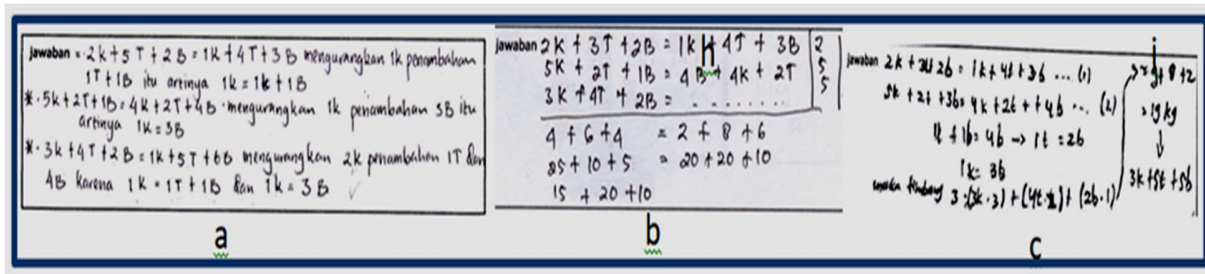
**Figure 7.** Questions 5 and 6 (mathematics tasks like PISA levels 5 and 6)

In addition, 46.81% solved the task correctly, 14.89% did not answer the question, and 21.28% could not construct the mathematics model since wrong answers were provided by guessing.

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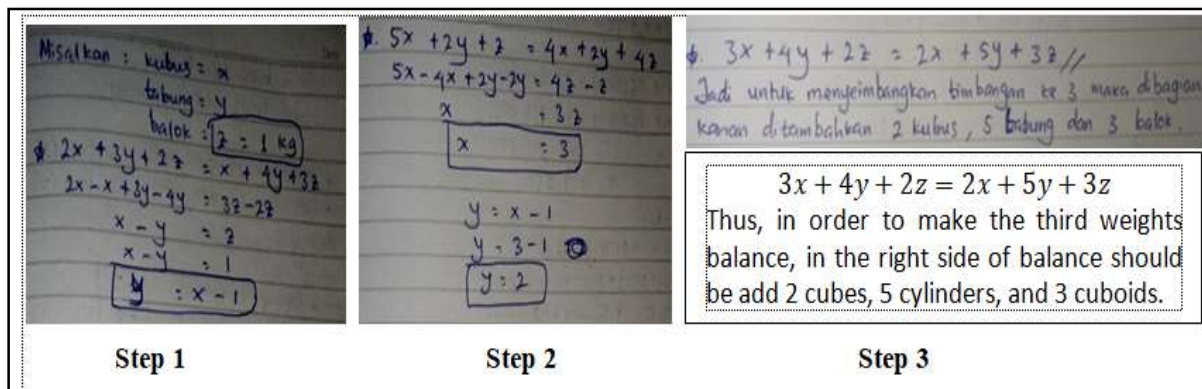
However, 17.02% construct mathematics models correctly but cannot continue finding the value of each variable to answer the question, as seen in Figure 8.



**Figure 8.** Students' incomplete answers for question number 5

The answers in Figure 8a showed that students can model the problem correctly and simplify it to the simplest expression. However, the value of  $B$  was not substituted to find  $k$  and  $t$ . The students can formulate the problem mathematically but fail to simplify the unstructured linear equations system. The interview section reported that the linear equation has three variables and two equations. Therefore, the equation system cannot be solved simultaneously to obtain the value of each variable. In performing the algorithm, different challenges were encountered. The last response showed that the students gave the correct answer for the weight of the things on the left side of the balance, but an incorrect final answer was provided.

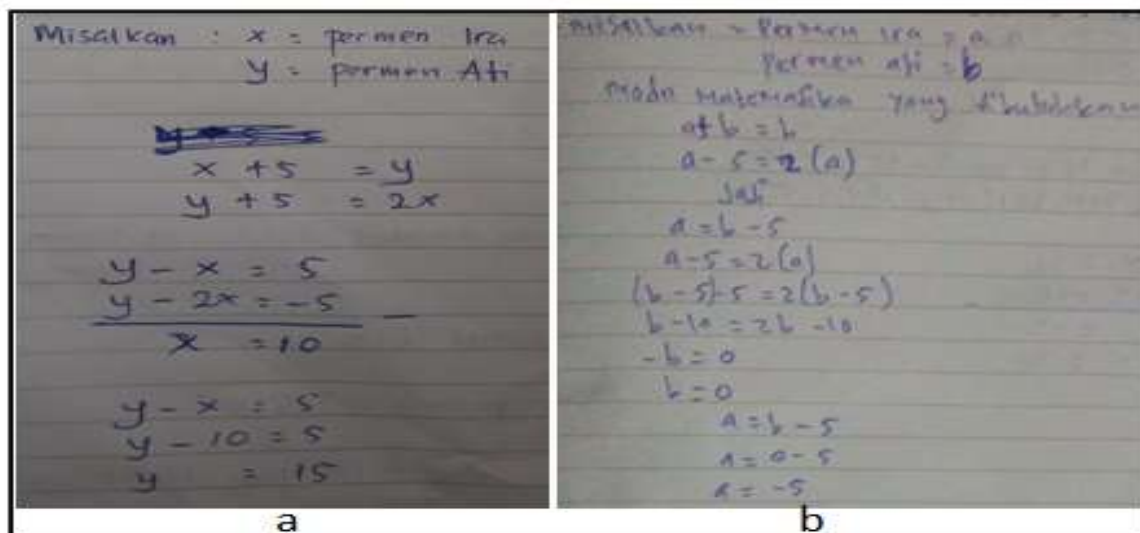
In contrast, students who failed to solve question number 2, displayed in Figure 2a, can solve this Mathematics task like PISA level 5, as shown in Figure 9.



**Figure 9.** The correct answer for question 1

The real-world contexts are translated to mathematics language by denoting *kubus* (cube) as  $(x)$ , *tabung* (cylinders) as  $y$ , and *balok* (cuboids) as  $z$ . Subsequently, each mathematics model was simplified, and the value of  $z$  was substituted with the simplest equation to obtain  $x$  and  $y$ .

Additionally, no students can correctly solve problems in level 6 since 63.83% did not answer, 21.28% guessed, and 14.89% made errors in modeling the problem. Students failed to transfer everyday language to mathematics because they misread and misinterpreted the problems, as shown in Figure 10.



**Figure10.** Students misreading or misinterpreting problem number 6

This study showed that students with a weak understanding of algebra concepts found difficulty solving routine mathematics tasks like PISA. However, some modeled a mathematics task like PISA and failed to solve the equation due to a lack of procedural knowledge and skill in manipulating and simplifying algebra expressions. As a result, they made some errors and misconceptions.

## DISCUSSION

Students' first type of error was assigned labels, arbitrary values, or verbs for variables and constants. This error contains several subcategories, and students tend to misinterpret a variable as a "label" and as a "thing" rather than a number. Misinterpreting letters as labels is a fundamental misconception that will lead to many other errors in algebra, and college students pursued similar interpretations of variables (Gunawardena, 2011; Widodo et al., 2018). Additionally, different interpretations of letters in different contexts may cause students to mix up and misinterpret the use of variables. Capraro and Joffrion (2006) stated that the variable is liable to change, especially suddenly and unpredictably. However, restricted solution sets are always provided when students encounter variables in algebraic situations. They are introduced as specific references to the value of a particular variable name. According to mathematical literacy, as defined for PISA, students can formulate situations because of misunderstanding the variable concept. It means that algebraic conceptual understanding supports the individual capacity to formulate situations mathematically.

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Furthermore, students' second type of misconception applied many illegal procedures in manipulating algebraic expression and equations, as seen in Figures 3a, 3b, 3c, 4a, 4b, 4c, 4d and 4f. Students should have a firm grasp of algebra's structure and characteristics to comprehend algebraic expressions. This is consistent with Van Lehn and Jone's (1993) study, where student misconceptions or gaps in conceptual knowledge of Algebra lead to incorrect and buggy procedures for solving problems. Capraro and Joffrion (2006) also stated that the procedural approach of translating from mathematical words to symbolic representations did not help students succeed on the items that required skills. Therefore, teachers should prepare students not to carry out algebraic procedures but to solve problems and represent situations. Procedures are almost meaningless without conceptual understanding. Connections make mathematics meaningful, memorable, and powerful. In conclusion, students fail to employ mathematical concepts, facts, procedures, and reasoning because they tend to manipulate algebraic expressions by memorizing given algorithms without deep conceptual understanding.

The third type of error was a misunderstanding of the algebra expression concept. Students' answers in Figures 4e and 4f showed that they struggled with algebra expression and equal sign concepts. This fact is in line with the Knuth et al. (2018) study, where one of the most common misconceptions in understanding equations is the significance of the equal sign ( $=$ ). Students forget that the equal sign means "operations equal answer" and are usually presented with the material in an "operations on the left-hand side of the equal sign" manner. As a result, they did not solve the problem by understanding the concept of the equation but used wrong rules that were persistently fixed in their minds. Another misconception was that they put an equal sign to separate some calculation steps. They were using the equal sign as a step marker and also violated the equivalence property by equalizing statements that were not equal to each other, as displayed in Figure 6a. Knuth et al. (2006) stated that students often think of the equals sign as an indicator of the result of operations being performed or the answer to the problem rather than the equivalence of two phrases. It means that students fail to employ mathematical concepts, facts, procedures, and reasoning because they lack an understanding of equal sign concepts.

The fourth type of error and misconception was manipulating and simplifying non-routine algebra expressions as shown in Figure 8. Booth and Koedinger (2008) said that using these incorrect strategies may persist because many of the procedure's students attempt to use will lead to a successful solution. Unfortunately, without adequate knowledge of the problem features, students cannot distinguish between the situations in which the strategy will work and the ones where it is not applicable. It means that some students can formulate the problem correctly, but they fail to employ the process because they cannot simplify unstructured linear equation system.

The fifth type of error was translating a real-world problem from natural to algebraic languages, as displayed in Figures 3a, 3b, 3c, 4a, 4b, 4c, 10a, and 10b. Furthermore, the error was caused by misreading and misinterpreting problems. The frequency of occurrence is very high in all types of

problems because they must interpret and convert everyday language into algebra. This is in line with Gunawardena (2011), where students need to do more than the other three conceptual areas of variables, expressions, and equations. This is because a word problem may contain concepts related to one or more of the above three areas.

## CONCLUSION

This study discovered that students encountered difficulties solving high-level mathematics tasks such as PISA, committed errors, and formed misconceptions due to limited comprehension of algebra's structural properties. Students that lack a conceptual grasp of algebra, such as expressions, equal sign concepts, and operation sign concepts, frequently answer mathematics problems using memorized procedures. They used many illegal procedures and made errors in manipulating algebra for easy or complex problems. Therefore, the struggle to solve PISA-Like mathematics problems for levels 5 and 6 was high. Teachers are recommended to facilitate the teaching and learning process with activities that can explore students' basic skills and encourage them to construct their understanding of algebra concepts before solving complex problems. According to Usiskin (1999), students should be introduced to the fundamentals of algebra and develop the context, not as meaningless symbols. Furthermore, they should involve all other mathematics as motivation for solving the algebra and as avenues for application. The most complex ideas should be broken down into subtopics instead of learning in one year. Some students with the capacity to formulate complicated mathematical problems cannot solve non-routine linear equations. Furthermore, those that can correctly solve PISA-Like mathematics problems for level 5 failed to solve conceptual questions because of the misconception of variable

## ACKNOWLEDGMENTS

The authors are grateful to the Polytechnic of Agriculture Kupang (Politeknik Pertanian Negeri Kupang) for funding and providing the opportunity and facility to conduct this study.

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## Students' Analytical Questions and Interaction Patterns in Group Discussion Facilitated with a Scientific Approach Learning

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**Abstract:** *Analytical questions are the types of questions that can lead students to gain an understanding of a concept and explore reasoning. This research is a descriptive, qualitative study investigating the emergence of analytical questions and their interaction patterns in group discussions facilitated by a scientific approach to learning. The subjects of this study were 30 students aged 14 years with heterogeneous mathematical abilities; they were distributed into five groups. Data were collected through observation with a video recorder as a tool. Method triangulation was carried out for the data validation process. The data were analyzed through data reduction, data presentation, and conclusion. The results showed that in group discussions: (1) observing is the dominant stage in raising analytical questions, (2) students with low mathematical abilities were dominant in triggering the emergence of analytical questions, and (3) dominant interactions occurred between students with low mathematical ability as the questioner and students with high mathematical ability as the answerer.*

**Keywords:** *analytical question, scientific approach, interaction pattern*

### INTRODUCTION

Asking is an activity that is very common in the learning process. Teachers often ask for various purposes, for example, to measure students' understanding, get information from students, stimulate students' thinking, and control the class (Kucuktepe, 2010; Widodo, 2006). Likewise with students. Students' questions during the learning process also have various purposes, for example, to get an explanation, express curiosity, or even to get attention (Widodo, 2006). No single theory denies the vital role of questions in the learning process (Almeida, 2012; Chin & Osborne, 2008; Graesser & Olde, 2003).

Although it is recognized that questions play an essential role in the learning process, there has not been much research on questions in the learning process. Cahyani et al. (2015) found that the types

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of questions used by teachers in learning were query questions, rhetorical, directing, direct information narrow questions, and centralized narrow questions. The types of questions used by students are comprehension and application questions. Faizah et al. (2018) and Yuliani et al. (2014) stated that the dominant types of questions raised by students in learning were questions on the cognitive dimension of understanding. Omari (2018) suggests that teachers can develop students thinking abilities by asking different types of questions to account for the individual differences among students.

For analyzing, questions are generally classified based on specific considerations. In the literature on questions, there are various classifications of questions. One of them is a question related to cognitive processes. In the revised version of Bloom's taxonomy (Anderson et al., 2001), a separation is made between the knowledge and the cognitive process dimension. One of the questions related to cognitive processes is an analytical question (Anderson et al., 2001; Chaffee, 1988). Analytical involves breaking down a problem into problem parts and determining how the parts are interrelated. Three kinds of cognitive processes were included in the analysis: differentiating, organizing, and attributing (Anderson et al., 2001). The analytical question is essential for students to ask to provide an understanding of a concept in detail.

Learning with a scientific approach is learning that is designed in five stages (observing, asking, experimenting, associating, and communicating) to encourage the development of attitudes, knowledge, and skills of students to be better by scientific principles (Hosnan, 2014; In'am & Hajar, 2017; Istungingsih et al., 2018; Prakoso et al., 2018; Tambunan, 2019; Wiyanto, 2017;). (1) *Observing*. It prioritizes the meaningfulness of the learning process. Through observation, students will feel challenged. (2) *Asking*. An effective teacher is a teacher who knows the student's competence and inspires students to grow up. When asking and answering questions, a teacher encourages his students to be good learners. (3) *Experimenting*. Students are required to apply their knowledge to solve problems. (4) *Associating*. Students are required to think systematically and logically to get a conclusion. (5) *Communicating*. Students are asked to present the findings or results obtained during the process. It can be done through group representatives if the learning process is carried out in groups.

The scientific approach also emphasizes the activity of asking questions by students during learning to be able to construct understanding optimally. Analytical questions are essential in this process (Chaffee, 1988; Paul & Elder, 2006). After observing, students are expected to have curiosity and ask analytical questions about the information that has been observed. These analytical questions can be asked of the teacher or other students.

The scientific approach requires students to be actively involved in learning. To demand active students, generally, learning is done through group discussions (Amran et al., 2016; Tesfaye & Berhanu, 2015). Ideally, the formation of groups pays attention to the heterogeneity of student characteristics (Herlina, 2018), for example, mathematical ability. This formation is intended to have a positive dependence between students in the group (Rosita & Leonard, 2015). Students

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with high mathematical abilities help students with low mathematical abilities understand a concept (Karsenty, 2020; Salido & Dasari, 2019; Yusupova, 2021). On the other hand, students with high mathematical abilities will increasingly understand the concept. During discussion activities, asking analytical questions helps students to understand a concept (Chaffee, 1988; Paul & Elder, 2006).

The literature review results indicate that no research focusing on the emergence of analytical questions posed by students and their interaction patterns in group discussions facilitated a scientific approach to learning. Therefore, it is crucial to do research that focuses on students' emergence of analytical questions during learning. Thus, this research aims to describe the emergence of analytical questions and their interaction patterns in group discussions facilitated by a scientific approach to learning. The research results become the basis for designing stimuli in learning so that students actively ask analytical questions. Giving the right stimulus will motivate students to understand the concept optimally.

## METHOD

This descriptive research with a qualitative approach was conducted by involving 30 students aged 14 years. Data were collected through observation with a video recorder as a tool. Observations were made on learning that applied a scientific approach to the topic of Cartesian Coordinates. The practical learning lasted two hours of learning with a video duration of 1 hour and 1 minute 49 seconds. The researcher only acts as an observer and does not intervene in implementing learning.

Learning with a scientific approach is carried out in groups. As many as 30 students were distributed into five groups so that each group consisted of 6 students. Each group consists of students with high, medium, and low mathematical abilities (Karsenty, 2020), each of which is two students. The teacher arranges the composition of students in a group. Data on students' mathematical abilities were obtained through essay tests. The scores obtained by students are used as the basis for classifying mathematical abilities. Refers to the average test scores of students ( $\bar{x} = 73.55$ ) with a standard deviation ( $s = 15.34$ ), the classification of students' mathematical abilities is based on Table 1.

Classification of Test Score ( $n$ )		Category of Mathematical Ability
$n > (\bar{x} + 1/2 s)$	$n > 81.22$	High
$(\bar{x} - 1/2 s) \leq n \leq (\bar{x} + 1/2 s)$	$65.88 \leq n \leq 81.22$	Medium
$n < (\bar{x} - 1/2 s)$	$n < 65.88$	Low

Table 1: The classification of students' mathematical ability

Learning with a scientific approach in a group setting is carried out through the following five stages.

### 1. Observing

At this stage, students are given a worksheet that contains a phenomenon related to the topic of Cartesian Coordinates. The worksheet illustrates a campground plan in Cartesian Coordinates, as shown in Figure 1.

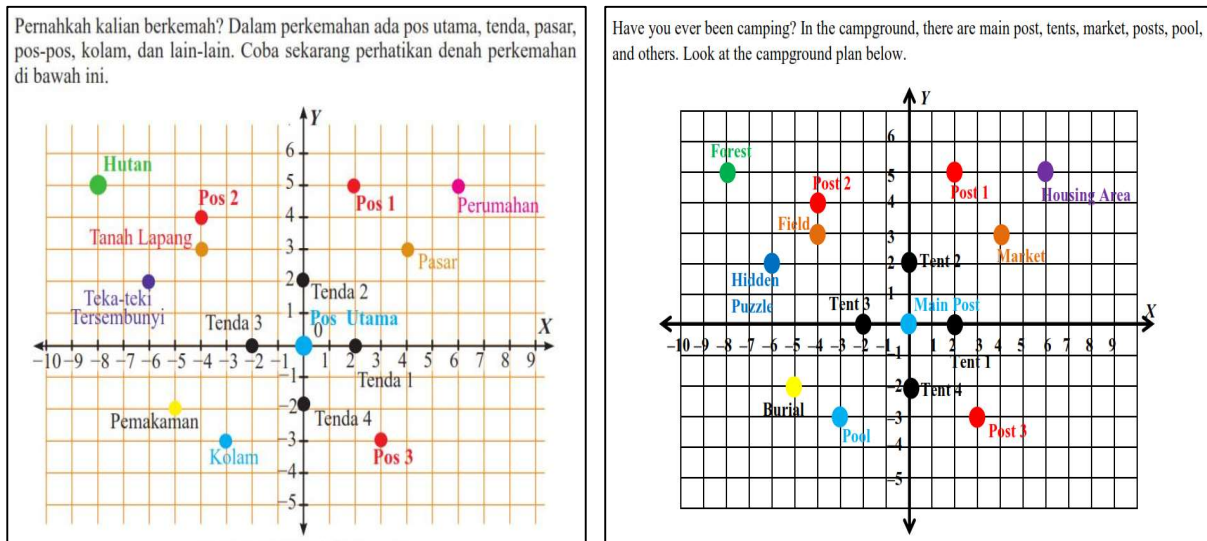


Figure 1: The illustration of a campground plan on a student worksheet

In practice, one group is only given one worksheet. Students are asked to observe the phenomena presented on the worksheet. In the observing activity, it is possible to have conversations between students in their groups.

### 2. Asking

At this stage, students can ask the teacher if they are confused about understanding the worksheet illustrations. The formulation of this question allows the emergence of conversations between students in groups. This conversation is because another student in the group can answer a student's question. Thus, the questions posed to the teacher were the only questions the students in their group could not answer.

### 3. Experimenting

At this stage, students are asked to solve the problems presented in the worksheet. This problem-solving process allows the emergence of conversations between students in groups. The result of this interaction will be the result of group work. The problems presented in the worksheet are presented in Figure 2.

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Setelah kalian mengamati denah perkemahan tersebut, coba lengkapi tabel berikut ini

**Tabel 2.4** Posisi tempat tertentu

No	Posisi dari titik asal (0, 0)		Posisi terhadap		
	Objek	Koordinat	Tenda 1 (2, 0)	Pos 1 (2, 5)	Pasar (4, 3)
1	Perumahan	(6, 5)	4 satuan ke kanan dan 5 satuan ke atas	4 satuan ke kanan dan 0 satuan ke atas	2 satuan ke kanan dan 2 satuan ke atas
2	Pemakaman	(-5, -2)	7 satuan ke kiri dan 2 satuan ke bawah	...	...
3	Pasar	(4, 3)	...	2 satuan ke kanan dan 2 satuan ke bawah	...
4	Hutan	(-8, 5)	...	...	10 satuan ke kiri dan 1 satuan ke bawah
5	Tenda 1	(2, 0)	...	...	...
6	Tenda 2	(0, 2)	...	...	...
7	Pos 1	(2, 5)	...	0 satuan ke kanan dan 0 satuan ke atas	...
8	Pos 2	(-4, 4)	...	...	...

After observing the campground plan, fill in the blanks in the following table.

**Table 2.4.** The Place Position

No.	The Position from Origin		The Position Against		
	Object	Coordinate	Tent 1 (2, 0)	Post 1 (2, 5)	Market (4, 3)
1	Housing Area	(6, 5)	4 units to the right and 5 units upward	4 units to the right and 0 units upward	2 units to the right and 2 units upward
2	Burial	(-5, -2)	7 units to the left and 5 units downward	...	...
3	Market	(4, 3)	...	2 units to the right and 2 units downward	...
4	Forest	(-8, 5)	...	...	10 units to the left and 1 units downward
5	Tent 1	(2, 0)	...	...	...
6	Tent 2	(0, 2)	...	...	...
7	Post 1	(2, 5)	...	0 units to the right and 0 units upward	...
8	Post 2	(-4, 4)	...	...	...

Figure 2: The problem in the student worksheet

#### 4. Associating

At this stage, students are asked to formulate conclusions from the problem on the worksheet. Formulating this conclusion allows the emergence of conversations between students in groups.

#### 5. Communicating

At this stage, group representatives are asked to present the results of their discussions in front of the class. The presentation of the results of this discussion allows the emergence of conversations between students in groups.

For data collection, observations were focused on conversations between students in group discussions to identify the emergence of analytical questions orally at each stage of the scientific approach. The analytical question indicators used in this study were modified from Rudsberg et al. (2016), as presented in Table 2.

Analytical Concept	Analytical Question Indicators
Encounter	Questioning an object in an event
Gap	Questioning objects that are not understood or make no sense
	Questioning the situation to be solved
Fast Stand	Questioning information that has been presented in an incident
Relations	Questioning the relationship of previous experiences with new information obtained
Meaning	Questioning things that might happen after a process occurs

Table 2: Concepts and indicators of analytical questions

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Data analysis was carried out through three stages: data reduction, data presentation, and concluding (Miles & Huberman, 2014). Before analyzing the data, the researcher transcribed the data from the video recording. After that, the researchers validated the data by triangulating the method. It was done by comparing the data of observations and the transcription of the video recordings. Next, the researcher reduced the data by focusing only on the analytical questions raised by the students. The data resulting from this reduction are then presented to obtain research conclusions.

## RESULT AND DISCUSSION

The results of this research indicate that group discussions on learning with a scientific approach raise analytical questions. These analytical questions arise at every stage by students with high, medium, and low mathematical abilities. The number of analytical questions at each stage is presented in Table 3.

Scientific Stage	Math Ability	Number of Analytical Questions	
		Answered	Not answered
Observing	High	0	2
	Medium	2	0
	Low	3	0
Asking	High	0	0
	Medium	2	0
	Low	1	0
Experimenting	High	0	1
	Medium	2	0
	Low	2	0
Associating	High	1	0
	Medium	1	0
	Low	2	0
Communicating	High	0	0
	Medium	0	0
	Low	2	0

Table 3: Number of analytical questions at each stage

Based on the data in Table 3, observing is the most dominant stage in raising analytical questions, which are seven questions. Students with low mathematical abilities at the observing stage raise the dominant analytical questions. In contrast to observing, communicating is the stage that raises the fewest analytical questions in group discussions. Students with low mathematical abilities remain the trigger in raising analytical questions at the communicating stage.

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If a thorough analysis is carried out, students with low mathematical abilities can trigger analytical questions. In group discussions, students with low mathematical ability raise ten analytical questions, students with medium mathematical ability raise seven analytical questions, and students with high mathematical abilities only raise four analytical questions. If paying attention to the pattern, the most dominant interaction occurs between students with low mathematical ability as the questioner and students with high mathematical ability as the answerer of questions. On the other hand, there was no interaction between students with high mathematical ability as the questioner and students with low mathematical ability as the answerer of questions. Data related to the interaction patterns of analytical questions in this group discussion are presented in Table 4.

Interaction Pattern	Math Ability	Number of Analytical Questions
Questioner	High	4
	Medium	7
	Low	10
Answerer	High	12
	Medium	4
	Low	2
Questioner – Answerer	High – Medium	1
	High – Low	0
	Medium – High	5
	Medium – Low	2
	Low – High	7
	Low – Medium	3

Table 4: The interaction patterns of analytical questions in group discussion

The research result indicates that observing is the dominant stage in raising analytical questions. At this stage, students are asked to observe a phenomenon related to the topic of Cartesian Coordinates presented on the worksheet. Observing is the initial stage in the scientific approach, which opens students' interest in participating in further learning activities (Azhar, 2015). Raising many analytical questions at this stage indicates that students are inquisitive about the concepts to be studied (Chaffee, 1988; Paul & Elder, 2006). Students with low mathematical abilities have triggered the emergence of analytical questions at this stage. Every question posed by students with low mathematical ability gets responses from other students in their group. It also motivates students to ask questions (Agustini & Sopandi, 2017). The examples of analytical questions that arise at this stage are as follows.

Question 1: What is this illustration about? (Encounter)

Question 2: What are the coordinates of the main post? (Stand Fast)

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At the asking stage, not as many analytical questions arise as in the observing stage. It occurs because students are confused, which triggers questions answered at the observing stage. Students with medium mathematical ability have triggered the emergence of analytical questions at this stage. The examples of analytical questions that arise at this stage are as follows.

Question 1: If I want to go to Market from Main Post, should I go to the right or upward first? (Gaps)

Question 2: Will the result differ if I move to the right or upward first? (Meaning)

The emergence of analytical questions again increases at the experimenting stage. Students with low and medium mathematical abilities raise analytical questions with the exact quantities at this stage. The examples of analytical questions that arise at this stage are as follows.

Question 1: The main post is at the center of Cartesian. Is  $(0, 0)$  the coordinate? (Relations)

Question 2: The sign is still positive if you move downwards, right? (Gaps)

Students with low mathematical abilities remain a trigger for the emergence of analytical questions at the reasoning stage. The examples of analytical questions that arise at this stage are as follows.

Question 1: At the coordinates  $(0, -2)$ , there is a Tent 4, right? (Encounter)

Question 2: The Main Post has the same distance to all tents, right? (Gaps)

Communicating is the stage that raises the least number of analytical questions. However, students with low mathematical abilities remain triggers for the emergence of analytical questions. The examples of analytical questions that arise at this stage are as follows.

Question 1: Where is Post 1 located? (Stand Fast)

Question 2: “2 units to the right and three units upward” and “3 units upward and two units to the right” will the result be the same? (Gaps)

In general, students with low mathematical abilities trigger the emergence of analytical questions. Students with high mathematical ability dominantly act as answerers to analytical questions. The low mathematical ability of students confuses understanding concepts (Manik et al. 2017). This confusion stimulates students to raise analytical questions (Sudarti, 2019). In addition, the composition of heterogeneous groups by paying attention to differences in mathematical abilities motivates students to dare to raise questions (Rosita & Leonard, 2015). It is because each question will receive a response from other students in the group. Questions that get responses can bring satisfaction to the questioner (Agustini & Sopandi, 2017). The results showed that only 3 of the 21 questions did not get responses from other students in the group and these three questions came from students with high mathematical abilities.

By paying attention to the pattern, the dominant interaction occurs between students with low mathematical abilities as questioners and students with high mathematical abilities. Students with low mathematical abilities are motivated to raise questions because they always get responses from other students (Agustini & Sopandi, 2017), both with high and medium mathematical abilities.

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Adequate mathematical ability becomes the capital to respond to analytical questions in group discussions. In contrast to the dominant interaction, the interaction did not occur between students with high mathematical ability as the questioner and students with low mathematical ability as the answerer. Learning in this discussion group creates a positive dependence between students in understanding a concept (Rosita & Leonard, 2015).

## CONCLUSION

In a scientific approach to learning, observing is the dominant stage in raising analytical questions, and communicating is the stage that is less able to raise analytical questions in group discussions. Students with low mathematical ability are dominant in triggering analytical questions, while students with high mathematical abilities are dominant in answering analytical questions. The dominant interaction occurred between students with low mathematical ability as the questioner and students with high mathematical ability as the answerer. Preferably, the interaction does not occur between students with high mathematical ability as the questioner and students with low mathematical ability as the answerer.

Based on the results of this research, teachers should provide stimulus to students in group discussions to raise analytical questions at each stage. However, this research contains weaknesses. It lies in the small number of subjects and the short duration of practical learning. Therefore, this research provides an opportunity to conduct further research by increasing the number of subjects and duration of learning. Besides that, the results of this study provide opportunities for further research related to characterizing analytical questions raised by students in group discussions on learning with a scientific approach. This characteristic is essential so that it is easy to stimulate students to formulate questions in the learning process.

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## The Students' Abilities on Mathematical Connections: A Comparative Study Based on Learning Models Intervention

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**Abstract:** *Mathematical connections are essential to emphasize in the learning process, to make students see mathematics as useful, relevant, integrated, and able to solve various mathematical problems. This study was conducted to analyze the comparison of achievement and improvement of students' abilities on mathematical connections based on learning model interventions. This comparative study used quasi-experimental types in three groups of students, namely 50 students who learned through the Connecting, Organizing, Reflecting, and Extending models with Realistic Mathematics Education (CORE RME), 49 students who learned through the CORE model, and 46 students who learned through the conventional model. The mathematical connections test is used as an instrument in this study. The finding in this study is that learning through the CORE RME model can facilitate students' mathematical connections abilities. This finding is based on the results of a survey that the achievement and improvement of the mathematical connections abilities of students who learned through the CORE RME model were better than the attainment and progress of the mathematical connections abilities of students who learned through the CORE model, and students who learned through the conventional model. Therefore, it is recommended for teachers to use the CORE RME model as an alternative to facilitate students' mathematical connection abilities.*

**Keyword:** Achievement, improvement, mathematical connections, learning models.

### INTRODUCTION

Mathematical connections allow students to see mathematics as an integrated subject, not as a collection of separate parts (Jaijan & Loipha, 2012). It is because mathematical connections include three aspects a) connections between different mathematical concepts or topics (Gamboa, Badillo, Ribeiro, & Sanchez-Matamoros, 2016); b) the connections of mathematical concepts with other scientific disciplines (Frykholm & Glasson, 2005), and c) connections of mathematical concepts with real-world phenomena (García-García & Dolores-Flores, 2018, 2020).

The ability of students to understand the three connections aspects is called mathematical connections abilities. García-García & Dolores-Flores (2018) define that mathematical connection

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abilities are the students' ability to connect mathematical concepts, mathematical concepts with other scientific disciplines, and mathematical concepts real-world phenomena. A student has mathematical connections ability if he can recognize and use connections among mathematical ideas, understand how mathematical ideas are interconnected, build on one another to produce a coherent whole, identify and apply mathematics in contexts outside of mathematics (NCTM, 2000).

Although mathematical connections abilities are essential for learning mathematics, students still face obstacles to master it. It is matched with the research of Kenedi, Helsa, Ariani, Zainil, & Hendri (2019) that students' mathematical connections abilities in solving mathematical problems are still relatively weak. Students still have poor mathematical connections abilities in understanding problems, performing operations by making symbols correctly, and applying mathematical concepts in daily life (Noto, Hartono, & Sundawan, 2016). Another study by Rahmawati, Budiyo, & Saputro (2019) and Siregar & Surya (2017) shows that secondary school students' mathematics connection abilities are categorized as very low. The low achievement percentage indicates it on the indicators of connections among mathematical concepts, connections between mathematical concepts and other scientific disciplines, also connections between mathematical concepts and daily problems.

The low abilities of students' mathematical connections are a very urgent problem and considered essential to overcome. Many stakeholders need to be involved in an attempt to resolve this problem, including teachers and researchers. Teachers should take roles as facilitators and mediators to facilitate students' mathematical connections abilities by providing challenging problems (Rahmawati et al., 2019). On the other hand, researchers should make students' mathematical connections capabilities one of the main variables in their research, either related to the causes of the students' poor mathematical connections abilities and how to overcome them. Research on this issue must be prioritized to be carried out and used as a basis for further study (Arjudin, Sutawidjaja, Irawan, & Sa'dijah, 2016).

Referring to the problems and suggestions, the researcher conducted interviews with several mathematics teachers at different schools around the study site. Most teachers said that they did not focus on facilitating students with mathematical connections. Some teachers said that they connect mathematical concepts in the learning process, connect mathematical concepts with other disciplines, and with real-world phenomena. However, when conducted evaluation, the results showed that students' mathematical connections abilities are still low. Based on the interview, the researcher conducted a test on the students' mastery of mathematical connections at one level above this study's subject. The test results showed that the students' abilities connections were relatively low. The average score of the students' mathematical connections abilities obtained was 42.88 of the maximum score of 100. It was far from expected.

Students will understand mathematical connections if the three connection aspects are highlighted and familiarized during their learning process. Teachers must teach subject matter to make the students recognize and understand mathematical connections (Mhlolo, Venkat, & Schfer, 2012). The teacher should develop these habits to promote the formation and strengthen the mathematical connections (Eli, Mohr-Schroeder, & Lee, 2013). Teachers may carry out such intervention by connecting mathematics with real-life problems and students' environment, mathematics with other subjects, and concepts or ideas in mathematics (Arthur, Owusu, Asiedu-Addo, & Arhin, 2018). They need to help the students connect conceptual and procedural knowledge because it plays a vital role in mathematical connections (Dolores-Flores, Rivera-López, & García-García, 2018). It is characterized as connections-rich knowledge (Rittle-Johnson & Schneider, 2015).

One of the student-centered learning models emphasizing the connections between old and new knowledge is the Connecting, Organizing, Reflecting, and Extending (CORE) learning model. This CORE learning model combines four main elements, i.e., connecting old and new information, organizing information to understand the subject matter, reflecting information obtained and extending knowledge (Calfee & Greitz, 2004). The learning process through the CORE model help students build their knowledge by connecting and organizing new and old knowledge, rethink about topics or concepts being studied, and expand their knowledge (Curwen, Miller, Smith, & Calfee, 2010).

The connecting element in the CORE model emphasizes connections among topics. A topic to be taught can be linked to other concepts, especially those learned and known by students. Connections describe the relationship between prior and new knowledge to build or strengthen understanding the relationship between ideas and mathematical concepts (Eli, Mohr-Schroeder, & Lee, 2011). In conjunction with the meaning of these connections, NCTM (2000) states that when mathematical ideas are interconnected with real-world phenomena, students will see mathematics as a valuable, relevant, and integrated concept and a compelling process in developing students' understanding of mathematics. NCTM statement implies that students' mathematical knowledge will be broader, more developed, and last longer if the learning process is carried out by developing connections with students' experiences, not only among mathematical concepts but also real-world phenomena. In the mathematics curriculum at school, a mathematical learning approach that places the actual context or real-world phenomena and student experience as the learning starting point is Realistic Mathematics Education (RME).

Freudenthal (2002), as a pioneer of RME, says that mathematic is a human activity. Learning mathematics requires learning activities and should use a real context around as a starting point because most of them play specific roles in learning mathematical concepts. The word is realistic in RME means (1) a natural context in daily life; (2) a formal mathematical context in the world of mathematics; and (3) an imagery context that is not contained in reality but can be imagined (Freudenthal, 2002., Heuvel-Panhuizen, 2003., Heuvel-Panhuizen & Drijvers, 2014). The three

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main principles underlying RME are guided reinvention, didactical phenomenology, and self-developed models (Gravemeijer, 1994).

Many researchers in Indonesia and other countries have conducted studies on the influence of the CORE model and the RME on the students' mathematical connections abilities. Findings of the study by Yulianto, Rochmad, & Dwidayati (2019) show that the achievement and improvement of mathematical connections skills of students who learn through the CORE model with scaffolding are better than the achievement and improvement of mathematical connections skills of those who know through the CORE model without framing. The CORE learning model can improve students' mathematical connections skills and result in better mathematical connections skills than similar skills of those who learn through the conventional model (Yaniawati, Indrawan, & Setiawan, 2019). A study on RME by Febriyanti, Bagaskorowati, & Makmuri (2019) concluded that students' mathematics connections skills taught with the RME approach were higher than those taught with conventional methods. Previous researchers have examined the effect of the CORE model and RME on students' mathematical connections abilities which treated separately. In this study, the authors combined the CORE model with the RME called the CORE RME learning model.

The CORE RME learning model is implemented by connecting, organizing, reflecting, and extending. Students were given real contexts related to their experience and real contexts around the connecting stage. The main principles of the connecting stage were prior knowledge, natural context, and interactivity principles. Students were allowed to reinvent and develop mathematical models based on the actual context given in the connecting phase in the organizing stage. The main focus of the organizing stage was guided reinvention, self-developed models, and interactivity principles. According to the subject matter, the reflecting stage was the stage of rethinking and seeing the relationship of non-formal mathematical models (models of) built by students with formal mathematical models (models for). The main principle of reflecting stage was metacognition, self-monitoring, and interactivity principles. The last phase was extending; it was a knowledge expansion to other real contexts. The main focus of extending phase was to develop a formal mathematical to another real context, intertwining, and interactivity principles.

The CORE RME learning model was done through syntax. As mentioned above, it could help students understand connections among mathematics concepts, connections between mathematics concepts with others discipline, and real-world phenomena. A research question was constructed as follows "Are there different achievement and improvement students' abilities in mathematics connections based on learning intervention model?".

## RESEARCH METHODS

This study applied a quantitative research method with a quasi-experimental approach. The reason is that the researcher did not regroup samples randomly but used classes that the school has formed. The research design used a non-equivalent comparison group design, which was better for all

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quasi-experimental research designs (Christensen, Jhonson, & Turner, 2015). In this study, there were two experimental groups, i.e., a group of students who learned through the CORE RME and the CORE models, while the control group was a group of students who learned through the conventional model.

The participant in this study consisted of 145 seventh-grade students in two state junior high schools (JHS) in Kefamenanu city-west Timor-Indonesia, in the 2018/2019 academic year details such as Table 1.

Learning models	Number of School students A	Number of School students B	Sum
CORE RME	30	20	50
CORE	27	22	49
Conventional	25	21	46
Total	82	63	145

Table 1: Research participants.

A and B schools were chosen by purposive sampling from six state JHS in Kefamenanu city. A and B schools were the earliest schools to apply the Indonesia national curriculum among the six state schools in the city.

This study used a mathematical connection test as the data collection instrument, which consisted of 5 essay test items. The mathematical connection tests were arranged based on the following indicators: (1) understanding the equivalent representation of the same concepts, (2) understanding the relationship of mathematical procedures of representation to equivalent procedure of representation, (3) using linkages between mathematical topics, (4) using linkages between mathematical topics with other topics in other disciplines, (5) using mathematics in everyday life. This instrument had been validated by several validators, and obtained an average score of 93,33, which showed that the mathematics connection test was in the good category. While trials on 20 students resulted in Cronbach's alpha score of 0,88; which means that the test items were reliable, and the Pearson correlation scores of the five questions were 0,89; 0,62; 0,93; 0,89; and 0,82 respectively, which means that these five questions are valid.

In this research, data analysis techniques were the normalized gain, one-way ANOVA, and post hoc Scheffe test. The normalized gain test was conducted to determine the improvement in students' mathematical connections. On the other hand, the one-way ANOVA test was carried out to determine the difference in achievement and advancement in mathematical connections between students who learned through the CORE RME, CORE, and conventional models. Additionally, the post hoc Scheffe test was a further test of the one-way ANOVA. The post hoc Scheffe test was conducted because this type of test was appropriate for all t-tests (Potthoff, 2012). The data source of the study showed the difference in meaning between achievement and improvement. The

students' mathematical connections achievement data was the mathematical connections post-test result data, while the students' mathematical connections improvement data was the normalization gain tests result. Both the prerequisite test and the hypothesis test in this study were analyzed using IBM SPSS Statistics 22.

## RESULTS AND DISCUSSION

### Results

The average scores of pre-tests, post-test, and normalized gain of mathematical connections abilities of students who learn through the CORE RME model, the CORE model, and the conventional model can be seen in Figure 1.

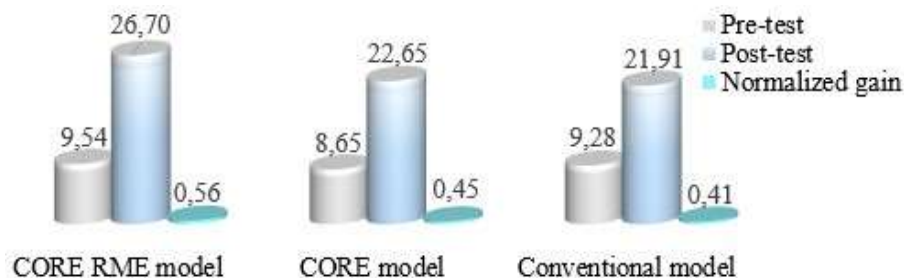


Figure 1. Average of pre-test, post-test, and normalized gain.

A comparison test of students' mathematical connections achievement was based on the post-test score showed that the average post-test score students' mathematical connections on the students who were learning through the CORE RME model were 26,70 out of a maximum score of 40 (Figure 1). Students who were learning through the CORE model was 22,65; students who were learning through the conventional model was 21,91. On the other hand, the comparison test of students' mathematical connections improvement based on the normalized gain score showed that the average normalized gain score of the students learning through the CORE RME model was 0,56; students learning through the CORE model was 0,45 (Figure 1). Finally, students learning through the conventional model was 0,41.

The conditions for using parametric statistical tests were normal homogeneous distribution data (Sarstedt & Mooi, 2019). The normality test results showed that the achievement and improvement data of the students' mathematical connections that learned through the CORE RME model, the CORE model, and the conventional model were normally distributed. The obtained homogeneity test results showed that the group data on achievement and improvement of students' mathematical connections were homogeneous.

Analysis results of difference in mathematical connections achievement between students who learned through the CORE RME model, the CORE model, and the conventional are presented in Table 2.

	Sum of squares	df	Mean square	F	Sig.	Ho
Between groups	648,72	2	324,36	6,83	0,00	Reject
Within groups	6745,25	142	47,50			
Total	7393,97	144				

Table 2. Test results for differences in students' mathematical connections achievement.

Table 2 shows that Ho was rejected. It shows a significant difference in mathematical connections between students who learn through the CORE RME model, the CORE model, and the conventional model. Since there was a considerable difference, the Scheffe post hoc test was conducted, which the results are presented in Table 3.

Learning models		Mean difference (I-J)	Std. error	Sig.	Ho
(I)	(J)				
CORE RME	CORE	4,05*	1,39	0,02	Reject
	Conventional	4,79*	1,41	0,00	Reject
CORE	Conventional	0,74	1,41	0,87	Accept

Table 3. Post hoc test results for students' mathematical connection achievement.

Based on the results of the post hoc test presented in Table 3, it can be concluded that at  $\alpha = 5\%$  then (1) There was a significant difference in mathematical connections achievement of students who were learning through the CORE RME model and those who were learning through the CORE model. Descriptively, the average of students' mathematical connections achievement who were learning through the CORE RME model was 26,70; and the average of students' mathematical connections achievement who were learning through the CORE model was 22,65. Because inferentially, there was a significant difference in students' mathematical connections achievement, which was  $26,70 > 22,65$ ; it can be concluded that students who were learning through the CORE RME model were better than mathematical connections achievement of students who were learning through the CORE model. (2) There was a significant difference in mathematical connections achievement between students who were learning through the CORE RME model and learning through the conventional model. Descriptively, the average of students' mathematical connections achievement who learned through the CORE RME model was 26,70; and the average of students' mathematical connections achievement who learned through conventional models was 21,91. Because inferentially, there was a significant difference in students' mathematical connections achievement with  $26,70 > 21,91$ ; it can be concluded that students who learned through the CORE RME model were better in forming mathematical connections than mathematical connections achievement of students who learned through the conventional model.

(3) There was no significant difference in mathematical connections achievement between students who learned through the CORE model and the conventional model.

The test result of mathematical connections improvement differences between students who learned through the CORE RME model, the CORE model, and the conventional are presented in Table 4.

	Sum of squares	df	Mean square	F	Sig.	Ho
Between groups	0,60	2	0,30	6,41	0,00	Reject
Within groups	6,69	142	0,05			
Total	7,29	144				

Table 4. Test results for differences in students' mathematical connections improvement.

Table 4 showed that Ho was rejected. It shows a significant difference in improving the mathematical connection between students who learned through the CORE RME model, the CORE model, and the conventional model. Considering that there was a significant difference in the students' mathematical connections improvement, the Scheffe post hoc test was conducted, and the results are presented in Table 5.

Learning models		Mean difference (I-J)	Std. Error	Sig.	Ho
(I)	(J)				
CORE RME	CORE	0,11*	0,04	0,04	Reject
	Conventional	0,15*	0,04	0,00	Reject
CORE	Conventional	0,04	0,04	0,70	Accept

Table 5. Post hoc test results for students' mathematical connection improvement.

Based on post hoc test results in Table 5, it can be concluded that at  $\alpha = 5\%$  then (1) There was a significant difference in the improvement of the mathematical connections between the students who learned through the CORE RME model and students who learned through the CORE model. Descriptively, the average of students' mathematical connections improvement who learned through the CORE RME model was 0,56; and the average of students' mathematical connections improvement who were learning through the CORE model is 0,45. Because inferentially, there was a significant difference in the improvement of the mathematical connections and  $0,56 > 0,45$ ; it can be concluded that the progress of the mathematical connections of students who learn through the CORE RME model was better than students who learned through the CORE model. (2) There was a significant difference in improving the mathematical connections between students who learned through the CORE RME model and students who learned through the conventional model. Descriptively, the average of students' mathematical connections improvement who learned through the CORE RME model was 0,56. The average of students' mathematical connections improvement who learned through the conventional model was 0,41. Because inferentially, there was a significant difference in the progress of the mathematical connections and  $0,56 > 0,41$ ; it can be concluded that the progress of the mathematical connections of students who were learning

through the CORE RME model was better than students who were learning through the conventional model. (3) There was no significant difference in improving the mathematical connection between the students who were learning through the CORE model and students who learn through the conventional model.

## Discussions

The finding of this study indicated that mathematical connections achievement and improvement of the students who were learning through the CORE RME model were better than mathematical connections achievement and improvement of students learning through the CORE model and students learning through the conventional model. This finding gives a positive effect of learning through the CORE RME model. It can facilitate mathematical connections aspects for students, in the connections among mathematical concepts, the connections between mathematical concepts and other science disciplines, also the connections with daily problems. Thus, students can gain the experience of a connection during the learning process. The learning process that facilitates students with mathematical connections will provide many connections experiences for students. Zengin (2019) said that the learning process based on intra-mathematical and extra-mathematical connections allows students to maintain their knowledge and gain a variety of connections experiences.

The results of this study indicate that the CORE RME learning model can facilitate students' mathematical connection. The application of the CORE RME model in the classroom is carried out through the stages as shown in Figure 2.

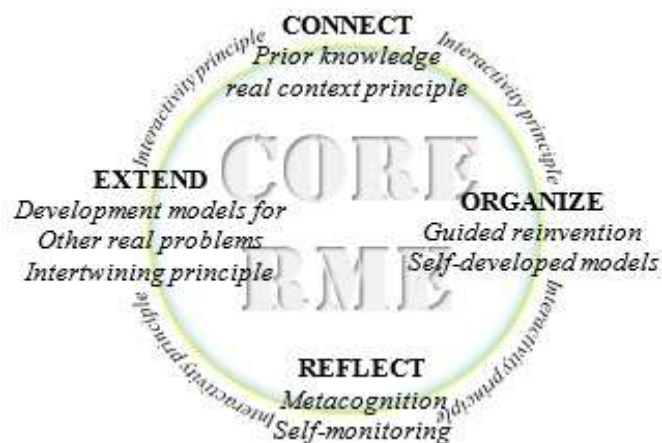


Figure 2. CORE RME models' cycle

Students were given real contexts on the connecting stage that have to do with their experiences, specifically real contexts around the students. The main principles of the connecting stage are prior knowledge of real context and the interactivity principle. Students' prior knowledge in mathematics

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is essential as a bridge for the target knowledge and plays a vital role in learning new mathematical material. Preparing students' previous knowledge of mathematics as a learning starting point functioned as a bridge for the target knowledge between prior knowledge and target knowledge in mathematics should be compatible, not conflict with one another (Rach & Ufer, 2020). In terms of the actual context principle, it should be recognized that students get a wealth of experience from their family and peer groups, all of which provide informal opportunities to develop mathematical concepts and skills (Clarke, Clarke, & Cheeseman, 2006). Such experience gained by students from families, social groups, and previous lessons is a potential basis for developing new knowledge (Taber, 2015).

Activity in the organizing stage allows students to reinvent and develop their mathematical models based on the actual context given in the connecting phase. The main principles of the organizing stage are guided reinvention, self-developed models, and interactivity. The reinvention process can facilitate students to use their experiences in developing non-formal mathematical models and connect them with formal mathematical models (Uzel & Uyangor, 2006., Selter & Walter, 2020) experienced the same process when mathematics was discovered. This cognitive process requires a guide and student interaction as a critical factor. The interaction among students and between students with teachers is significant to allow students to reinvent mathematical objects, ideas, concepts, and strategies (Abrahamson, Zolkower, & Stone, 2020). Students will achieve a cognitive experience that helps them see the connections between mathematics and problems in real-world phenomena when students discover objects, ideas, concepts, and formal mathematical strategy from an authentic context.

Furthermore, developing non-formal mathematical models (horizontal mathematics) and connecting with formal mathematical models (vertical mathematics) provides an experience for students to understand the connections between ideas, concepts, and topics in mathematics. Students use prior knowledge to develop their conceptual and procedural knowledge because they need to create mathematical connections, both intra-mathematical and extra-mathematical connections. Dolores-Flores et al. (2018) say that conceptual and procedural knowledge plays a vital role in mathematical connections, and both are positively correlated. The intended relationships include facts and propositions so that all information is related one to another. Conceptual and procedural knowledge are characterized most clearly as the rich knowledge in its relationships (Rittle-Johnson & Schneider, 2015).

The reflecting stage is rethinking and seeing the relationship between non-formal mathematical models (model of) built by students with formal mathematical models (model for). The main principles of the reflecting stage are metacognition, self-monitoring, and interactivity. Learning through reflection encourages students to look back and reflect on their learning process (Selter & Walter, 2020). Through metacognitive reflection, students can evaluate the right or wrong mathematical models they have developed and guide students' thought processes to self-

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monitoring. It immediately corrects if there are still errors in their mathematical process. As said by Stillman (2011), it is essential for students that metacognitive reflection on the processes and results in mathematics learning plays a vital role in students' abilities to evaluate mathematical models that they have developed. In this stage, the students presented the impact of their discussions or having discussions in each group that involved students' active participation. Actively contributing to class discussions or listening to the questions and answers sessions helped to develop metacognitive skills of reflective thinking (think about one's thoughts, and think about the relationship of models of and models for), which is an essential step towards developing concepts of new mathematical (Taber, 2015).

The final stage of the CORE RME learning model is the extending stage. It is the stage of expanding knowledge through different and challenging real contexts. The main activity is to accommodate the students to develop their understanding through other real contexts. On this occasion, students applied formal mathematical models that they had understood, using their conceptual and procedural knowledge to formulate and solve mathematical models from the other real contexts. This cognitive process facilitated and provided experiences for students to understand intra-mathematical and extra-mathematical connections. Students see mathematics as a separate science but as relevant and integrated, practical, and closely related to real-world phenomena (NCTM, 2000).

This study's findings explicitly showed that the CORE RME learning model could facilitate students' mathematical connections, both intra-mathematical connections and extra-mathematical connections. Therefore, it certainly could positively impact students, including the development of student interest in learning mathematics. The learning process that facilitates students with intra-mathematical and extra-mathematical connections can develop students' interest in learning mathematics (Arthur et al., 2018., Rellensmann & Schukajlow, 2017). Besides, it enhances students' abilities to adapt to unknown situations, increase students' intrinsic motivation to learn mathematics, and stimulate student development to become lifelong independent learners (Ormond, 2016).

## CONCLUSIONS

The conclusions obtained from this study is that the achievement and improvement of students' mathematical connections abilities through learning from the CORE RME model was better than students who learned through the CORE model comparing to students who learned through the conventional model. Besides, the achievement and improvement of students' mathematical connections abilities through learning from the CORE model and students who learned through the traditional model have no significant difference. These two concluding statements did not mean that the CORE and Conventional learning models did not facilitate students' abilities in mathematical connections. However, these two learning models could boost students' mathematics connections, like Yaniawati et al. (2019) 's research that CORE learning could improve students'

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mathematics connections. Learning through the quality CORE model helps students enhance their mathematics connections skills (Konita, Asikin & Asih, 2021). Nevertheless, compared to the CORE RME Model, it resulted that the achievement and improvement of students' mathematical connections through learning from the CORE RME model was better than students who learned through the CORE model and students who learned through the conventional model.

The findings as a substantive generalization from this study is a student can master mathematics connections if the mathematics learning uses real context, which could be imagined by students as starting point with its phase as follows: 1) Connecting, emphasizing in the natural context, prior knowledge, and interactivity principles, 2) Organizing, emphasizing in the guided reinvention, self-development models, and interactivity principles, 3) Reflecting, emphasizing in the metacognition, self-monitoring, and interactivity principles, 4) Extending, emphasizing in the develop a formal mathematical to another real context, intertwining, and interactivity principles. Why is real context essential to be made as a starting point in learning? Freudenthal (2002) says that something else around us has a role in the mathematics concept of learning. We must admire those students who have many experiences in their family and their peer groups, giving them informal opportunities to develop mathematical concepts and skills (Clarke et al., 2006). Students' experiences from their homes, society, or past could be taken as a chance to build up their new mathematics knowledge (Taber, 2015).

Based on this study's findings, it offers the CORE RME learning model as a solution to develop students' mathematics connections. Suggestion for the teachers to use this learning model as a learning intervention form to facilitate students' mathematical connections. In line with Mhlolo et al. (2012) 's recommendation, teachers must teach subject matter in ways that make the students recognize and understand the mathematical connections better. Teachers must build up this habit to promote and strengthen mathematical connections (Eli et al., 2013). The teacher could use the intervention to relate mathematics with actual daily life problems and environment near the students, and the other was scientific and between concepts or ideas in mathematics (Arthur, et al., 2018).

The learning which could facilitate mathematical connections can help students to correlate procedural knowledge and conceptual. Procedural and conceptual understanding play essential roles in mathematical connections (Dolores-Flores et al., 2018). These bits of knowledge are correlated positively and identified clearly as rich knowledge with connections (Rittle-Johnson & Schneider, 2015). The positive correlation has caused improvement in procedural knowledge or vice versa. Therefore, mathematics learning needs to emphasize these two abilities to improve students' mathematics connections.

The results of this study have proven that learning through the CORE RME model can enrich students' mathematical connections. Therefore, it is recommended for teachers to use the CORE RME model as an alternative to facilitate students' mathematical connection abilities.

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## Increasing the Efficiency of Teacher's Work: The Case of Undergraduate Mathematics Mid-Term Assessment

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**Abstract:** *Computer-aided assessment is an important tool that reduces the workload of teachers and increases the efficiency of their work. The multiple-choice test is considered to be one of the most common forms of computer-aided testing and its application for mid-term has indisputable advantages. For the purposes of a high-quality and responsible assessment process, it is necessary to provide a sufficiently extensive databank of test items, especially when a large number of students is involved in the examination. In this paper we deal with the issues of automatic generation of such test items for undergraduate mathematics mid-term assessment. We describe the techniques and further circumstances related to designing test items, including the incorrect answers offered. The text also includes a case study dealing in detail with the creation of test items for a particular type of mathematical problem. Finally, efficiency evaluation of automatic item generation in comparison with paper-based questions is presented. Although the results of the article are based on experience influenced by the local conditions of the institution concerned, the ideas suggested remain generally applicable.*

### INTRODUCTION

The expansion of computer technology into an ordinary university facility has brought with it new possibilities in the assessment process. An appropriate information system and a large capacity computer lab at one's disposal allows the modification of traditional assessment and evaluation approaches and significantly improves the efficiency of teachers' work. In particular, educational institutions organising courses for hundreds of students make use of the benefits of technology very often, no matter what fields of study they provide, including undergraduate mathematics. It is true that mathematics and applied mathematics have their specifics and this often lead in extensive discussions on computer-based assessing students' knowledge (see, e.g., Bennet et al., 2008; Clariana and Wallace, 2002; Oates, 2011). Various research papers about the advantages and disadvantages of these methods in mathematics have been written (Croft et al., 2001; Rasila,

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Malinen and Tiitu, 2015; Rønning, 2017). Computer assisted assessment has also been subject to gender analysis for a long time (Akst and Hirsh, 1991; Bennet et al., 2008; Clariana and Wallace, 2002; Goodwin, Ostrom and Scott, 2009). Lee (2011) is of the opinion that today's teachers must be creative and combine the application of information technology with innovative teaching methods and strategies. It is interesting, but also expected, that the use of technology is much more discussed in tertiary mathematics education than in lower mathematics education. Even the research of Foster and Inglis (2018), which analyses two leading UK mathematics teacher professional journals targeting mainly at secondary education, concluded that there is "a smaller decline in discussions relating to [...] technology".

The issue for computer aided assessment (CAA), or also e-assessment, follows the development of technology, to which available forms of the assessment process correspond. Journal papers commonly focus on two formats, multiple-choice (MC) and constructed response (CR). The MC format has a lot of critics, but its advantages are indisputable. Its application does not require students to enter their mathematics in fill-in-the-blank items. Sangwin (2013) admits, that "syntax presents the most significant barrier to students' successful use of computer aided assessment, particularly when the stakes are high". There is also great flexibility in use of MC independent of the learning management system. The strong position of the MC format is evidenced by a long-term research (Goodwin, Ostrom and Scott, 2009; Kosh, 2019; Mitkov, Ha and Karamanis, 2006; Sims Williams and Barry, 1999) and the fields of psychometrics associated to it, especially so-called item response theory (IRT) (Andrich and Styles, 2011; Embretson and Kingston, 2018; Hoppe, 2016). On the other hand, the usage of the CR format belongs to an important topic of CAA research. The CR items removes guessing, prevents the "reversible mathematical processes" resulting in back substitution, see Sangwin and Jones (2017), and provides detailed feedback to teachers and students. The applications of IRT to the CR items can be found e.g. in Holling, Bertling and Zeuch (2009) or Maxwell and Gleason (2019). Leaving aside the problems of CR format with entering a mathematical expression by students into a machine, there are requirements for the university learning management system to interact with the computer algebra system used to design the test items.

However, this paper does not aim to analyse the pros and cons of MC and CR but focuses rather on the issue of generating test items, more precisely on automatic item generation (AIG). Kosh (2019) presents AIG as an item development process that can be used to supplement item writing efforts and defines it as a three-stage process. In the first stage the cognitive model is developed, i.e. the mathematical knowledge and skills needed to solving items is characterized. The second stage is developing item model encompassing specifications detailed enough to produce items in the last stage through algorithmic means. Kosh (2019) emphasizes that a crucial stage is "developing item models. Despite item model creation serving as a critical methodological component of AIG, to my knowledge there are no published methods that describe the principles or standards used to create AIG item models. Instead, researchers merely present the cognitive model and item models they used without describing how those item models were created, ...".

Kosh (2019) distinguishes five steps in the item model creation process, not necessarily in a linear arrangement, as follows: 1. Identification of the schema representing the desired content of the items, 2. Identification of features drawing on existing theory and impacting cognitive

complexity of the schema (i.e., structural complexity, contextual support, extent of generalization), 3. Determining item type (MC, CR, etc.), 4. Specification of so-called item meta-model (a model for the item model), 5. Writing item models. The item meta-model in the fourth step characterizes the structure of the item models and their variability aligned to the given cognitive complexity. It defines components that vary across item models, including a list of typical student mistakes and associated distractors. The variability of item models is given by already predefined combinations of features that maintains the same sufficient level of all test items.

The matter of AIG must be viewed from two aspects, not only quality but also quantity. Quantity is an increasingly important requirement in order to create a large variety of tests (see Sims Williams and Barry, 1999) because of the large numbers of students. A substantial item bank prevents unacceptable student collaboration during a test as well as making it harder to illegally share test items already used. The continual supply of new test items also allows to generate an individual test for each student.

Incorporation of AIG into pedagogical practice has positive impact on a teachers' work efficiency. Efficiency is understood here in the classical sense of meaning as a measure of the amount of resources required in order to meet the goals. De Witte and López-Torres (2017) provide an extensive review of the literature on efficiency in education including the 'economics of education' literature. The review deals with different levels of analysis as university level, school/high school level, district level, national level, student level and classroom level, but there are no research papers on the efficiency of the teacher's work, even though the integration of technologies into the educational process is directly offered for such analyses. In particular, it is assumed that an effective way of assessing students' knowledge saves resources such as time and manpower.

In this context, it is worth mentioning that the issue of efficiency needs to be distinguished from the topic of teacher effectiveness and effective teaching. Teacher effectiveness is defined, in the narrowest sense, as a teacher's ability to produce gains in student achievement scores. It is pertinent to note here that such effectiveness in mathematics education at the undergraduate level is closely related to the effectiveness of secondary schools. In order to achieve sufficient effectiveness, universities are looking for various ways to fill the gaps in students' mathematical knowledge from their previous education, see e.g. Hampel and Viskotová (2021) or Dagan, Satianov and Teicher (2019).

A practical guide to evaluating effectiveness can be found in Little, Goe and Bell (2009), other studies as Gurney (2007) and Yue (2019) describe factors and methods for effective teaching. Burden and Byrd (2019) in their work deal with effective teaching in all its complexity. Harris and Sass (2014) involve principals' evaluation of teachers' cognitive and non-cognitive skills in their analyses and touch on the issue of cost-effective methods of measuring teacher productivity. Research on measurements of teacher productivity can be find also in Sass, Semykina and Harris (2014). Ilkovičová, Ilkovič and Špaček (2017) look at teaching efficiency from three different perspectives, one of which is viewed from a teacher and reflects on the rationality of the pedagogical time given. Levin (1997) approaches the issue of raising school productivity using x-efficiency, see Leibenstein (1966).

Another notion that often appears in the literature is the so-called teachers' self-efficacy as a socio-affective concept, different from the concept of teachers' efficiency. Nevertheless, the connection can be traced, the influence of self-efficacy on teachers' performance is obvious. Alibakhshi, Nikdel and Labbafi (2020) describes all the consequences of self-efficacy and, among other things, point to the impact of self-efficacy on the using computers and technology.

The aim of this paper is to present the case of increasing efficiency of teachers' work in the process of assessment, specifically the methodology of computer-aided generation of MC test items with one keyed answer for the purpose of mid-term assessment in undergraduate mathematics. In the following, we propose selection of mathematics topics suitable for computer processing, suggest useful techniques for creating source text files containing test items, discuss related problems that may occur and briefly evaluate teachers' work efficiency of the proposed solution.

## MATERIAL AND METHODS

In this section, we present the entry conditions given by the content and enrolment characteristics of the courses where automated test generation is employed, as well as the conditions given by the university information system environment. The courses of undergraduate mathematics at the Faculty of Business and Economics of Mendel University in Brno cover typical topics, that are also found in classical textbooks such as Sydsæter et al. (2016). Namely, these are the issues of linear algebra (matrices and matrix operations, determinant, the inverse of a matrix, matrix equations, systems of linear equations), differential calculus of function of one or more variables (definition of a function and its properties, limits and continuity, derivatives and their application) and integral calculus of one variable (indefinite, definite and improper integral). These classes have the important characteristic that a large number of students (up to 1,000) are enrolled. Thus computer-based assessment has become the standard means of mid-term assessment and has a tradition of more than fifteen years. For this purpose, a computer lab with a capacity of 80 workstations is used which enables handling of the assessment quite quickly, in two days.

The university information system provides the application, which allows the teacher to create a unique test for each student registered for the exam date. The properties of a test, such as its duration, the number of test items, the number of points assigned for each item, and the link to the repository of items, all depend on application settings given by the examiner. To deter students from guessing, it is recommended that incorrect answers be penalised with negative points. Test items have to be imported in advance from text files of a given structure into the test item bank of the information system.

The importable form of items saved in a text file follows the syntax of the LaTeX typesetting system. Each item consists of several text lines; the first line corresponds to the stem, the second line starts with the plus symbol and the keyed answer follows, the other lines (their quantity is in line with the number of responses being offered) contain the minus symbol and the

distractor. Each test item is separated by an empty line. As an example, the problem of calculation of a determinant can be typed as mentioned in Figure 1.

LaTeX form of the item in the source text file	Final form of the item in the test
<pre>Calculate the determinant of the ... ... matrix \$\left(\begin{array}{rrr}5&amp;1&amp;0\\7&amp;-1&amp;-2\\1&amp;0&amp;-1\end{array}\right)\$. +\$10\$ -\$5\$ -\$11\$ -\$-4\$ -\$-1\$</pre>	<p>Calculate the determinant of the matrix</p> $\begin{pmatrix} 5 & 1 & 0 \\ 7 & -1 & -2 \\ 1 & 0 & -1 \end{pmatrix}.$ <p> <input type="radio"/> 10      <input type="radio"/> 11  <input type="radio"/> 5      <input type="radio"/> -4      <input type="radio"/> -1 </p>

Figure 1: Text format of the test item ready for import and its preview in a test

General approaches as parametrization of the problem are used for development of the resulting automated test item generation methodology. The computational system MATLAB R2020b with Symbolic Math Toolbox was chosen as the tool for generation of the test items.

## RESULTS

Within this section, we propose classification of items with respect to possible parametrization approaches and appropriate techniques for generation of stems and distractors. The case study with complete description of the generating procedure is elaborated. Finally, the differences between the efficiency of automatic generation and the paper-based creation of particular item types are presented.

Those issues of undergraduate mathematics are being processed whose solutions are of “a technical nature”, when the complexity of the structure of the meta-model does not outweigh the benefits of AIG, and therefore these issues are suitable for midterm technology-assisted assessment. Computer-aided generation has been used for the following tasks: the product of matrices, the determinant of a square matrix, the eigenvalues of a square matrix, the inverse of a square matrix, solvability of the system of linear equations, the derivative of a function of one variable (product rule, quotient rule, chain rule, derivative at a given point), the indefinite integral of rational functions.

### Computer-aided generation of stems

Computer-aided generation of a test item for a given area is based on the idea of parametrization of an example test item, where parameters are being considered instead of numerical values in the stem. It is possible to distinguish two techniques for assigning values to parameters:

- (1) The technique based on the pseudo-random generation principle is undoubtedly applicable. Pseudo-random generation means that randomly generated values in the test item must

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meet pre-determined conditions, otherwise the non-compliant values are dropped and generated again.

- (2) The parameters are gradually assigned all values from the given numeric set, usually all the integers from an interval. It is convenient to check whether the new values of the test item correspond to the requirements for the difficulty of calculation and the final form of the result.

The first approach finds wide application in linear algebra, the second approach is more likely to be applied in calculus to test items concerning the calculation of derivatives and integrals.

Random value generation can generally pose the problem of repeated test items. A way to prevent this is to store already generated values in an auxiliary matrix, where each row of the matrix represents the numeric values of one generated item. If the sequence of new values differs from all rows of the matrix, the corresponding item is included in the test bank and the auxiliary matrix is updated. Otherwise, random generation must be performed again.

Determining the appropriate conditions for generating a new test stem is essential to achieve adequate test difficulty. To illustrate the situation, consider a  $3 \times 3$  determinant calculation. The conditions might be such that the matrix contains a given number of zeros and the absolute value of the determinant does not exceed the specific number.

The thought-out change of conditions in the generating algorithm creates a new group of test items within one issue. For example, a well-chosen variety of conditions of the eigenvalue problem results in separate groups of test items with different number of unique eigenvalues. Variability of conditions may cause different difficulty level of test items, which can be solved by changing the range of the generating interval from case to case.

Fulfillment of given conditions on issues with many numerical values can cause very long program calculation times. Then, it is advantageous to generate only a part of the parameter values of the test item and verify that they meet the conditions, and in a second step use these values to derive by means of a random variable the values of the remaining parameters. For example, the generation of test items for linear equation systems, depending on the required degree of freedom and number of equations and unknowns, can be accelerated as follows. First, the coefficients of as many equations as the expected rank of the coefficient matrix are generated, and secondly, the remaining equations are given by random linear combinations of previously obtained equations. Finally, to achieve a smarter appearance of the system it is more than convenient to test the linear dependence of all possible pairs of equations and accept the system only in the case of independence. To enlarge the variability of test items it is sometimes feasible to generate coefficients so that the corresponding submatrix formed of the first  $r$  columns of the coefficient matrix has the rank lower than  $r$ , provided that  $r$  is the rank of the coefficient matrix.

There are mathematical problems where it is more convenient to generate test items based on the desired result and derive the stem from the correct answer. Suppose that the goal is to create the test item on integration of a proper rational function involving a quadratic in the denominator. Due to the application of appropriate anti-derivative formula, calculation of square roots is often necessary, which makes some of the results too complicated. Therefore, it is preferable to determine the expected result first and then to calculate the stem.

### Computer-aided generation of distractors

The issue of computer-aided generation of distractors is primarily solved using algorithms simulating common student mistakes. Similar to the situation where stems are generated, it is necessary to verify the difference of the new distractor from the keyed answer and from already generated incorrect answers. In the case of a match, the distractor is generated randomly and is verified again. As an example of a good distractor on  $3 \times 3$  determinant calculation seems to be the value which is obtained by exchanging the signs of the part of associated minors.

It sometimes occurs that the aspects of creating proper incorrect answers are more extensive in some areas of mathematics. Distractors on calculation derivatives of functions requires more than just changing constants in the result, it needs in addition a change in the function type in the distractor. To prevent students from guessing the answers, a clever composition of offered answers can be used. The idea is that each test item randomly varies the number of wrong answers differing from the keyed answer only in the constants, and the number of incorrect answers differing fundamentally, for example by a different number of roots in an eigenvalues calculation or a different type of function in an antiderivative calculation.

Another idea to discourage guessing can be presented in the case of calculating the elements of the 3-by-3 inverse matrix. To avoid situations where students guess the correct answer from the first few elements of the matrix, it is advisable to list only 3 selected elements in the offered answers, preferably so that each element lies on a different row and column. The choice of listed elements may be influenced by random index generation, which increases the diversity of the answers being offered.

### Strings in the source text files

As mentioned in the Entry conditions section, the source text file for importing test items into the test bank must follow the structure given by the relevant information system administering the tests. This requires construction of variables in the format of a string array, which serves to write each test item in the text file. The circumstances of creating these strings depend on the chosen software, but the principle always lies in concatenating shorter strings. These arise as results (converged into LaTeX format) of the generation and calculation given by software, or represent fixed parts of the test item, such as texts or some mathematical symbols. This manipulation of text strings brings a number of difficulties, for example the problem of the presence of constants 1, -1 or 0 at the beginning of the equation, or the format of generated fractions  $3/1$ ,  $0/3$  etc.

### Case study: inverse of $3 \times 3$ matrix

Let's deal with the issue of an inverse matrix of order 3

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

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The matrix elements are randomly generated integers from the interval  $\langle -5, 5 \rangle$ . The order 3 guarantees calculation by students in a reasonably short time. It is necessary to bear in mind that students may choose from the two most common methods, i.e. the method based on using elementary row operations and the method of calculating the determinant and the adjoint matrix. Considering aspects of these techniques, three types of test items have been generated:

(1) The conditions for generating the items of the first type have been set as follows:

- the number of zero elements is a value from the set  $\{1, 2\}$ ;
- $|a_{11}| = 1$ ;
- $|\det \mathbf{A}| \in \{2, 3\}$ ;
- $\left| a_{22} - \frac{a_{12}a_{21}}{a_{11}} \right| = 1$ , which ensures that after the first step of Gauss-Jordan elimination the absolute value of the second element of the main diagonal equals 1;
- $\text{mod} \left[ a_{13} - \left( a_{23} - \frac{a_{13}a_{21}}{a_{11}} \right) \frac{a_{11}a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, a_{33} - \frac{a_{13}a_{31}}{a_{11}} - \left( a_{23} - \frac{a_{13}a_{21}}{a_{11}} \right) \frac{a_{11}a_{32} - a_{12}a_{31}}{a_{11}a_{22} - a_{12}a_{21}} \right] = 0$  and  $\text{mod} \left[ a_{23} - \frac{a_{13}a_{21}}{a_{11}}, a_{33} - \frac{a_{13}a_{31}}{a_{11}} - \left( a_{23} - \frac{a_{13}a_{21}}{a_{11}} \right) \frac{a_{11}a_{32} - a_{12}a_{31}}{a_{11}a_{22} - a_{12}a_{21}} \right] = 0$ , i.e. once row operations give zeros in all non-diagonal elements of the first two columns, the first and the second element of the third column are divisible by the element in the third row of this column.

(2) The second type of the inversion matrix items satisfies these demands:

- the number of zero elements is a value from the set  $\{1, 2\}$ ;
- $|a_{11}| = 1$ ;
- $\det \mathbf{A} = -1$ ;
- $\left| a_{22} - \frac{a_{12}a_{21}}{a_{11}} \right| = 0$ , which ensures that after the first step of Gauss-Jordan elimination the second element of the main diagonal equals 0;
- $\left| a_{32} - \frac{a_{12}a_{21}}{a_{11}} \right| = 1$ , which ensures that after the first step of Gauss-Jordan elimination the absolute value of the last element of the second column equals 1.

(3) The third type of test items corresponds to singular matrices, i.e.  $\det \mathbf{A} = 0$ . The matrix is generated with exactly one zero element with arbitrary position and its singularity is verified by calculating the determinant of the matrix.

The format of the answers (the list of the three selected elements of the resulting inverse matrix, where each element lies on a different row and column) has been already mentioned above. In the case of regular matrices, the construction of distractors is based on typical student mistakes. The first distractor arises from the omission of the determinant in calculation using an adjoint matrix. The initial conditions mentioned above consider  $\det \mathbf{A} \neq 1$ , so the keyed answer is always

different from this distractor for non-zero elements. The second distractor corresponds to the situation when transposing is forgotten when calculating an adjoint matrix. The third distractor has been realized by replacing one of  $a_{31}$ ,  $a_{32}$ ,  $a_{33}$  by a random generated value. The fourth incorrect answer is the sentence “The inverse of the matrix does not exist.”. Of course, the need for a difference of the answers offered must be verified. As regards test items with singular matrices, distractors are generated randomly. Due to the different selection of elements for each answer, there is no problem with equality of the distractors in the case of singularity.

### Efficiency evaluation

Evaluating the effectiveness of AIG depends, of course, on the teachers’ ability to algorithmize the problem and master programming techniques and the programming language. The following analysis assumes advanced ICT skills, reported times include the implementation of the correct conversion to LaTeX format with all its difficulties. It should be emphasized that the figures given are average estimates.

Table 1 shows that test item creation using AIG requires different times according to the type of problem. It took approximately 9 hours to prepare and check a script for computer-aided generation of test items corresponding to one type of a problem within the topic of linear algebra and indefinite integral of rational functions. In the case of differentiation, covering the whole range of problems required an adequate amount of scripts (typically related to particular function form) that could not be directly included in a single script. A total of 20 shorter scripts were needed to process one type of question to apply all the elementary functions and the related phenomena. Calculated in time, it took 60 hours to create these scripts for one type of differentiation test item with the appropriate variety.

Topic	The scripts for AIG using MATLAB	Paper-based creation of 20 test items	Paper-based creation of 100 test items
Linear algebra	9 h	11 h 40 min	58 h 20 min
Differentiation	60 h	13 h 20 min	66 h 40 min
Indefinite integral of rational functions	9 h	10 h 0 min	50 h 0 min

Table 1: Estimated number of working hours spent for one type of test item creation.

Considering the experiences with the paper-based test item creation, the linear algebra test item is created and inserted into the information system in about 30 minutes, the test item considering differentiation in about 35 minutes and the test item concerning the indefinite integral of rational functions in about 25 minutes. In all mentioned paper-based cases, another 5 minutes



are devoted to the final inspection. In Table 1 we can find working time employed to create 20 paper-based test items as minima per 1 semester for one type of the question.

Regarding linear algebra and indefinite integral, it can be argued that the time savings due to AIG are already apparent during the first semester. As for differentiation, AIG compared to paper-based work places great demands on the spent time and is not be paid until the fifth semester. However, it is necessary to keep in mind, that AIG allows to generate an extensive database of given types of questions and provide individual test assignments for each student. The times needed to accomplish such goal with paper-based test items are given in the last column of Table 1.

In the course of undergraduate mathematics AIG has been used to generate 5 types of test items from linear algebra, 4 types from differentiation and 3 types on indefinite integral. In total, the creation of scripts for generating large databank took 313 hours. Paper-based database creation of the same content for single semester, e.g. 20 items per type, would last 158 hours. It is evident that the time invested in AIG pays off after only two semesters. Likewise, it is easy to see that AIG reduces teacher workload in a particularly fundamental way when processing issues of linear algebra and indefinite integral of rational functions.

## DISCUSSION

Our approach to the computer-aided generation of test items for selected mathematical problems seems to be successful and has enabled us to create a large test databank of required quality. Thanks to the selection of appropriate topics and the suggested formulations of tasks, the problem of reversibility of mathematical processes, described e.g. in Sangwin and Jones (2017), was suppressed. It can be stated that back substitution is only possible for the items concerning the calculation of indefinite integrals of rational functions. But even here, the back substitution is very complicated, students have to differentiate up to five times including tough simplification of expressions, which overweighs the difficulty of the direct finding an antiderivative. Back substitution strategy problem is being investigated from different points of view; for example, Goodwin, Ostrom and Scott (2009) focused their attention on gender differences in the tendency to use the back substitution strategy in a multiple-choice mathematical test and found that this relationship was not significant.

The fact that the test items have been generated in the LaTeX format enables – beside primary use with the learning management system – creating pdf files with the test items in an easy way. This is important in cases of a technical failure of the information system or Internet outages. Herbert, Demskoi and Cullis (2019) also take advantages of the independence of the generated test from the learning management system. However, when importing, they do not work directly with the LaTeX format, but already use its conversion to PDF format.

The MATLAB computational system with the Symbolic Math Toolbox has proved to be suitable software for generating test items for mathematics assessment. However, considering availability of software and the various tools on offer, many of these can be used, e.g. Maple (see Herbert, Demskoi and Cullis, 2019) or Maple T.A. (see Jahodová Berková, 2017; Rønning, 2017). Of course, the ability of the software to calculate corresponding mathematical problems to get correct answers and distractors according to a given algorithm is assumed. A tool for generating

uniformly distributed random numbers is necessary for setting numeric values in all parts of the test items, as well as for random generation of the type of distractor. Furthermore, it is necessary to handle text strings and write them to source text files. The ability of software to return the LaTeX form of symbolic math expression is certainly an advantage.

Computer-aided generation of test items has a huge benefit that once adequate program code is created in a given software, then a lot of items arise in a simple manner. We observe high efficiency of coding for many topics. Gierl and Lai (2016) arrived at similar results and confirmed the high efficiency of AIG, applying their research to test item generation not only in mathematics but also in health science. Moreover, it is realistic to obtain a sufficiently extensive test bank to guarantee, together with the personalized test items for each student, cheat-resistant assessment. Unfortunately, as Manoharan (2019) confirms, personalization approach does not mitigate contract-cheating or cheating using communication devices.

Responsibly designed technology-assisted testing finds its application during the semester as a time-saving tool that allows only promising students to take the final exam. This approach to applicability just for mid-term assessment is fully in accordance with Hoogland and Tout (2018). They point to the important fact that there are two opposing pressures when considering the assessment in mathematics education. On the one hand, emphasis needs to be placed on the higher-order thinking skills and knowledge, and on problem solving and reasoning. On the other hand, the trend of focusing on technology and efficiency results in risk of an assessment of lower-order goals, based on reproduction of calculation procedures.

The applicability of the presented concepts primarily focused on a summative assessment is also expanded to the form of formative assessment. The provision of dozens of items, randomly drawn from an extensive item bank, provides an efficient mechanism for students to practice skills they have learned from their classes. In their paper, Morphew et al. (2020) prove that more frequent testing schedule (for both MC and CR formats) in undergraduate engineering lead to better student outcomes. Similarly, in the case of formative type of assessment, self-testing improves students' learning. Moreover, students find online formative feedback enjoyable and useful (Acosta-Gonzaga and Walet, 2018). Formative assessment also contributes to their ability of self-assessment as a feedback mechanism for improving own learning strategies (Hosein and Harle, 2018). There may arise a discussion about whether or not to grant students the repository of test items in full. This will undoubtedly provide students with the maximum amount of right material to practice and, in addition, any student ambitions to copy or record test items will have no sense.

One of the limitations of our approach is the fact that there are still mathematical topics convenient for assessment in the form of a multiple-choice test, but their computer generation brings many troubles due to the non-uniform structure of test items. Therefore, such items have been made up "manually". Specifically, these were items related to areas like the function and its properties, the limit of a function, integration by parts, integration by substitution, the definite integral, first-order and second-order partial derivatives. Adji et al. (2018) made research on AIG within high school mathematics and found out that 45 % of mathematical questions for the local National Exam cannot be processed by AIG.

Automatic generation of test items, despite efforts to treat all possible situations, can give rise to problematic items where the solution does not match the required level. In addition to

insufficient difficulty of test items, computer-aided generation can bring with it other undesirable effects that cannot always be predicted in advance. It is therefore advisable to check a random sample of test items with students whose feedback is significant. Gierl and Lai (2016) also emphasize the importance of external review. In their case, the review is performed by subject-matter experts who did not develop AIG models and is based on the so-called standardized rating rubric.

Once the generated test items are incorporated into the assessment, the information system provides a basic summary of the average scores relating to particular items. Examiners can view these and distinguish items with outlying scores and assess their inclusion in the test, potentially replacing them with newly created items. Future research could focus on evaluation with respect to item response theory. In mathematical education, the application of this methodology can be found, for example, in Bolondi, Branchetti and Giberti (2018).

It should be taken into account that the computer-based assessment place demands on ICT skills of teachers (see, e.g., Tondeur et al., 2019). TALIS (The OECD Teaching and Learning International Survey) repeatedly reports that teachers themselves are aware of this need. OECD (2020) points out that teachers cite ICT skills as the second most urgent area in further professional development. However, the situation about Covid-19 has an accelerating effect on the growth of teachers' and students' ICT abilities (fully online mathematics teaching is discussed in Trenholm and Peschke, 2020) and increases the importance of research in the field of effective technology integration not only into assessment, but also in education generally (see, e.g., Cardoso Espinosa, Cortés Ruiz and Cerecedo Mercado, 2021).

## CONCLUSIONS

Our methodology for the computer-aided generation of test items has proved to be useful and has enabled us to create a large test databank containing questions on undergraduate mathematics of various topics and required difficulty. Further, the MATLAB computational system with the Symbolic Math Toolbox seems to be software entirely suitable for generating test items for mathematics assessment. Of course, it is necessary to use the system in a responsible way and to verify the quality and level of individual questions in real testing of students' knowledge.

Automatic generation of test items has brought a new insight to the mid-term assessment, when the huge variety of test items brings teachers a strong tool for the implementation of a fair automatic assessment process. As part of the institution's teaching innovations, the test bank has recently been expanded by over 1,500 test items, now using automatic test item generation. Through minor adjustments, the system of computer-aided generation of test items so introduced can be utilized repeatedly to construct more items already implemented of a given quality as well as new types of items.

Considering AIG in mathematics, there is an opportunity to expand the issue of AIG by test items accompanied by automatic generated graphs of functions. This issue does not occur in the literature, although its applicability is undoubtedly obvious. This identified literature gap may point out additional research areas.

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## Teachers' Conceptions in Training on Mathematics of Medical Students

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**Abstract:** *The goal of the paper is to pay attention to some important techniques and approaches including adequate designations as a tool for unambiguous understanding and a key to success in solving problems, vivid visual images as a mnemonic techniques, and special formulas as a universal tool for solving typical problems, when teaching medical students of mathematics.*

*The motivation for this paper is to help non-mathematics students understand complicated mathematical topics in an easy, natural, and simple way.*

### 1. INTRODUCTION

Mathematics is the fundamental science giving language means to other sciences. This has been noted by many outstanding scholars who asserted that “The book of nature is written in the language of mathematics” (Galileo Galilei), “In any science it is so much true, how many in it of mathematics” (Immanuel Kant), “Mathematics is a basis of all exact natural sciences” (David Hilbert).

Today a great variety of mathematical methods is applied in biology, medicine, and other biological sciences. Using mathematics in public health services in world space occurs promptly, new technologies and the methods based on mathematical achievements in the field of medicine are entered. Mathematical methods are widely applied in medicine. The modern medicine cannot do without the most complicated techniques, therefore the role of mathematics appreciably grows.

An understanding of mathematical calculus leads to better comprehension of chemistry and physics because calculus offers new ways of thinking that are quite useful to a medical practitioner. A knowledge of different parts of mathematics is very important for most future physicians (Nusbaum, 2006). But, unfortunately, medical students often find it difficult to study mathematics (Chasteen-Boyd).

Real-life applications of mathematics provide a great deal of stimulation for various kinds of research in the subject matter field, involving professional mathematicians and students of different majors alike (Abramovich & Grinshpan, 2008).

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In this paper, we would like to share the operational experience at carrying out of employment on mathematical disciplines. The data reported in this paper occurred in a mathematical course taught by the authors at the Immanuel Kant Baltic Federal university (IKBFU) in Kaliningrad (Russia) for medical students. Students seeking a medical degree were required to complete a course in mathematics, which is typical of university degree programs in medicine (Khobragade & Khobragade, 2015; Voltmer et al., 2019). We will describe some features and the curious moments which arose from authors' experiences while teaching medical students at IKBFU.

It is important for a teacher to help a medical student master the necessary sections of mathematics in a short time. And we suggest using methods that are successfully applied in our classes for teaching mathematics to medical students.

The purpose of this study is to characterize the best moments in the growth of medical students' mathematical understanding. This is very relevant since the need for mathematics in medical research is growing rapidly.

Of course, at our lectures and practical classes we have no intention to distract the medical students from their main field of activity and to train them as competent mathematicians. Our aim is rather to prepare them for an understanding of the basic mathematical operations and to enable them to communicate successfully with mathematicians in case they need help of the last.

Usually we use many illustrations and some historical notes to encourage the medical and biological students who are perhaps somewhat reluctant to be involved with the abstract side of mathematics. This is not surprising, after all, the course of Mathematics includes the following sections: Precalculus, Linear algebra, Limits, Differentiation, Integration, Differential equations, Probability, and Statistics (Batschelet, 1979).

Thirty students in two groups from India enrolled in the one-year course in Mathematics and Computer Science for medical students at Immanuel Baltic Federal University (Fig. 1 and 2). At the beginning of the first semester, all students were told that research was being conducted on this experimental class; all students agreed that the data that was collected in the study could be used for research purposes.

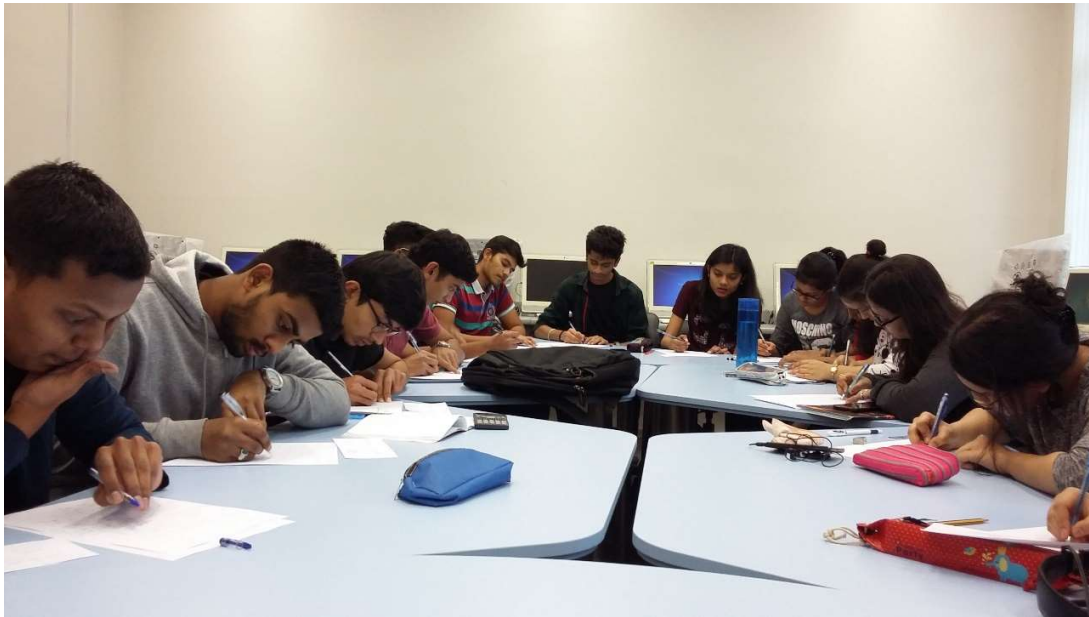


Figure 1: Medical students from the group 1

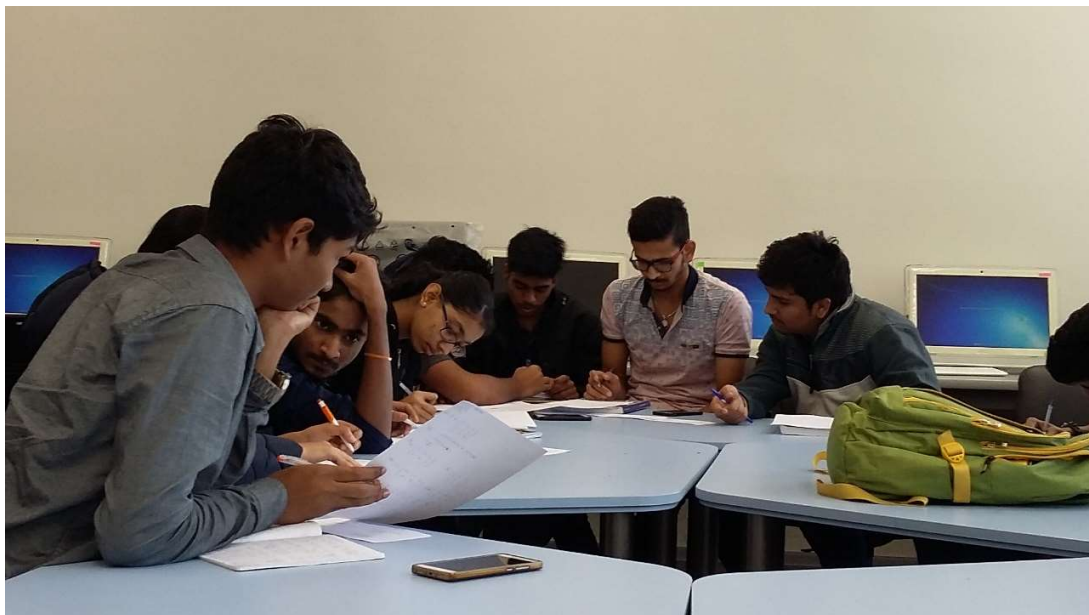


Figure 2: Medical students from the group 2

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## 2. ADEQUATE DESIGNATIONS AS A TOOL FOR AN UNAMBIGUOUS UNDERSTANDING AND A KEY TO SUCCESS IN SOLVING PROBLEMS

It is very important to respond to pedagogical situations around notations that students might encounter. This study explored an importance of notations of logarithmic function and inverse trigonometric functions in their calculations.

### 2.1. Logarithms $\ln x$ vs $\log_a x$

Researchers continue to report that many students struggle to develop coherent understandings the idea of logarithm (i.e., logarithmic notation, logarithmic properties, the logarithmic function). Of course, developing students' understanding of the idea of logarithm requires much more than being introduced to and applying Euler's definition (Kuper & Carlson, 2020; Hirsch & Pfeil, 2012).

Recall the definition of logarithm.

**Definition 1.** If  $a$  is any number such that  $a > 0$  and  $a \neq 1$  and  $x > 0$ , then  $y = \log_a x$  is equivalent to  $a^y = x$ . We usually read this as “log base  $a$  of  $x$ ”.

If students do not hold the foundational understanding, they may struggle to envision the relationship between  $a$  and  $x$ .

It is also important to mention that there are special notation for the logarithmic function for different bases: the common logarithm  $y = \log_a x$ ; the natural logarithm  $y = \ln x = \log_e x$ ; the Briggsian logarithm  $y = \lg x = \log_{10} x$ .

**Remark 1.** Of course, if it is necessary, one can always change the base of a logarithm. The logarithm  $\log_a b$  can be computed from the logarithms of  $b$  and  $a$  with respect to an arbitrary base  $c$  using the following formula:

$$\log_a b = \frac{\log_c b}{\log_c a}. \quad (1)$$

There have been a number of studies that have examined students' difficulties in understanding of logarithms. Our data collection and analysis focused on understanding and characterizing the meanings students constructed as they engaged in tasks and responded to questions that provided opportunities for reflection.

Our medical students from India, for the most part, prefer to write more short notation  $\ln x$  for all logarithms and do not use more long notation  $\log_a x$ . If  $c = e$  formula (1) has the form  $\log_a b = \frac{\ln b}{\ln a}$ . But it can make formulas and expressions very bulky. Undoubtedly, the common logarithm is more universal and convenient and students should not avoid it.

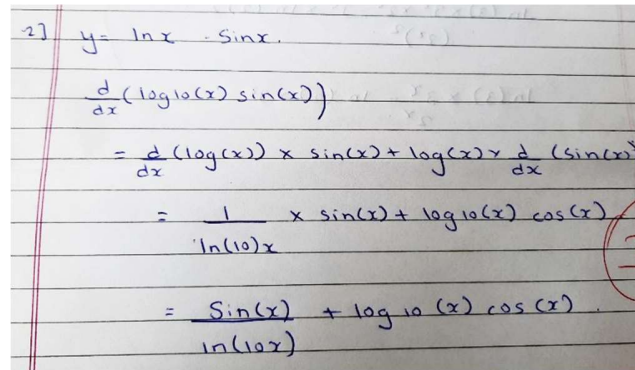
Some students use incorrect notation “log” without any base at all (Fig. 3a) or confuse designations “ln” and “lg” (Fig. 3b)

$$\int_3^8 \frac{1}{u} du$$

$$= [\log u]_3^8$$

$$= \log 8 - \log 3.$$

a)



2]  $y = \ln x - \sin x$

$$\frac{d}{dx} (\log_{10}(x) \sin(x))$$

$$= \frac{d}{dx} (\log(x)) \times \sin(x) + \log(x) \times \frac{d}{dx} (\sin(x))$$

$$= \frac{1}{\ln(10)x} \times \sin(x) + \log_{10}(x) \cos(x)$$

$$= \frac{\sin(x)}{\ln(10x)} + \log_{10}(x) \cos(x)$$

b)

Figure 3: Students' works

## 2.2. Inverse trigonometric functions ( $\sin^{-1}x$ vs $\arcsin x$ )

Inverse function is an important concept in secondary and post-secondary mathematics (Paoletti, 2020). Although inverse functions play an important role in many secondary mathematics curricula, but unfortunately students' understanding of inverse functions is limited. Many authors try to give the mental constructions using a unit circle approach to the sine, cosine, and their corresponding inverse trigonometric functions (Martinez-Planell & Delgado, 2016). Inverse trigonometric functions do the opposite of the “regular” trigonometric functions. For example, for the regular function  $y = \sin x$  (Fig. 4) the inverse function is  $y = \sin^{-1}x$  (Fig. 5) (see, e.g., (Weber et al., 2020)).

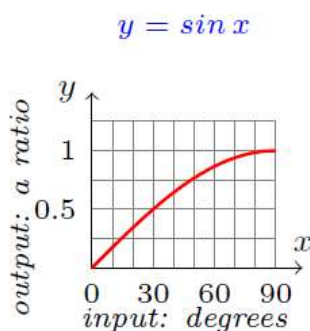


Figure 4: The graph of the function

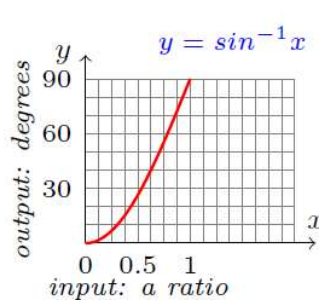


Figure 5: The graph of the inverse function

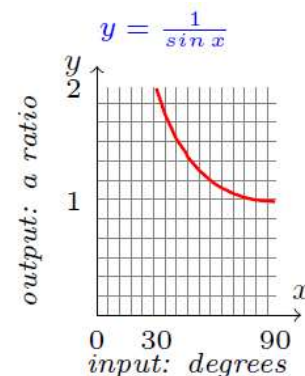


Figure 6: The graph of the function

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$$y = \sin x$$

$$y = \sin^{-1}x$$

$$y = \frac{1}{\sin x}$$

One particular source of confusion of students is the symbol “−1” in the inverse notation  $f^{-1}$ . Many students simply conflate the meanings of the superscript “−1” in functional and numerical settings, regarding  $f^{-1}(x) = 1/(f(x))$  (see, e.g., (Zazkis & Kontorovich, 2016)).

**Example 1.** *It is well known that  $\sin 30^\circ = 0,5$ . Consequently, we have the true equalities*

$$\sin^2 30^\circ = (\sin 30^\circ)^2 = 0,25.$$

*However, the inverse sine does not work that way:  $\sin^{-1} 30^\circ \neq (\sin 30^\circ)^{-1}$  because  $\sin^{-1} 30^\circ = \text{error}$  whereas  $(\sin 30^\circ)^{-1} = 2$ .*

There are two alternate notations for inverse trigonometric functions. The inverse sine can be expressed as  $\arcsin x$  or  $\sin^{-1}x$ . It means “The inverse sin of  $x$ ”. The expression  $\sin^{-1}x$  is not the same as  $\frac{1}{\sin x}$  (Fig. 6). In other words, the −1 is not an exponent. Instead, it simply means inverse function.

The inverse trigonometric functions are also called arcfunctions, as they return the unit circle arc length (in radians) for a particular value of sine, cosine, etc. and denoted by  $\arcsin x$ ,  $\arccos x$ , etc. These notations enable us to avoid the above-mentioned ambiguity.

In Table 1 one can see inverse trigonometric functions, which are used more often.

Notation 1	Notation 2
$\arcsin x$	$\sin^{-1}x$
$\arccos x$	$\cos^{-1}x$
$\arctan x$	$\tan^{-1}x$
$\text{arccotan} x$	$\text{cotan}^{-1}x$

Table 1: The table of various designations for inverse trigonometric functions

**Remark 2.** Carl Gauss also objected to this particular notational inconsistency. He proposed that  $\sin^2 x$  ought to mean  $\sin(\sin x)$ , whereas  $(\sin x)^2$  should be written in that way.

In our opinion, the form of these functions when denoted by Notation 1 (the first column in the Table 1) is much more unambiguous and correct than in the second. If we use Notation 2 we must say that here −1 is not an exponent, but it is not always convenient. And it can create additional problems, as if −1 is an exponent then  $\tan^{-1}x = \text{cotan} x$ , and  $\text{cotan}^{-1}x = \tan x$ . Another convention used by a few authors is to use a upper-case first letter along with a −1 superscript:  $\text{Sin}^{-1}x$ ,  $\text{Cos}^{-1}x$ ,  $\text{Tan}^{-1}x$ ,  $\text{Cotan}^{-1}x$ . This potentially avoids confusion with the multiplicative

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inverse, which should be represented by  $\sin^{-1}x$ ,  $\cos^{-1}x$ , etc. But it is not so successful way out as the difference in upper-case and lower-case letters can be not so appreciably in hand-written texts of students. Teachers should be careful with the notation for inverse trig functions.

**Remark 3.** The notations  $\arcsin x$ , etc. are common in computer programming languages. Moreover, this notation arises from the following geometric relationships: When measuring in radians, an angle of  $x$  radians will correspond to an arc whose length is  $rx$ , where  $r$  is the radius of the circle. Thus, in the unit circle, “the arc whose sine is  $x$ ” is the same as “the angle whose sine is  $x$ ”, because the length of the arc of the circle in radii is the same as the measurement of the angle in radians.

**Remark 4.** The notations  $\sin^{-1}x$ ,  $\cos^{-1}x$ ,  $\tan^{-1}x$ , and  $\cotan^{-1}x$  introduced by John Herschel are often used as well in English-language sources.

**Remark 5.** In the mathematical literature of Russia it is used  $tg x$  and  $ctg x$  instead of  $\tan x$  and  $\cotan x$ , and named inverse trigonometric functions using an arc- prefix:  $\arcsin x$ ,  $\arccos x$ ,  $\arctg x$ , and  $\text{arcctg } x$ .

We develop a more productive understanding of inverse functions in our work with students and draw students’ attention to the best ways to denote inverse functions. In order to avoid ambiguity and not to be confused it is better to use the notation for the inverse functions with the prefix “arc” (see., e.g., Edelstein-Keshet, 2020; Brenner & Lacay, 2016/17).

From our pedagogical experience, when studying the actions of students with inverse functions, we received the following pictures:

1. Before an explanation for what the *arcsine/arccosine* function represents (Fig. 7a).
2. After an explanation for what the *arcsine/arccosine* function represents (Fig. 7b-7e).

a)

$$\begin{aligned}
 y &= \cos^{-1} \sqrt{-3x+1} \\
 &= \frac{-1}{\sqrt{1-(-3x+1)^2}} \cdot \frac{1}{2} (-3x+1)^{-1/2} \cdot (-3) \\
 &= \frac{-1}{\sqrt{1+3x-1}} \cdot \frac{1}{2} (-3x+1)^{-1/2} \cdot -3 \\
 &= \frac{-1}{\sqrt{3x}} \cdot \frac{1}{2} \frac{(-3x+1)}{\sqrt{(-3x+1)}} \cdot -3
 \end{aligned}$$

b)

$$y = x \times \arcsin x.$$

$$(\sin^{-1}x)' = (\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$= 1 \times \arcsin x + \frac{1}{\sqrt{1-x^2}} \times x$$

$$= \arcsin x + \frac{x}{\sqrt{1-x^2}}$$

c)

$$y = \arctan \sqrt{e^x - 1}$$

$$= (\arctan \sqrt{e^x - 1})' = \frac{1}{1 + (\sqrt{e^x - 1})^2} \cdot (\sqrt{e^x - 1})'$$

$$= \frac{1}{1 + (e^x - 1)} \cdot \left[ \frac{1}{2} (e^x - 1)^{-1/2} \right] \cdot (e^x - 1)'$$

$$= \frac{1}{e^x} \cdot \frac{1}{2\sqrt{e^x - 1}} \cdot e^x = \frac{1}{2\sqrt{e^x - 1}}$$



d)

$$\textcircled{2} \int \arctan \sqrt{2x-1} \, dx = \left. \begin{array}{l} \sqrt{2x-1} = t \\ \frac{2}{2\sqrt{2x-1}} dx = dt \\ \frac{1}{t} dx = dt \\ dx = t dt \end{array} \right\} = \int t \cdot \arctan t \, dt$$

$$u = \arctan t$$

$$dv = t \, dt$$

$$du = \frac{1}{1+t^2} dt$$

$$v = \frac{t^2}{2}$$

$$vu = \int v \, du$$

$$\frac{t^2}{2} \arctan t - \int \frac{t^2}{2} \cdot \frac{1}{1+t^2} dt$$

$$\frac{t^2}{2} \arctan t - \frac{1}{2} \int \frac{t^2 + 1 - 1}{1+t^2} dt$$

$$\frac{t^2}{2} \arctan t - \frac{1}{2} \int 1 \, dt - \frac{1}{2} \int \frac{1}{1+t^2} dt$$

$$\frac{t^2}{2} \arctan t - \frac{1}{2} t - \frac{1}{2} (-\arctan t) + C$$

$$\frac{1}{2} (t^2 + 1) \arctan t - \frac{1}{2} t + C$$

$$\frac{1}{2} ((\sqrt{2x-1})^2 + 1) \arctan \sqrt{2x-1} - \frac{1}{2} \sqrt{2x-1} + C$$

$$x \arctan \sqrt{2x-1} - \frac{1}{2} \sqrt{2x-1} + C$$

e)

$$y = \sqrt{2x} \cdot \arccos x$$

$$y' = \frac{1}{2\sqrt{2x}} \cdot \arccos x + \sqrt{2x} \cdot \left( -\frac{1}{\sqrt{1-2x^2}} \right)$$

Figure 7. Inverse trigonometric functions in students' works

### 3. VIVID VISUAL IMAGES AS A MNEMONIC TECHNIQUE

Many authors (Arcavi, 2003; David & Tomaz, 2012; Kadunz & Yerushalmy, 2015) investigate how visual representations can structure maths activity in the classroom and discuss teaching practices that can facilitate students' visualization of mathematical objects. Some concepts simply should be presented visually rather than verbally (Clements, 1982; Leppink, 2017).

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We would like to highlight the importance of various visual mathematics approaches that are effective in mathematical education, e.g., bright memorable images for complicated mathematical notions. Some mathematical concepts are difficult to understand by words. Everyone knows that a picture is worth a thousand words. For students it is very important to understand mathematics intuitively. Ways must be found for them to learn mathematics that will promote intuitive understanding (Aso, 2001).

A theory of Howard Gardner (Gardner, H. (1983)) about multiple intelligences suggests that people have different approaches to learning, such as a visual, kinesthetic or logical approach. Thus, along with rigorous mathematical methods including formulas and proofs, mathematics teachers should use visuals, manipulative and motion to enhance students' understanding of mathematical concepts. All these aids help learners to boost their confidence and performance in maths (Boaler et al., 2016).

Maths classes are often composed entirely of symbol manipulation and the idea that visuals or manipulative are a mere prelude to abstract mathematics becomes instantiated. Calculus is often taught as a technical subject with rules and formulas (and occasionally theorems). Students are made to memorize maths facts, and plough through worksheets of numbers, with few visual or creative representations of mathematics or invitations to work visually. When non-mathematics students learn through visual approaches they are given access to deep and new understandings. Most of students reported that the visual activities enhanced their learning of mathematics. Normally, most of our students tell that they feel that mathematics is inaccessible and uninteresting when they are plunged into a world of abstraction and numbers. Someone might develop the idea that visuals and manipulative are babyish, and mathematical success is about memorizing numerical methods, but undoubtedly, visual aids are highly effective. A goal of any teacher is to make every student (especially non-mathematician) see that mathematics is not just a subject of numbers and symbols (Boaler et al., 2016; Matic, 2014).

### 3.1. Limits (Squeeze Theorem vs Sandwich Rule)

To calculate limits, we need the following

#### **Theorem 1.** *Squeeze Theorem*

If  $f(x) \leq g(x) \leq h(x)$  for all  $x$  in an open interval that contains  $x_0$  (except possibly at  $x_0$ ) and

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} h(x) = A,$$

then

$$\lim_{x \rightarrow x_0} g(x) = A.$$

#### **Proposition 1** *Sandwich Rule*

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For the best remembering Theorem 1 it is possible to use a picture of a sandwich (see Fig. 8). Here two functions  $f(x)$  and  $h(x)$  are pieces of bread, and  $g(x)$  is ham.

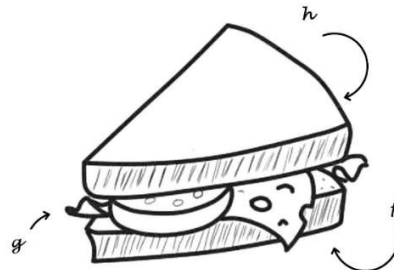


Figure 8: Sandwich

### 3.2. Differentiation of composite function (Chain Rule vs Nesting Doll Rule)

Differential calculus is an indispensable tool in every branch of science and engineering. Differential calculus is about describing in a precise fashion the ways in which related quantities change. In day to day life we are often interested in the extent to which a change in one quantity affects a change in another related quantity. This is called a rate of change.

The major motivations for introducing the differential calculus are problems of growth rate, reaction rate, concentration, velocity, and acceleration. There exists another group of problems, equally important for life scientists, which leads to the integral calculus. An integration in medicine is applied for describing, for example, cardiac output and Poiseuille's law.

Differential calculus is a procedure for finding the exact derivative directly from the formula of the function, without having to use graphical methods. In practise we use a few rules that tell us how to find the derivative of almost any function that we are likely to encounter (Thomas, 1997).

The composite function rule (also known as the chain rule) reads as follows: differentiate the "outside" function, and then multiply by the derivative of the "inside" function, i.e., if  $y$  is a function of  $g$  and  $g$  is a function of  $x$  then

$$\frac{dy}{dx} = \frac{dy}{dg} \cdot \frac{dg}{dx}$$

This makes the rule very easy to remember. The expressions  $\frac{dy}{dg}$  and  $\frac{dg}{dx}$  are not really fractions but rather they stand for the derivative of a function with respect to a variable. However, for the purposes of remembering the chain rule we can think of them as fractions, so that the  $dg$  cancels from the top and the bottom, leaving just  $\frac{dy}{dx}$ .

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Of course, the first step is always to recognise that we are dealing with a composite function and then to split up the composite function into its components.

To find the derivative of composite functions, we need the *Chain Rule*.

**Theorem 2 Chain Rule**

If  $g$  is differentiable at  $x$  and  $f$  is differentiable at  $y = g(x)$ , then the composite function

$$(f \circ g)(x) = f[g(x)] \quad (2)$$

is differentiable at  $x$ , and

$$(f \circ g)'(x) = f'[g(x)]g'(x). \quad (3)$$

In (Bittinger, M.L., Ellenbogen, D.J., Surgent, S.A. (2012)), authors give a very interesting visualization the composition of functions as a composition machine for functions  $f$  and  $g$ . To find  $(f \circ g)(x)$  we substitute  $g(x)$  for  $x$  in  $f(x)$ . The function  $g(x)$  is nested within  $f(x)$ .

Sometimes for non-mathematicians it is not so easy for comprehension. For better understanding and visual memorization, we propose to use the following rule. We call this rule the *Nesting Doll Rule*. As is known, a nesting doll (Russian Dolls, Stacking Dolls, Matryoshka) is a Russian wooden toy as a painted doll inside which there are dolls of the smaller size similar to it (see Fig. 9).

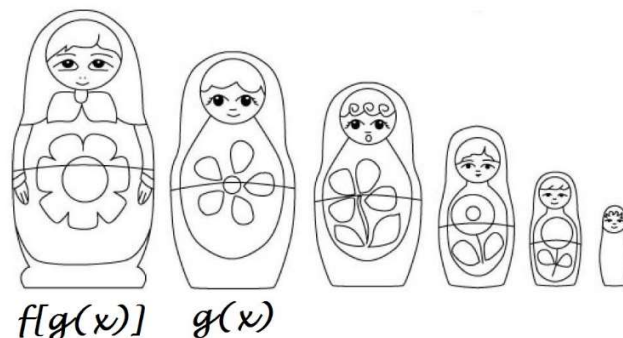


Figure 9: Nesting doll

Readers can ask: what is the connection between mathematics (namely, differentiation) and Russian souvenir? Let us consider an example.

**Proposition 2 Nesting Doll Rule**

Let in (2)  $f$  be the first (the biggest) nesting doll and  $g$  is the second one. First of all we can see only the biggest nesting doll. And we must take derivative of the function  $f$ . Then we open the first nested doll and we see the second one. And we must take derivative of the function  $g$ , etc. We should multiply all founded derivatives, thus we use formula (3).

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**Example 2** Find the derivative of the function  $y = \sin(3x + 1)$ .

Step 1. The function  $y = \sin(3x + 1)$  is a composite function. Let  $\sin(\dots)$  be the first nesting doll. The derivative of  $\sin(\dots)$  is equal to  $\cos(\dots)$ .

Step 2. And  $3x + 1$  is the second nesting doll. We have  $(3x + 1)' = 3$ .

Step 3. Finally we get:  $y' = (\sin(3x + 1))' = \cos(3x + 1) \cdot 3$ .

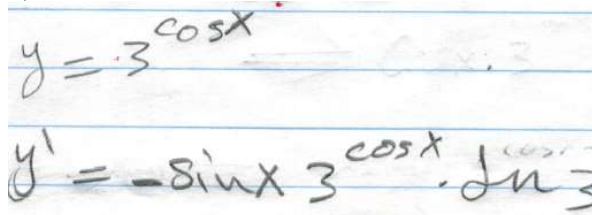
Mathematically the answer will be more correct in the form:  $y' = 3\cos(3x + 1)$ .

In other words, it looks like the outside function is the sine and the inside function is  $3x + 1$ , i.e.,

$$y' = \underbrace{\cos}_{\substack{\text{derivative of} \\ \text{outside function}}} \underbrace{(3x + 1)}_{\substack{\text{leave inside} \\ \text{function alone}}} \cdot \underbrace{3}_{\substack{\text{derivative of} \\ \text{inside function}}}$$

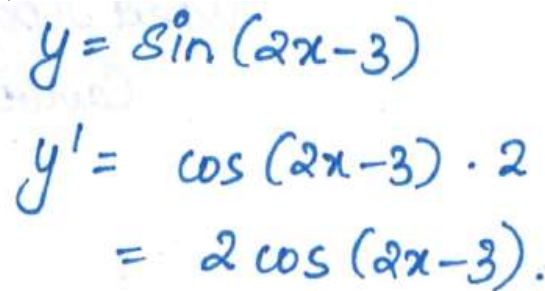
Students showed good results using the Nesting Doll Rule. Their results are presented in Fig. 10.

a)



$y = 3 \cos x$   
 $y' = -\sin x \cdot 3$

d)



$y = \sin(2x - 3)$   
 $y' = \cos(2x - 3) \cdot 2$   
 $= 2 \cos(2x - 3)$

b)

e)

$$4) \ y = \sqrt{1+2x}$$

$$y = (1+2x)^{\frac{1}{2}}$$

$$y' = \frac{1}{2}(1+2x)^{-\frac{1}{2}} \cdot 2$$

$$= (1+2x)^{-\frac{1}{2}}$$

$$3) \ y = \sin(4x+3)$$

$$\frac{dy}{dx} = \frac{d}{dx} [\sin(4x+3)]$$

$$\frac{dy}{dx} = \cos(4x+3) \cdot 4$$

$$\frac{dy}{dx} = \cos(4x+3) \cdot 4$$

c)

$$4) \ y = 2\sqrt{\sin x}$$

$$= 2(\sin x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 2 \left[ \frac{1}{2} (\sin x)^{-\frac{1}{2}} \cdot \cos x \right]$$

f)

$$y = \sin^2(7x+3)$$

$$y = 2\sin(7x+3) \cdot \cos(7x+3)$$

Figure 10: Students apply the Nesting Doll Rule

### 3.3. Concave functions (Concavity Theorem vs Smile Rule)

Graphs are a common method to visually illustrate relationships in data. Skill to read graphs is very important for the future physicians. It is known, that the second derivative of a function is related to the shape of its graph.

The important result that relates the concavity of the graph of a function to its derivatives is the following one.

#### Theorem 3 Concavity Theorem

Let a function  $f$  be twice differentiable at  $x = x_0$ . Then the graph of  $f$  is concave upward at  $(x_0, f(x_0))$  if  $f''(x_0) > 0$  and concave downward if  $f''(x_0) < 0$ .

A good way for explaining the notions of concavity up and concavity down (or convexity down and convexity up), which are very confusing, is to recall a parabola. Any student knows the

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branches of the parabola  $y = x^2$  look up. Indeed, it is well-known if the coefficient of  $x^2$  is positive, then the branches look up. On the other hand, it is easy to evaluate the first derivative  $y' = 2x$  and the second one  $y'' = 2$ . That is, the positive second derivative implies the branches look up, i.e., “ $\cup$ ”. Similarly, the negative second derivative implies the branches look down, i.e., “ $\cap$ ”. Positive (+) means up ( $\cup$ ), negative (−) means down ( $\cap$ ).

The best way for understanding distinctions between the sign of the second derivative (positive “+” or negative “−”) and the direction of concavity (up “ $\cup$ ” or down “ $\cap$ ”) is the following *Smile Rule*.

**Proposition 3 Smile Rule**

*If the second derivative of a function  $y = f(x)$  is positive on  $(a, b)$ , then the graph of  $y = f(x)$  on this interval “smiles”, i.e., “ $\cup$ ”.*

*If the second derivative of a function  $y = f(x)$  is negative on  $(a, b)$ , then the graph of  $y = f(x)$  on this interval “frowns”, i.e., “ $\cap$ ”.*

Students remember distinctions between two types of concavity if they use the “smile” and “frown” pictures (see Fig. 11).

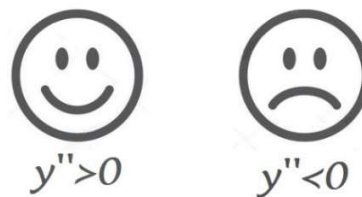


Figure 11: a) Concavity up    b) Concavity down

Indeed, positivity of the second derivative corresponds to positive emotion, since the symbol “ $\cup$ ” for concavity up is similar to “smile” (Fig. 11a). Negativity of the second derivative corresponds to negative emotion, since symbol “ $\cap$ ” for concavity down is similar to “frown” (Fig. 11b).

**4. SPECIAL FORMULAS AS A UNIVERSAL TOOL FOR SOLVING TYPICAL PROBLEMS**

At last, we promote universal formulas replacing lengthy multi-step algorithms.

Algebra provides us with the ability to deal with formulas that always work. This can relieve us from the burden and messiness of having to muck about with the numbers every single time we do the exact same thing. The examples of square equations and systems of linear differential equations demonstrate the advantage of the formula over the algorithm for medical students.

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#### 4.1. Factorization (Grouping vs Discriminant)

All the way through tertiary level mathematics, quadratic expressions routinely appear and so being able to quickly factor them is a basic skill.

Quadratic equations are considered important in mathematics curricula because they serve as a bridge between different mathematical topics. In general, for some students, quadratic equations create challenges in various ways such as difficulties in algebraic procedures (particularly in factoring quadratic equations) (Didis & Erbas, 2015; Kachapova et al., 2007).

The general quadratic function has the form

$$y(x) = ax^2 + bx + c \quad (a \neq 0) \quad (4)$$

with three constants  $a, b, c \in \mathbb{R}$ . The right-hand expression in (4) is a polynomial of the second degree in  $x$  three termed quadratic (i.e., trinomial). The graph of this function is a parabola.

Sometimes it is necessary to find for what values of  $x$  the equality  $y(x) = ax^2 + bx + c = 0$  holds? Or, where does the quadratic parabola intersect the  $x$  axis? In order to answer these questions we have to solve the quadratic equation

$$ax^2 + bx + c = 0 \quad (a \neq 0). \quad (5)$$

In many mathematical tasks it is important to represent a quadratic polynomial as a product of two linear polynomials. To factorize a quadratic equation is to find what to multiply to get the quadratic one. There are a number of different techniques for factoring this type of expression.

1. The solutions of (5) can be found by the Quadratic Formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (6)$$

The expression the square root of which must be taken is called *the discriminant* of the quadratic equation. It is denoted by  $D$ . Thus,  $D = b^2 - 4ac$ .

When the discriminant  $D$  is

1. positive, there are two different real solutions  $x_1 \neq x_2$ ;
2. zero, there are two equal real solutions  $x_1 = x_2$ ;
3. negative, there are two complex conjugate solutions  $x_1$  and  $x_2$ .

**Remark 6.** Vieta's formulas can be also helpful. The roots  $x_1$  and  $x_2$  of the quadratic polynomial (4) satisfy the following relations:



$$x_1 + x_2 = -\frac{b}{a}, \quad x_1 \cdot x_2 = \frac{c}{a}. \quad (7)$$

From (7) we can find the roots  $x_1$  and  $x_2$  of polynomial (4).

Using (6) one can find two roots  $x_1$  and  $x_2$  and rewrite (4) in the following way

$$y(x) = ax^2 + bx + c = a(x - x_1)(x - x_2).$$

2. The factorization (6) can be accomplished also by grouping. First of all it is necessary to check if there exist any common factors. But it is not always easy.

We may apply the following method: to find two numbers that multiply to give  $ac$ , and add to give  $b$ . Let  $\alpha$  and  $\beta$  be the real numbers. Then

$$ax^2 + bx + c = ax^2 + \alpha x + \beta x + c.$$

We factor first two and last two terms and find the common factor.

**Example 3** Factorize  $6x^2 + 5x - 6$ .

Step 1.  $ac = 6 \cdot (-6) = -36$  and  $b = 5$ . List the positive factors of  $ac = -36$ : 1, 2, 3, 4, 6, 9, 12, 18, 36. One of the numbers has to be negative to make  $-36$ , so by playing with a few different numbers we find that  $-4$  and  $9$  work nicely:  $-4 \cdot 9 = -36$  and  $-4 + 9 = 5$ .

Step 2. Rewrite  $5x$  as  $-4x$  and  $9x$ :  $6x^2 - 4x + 9x - 6$ .

Step 3. Factor first two and last two terms:  $2x(3x - 2) + 3(3x - 2)$ .

Step 4. Common factor is  $(3x - 2)$ , that is  $(2x + 3)(3x - 2)$ .

The nice thing about the Quadratic Formula is that the Quadratic Formula always works. As compared to completing the square, we're just plugging into a formula. There are no "steps" to remember, and thus there are fewer opportunities for mistakes.

But it is important to apply the Quadratic Formula correctly. For this purpose there are the following helpful recommendations for students. 1) Take care not to omit the  $\pm$  sign in front of the radical. 2) Don't draw the fraction line as being only under the square root, because it is under the initial  $-b$  part, too. 3) Don't forget that the denominator of the Formula is  $2a$ , not just 2. That is, when the leading term is something like  $5x^2$ , you will need to remember to put the  $a = 5$  value in the denominator. 4) Use parentheses around the coefficients when you're first plugging them into the Formula, especially when any of those coefficients is negative, so you don't lose any "minus" signs. 5) When using the Formula, take the time to be careful because, as long as you do your work neatly, the Quadratic Formula will give you the right answer every time.

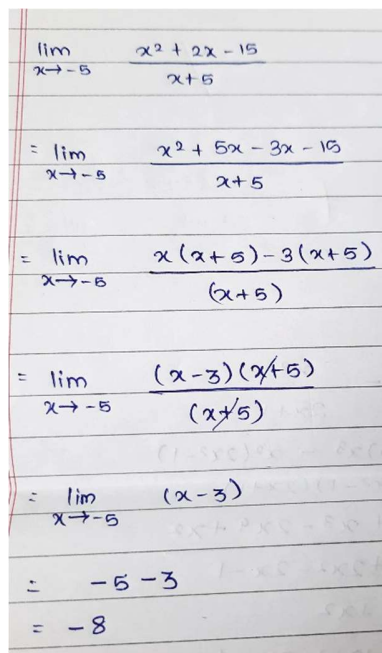
Specially for auditory learners there is a song to help remembering the Quadratic Formula, set to the tune of “Pop Goes the Weasel”:

- $x$  is equal to negative  $b$
- Plus or minus the square root
- Of  $b$ -squared minus four  $ac$
- All over two  $a$ .

In the real world though, we always use the quadratic formula. Factoring by grouping is almost always completely useless in any kind of an experimental or scientific scenario. The reason why we teach factoring by grouping is to give students at least some exposure in high school to Diophantine Equations. As such, it is an important part of their education.

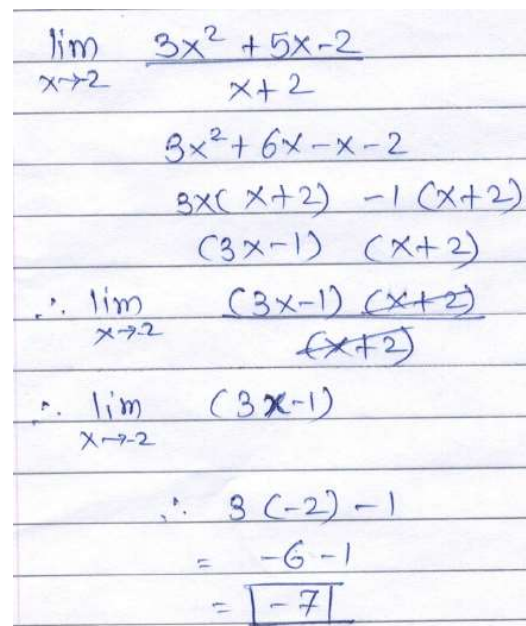
**Remark 7.** Unlike Russian tradition our medical students from India prefer finding the common factor for computing  $D$ .

More than 90 percents of Indian students from the selected group do not find roots of a quadratic function for its factorization (compare Fig. 12 and 13).



$$\begin{aligned} \lim_{x \rightarrow -5} \frac{x^2 + 2x - 15}{x + 5} \\ &= \lim_{x \rightarrow -5} \frac{x^2 + 5x - 3x - 15}{x + 5} \\ &= \lim_{x \rightarrow -5} \frac{x(x + 5) - 3(x + 5)}{(x + 5)} \\ &= \lim_{x \rightarrow -5} \frac{(x - 3)(x + 5)}{(x + 5)} \\ &= \lim_{x \rightarrow -5} (x - 3) \\ &= -5 - 3 \\ &= -8 \end{aligned}$$

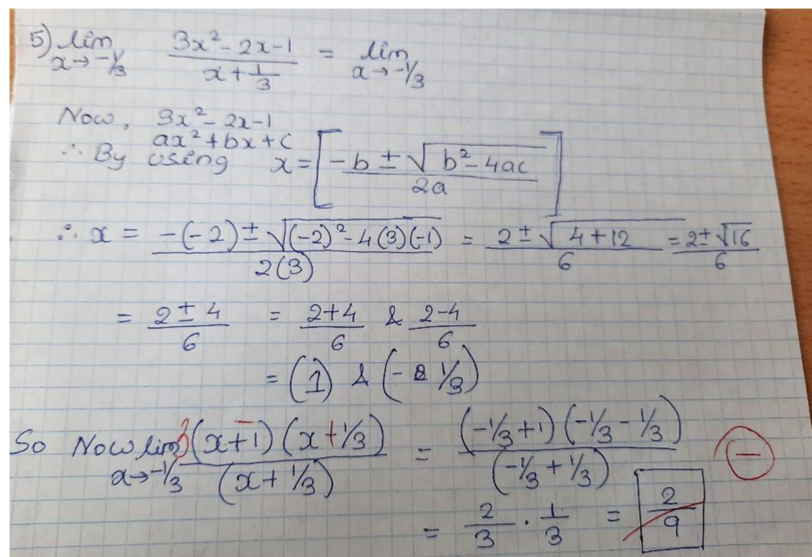
a)



$$\begin{aligned} \lim_{x \rightarrow 2} \frac{3x^2 + 5x - 2}{x + 2} \\ &= \frac{3x^2 + 6x - x - 2}{x + 2} \\ &= \frac{3x(x + 2) - 1(x + 2)}{(x + 2)} \\ &= \frac{(3x - 1)(x + 2)}{(x + 2)} \\ \therefore \lim_{x \rightarrow 2} (3x - 1) \\ &= 3(-2) - 1 \\ &= -6 - 1 \\ &= \boxed{-7} \end{aligned}$$

b)

Figure 12: Usual solutions for Indian students



5)  $\lim_{x \rightarrow -\frac{1}{3}} \frac{3x^2 - 2x - 1}{x + \frac{1}{3}} = \lim_{x \rightarrow -\frac{1}{3}}$

Now,  $3x^2 - 2x - 1$   
 $\therefore$  By using  $ax^2 + bx + c = \left[ \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right]$

$\therefore x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-1)}}{2(3)} = \frac{2 \pm \sqrt{4 + 12}}{6} = \frac{2 \pm \sqrt{16}}{6}$

$= \frac{2 \pm 4}{6} = \frac{2+4}{6} \text{ \& } \frac{2-4}{6}$   
 $= (1) \text{ \& } \left(-\frac{1}{3}\right)$

So Now  $\lim_{x \rightarrow -\frac{1}{3}} \frac{(x+1)(x+\frac{1}{3})}{(x+\frac{1}{3})} = \frac{(-\frac{1}{3}+1)(-\frac{1}{3}-\frac{1}{3})}{(-\frac{1}{3}+\frac{1}{3})}$  -

$= \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$

Figure 13: Unusual solution for Indian students

#### 4.2. Differential equations in medicine and biology

Almost every real-world system can be modelled by differential equations (Chasteen-Boyd). Differential equations occur frequently in the analysis of physiological systems and of ecological systems (see Table 2).

The form of equation	Application
$y' = ay$	growth of a cell, a birth process, a birth-and-death process, radioactive decay, living tissue exposed to ionizing radiation, radioactive tracer, dilution of a substance, chemical kinetics;
$y' = ay + b$	restricted growth, a birth-and-immigration process, cooling, a diffusion problem, nerve excitation;
$y' = ay^2 + by + c$	restricted growth, spread of infection, chemical kinetics, autocatalysis;
$\frac{dy}{dx} = k \frac{y}{x}$	relative growth of parts of a body, metabolism, dose-response problems, racial differences, evolutionary history;

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<p>A system of linear differential equations</p> $\frac{dx}{dt} = ax + by,$ $\frac{dy}{dt} = cx + dy,$ <p><math>a, b, c, d</math> are given constants.</p>	<p>ecology, passage of food in ruminants, excretion of a drug.</p>
------------------------------------------------------------------------------------------------------------------------------------------------------------	--------------------------------------------------------------------

Table 2: On application of differential equations in medicine

There are lists of differential equations and their solutions available (Kamke, 1942, Kamke, 1956). Solutions of some differential equations cannot be written in a manageable form. We teach our student to solve these equations by computers.

#### 4.3. Systems of linear differential equations (Substitutions vs Determinants)

Let us consider a systems of linear first-order differential equations with constant coefficients

$$\begin{aligned} y' &= a_1y + b_1z + f_1, \\ z' &= a_2y + b_2z + f_2, \end{aligned} \quad (8)$$

where  $y = y(x)$ ,  $z = z(x)$  are unknown functions,  $a_1, a_2, b_1, b_2$  are constants,  $f_1 = f_1(x)$ ,  $f_2 = f_2(x)$  are given functions. Differentiating the first equation with respect to  $x$  we obtain

$$y'' = a_1y' + b_1z' + f_1'.$$

By substituting the expression of  $z'$  (8) we can eliminate  $z'$  from the last equation

$$y'' = a_1y' + b_1(a_2y + b_2z + f_2) + f_1'. \quad (9)$$

Express  $z$  from the first equation of (8)

$$z = (y' - a_1y - f_1)/b_1. \quad (10)$$

Eliminate  $z$  from equation (9) substituting (10) into equation (9). Then

$$y'' = a_1y' + b_1a_2y + b_1b_2(y' - a_1y - f_1)/b_1 + b_1f_2 + f_1'.$$

Regroup the terms and rewrite the last equation as follows

$$y'' - (a_1 + b_2)y' + (a_1b_2 - b_1a_2)y = f_1' + b_1f_2 - b_2f_1. \quad (11)$$

This algorithm can be applied for every system of the form (8). Also for system (8) written in matrix notation  $\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$  one can easily make up the linear non-homogeneous second-order differential equation with constant coefficients (11) written in the form

$$y'' - (a_1 + b_2)y' + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} y = f'_1 + \begin{vmatrix} b_1 & f_1 \\ b_2 & f_2 \end{vmatrix}, \quad (12)$$

based on the coefficients and functions of system (8). For finding  $z$  we use (10).

Hence, in order to solve (8) one can either take all the above-mentioned multi-step algorithm obtaining (9), (10), and (11) consistently or use the universal formula (12) in the stated below example. The first path is no doubt longer, more tricky and bulky than the second one based on (12). The formula (12) is easier than the multi-step algorithm (9)–(12).

**Example 4** Solve the system  $y' = 8y - 9z + 3x$ ,  $z' = 7y - 8z + 2x$ .

Our matrices have the form  $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 8 & -9 \\ 7 & -8 \end{pmatrix}$ ,  $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 3x \\ 2x \end{pmatrix}$ .

Calculate all coefficients for the final equation (12):

coefficient of  $y'$  and  $y$ :  $a_1 + b_2 = 8 - 8 = 0$  and  $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 8 & -9 \\ 7 & -8 \end{vmatrix} = -1$ ,

the right-hand side:  $f'_1 + \begin{vmatrix} b_1 & f_1 \\ b_2 & f_2 \end{vmatrix} = 3x' + \begin{vmatrix} -9 & 3x \\ -8 & 2x \end{vmatrix} = 6x + 3$ .

Make up the final equation  $y'' - y = 6x + 3$ .

Then one should write auxiliary (or characteristic) equation  $k^2 - 1 = 0$ , find the general solution for the homogeneous equation  $y'' - y = 0$  and a particular solution for non-homogeneous equation  $y'' - y = 6x + 3$ , and at last the general solution for the non-homogeneous equation (see (Adams & Essex, 2013)).

In particular, for solving a system of homogeneous linear first-order differential equations with constant coefficients

$$y' = a_1y + b_1z,$$

$$z' = a_2y + b_2z$$

one can easily make up the linear homogeneous second-order differential equation with constant coefficients

$$y'' - (a_1 + b_2)y' + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} y = 0.$$

The last formula is memorized easily and it is very simple to use.

## 5. RESULTS AND DISCUSSION

In teaching medical students, different methods of explanation were chosen in two different groups. The content of the first method was strictly mathematical, while the second method allowed and welcomed all kinds of funny images from the world around us. Our students struggled to understand the concepts through definitions, but that embodied, visual ideas proved a valuable adjunct to their thinking. The results are presented in the form of the histogram (see Fig. 14). Here “Traditional explanation” means theorems prevail over vivid rules, and “Special explanation” means vivid rules prevail over strict theorems.

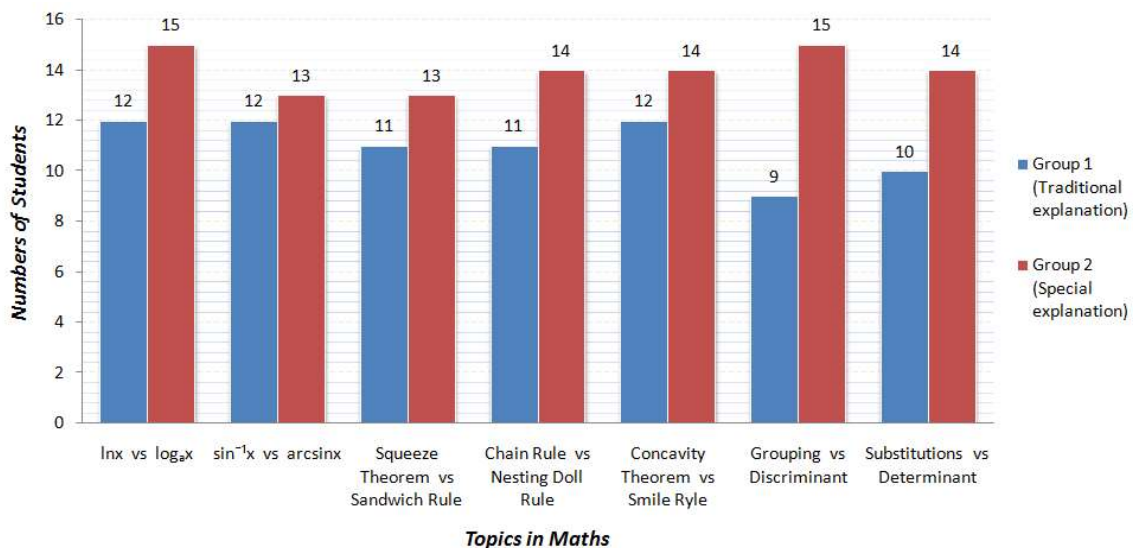


Figure 14: Comparative analysis of the results of mastering topics

93.3% of our students were able to solve problems using some special techniques, as compared to 73.3% for students who followed the standard course of study. In the first group we had taught maths in the conventional way. In the second group, we had implemented, among other things, images which encouraged students to favour visualization. Thus, we see a difference of 20%.

## 6. CONCLUSIONS

The main purpose of teaching is to stimulate students to perceive mathematics as an indispensable part of the medical curriculum and profession and to provide an education adapted, as far as possible, to the needs and demands of the students and to interest as many of them as possible. It

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requires from teachers a steady improvement in their own culture and teaching methods. This, in turn, calls for the regular updating of the content of the teaching and the regular exchange of experiences.

Our study has shown that performance of students of the first group was poor in mathematical calculations compared to the students from the second group. It was found that students in the second group demonstrated the better mathematical knowledge. Thus, the alternative ways of memorizing formulas and procedures improve multimodal learning abilities of students.

From this comparative analysis, it can be concluded that all kinds of techniques and approaches including adequate designations, vivid visual images and universal formulas enhance students' understanding and improve their performance in the study of mathematics (see Kachapova et al., 2007).

Mathematics plays a significant role in the advancement of medicine. Mathematical reasoning can contribute to medicine in many ways. It may enable physicians, on the one hand, to obtain quantitative estimates in situations where their information has previously been only qualitative, and on the other hand, to find qualitative interpretations for purely quantitative measurements. And sometimes pure mathematical research may produce beneficial practical applications, some of which may lead to new innovations and even life-saving technologies. For this reason it is very important for medical students to study mathematics. And an essential and crucial factor is to give mathematical knowledge in the vivid, accessible, and clear image. We hope that our experience, a small collection of techniques and amusing tricks presented above will turn out to be useful to colleagues and students. Now we know the answer to the question: "How mathematics should be taught to non-mathematicians?"

The authors' experience indicates that famous theorems and conjectures with origins in both pure and applied mathematics have the potential to trigger the imagination and thought process of those whose minds are open to challenge, and thus can be utilized appropriately as useful didactical tools (Abramovich & Grinshpan, 2008).

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## The Problem Corner

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The Purpose of **The Problem Corner** is to give Students and Instructors working independently or together a chance to step out of their “comfort zone” and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor [iretamoso@bmcc.cuny.edu](mailto:iretamoso@bmcc.cuny.edu) stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem as an attachment to The Problem Corner Editor [iretamoso@bmcc.cuny.edu](mailto:iretamoso@bmcc.cuny.edu) stating your name, institutional affiliation, city, state, and country.

Hello Problem Solvers, I got solutions to **Problem 3**, and I am happy to inform that they were correct, interesting, and ingenious. By posting different solutions, I hope to enrich and enhance the mathematical knowledge of our international community.

Solutions to **Problem** from the Previous Issue

### Interesting Geometric Problem with a surprising solution.

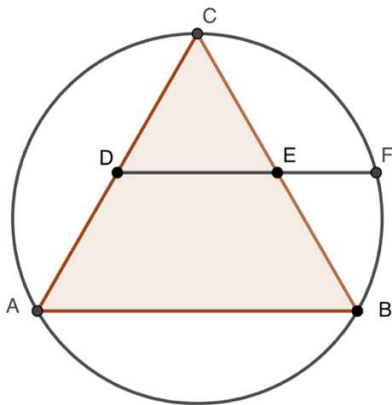
Proposed by Aradhana Kumari Borough of Manhattan Community College, City university of New York, USA

#### Problem 3

Triangle ABC is an equilateral triangle inscribed in a circle. D and E are the mid points of sides AC and BC respectively. Find the ratio, length DF : length DE?

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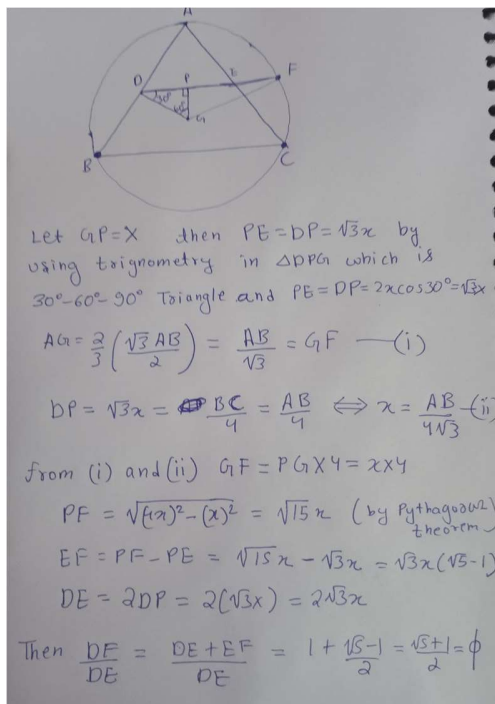


### Solution 1

by Jayendra Jha, Arihant Public School, India

and Sankalp Savaran, Shiv jyoti Senior Secondary School, India.

This solution, interestingly, combines Geometry (Centroid property and the Theorem of Pythagoras) and Trigonometry (sine and cosines of angles  $30^\circ$ , and  $60^\circ$ ), and some basic algebra, ending up with the solution in exact form.



Let  $GP = x$  then  $PE = DP = \sqrt{3}x$  by using trigonometry in  $\triangle PGE$  which is  $30^\circ-60^\circ-90^\circ$  Triangle and  $PE = DP = 2x \cos 30^\circ = \sqrt{3}x$

$$AG = \frac{2}{3} \left( \frac{\sqrt{3} AB}{2} \right) = \frac{AB}{\sqrt{3}} = GF \quad \text{--- (i)}$$

$$DP = \sqrt{3}x = \frac{BC}{4} = \frac{AB}{4} \Leftrightarrow x = \frac{AB}{4\sqrt{3}} \quad \text{--- (ii)}$$

from (i) and (ii)  $GF = PG \times 4 = 4x$

$$PF = \sqrt{(4x)^2 - (x)^2} = \sqrt{15}x \quad \text{(by Pythagoras theorem)}$$

$$EF = PF - PE = \sqrt{15}x - \sqrt{3}x = \sqrt{3}x(\sqrt{5}-1)$$

$$DE = 2DP = 2(\sqrt{3}x) = 2\sqrt{3}x$$

Then  $\frac{DE}{EF} = \frac{DE+EF}{DE} = 1 + \frac{\sqrt{5}-1}{2} = \frac{\sqrt{5}+1}{2} = \phi$

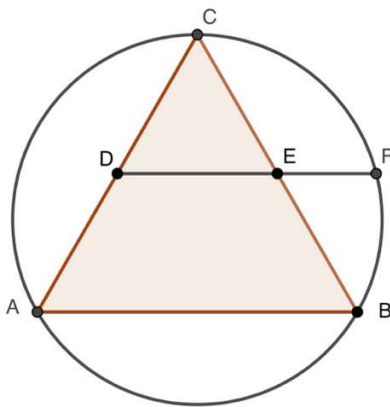
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### Solution 2

by Aradhana Kumari, Borough of Manhattan Community College, City university of New York, USA (The proposer).

This solution is based on a clever change of variable, an auxiliary extension of a segment together with the Intersecting chords theorem.



Let the length of sides of the equilateral triangle ABC as  $2x$ . Since D is the midpoint of CA and E is the midpoint of CB therefore the length of CD and CE is  $x$ .

In the triangle CDE,

length CD = length CE =  $x$

hence *angle* CDE = *angle* CED

since the angle DCE is  $60^\circ$

*angle* CDE + *angle* CED +  $60^\circ = 180^\circ$

*angle* CDE + *angle* CDE +  $60^\circ = 180^\circ$

$2 \times \text{angle CDE} = 120^\circ$

Angle CDE =  $60^\circ$

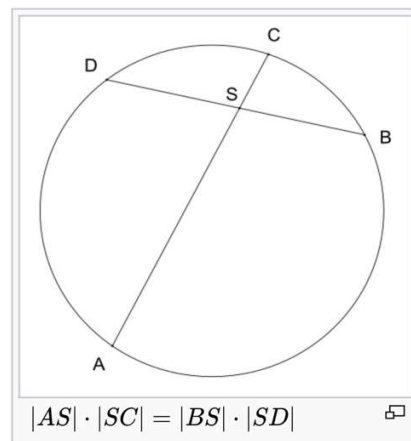
Hence triangle CDE is an equilateral triangle with side lengths  $x$ .

Let the length of EF =  $y$  then length of DF =  $x+y$  and length of DG =  $y$ .

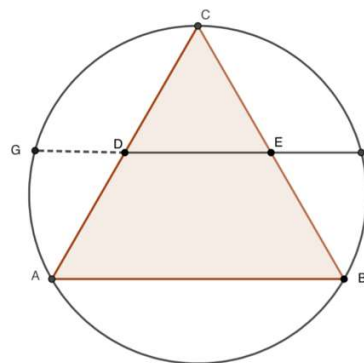
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Intersecting chord theorem: If two chords intersect in a circle, then the products of the measures of the segments of the chords are equal. (The below picture is taken from Wikipedia.)



In the below diagram the two chords GF and CA are intersecting at D.



Hence by intersecting chord theorem, we have

length of GD × length DF = length CD × length DA

$$y(x+y) = x \cdot x$$

$$\frac{(x+y)}{x} = \frac{x}{y}$$

$$\frac{x}{x} + \frac{y}{x} = \frac{x}{y} \dots\dots (1)$$

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Substitute  $\frac{x}{y} = \alpha$  in the above equation given by (1)

We get  $1 + \frac{1}{\alpha} = \alpha$

After simplifying we get  $\alpha + 1 = \alpha^2$

or  $\alpha^2 - \alpha - 1 = 0$

therefore  $\alpha = \frac{1+\sqrt{5}}{2}$  hence  $\frac{x}{y} = \frac{1+\sqrt{5}}{2}$

Therefore

$$\frac{\text{length } DF}{\text{length } DE} = \frac{x+y}{x} = \frac{x}{y} = \frac{1+\sqrt{5}}{2}$$

Note:  $\frac{1+\sqrt{5}}{2}$  is also known as golden ratio.

### Solution 3

by Ivan Retamoso, Borough of Manhattan Community College, USA (Editor of *The Problem Corner*).

This solution uses an auxiliary line and exploits the symmetry and the independence of the units of measurements, since the solution is a ratio.

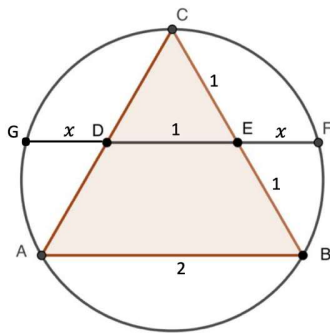
Since we are looking for a ratio, without loss of generality, let the side length of the equilateral triangle be 2 units.

Then  $\overline{AB} = 2$ ,  $\overline{DE} = 1$ ,  $\overline{EC} = 1$ , and  $\overline{EB} = 1$

Let's extend  $DE$  to the left, where  $DE$  meets the circle let's call this point  $G$ , let  $x$  be the length  $EF$  and  $GD$  which are the same due to Symmetry as shown in the figure below

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By The Intersecting Chords Theorem

$$\overline{GE} \cdot \overline{EF} = \overline{CE} \cdot \overline{EB}$$

$$(x + 1) \cdot x = 1 \cdot 1$$

$$x^2 + x = 1$$

$$x^2 + x - 1 = 0$$

$$x = \frac{-1 + \sqrt{5}}{2}$$

Then

$$\frac{\overline{DF}}{\overline{DE}} = \frac{1 + \frac{-1 + \sqrt{5}}{2}}{1}$$

Then

$$\frac{\overline{DF}}{\overline{DE}} = \frac{1 + \sqrt{5}}{2}$$

Note:

The number  $\frac{1 + \sqrt{5}}{2}$  is “The Golden Ratio”, amazing!

Dear Problem Solvers,

I really hope you enjoyed solving Problem 3 as much as I did, below are the next two problems, I am happy to tell you that a Canadian professor has proposed a “proof” problem, I must warn you it is a little advance, but it is accompanied with hints and graphs as help.

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#### Problem 4

Proposed by Ivan Retamoso, BMCC, USA

In a cartesian plane, between the half of the parabola  $y = \frac{x^2}{2}$  for  $x \geq 0$  and the  $x$  – axis there is a circle tangent to the parabola at the point  $(2,2)$  and to the  $x$  – axis, find the radius of the circle.

#### Problem 5

Proposed by Mohsen Soltanifar, Adjunct Instructor, Continuing Studies Division, University of Victoria, Victoria, BC, Canada

Let  $f(x) = x^x$  ( $x > 0$ ) be the second tetration function. Prove that  $f$  is continuous merely using the  $\epsilon - \delta$  definition.

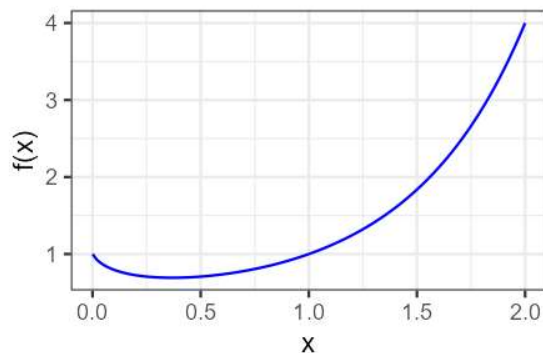


Figure 1: The plot of the second tetration function  $f(x) = x^x$  ( $x > 0$ ).

#### Hint:

Step (i) Prove that the logarithm function  $\ln(\cdot)$  is continuous using the  $\epsilon - \delta$  definition, and save  $\delta = \delta(\epsilon)$ .

Step (ii) Prove that the exponential function  $\exp(\cdot)$  is continuous using the  $\epsilon - \delta$  definition, and save  $\delta = \delta(\epsilon)$ .

Step (iii) Prove that if the function  $g(\cdot)$  is continuous at  $x = a$  and the function  $f(\cdot)$  is continuous at  $y = g(a)$ , then the function  $f \circ g(\cdot)$  is continuous at  $x = a$ , using the  $\epsilon - \delta$  definition.

Step (iv) Use steps (i),(ii) and (iii) for  $f(x) = \exp(x)$ , and  $g(x) = x \ln(x)$  in reversed method to prove the statement for the second tetration function.



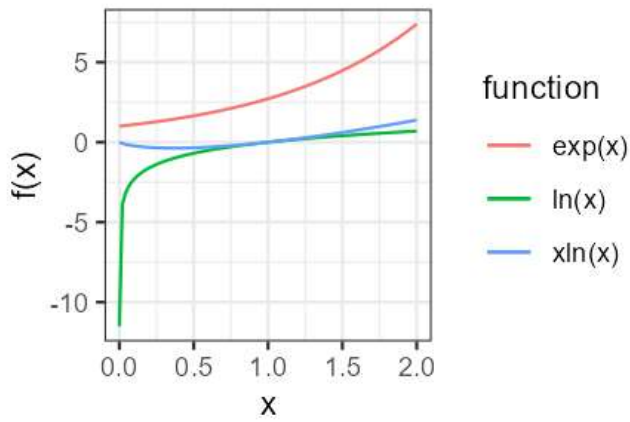


Figure 2: The plot of the three functions  $f(x) = \exp(x), \ln(x), x \ln(x), (x > 0)$ .

## The Theory on Loops and Spaces. Part 1.

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*Abstract: We are all fascinated by loops and their formation in space. When a line cuts itself, it forms an intersection point and creates a space. This is an experimental study done by analyzing several loops, forming a concrete formulation by visualizing the patterns observed, and then proving the formulations proposed using the known standard mathematical methods. This piece of mathematics is studied under graph theory and forms the basis for understanding and developing thinking of the graph theory at the elementary level of mathematics. This article develops the thinking behind how to analyze patterns in nature and write them in the form of mathematical statements or formulas. This article has been inspired by a YouTube video posted by the mathematician Dr. James Tanton on 27th Sept. 2021 on his YouTube Channel.*

### 1. INTRODUCTION

We have all been fascinated by loops and curves in space. “A loop is a path whose initial and terminal point is the same.” It may or may not cut itself. We do not need to lift the pencil while drawing a loop. Now when two lines intersect at a point is called an Intersection Point. So, while drawing a loop, it is possible that a line can cut itself at one or more than one point. It is equally probable that a line may cut the exact intersection point multiple times. This leads to a few interesting questions in our mind!

### 2. QUESTIONS/ PROPOSALS:

1. Is the theory being, Pieces + Intersection = Spaces?
2. Can we prove that if we draw a group of several loops that intersect at least once such that no loop can be isolated, then we can draw the entire picture of loops without lifting our pencil?
3. Can we prove that every picture we draw can be two colorable such that no two regions can share a section of boundaries of the same color?
4. Can we find the relation that while tracing a loop, we pass through an intersection point ‘P,’ then the number of intersections passed before reaching ‘P’?
5. Can we find the relation to find the sum of intersection points we have passed starting from P and reaching P again?
6. Can the above results work in Higher Dimensions also?

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Before we begin, let us introduce “The value of a point” and a “Piece.”

### 3. VALUE OF A POINT AND A PIECE:

If we consider a point as a source of two or more rays emanating from it in opposite directions, like if we take a point on a line, then we have two rays emanating from that point. Similarly, a point with two intersecting lines will have four rays emanating from it. Now the Value of a Point, ‘P,’ will be evaluated as:

$$V(P) = 1 + \frac{n-4}{2} = \frac{n-2}{2} \quad (1)$$

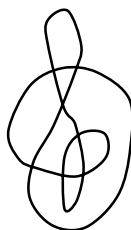
Where n is the number of rays emanating from the point ‘P.’ The Value of a point is non-zero only at intersection points; the rest everywhere is zero.

We define a “Piece” as a single loop or collection of loops such that no loop can be isolated from the group.

We will always restrict ourselves to only “one-piece,” as calculating spaces for one piece and then adding to get the final number of spaces is more effective than calculating spaces for multiple pieces simultaneously.

### 4. TESTING THE THEORY: PIECES + INTERSECTION = SPACES:

Whenever we draw a loop, we either cut a line to form an intersection point or form a closed curve without any intersection point (e.g., Circle). Let us take the following examples:



No. of pieces: 01  
No. of Intersection points: 07  
No. of Spaces:  $7+1 = 8$

Figure 1: Example 1



No. of pieces: 01  
No. of Intersection points: 06  
No. of Spaces:  $6+1 = 7$

Figure 2: Example 2

#### 4.1 HYPOTHESIS:

From the above example, we can find a relation as:

Given a loop, let the value of intersection points be  $V_1, V_2, V_3 \dots V_n$ , also let No. of intersection points with values  $V_1, V_2, V_3, \dots V_n$  be  $N_1, N_2, N_3, \dots N_n$  respectively, then:

$$\text{Number of Spaces (S)} = \text{No. of Pieces (1)} + N_1V_1 + N_2V_2 + N_3V_3 + \dots + N_nV_n$$

$$S = 1 + \sum_{k=1}^n N_k V_k \quad (2)$$

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Before we begin the proof, let us look at the following postulate:

#### 4.1.1 POSTULATE

The Value 'V' of an Intersection Point 'P' must be a Natural Number.

$$V(P) \in \mathbb{N} \quad (3)$$

#### 4.1.2 PROOF OF HYPOTHESIS 4.1, PROOF BY CONSTRUCTION:

Suppose we draw a closed loop with no intersection point (E.g., a Circle), then we have created one space inside the Loop (Figure 3).

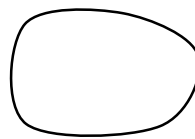


Figure 3: A closed loop with no intersection point having one space within it

If we draw another loop with one intersection point with value  $V_1$ , we will observe that we have created  $(1 + V_1)$  Spaces (Figure 4).

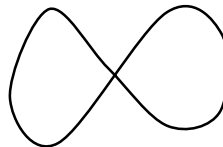


Figure 4: Example of a closed loop having 2 Spaces, ( $N_1 = 1$ ;  $V_1 = 1$ )

Similarly, if we draw another loop with  $N_1$  intersection points, each having value  $V_1$ , we will observe that we have created  $(1 + N_1 V_1)$  Spaces (Figure 5).

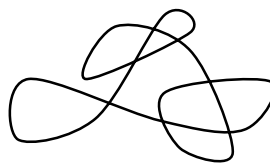


Figure 5: Example of a closed loop having 8 Spaces, ( $N_1 = 7$ ;  $V_1 = 1$ )

If we generalize it more by drawing another loop having  $(N_1 + N_2)$  Number of Intersection points with  $N_1$  points having value  $V_1$  and  $N_2$  points having value  $V_2$ , we will observe that we have created  $(1 + N_1 V_1 + N_2 V_2)$  Spaces (Figure 6).

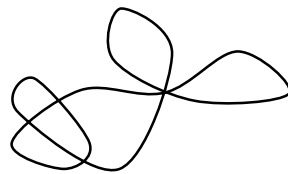


Figure 6: Example of a closed loop having 6 Spaces, ( $N_1 = 3$ ;  $V_1 = 1$ ;  $N_2 = 1$ ;  $V_2 = 2$ )

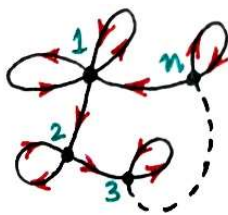


Figure: 7

Now, let us simplify things radically further. On a Blank Piece of paper, we slowly start drawing a self-intersecting loop. Every time the loop intersects itself, we write +1 in the tally of intersection points whether it has created a new intersection point or crossed the same intersection point again, +1 in the tally of value of point, and +1 for the tally of regions, as one intersection point will create at least one region. When we complete one loop, we write +1 for loops and +1 for regions, as when we connect the loop to the starting point, one more region is created, and

we have completed one loop. We will also consider the value of the intersection point in the following table that corresponds to the Loop in Figure 7. Here, the dotted line represents that the loop continues further, but the initial few intersection points are shown out of total ‘n’ intersection points. Let us see it in the following table:

Loop/Piece	Intersection Point	Region/ Space	The number assigned to that Intersection Point	Value
-	+1	+1	1	+1
-	+1	+1	1	+1
-	+1	+1	2	+1
-	+1	+1	3	+1
-	...	...	...	...
-	+1	+1	n	+1
+1	0	+1	-	-

Table 1: Tally of Number of Regions, Intersection Points, Loops and Value of Intersection Points

Here, we observe that, for each intersection point, the value is determined by the sum of all the values corresponding to that point. E.g., the total Value of Intersection Point (1) is equal to 2 (+1+1), the total Value of Intersection Point (2) is +1, and so on. We get +1 when we connect the loop to the initial point, which completes one loop, so +1 for Loop/Piece. Therefore, the total number of regions produced equals 1+ (sum of all the values in the last column). The sum of the values in the last column can be simplified to a total sum of  $(1 \times \text{Number of times loop passes an intersection point})$ . e.g., from the above table, the number of spaces can be given as  $1 +$

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$[(1 \times 2) + (1 \times 1) + (1 \times 1) + \dots + (1 \times 1)]$ . Here, we have assumed that the intersection point of a particular value,  $V$  occurs only once. Now, let us generalize the above observations.

#### 4.2 GENERALIZATION:

“If we have given a loop or a collection of connected loops such that no loop can be isolated from the group, having intersection points of values  $V_1, V_2, V_3, \dots, V_n$  also let No. of intersection points with values  $V_1, V_2, V_3, \dots, V_n$  be  $N_1, N_2, N_3, \dots, N_n$  respectively, then the number of Spaces ( $S$ ) created will be given as:”

$$\text{No. of Spaces } (S) = 1 + \sum_{k=1}^n N_k V_k \quad (4)$$

### 5. CLASSIFICATION OF INTERSECTION POINTS:

#### 5.1 PURE INTERSECTION POINTS:

These points are inherent to the original loop.

#### 5.2 MIXED INTERSECTION POINTS:

These are the intersection points created/ formed when one loop intersects with the other at a minimum of 2 points. These intersection points are formed by the Intersection of one Loop with one or more than one loop.

### 6. CONNECTING THE LOOPS:

Let us prove that if we draw several loops that intersect, we can draw the entire picture without lifting our pencil from the page (as though it were one loop).

To begin with, let us consider the loops  $P_1, P_2, P_3, \dots, P_k$  having  $p_1, p_2, p_3, \dots, p_k$  ‘pure’ Intersection points, respectively. Now let all these loops from  $P_1$  to  $P_k$  intersect such that no loop can be isolated from the rest.

Let us suppose that  $P_i \{1 \leq i \leq k ; i \neq j\}$  intersects with  $P_j \{1 \leq j \leq k ; j \neq i\}$  at  $K_m$  Number of intersection points. Where  $K_m = \{K_1, K_2, K_3, \dots, K_n ; 1 \leq m \leq n ; K_m \in \mathbb{Z}^+\}$ . Then, in this case, the total number of intersection points will be given as:

Total No. of Intersection Points:  $\{\text{No. of pure Intersection Points}\} + \{\text{No. of Mixed Intersection Points}\}$

Hence,

Total number of Intersection points when all loops from  $P_1$  to  $P_k$  intersect will be given as:

$$\{p_1 + p_2 + p_3 + \dots + p_k\} + \{K_1 + K_2 + K_3 + \dots + K_n\} \quad (5)$$

Therefore,

Number of intersection points :=  $\{p_1 + p_2 + p_3 + K_1 + K_2 + K_3 + \dots + p_k + K_n\}$

Now, it is possible to draw a new loop (L) with  $\{p_1 + p_2 + p_3 + K_1 + K_2 + K_3 + \dots + p_k + K_n\}$  number of intersection points, and there is only one way in which these intersection points can be placed in the same orientation as they were in original small loops ( $P_1, P_2 \dots$  and so on.). So, from the definition of loops (Introduction), if we go in reverse as we already have intersection points and then trace the bigger loop through those intersection points, it is possible to draw the entire picture without lifting our pencil.

### 6.1 A SPECIAL CASE:

It might be possible that pure and mixed intersection points overlap. Then, in that case, we will only increase the value of that intersection point and eventually increase the number of spaces, but the relative orientation of intersection points will remain the same, and there is only one way these intersection points can be fixed in space. Therefore, it is still possible to define a new loop L with the same orientation of intersection points in space, and hence it is possible to draw the entire picture without lifting our pencil.

Hence, the above statement is proved.

## 7. TWO COLORABLE:

Let us now prove that every picture we draw “can be” two-colorable, meaning that we can color the spaces blue and yellow (for example) so that no two regions that share a section of the boundary are of the same color.

To build this, if we look closely at any arbitrary intersection point then, we can “separate” any intersection point in the following ways:

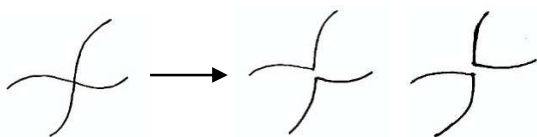


Figure 8: Separating Intersection Points

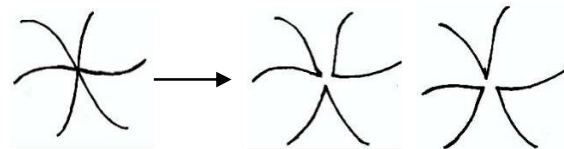


Figure 9: Separating Intersection Points

Now, from the above observation, we can form the following postulate:

### 7.1 POSTULATE:

For any given intersection point in a loop, we can separate in “only two” possible ways. Figures 8 and 9 can be considered as visual proof for this postulate.

Now, we will extend this idea to the bigger picture of loops as:

For any given loop ‘L,’ we can separate every intersection point to create disjoint loops with no intersection points and do not share any common boundary. Then from postulate 7.1, there are only two ways to separate them, which will result in ‘only two’ distinct figures and can be colored with only two distinct colors. When we combine both the distinct figures to get the original loop, we will get the loop colored so that none of the space shares the boundary of the same color.

### 7.2 ILLUSTRATION:

The above argument can be illustrated as follows:



Two ways to separate intersection points

Figure 10: Illustration that two coloring schemes do exist!

Hence, from the above discussion, it is established that “It is possible that every picture we draw “can be” two-colorable so that no two regions that share a section of the boundary are of the same color.

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## Stimulating Reflection through Self-Assessment: Certainty-based Marking (CBM) in Online Mathematics Learning

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**Abstract:** *This collaborative action research highlights the need for developing students' evaluative competence and self-reflection by embedding self-and-peer assessment into online instruction. Over the course of a semester in an online master program in mathematics and computer sciences, students conducted research on assigned topics, held presentations, formulated meaningful questions for peer-assessment, and finally engaged in Certainty-based Marking (CBM) by rating how certain they are that their answer is correct. The goal of using CBM was to foster students' careful reflection and provide feedback to teachers about students' status of knowledge. A mixed-method approach was used to triangulate data from two sources: (a) assessment artifacts, i.e., student-generated questions and CBM, as evidence of learning, and (b) students' attitude captured through 'Task Perception Questionnaire'. Assessment data were analyzed by three domain experts based on their judgement of 'quality' and Kappa measure was used to assess inter-rater consistency. Quantitative analysis of questionnaire data, coupled with instructors' observation, indicated positive attitudes (engaging and useful) towards CBM among students. We conclude with a discussion of limitations as well as implications of this classroom research project.*

**Keywords:** Certainty-Based Marking (CBM), Online Mathematics Assessment, Self-and-Peer Assessment

### 1. Introduction

Empirical evidence from comparative international tests, e.g., PISA, TIMS, as well as frequent failed national educational reforms raise alarm about inadequate mathematics performance and increasing STEM disengagement, e.g., high drop-out and low enrollment rates among students. Supporting the meaningful learning of mathematical procedures and developing robust fluency with mathematical skills is an urgent priority for Western mathematics education (Foster, 2016). Teachers are supposed to play a critical role in mitigating the rift between policy, research, and practice by engaging in 'evidence-based STEM education' (Milner-Bolotin, 2018) and data-driven decision-making: to collect, analyze and use research-based data to improve education (Maxwell, 2021). This study is a classroom intervention, conducted by teacher-researcher, who used 'assessment' to simultaneously generate evidence and foster students mathematical learning.

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Assessment is at the core of the learning process: it shapes how students learn and provides observable evidence of learning achievement. Traditionally, higher education focused on ‘*Assesment-of-Learning*’ (AoL): formal, summative tests at the end of a course to measure how much students have learned. However, such once-a-year tests can not help teachers make crucial instructional decisions which need moment-to-moment information about students’ progress (Stiggins, 2002). ‘*Assessment-for-Learning*’ (AfL), on the other hand, is conducted in the classroom formatively and continuously, with the aim of supporting and improving learning through diagnosing weaknesses and problems (Wiliam, 2011). Although AfL is mostly performed by teachers, there is a call for engaging students more in assessment to become progressively independent of their teachers, e.g., Sadler (2010) urged higher education institutes to develop ‘*evaluative judgement*’ in their graduates, the ability to judge the quality of one’s own and others’ work, as a sustainable life-long skill which is necessary both within and beyond higher education settings (e.g., professional jobs).

Self- and peer-assessment (SAP) is an AfL method which has the capacity to engender evaluative judgement. SAP assumes that by handing over assessment responsibility to students, they engage in active learning and become more reflective through understanding and appraising quality/standards/criteria related to work (Boud and Soler, 2016). Furthermore, interacting with criteria helps to close the gap between the current and the expected performance level. In this study, we used two SAP strategies: Student-generated Questions (SGQs) and Certainty Based Marking (CBM). By requiring students to generate meaningful, quality questions and indicate their degree of certainty (c) about the answer they choose, learners will be encouraged to reflect and self-assess their knowledge (Gardner-Medwin, 2006). This classroom study sought to answer the following questions:

- Q.1. How competent are students in producing higher-order questions for peer-assessment?*
- Q.2. How confident are students in answers they choose in self-assessment?*
- Q.3. What are the attitude and perceptions of mathematics students towards CBM?*

## 2. Literature Review: Certainty-based Marking (CBM)

Multiple-choice Questions (MCQs) is a widely used assessment technique which provides prompt feedback on students’ learning. Although students who get the right answer might think they have knowledge and know the answer, responses to multiple-choice tests can be an evidence of knowledge as well as a pure ‘*lucky guess*’ without any knowledge or an ‘*educated guess*’ based on partial, uncertain knowledge. Both guesses introduce error variance into the test score and affect reliability negatively (Lindquist & Hoover, 2015). Furthermore, such chance response encourages an uncritical habit of mind in students.

To remedy this inherent problem with MCQs based on a single-best answer method, *Certainty Based Marking (CBM)*, formerly known as Confidence-based Marking, assumes knowledge is not a binary thing (you know it or don’t know it), i.e., by asking ‘how sure, confident, certain are you?’, students start to think more carefully and look for justification and reservations. It also provides a more refined differentiation of students’ knowledge levels.

Students are posed with multiple choice items. After answering, they should choose from a 3-point scale: 1 (low), 2 (mid) or 3 (high), the degree of certainty (c) about the correctness of their answers. Therefore, item score is a product of both correct answer and certainty level. Based on the reported degree of certainty, different rewards and penalties are assigned: i.e., a confident, wrong answer gets the highest penalty (see Table 1). Therefore, CBM differentiates between students who choose the same correct answer by rewarding those who can distinguish their more reliable and less reliable answers.

**Table 1. Mark scheme for CBM**

Degree of Certainty (c)	C=1 (low)	C=2 (mid)	C= 3 (high)
Score (correct answer)	1	2	3
Penalty (wrong answer)	0	-2	-6

Certainty-based Marking aims at (a) identification of uncertainty, (b) rewarding accurate judgement of reliability, (c) reducing biases due to over-confidence and hesitation, and (d) even diminishing unwarranted self-confidence (Gardner-Medwin, 2006).

Although CBM is used extensively in Medicine to discourage guessing in life-or-death matters (Gardner-Medwin, 2019; Nathaniel et al., 2021), several other areas also embed it in their pedagogical practices. Hassmén & Hunt, (1994) found that CBM can enhance test validity by reducing gender biases. Ehrlinger et al., (2008) studied how ‘*illusory over-confidence*’, in which low-ability students over-estimate their competence because they ‘do not know what they do not know’, could be calibrated through consistent use of CBM. In another study, Yen et al, (2010) examined the correlation between students’ ability and their confidence in computer-administered MC tests. In addition to a positive association, CBM was found to be more efficient compared to traditional MC tests, because it needs fewer items to estimate test-takers’ knowledge level. However, some research failed to find any positive effect on outcomes such as achievement, e.g., Foster (2021) examined the effect of repeated and formative use of CBM on summative mathematics attainment across four schools ( $N=475$ ). A Bayesian meta-analysis of the effect sizes showed no effect on students’ mathematics achievement. It was concluded that CBM cannot cause a quick, easy and visible raise in gain scores in the short time. Wu et al. (2021) suggested that CBM could be affected by individual difference variables, such as gender or risk-attitude, that are not related to the main construct (e.g., ability or knowledge).

### 3. Method

This action research was carried over 10 months in three phases: planning, preparation, and data collection. Although the planning phase is explained briefly, the focus of this paper will be on the ‘classroom action research’, as conducted by the instructors during preparation and data collection phases.

#### 3.1. Planning phase: Collaborative Action Research

This study was conducted as a part of Faculty Professional Development Program in SKILL.de project, Germany during 2021. The goal of Evidence-based Evaluation in SKILL.de is to enhance instructors’ Data Literacy: ability and competence in collecting and analyzing empirical data about students’ learning in order to improve instructional decision-making. During this phase,

the action research coach, a researcher in empirical learning sciences, worked collaboratively with the course instructors, a professor and her two Teaching Assistants (TA). Based on the course goals, i.e., Self-regulated Learning, they designed an action research study which embeds formative self-and-peer assessment into learning activities. A critical consideration in this phase was ‘ecological validity’: to make sure that intervention is a naturalistic trial, easy and low-cost to implement, without imposing any new system from outside or re-designing the whole course (Neumark, 2019).

### 3.2. Preparation phase: training students in assessment

This small-scale classroom research was conducted during Corona-pandemic in the online seminar "Applied Mathematics in the Math Museum", over a 14-week semester at university of Passau, Germany. The Passau Mathematics Museum encourages students to design an exhibit (i.e., applet) that communicates a mathematical concept to the visitors of the math museum in addition to delivering a scientific presentation. Course delivery was through Stud.IP (Learning Management System) as well as synchronous Zoom meetings. Participants consisted of five students in bachelor and master of mathematics and computer science. To help students become more self-regulated and control their own learning, they were asked to choose a topic from an assigned list, do research and reading on the topic, develop some questions and deliver an oral presentation. Developing students’ competence to ask meaningful, quality questions and reflect deeply when answering questions are at the heart of mathematical scientific literacy. However, the results of our past study showed that students are not familiar with generating quality questions (Caspari et al, 2021).

Therefore, the first session was spent on introducing the project, getting students’ consent, and instructing them about Student-generated Questions. They had no prior experience in systematically formulating questions about a topic. They were introduced to ‘worked examples’, a sample of questions with different quality levels (lower-order and higher-order), were encouraged to discuss what makes a good multiple-choice question (both form and functions) and were asked to judge attributes of a strong and a weak MCQ. It should be noted that not all quality features can be communicated through explicit criteria; some will remain tacit and embodied (Hudson et al. 2017). Quality levels (lower or higher) were measured with reference to Bloom’s Taxonomy (1958) which stipulates different levels of cognitive complexity involved in answering the questions.

*Table 2. Bloom’s Taxonomy of Cognitive Levels*

	Cognitive domains	Cognitive levels	Actions required
Lower-Levels	Remembering (knowledge)	Low	Recognition, recall, name, list
	Comprehension	Low	Describe, explain, summarize, visualize
	Application	Low	Use, practice, solve, manipulate
Higher-Levels	Analysis	High	Compare, deduce, analyze, infer
	Synthesis	High	Synthesize, plan, design, construct,
	Evaluation	High	Judge, criticize, estimate, justify, defend

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Certainty Based Marking (CBM) was also introduced later in the course. Some studies (Bar-Hillel, Budescu, & Attali, 2005) showed that students' choices of a certainty level were affected by their risk attitudes: when students have a high success probability on an item, they become risk averse (under-reporting of their certainty) and conversely become risk-taking if there is a low success probability (over-reporting of their certainty). To avoid 'demotivating' of students, we decided not to assign any score as 'penalty and reward' to certainty level. The students were asked simply to indicate their certainty level on a 3-point scale: 1(*low*)= unsure/not confident; 2(*mid*)= relatively confident; 3(*high*)= highly confident.

### 3.3. Data collection phase

Students conducted self-study on a topic, prepared a presentation and formulated two MCQs which were presented at the end of their lecture. The class answered and indicated their certainty in answers (see Appendix A). There were subsequent discussions about questions (levels, ambiguity, etc.) during the whole process, instructors took some field-notes about their observation.

Students' perspectives and attitudes towards CBM were captured at the end of semester through a questionnaire, Task Perception Questionnaire (TPQ), developed by the authors. First, we reviewed existing related literature and developed an initial 7-item scale based on selective adoption of the Self-determination Theory Framework (Deci & Ryan, 1991) and Technology Acceptance Model (TAM), which are used widely to assess digital competence and acceptance (Venkatesh and Davis, 2000). The scale was reviewed by two experts (Mathematics Professor and learning science researcher) and was refined again. The final version of the TPQ is composed of five questions, on a four-point Likert scale ("strongly disagree" to "strongly agree"), measuring three aspects of a task perception, (a) *usability*: the perceived ease or difficulty in performing the task, (b) *engagement* with the task, and (c) *intention to use* in future (see Appendix B). The questionnaire was administered online and anonymously.

## 4. Analysis

Three mathematics instructors were instructed to code the quality of SGQ based on a two-dimensional rubric: (a) the overall quality of a question, and (b) the cognitive demand involved in a question. The overall question's quality was assessed based on its content coverage, clarity, relevance, and plausibility on a rating scale of 1-3 (1= poor, 2=good, 3= excellent). Both stems and distractors were considered. A question was rated as '*Poor=1*' if it was ambiguous, had irrelevant alternatives and very little topic coverage (Caspari et al., 2021). The cognitive demand of SGQs was measured with reference to Bloom's Taxonomy or levels of cognitive complexity (e.g., remembering; understanding; applying; analyzing; synthesizing and evaluating). The inter-rater reliability among three subject-matter experts was calculated, resulting in an overall Cohen's kappa value of  $d= 0.68$  among all raters. Next, all raters and the moderator (action researcher coach) met to negotiate discrepancies. Discussions continued until consensus was reached on all codes.

## 5. Results

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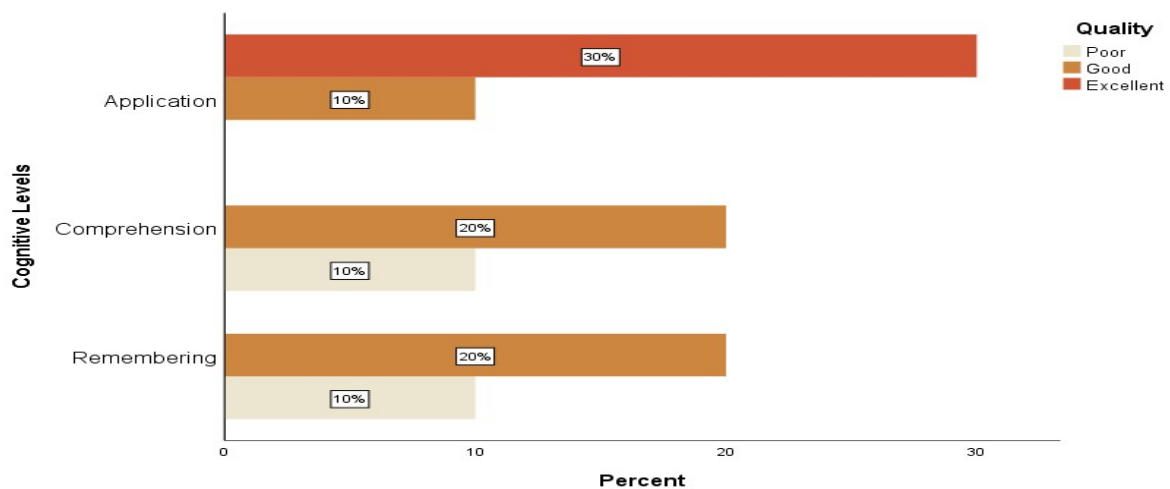


### 5.1. SGQ

#### Q.1. How competent are students in producing higher-order questions for peer-assessment?

Results of SGQ showed all questions authored by students were at the so-called ‘lower-levels’ of cognitive complexity, namely 30% at Level 1= *Remembering*, which requires mere retrieval of facts and information, 30% at Level 2= *Comprehension*, which requires understanding of materials, and eventually 40% targeting Level 3= *Application*, which necessitates the use of knowledge to perform or solve problems. None of SGQs reached ‘higher-levels’ of cognitive complexity, such as analysis, synthesis, or evaluation. Quality-wise, 30 % of produced questions were rated as ‘*excellent*’, with another 50% as ‘*good*’ and only 20% were assessed as ‘*poor*’.

Figure 1. SGQ Quality



### 4.2. CBM

#### Q.2. How confident are students in answers they choose in self-assessment?

24 Out of 40 answers to all SGQs were correct. In reporting their degree of certainty in the correct answers, 30% expressed a low level of confidence, while 54% were almost/relatively sure about the correctness of their answers. Only 16% had a high degree of certainty. None of the students expressed full assurance (being 100% confident) in answers they selected (see table 3).

Table 3. Degree of Certainty in correct answers

Certainty levels	C=1 (low)	C=2 (mid)	C= 3 (high)
Assessment of correct answer	30%	54%	16%

1= low confidence/unsure; 2 = average confidence/relatively confident; 3= high confidence

### 4.3. Task Perception Questionnaire (TPQ)

#### Q.3. What are the attitude and perceptions of mathematics students towards CBM?

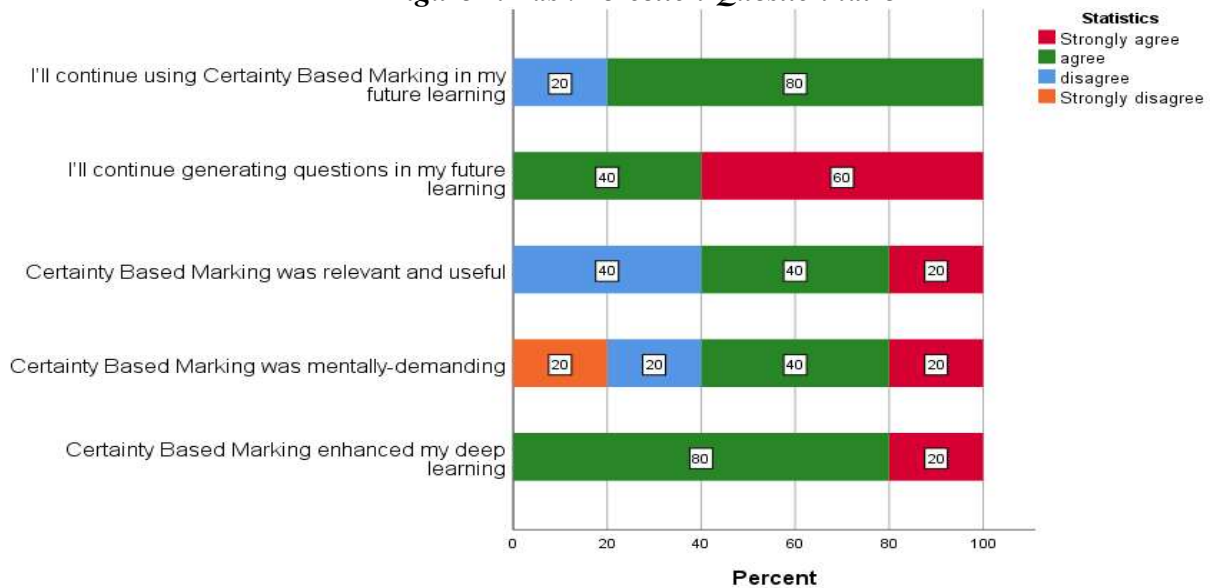
Figure 2 shows students’ responses from the online anonymous Task Perception Questionnaire (TPQ). ‘Usability of CBM’ received a mixed reaction: although 60% considered it as a difficult

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and mentally demanding task, the rest believed it was easy and manageable. Most of the students (60%) viewed the activity as relevant and useful. In terms of ‘engagement’, or to what extent CBM involved students in self-reflection, all students (80% agree, 20% strongly agree) believed CBM enhanced their deep learning and reflection. Majority of cohort expressed their positive attitude toward ‘*intention to use*’: with 80% agreed that they’ll continue using CBM in their future learning, while all students either strongly agreed (60%) or agreed (40%) about continuing the use of SGQ for future learning.

**Figure 2. Task Perception Questionnaire**



## 5. Discussion

The analysis of data, presented in section 4, indicates that this small-scale intervention could enhance students’ participation in assessment. Survey results imply a positive attitude and students’ increased motivation to take charge of their own-and-peer assessment, a sustainable skill which is transferable to other contexts.

We found a slight improvement in the quality of SGQ compared to our previous classroom research (Caspari-Sadeghi et al, 2021). It might be tempting to ascribe this enhancement to some explicit actions taken by instructors, such as direct instruction about quality of MCQs or assigning few scores to motivate students’ serious involvement. However, due to its naturalistic design and inherent lack of control of pre-existing variables, e.g., background knowledge, action research avoids establishing any cause-and-effect relationship. Even though students failed to produce questions at higher-levels of cognitive complexity, a closer look into the existing literature and the nature of SGQ can shed some lights on this phenomenon. In a large-scale review of MCQs across the U.S. biology courses, Momsen et al., (2010) found that 90% of items are at the lowest two levels of the Bloom’s taxonomy, namely remembering and understanding. This could be partly attributed to the ‘nature’ of such questions: MCQs are often criticized for their inability to target ‘*conceptual understanding*’ and being mostly focused on recall of factual knowledge (Biggs &

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Tang, 2011). Additionally, developing such higher-order competences requires more time, practice, and a shift in the culture of educational systems.

Results of CBM revealed that majority of cohort (70%) were certain and sure about the correctness of their answers. It should be also mentioned that in our small sample ( $N=5$ ), we couldn't observe students who gave incorrect answers and expressed a high certainty about their incorrect belief. Overall, our findings are in line with other studies (e.g., Sparck, Bjork, & Bjork, 2016) that suggest CBM as a useful and efficient self-test provided that it is used continuously in the classroom. There were some limitations to this study.

5.1. For reliability purposes, it would have been better to develop a longer questionnaire.

For pragmatic reasons, authors decided against this, e.g., the intervention was supposed to be non-invasive and small-scale. Furthermore, the students were already assigned to several other tasks (i.e., presentation, SGQ, CBM, digital exhibits, etc.) as well as participating in a university-led survey.

5.2. This exploratory case study is more like a formative experiment carried over a short period of time. Authors make no claim over generalizability or causality of such a small-scale intervention. Cautions should be taken in attempting to replicate in other contexts.

## 6. Conclusion

This case study aimed to explore the development of evaluative judgement through self-and-peer assessment. Based on a classroom action research, we examined implementation, uptake as well as students' attitude towards effectiveness of CBM and SGQ as efficient techniques to engage students with assessment. It's safe to say both instructors and students believed this formative intervention effectively enhanced their learning. Although the results of the survey revealed positive attitude, we could not establish the extent to which SGQ and CBM improved students' mathematics attainment (i.e., final score). There is a need for more research on several aspects of CBM that we didn't cover in this study, e.g., Novak (2017) asserted Asian cultures find it quite unnatural to rate themselves above the average. It might be interesting to examine if other demographic variables such as 'discipline' or 'socio-economic class' might have any implications for using CBM.

**Acknowledgement:** Funding was provided by German's Federal Ministry of Education and Research (BMBF).

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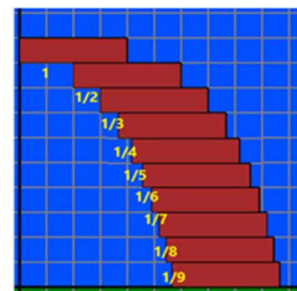
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### Appendix A Student Generated Questions (SGQ)

1. The sequence of the partial sums of a series  $\sum_{i=1, \dots, n} a_i$  is defined as the x-coordinate of the lower left corner of the n-th brick of a tower (brick 0 is at the top and lies at  $x = 0$ ). For an element of the sequence  $(a_n)_{n \in \mathbb{N}}$ ,  $a_i$  is the difference between the coordinates of the i-th and the i-1-th brick. Which of the following statements is / are correct?  
[Assumption: The size of the brick remains unchanged.]

- a) If you can build a tower with an infinitely large ledge based on the partial sums, the series diverges.  
b) If the tower topples over, the series diverges.  
c) If you can build a tower, the series converges.  
d) If the series converges, you can build a tower.

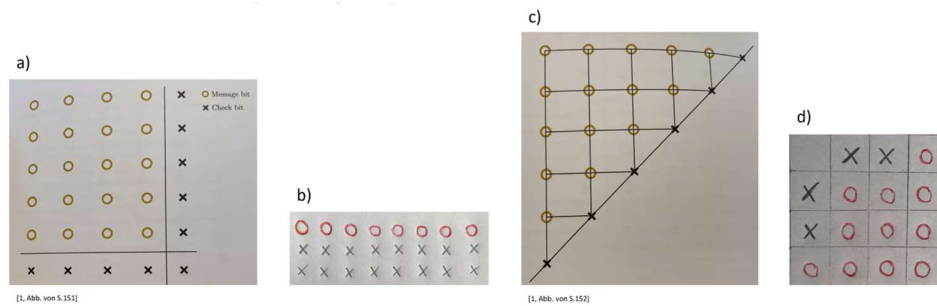


2. Which of the following approaches is the most robust one with regard to error correction? -  
Order them from the most to the least robust.

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- a) Order c,d,a,b
- b) Order d,c,a,b
- c) Order d,a,c,b
- d) Order d,b,a,c



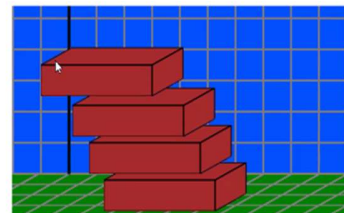
3. Magic Mike says: “After the member of the audience has shuffled the cards if it can absolutely happen that 2 hearts or also 3 black cards are lying together. But this is no problem!”

Which of the following answers to Magic Mike’s comments would be correct?

- a) “As the cards are always presented block by block, the audience wouldn’t notice.”
- b) “The fact that  $k$  cards leave the remainders  $\{0, \dots, k-1\}$  when dividing by  $k$  means that there are  $k$  different cards in each of the blocks shown.
- c) “You didn’t get the trick because ...”
- d) “It cannot be that 3 black cards are next to each other– but it can certainly happen with features that appear in more than two variations (for example card value).”

4. Is it possible to build an infinite ledge in Two directions on the tower?

- a) Yes, because the coordinates of the barycenter can be calculated separately for every coordinate direction.
- b) No, the tower topples over.
- c) Yes, if the corner point lies exactly under the barycenter of the tower on top.
- d) No, because the downwards shift also influences the horizontal coordinate of the barycenter.



5. Is it possible to distort an image at 360° for a cylindrical mirror?

- a) Yes, the image will be brought to focus at the front of the mirror anyway.
- b) No, only works for 2 images.
- c) No, there is no way to get the image back in focus.
- d) Will not work with AnamorphMe, but can be done using grids.

6. Can we have more than two image distortions on the same anamorphic plane? For example, is it possible to distort 3 or 4 images, to be projected on the same cylindrical mirror?

- a) Yes, then the images will be close together on the mirror.

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- b) Yes, though the images will overlap on the mirror.
  - c) Depends on method of distortion being used.
  - d) No, we can only have a maximum of 2 images.
7. Which of the following properties is true for the extended Hamming Code?
- a) It detects all errors, but it can only correct one of them.
  - b) It detects all even errors and can correct one bit if the error is a single error.
  - c) It detects all errors and can correct all even errors.
  - d) If there are an odd number (larger than 1) of errors, then neither the error detection nor the error correction works.

## Appendix B

### Task Perception Questionnaire (TPQ)

1. Certainty Based Marking was a relevant and useful activity.  
(a) Strongly agree (b) agree (c) disagree (d) strongly disagree
2. Certainty Based Marking was mentally very demanding.  
(a) Strongly agree (b) agree (c) disagree (d) strongly disagree
3. Certainty Based Marking made me think deeper (more reflective).  
(b) Strongly agree (b) agree (c) disagree (d) strongly disagree
4. I will continue producing questions when I learn new materials in the future.  
(a) Strongly agree (b) agree (c) disagree (d) strongly disagree
5. I will continue re-assessing my answers to become more confident.  
(b) Strongly agree (b) agree (c) disagree (d) strongly disagree

## Teachers' Skills for Attending, Interpreting, and Responding to Students' Mathematical Creative Thinking

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**Abstract:** *This study aimed to explore the mathematics teachers' skills in attending, interpreting, and responding to students' mathematical creative thinking. The data of this study comprised of the teachers' skills in attending, interpreting, and responding to students' mathematical creative thinking gained from observing a recorded video of the teachers' teaching enactment. The data were collected inductively with open coding to examine classroom teaching. Findings suggest that the mathematics teachers raise the attending skills in two categories: activities directly related to students' mathematical creative thinking and activities that support students' mathematical creative thinking. They interpret mathematical understanding in various ways: excluding the justification of right or wrong answers; focusing on right or wrong; focusing on the only correct solution; focusing on deficiencies in the student's working process; focusing on how they wrote and drew on grid paper and anything else indirectly connected to mathematical thinking; stating that working on the problem is easy; comparing the thinking process of the teachers and students. The mathematics teachers responded by giving comments or questions about their students' knowledge, idea, procedure, or mathematical thinking and based on mathematical creative thinking; giving general comments or questions; giving comments or questions to trigger students to share their opinions; asking other students to comment or ask questions about certain students' thinking ideas; asking other students to explain certain students' thoughts; giving comments or questions about students' mathematical creative thinking/open-ended problem/critical thinking.*

## INTRODUCTION

The skills of attending, interpreting, and responding to students' thoughts have been the focus of educational research. Those three skills are vital learning components to support students' learning so that the effort to comprehend and develop them becomes one of the focuses of mathematics learning (Luna & Selmer, 2021). Those essential skills need to be developed during teacher

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training since they affect the effectiveness-based learning and improve students' mathematics competency (Sánchez-Matamoros et al., 2019). Those skills are described as practices with challenging development, yet they could be learned (Tyminski et al., 2021), could be developed from time to time (Jacobs et al., 2010; van Es & Sherin, 2008), and one of the essential components of teaching mastery and learning quality (M. Y. Lee, 2020). Those skills are the tools to assess someone's teaching practice and improve learning (Barnhart & van Es, 2015).

Some experts proposed the connections between the skills of attending, interpreting, and responding to teachers' competencies. Teachers need to attend to specific mathematics ideas on students' papers and produce logical feedback to interpret students' thoughts used later for responding (Krupa et al., 2017). Most teachers showed evidence of attending to the students' thoughts, but they showed a fewer evidence of interpreting students understanding as well, much less evidence of how to respond to students' thoughts based on their understanding (Larochelle et al., 2019). Responding skills seem to be the toughest skills to develop (Barnhart & van Es, 2015; Jacobs et al., 2010; Tyminski et al., 2014). In other words, teachers tend to attend and interpret their students' mathematical thinking instead of responding to them (Land et al., 2019).

Many studies of attending, interpreting, responding to students' mathematical thinking skills have been widely carried out. Research related to these three skills by mathematics teachers in various fields of mathematics studies has also been widely carried out (such as Jacobs et al., 2010; Kiliç & Masal, 2019; Nagle et al., 2020; Sánchez-Matamoros et al., 2019; Walkoe, 2014). Research by Jacobs et al. (2010) focused on integer operations. Previous research by Walkoe (2014) revealed that using video clubs helps teachers be more consistent in following the substance of students' algebraic thinking and reasoning about students' algebraic thinking. Research by Sánchez-Matamoros et al. (2019) used derived material and the result was that students connected the rate of change to the slope of the line and the instantaneous rate of change to the slope of the tangent. Research by Kiliç and Masal (2019) used algebraic material. Research by Nagle et al. (2020) employed statistical material and the results are teacher interpretations that are often evaluative and tend to describe student processes. Research on the three skills is reviewed from a variety of strategies (such as Araujo et al., 2015; Kristinsdóttir et al., 2020; Krupa et al., 2017; Nagle et al., 2020; Roller, 2016; Sánchez-Matamoros et al., 2019). Research by Sánchez-Matamoros et al. (2019) used students' written answers to explore the relationship between the three skills to prospective high school mathematics teachers on students' mathematical understanding. Research by Krupa et al. (2017) designed a curriculum module consisting of pre-and post-assessment, reading, class discussions, structured interviews with high school students, and written reflection to develop pre-service teacher attention in students' mathematical thinking. Research on the three skills involved both teachers and prospective secondary school mathematics teachers (such as Baldinger, 2020; Dyer & Sherin, 2016; Krupa et al., 2017; Larochelle et al., 2019; Nickerson et al., 2017; Roller, 2016; Sánchez-Matamoros et al., 2019; Simpson & Haltiwanger, 2017; Styers et al., 2020; Wallin & Amador, 2019). Research by Dyer and Sherin (2016) identified three types of

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instructional reasoning about the interpretation of students' thinking used by teachers: (a) making connections between certain moments of student thinking, (b) considering the relationship between students' mathematical thinking and the structure of mathematical tasks, and (c) develop students' thinking tests.

Many researchers define creative thinking from various points of view. Creative thinking is the ability to generate novel ideas or solutions in a problem-solving process (Hadar & Tirosh, 2019), as a mental activity that is used to construct an idea or notion of the "new" (Siswono, 2014), as the ability to generate new ideas or solutions and select unique or the most useful idea or solution to develop or apply in action (Tran et al., 2017). Mathematical creative thinking is the competence to engage productively in the learning, evaluation, and improvement of ideas that can result in original, practical solutions (Suherman & Vidákovich, 2022). The researchers use different indicators in their creative thinking research. Research conducted by (Leikin & Lev, 2013), (Elgrably & Leikin, 2021), and (Levenson, 2022) uses indicators of fluency, flexibility, and originality. The researchers (Sahliawati & Nurlaelah, 2020) used concepts of fluency, flexibility, elaboration, and originality.

In this research, mathematical creative thinking is defined as the ability to generate ideas in solving mathematical problems with indicators of fluency, flexibility, and originality. In this research, fluency in mathematics is a person's skill produces many mathematically correct answers that is not duplicated, and can generate many meaningful ideas/possibilities/approaches in solving problems. Flexibility is a person's ability to change focus, use different thinking strategies, use various representations, or relate different mathematical topics and is measured based on the classification of student completion in categories and then the number of categories with correct answers is calculated or measured through the number of different methods carried out in solving problems. Originality is a person's ability to produce problem-solving using insights that are new to him.

Creative thinking is closely related to mathematical thinking. Creative thinking is a subcomponent of mathematical thinking (Kattou et al., 2013). On the other hand, Schoevers et al. (2020) noted that general creativity and mathematical ability could predict mathematical creativity better than general and mathematical creativity. Teaching creative mathematics has become something needed (Luria et al., 2017). Teachers cannot teach creativity and the more teachers teach, the less opportunities students have for creative thinking (Baker et al., 2020). Some studies on mathematics teachers' competency related to mathematical creative thinking are conducted by Levenson (2013), Luria et al. (2017), and Levenson (2021). When deciding the tasks to promote students' mathematical creative thinking, the teachers' considerations should be based on values. The practical value of mathematical creative thinking is various ways to solve mathematics problems (E. S. Levenson, 2021). Teachers could implement a strategy to promote students' creative thinking in the classroom and develop equity principles. The principles include presenting open-

ended problems, modeling and encouraging risk-taking, discussions, debating mathematical concepts, concept-based learning, divergent thinking strategies, and incorporating cultural awareness and creativity into curricula and classroom environments (Luria et al., 2017).

The research examines mathematical creative thinking in terms of various points of view. Creative thinking is an educational goal (for example Hadar & Tirosh, 2019; Pendidikan et al., 2018; Tabach & Friedlander, 2017). Based on the Indonesian Ministry of Education and Culture Regulation No 36 of 2018 cited from Pendidikan et al. (2018), Indonesia also lists "creative" as the goal of the 2013 Curriculum. Creative thinking research is competency and stimulus for teachers and prospective teachers (for example, Ayele, 2016; Kaiser et al., 2015; K. H. Lee, 2017; Sánchez et al., 2021; Siswono, 2015). Research by Sánchez et al. (2021) investigated the development of creativity in mathematics classes for pre-service teachers of a secondary school teaching master's program, who were not trained on how to develop creativity. Research on creative thinking as a stimulus to student creativity (for example Bicer et al., 2020; Elgrably & Leikin, 2021; Kurniasih et al., 2020; Molad et al., 2020; Sánchez et al., 2021) as well as research on a person's typology is said to be creative thinking (eg. Aljarrah, 2020; Lassig, 2020). Research by Kurniasih et al. (2020) found that problem posing, asking questions, and using songs were used by 5<sup>th</sup>-grade elementary school teachers in mathematics lessons to facilitate students' mathematical thinking. Research by Lassig (2020) revealed that there are three types of creativity, namely creative personal expression, boundary-pushing, and task achievement.

A preliminary study for the present research involving junior high school mathematics teachers in Central Java was carried out on May 31, 2021, using PISA questions which were used to measure mathematical creative thinking and the answers of 2 8th grade students. PISA questions, for example in PISA 2012 were used to measure students' creative problem solving abilities (Yang & Fan, 2019) and PISA questions could be used to measure students' creative knowledge in everyday life (Komatsu & Rappleye, 2021). Five teachers were asked to answer questions with open responses related to the skills of attending, interpreting, and responding to students' creative thinking based on PISA questions and 2 students' answers.

Based on the teachers' response to question 2 from the preliminary study with the question "explain in detail, according to you, what each child did in response to the problem", it was known that they attended students' thinking by explaining their activities and their thinking processes (one teacher), writing down the ideas the students chose to solve the problems (three teachers), explaining students' activities when solving the problems (one teacher). Question 3 was related to how they interpreted students' thinking. The results were that the teachers interpreted students' thinking by knowing their deficiency during the process of doing the problems (one teacher), interpreting their thinking process (three teachers), and interpreting their personalities (one teacher). Question 4 related to how teachers responded to students' thinking. They brought up new tasks asking students what steps they took to solve the problems (three teachers) and created new assignments that

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helped students evaluate their thinking process (two teachers). One of the teachers' responses to answer question 3 shown in Figure 1.

3) Siswa A model siswa yang pintar, rutin, normatif dan rasional terlihat dari cara menjawab  
Siswa B model siswa kreatif, bebas, cerdas, perlu penanganan khusus untuk pembelajaran untuk siswa ini.

3) Student A is a student who is smart, orderly, normative, and rational as seen on the answers. Student B is a student who is creative, free, smart, and need personal touch.

Figure 1: The teacher's response to question 3

The teacher argued that student B was creative, free, intelligent, and needed personal touch (see Figure 1). This indicated that the teacher interpreted creative thinking by stating that student B was creative. In other words, teachers interpret the personalities of student B. However, the teacher did not provide more explanation to support the statement.

During preliminary research, the teachers demonstrated the activities of attending and interpreting but their comments were not directed to students' mathematical creative thinking. While the problems asked students to conduct creative thinking. One teacher responded by commenting that student B was creative, but it was not supported by any evidence (see Figure 1). Hence, further research is needed to study the teachers' attending, interpreting, and responding skills toward students' creative thinking in learning mathematics in class.

The research objective is to explore the teachers' attending, interpreting, and responding skills to students' mathematical creative thinking. The results of this study are expected to be the first step for further research involving training to identify the characteristics of attending, interpreting, and responding to students' mathematical creative thinking in the setting of teacher professional development. It is supported by Yaakob et al. (2020) that effective teacher professional development is carried out in the form of training.

## RESEARCH METHODS

This study applied a qualitative approach with a grounded theory research design. The subjects were three junior high school teachers from different cities in Central Java Province, Indonesia. One teacher had taught for more than 20 years, coded with P1. Another teacher who had taught for 10 to 20 years are coded with P2. Next, a junior teacher who had taught for less than five years is coded with P3. The three junior high school mathematics teachers stated that they were willing

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to participate in this research. The main requirement for teacher involvement is a teacher who has experience in teaching mathematical creative thinking. P1 is an administrator of the Association of Mathematics Teachers in Semarang, Central Java, and is actively involved in research activities on creative thinking with the first author of this article. P2 is the coach of the student mathematics Olympiad at the school where she teaches. P2 is used to invite Olympiad fostered students to think creatively in solving Olympic mathematics problems. P3 is a teacher who has taught mathematics for less than 5 years. He is a graduate of the mathematics education study program at one of the universities in Central Java and his final thesis has the theme of mathematical creative thinking.

The data is the description of the practice of attending, interpreting, and responding to students' creative thinking carried out by the three teachers. The three teachers carried out mathematics learning in 3 meetings each and the lessons were recorded on video. P2 and P3 taught function and linear equations, while P1 taught the Pythagorean Theorem.

Data analysis in this research was carried out following Miles et al.'s (2014) framework. First, researchers critically watched the recorded video and transcribed it into written texts. Next, the researcher reduced the data by choosing information related to the teachers' activities of attending, interpreting, and responding to students' creative thinking. The activities of the teacher attending, interpreting, and responding to students' creative thinking are grouped. The researcher did the coding by compiling an inductive code based on the data that appeared in the learning by each teacher as a category and subcategory of each competency. The learning carried out by teachers involves the interaction of individual students with teachers, groups of students with teachers, and all students with teachers. So, the general coding for students' interaction with the teacher was A1, students in groups with the teacher was A2, and whole students with the teacher was A3. The skills of attending, interpreting, and responding are symbolized by the letters A, I, and R, respectively. The description of the categories and subcategories of each inductively acquired skill is presented in Table 1.

Noticing Component (Code)	Category (Code)	Subcategory (Code)
Attending (A)	Activities that are directly related to students' mathematical creative thinking (X)	Attending the process/result of students' creative thinking (X1)
		Asking for an explanation about their thinking steps (X2)
		Detailing or not detailing students' thinking strategies (X3)
		Asking for justification for their reasoning (by giving guided questions, giving hints/keywords, bringing up sentence

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	phrases, asking students using “How and Why”) (X4)
	Reminding the concepts and principles they had learned or the relevant ones (X5)
	Emphasizing what they had done (X6)
Activities that support students' mathematical creative thinking (Y)	Attending their articulation, how students draw, and gesture when students explain their thinking (Y1)
	Allowing their mathematical reasoning development (provide the widest opportunity for students to explore mathematical ideas, provide opportunities for students to state true or false the results of thinking students or other groups of students, say that students can get various answers) (Y2)
	Asking them if they had finished the tasks (Y3)
Interpreting (I)	excluding justification of right or wrong answers (B)
	Focusing on the correctness or incorrectness of a solution (showing which one was correct or incorrect, putting a checkmark on the right solution, using words to interpret correct or incorrect solution implicitly, or showing a smiley face) (C)
	focusing on the only correct solution (D)
	focusing on the deficiency of students' working process (E)
	Adding important information missing (E1)
	Asking for the clarification of students' statements (E2)
	Giving questions or comments about students' reasoning to check whether the student's answer is correct or not (E3)
	Pointing out students' mistakes related to the procedures of doing the problems (E4)
	Asking if their students were aware of their mistakes made in the process (E5)

	focusing on how they wrote and drew on grid paper and anything else indirectly connected to mathematical thinking (F)	-
	stating that working on the problem is easy (G)	-
	comparing the thinking process of the teachers and students (H)	-
Responding (R)	giving comments or questions about their students' knowledge, idea, procedure, or mathematical thinking and based on mathematical creative thinking (K)	Giving comments/questions to examine students' thinking (such as using why and how questions) (K1)
		Giving questions/guided comments to help students think creatively about mathematics (K2)
		Giving follow-up questions to confirm students' mathematical reasoning (K3)
		Giving comments/questions about students' thinking mistakes (K4)
		Giving comments about the relevant concepts/principles/calculations (K5)
	giving general comments or questions (such as any question, do you understand, can you do it, and what is the conclusion) (L)	-
	giving comments or questions to trigger students to share their opinions (M)	-
	asks other students to comment or ask questions about certain students' thinking ideas (N)	-
	asking other students to explain certain students' thoughts (O)	-
	giving comments or questions about students' mathematical creative thinking/open-ended problem/critical thinking (P)	-

Table 1: Categories and subcategories of attending, interpreting, and responding to students' creative thinking raised by the three teachers and obtained inductively

## RESULTS AND DISCUSSION

P1 delivered learning material about the Pythagorean Theorem. As shown in Figure 2 below, he applied various attending, interpreting, and responding patterns.

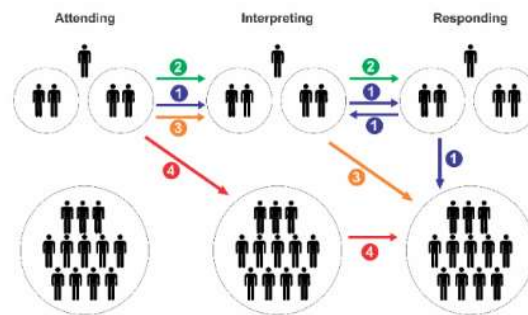


Figure 2: P1's Patterns

P1 performed four patterns (see Figure 2). Pattern 1 was AA2-IA2-RA2-IA2-RA2-RA3 as the dominant pattern done by him. On pattern 1, the interpreting skills were followed by repeatedly responding to the interaction between a group of students with P1 until they understood. This process was ended with the teacher classically responding to students. Pattern 2: AA2-IA2-RA2. Pattern 3: AA2-IA2-RA3. Pattern 4: AA2-IA3-RA3.

In learning meeting 1 with the material on Proving the Pythagorean Theorem, 4 kinds of mathematical problems are provided to prove the Pythagorean Theorem. Every 2 groups of students get 1 task to prove the Pythagorean Theorem. However, in presentations by group representatives and classical discussions, only 2 ways of Proving the Pythagorean Theorem are discussed in class. The results of the work of proving the Pythagorean Theorem by one group of students are presented in Figure 3 below.



Conclusion:  
Square of  
hypotenuse =  
sum of the  
square of the  
other sides

Figure 3: The results of the work of proving the Pythagorean Theorem

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Based on Figure 3, P1 performs the skills of attending, interpreting, and responding to students' thinking focus on the interaction of a group of students with the teacher by the pattern 2: AA2-IA2-RA2. In attending, P1 performs the activities coded by AA2, the category for attending coded by X and Y. The subcategories of X that are done by P1 were X2, X3, and X4. The subcategory of Y is done by P1 only Y1. P1 asked one of the groups to write down the results of their work on the blackboard as shown in Figure 3 above. After finishing the work, P1 asked one of the group representatives to explain the results of their work. P1 applies the skill of attending, coded X2 by saying, "Now you read it. Explain this one. (pointing to the results of student work)". The following is an excerpt from the conversation between P1 and student representatives in a group (PS).

P1: Come on now you read. Explain this one.

PS: Square...(silent)

P1: Explain this. Come on, explain. The ABPQ square is formed by? (points to the image as shown in Figure 3)

PS: ABPQ square is formed...

P1: formed from...

PS: The ABPQ square is formed from...(silent)

P1: How many ways did it come from? (points to picture)

PS: Five

P1: Explain. There. Explain there (ask students to explain to their friends). If you look at the picture...

PS: ABPQ consists of 5 shapes

P1: That is?

PS: Triangle ABC

P1: How much?

PS: As much as 4

P1: Yes go on. Continue

PS: Square CDEF as much as 1.

It appears in the conversation above, the student was asked to prove the Pythagorean Theorem using some of the rectangle and right triangles used. P1 asks for X4 in reaching conclusions by providing guided questions (e.g., How many shapes are there?, How many?). P1 also provides hints/hints/keywords (ABPQ square formed by ?). P1 brings up sentence phrases (for example, when you look at a picture). For the Y activity, P1 asks students to point to pictures when explaining. This means P1 attending by coded Y1. In another conversation, it was also seen that the P1 activity asked students to point to the picture when explaining. The following is an excerpt of the conversation (the P1's statement in bold).

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P1: **Show. Show with your hands**

PS: side AB equals c, side AC equals b, side BC equals a (while pointing to the picture)

P1: Yes. Continue. BD?

PS: BD equals AC

P1: Yes go on

PS: CD equals BC minus BD

P1: **Where's BD? (ask students to point to the picture to show the position of BD). The result (pointing to the calculation result) is?**

PS: CD equals BC minus BD equals a minus b (while pointing to the work result)

In interpreting, P1 performs the activities coded by IA2. The categories for interpreting were coded by B, C, and E. The subcategory for E that is done by P1 was E2. P1 interprets students' understanding with code C by speaking true or false and uses words/sentence phrases as implicit interpretations of true or false answers. When the student's explanation is correct, P1 says "Yes, that's right". The P1's statement means to say the correct answer. P1 also uses the word as an implicit interpretation of the correct answer by saying "Yes go on", as seen in the two conversations above. Based on the student's writing, P1 asked twice why there were  $-2ab$  and  $2ab$  in one equation that disappeared in the next solution step. Group representatives always answer because the types of variables are the same. This student's answer is wrong. However, the teacher did not state the answer was right or wrong. This shows that P1 interprets with code B. The interpretation of code E is carried out with subcategories E2. The conclusion written by a group of students is "*Square of hypotenuse = sum of squares of other sides*". P1 asks "*Where does the square of the hypotenuse come from? What is the square of the hypotenuse?, Yes, where did that come from? There is the sentence square of the hypotenuse*".

In responding, P1 performs the activities coded by RA2. The categories for responding were coded by K, L, M, N, O, and P. The subcategories of K that are done by P1 were K1, K2, and K3. For the response activity in code K, P1 asks questions to explore students' thinking (using Why?), coded by K1. When the group representative said that the area of ABPQ was equal to c squared, P1 asked the students "*Why is c squared?*". On another occasion, after the group representative explained the results of their work, the teacher asked the students "*Let the children see this. How can it be missing  $-2ab$ ,  $2ab$ ? Why is it suddenly like this? Please explain.*" (points to  $-2ab$ ,  $2ab$ , and  $c^2 = a^2 + b^2$ ). Here's the conversation, P1 responding by "Why".

P1: So that we find that c squared is equal to a squared plus b squared. Now let's take it to the triangle ABC earlier. We bring c here (pointing to side AB). What is a in a right triangle?

PS: hypotenuse

Q1: Why does it say hypotenuse? It's not tilted though? (Points to an image)

PS: Because it is in front of a right angle.

P1 also responds with code L (say hello, got it?). P1 responds with code K2, continued with K3, and ends with K2. This fact can be seen in the following conversation between P1 and student representatives in the group.

P1: (takes a marker and writes on the whiteboard). The variable type is the same. How much is  $3a$  plus  $5a$  ?

PS:  $8a$

P1: How come it doesn't disappear? It says the type of variable is the same

PS: since it's a plus, it's a minus in the sum (pointing to  $-2ab$  and  $2ab$ )

P1: So what is the total result?

PS: zero

Pattern 4: AA2-IA3-RA3 appears in the lesson by P1 at meeting 2 with the material determining the type of a triangle based on the length of its sides and applying the Pythagorean Theorem. Each group of students is given an investigation sheet to determine the type of triangle based on the existing side length measurements, measure the angle of the triangle using an arc, and compare the squares of the length of the side of the triangle to conclude whether the triangle is acute, right or obtuse. The following Figure 4 is the result of the work of one group of students according to the problem in attachment 2 which is written on the whiteboard.

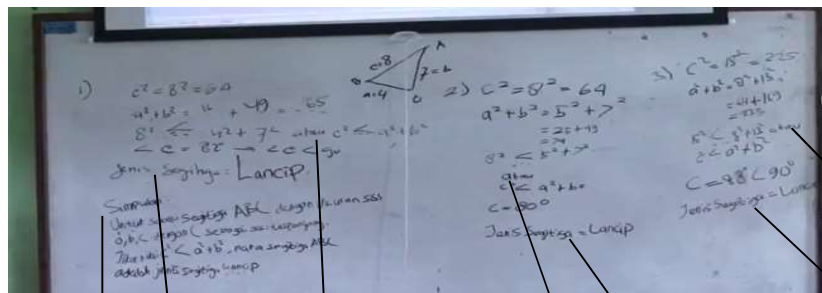


Figure 4: one of the results of the work of a group of students meeting 2 in class P1

Type of triangle: acute

Or

Or

Type of triangle: acute

Type of triangle: acute

Conclusion: For a triangle ABC with sides  $a$ ,  $b$ ,  $c$  (with  $c$  as the longest side). If the value  $c^2 < a^2 + b^2$  then triangle ABC is an acute triangle

The activity of attending in pattern 4, P1 the interaction of a group of students, coded AA2, in pattern 4 was carried out in category X, subcategory X1, and category Y, subcategory Y2. P1 knows that students have difficulty writing problems on the whiteboard so P1 assists by writing down some information that students must write on the whiteboard (see Figure 4, work 1). As long as the group representatives write the results of their work on the whiteboard, P1 attended to the



results of their work. This fact shows that P1 performs AA2 category X subcategory X1. Then P1 asked another group to state whether the work in Figure 4 was true or false and the group working on the same problem stated that the work was correct. This fact shows that P1 performs AA2 in category Y and subcategory Y2.

The activity of interpreting in pattern 4 is done by P1 in code IA3. P1 interprets students' thinking classically in category E, subcategory E1. P1 asked all students to add important information that should exist but have not been written by students, namely  $a$ ,  $b$ , and  $c$  as the sides of the triangle and  $c$  as the longest side of the triangle.

The activity of responding to P1 in pattern 4 is carried out in code RA3. P1 responded by giving general questions, category L to all students. P1 asked, "Do you agree or disagree? Did you understand? There are 3 conclusions, what are the conclusions?". P1 also responds classically in category K and subcategory K5. P1 writes one of the conclusions and students are classically asked to read the other conclusions together. The following is a conversation between P1 and all students (S).

P1: For a shape (meaning a triangle) whose sides are known,  $a$ ,  $b$ , and  $c$  with  $c$  the longest side.

One. If applicable  $c^2 < a^2 + b^2$  (read  $c$  squared less than  $a$  squared plus  $b$  squared) or in general language, if the square of the longest side is less than the sum of the squares of the other sides, which triangle is formed?

S: acute triangle (all students answered in unison).

P1: Let's read another conclusion

S: Two. If applicable  $c^2 > a^2 + b^2$  (read  $c$  squared more than  $a$  squared plus  $b$  squared) or in general language, if the square of the longest side is more than the sum of the squares of the other sides, the triangle formed is obtuse (all students answered in unison).

During the third meeting, the teacher discussed the kinds of triangles (acute, right, and obtuse) with the Pythagorean Theorem. The teacher gave three problems shown in Figure 5.

Mr. Danar is a mathematics teacher. To teach about two-dimensional figure in the classroom, he would make a teaching instrument of triangle. Based on the previous design, the triangle would have a perimeter of 12 meters.

- a. Find out the triangle's sides to make an acute triangle!
- b. Find out the triangle's sides to make a right triangle!
- c. Find out the triangles' sides to make an obtuse triangle!

Figure 5: Problem 2 in P1's class

A group representative wrote their solution for a problem in Figure 5. The P1's pattern for attending, interpreting, and responding for solution 2a was Pattern 3. Figure 6 below shows the results of the work of one group of students and P1 shows the mathematical principles used.

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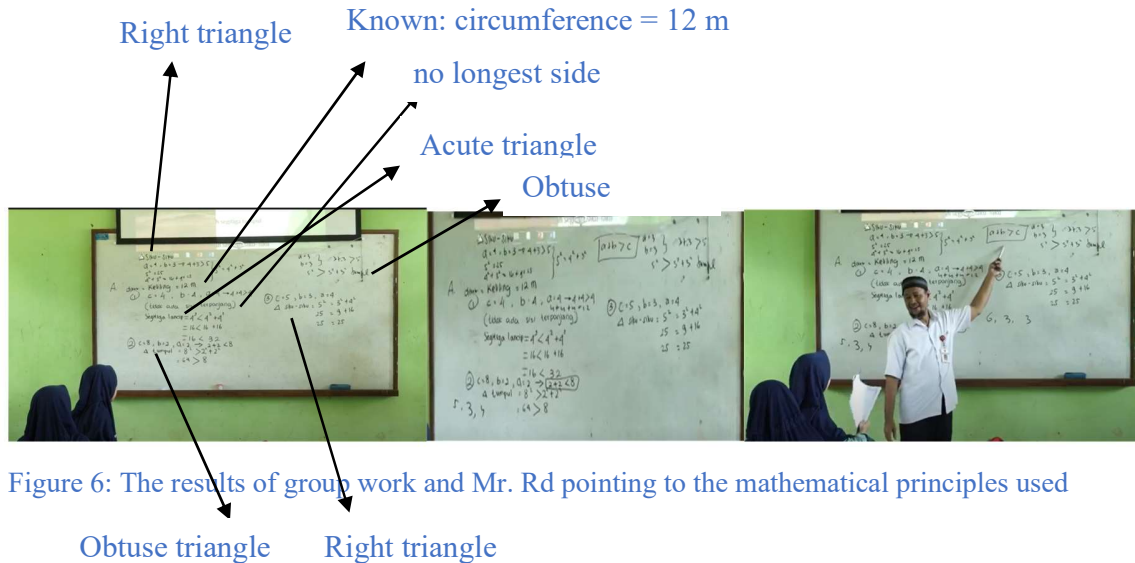


Figure 6: The results of group work and Mr. Rd pointing to the mathematical principles used

The excerpt of the discussion below was between teacher (P1) and student (S)

P1: First, they (he meant the group that wrote the solution) choose 4, 4, 4 with a requirement to not use the longest side. Yes. Is  $4 + 4 + 4 = 12$ ?

S: Yes

P1: Okay. It means... can I borrow the marker? (The teacher wrote  $4 + 4 + 4 = 12$ ). This is the requirement for the perimeter. It needs the perimeter requirement. The perimeter is 12, then,  $4 + 4$  equals what?

S: 8

P1: 8 is more than 4. The requirement of inequality of  $a + b > c$  ( $a$  plus  $b$  greater than  $c$ ). Right? (Writing  $a + b > c$  on the board)

S: Yes.

P1: You need to remember this. You can't do it randomly. You need to use this postulate. You can't choose any random number without meeting this requirement (Pointing to  $a + b > c$ . Very well. The requirement is to fulfill  $4 + 4 > 4$ . Right? (Pointing to  $4 + 4 > 4$ )

S: Correct.

The excerpt shows the attending skills coded AA2 (attending the interaction between students' group and the teacher), category X and subcategory was X3 (in this case not in detail), and X5. P1 reminded triangle inequality and perimeter conditions problem in Figure 5, coded as X5. P1 interpreted the groups' work with the code of IA2, with the category coded was E and the subcategory coded was E1 (the perimeter requirements). Next, P1 responded with the code RA3. The category code was K and the subcategories were K3 and K5. P1 asked if the three measurements (4, 4, 4) fulfill the perimeter requirements, if  $4 + 4 > 4$  fulfill the inequality requirement of side lengths to students classically, coded as K3. The P1's responding skills were shown by a classical comment about the relevant concepts of the requirements of perimeter and inequality of a triangle measurement, coded by K5.

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Dealing with works on the board in Figure 6 as a solution problem part c in Figure 5, P1 also applied the skills of attending, interpreting, and responding with pattern 1 (AA2-IA2-RA2-IA2-RA2-RA3). P1 used attending skills with the interaction code between a group of students and the teacher. The category of attending was coded by X and the subcategory of attending was X1. Next, the teacher applied interpreting skills of the interaction between a group of students and the teacher (IA2). This category was E and the subcategory E3. P1 asked the groups if (8, 2, 2) fulfill the perimeter and triangle inequality requirements. It turned out that the groups were aware that (8, 2, 2) fulfill the requirements of triangle perimeter but did not fulfill the requirement of inequality. P1's response had the interaction code of RA2 with the category coded by K and the subcategory was coded by K4. This interview excerpt shows P1's question, "*Why do you choose?*" The next process was the skills of interpreting coded IA2, and the subcategory E1. That was the information on the instructions written on the discussion paper, the requirements of triangle sides inequality that if the longest side was  $c$ , then the formula should be  $a + b > c$ . The next skills were the responding coded RA2, the category coded by M. The P1 asked other groups about the lengths of triangle sides to make an obtuse triangle. One of them answered (6, 3, 4). The P1 did not say it correctly or wrong, but applied interpreting skills code IA2, the subcategory E3. The P1 asked other groups if (6, 3, 4) fulfilled the triangle inequality and perimeter requirements or not. The following process was the skills of responding coded RA2, the category coded by M. Another group answered (6, 3, 3). The teacher interpreted it with code IA2 and the subcategory E3. The P1 commented that (6, 3, 3) fulfill triangle perimeter requirements and asked if the numbers fulfill the triangle inequality requirements. The processes of IA2 and RA2 were repeated in students' solutions of (7, 2, 3); (6, 4, 2); (6, 5, 1); (5, 5, 2). Finally, the teacher provided a response code R3 (classical interaction between students and teacher) category coded by P. See the discussion excerpt between P1 and a group's representative (KS).

P1: Now about this problem (Meaning problem part c as shown in Figure 5 ).  $8 + 2 + 2$  equals what?

KS: 12

P1: It meets the requirements of triangle perimeter, doesn't it?

KS: Yes

P1: What about the requirements of the triangle?

KS: No

P1:  $2 + 2 < 8$ . Is it one of the triangle requirements?

KS: No

P1: Hello...Is it one of the requirements? (Looking at other groups)

KS: No

P1: Why do you choose? (Looking at the group that did the problem)

P1: What? It was ... what was the information? It was (The P1 referred to the investigation sheet) information about what you used, right? You used the requirements of triangle sides inequality. If  $c$  is the longest side, it should be  $a + b$  ( $a$  plus  $b$ ) more than?

KS:  $c$

P1:  $a + b > c$ . That's your guide. That's the formula. Okay, what about other groups? What about yours?

KS: (6, 3, 4)

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P1: According to group 4, it should be (6, 3, 4) (Writing 6,3,4 on the board). 6 + 3 + 4 equals what?  
(Asking to group 4)

KS: 13

P1: What? 13? What is the perimeter? (Asking other groups)

KS: 12

Research findings on P2's skills in attending, interpreting, and responding to students' mathematical creative thinking. P2 delivered the lesson on the function and equation of the straight line. P2 applied two patterns of skills; pattern 1 was AA2-IA2-RA2, while pattern 2 was AA1-IA1-RA1-RA3, as shown in Figure 7.

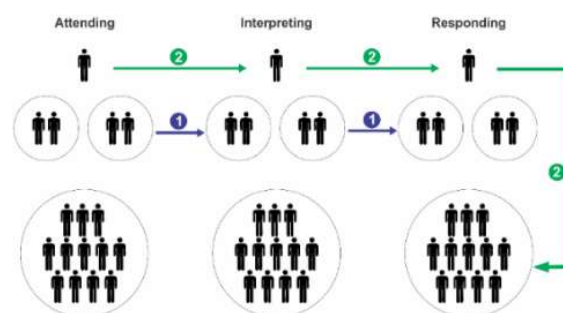


Figure 7: Patterning process of three skills by P2

ht

line. Two problems were given. One of which is shown in Figure 8 below.

1. It is known that point  $P = (0,5)$  and  $Q = (5,0)$ 
  - a. Draw line  $g$ , straight line through point P and Q
  - b. Formulate the equation of line  $g$  through P and Q
  - c. Draw other straight lines parallel to line  $g$  (at least three lines)
  - d. Create the formula for straight lines made of question c). Explain

Figure 8: Problems discussed by P2 at class

P2 asked students to work the problems in pairs and went around to see their discussion process. P2 also gave questions to trigger each group to reconstruct the relationship between the line equations, parallel line, gradient of the line, and gradient of parallel lines. The short academic hour made the application of attending, interpreting, and responding to students' creative thinking get started when a group started to finish their task even if it was incomplete. So, those skills were not applied to the finished answer.

During the first meeting, P2 used pattern 1 of AA2-IA2-RA2. The skill of attending with a code of AA2 was applied with category X and subcategory X2. Some pairs responded, "by drawing."

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P2 interpreted the statement by saying, "Yes," and coded it IA2. This showed that the P2 interpreted the category C. P2 responded with code RA2, category K, and subcategory K2. P2 demonstrated the steps for drawing a straight line through the points of P and Q and called it the line g. When students were drawing it too, P2 attended to her students' working process and result. This was coded AA2, category X, and subcategory X1. The skills of interpreting were coded IA2, the category C. P2 do it often. Next, the responding skills were coded RA2 with the category K and the subcategory K4. P2 commented that some students made mistakes when writing the letter g. Some students wrote it with capital G, but it should be lowercase since it represented a line. The next mistake was that some students drew a line that looked like a line segment, so P2 asked them to draw a long line to show their difference. During the first meeting, P2 frequently applied pattern 1 for questions b, c, and d. The excerpt below was the discussion between the teacher (P2) and some pairs of students (PS) when doing problem 1a in Figure 8.

P2: How? How to draw a line g? The line g must pass through the points P and Q. How?

PS: Drawn

P2: Yes. Stay connected to point P and point Q. Connect point P and point Q to form a straight line. Name the line. Yesterday I had told you how to name lines. Name the line. Maybe at the bottom, maybe at the top. Name the line.

PS: (Drawing a line). We think we made a mistake....

P2: (Attending the drawing process and seeing the result of the line g) That is correct. The letter g should be lowercase. Make it long, don't make it too short, just like that. Line g could be above it or below. (Referring to the position of letter g while pointing to the line they drew)

Students in the group have difficulty solving problem 1 part c in Figure 8. This makes P2 detail what strategies the students should make. P2 attended to category X and subcategory X3. P2 stated *"Take a look first. You have found the equation for the line g. If you have trouble answering question c, what does question c mean? You are asked to draw parallel lines. Parallel lines have the same gradient. Now, the line you will make later the gradient, the slope is the same. You've learned to make gradients. Remember the formula is the value of y per value of x"*.

In general, in interpreting the results of student work on all problems in the first meeting, P2 often said that the process or the result of student work was right or wrong, either explicitly or implicitly (saying yes right, yes right, wrong, wrong, okay, yes, okay continue ). This shows P2 interprets in category C. P2 also frequently interpreted the interaction between students' groups with the teacher in category F. Some of the P2 statements are written below.

"Your drawing is too small."

"Make the line precise. Draw it long. Don't limit it. Okay, do it again."

Pattern 2 is applied by P2 with the same categories and subcategories as pattern 1. The difference is only in its application in individual student interactions with the teacher.

The research results of P3's skills in attending, interpreting, and responding to students' mathematical creative thinking. P3 delivered the learning material of function and the equation of a straight line. He applied those three skills in two patterns, as seen in Figure 9. Pattern 1 was AA2-IA2-RA2, while pattern 2 was AA2-IA2-RA2-RA3. P3 dominantly applied both patterns.

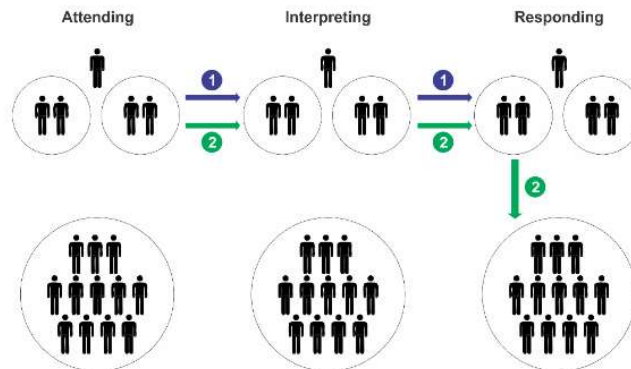


Figure 9: The third pattern of those three skills by P3

In general, P3 conducted the learning process by doing pair discussions as P2 did. When students discussed in pairs, P3 approached each pair. P3 helped them think by asking guided questions and reminding them about their learned concepts. The limited learning time available made the application of attending, interpreting, and responding to students' creative thinking get started when a pair started finishing the problems even if it was incomplete.

During the second meeting, P3 delivered the material of the equation of a straight line. He evenly distributed three problems to six pairs of students, so each problem was done by two different pairs. Figure 10 below shows problem 3 in the P3 class.

#### Problem 3

Draw a rectangle  $ABCD$  with the length of 5 units and the width of 3 units on a Cartesian Coordinate

- Formulate the equation for line segments  $AB$ ,  $BC$ ,  $CD$ , dan  $DA$
- Decide the domain and range for each equation in question a)
- Describe the connection between each line (For example, based on its gradient)

Figure 10: P3's mathematics problem

P3 asked one pair with problem 3 in Figure 10 to write down their work on the board. It seemed that P3 applied pattern 2 (AA2-IA2-RA2-RA3) of the process of attending, interpreting, and responding to students' mathematical creative thinking. In the skills of attending, the interaction was between students in group and teacher (AA2), the category X with the subcategory X3 (in this

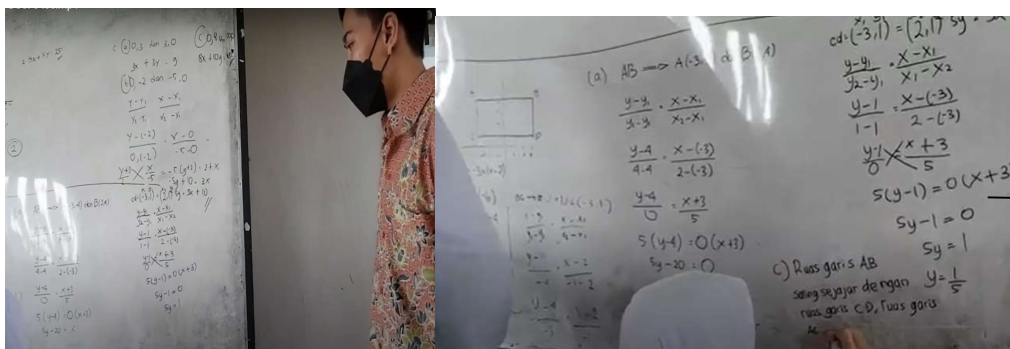
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case in detail). The statement of P3 is written below. P3 is also attended by category Y with subcategory Y2 (for the last statement).

*"This pair draws a rectangle on a Cartesian Coordinate. The length is 5 units, and the width is 3 units. The position is not limited. The first pair draws it like this. So, you can draw it anywhere (Meaning everywhere on the Cartesian Coordinate) as long as the length is 5 units and the width is 3 units...The drawing does not need to be the same. I remind you that the drawing does not need to be the same. The drawing is like this. Then, the equation of each group would be different.*

In Figure 11 below, the student representatives wrote down the process of finding the equation of the line CD through C(-3,1) and D(2,1) by writing  $\frac{y-1}{0} = \frac{x+3}{5} \leftrightarrow \frac{-5}{0}$ . P3 does not immediately blame the student's work but asks the students to multiply the cross and write it down to be  $5(y-1) = 0(x+3)$ . Students are then asked to describe it and produce  $5y-1=0$ ,  $5y=1$ . The job is wrong, it should be  $5y-5=0$ ,  $5y=5$ ,  $y=1$ . P3 does not blame, but allow other students the opportunity to comment. When the student representative looks for the equation of the line AB with A(-3,4) and B(2,4), she wrote  $5(y-4) = 0(x+3)$ ,  $5y-20=x$ , P3 does not blame or justify but ask the question "0 when multiplied by x how much?". The student realizes her mistakes and justifies the next calculation process. This shows that P3 interprets in category E and subcategory E3. P3 always interprets by giving a checkmark on the right solution. This shows that P3 interpret in category C.



The line segment AB is parallel to the line segment CD, the line segment AC

Figure 11. Written answers of group representatives in class P3

P3 applied the skills of interpreting in the form of interaction code IA2, category C (just as P2), and category E with the subcategory E3.

A pair asked P3 about problem 3 part c in Figure 10. P3 guided them by asking them a question.

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P3: Now, look. The line segments AB and AC are what?

KS: Perpendicular

P3: Yes, they are perpendicular. You may choose. AB and CD are parallel. AB and AC are what? You may choose. The c. It's the example. You may take another one.

After the pair understood, one of them wrote on the board. P3 demonstrated the skills of responding to code RA2, category K with the subcategory K2. P3 then applied the skills of responding to the creative thinking coded RA3 by showing various possibilities of rectangle shapes and available solutions based on the rectangles made by students. P3's statement below showed category P, mainly associated with the open-ended problem and creative thinking.

*This is the solution of group 1. For the other groups, your solution does not need to be the same. Yes. Drawing a rectangle. I will show you some examples. It could be like this. (P3 drew some possible rectangles on the Cartesian Coordinate) You may put your rectangle here, as long as the length is 5 units and the width is 3 units. You may also you're your rectangle here (P3 drew a rectangle) ... The most important thing is that the line segment of AB and CD are parallel. AC and BD are also parallel. AC and AB are perpendicular. CD and BD are also perpendicular too.*

Pattern 1 is applied by P3 with the same categories and subcategories as pattern 2. However, P3 does not respond classically.

The three teachers applied the skills of attending, interpreting, and responding in various ways, as shown in Table 1. The similarity and differences practice of three skills by three teachers shown in Table 2. Based on Table 2, the three teachers applied the skills of attending students' creative thinking in category X and its subcategory was X3. The teachers detailed the thinking strategy their students used with an expectation that they would understand the concept and thinking strategy better to solve the mathematics problems. According to Jacobs et al. (2010), the strategy in detail is essential as it could be a window of children's understanding. In mathematics class, attending students' strategies effectively involved the teachers in tracing all strategies applied by students with a focus on the essential mathematics components Styers et al. (2020). The teachers' attending skills of detailing or not detailing students' creative thinking strategies were used as the fundamental of interpreting students' thinking. This statement is supported by Amador et al. (2016) that itemizing students' thinking strategy is one of the characteristics of advanced skills of attending, and it underlies the teachers to interpret students' mathematical thinking. P1 frequently allowed other groups to assess the correctness or incorrectness of the work of a group written on the board. P3 did the same and also said that students could have various solutions for problem 3 depending on the shapes of the rectangle they drew. P1 and P3 attend by category Y and the subcategories Y2. Their decision not to tell them the correct solution encouraged them to validate their argument and finally led them to be independent and confident learners (Francisco & Maher, 2011).

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Noticing Component	Similarity	Difference
attending	All teachers, attending in category X. Its subcategory was X3.	P1 and P3, attending by category Y and the subcategories Y2 P1 and P2 attending by category X and the subcategories X1 and X2 P1 attending by category X and the subcategories X4 and X5 P1 attending by category Y and subcategory Y1
interpreting	All teachers interpreting in the category C	P1 and P3, interpreting by category E and subcategory E3 P1, interpreting by category B, and category E and sub-categories E1 and E2 P2, interpreting by category F
responding	All teachers responding in category K with subcategory K2	P1 and P2 responding by category K with subcategory K4 P1 and P3 responding by the category P, mainly associated with the open-ended problem and creative thinking P1 responding by category K with subcategories K1, K3, and K5; and category L, M, N, and O.

Table 2: Comparison of noticing between teachers with different teaching experiences

P2 and P3 often stated the correctness or incorrectness of students' work. P3 also applied checkmarks to point out the correct solution. From the discussion excerpt, it was known that P1 also did it by smiling. It showed that those three teachers interpreted their students' understanding with a category C. The teachers' comment on the correctness of the solution was considered to show the characteristics of students' solution, while the incorrect solution showed students' confusion, wrongdoing, or misunderstanding (Crespo S, 2000).

P1 and P3 interpret by category E and subcategory E3. This demonstrated that P1 and P3 did not quickly claim that their students understood or not, but they tried to explain further the mathematical meaning of solving a mathematics problem. P1 reduced the interpretation of the correctness or incorrectness of their solution and changed the discourse into the activity of finding out the information about the mathematics information and concept his students missed when solving the given problem. Based on Crespo et al. (2000) stated that the teachers' experience in interacting with students helps them change their students' understanding.

P2 discussed with her students by pointing out their mistakes when drawing the line and Cartesian Coordinate. P2 commented on their work directly by telling them to draw long lines to differentiate it from line segment and asking them to draw a large Cartesian Coordinate. P1's questions

confirmed students' mistakes, and finally, they became aware of their thinking mistakes. P1 ended the question-and-answer session by saying, "Why did you choose it?". P2 and P1's follow-up action was asking students to revise their mistakes. This illustrated that teachers could apply various ways to point out their students' mistakes and fix them. Those mistakes could be helpful for the teachers. Based on Shaughnessy et al. (2021) argued that students' mistakes help teachers learn their students' thinking patterns and get them to learn the mathematics contents better that could be used to build interactions with students.

The teachers applied the skills of responding during the interaction between students in groups. It was always ended with responding to the classical interaction between students and them. They also demonstrated the skills of responding by supporting students' thinking. The way is the teachers brought up guided questions to help students solve the problems. P1 responded to his students' thinking by widening their ideas. P1 widened his students' thinking by giving a follow-up question, "Is there any side length that makes an obtuse triangle?".

Responding children's mathematical thinking could be done by asking questions designed to support or broaden students' thinking (Jacobs & Ambrose, 2020), redirect (Lineback, 2015), or introduce "further problem" (Jacobs et al., 2010), and pose the problem responsively (Land et al., 2019). The teachers' activities to support students' thinking were to make sure that their students understand the problem, change the mathematical problem to suit students' understanding, explore anything they had done, and remind them to apply other strategies (Jacobs & Ambrose, 2020). Some of the teachers' activities to widen students' thinking were to support the reflection of the strategy they had applied, to find out more strategies and discover the relationship between them, make connections between their thinking with symbolic ideas, and bring up further problems related to the problem they had just solved (Jacobs & Ambrose, 2020).

## CONCLUSIONS

The mathematics teachers raise the attending skills in two categories, namely, activities that are directly related to students' mathematical creative thinking and activities that support students' mathematical creative thinking. The subcategories for activities that are directly related to students' mathematical creative thinking are attending the process/result of students' creative thinking; asking for an explanation about their thinking steps; detailing or not detailing students' thinking strategies; asking for justification for their reasoning (by giving guided questions, giving hints/keywords, bringing up sentence phrases, asking students using "How and Why"); reminding the concepts and principles they had learned or the relevant ones; and emphasizing what they had done. The subcategories for activities that support students' mathematical creative thinking are attending their articulation, how students draw, and students gesture when they explain thinking; allowing their mathematical reasoning development (provide the widest opportunity for students

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to explore mathematical ideas, provide opportunities for students to state true or false the results of thinking students or other groups of students, say that students can get various answers); and asking them if they had finished the tasks.'

The mathematics teachers interpret students mathematical creative thinking in various ways, namely excluding justification of right or wrong answers; focusing on right or wrong; focusing on the only correct solution; focusing on deficiencies in the student's working process; focusing on how they wrote and drew on grid paper and anything else indirectly connected to mathematical thinking; stating that working on the problem is easy; and comparing the thinking process of the teachers and students. The subcategories for focusing on the deficiency of students' working process are adding important information missing; asking for the clarification of students' statements; giving questions or comments about students' reasoning to check whether the student's answer is correct or not; pointing out students' mistakes related to the procedures of doing the problems; and asking if their students were aware of their mistakes made in the process.

The mathematics teachers responded by giving comments or questions about their students' knowledge, idea, procedure, or mathematical thinking and based on mathematical creative thinking; giving general comments or questions (such as any question, do you understand, can you do it, and what is the conclusion); giving comments or questions to trigger students to share their opinions; asks other students to comment or ask questions about certain students' thinking ideas; asking other students to explain certain students' thoughts; giving comments or questions about students' mathematical creative thinking/open-ended problem/critical thinking. The subcategories for giving comments or questions about their students' knowledge, idea, procedure, or mathematical thinking and based on mathematical creative thinking are giving comments/questions to examine students' thinking (such as using why and how questions); giving questions/guided comments to help students think creatively about mathematics; giving follow-up questions to confirm students' mathematical reasoning; giving comments/questions about students' thinking mistakes; and giving comments about the relevant concepts/principles/calculations.

Senior teachers and junior teachers responded with teacher comments/questions related to mathematical creative thinking/open-ended questions. Teacher responses in the form of teacher follow-up questions to confirm students' mathematical reasoning were carried out by senior teachers. Senior teachers and young teachers respond with teacher comments/questions related to students' thinking mistakes. So, there is a need for further research to explore why students make mistakes in mathematics as part of the teacher's skills in interpreting students' mathematical understanding. Teachers in this study did not receive training in the skills of attending, interpreting, and responding to students' mathematical creative thinking. The follow-up is to provide training to mathematics teachers in these three skills as their professional development and to further investigate the implementation of these three skills in the professional development of mathematics teachers. Prospective mathematics teachers can learn these three skills so it is

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necessary to do research of these three skills by involving prospective mathematics teachers so that they have the initial experience to improve the quality of the implementation of the three skills in their teaching practice in the future.

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## A New Tool for the Teaching of Graph Theory: Identification of Commognitive Conflicts

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**Abstract:** *In this exploratory work, the discourse of first-year computer engineering undergraduate students of graph theory was analyzed with the aim of improving the teaching of this branch of mathematics. The theoretical framework used is the theory of commognition, specifically, we focus on commognitive conflicts because they are learning opportunities since they foster the learning process when resolved by students, and so teachers should consider them in their practice. A qualitative analysis of the written responses to a questionnaire dealing with definitions and the concepts of path and cycle graphs was performed. Thus, several commognitive conflicts were found, coming from the confluence of discourses governed by different discursive rules. Furthermore, the conflicts encountered were classified according to their origin, into object-level and metalevel conflicts. Concretely, the object-level conflicts had to do with the school discourse of geometry or sequences, and with the discourse of directed graphs; the metalevel commognitive conflicts were associated with the school discourse of the mathematical practice of defining. Finally, our findings are contrasted with related works in the literature, and also a series of implications for the teaching of graph theory are presented.*

**Keywords:** Graph theory, written discourse, commognitive conflict, object-level, metalevel, teaching

### INTRODUCTION

Discrete mathematics is dedicated to the study of discrete structures, i.e., finite, or numerable sets. It is of growing interest in society because although it is a relatively new branch of mathematics, its recent development is “linked to the evolution of the society and also other disciplines such as computer science, engineering, business, chemistry, biology, and economics, where discrete mathematics appears as a tool as well as an object” (Ouvrier-Buffet, 2020, p.182). Discrete mathematics comprises several areas, among which graph theory stands out. In few countries, graph theory is studied in secondary education, but in most countries, it appears for the first time in undergraduate courses, mainly in mathematics or engineering (González et al., 2019; Milková, 2009; Vidermanová & Melušová, 2011), as it is a powerful tool for modeling reality. Authors like

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Rosenstein (2018) advocate for their inclusion in the school curriculum. Apart from their usefulness in modeling, other reasons they present are that many of their concepts and problems can be understood without having great knowledge in mathematics and that they allow the development of mathematical practices like those of professional mathematicians (Balsim & Feder, 2008; Ouvrier-Bufferet, 2020).

Regarding the teaching of graph theory, there are research works that describe didactic sequences or resources to be used in the classroom (Cartier, 2008; Costa et al., 2014; Geschke et al., 2005; Hart & Martin, 2018; Smithers, 2005; Wasserman, 2017). However, all these works do not deepen into students' reasoning. In fact, there are only a few studies considering this issue such as the work by Hazzan and Hadar (2005), who analyzed understanding from the perspective of the reduction of abstraction. Also, Medová et al. (2019) studied the errors made by students when using algorithms, and more recently, González et al. (2021) extended the Van Hiele model to characterize students' reasoning in graph theory.

Gavilán-Izquierdo et al. (2021) propose a new perspective to consider students' reasoning during the teaching process of graph theory: the sociocultural theory of commognition (Sfard, 2008). This framework has as its focus the study of discourse and its change. This discourse can be spoken or written, and learning is perceived as a change in discourse. Thus, learning can occur when commognitive conflicts appear, that is, when participants in the discourse act according to different discursive rules. The resolution of these conflicts results in learning.

This paper aims to investigate the written discourse of first-year undergraduate students of graph theory, thus identifying possible commognitive conflicts that teachers may use in their classrooms. In addition, based on our findings, we intend to provide a series of guidelines for the teaching of this area.

### *Literature Review*

There are several recent works on commognitive conflicts. We present a synthesis of these works in chronological order, focusing on those studying the secondary-tertiary transition (Gueudet & Thomas, 2020) since they will be helpful in the discussion of our results.

Research has revealed different commognitive conflicts in a variety of concepts. Jayakody (2015) identified several commognitive conflicts in university students when they approached the concept of continuous functions. In his data collection instrument, she included a questionnaire in which they had to first describe and then define what a continuous function is. Two types of commognitive conflicts were identified: A first one, interpersonal, in different uses of the word "domain" and a second one, intrapersonal, arising from different inconsistent definitions extracted from textbooks. This author concludes that the identified conflicts provide information about the thinking of functions and thus have implications for teaching. Also, Ioannou (2018) identified commognitive conflicts in first-year university students when studying group theory. The first

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conflict has its origin in that these students at the university tend to consider that all sets have an internal composition law defined (e.g., the set of integers with addition). This author points out that “in the secondary education mathematics discourse, mathematical sets have algebraic structure, and, in particular, a binary operation with some properties. The notion of a set without an operation is new for novice undergraduate students” (Ioannou, 2018, p. 140). He also identified a second commognitive conflict in relation to the characteristic that a set can be empty, which does not occur in high school since all sets that appear in this stage have at least one element. Moreover, in high school sets are usually described in terms of their elements and not axiomatically, as often happens at university. Then both conflicts are closely related to the transition between school mathematics and university mathematics. The author concluded that his findings have some teaching implications, such as the need for these teachers to guide their students when using old ideas of school discourse in a new tertiary discourse.

Thoma and Nardi (2018) focused on manifestations of unresolved commognitive conflicts in first-year university students, such as the absence of a specification of the appropriate numerical context in tasks; the confusion between the symbols and the rules of school algebra and set theory discourses; the symbols of probability and set theory discourses; and between the symbols and rules of the probability theory discourse. They deduced that teachers should make a more explicit and systematic presentation of the differences between discourses and facilitate flexible movements between them. Regarding applied mathematics, Viirman and Nardi (2019) detected some commognitive conflicts in undergraduate biology students when performing mathematical modeling tasks. Some conflicts were classified as intramathematical (relating to what is understood as a math task) and others as extramathematical (relating to what constitutes a valid solution to the tasks in the biological sense). These authors believe that considering these conflicts in the teaching of this subject would be beneficial for learning.

Schüler-Meyer (2019) investigated conflicts in upper secondary students on the topic of elementary number theory and stated that “difficulties in learning processes in transition can be conceptualized as the students’ attempts to communicate in secondary discourses while being engaged in tertiary discourses.” (Schüler-Meyer, 2019, p. 165). His results illustrate the convenience of dealing with and explaining metanarratives when teaching. Fernández-León et al. (2021), agreeing with Sánchez and García (2014), also indicated that it is possible to consider conflicts between the students’ discourse and the academic discourse of mathematicians. Specifically, they identified conflicts in the discourse of undergraduate students when constructing definitions, due to differences between the metarules of the students’ activities and the sociomathematical norms.

González-Regaña et al. (2021) identified different commognitive conflicts in first-year undergraduate students (of the bachelor’s degree in primary school education) when describing and defining geometric solids. They classified them as object-level and metalevel commognitive

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conflicts. An example of a commognitive conflict is described as follows: “The passage from a mathematical discourse that allows describing geometric objects [...] to one that is capable of elaborating formal definitions of these objects entails a development at the metalevel of the first discourse” (González-Regaña et al., 2021, p. 93). On the other hand, an example of an object-level conflict they found is the use of 2D geometry metarules to solve 3D situations. Thus, in the first case, there is a metalevel conflict between discourses about the mathematical practice of defining, and in the second there is an object-level conflict between the concepts of 2D and 3D geometry.

Finally, Kontorovich (2021), using the construct of precedent pockets as prior experiences in learning, analyzed the intrapersonal commognitive conflicts that appear in preuniversity students when using the concept of square root. He recommends that teachers provide students with tasks that are sufficiently varied and well thought out to be able to detect these commognitive conflicts.

### ***Theoretical Framework***

#### *Fundamentals of Graph Theory*

We first state the concepts of graph theory (Rosen, 2019) that will appear in this paper to ensure that it is self-contained. We say that a *graph*  $G$  is a pair  $(V, E)$ , where  $V$  is any set (called the set of *vertices*), and  $E$  (the set of *edges*) is a set of unordered pairs of vertices. Two vertices are *adjacent* if they form an edge, and the *degree* of a vertex,  $d(v)$ , is defined as the number of vertices adjacent to it, i.e.,  $d(v) = |\{u \in V \text{ s. t. } \{u, v\} \in E\}|$ . Although there are several systems of graph representation, one of the most common consists of associating points in the plane to the vertices that are joined by lines, provided that the corresponding vertices are adjacent (*pictorial representation*). This type of representation is not unique, as shown in Figure 1 for the graph  $G_1 = (\{a, b, c, d, e\}, \{\{a,b\}, \{b,c\}, \{c,d\}, \{d,e\}\})$ .

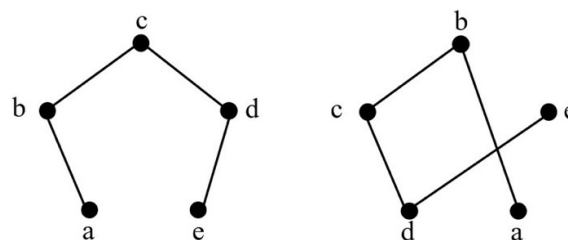


Figure 1: Two pictorial representations of graph  $G_1$

There are other graph representation systems besides the set representation and the pictorial representation, such as matrices, degree sequences, intersections of objects, etc.

Formally, two graphs are said to be *isomorphic* if there exists a bijection between their vertex sets such that it preserves the edges. In Figure 2, two isomorphic graphs with bijection  $a \leftrightarrow w, b \leftrightarrow x, c \leftrightarrow y, d \leftrightarrow z$  can be seen.

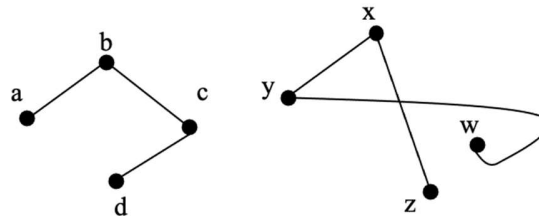


Figure 2: Two isomorphic graphs

A graph  $G = (V, E)$  contains a graph  $G' = (V', E')$  as a *subgraph* if  $V' \subseteq V$  and  $E' \subseteq E$ . On the other hand, a graph is said to be *connected* if any pair of vertices can be joined by a sequence of adjacent vertices. This property allows us to define classical families such as *paths*, which are connected graphs having two vertices of degree one and the rest of degree two, and *cycles*, which are connected graphs with all their vertices of degree two. Finally, a graph is said to be *directed* if its set of edges is a set of ordered pairs of vertices.

### *The theory of commognition*

We have selected the sociocultural theory of commognition (Sfard, 2007, 2008) to analyze our data. As Presmeg (2016) stated, this is a framework with a high potential to consider issues of teaching and learning of mathematics. The word *commognition* derives from the words “communication” and “cognition”. This theory holds that thinking does not exist without discourse, and reciprocally. Therefore, changes in mathematical discourse produce changes in mathematical thinking and, likewise, changes in thinking about mathematics produce changes in discourse, that is, in the way students communicate mathematically. According to this theory, mathematical *learning* occurs when the discourse is modified, extended, and enriched academically, in short, when the discourse changes, and teaching involves facilitating these changes. This theory offers a holistic view of mathematical learning since it considers the types of change that result from learning, the process followed by the participants (students and teachers) to achieve the change, and the expected results of the change (Sfard, 2007).

According to the theory of commognition (Sfard, 2007, 2008), mathematics is a type of discourse, and discourse is mathematical when it does not refer to material, tangible objects but to mathematical objects, which are abstract discursive objects, constructed in discourse, with

signifiers considered mathematical (numbers, geometric figures, etc.). Mathematical discourse is characterized by these four properties:

-*Word use*. It encompasses the use of terms that are specifically mathematical (e.g., subgraph) and the use of common language words that may have mathematical meanings (e.g., degree).

-*Visual mediators*. These are the means through which discourse participants identify the objects to which they are referring and coordinate their communication. While in colloquial discourse visual mediators are concrete material objects (which may be present or mentally visualized), more specialized discourses often involve symbolic artifacts, created specially for this form of communication. Some examples are mathematical formulae or diagrams.

-*Narratives*. These statements report the characteristics of mathematical objects, their properties, or relationships between them, and are subject to acceptance or rejection by the community. If they are considered true because they have been substantiated by the community, they are called endorsed narratives. An example in mathematical discourse is the statement “all vertices in a cycle have degree 2”.

-*Routines*. These are delimited and identifiable recurrent patterns in the discourse. They can be inferred from the discourse by observing word use, visual mediators, and analyzing how narratives are created and endorsed. Examples of routines specific to mathematics are defining, conjecturing, proving, executing an algorithm, etc.

As we have pointed out above, in the theory of commognition, learning mathematics means changing the discourse (i.e., its properties). Two types of mathematical learning can be distinguished. On the one hand, at the object level (object-level learning), which is expressed in the enrichment of the discourse through vocabulary expansion, construction of new routines, and production of new endorsed narratives, and, on the other hand, at the metalevel (metalevel learning), which implies changes in the metarules of the discourse. That is, some usual tasks, such as defining an object or identifying geometric figures, are done in a different way than usual, and some words used so far may change their meaning (Sfard, 2008).

Learning occurs mainly when the learner encounters a new discourse. If this new discourse is governed by different rules than the ones he/she knows so far, what Sfard (2008) called a *commognitive conflict* arises in the learner. The resolution of a commognitive conflict results in learning. We will adopt here the classification of commognitive conflicts used by González-Regaña et al. (2021), who distinguish between *object-level* and *metalevel* commognitive conflicts. Object-level commognitive conflicts are those that, when resolved, produce object-level learning, that is, those related to mathematical concepts. Similarly, metalevel conflicts are those that, when resolved, produce metalevel learning, that is, those related to mathematical practices, such as defining, conjecturing, or proving.

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Thus, in this work, of an exploratory nature, we will focus on the construct of commognitive conflict. As mentioned above, it plays a relevant role in discourse change. Specifically, the research objectives addressed are as follows:

- Identify evidence of commognitive conflicts in the written discourse about graph theory of first-year computer engineering students.
- Classify these commognitive conflicts, and, if possible, determine their origin.
- Present some general recommendations for the teaching of graph theory based on the identified commognitive conflicts.

## METHOD

Considering the nature of our data and the research objectives, an interpretive qualitative methodological approach is used. The participants and the context of the study, the instrument used for data collection, the data collection process, and how the data analysis was carried out are described below.

### *Participants and context*

The study involved thirty-nine students (numbered from 1 to 39) from a group of the basic first-year course “Logic and Discrete Mathematics”, taught in the first semester, with a duration of 60 hours. This group included students from different computer engineering degrees from the Polytechnic University of Madrid in Spain. The previous training required to take this subject is the one that any student who has taken a technology or health sciences baccalaureate is supposed to have acquired. No previous knowledge of the subject is required since most of the syllabus consists of topics that are developed in a self-contained manner. Specifically, the subject deals with 6 topics: 1. (Introduction) Sets, applications, and relationships; 2. Propositional and predicate logic; 3. Induction and recursion; 4. Combinatorics; 5. Binary relationships; 6. Graphs and digraphs. The most extensive is the one dedicated to logic since it is intended to be an instrument that facilitates reasoning and formalization in all subjects of the degree. The rest of the subjects are presented more briefly, emphasizing the formal aspects, since in later subjects the aspects more related to computer science (such as algorithm programming) are taken up and dealt with.

### *Data collection instrument*

We used a written questionnaire to collect data, as other questionnaires designed to determine commognitive conflicts (Kontorovich, 2021). Indeed, according to the commognitive approach (Sfard, 2008), communication about mathematics in written or verbal responses is not a window to thinking, but an inseparable part of it. Thus, in this study, students’ responses to this questionnaire are considered acts of communication and, therefore, part of their meaning making

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(Biza, 2017, p. 1994). Besides, as Dimitrić (2018) stated, students' written work can be used to discover how they understood mathematical concepts and to suggest several methods to improve teaching.

Specifically, our questionnaire consisted of open-ended questions instead of multiple-choice items, like in the work by Manero and Arnal-Bailera (2021), because in this way more of the discourse and reasoning of the students will be shown.

The tasks of the questionnaire are presented below. All of them were related to the formulation of definitions of the concepts of cycle and path graphs.

1. Define a 6-vertex cycle. Are there any properties that you can eliminate from your definition of a 6-vertex cycle so that it remains equally valid? If so, indicate which one(s).
2. Define any path. Are there any properties that you can eliminate from your definition of a path so that it remains equally valid? If so, indicate which one(s).
3. Could you give a definition of a path equivalent to the one you have already given but containing the concept of a cycle?

Among these items, we find questions that can be classified as object-level tasks, i.e., they refer mainly to the mathematical objects of discourse, and it is not necessary that metalevel learning has been produced or need not be produced to solve them correctly; for example, when a definition of a 6-vertex cycle or path is requested. On the other hand, we find questions in the metalevel, i.e., associated with mathematical practices, specifically referring not to mathematical objects but to the definition of these objects. In our case they ask to find other definitions equivalent to those given before. Solving these metalevel questions correctly requires that the student either knows or learns the metarules that dominate the construction of equivalent definitions. These metalevel questions may cause the students to go back and ask themselves if the definition given at the beginning is correct and equivalent to the one given later, so they have the potential to produce metalevel learning.

### ***Data collection***

The participants answered the questionnaire individually and in writing during the last class of the course. The written responses are the data of our study, and the students gave their consent for them to be used for this purpose. These assignments were also part of the course assessment, and no notes or bibliographic material was allowed to be used to answer them.

### ***Data analysis***

Data analysis consisted of two phases. In the first phase, we focused on identifying the properties of the discourse: word use, visual mediators, narratives, and routines. In the second phase, based on the results of the first phase, we identified evidence of unresolved commognitive conflicts. In

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both phases, each researcher first analyzed the written responses individually, followed by sharing sessions with all members of the research team in which discrepant cases were discussed until consensus was reached. It should be noted that all the responses of the 39 students were analyzed, but for reasons of space, we only show evidence of the most representative ones in the findings section.

## FINDINGS

The results are presented using vignettes. Each vignette is characterized by the identification of a particular type of commognitive conflict. We have classified them as object-level or metalevel commognitive conflicts. Within the object-level type, we have distinguished between the subtypes *intra* (within graph theory) or *inter* (in relation to other areas of mathematics, such as geometry, or other mathematical concepts, such as sequences). The metalevel commognitive conflicts found are mainly related to the mathematical practice of defining. In each vignette, the empirical data are shown together with the item being answered and the analysis performed. Below the students' answers, we show its literal translation into English.

### *Object-level commognitive conflicts*

*Vignette 1. Object-level inter commognitive conflicts, between the discourse of graph theory and the discourse of Euclidean geometry.*

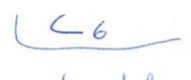
This subtype of commognitive conflicts has its origin in the knowledge that students have of plane Euclidean geometry when they begin their study of graph theory at university. In fact, there is a certain analogy between plane geometric figures and graphs, since many graphs resemble geometric figures in their pictorial representation, and also possess vertices and edges. Moreover, rigid movements, which preserve the shape and size of geometric figures, are a particular case of topological transformations, which do not alter the sets of vertices and edges of the graph (González et al., 2021).

In Figure 3, student 12 is asked to define a six-vertex cycle. The first sentence that appears is a correct definition. In the second sentence, he/she intends to give an alternative definition using the word "circular". This definition is incorrect since a correct mathematical definition, according to Zaslavsky and Shir (2005), should not depend on the chosen representation, and a cycle does not necessarily have a circular shape. It would suffice to consider a pictorial representation with edges that intersect at points that are not vertices. Thus, this definition is clearly dependent on the chosen representation and is not useful to recognize cycles in other representation systems. In addition, the use of this definition may lead to conflict when the student encounters pictorial representations of six-vertex cycles that have no circular shape. This could lead him to conclude, for example, that both graphs are not the same graph (or isomorphic), because he/she takes into account only the shape instead of the combinatorial information that the graph possesses, which is precisely what defines it.

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Figure 3: Student 12 response

<p>Define un ciclo de 6 vértices:</p> <p><i>Conjunto de 6 vértices adyacentes y 6 pares de vértices</i></p> <p><i>Gráfico formado por un conjunto de 6 vértices y 6 aristas tal que <math>v_1</math> es adyacente con <math>v_2</math>, <math>v_2</math> con <math>v_3</math>, <math>v_3</math> con <math>v_4</math>, <math>v_4</math> con <math>v_5</math>, <math>v_5</math> con <math>v_6</math> y <math>v_6</math> con <math>v_1</math>. Es un camino circular.</i></p> 
<p>Define a 6-vertex cycle:</p> <p>“Graph formed by a set of 6 vertices and 6 edges such that <math>v_1</math> is adjacent to <math>v_2</math>, <math>v_2</math> to <math>v_3</math>, <math>v_3</math> to <math>v_4</math>, <math>v_4</math> to <math>v_5</math>, <math>v_5</math> to <math>v_6</math> and <math>v_6</math> to <math>v_1</math>. It is a circular path.”</p>

In Figure 4, the student is asked to define any path. It can be understood that this student answers that the number of edges must be equal to the number of vertices minus one, although he/she uses incorrect visual mediators (symbols) (since it is not specified that it refers to the cardinality of both sets). This student did not give any information about how the connections between vertices must be in this type of graph, although there exist many different graphs given a specific number of edges and vertices. We think that the error could have its origin in a conflict with the discourse of geometry, with some definitions of polygons. For example, a pentagon is defined as a shape with five angles and five sides, and only with these two data (number of angles and sides) this type of shape is univocally determined, unlike what happens in graphs.

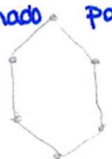
<p>Define un camino cualquiera:</p> <p><i><math>G(V, A)</math> tal que <math>A = V - 1</math></i></p>
<p>Define any path:</p> <p>“<math>G(V, A)</math> such that <math>A = V - 1</math>”</p>

Figure 4: Student 10 response

Figure 5 shows a different conflict from the previous one, also with its origin in plane geometry. As in Figure 3, the student is asked to define a six-vertex cycle. The first sentence is redundant, and the set of edges is missing. The second sentence may be ambiguous, but it can be understood, according to the example drawn, that he/she refers to the fact that each vertex is joined (*at least*) to another by an edge, that is, there are no isolated vertices. In the third sentence, he/she indicates that this type of graph is closed. We believe that this use of words (“closed”) together with the pictorial representation given may indicate a conflict with the discourse of geometry, since the word closed is used in the discourse of geometry to indicate a closed polygonal chain or a closed

curve, but it is not used in the discourse of graph theory to mention a property of a graph. Note that the given definition is incorrect because it includes graphs that are not six-vertex cycles (e.g., the disjoint union of two 3-vertex cycles or the graph in Figure 6, left). We also think that some very different pictorial representations of six-vertex cycles might not be identified as such by this student (e.g., see Figure 6, right). This is because there is some evidence that he/she thinks in a specific geometric shape (hexagon); even from the position of the representation and the text in the answer (Figure 5), he/she seems to have drawn this pictorial representation first and then described it.

Define un ciclo de 6 vértices:  
 Un ciclo de 6 vértices es el que está formado por 6 vértices  
 y cada uno de ellos está unido a otro por 1 arista.  
 Está cerrado.



---

Define a 6-vertex cycle:  
 "A 6-vertex cycle is the one that is formed by 6 vertices and each of them is connected to another by an edge. It is closed."

Figure 5: Student 19 response

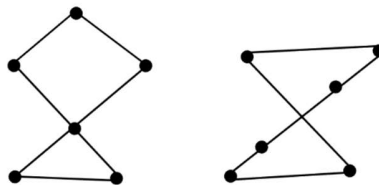


Figure 6: Six-vertex graphs

*Vignette 2. Object-level inter commognitive conflicts, between the discourse of graph theory and the discourse of sequences.*

This conflict occurs mainly when graphs appear in their set representation. Thus, in Figure 7, the student defines a six-vertex cycle by giving only the set of vertices as a set of elements where the subscripts of the elements indicate order. He/she also states that the last element that appears is equal to the first. This definition has two fundamental errors: first, if one vertex is equal to another, then by convention in set theory this set has five elements instead of six. Furthermore, it does not

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mention the set of edges, that is, the connections between these vertices. We believe therefore that there is a conflict with the discourse of sequences, since a sequence is defined precisely as an ordered set of elements and these elements are not related.

<p>Define un ciclo de 6 vértices: Un ciclo de 6 vértices es un grafo  <math>G(v) = \{v_1, v_2, v_3, v_4, v_5, v_6\}</math> donde <math>v_1 = v_6</math>.</p>
<p>Define a 6-vertex cycle:          "A 6-vertex cycle is a graph <math>G(v) = \{v_1, v_2, v_3, v_4, v_5, v_6\}</math> where <math>v_1 = v_6</math>."</p>

Figure 7: Student 30 response

Figure 8 shows a similar conflict between the discourse of graph theory and the discourse of sequences. Here the student defines a path as an ordered set of vertices, writes that there is an initial and a final vertex and does not mention the edges.

<p>Define un camino cualquiera:          un camino es aquel que muestra el conjunto de vértices que forma el grafo, de manera que estos sí se pueden repetir y <math>v_1 \neq v_n</math>, siendo <math>v_1</math> el vértice inicial y <math>v_n</math> el vértice final.</p>
<p>Define any path:          "A path is one that shows the set of vertices that form the graph in such a way that they can be repeated, and <math>v_1 \neq v_n</math>, where <math>v_1</math> is the initial vertex and <math>v_n</math> is the final vertex."</p>

Figure 8: Student 25 response

*Vignette 3. Object-level intra commognitive conflicts, within the discourse of graph theory itself.*

Figure 9 shows a student's answer in which he/she uses the words "one in-edge in and one out-edge" (sic). These concepts do not apply for graphs, but there are others that have a similar name and are specific to the discourse of directed graphs, such as "in-degree / out-degree" of a vertex. In addition, we have found more evidence of this conflict, for example, in student 37, but in that case, he/she did not use the words mentioned but the associated visual mediators,  $g^+(v)$  and  $g^-(v)$  (see Figure 10).

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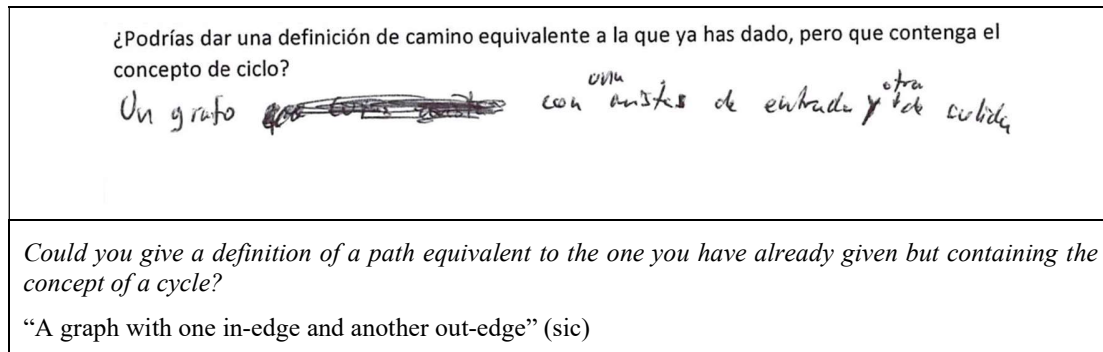


Figure 9: Student 24 response

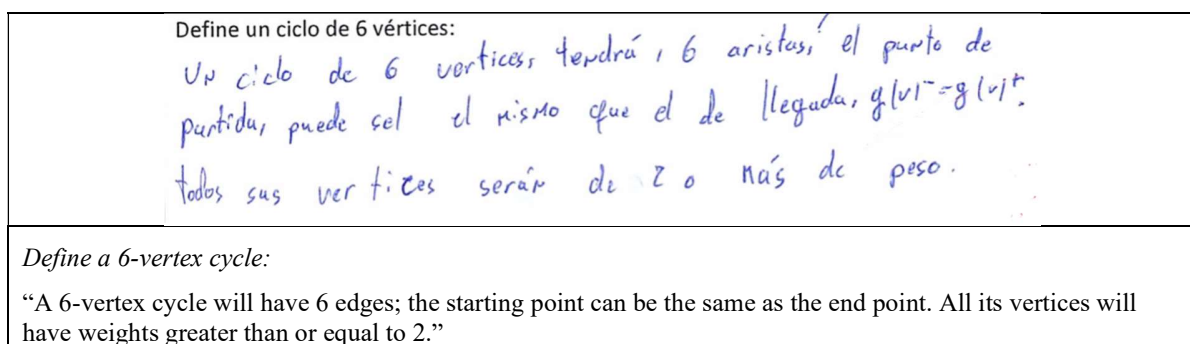


Figure 10: Student 37 response

Figure 11 shows another conflict of type intra in the use of words between the discourse of graphs and the discourse of directed graphs. Note that, by general convention in graph theory, unless otherwise explicitly stated, when we say “graph” we refer to an undirected graph. The student here used the words “ordered pairs” to refer to the edges, as happens in directed graphs and contrary to undirected graphs, where the pairs are considered unordered. Several more students showed this conflict in their use of visual mediators: they denoted edges in the form  $(a,b)$  instead of  $\{a,b\}$  (see Figure 12).

<p>Define un camino cualquiera:</p> <p><i>un camino de n vértices consiste en una serie de aristas que el conjunto de aristas este formada por pares ordenados de manera que el final de una arista sea el comienzo de la siguiente y que <math>v_i \neq v_j</math></i></p>
<p>Define any path:</p> <p>“A path of n vertices consists of the set of edges being made up of ordered pairs such that the end of one edge is the beginning of the next”</p>

Figure 11: Student 3 response

<p>Define un ciclo de 6 vértices:</p> <p><math>V(G) = \{1, 2, 3, 4, 5, 6\}</math></p> <p><math>E(G) = \{(1,2), (2,3), (3,4), (4,5), (5,6)\}</math></p>
<p>Define a 6-vertex cycle:</p> <p><math>V(G) = \{1, 2, 3, 4, 5, 6\}</math></p> <p><math>E(G) = \{(1,2), (2,3), (3,4), (4,5), (5,6)\}</math></p>

Figure 12: Student 6 response

Figure 13 shows a conflict of type intra related to word use. Thus, some of the students, such as student 15, understood that the item asked to define a path as a subgraph within a graph instead of a path graph.

<p>Define un camino cualquiera: <math>P_n</math>: notación</p> <p><i>un camino une dos vértices cualesquiera de un grafo pasando más de una vez por cada arista o vértice en caso de ser necesario y siendo todas las aristas pertenecientes al grafo, en caso de que no haya una arista uniendo dos vértices dichos vértices no se podrán unir de forma directa</i></p>
<p>Define any path:</p> <p>“<math>P_n</math>: notation</p>

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A path joins any two vertices of a graph passing more than once through each edge or vertex if necessary and being all the edges belonging to the graph, in case there is no edge joining two vertices, these vertices cannot be joined directly.”

Figure 13: Student 15 response

It should be noted that several students in the sample, such as student 15 (Figure 13), defined a path or a six-vertex cycle referring to the idea of physically traversing the graph, which was evidenced by their use of expressions such as “passing through each vertex/edge” or “starting to traverse it from one vertex, it ends at another/the same vertex”.

### *Metalevel commognitive conflicts*

#### *Vignette 4.- Metalevel commognitive conflicts, related to the mathematical practice of defining*

Several responses from student 16 are shown below. When asked to define any path, he/she responds by giving an example of a (directed) path with four vertices in its set representation (Figure 14). The metalevel conflict may have its origin in the differences between the discourses of the practices of defining and proving, since in the latter, examples are usually given to prove the existence or counterexamples to prove that something is not true. We think that a conflict in the use of the word “any” may also have contributed to this error. The student may have interpreted the question as follows: “choose any path and define it”, which corresponds to a non-formal or everyday use of the word “any”, as opposed to its use in formal mathematics, where the proposed statement means: “give a valid definition for any path”. Apart from this, we can observe in the second question that the student thinks of an edge as a property of the graph (instead of as an element of the graph), thus trying to “avoid” the task as it is presented, which is a metalevel task, and therefore more complicated. The purpose of this task was for the student to define the concept of a path graph using a set of necessary and sufficient properties; however, he/she answered by moving the task to the object level, referring to the concept “subgraphs of paths that are also paths”. Finally, in the last task (Figure 15), the student did the same as in the previous one, he/she interpreted a metalevel task, in which he/she was asked to provide an equivalent definition of a path including the concept of cycle, as an object-level task using this time the concepts of cycle, path and subgraph, Specifically, he/she might have interpreted the sentence “contain the concept of cycle”, as “a cycle is a subgraph of a path”, as we can infer from his answer.



Define un camino cualquiera:

$$V(G) = \{a, b, c, d\}$$

$$E(G) = \{(a, b), (b, c), (c, d)\}$$

¿Hay alguna propiedad que puedas eliminar de tu definición de camino de manera que siga siendo igualmente válida? Caso afirmativo, indica cuál o cuáles.

Podría eliminar cualquier arista menos (b,c) y seguiría existiendo un camino.

---

Define any path:

“ $V(G) = \{a, b, c, d\}$   
 $E(G) = \{(a, b), (b, c), (c, d)\}$ ”

Are there any properties that you can remove from your definition of a path so that it is still valid? If so, indicate which one(s).

“I could remove any edge except (b,c) and a path would still exist.”

Figure 14: Student 16 responses

¿Podrías dar una definición de camino equivalente a la que ya has dado, pero que contenga el concepto de ciclo?

No, puesto que ~~para~~ un ciclo si puede contener un camino pero no a la inversa

---

Could you give a definition of a path equivalent to the one you have already given but containing the concept of a cycle?

“No, since a cycle can contain a path, but not vice versa.”

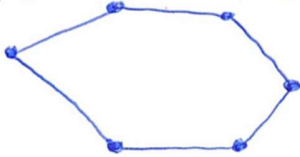
Figure 15: Another student 16 response

Figure 16 shows a response of student 11 to the task of defining a six-vertex cycle. He/she presents an example of this graph in its pictorial representation and gives a series of characteristics: “it has 6 vertices, it is connected, and cyclic”. We can identify a conflict in the mathematical practice of defining because he/she uses in his definition a word derived (*cyclic*) from the one it is defining (*cycle*), i.e., there is circularity in the definition (Zaslavsky & Shir, 2005). Furthermore, if we omitted this word, there would not be a set of necessary and sufficient properties, thus skipping another condition that a correct mathematical definition must have (Zaslavsky & Shir, 2005).

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Define un ciclo de 6 vértices: *grafo con 6 vértices, conexo y cíclico*



Define a 6-vertex cycle:  
"Graph with 6 vertices, connected and cyclic"

Figure 16: Student 11 response

Finally, in Figure 17, in the same task, student 28 presents another conflict different from the previous ones in this vignette, also in the discursive activity of the mathematical practice of defining. This student shows only a concrete pictorial representation of a six-vertex cycle and does not refer to mathematical properties. He/she also did the same when asked to define the concept of a path. Moreover, in this case he/she provided a pictorial representation of a path with a concrete number of vertices, instead of somehow indicating that it can have any number of vertices, this is, in addition to not having understood the task of defining, he/she did not understand the generalization that was asked for. This may be due to a misinterpretation of the word "any", just as it happened to student 16 in Figure 14.

Define un ciclo de 6 vértices:



Define un camino cualquiera:



Figure 17: Student 28 responses

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Now we present a quantitative summary of our study. If we consider the variable “number of conflicts per student”, its mean is 0.9487 and its standard deviation is 0.5969. Specifically, there were 8 students who did not show evidence of any type of conflict, 25 students showed evidence of only one type of conflict, and 6 students showed evidence of two types of conflict.

Table 1 shows the number and the percentage of students who presented each type of commognitive conflict. We have identified the vignette number in the results with the type of conflict, i.e., type 1 corresponds to the conflicts of vignette 1, and so on. The percentages do not add up to 100% because there are students who presented more than one type of conflict and students in whom no evidence of any type of conflict was found. It can be observed that the most frequent conflicts are those of types 3 and 4, that is, those related to the graph theory itself and the metalevel ones related to the characteristics of a mathematical definition. The number of students who presented type 4 conflicts is not so large, which does not indicate that all those who did not present this type of conflict knew the characteristics of a mathematical definition, since many students left blank items that could bring up this type of conflict.

	Number of students (percentage)
Type 1	5 (12.82 %)
Type 2	2 (5.13 %)
Type 3	22 (56.41 %)
Type 4	8 (20.51 %)

Table 1: Percentage of students presenting each type of commognitive conflict

## DISCUSSION AND CONCLUSION

This study provides findings applicable to improving the teaching and learning of graph theory. Responding to the need pointed out by Hazzan and Hadar (2005) and Ouvrier-Buffet et al. (2018) to employ appropriate theoretical frameworks to investigate the teaching and learning of graph theory, we selected the commognitive theoretical framework (Sfard, 2008). The use of this framework has allowed us to identify commognitive conflicts, which is relevant for teaching and learning because when they occur, they generate learning opportunities, and their resolution results in learning. Specifically, in this article we have analyzed the written responses of first-year university students about basic concepts of graph theory. We have found commognitive conflicts that arose from the confluence of discourses governed by different rules. These conflicts have been classified using the same classification as González-Regaña et al. (2021). That is, they have been classified as object-level commognitive conflicts (i.e., related to the mathematical content) and metalevel commognitive conflicts (i.e., related to mathematical practices). Within the object-level ones, we have distinguished the subtypes inter, in our case between the discourse of graph theory

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and other mathematical areas or concepts, such as geometry or sequences; or intra, which appear within the discourse of graph theory itself. The identified metalevel conflicts are related to the mathematical practice of defining.

Regarding the conflicts found between the discourse of graph theory and that of Euclidean geometry, the data show the erroneous use or extrapolation of properties of the discourse of Euclidean geometry in the context of graph theory, due to the analogy between these two areas (González et al., 2021). It is influenced by the common use of words in the two areas; for example, the word “vertices” is used in geometry and graphs, and the word “edges” is also used in both graphs and geometry. As for visual mediators, there are similarities between the pictorial representation of graphs and 2D and 3D geometry; that is, similar visual mediators are used, but they have different meanings. Thus, this analogy between mathematical objects, which sometimes proves to be beneficial, for example, in solving problems, at other times leads to errors when the differences between the objects are not considered.

We have also presented evidence of conflicts between the discourse of graph theory and that of sequences. Several students defined cycle or path graphs as ordered sets of vertices, which resembles the definition of sequences as ordered sets of elements. Some of them also omitted the set of edges in their definition, which can be related to the work by Ioannou (2018). Indeed, this author states that, in university discourse, unlike what happens in secondary school, the sets may or may not have a binary relationship defined, which in some cases can confuse students who assume that they always have it defined or, on the contrary, that they never have it.

We have also observed that students frequently refer to the idea of physically traversing the cycle or path graph, which does not usually happen with other families of graphs. The origin of this type of reasoning can be found in the name of these graphs, since the words “cycle” or “path” are used in common language associated with the meaning of a route. We can also think that this emphasis on the traversal process may have its origin in real problems that are commonly used to introduce graph theory. For instance, the famous problem of the Königsberg bridges, which consists of finding whether a certain traversal is possible in a city represented by a graph. Thus, many students do not see these types of graphs as static objects that have a series of properties, but as elements that are associated with an action or process (traverse them). If we take into account that the students analyzed are taking their first steps in graph theory, this agrees with process-object theories such as Sfard’s (1991) reification theory and Dubinsky’s (1991) APOS theory, which state that procedural conceptions (processes) precede structural ones (objects). This may also be an indicator that students use the reduction of abstraction in the sense of Hazzan and Hadar (2005), which states that when they find it difficult to solve a mathematical problem due to its degree of abstraction, they unconsciously reduce it. Note that the association of cycles or paths with routes can also be related to the conflicts found in relation to the use of words and visual mediators of

directed graphs in undirected graphs, since a route can be identified in some way with a certain direction on the edges of the graph.

Regarding now the commognitive conflicts in relation to the mathematical practice of defining, other works have also dealt with students' difficulties in this practice (Fernández-León et al., 2021; Zaslavsky & Shir, 2005), although in relation to other areas of mathematics, such as analysis or geometry. However, the conflicts found in our case are similar to those of these works, since many of them are related to the lack of knowledge of the characteristics that a correct mathematical definition should have: noncontradicting, unambiguous, invariant (under change of representation), hierarchical, noncircular, and minimal (Borasi, 1992; Zaslavsky and Shir, 2005). Therefore, there is evidence to deduce that several students in the sample do not understand the concept of definition given by a set of necessary and sufficient conditions and its usefulness in mathematics, which in the framework proposed by González et al. (2021) would be similar to stating that they have not reached Van Hiele's level 3 in graph theory. There were also conflicts related to the misinterpretation of metalevel tasks as object-level ones. Specifically, in these tasks they were asked about the definition of a type of graph, and some students gave answers that only referred to the type of graph defined. This again presents an analogy with the theoretical framework of the reduction of abstraction used by Hazzan and Hadar (2005).

To sum up, all these conflicts can be related, on the one hand, to the difference of discourses in the transition from high school to university education (Gueudet & Thomas, 2020; Ioannou, 2018; Schuler-Meyer, 2019), which is a critical point in teaching and learning because the rules of discourse change, and often the new rules of discourse are tacit, that is, not endorsed explicitly by the teachers (Sfard, 2008). On the other hand, these difficulties have to do with the passage from elementary mathematical thinking to advanced mathematical thinking (Tall, 1991); in our context, it is the passage from describing objects to providing mathematical definitions that meet all the characteristics indicated above.

Finally, Sfard (2007) stated that Van Hiele's cognitive theory and her sociocultural theory of commognition can complement each other. In her words "Van Hiele's levels can be interpreted as a hierarchy of mutually incommensurable geometric discourses" (p. 597), that is, they differ in the use of their properties. This complementarity was addressed by Wang and Kinzel (2014), who proposed that at each Van Hiele level (in the geometric case) there can be different types of discourse. The proposal by González et al. (2021) of Van Hiele levels of reasoning for graph theory suggests that object-level inter commognitive conflicts related to geometry may be linked to Van Hiele level 1 (recognition) since they come from descriptions of graphs supported by visual referents and the use of words that refer to their shape rather than to their topological or combinatorial properties, the latter being more typical of a higher level (González et al., 2021).

### *Implications for teaching*

The relevance of our results is explained by the fact that the commognitive conflicts found give an idea of the way of thinking and the difficulties of the students when they begin the study of graph theory, and therefore they can be used by teachers for the design of hypothetical learning trajectories in the sense of Simon (1995). According to this author, hypothetical learning trajectories have three components: the learning objectives, the learning activities, and the hypothetical learning process (i.e., a prediction of how students' thinking and understanding will evolve). We can then examine this evolution in practice by analyzing their discourse using the commognitive framework, and based on this analysis, modify or design a new hypothetical learning trajectory. After all, hypothetical learning trajectories are not static but may undergo continuous modifications, that are mainly caused by the results of the assessment of student learning and the changes that occur in teachers' knowledge about the teaching-learning process.

All this leads us to suggest that in the early stages of the teaching of graph theory, sequences of tasks could be proposed in order to bring to the surface the identified commognitive conflicts since their resolution will facilitate progression through the Van Hiele levels. Furthermore, as Thoma and Nardi (2018) stated, during teaching it is important to make explicit reference to the change in the rules of discourse, in our case both the differences between the discourse of graph theory with other areas of mathematics such as geometry, and the difference between the rules of school and university discourse, or elementary and advanced mathematics discourse. However, before making the differences between discourses explicit, we believe that it is essential to promote in class situations in which these conflicts arise, so that students experience the need for discourse change. As an example, when students are asked to provide a definition, we can present examples of several graphs that meet their definition but are not of the class of the defined object, or vice versa, we can present examples of graphs that do not meet their definition but are of the class that was asked to be defined. This triggers a metalevel commognitive conflict, which, when resolved, leads to learning.

It is also important to try to avoid the excessive use of visualization or visual-based reasoning by learners. Several authors (e.g., Hershkowitz (1989)) pointed out the limitations of purely visual reasoning in the context of geometry, which is favored because often only prototypical examples are presented to students when they try to learn a concept. Similarly, as we have said above, Hazzan and Hadar (2005) interpret visualization as a mechanism of reduction of the level of abstraction in graph theory that leads students to erroneous reasoning or poor understanding of concepts. To prevent this, we recommend teachers to present, when introducing concepts and proposing tasks, both prototypical and non-prototypical examples of graphs in their pictorial representation, as well as examples in other systems of representation: intersection of objects, set representation, matrices, degree sequences, etc. This is also relevant because various authors (e.g., Dagan et al., 2018) found that the use of different representations enhances cognitive interest, creative thinking, and a deeper

understanding of the concepts involved. To conclude, it is important to remark that the findings of this study can be used to guide teachers in the development of learning materials for graph theory.

### *Examples of application of the tool in the classroom*

As a first example, we describe a situation that could arise in the class when the teacher asks to define a 6-vertex cycle. We suppose that a student replies that it is a graph with 6 vertices and 6 edges. This response provides evidence that the student presents a type 1 conflict, such as the one in Figure 4. The teacher could then provide pictorial representations of various ad hoc selected graphs (like the ones in Figure 18) selected ad hoc to collect further evidence of this conflict. Thus, the teacher would ask if these representations corresponded to six-vertex cycles. We assume that the student says that the graph in Figure 18(a) is a 6-vertex cycle because it has 6 vertices, six edges, and it is hexagon-shaped. Regarding the graph in Figure 18(b), the student could say that it is not a six-vertex cycle because although it has 6 vertices and 6 edges, it is not hexagon-shaped. The teacher would thus confirm that there is a type 1 conflict. We assume that this student also states that the graphs of Figures 18(c) and Figure 18(d) are not six-vertex cycles. This would also reveal signs of a type 4 conflict related to the mathematical practice of defining. The teacher could tell the student that according to his/her definition, Figure 18(c) and Figure 18(d) are six-vertex cycles and could then explain to him/her the function of mathematical definitions to classify objects. The teacher would also tell the student that indeed the graphs in Figure 18(c) and Figure 18(d) are not six-vertex cycles, but they meet his/her definition and therefore his/her definition is wrong. The teacher would state the correct definition for a six-vertex cycle and would remark that it makes no reference to the shape of the graph and that graph in Figure 18(b) meets it, therefore, it is a six-vertex cycle, thus resolving the type 1 conflict. The teacher could further explain to the students the characteristics that correct mathematical definitions must have, for example, by focusing on invariance by change of representation, using graphs in Figure 18(a) and Figure 18(b). They could also deal with minimality using the correct definition of a six-vertex cycle by adding or removing properties from the definition and asking for or presenting examples that meet the definitions resulting from these modifications. Here again, more type 4 conflicts could arise, which could be resolved with the help of examples and counterexamples of the modified definition.

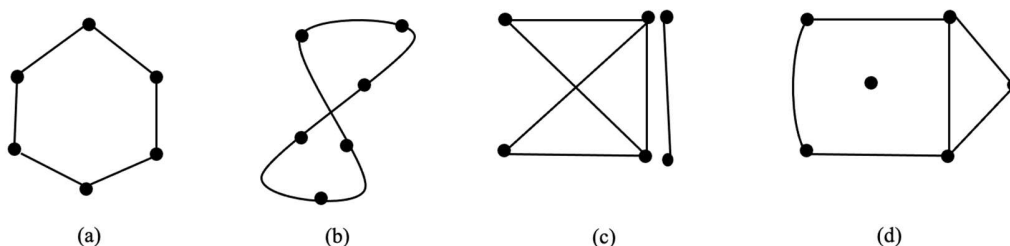


Figure 18: Some graphs having 6 vertices and 6 edges

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As a second example, the teacher could ask the students to define and represent a path graph with four vertices and we assume that one of them gives this expression  $G = \{v_1, v_2, v_3, v_4\}$  and the representation in Figure 19(a). The teacher then asks this student if the expression  $G = \{v_2, v_1, v_4, v_3\}$  also represents a path graph and, in such case to represent it. We assume that the student draws a path graph like the one in Figure 19(b). The teacher would therefore detect a conflict of type 2, as the student considers the order of the vertices when representing and does not detect that his/her “definition” is wrong because the set of edges is missing. The teacher would ask him/her if the graphs he/she has represented in Figure 19(a) and Figure 19(b) are the same graph, and we assume that the student says they are not. The teacher could then ask the student to represent the graph in Figure 19(c) in a similar way. We assume that he/she represents it like this:  $G = \{v_1, v_2, v_3, v_4\}$ . The teacher could ask if the latter graph is the same as graphs in Figure 19(a) or Figure 19(b). If the student says no, the teacher could answer that indeed, they are not the same graph and could point out that the set representation he/she has given for the three graphs is the same because in graph theory the vertex sets are not ordered, and therefore the given expressions do not univocally determine those graphs. The teacher might add that, in fact, graphs in Figure 19(a) and Figure 19(b) are not the same path graph, although they are isomorphic, and Figure 19(c) is not even a path graph. In conclusion, students should learn that in the set representation of graphs it is essential to give the set of edges, since there is an infinite number of graphs with the same set of vertices, thus resolving the conflict of type 2.

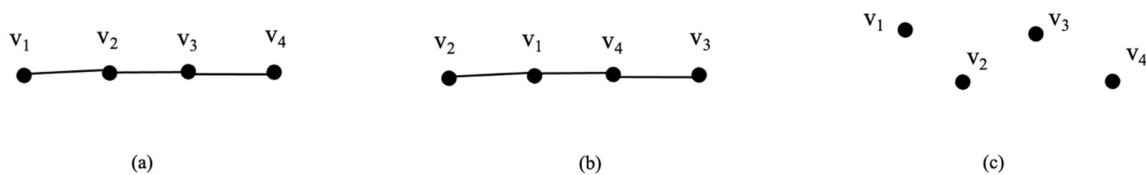


Figure 19: Some graphs having 4 vertices

Finally, this exploratory study is limited by the size of the sample, but it has allowed us to make a first approximation to the written discourse of graph theory students. In future works, we intend to complete the study by expanding the sample and taking data also from spoken discourse, both in class and in individual interviews. We also would like to identify commognitive conflicts relating to other concepts and other mathematical practices within graph theory.

### *Acknowledgments*

The first author was partially supported by the “VI Plan Propio de Investigación y Transferencia: Proyectos precompetitivos (PP2020/IV.4/01)” of the Universidad de Sevilla (Spain). The third author would like to thank to Universidad de Almería for its support during his stay in Almería in September and October 2021. The first, second and third authors belong to the Research Group in

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## A Review of Math Girls 3: Gödel's Incompleteness Theorem by Hiroshi Yuki and How It Can Be Used to Teach Introductory Proofs

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*Abstract: Math Girls 3: Gödel's Incompleteness Theorem is a student led adventure through advanced mathematics topics, proof writing, and finally the proof of the title theorem itself. We offer a review of the book from the perspective of both the casual read and its potential for use in an undergraduate Introduction to Proofs course. We discuss both how a teacher trying to connect disparate topics in an Introduction to Proofs course may consider using the book, as well as the academic and emotional guidance it can give a student working through understanding proof writing for the first time.*

### Introduction

Hiroshi Yuki has branded a new type of mathematics novel where the author teaches mathematics to the reader through interactions between three female students and one male narrator. His first two Math Girls books on infinite series (Yuki, 2011) and Fermat's last theorem (Yuki, 2012) were so well received when translated into English from Japanese by Bento Books that he created three additional books more accessible to a younger audience on integers, [equations and graphs](#) (Yuki, 2014), and [trigonometry](#) (Yuki, 2014). We consider his third Math Girls book on Gödel's Incompleteness Theorem in this review and are excited that the remaining Math Girls books on Randomized Algorithms, Galois Theory, and the Poincare Conjecture are now available in their English translations as well. Many mathematical novels as well as nonfiction mathematical stories exist (Doxiadis 2000, Jackson 2000) weaving the stories of mathematicians, their struggles and obsessions, as well as descriptions of famous problems together. Few strive to teach the reader mathematics from the ground up in a story telling fashion as Yuki does. We describe the book and some specific topics it covers, then discuss potential impacts on a student struggling in mathematics and conclude with how the book could be used in an Introduction to Proofs course.

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## Book Topics Description

Math Girls 3 starts with entertaining riddles and takes the reader into the world of math from a student perspective. While constructing a compelling narrative of the narrator's emotional turmoil during the school year with his study group (Yuri, Tetra, and Miruka), Yuki is able to teach Gödel's Incompleteness Theorem and the necessary preliminary mathematical topics. The story starts with the narrator and his cousin, Yuri, walking through logic problems together and slowly builds an understanding of surrounding topics until the reader is ready to grasp Gödel's Incompleteness Theorems. While the book is centered around Gödel's Incompleteness Theorems, it is not necessary for the reader to fully devote themselves to learning Gödel's Incompleteness Theorem to be able to gain meaningful insight from Hiroshi Yuki's, "Math Girls 3: Gödel's Incompleteness Theorem." Advanced concepts covered in this book include Peano's axioms (Chapter Two), set theory (Chapter Three), limits (Chapter Four), formal systems (Chapter Five), proof theory (Chapter Five), epsilon-delta proofs (Chapter Six), diagonalization (Chapter Seven), and trigonometry (Chapter Nine).

## Use in an Introduction to Proofs Course

When constructing an Introduction to Proofs course for undergraduate students, there are varied, seemingly disconnected topics that must be covered before being able to prove anything. Set theory must be known to show that the set of two things are equal and notation that is used in proofs must be known to be able to read proofs. Jumping from topic to topic before really getting into understanding what constitutes a proof can make students struggle to find connections between the concepts, which may cause them to feel developing an expertise in proof writing is unattainable. While unlikely to be sufficient as a stand-alone text for an Introduction to Proofs course, the book can serve as a narrative for students to read throughout the semester and a more standard textbook may be used to go into more depth to each topic. Students primarily use textbooks to look for examples of how to do specific types of problems (Benesh, 2006) and often skip reading the narrative, if it is present, on how concepts connect to each other. Math Girls 3 is less dense than a traditional textbook and can serve to motivate students on why they are learning a particular concept or proof technique (Hembree, 1990). Additionally, few math and science books adequately represent female students (Becker 2021), which Math Girls 3 refreshingly flips. Storybooks have been shown to have a positive impact on mathematical vocabulary and problem-solving skills in elementary age students (Hassing-Das 2015, Wangid 2021) but is seldom used at higher educational levels. Furthermore, an Introduction to Proofs course is a significant transition for mathematics students between the problems-based courses such as Calculus and Differential Equations to proof based courses such as Analysis and Abstract Algebra. This transition can cause students to become disheartened about mathematics due to their struggles to understand elements of proof. Math Girls 3 stresses the importance of learning how to embrace the struggle of learning new topics and the reader will find themselves fitting in with the characters

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that each serve a niche in the classroom. For those who are able to grasp concepts quickly, dialogue in the study group will give different perspectives of the same topic, providing a deeper understanding of the topic. For those who feel disparaged because they need to work harder to grasp the same concepts, the narrative provides constant reassurance of the hard work required to understand material. With each topic that is taught, Yuki provides a character who fits different types of students in the classroom, producing a moment of clarification to ensure a secure understanding is being built.

Yuki is able to build empathy with the reader by illustrating the anguish that all mathematicians at all levels have felt when it seems that the problem at hand is too overbearing to be able to conquer. The book provides opportunities to gain new perspectives from “peers” when a student is unable to articulate their struggles or feelings of inadequacy to others. Students may find themselves becoming study partners with the narrator, Tetra, Miruka, and Yuri with well-constructed dialogue that allows students a window to what a constructive partnership with classroom peers can be.

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