Geometric Reasoning to Reinventing Quadratic Formula: The Learning Trajectory on Realistic Mathematics Education Principles

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Abstract: This study aims to establish a knowledge base on how to support students in learning. We develop an initial hypothetical learning trajectory by formulating learning activities and predicting the development of students' thinking and understanding. The methodological framework employed in this study is design research, which seeks to generate actionable knowledge for achieving various educational goals through design. The design process incorporates the Hypothetical Learning Trajectory (HLT) thought experiment and the teaching experiment based on the six Realistic Mathematics Education (RME) principles. By integrating individual learning activities with the context and principles of the RME approach, students can construct their knowledge and rediscover the quadratic equation formula. Through comparing the HLT with observed learning outcomes, we redesign the process, revise our HLT, and provide answers to research questions regarding the attainment of specific learning objectives. Similar to our retrospective data analysis, we create a revised HLT to reinforce the concept of the completing the square and reinvent the quadratic formula.

Keywords: Quadratic formula, geometric reasoning, realistic mathematics education, design research

INTRODUCTION

Students struggle to solve quadratic equations, for they do not adequately develop their understanding of the concept, leading to the lack of justification of their knowledge (Allaire & Bradley, 2020; Fachrudin et al., 2014; Gözde & Kabar, 2018; Thomas & Mahmud , 2021; Utami

& Jupri, 2021; Yahya & Shahrill, 2015; Zakaria et al., 2010). These may include struggles in solving symbolic problems, challenges in manipulating operations, and weaknesses in mastering related concepts, such as arithmetic, which we provide further details in the literature review section below. Furthermore, students also encountered challenges with comprehending the quadratic formula as a viable approach to solving quadratic equations (Mahmood, 2021; Picciotto, 2008; Yahya & Shahrill, 2015; Zakaria et al., 2010). The quadratic formula has long been a fundamental component of the introductory Algebra course. However, it is unfortunate that students are introduced to quadratic formulas as their initial encounter with relatively complex formulas they memorize (Loh, 2019). Many often study it as a systematic alternative to the guess-and-check method, limited to factoring certain quadratic equations. These circumstances contribute to our assumption that there is a lack of understanding regarding the construction and utilization of the quadratic formula.

For many students, developing algebraic skills in the classroom often marks the beginning of their sense of mathematical inadequacy. This is because students frequently struggle to transition from arithmetic to algebra (Star et al., 2015). Moreover, they find formalistic or early abstract approaches challenging and uninteresting, whereas delving into context-related subjects proves more meaningful (Wittmann, 2021). Such difficulties also arise when learning quadratic equations and quadratic formulas, which are more effectively understood when presented with a geometric approach (Allaire & Bradley, 2020; Fachrudin et al., 2014; Maharaj & Takisa, 2012). Besides, from a historical standpoint, the foundation for solving quadratic equations is rooted in geometric principles (Güner & Uygun, 2016; Hungerbühler, 2020; Irving, 2020; Mehri, 2017;

Milne, 1927; Owen, 1991). In his book "Hisob Al-Jabr wa'l Muqabalah," Al-Khwarizmi (in Ben-Ari, 2022; Hungerbühler, 2020; Mehri, 2017) describes the geometric proof of solving quadratic equations. It has also been popular that students use the Babylonian geometry method to solve quadratic equations, a concept identified by J. Høyrup as Naïve Geometry (Høyrup, 1990; Radford & Guérette, 2000).

In the discursive processes of expanding knowledge, explaining concepts, and providing proof, Duval (1998) presents a cognitive model of geometric reasoning. According to this model, visualization can assist reasoning by aiding in the search for evidence. The model comprises three distinct cognitive processes: visualization, construction (using tools), and reasoning. Geometric reasoning, as described by the figural concept, involves the interaction between two aspects: figural and contextual. Therefore, geometric reasoning facilitates student learning by allowing them to expand their knowledge through visualizing and exploring geometric situations and engaging in construction processes. Furthermore, as geometric reasoning enables students to deduce and draw conclusions based on the properties of geometric shapes, this approach can be utilized in student learning to help them rediscover the quadratic formula. In practice, we can present instructional designs incorporating integrated learning activities, enabling the application of geometric reasoning throughout the student learning process.

The domain-specific instruction theory Realistic Mathematics Education (RME, den Heuvel-Panhuizen, 2020; Freudenthal, 1973; Gravemeijer, 2020; Gravemeijer & Stephan, 2002; Treffers, 1987) suggests that students learn mathematics most effectively when actively building their understanding through problem-solving activities and mathematical discourse. RME theory provides a broader perspective and meaning on real-world problems, going beyond the challenges students encounter in everyday life. Instead, students are presented with problem situations they can relate to, such as the imaginative realm of fairy tales or the formal world of mathematics, as long as the problem holds experiential significance. This principle, referred to henceforth, pertains to how geometric approximation is presented while preserving all other RME principles.

This study aims to contribute knowledge on supporting students in learning. In this study, we present a novel approach to geometric reasoning by focusing on the use of not only visual but interactive squares. This approach deviates from the previous existing procedure, which relied on a geometric approach and visualization of the quadratic formula using the static completing the square procedure (Fachrudin & Putri, 2014; Utami & Jupri, 2021; Zakaria et al., 2010) . We emphasize the importance and urgency of fostering independent student learning as the issue of quadratic equations, particularly the quadratic formula, persists. We encourage students to develop their understanding of mathematics by formulating quadratic equation formulas through teaching experiments guided by RME principles. Initially, we constructed a hypothetical learning trajectory (HLT, Akker et al., 2006; Bakker & Smit, 2018; Simon, 1995) by formulating learning activities and predicting how students' thinking and understanding would evolve. This HLT incorporates assumptions about students' potential to develop quadratic formulas through approaches that facilitate geometric reasoning and how teachers would support them in rediscovering quadratic equation formulas. Subsequently, the modified HLT was derived based on the conclusions drawn from a retrospective analysis of the teaching experiments.

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

Challenges Faced by Students in Quadratic Equations and the Quadratic Formula

In learning situations, several problems related to quadratic equations have been identified. Despite facing difficulties in solving symbolic quadratic equations and quadratic word problems, students demonstrated better performance when dealing with symbolic equations as compared to word problems (Szczerba & Krygowska, 2007). Students' challenges in solving symbolic problems stem from arithmetic and algebraic manipulation errors. Gözde & Kabar (2018) revealed that students had a limited understanding of the quadratic equation concept and were predominantly influenced by the idea of factoring. French (2002) discovered that common mistakes made by students included assuming that $(a+b)^2$ is equal to a^2+b^2. Zakaria et al., (2010) found that most students made errors in transformation and process skills while solving quadratic equations. These errors were attributed to weaknesses in grasping fundamental concepts such as algebra, fractions, negative numbers, and algebraic expansion. Yahya & Shahrill (2015) conducted research that analyzed students' errors in solving quadratic equations, highlighting patterns and causes such as selecting an incorrect multiplication factor when factoring quadratic expressions. (Akgul & Yilmaz, 2023) discovered that advanced participants struggled with interpreting the square root of a squared number when presented in exponential form.

In this study, we examine various sources of error, including the inability to recall the correct quadratic formula while solving quadratic equations due to a flawed comprehension of the formula. Additionally, we address the issue of incorrectly manipulating operations when attempting to change the subject of a given formula.

Geometric Reasoning

Fischbein (1993) argues that a geometrical figure, such as a square, is not only a concept but also an image. He states that these figures possess both conceptual and figural characteristics. Figural properties pertain to mental representations of space, unique to images, and not found in ordinary concepts. Thus, all geometric figures are mental constructions with conceptual and figural properties. Fischbein's notion of figural concepts suggests that geometric reasoning involves the interaction between these two aspects: figural and conceptual. Another extensively researched model describing the development of geometric reasoning is van Hiele's model of thought in geometry (Usiskin, 1982; Van Hiele, 1986). Additionally, we incorporated Duval's framework ((Duval, 1998)) in this study since it encompasses a process for knowledge extension, accommodating the formation of other understandings, including algebra. The framework proposes that geometric reasoning involves three cognitive processes that serve specific epistemological functions. These processes are as follows:

1. Visualization processes, such as visually representing a geometrical statement or heuristically exploring a complex geometrical situation.
2. Construction processes, involving the use of tools.
3. Reasoning processes, particularly discursive processes for knowledge extension, explanation, and proof.

Duval emphasized that the different processes involved in geometry can be conducted independently. For instance, visualization is not necessarily dependent on construction, and the construction process relies solely on the relationship between relevant mathematical properties and the tools' limitations, even if it leads to visualization. While visualization can assist in reasoning and evidence finding, it can sometimes be misleading. Nevertheless, these three cognitive processes are closely interconnected.

By visualizing various geometric representations of the quadratic formula, students are expected to develop a better understanding of the underlying mathematical properties and relationships. Through this reasoning process, students can approach the study of the quadratic formula with a more comprehensive and integrated perspective, thereby enhancing their understanding of the topic.

The RME Theory

RME is a domain-specific instructional theory for mathematics education (den Heuvel-Panhuizen, 2020; Freudenthal, 1973; Gravemeijer, 2020; Gravemeijer & Stephan, 2002; Treffers, 1987).

Initially conceptualized by Treffers (1987), the theory aimed to distinguish the realistic approach from other approaches, such as structuralistic, empiricistic, and mechanistic. Treffers intended RME to be a descriptive theory but has since expanded to include instructional design heuristics, guided reinvention, didactic phenomenology, and emergent modeling.

The development of RME theory was based on Freudenthal's ideas, which argue that mathematics education must be viewed as a human activity. One of the main principles of RME is guided reinvention, which emphasizes that teachers and assignments should guide students to reinvent mathematics and experience it as a human activity. This heuristic aims to help students develop a deep understanding of mathematical concepts and foster their creativity and problem-solving skills. Another central instructional design heuristic in RME theory is didactic phenomenology. This heuristic is based on Freudenthal's notion that organization is a key characteristic of mathematical activity. Students are encouraged to analyze which phenomena are organized and how they are organized through their mathematical thinking. This analysis can lead to situations that require regulating such phenomena and generating appropriate thinking. The third heuristic in RME theory is the emergent modeling design heuristic, which aims to support the incremental process by which mathematical models and concepts develop together. The central idea of this heuristic is the use of a series of sub-models that collectively strengthen the overall model. This overarching model evolves from an informal model of mathematical activity to a more formal model of mathematical reasoning. These three heuristics are also represented in six core principles: activity, reality, level, intertwinement, interactivity, and guidance. (Heuvel-panhuizen et al., 2020).

The adjective “realistic” accurately reflects the approach to teaching and learning mathematics in RME, but it can also be confusing. In Dutch, the verb “zich realiseren” translates to “imagine,” meaning that "realistic" refers more to the idea that students should be presented with problem situations they can imagine rather than the authenticity of the problem. However, this does not diminish the importance of relating mathematics to real-life scenarios. The problem context need not be limited to real-world situations; even fantasy worlds or formal mathematics are suitable as long as they are perceived as "real" by students. It emphasizes the importance of creating a mental connection between students and the problems presented.

We provide comprehensive background information on RME theory and its role in this research. Our discussion emphasizes two key mathematization methods that are integral to RME, the various levels of understanding that shape the learning process, the potential for students to participate in model development actively, and the dynamic nature of models throughout teaching and learning to enhance higher levels of understanding. We apply this foundational knowledge to the specific domain of quadratic equation formulas, examining how geometric models can effectively bolster geometric reasoning in constructing these concepts.

METHOD

Research Design

This study used design research as a methodological framework to develop actionable knowledge. This knowledge was condensed into design principles, conjecture maps, and hypothetical learning trajectories (Akker et al., 2006; Bakker, 2018; Cobb et al., 2017; Confrey & Maloney, 2015 ; Plomp, 2013). Throughout the study, we formulate the findings in the form of a revised hypothetical learning trajectory

The study consisted of three phases: (1) Preparation and implementation of teaching experiments: Anticipatory thought experiments were conducted to envision how the proposed instructional activities could be utilized in the classroom and predict what students might learn through participation. We aimed to anticipate students' learning processes during a pilot experiment. Subsequently, we determined the design of the learning activities and the type of knowledge to be developed in the initial hypothetical learning trajectory (HLT). (2) Teaching experiments—Trials with individuals: Weekly one-hour sessions were held outside of the regular school schedule, where the teaching experiments took place. The process was carefully documented and recorded. A total of six teaching experiment sessions focused on fostering an understanding of quadratic equations by reinventing the quadratic formula. At this stage, the HLT primarily guided the implementation of teaching experiments and data collection regarding the learning process, mechanisms, and the revised HLT. (3) Retrospective analysis: The actual learning process of the students was compared to the HLT. Transcripts were read aloud, and video footage was reviewed chronologically, episode by episode. Using the HLT and research questions as a guide, conjectures about student learning and perspectives were formulated, documented, and tested against other episodes and data materials such as student work, field notes, and assignments.

Participants and Data Collection

We recruited two students, Kenia and Rara (pseudonyms), from different junior high schools (SMP) in Indonesia to participate in one pilot experiment and three teaching experiments. They performed above average compared to their peers based on their semester grades. However, they showed apparent weaknesses in their understanding of mathematics, particularly in algebra and quadratic equations, specifically factoring quadratic equations and quadratic formulas. We outline their initial knowledge in the results section. We selected them as subjects in an assessment of their prior knowledge, which then guided the development of the initial HLT.

The data we present were collected during three separate learning sessions, each involving Kenia and Rara. Prior to the teaching experiments, we engaged in anticipatory thought experiments, imagining how the proposed instructional activities could be used in the classroom and what students might learn from participating. Additionally, video recordings of the teaching experiments and student worksheets serve as primary data that are retrospectively analyzed according to the HLT. We analyze all of these elements simultaneously in the results and discussion section.

Data Analysis

We present and analyze the three phases in parallel order for each activity performed. First, we outline each initial HLT component and then analyze it based on the implementation in the lesson. We examine why and how our design worked and use our findings to evaluate and revise the HLT. The main results of this study depend not on the design itself but on the underlying principles that explain how and why the design works as the goal of the study is preserved.

In addition, we analyze the activities carried out by students based on the three components of the geometric reasoning framework: visualization processes, construction processes (using tools), and reasoning processes. These components are integral to the geometric reasoning experienced by students.

RME as Design Principle and Heuristic

We employ six RME principles as the foundation for our design process, encompassing the HLT thought experiment and the teaching experiment.

To grasp quadratic equations, students participate in a series of learning activities. These activities are practically implemented by utilizing geometric shapes that can be adjusted to meet the objectives of each activity. Furthermore, students actively engage in collaborative exercises with the teacher, demonstrating how geometric reasoning can be applied. This process facilitates the progression of students' comprehension from a situational understanding to a formal knowledge throughout the activities. Additionally, we ensure that each geometry concept is interconnected with a concept in quadratic equations, promoting an intertwined understanding. Lastly, as part of the teaching experiments, we guide students in constructing knowledge of quadratic equations and encourage them to rediscover the quadratic formula as a guiding principle.

TASK, RESULT, AND ANALYSIS

Prior Knowledge of Kenia and Rara.

Prior to engaging in instructional interventions, and to develop the initial HLT, we assessed the mathematical abilities of Kenia and Rara in solving quadratic equations. We provided a quadratic equation, $2x^2 + 4x - 12 = 0$, which would later be used in teaching experiments. Based on the findings and our interviews, conjectures were made about each's knowledge of quadratic equations. Kenia was found to have limited understanding of constructing quadratic formulas and struggled with algebraic operations, but she showed proficiency in geometry. On the other hand, Rara also lacked knowledge of quadratic formulas and focused more on geometric concepts but demonstrated better skills in algebraic manipulation.

Utami and Jupri (2021) discovered that most students used a structure sense strategy, although some were categorized as having partial structure sense. Building upon these findings, the researchers developed HLT, which included instructional objectives, hypothesized learning processes, and task sequences. The following sections will discuss HLT and the experiments in more detail.

Activity 1: Let's Complete the Square.

Activity 1 focuses on solving quadratic equations using diagrams. The approach involves using the concept of equal areas of squares to simplify the equations by stating that the lengths must be equal.

Picciotto (2008) presented how the elements of quadratic formulas can be derived from graphs or images. Using geometric ideas (reshaping into squares) can assist students in performing symbolic operations (Fachrudin et al., 2018; Gözde & Kabar, 2018). In the visualization process, students use the interactive tool to construct the shape find a solution by changing the size of the square on the right and considering its area (Figure 2). Students are invited to consider that in order to create an almost square total area, certain shapes can be rearranged using the small circular button located in the corner, although it may seem as though a piece is missing. This process eventually leads to the completion of the square, and showing how one side of the square equals to the other.

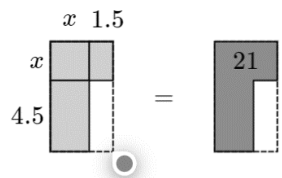
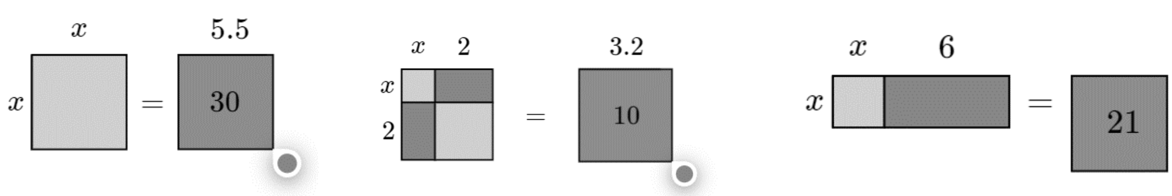


Figure 1. Interactive button

In this learning activity, students are invited explore how to use an interactive tool to solve equations involving squares and their areas. For example, they consider the equation $x^2 = 30$ (Figure 2a) and discuss how to find a number that, when multiplied by itself, yields 30. They note that there is no integer solution to this equation, but a different type of number that works. Specifically, the solution is the side length of a square whose area is equal to 30. The interactive tool helps students visualize this and find an approximate solution to the equation.



a b c

Figure 2. The phase of Activity 1

The activity also covers another problem involving areas and squares of the equation $(x + 2)^2 = 10$ (Figure 2b). Students are also asked to determine the equation represent from the diagram that is not a square (Figure 2c), that eventually become a square. This time they were given multiple choices giving them help that they have to finally emerge conjecture the $6x$ rectangle to be moved half.

Our prediction for Activity 1 is that students will learn how to reason about the area of a square using two different visuals side by side. This will help them determine the value of one side of a square in algebraic notation. By recognizing that the lengths must be equal, they can simplify the problem and find a solution, as shown in Figure 3. However, relying solely on geometry may not be sufficient for all problems. Students may struggle to understand that the area of a square is determined by the product of x, resulting in x^2. They may only estimate the area based on one side of x and one side of the square, which is acceptable since it yields the correct value of x. Nevertheless, they will also need to comprehend square roots because they will encounter variables rather than numbers at the end of the activity.

In this activity, Kenia first determines the relationship between a square with a side length of x and another square with an area of 30. Initially, Kenia used a circle to adjust the area to 30. Then, we asked what the approximate value of x is. (Note: "R" stands for the researcher, "K" for Kenia, and "A" for Rara).

R: *Please use the mouse to adjust the button to change the size of the square. Based on the current situation, what is the value of x?*

K: [Adjusting the button to make the area of the square 30] *Looking at the square with side x, the value of x changes as I move the button. Therefore, the value of x is* 5.5.

R: *How do you calculate the area of the square with side x?*

K: *The area of the square is x multiplied by x, which is x^2.*

Meanwhile, Rara gives a more specific conjecture, namely $\sqrt{30}$, which shows how he derived the value of x from an area of 30. Here is how he solved the problem:

A: [Adjusts the button to make the area of the square 30] *The value of x is $\sqrt{30}$, which is* 5.5.

R: *Can you explain how you arrived at $\sqrt{30}$?*

A: *Sure. The area of the square is x multiplied by x, which is x^2. We know that the area of the square is* 30*, so $x^2 = 30$. By taking the square root of both sides, we get $x = \sqrt{30}$, which is equal to* 5.5.

At this stage, although students arrive at the quadratic form in different ways, they both obtain similar results for the value of x. However, in the next scenario, where the square has an area of $(x+2)^2$, they both make an error in determining the value of x. To resolve this, they manipulate the circle until the area of the square on the right is 10, creating a visual representation of the equation $(x+2)^2 = 10$. From this, they determine that the side length of the square is approximately 3.2 when $x + 2$ is equal to the square's side length. By visually inspecting the square, students can see that the value of 3.2 represents the length of one side of the square, which is equivalent to $x + 2$ in the other square. Therefore, x can be calculated as approximately 1.2, that is by $3.2 - 2$. We confirm their answer and reinforce the concept of the length of the other side of a square. They arrived at the same conclusion when they realized the relationship between the length of 3.2 in one of the squares and the length of $x + 2$ in the other square.

R (to both): *Are you certain that the length of this square is $x + 2$ and that x has a value of* *approximately 3.2?*

K: [pausing to examine the picture] *Uh...no, I apologize. The value of x is actually 3.2-2, which is equal to 1.2.*

Meanwhile, in a separate teaching experiment, Rara states:

A: *It's 1.2,* [pausing briefly to examine the picture] *I just realized.*

R: *How did you arrive at that conclusion?*

A: *Since the length of the square is* $x + 2$, *and we know it equals* 3.2, *we can subtract* 2 *from* 3.2 *to get* x, *which equals* 1.2.

We continue working on determining the equation of the third shape (Figure 2c). One of the challenges students face is dealing with non-square areas and attempting to make them more square-like.

This situation resembles the one observed by Zakaria et al. (2010) in which students demonstrated a tendency to make errors in transformation and process skills while solving quadratic equations. Reshaping a non-square area into a square or a shape that closely resembles a square can aid in the understanding of the relationship between side lengths and areas. To accomplish this, students may need to rearrange some parts of the shape until the total area is nearly a square. Previous research has shown that students often struggle due to the teacher's insufficient focus on fostering a deep understanding of mathematical language and the necessary skills (Zakaria et al., 2010).

In this instance, students are required to adjust the button to create an almost square shape and determine the missing area. Subsequently, they were tasked with determining the equation resulting from the sum of the two squares. During the process, the two students had no trouble reshaping the non-square shape into a more square-like one. However, they encountered difficulties answering the equation question. The following describes the situation experienced by both of them.

K: *I'm not entirely sure, but what is the equation supposed to look like?*

R: *Think about this*: *the area of the square on the left is equal to the area of the square on the right.*

K: [pauses] *So, can I just multiply it and get* $6x^2$*?*

R: *Well, there are two shapes there, so you need to make it correct.*

K: *Let me see, this one* [points to the square with side x] *must be* x^2*, and this one* [points to the rectangle] *is* $6x$*, so it's* $x^2 + 6x$*.*

R: *Equal to?*

K: *Equal to* 21.

R: *When you finally make it into a square, what is the area of the missing square?* [points at the square]

K: *It's* 9*, from* 3×3.

R: *And what is the total area and the new side length?*

K: *The new side length is* $x + 3$*, so* x *is* 6*. The total area is...* 30.

Rara worked diligently on this case. He immediately started reshaping the rectangle into an almost square shape, and only then did he determine the equation by squaring both sides, resulting in $(x + 3)^2$. However, he encountered the same problem here and was initially confused by the resulting equation. Our approach remains the same, by providing instructions in the form of a left square being equal to a right square.

R: *I give you an idea, the area of square in the left* ***is equal to*** *the one in the right.*

A: *So, it is equal to* 21*?*

R: *Take it carefully, you have your new shape, there is a missing square, no?*

A: *Yes, but I have no idea about the area of the missing square.*

R: *Look at the left, there is a length in it.*

A: .... [pausing briefly], *maybe it is* 9

R: *Why?*

A: 3×3.

It was also noted that a significant number of students obtained an inaccurate final solution. At this point, we need to add instructions to our HLT that explain how the two shapes are related to each other so that students can generate their ideas more effectively. However, both students were generally able to complete this activity well, and we also discovered a new conjecture about the area of $6x^2$ that we had yet to anticipate before. The trajectory of how to complete the square in Activity 1 is illustrated in Figure 3 below. Additionally, we have found that we may need to emphasize to students the importance of "dividing by half" when they modify and reshape a non-square shape into the HLT, as it affects their geometric reasoning in the next activity. To arrange a non-square shape into a square, students must divide it into two parts.

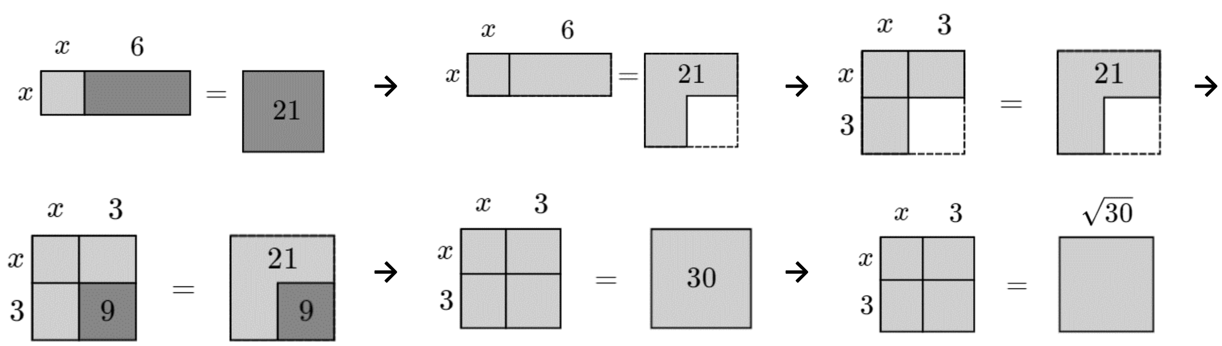


Figure 3. Activity 1 on completing the square

Although students may not realize it yet, we just come up with our goal for solving lots of quadratic equations —completing the square— and reinventing the quadratic formula.

Activity 2: How about different shape?

Activity 2 involves solving an equation that is still quadratic but slightly different from the one they solved earlier, that is $2x^2 + 4x - 12 = 0$.

The purpose of employing these diverse equations is to enhance students' comprehension by training them on more intricate problems, thereby elevating their understanding to a more advanced level. According to Utami and Jupri (2021), the capacity to perceive patterns and solve quadratic equations is more intricate compared to factoring algebraic expressions. In this section, the visualization process remains connected to the preceding activity, namely, the utilization of interactive boxes.

Here, the students are encouraged to apply the same technique they used on the previous problem to solve the current equation. However, this time, the students are given multiple choice options that lead them to divide every rectangle's width by 2, while ensuring that the equation remains balanced (see Figure 4). While manipulating them, students explore these geometric scenarios to ascertain the equilibrium between algebraic and geometric shapes.

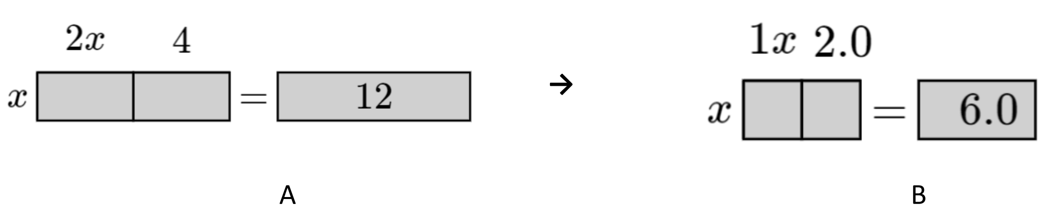


Figure 4. Illustration of dividing rectangle into a half.

Upon constructing in this manner, this division modifies the equation, while the value of x remains constant. Subsequently, the instructor directs the students to complete the square as used in the previous task to solve the current equation logically. Finally, the students are expected to approximate the value of x, similar to their approach in activity 1.

During this phase, students may encounter fewer difficulties than before, provided that they have fully understood the entire procedure of activity 1. The potential challenge they might face is when they come across division of the rectangle, prompting them to question why the side of the rectangle is being divided by two (Figure 4). However, we anticipate that this is simply a way to determining the value of x.

We began by posing this task (R stands for the researchers, and K and R for the students, Kenia and Rara). Kenia and Rara started by looking at a non-square shape (Figure 4a). Both felt something was different but they still made the same equation: $2x^2 + 4x = 12$. Our anticipation makes them think with the question whether determining x will be easier or more difficult if what we have is $2x$ instead of x, they have the same answer which is more difficult. So, here are the ideas they have for the problem:

R (to both): *So, what do you think is the solution of this?*

K: *We should make it into a square first, so we may be squaring or dividing two times here. All dividing by* 2.

Meanwhile, Rara sticks with his square root idea.

A: *The length of the side must be* x *because we need to find the square root of* x^2

R: *How do you do that.*

A: *Just…divide them into the half of it.*

They were unable to complete the task correctly or provide an explanation of the steps until the value of x was determined. Based on this analysis, our evaluation in Activity 2 serves more as a guide for what students should do at the beginning, leading to a tendency for the given rectangles to not have a length of x. As a result, students are immediately encouraged to solve these problems without going through interviews.

Furthermore, Kenia presented an interesting idea stating that "I think we may have to divide by two here," indicating that she needs to square the first rectangle to turn $2x$ into x, along with the other value. She also needs to reshape the rectangle with the side x to make it a square. We have this conjecture into our revised HLT.

Finally, the path of how to complete the square in activity 2 is illustrated in Figure 5 below.

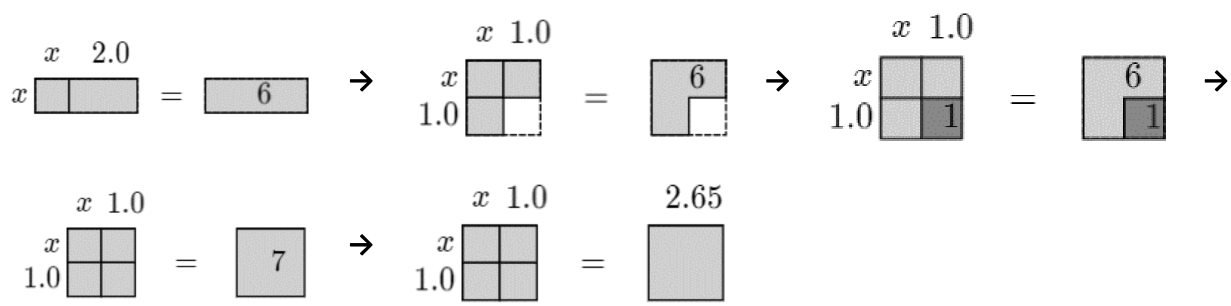


Figure 5. Activity 2 on completing the square

Activity 3: Let's generalize.

Gözde & Kabar (2018) emphasized the significance of knowing and comprehending the formal definition of a quadratic equation to gain a conceptual understanding of the relationships involved beyond mere symbolic calculations when solving quadratic equations. This activity invites students to solve any quadratic equation containing an x^2 term and an x and a constant term. Although Activity 3 requires a deeper understanding, it still relies on the student's ability to complete the square. Students asked to represent all quadratic equations simultaneously by rearranging the equation to the form $ax^2 + bx + c = 0$, where $a, b,$ and c are numbers. The students are then tasked with determining the values of $a, b,$ and c and retracing their steps to solve $ax^2 + bx + c = 0$. Although we presented the algebraic manipulations, we only showed the changes in geometric approach and algebra culminating in the quadratic formula, as the operations are too advanced for their grade level.

Start by subtracting c from both sides and dividing every horizontal length by a (Figure 5), to determine the side lengths of the left side shape when it becomes a square. The students then need to find $x + \frac{b}{2a}$ by splitting the $x + \frac{b}{a}$ rectangle in half. The missing square's side length is then $\frac{b}{2a}$, so its area is $\left(\frac{b}{2a}\right)^2$. Adding $\frac{b^2}{4a^2}$ makes the left-hand side of the equation a square, which lets the students finish solving for x. The full steps are presented in Figure 5.

We conjecture students may not have any difficulty in determining the values of $a, b,$ and c. However, the process of deriving the quadratic formula using a geometric approach may raise some questions for students, particularly in the algebraic expressions involved. Therefore, we aim to assess their understanding by asking about the changes in the geometric and algebraic aspects of the process (see Figure 5).

Both individuals did not encounter any difficulties in determining the values of $a, b,$ and c as they progressed toward the general form of the quadratic equation. Although they initially failed to recognize that they had halved before, in activity 2, both students halved again in this activity. However, they faced an obstacle in the last question. When Rara was asked to determine the length of side $x + \frac{b}{a}$ after it was squared, they responded with $\frac{x+\frac{b}{a}}{2}$, indicating their need to understand the halving concept introduced in the previous activities. When we intervened, they immediately recognized their mistake. As a result, we consider this error to be a form of inaccuracy. On the other hand, Kenia provided the correct answer, which was $x + \frac{\frac{b}{a}}{2}$.

One noteworthy aspect was their realization of how the quadratic formula is derived after knowing that the sides of a square are divided by two (the complete process is shown in Figure 5). We also asked them to share their insights on the process and how they interpreted the plus or minus notation in the quadratic formula.

R: *Did you find this process meaningful? What do you think the plus-or-minus notation means there?*

K: *The value may be either positive or negative.*

R: *The value of what?*

K: *I mean… this operation, basically you can just use the minus one or just the plus one, so after "$-b$" we do it with plus or minus.*

R: *How about using both?*

K: *I don't know.*

Meanwhile, unlike Kenia, Rara has actually understood the meaning of "or" there.

A: [paying attention to the process in Figure 5]

R: *What do you think of it, why does it have* $\frac{b^2}{4a^2}$*?*

A: *That's the multiplication of* $\frac{b}{2a}$ *times* $\frac{b}{2a}$

R: *What do you think the plus-or-minus notation means there?*

A: *It will be used later when determining the final result, use it with plus or minus.*

R: *So, if for example* $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$*, how many operations are there? [pointing at the plus-or-minus or operation at the quadratic formula].*

A: *Two, plus and minus*

R: *When you mention "and", does that mean you have to operate both?*

A: *No, only one.*

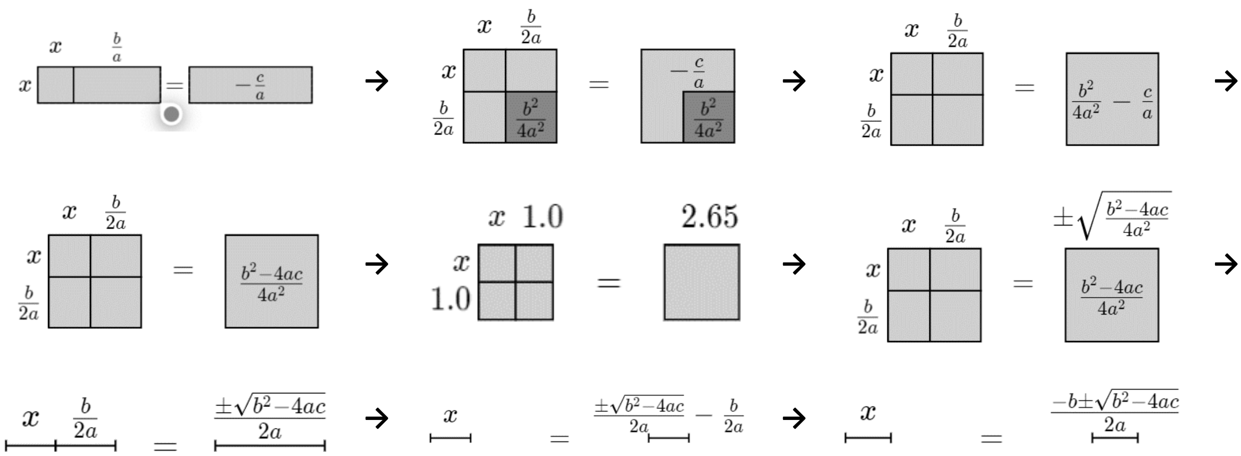


Figure 5. Activity 3 on completing the square and reinventing the quadratic formula

Our analysis indicates that an important aspect of this lesson, algebraic notation, is missing.

Tall et al. (2014) stated that teachers also highlight the quadratic formula, as it is applicable to all quadratic equations. When we present the real form of the area of a square and ask students to determine the length of one side (x), they often assume that finding the value of $x^2 = a$ involves taking the square root. However, there are actually two different results, and we take the positive one because it represents the length of the side of the square. Therefore, we need to add steps related to determining the result of $x^2 = a$ through factorization to follow the process that leads to the positive and negative results. Nonetheless, students may still see the plus-or-minus notation as representing two operations, even though only one of the results is used.

CONCLUSION

Concluding Remarks

Recalling this study aims to examine the impact of learning activities on students' ability to formulate a quadratic equation, the analysis of the z presented above and the explanation of the teaching experiment processes provide evidence of a promising trajectory for students learning. The study design emphasizes students' active engagement in learning and its correlation with quadratic equation concepts. Through individualized learning activities that integrate the context and principles of the RME approach, students can construct their knowledge and reinvent the quadratic equation formula.

In conclusion, activities designed in such a way resulted in students developing their knowledge. The results indicate that students can learn according to the designed learning trajectory. Likewise, the situation that occurs in this study may well be different depending on students' initial abilities.

Trajectory Experience, Conjectures, and the Revised HLT

From the comparison of HLTs and observed learning, we do the redesign process, revising our HLT, and affords answers to research questions that ask how particular learning goals could be reached. Following our retrospective analysis of the data, we created a revised HLT for fostering quadratic equation concept and reinventing quadratic equation formula.

Activity 1 — This activity focuses on solving quadratic equations using diagrams. The approach involves using the concept of equal areas of squares to simplify the equations by stating that the lengths must be equal. Students use the interactive tool to find a solution by changing the size of the square on the right and considering its area. The two shapes are related to each other so that students can generate their ideas more effectively. Students are invited to explore how to use an interactive tool to solve equations involving squares and their areas. For example, they consider the equation $x^2 = 30$ and discuss how to find a number that, when multiplied by itself, yields 30. Next, the activity covers another problem involving areas and squares of the equation, that is $(x + 2)^2 = 10$.

Our prediction for Activity 1 is that students will learn how to reason about the area of a square using two different visuals side by side. This will help them determine the value of one side of a square in algebraic notation. By recognizing that the lengths must be equal, they can simplify the problem and find a solution. However, relying solely on geometry may not be sufficient for all problems. Students may struggle to understand that the area of a square is determined by the product of x, resulting in x^2. They may only estimate the area based on one side of x and one side of the square, which is acceptable since it yields the correct value of x. Nevertheless, they will also need to comprehend square roots because they will encounter variables rather than numbers at the end of the activity.

Activity 2 — In this activity, the aim is to solve a equation that is still quadratic but slightly different from the one they solved earlier, that is $2x^2 + 4x - 12 = 0$. Students are asked to apply the technique they used in the previous activity to solve it. The student presented with options that require them to divide every rectangle's width by 2 while maintaining a balanced equation. After that, the student is guided to use the same method of completing the square as they did in the previous activity to solve the current equation. They will be expected to approximate the value of x as they did in the first activity. This will help them consolidate their understanding of the process and how it works.

During this activity, the student may encounter fewer difficulties than before, especially if they have fully understood the entire procedure of the previous activity. However, the potential challenge they might face is when they come across the division of the rectangle, prompting them to question why the side of the rectangle is being divided by two. Therefore, the student will need to ask the instructor to explain the rationale behind this step to ensure that they understand its purpose.

Activity 3 — This activity invites students to solve any quadratic equation containing an x^2 term and an x and a constant term. Activity 3 still relies on the student's ability to complete the square. Students are asked to represent all quadratic equations simultaneously by rearranging the equation to the form $ax^2 + bx + c = 0$, where $a, b,$ and c are numbers. The students are then tasked with determining the values of $a, b,$ and c and retracing their steps to solve $ax^2 + bx + c = 0$. Students only showed the changes in geometric approach and algebra culminating in the quadratic formula, as the operations are too advanced for their grade level. We conjecture students may not have any difficulty in determining the values of $a, b,$ and c. After that, students doing steps related to determining the result of $x^2 = a$ through factorization need to follow the process that leads to the positive or negative results.

Students may still see the plus-or-minus notation as representing two operations, even though only one of the results is used. We acknowledge that this may raise some questions for students, particularly in the algebraic expressions involved. The process of deriving the quadratic formula using a geometric approach may be challenging for some students. Therefore, additional support and guidance is required to help students understand the process.

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