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From Małgorzata Marciniak, Managing Editor of MTRJ
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Due to the large volume of submitted and accepted papers, the Editors decided to increase the number of issues published per year. Thanks to that decision, now the journal appears with its first Early Spring edition. The weather during the season of Early Spring is usually unpredictable and may equally likely bring snow or shine. Coincidentally, this journal issue happens to be about the weather-like nature of the mind observed via metacognition. Thus, self-awareness of the math-related states of mind is the main theme of the current journal issue.

Vol 15 no 1 opens with three metacognitive papers in which each one focuses on a different aspect of metacognition. The first one, **Thinking Beyond Thinking: Junior High School Students' Metacognitive Awareness and Conceptual Understanding of Integers** submitted by authors Janina C. Sercenia, Edwin D. Ibañez, Jupeth T. Pentang from the Philippines, discusses the role of metacognition for the purpose of improving math skills. Student interactions within a group as a tool for regulating self-awareness are studied in the paper **The Emergence and Form of Metacognitive Regulation: Case Study of More and Less Successful Outcome Groups in Solving Geometry Problems Collaboratively** authored by Anis Farida Jamil, Tatag Yuli Eko Siswono, Rini Setianingsih from Indonesia. Another team of authors from Indonesia, Alifiani, et al, in their paper **Metacognitive Intervention: Can It Solve Suspension of Sense-Making in Integration Problem-Solving?,** discuss the possibility of using metacognition for finding the weaknesses of the process of integration.

Psychological aspects of problem-solving are analyzed in the paper **High School Students' Beliefs about Mathematical Problem Solving: A Cluster Analysis** by South African authors Edgar J. Sintema, Mogege Mosimege. Their questionnaire is particularly interesting. Practical problem solving is demonstrated by our Corner Editor, Ivan Retamoso, in the Problem Corner, where new solutions and new problems are posted.

The next two papers in the issue are devoted to analysis of the thinking process applied in abstract algebra courses. Authors Nihayatus Sa'adah, et al. from Indonesia in their paper **Students' Mathematical Thinking Process in Algebraic Verification Based on Crystalline Concept** discuss examples of students' work while solving abstract algebra problems. While the authors Indriati Nurul Hidayah, et al from Indonesia in **Creative Conjecture: Abductive Reasoning to Generate Some Ideas in Algebra** analyze a particular way of thinking in mathematics which does not use direct implications. This approach may be confusing to students and thus needs increased attention.

In the paper, **Building on Students' Prior Mathematical Thinking: Exploring Students' Reasoning Interpretation of Preconceptions in Learning Mathematics,** authors Robert Wakhata (Rwanda), Sudi Balimuttajjo (Uganda), Védaste Mutarutinya (Rwanda) present a study on 11th grade students' mathematical preconceptions and misconceptions.

The two papers that follow analyze early development of students in primary and secondary levels. Halil Önal from Turkey presents **Primary School Students' Understanding of Four Operation Symbols (+, −, x, ÷, =) and Using Them in Arithmetic Operations and Word Problems,** while Elif Nur Akgul and Rezan Yilmaz from Turkey present **Secondary School Students' Construction Processes of Square Root Concept with Realistic Problems: An APOS Perspective.** The abbreviation APOS means Action-Process-Object-Schema (APOS) and the theory is rooted in Piagetian theory of cognitive development. The issue closes with a metacognition-like view on teaching topology in which the authors analyze how the literature of pedagogy has been treating the teaching of topology. This statistical approach entitled, **Teaching of Topology and its Applications in Learning: A Bibliometric Meta-Analysis of the Last Years from the Scopus Database,** is delivered by authors Diego Vizcaíno, Victor Vargas, and Adriana Huertas from Colombia.

Contents

Thinking Beyond Thinking: Junior High School Students' Metacognitive Awareness and Conceptual Understanding of Integers

Janina C. Sercenia[1], Edwin D. Ibañez[2], Jupeth T. Pentang[3,4]

[1]De La Salle University, Manila, Philippines,

[2]Central Luzon State University, Science City of Muñoz,

[3]Western Philippines University, Puerto Princesa City, Philippines,

[4]Wesleyan University - Philippines, Cabanatuan City, Philippines

janina_sercenia@dlsu.edu.ph, edwindibanez@clsu.edu.ph, jupeth.pentang@wpu.edu.ph*

Abstract: The potential benefits of cognitive skills in enhancing mathematics ability have been claimed by numerous researchers. Since mathematics requires a complete understanding and grasp of abstract concepts, it is essential to explore how learning with metacognitive skills affects mathematics learning. Thus, the study investigates the students' metacognitive awareness and conceptual understanding of integers. A descriptive-correlational method approach was utilized, and it was carried out on 303 seventh-grade students. The data were obtained using a metacognitive awareness inventory and achievement test on integers. It was further revealed that the students have average metacognitive awareness and performed well in the fundamental operation of integers. Follow-up qualitative analysis revealed that students who were high achievers had the best understanding, average achievers had corrected or incomplete understanding, and low achievers had a functional misconception of integers. Moreover, the student's metacognitive awareness was significantly related to their conceptual understanding of integers. This indicates that student's higher-order thinking skills, such as metacognition, are essential since they are associated with building conceptual skills. Thus, teachers should encourage students' metacognitive awareness to improve students' conceptual understanding of integers. The study provides relevant information for educational managers on the potential factors to be considered in improving mathematics education practices, particularly in promoting metacognition among high school students.

INTRODUCTION

Students' ability to control, monitor, and comprehend their learning process has become one of the focused concepts in education. Students' acquisition of metacognition and conceptual understanding is essential, especially in mathematics (Ibañez & Pentang, 2021; Mariano-Dolesh et al., 2022). With this, educators recognize the importance of higher-order thinking skills such as metacognition and conceptual understanding of students in learning. Evaluating students' skills is best understood by determining how individuals acquire them. Metacognition was defined as "knowledge and regulation of one's cognition," "thinking about thinking", or "learning how to learn" (Flavell, 1979). Metacognition can be seen when a person involves active awareness and control over the cognitive processes engaged in learning. Being aware of one's cognition involves metacognitive awareness. Metacognitive awareness entails self-reflection on one's thought patterns to comprehend and develop them. However, young children are entirely unaware of their thinking and other aspects of metacognition. Furthermore, metacognition is associated with concepts, namely, metacognitive knowledge and metacognitive regulation. These two components were hypothesized to be connected (Brown, 1987; Schraw & Dennison, 1994).

On the other hand, mathematics educators emphasize conceptual understanding among learners as a critical component of mathematical ability. Conceptual understanding is associated with profound and flexible knowledge of abstract principles (Mariano-Dolesh et al., 2022; Star & Stylianides, 2013). It is evident as students become successful problem solvers, determined to persist until a rational solution is attained (Ibañez & Pentang, 2021; Santos et al., 2022). Conceptual knowledge enables students to apply and potentially adopt certain learned mathematical concepts to new contexts (Qetrani & Achtaich, 2022). Moreover, it was concluded that students face difficulties operating integers (Pentang, 2019). The errors seemed to be caused by a lack of access to mediating objects such as number lines or other real-world contexts when learning to work with integers. Makonye and Fakude (2016) also found that students struggled comprehending negative numbers or operations containing negative integers. Hence, exploring the conceptual understanding of integers and their operations among learners must be encouraged.
Several studies indicated the relationship between metacognition and conceptual understanding of learning (Fleming et al., 2012; Gunstone & Mitchell, 2005). Metacognition was used to explain an individual's ability to influence their learning strategically. Conceptual understanding entails identifying current conceptions, assessing them, and determining whether to create and update, both of which include sufficient metacognitive awareness and control. Still, the conceptual understanding of integers of Grade 7 students in the Philippines remains unexplored. Students nowadays show various levels of knowledge and understanding in different learning situations. Some may show awareness of their learning and know-how to monitor and reflect on their thinking. Others might have poor performance and difficulty monitoring and understanding their learning. Hence, the students possess different levels of metacognition and conceptual understanding, and conceptual knowledge or understanding is essential for explaining students' performance as metacognition does.

Relative to this condition, the role of both metacognition and conceptual understanding in the learning process has been identified. However, other researchers have claimed that the effects of metacognition are unrelated to academic success. More research needs to be done on the relationship between metacognitive awareness in mathematics instruction. The pedagogical grasp of metacognition needs to be improved, contributing to the gap between theory and practice. Thus, there is a need to reassess the relationship of metacognition in mathematics performance to serve as a guide and basis for conducting new studies, making institutional policies, and reviewing educational courses. With these, this study aimed to determine the metacognitive awareness and conceptual understanding of Junior High School students on integers and the relationship between them. This study provides teachers with insights and valuable information about its association with student conceptual understanding and integrating and implementing metacognition in mathematics education.

LITERATURE REVIEW

Metacognition has been conceived as a crucial component of learning. Many researchers have asserted its effects on academic performance. For instance, experts conducted several studies on cognitive abilities such as metacognition and what it could bring to mathematics performance. Learning mathematics involves critical thinking, problem-solving, analytical thinking, and reasoning. Learning mathematics effectively requires a deep understanding and comprehension of mathematical ideas and skills such as metacognition. With these scenarios, researchers concluded its association with learning mathematics.

Several researchers found that metacognition is associated with mathematics performance (Bernard & Bachu, 2015; Schraw & Dennison, 1994). Metacognitive knowledge and mathematical intelligence substantially impact academic performance, and there is an association between metacognitive knowledge and mathematical intelligence (Chytry et al., 2020). There is a strong positive association between metacognition and achievement in mathematics (Özsoy, 2011), indicating that students with higher metacognition tend to have excellent performance in mathematics, and students with low metacognition will likely have poor achievement in mathematics. This notion is also supported by Desoete and De Craene (2019), who found that poor learners were less accurate in metacognitive terms and more often underestimated their performances. Some misunderstandings were identified as logical shortcomings, mathematical confusion, misinterpretation of problems, and poor problem-solving skills. In addition, when the students face a test case, they have a significant challenge, which can be due to their weak math ability and inability to deal with complex situations (Pentang, 2019). It might be argued that teachers and educators should emphasize metacognition to develop and increase learners' mathematical performance abilities. The metacognitive component is crucial for learning since it aids students in organizing, monitoring, and evaluating their thought processes (Naufal et al., 2021). Students' mathematics performance is significantly and positively affected when teachers apply metacognitive strategies in their teaching approach (Alzahrani, 2017). Besides, metacognition was involved in conceptual change (Tickoo, 2012). Since metacognition includes planning, evaluating, and monitoring problem-solving activities, these processes are central to intelligence. The benefits of metacognition include increased awareness of mathematical instruction and improved learning outcomes (Salam et al., 2020). Metacognitive learners also include self-aware and reflective learners. Reflecting on a lesson's relevance gives didactic mathematical knowledge to aid the instructor in making judgments (Navarro & Céspedes, 2022). A learner who is metacognitively aware has a method of figuring out what they need to know.

Moreover, metacognition is beneficial in predicting academic achievement. According to Young and Fry (2008), "if the students have well-developed metacognitive knowledge and metacognitive regulatory skills and use their metacognition, they will excel academically" (p.2). Metacognitive ability proficiency in analyzing one's thought processes is related to correctly assessing an individual's mastery of a task (Dunning et al., 2003). Also, Asy'ari et al. (2019) emphasize the importance of declarative knowledge within the inquiry learning model, as these could potentially aid in constructing components that create thorough metacognition awareness. With these findings, studies have provided how metacognition affects mathematics performance. The studies also emphasize the importance of integrating strategies to improve metacognition since it impacts learning mathematics.

However, despite the positive effects of metacognition on students' academic performance, some studies asserted that the effects of metacognition are not related to academic performance. It has been found that being metacognitively aware of one's cognitive knowledge does not necessarily translate into higher academic performance, and mathematical performance cannot be predicted by metacognitive awareness levels (Smith, 2013). Educators have also included metacognitive strategies in their teaching to determine the effect of metacognition in mathematics. However, it was resolved that the average learners who do not perform well in mathematics do not improve from instructions integrated with metacognition (Artlet & Schneider, 2015). Students do not usually acquire metacognitive ability through instruction (Ahmad et al., 2018). It is due to the limited framework and planned cognitive activities in teaching. Additionally, Zohar and Barzilai (2013) discovered numerous instructional strategies available to promote metacognition in the classroom after reviewing 178 studies for their systematic review analysis. It brought attention to the growing role of metacognition in education. However, little is known about effective metacognitive strategies used in mathematics education. Aside from that, there is a gap between theory and practice because many teachers lack a pedagogical understanding of metacognition (Wilson, 2010). With this, several research has reached varying conclusions on the impact of metacognition on student performance.

METHODS

Research Design and Participants

This study employed a descriptive-correlational research design. The descriptive design focused on the quantitative assessment of the respondents' metacognitive awareness and conceptual understanding of integers. The correlation analysis determined the statistical relationship between the variables. Furthermore, the qualitative assessment focused on the follow-up analysis of the student's conception of operating integers.

A total of 303 Grade 7 students were chosen as the study's respondents using random cluster sampling. They were classified into three categories based on their mean percentage score on the summary of the first quarter's test results in mathematics: high achiever, average achiever, and low achiever. Moreover, the majority of these students were females (56.77%) and had an average of 12 years old. Most respondents also graduated from elementary public school (93.07%), and the mean grade of the respondents on mathematics subjects for the first quarter was 81.80, with a standard deviation of 5.285. Finally, for the follow-up interview, the researchers selected three students in each level (low, average, and high achiever), totaling nine students. This allowed the researchers to make conclusions about their conceptual understanding of integers.

Research Instruments

The study utilized a questionnaire with an adopted inventory and achievement test as an instrument, with their permission. The Metacognitive Awareness Inventory was adapted from Schraw and Dennison (1994). The inventory consisted of 52 statements that the students rated as 1-never, 2-seldom, 3-sometimes, 4-often, and 5-always (see Appendix). The items were translated into the Filipino language by a Filipino professor to be easily understood by the respondents. The internal consistency found the Cronbach alpha 0.869 and was considered reliable. The achievement test on the fundamental operation on integers was also adapted from Rubin et al. (2014). It was used to evaluate students' conceptual understanding of integers. The achievement test consisted of 40 items based on the fundamental operations on integers. The items were divided into seven (7) parts: definition of integers (items number 1 to 6), the concept of the number line (items number 7-11), comparing integers (items number 12-14), real-life application of integers (items number 15-16), operation of integers (item number 17-32), integers property (items number 33-35), and rules on operating integers (item numbers 36-40). The achievement test was developed using the table of specifications following the Department of Education's Minimum Learning Competencies on integers. Two College Mathematics Professors and a pilot validated the test tested on 40 Grade 7 students. The instrument was considered reliable, with a Cronbach Alpha reliability coefficient of 0.859.

Data Collection and Analysis

The researchers obtained prior approval from the school authorities and consent from the parents and students. Data were gathered in person for a week while the follow-up assessment was conducted the following week. The study utilized descriptive statistics such as percentage, mean, standard deviation, and Pearson's r. Metacognitive awareness and conceptual understanding of the integers of the respondents were described using percentage, mean, and standard deviation. Pearson product-moment correlation coefficient (Pearson's *r*) was utilized to determine the interrelationship between the socio-demographic characteristics of the respondents and their metacognitive awareness. This also determined the relationship between respondents' socio-demographic characteristics and their conceptual understanding of integers, as well as the relationship between respondents' metacognitive awareness and their levels of conceptual understanding.

Furthermore, the researchers analyzed the qualitative responses from the interview. The follow-up interview was conducted with the selected respondents to clarify their solutions or reasons on how and why they came up with their answers. This also served as a reflection on the performance of the respondents. The interviews were done after the achievement test was administered. A conceptual trace analysis based on Jensen and Finley's theory was used to determine the conceptual understanding of the respondents. It was done on the respondents' response and their corresponding solutions or explanations for the questions in the achievement test. These data were analyzed based on the following: (1) Best understanding (BU) when the respondent has a correct answer accompanied by a correct and complete explanation; (2) Partial understanding (PU) involves a correct answer but with incomplete reason; (3) Correct or incomplete understanding is observed when the answer is wrong but with correct and incomplete solution/ reason; (4) Functional misconception (FM) when the answer is correct but has incorrect solution or reason; (5) No understanding (NU) when the answer is wrong, accompanied by incorrect solution or reason. In the study conducted by Ibañez (2009), he described the solution or reason of the respondents to each item based on the following descriptions: (1) Complete correct solution/ reason when the answer is complete and correct, and all parts of the question are addressed; (2) Correct but incomplete solution/ reason when the respondent gives a partially correct answer, or task is incomplete; and (3) Incorrect solution/reason when the respondent does not address task or has no answer.

RESULTS

Metacognitive Awareness of the Respondents

The overall mean of the respondents' metacognitive knowledge was 3.27 *(SD = 0.64)*, which is described as average (Table 1). The respondents' declarative knowledge (*Mean = 3.23, SD = 0.67*), procedural knowledge (*Mean = 3.21, SD = 0.74*), and conditional knowledge (Mean = 3.36, SD = 0.84) were also average. This result implied that the students occasionally considered metacognitive knowledge in its component when doing their schoolwork or homework.-Also, the student's metacognitive regulation was average (*Mean = 3.33, SD = 0.63*). Their planning ability was above average (*Mean = 3.51, SD = 0.83*). However, the students have average regulation regarding information management strategies (*Mean = 3.29, SD = 0.69*), comprehension monitoring *(Mean of 3.28, SD = 0.74*), debugging strategy (*Mean = 3.37, SD = 0.76*), and evaluation (*Mean = 3.24, SD = 0.71*). It can be noted from the findings that the respondents occasionally monitored and assessed their knowledge.

Metacognitive Awareness	Mean	SD	Description
Metacognitive Knowledge	3.27	0.64	Average
Declarative Knowledge	3.23	0.67	Average
Procedural Knowledge	3.21	0.74	Average
Conditional Knowledge	3.36	0.84	Average
Metacognitive Regulation	3.33	0.63	Average
Planning	3.51	0.83	Above Average
Information Management Strategies	3.29	0.69	Average
Comprehension Monitoring	3.28	0.74	Average
Debugging Strategies	3.37	0.76	Average
Evaluation	3.24	0.71	Average

Table 1. Respondents' metacognitive awareness level (Legend: 4.21-5.00 = High, 3.41-4.20 = Above Average, 2.61-3.40 = Average; 1.81-2.00 = Below Average; 1.00-1.80 = Low)

Conceptual Understanding of Integers of the Respondents

Findings showed that the students' mean scores on the achievement test were 17.07 (*42.68%),* described as a "good" remark (Table 2). Besides, students did best in defining the integers with a mean score of 3.80 (*63.33%)* with a remark of "very good". Comparing the integers also got a "very good" remark with a mean score of 2.04 (*68.00%).* Also, results revealed that the ability to distinguish concepts in number lines got a remark of "good" with a mean of 2.01 (*40.20%*). The integer's real-life application was also considered "good", with a mean of 1.09 (*54.50%*). In terms of operating integers, students' performance in adding integers was good, with a mean score of 1.9 (*47.50%),* the ability to multiply integers with a mean score of 1.87 (*46.75%),* and the ability to divide integers with a mean score of 2 (*50%).* However, the ability of the students to subtract integers got a remark of "fair" with a mean score of 1.4 (*35%*). On the other hand, students performed poorly in applying integer properties (*0.50 or 16.67%)* and rules on the operation of integers (*0.40 or 8%).*

Fundamental Operation on Integers	Mean	Percentage	Description
Definition of Integers (Items 1-6)	3.8	63.33	Very Good
Concept of Number Line (Items 7-11)	2.01	40.20	Good
Comparing Integers (Items 12-14)	2.04	68.00	Very Good
Real-Life Application of Integers (Items 15-16)	1.09	54.50	Good
Addition of Integers (Items 17-20)	1.90	47.50	Good
Subtraction of Integers (Items (21-24)	1.40	35.00	Fair
Multiplication of Integers (Items 25-28)	1.87	46.75	Good
Division of Integers (Items 29-32)	2.00	50.00	Good
Integer Properties (Items 33-35)	0.50	16.67	Poor
Rules on Operation of Integers Items (36-40)	0.40	8.00	Poor
Total	17.07	42.68	Good

Table 2. Respondent's conceptual understanding of fundamental operation on integers (Legend: 80.01– 100.00% = Outstanding; 60.01 – 80.00% = Very Good; 40.01 – 60.00%= Good; 20.01 – 40.00% = Fair; 00.00 – 20.00% = Poor)

The researchers conducted a follow-up assessment to discuss the respondents' conception further. The individual responses of the respondents were interpreted based on Jensen and Finley's (1995) Conceptual Trace Analysis (Table 3). Results show that the student respondents' level of metacognitive awareness was "average metacognition", with an overall mean of 3.34 and a standard deviation of 0.63. Specifically, the results also showed that the students in the high achiever group had the best understanding of integer concepts, with an overall mean of 3.27. The average achiever group showed a correct or incomplete understanding of integer concepts with an overall mean of 2.23. Students in the low achiever group had functional misconceptions with an overall mean of 0.80.

Respondents	Level of Understanding	
	Mean	Description
High Achiever	3.27	Best Understanding
Average Achiever	2.23	Correct/Incomplete
Low Achiever	0.80	Functional Misconception

Table 3. Respondents' Level of Conceptual Understanding of Integers (Legend: 3.20-4.00 = Best Understanding; 2.40-3.19 = Partial Understanding; 1.60-2.39 = Correct/Incomplete; 0.80-1.59 = Functional Misconception; 0.00-0.79 = No Understanding)

Summary of the Analysis of the Interview Results

The first part of the achievement test was about the definition of integers. Student 1 answered all the items on the first part of the questionnaire correctly with the corresponding reason or explanation. She explained her answer correctly on the first item as she stated, "*Integers include the natural number and their negatives; hence integers can be both positive and negative*". She also added, "*Integer is a rational number with no fraction...examples of numbers that are not integers are 1, 2, 3,...*". On the second item, she described that opposite numbers have the exact distances from zero because it deals with distances. The distance can never be negative; she stated, "*a straight line has two directions and can be used as a scale so that equal distances on the line always correspond to equal differences between numbers*". In item number 3, she explained that absolute value was always positive as she described the absolute value as "*it is like a number between 0 and a number and was denoted by two vertical lines*", but she never mentioned that the concept of absolute value was about measurement or distances. She also explained how to subtract integers and that the division and multiplication of integers have the same rule as she mentioned, "division and multiplication of integers are the inverse operations of one another". However, she explained item 5 using an example: "-(-5) – (3), and she said, "When you see a minus sign followed by a minus sign, the sign will turn into plus (+5), and then add the two numbers". Meanwhile, Student 2 from the high achiever group had the best understanding of the items in the first part except for item number 3, she got a correct answer, but stated that, "*absolute value was about weighing the numbers, and I just got confused with the statement*". On the other hand, Student 3 had the best understanding of the items except for items 2, 3, and 5. Student 3 discussed item 2 by giving examples and drawing a number line; he said they have the same scale. In item number 3, student 3 had the same explanation as student 1, and never mentioned the concept of distances or measurement. In item 5, he gave examples and illustrations like 5- (-1) = 6, 1- (-3) = 4 and said, "*when you subtract a negative from a positive, the subtrahend (-1) will become positive since the multiplication of like signs is positive, then add the numbers, and the answer will be positive 6*".

The second part of the achievement test was about the concept of a number line. Students 1, 2, and 3 best understood most of the items. They had almost the same explanation about the number line. Student 1 said, "*number line is a straight line with 0 between them, and if you move to the left of zero, that is a negative number, and to the right are the positive numbers*". Student 2 discussed that "*using the number line, if you move to the left of zero, the value of a number decreases, and if* *you move to the right, the number increase*". In item 7, students 1 and 3 discussed the absolute value again without mentioning the concept of measurement and distance, while student 2 incorrectly answered the item but discussed the concept of absolute value.

Part three of the achievement test was about comparing the integers. Students 1, 2, and 3 answered all of the items correctly and had the best understanding. Student 1 explained, "*Negative numbers are always greater than the positive number and using the number line, it can be seen that the number to the right is greater than the number to their left*". Student 2 explained, "*In positive numbers, the numbers to the right have bigger value than the numbers to the left same with the negative number*". Student 3 added, "*the negative sign was powerful, for example, in numbers 100 and 1… 100 is greater than 1, right*?... but if you put a negative sign to a 100 (-100), it decreases its value because negative means a loss... but in the case of 0 and a negative, we can always figure that in temperature -12 degrees Celsius was colder than 0 degree Celsius".

Part four was about the real-life application of integers. Students 1, 2, and 3 got almost the items correctly and had a partial understanding. Student 1 explained, "*a loss means to decrease or minus, and a deposit means to increase or plus*". Student 2 had the same explanation as Student 1. However, student 3 got item number 16 correctly but got confused and explained, *"₱2500 deposit in a bank means putting money in a bank, and this means that he will loss ₱2500*".

Part five of the achievement test was about operating integers. This was divided into four parts: addition, subtraction, multiplication, and division of integers. In addition to integers, students 1, 2, and 3 had either best, partial, correct, or incomplete understanding. In item numbers 17 and 18, student 1 got the item correctly and discussed, "*adding a negative number from a positive number is like subtracting the two numbers and then getting the sign of the number with the highest value*". Student 2 also got the correct answer and discussed, "*when you see a (+) plus sign followed by a (-) minus sign, turn the sign into a (-) minus, then subtract*". Student 3 got the answer incorrectly in the said items because she got confused with the sign but had the same explanation as student number 2. In item number 19, students 2 and 3 got the answer correctly and explained, "*Addition of like signs is just adding both the numbers and getting the sign of the number with the highest value*". However, student 1 got the item incorrectly because she was confused with the signs and had the same explanation as students 2 and 3. In item 20, students 2 and 3 showed the best understanding. In contrast, student 1 had partial understanding since she got the item correctly but explained, "*adding a negative number to a zero is like subtracting the number to zero by changing (+) plus sign to minus (-) sign*". In the subtraction of integers, students had different conceptions. In item 21, students 2 and 3 partially understood the answer correctly and explained, "*when a (-) minus sign was followed by a negative sign, turned the signs into a plus sign, then add the two numbers*". While student 2 got the item correctly and explained, "*In subtracting unlike signs, add the two numbers and get the sign of the number with the highest value*". In item 22, student 3 got the answer correctly and explained, "In subtracting positive to positive is just the usual subtraction". However, students 1 and 3 got the answer incorrectly because they got confused with the sign and had the same explanation as student 2. In item 23, students 2 and 3 got the answer correctly and explained, "*Subtraction of unlike sign is adding the two numbers and getting the sign of a number with the highest value*". While student 1 got the item correctly and explained, "*Adding a number to a zero is just getting the number added to the zero*". In multiplication and division of integers, students 1, 2, and 3 almost had the best understanding of each item. However, students 2 and 3 got confused with dividing and multiplying a 0 from a number. Student 2 explained that 0 has no value; hence her solution was "*0(-9) = -9" and "0† (-2) = 2*". Also, Student 3 discussed, "*0 has no value, and no number can be divided or multiplied to 0*".

Part six of the achievement test involved the application of the properties of integers. Students 1 and 3 partially understood item 33 since they correctly answered it and explained, "*The distributive property is the distribution of the number outside the bracket before adding the two numbers*". However, student 2 got the item incorrectly because she was confused by solving it instead of rewriting it using the distributive property. She explained, "*Distributive property usually use because the two terms inside the parentheses cannot be added because they are not like terms*". In items 34 and 35, student 1 showed partial understanding since she answered the items correctly and discussed briefly, "*The process of commutative is just swapping the two numbers, and they still have the same value, and the associative property was based on the concept of grouping or regrouping*". While student 2 got the item incorrectly by solving it instead of rewriting it in commutative form, but she discussed it briefly. However, Student 3 had no understanding of items 34 and 35.

Relationship between the Respondent's Socio-Demographic Characteristics and Metacognitive Awareness Respondents

Table 4 presents the relationship between the socio-demographic characteristics and the metacognitive awareness of the respondents. Results show that the respondents' first-quarter math grade was significantly related to the respondents' metacognitive awareness ($r = .267, p < .01$). The father's educational attainment was also related to metacognitive awareness ($r = .125, p < .05$), indicating that the students whose fathers have the highest educational attainment have more heightened metacognitive awareness. The respondents' monthly family income was also positively related to metacognitive awareness ($r = 0.131, p < .05$), implying that students with high monthly family income have more heightened metacognitive awareness. However, other characteristics such as age, sex, and the type of elementary school attended were unrelated to the student's metacognitive awareness.

Socio-Demographic Characteristics	Metacognitive Awareness	p-value
Age	-.099	.212
Sex	-.076	.137
Type of Elementary School Attended	.021	.276
Monthly Family Income	.131	.034*
First Quarter Math Grade	.267	.000**

Table 4. Relationship between Socio-demographic Characteristics and Metacognitive Awareness

Relationship between Socio-demographic Characteristics and Conceptual Understanding

Table 5 presents the relationship between socio-demographic characteristics and the conceptual understanding of the integers of the respondents. The respondents' type of elementary school attended ($r = .195$) was related to the respondents' conceptual understanding of integers at 0.01 levels of significance. Monthly family income was also significantly associated with the conceptual understanding of the integer ($r = .174, p < .01$). Moreover, the first quarter math grade of the respondents was positively related to the student's conceptual understanding of integers ($r = .711, p < .01$). This implied that students who performed well in their first quarter math subject had a high conceptual understanding of integers. Meanwhile, the respondents' age was negatively related to their conceptual understanding of integers ($r = -.127, p < .05$).

Socio-Demographic Characteristics	Conceptual Understanding	p-value
Age	-.127	.027**
Sex	-.010	.864
Type of Elementary School Attended	.195	.001**
Monthly Family Income	.174	.002**
First Quarter Math Grade	.711	.000**

Table 5. Relationship between socio-demographic characteristics and conceptual understanding of integers

Relationship between Students' Metacognitive Awareness and Conceptual Understanding of Integers

Primarily, the study determined the relationship between metacognitive awareness and conceptual understanding of integers in junior high school students (Table 6). The respondents' metacognitive awareness was significantly related to respondent's conceptual understanding of integers ($r = .407$, $p < .001$). Specifically, the metacognitive knowledge ($r = .410$) and metacognitive regulation ($r = .407$) were significantly related to respondents' conceptual understanding of integers.

Metacognitive Awareness	Conceptual Understanding	p-value
Metacognitive Knowledge	.410	.000
Declarative Knowledge	.343	.000
Procedural Knowledge	.394	.000
Conditional Knowledge	.400	.000
Metacognitive Regulation	.407	.000
Planning	.352	.000
Information Management Strategy	.375	.000
Comprehension monitoring	.287	.000
Debugging strategy	.411	.000
Evaluation	.322	.000
Total Metacognitive Awareness	.407	.000

Table 6. Relationship between metacognitive awareness and conceptual understanding of integers

DISCUSSION

This study investigated the students' metacognitive awareness and conceptual understanding of integers. It was revealed from the study that students had average metacognitive awareness. Specifically, they had an average level of metacognitive knowledge in terms of declarative knowledge, procedural knowledge, and conditional knowledge. This indicates that the respondents demonstrate average awareness of their tasks, thinking abilities, and ability to self-manage their associated cognitive responses. They also demonstrate occasional metacognitive knowledge when doing their schoolwork or homework, which shows that the students could reflect while learning their lessons which can be related to Jaleel and Premachandran (2016). The students in this study have shown critical awareness of their thinking and learning, showing their metacognitive knowledge as thinker-learner (Chick, 2013). Meanwhile, the student's metacognitive regulation was average. Their planning ability was above average, and their information management strategies, comprehension monitoring, debugging strategy, and evaluation were average. The respondents occasionally monitored and assessed their knowledge. The students are aware but not fully informed of the metacognitive strategies they can use in studying. In addition, they were characterized as possessing the capacity to prepare, track, and evaluate their comprehension and performance, as well as a critical knowledge of "one's thought and learning" and "oneself as a thinker and learner." This result is similar to the study of Yakubu et al. (2022), where students recognized these metacognitive regulations in solving problems. Moreover, the average metacognitive awareness of the students is attributed to their inadequate knowledge and regulation of cognition. They have limited skills to think beyond thinking and self-regulatory processes, thus, challenging them to develop understanding. Given that metacognition can be taught, math educators must focus on assisting their students in achieving a higher level of metacognitive awareness.

Relative to students' conceptual understanding, it was demonstrated based on their performance in the fundamental operation of integers. Based on the results, it was revealed that they have a good performance on the fundamental operation of integers. Specifically, students had very good remarks about defining and comparing the integers. This may be because the students acquire basic knowledge, such as defining and comparing concepts, before applying certain information or ideas. This result was followed by the ability to distinguish concepts in number lines and integer's real-life application, which got a "good" remark. This implies that one popular tool for teaching about numbers is the number line, and it may fit for early teaching of operations involving negative numbers. Also, students strongly prefer mathematics problems associated with mystery when they can relate them to their everyday lives (Premadasa & Bhatia, 2013). In terms of operating integers, students' performance in adding, multiplying, and dividing integers was good. However, the ability of the students to subtract integers got a remark of "fair".The study concluded that the most common error in operating integers fell under subtraction. This agrees with Vlassis (2002), who found that the common mistakes made when solving equations were caused by negative numbers or unlike signs and implied that negative numbers created a degree of abstraction. On the other hand, students had poor performance in applying integer properties and rules on the operation of integers. Errors in applying properties of integers occurred when the students understood what the question asked, and still, they could not identify the operation or sequence of processes needed to solve the problem (Ryan, 2007). This shows that the students struggle even with the basic knowledge they must have attained during the formative years of schooling. Their difficulty grasping concepts will eventually determine their future struggles with more complex math problems.

For further analysis, the researchers conducted a follow-up assessment on six selected respondents to discuss the respondents' conception of integers. The results showed that the students in the high achiever group had the best understanding of integer concepts, the average achiever group showed a correct or incomplete understanding of integer concepts, and the students in the low achiever group had functional misconceptions. This indicates that high-achiever students have established their conceptual understanding. Meanwhile, low to average achievers may have difficulties with conceptual knowledge due to their poor elementary mathematics background. Per Santos et al. (2022), this may be attributed to their poor number sense competency. The generated responses from the interview provide further evidence of the student's strengths and struggle with the fundamental operation of integers. Students signified their understanding based on their verbatim responses on defining integer and number line concepts, comparing numbers, and real life-applications of integers. They also showed difficulty subtracting integers as negative numbers confused them when combined with positive numbers. Moreover, the respondents were quite familiar with the properties of the integers. The students were not given proper examples and real-life applications of closure, commutativity, associativity, distributive and identity property of integers. It may be because properties on integers were not correctly introduced in high school mathematics and were just emphasized to students taking higher mathematics. Since these students have misconceptions or incomplete understanding, remediations, and differentiated instruction must be conducted. Teachers must understand the students' conceptual knowledge challenges while employing appropriate interventions to help them with their difficulties.

For further analysis, the researcher explores students' socio-demographic profiles. The study investigated the relationship between students' socio-demographic profiles and metacognitive awareness. Results show that the respondents' first-quarter math grade was significantly related to the respondents' metacognitive awareness. This indicated that students with high first-quarter math grades are related to their metacognitive awareness. This supported the findings of Young and Fry (2008), who concluded that correlations were found between metacognitive awareness and course grades. Baltaci et al. (2016) also revealed a statistically significant relationship between metacognitive awareness levels and grades in mathematics. The father's educational attainment was also related to metacognitive awareness, indicating that the students whose fathers have the highest educational attainment have more heightened metacognitive awareness. The respondents' monthly family income was also positively related to metacognitive awareness, implying that students with high monthly family income have more heightened metacognitive awareness. This finding contradicted the study of Narang and Saini (2013), where the impact of socioeconomic status on metacognition was non-significant, which indicated that metacognition had other impacting factors apart from socioeconomic status, which separated the children crossways diverse levels of metacognition. The students' first-quarter math grades and monthly family income must be considered when dealing with metacognitive knowledge and regulation. Teachers with high hopes of empowering metacognitive awareness among their students must see to it that these personal characteristics must be considered. Nevertheless, other characteristics such as age, sex, and the type of elementary school attended are unrelated to the student's metacognitive awareness. In terms of the relationship between socio-demographic characteristics and the conceptual understanding of the integers of the respondents, the respondents' type of elementary school attended was related to the respondents' conceptual understanding of integers. The result exposed that students from private institutes showed a more conceptual understanding of integers than those from public schools. This result coincided with Lubienski and Lubienski's (2005) findings, who found that Mathematics appeared to be a subject where public-school students outperformed their private school peers. Monthly family income was also significantly related to the conceptual understanding of the integer. This denoted that students with a high monthly family income had a high conceptual understanding. This finding has the same result as Kirkup (2008), who revealed that students with a higher socioeconomic status outperform those with a lower socioeconomic status. Moreover, the first quarter math grade of the respondents was positively related to the student's conceptual understanding of integers. This implied that students who performed well in their first quarter math subject had a high conceptual understanding of integers. This was related to the findings of Zakaria et al. (2010). They revealed that mathematics grade was related to conceptual understanding since Mathematics requires understanding certain principles and processes and the practice of carrying out practical activities and operations. Meanwhile, the respondents' age was negatively related to their conceptual understanding of integers. This indicated that younger students tend to have a higher conceptual understanding of integers than older students. This was contradicted by Shute et al. (2011), who concluded that older children fared better academically than younger ones. This result might explain why the older students are students who come from cases of dropouts, a temporary stop in school, and irregular academic standing. Similar to the results above, the teacher must learn from the background of their students while dealing with their conceptual knowledge. Investigating the profile of the learners while aiming to develop their conceptual understanding would be necessary. Still, sex characteristic is unrelated to the student's conceptual understanding, indicating that sex might not be a factor in obtaining a higher conceptual understanding.

Finally, the study determined the relationship between metacognitive awareness and conceptual understanding of integers in Junior High school students. The respondents' metacognitive awareness was significantly related to respondent's conceptual understanding of integers. The result indicated that the students with high metacognitive awareness have a high conceptual understanding of integers. This result was similar to Young and Fry (2008), who stated that students with well-developed metacognition would excel academically. Tickoo (2012) also concluded that metacognition has been heavily involved in the desire to create a conceptual change, while Gunstone and Mitchell (2005) revealed that the connections between conceptual change and metacognition seem to be an apparent result of the conceptual change definition. Recognizing prior conceptions and deciding whether to reconstruct and perform a self-evaluation requires metacognition comprehension and control. Specifically, the metacognitive knowledge and metacognitive regulation were significantly related to respondents' conceptual understanding of integers. The students' conceptual understanding can be enhanced by developing a firm metacognitive knowledge and regulation among them at a young age. When their metacognitive awareness is established in their primary years, they tend to display complete conceptual understanding. From lower to higher-order thinking, math teachers need to reframe their practices to help the students learn best about integers and other foundational topics in mathematics. Morover, with these associations, teachers should emphasize the use of metacognitive techniques in teaching mathematics. Teachers must help students become more aware of themselves by providing metacognitive exercises that prompt reflection on what they know and care about and can also provide valuable information for teachers. Such introspective exercises are additions that impede continuing analysis, revision, and planning, as well as strategic thinking. Teachers can have a long-lasting effect on how their students learn long after they leave the classroom by making learning and problem-solving processes apparent and assisting students in identifying their strengths and strategies.

CONCLUSIONS AND RECOMMENDATIONS

The study investigated the relationship between metacognitive awareness and students' conceptual understanding of integers. From the analysis of the data gathered, the following major findings were drawn: (1) students had an average level of metacognitive awareness in terms of metacognitive knowledge and metacognitive regulation, (2) students' had good performance in fundamental operation on integers, which indicates their good conceptual understanding of integers concepts, and (3) students' metacognitive awareness were significantly related with their conceptual understanding on integers. These results conclude that students in primary school have not fully developed their metacognitive awareness while learning mathematics. Consequently, it can be said that learning practices among students are still in the early stages and still requires improvements. Teachers could not help students in their primary years to use thinking strategies while learning. For instance, it is strongly recommended that methodologies, exercises, models, or learning modules integrated with thinking techniques based on metacognitive knowledge and regulation strategies be created to suit the needs of the 21st-century teaching approach. This will boost 21st-century learning and the student's ability to think while learning. Technology integration can also improve learning efficiency, increase student competence, and alter the learning environment.

Regarding their conceptual understanding, the result also revealed gaps in students' conceptions and skills in mathematics. Despite several studies on improving students' skills in mathematics, students still have misconceptions and errors in basic fundamental operations on integers. This is alarming as operating on integers is a prerequisite for higher mathematics. This may be because mathematics was still taught procedurally without helping students understand how these topics relate to one another and without necessary justifications for why particular concepts are implications for others. It is particularly challenging when rules and processes are presented to students extremely abstractly without using models to help students understand the concepts. Moreover, the significant relationships between metacognitive awareness and conceptual understanding of integers are exhibited as one factor teachers can consider in teaching mathematics. This demonstrated how metacognition strongly impacted the ability to make a conceptual change. In addition, deep understanding and flexible knowledge of integers may require appropriate metacognitive knowledge, awareness, and control. This only denoted that metacognition plays a significant role in successful learning. With this finding, students are suggested to learn how to be skilled thinkers and use their knowledge in innovative contexts to aid in learning and develop logical reasoning, decision-making, and, thus, conceptual understanding. Considering the relationship between metacognitive awareness and students' conceptual understanding, teachers and educators might adapt strategies and pedagogical metacognitive approaches to improve students' skills in mathematics learning. Additionally, further studies on exploring metacognition in different mathematics skills and emotional aspects of learners might be conducted for further conclusion. Future research may use a larger sample to investigate metacognition, and conceptual understanding since the generalizability of the present study's findings is constrained due to the short sample size. Other researchers might be considered drawing more in-depth analyses based on qualitative results from other mathematics disciplines. They could also be thoroughly investigated using an experimental design that includes both the control and experimental group to generate substantial results of the factors that might help improve mathematics teaching and learning practices.

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Appendix

Metacognitive Awareness Inventory

Directions: Check which is appropriate.

STATEMENTS	Always	Often	Sometimes	Seldom	Never
1. I ask myself periodically if I am meeting my goals.					
2. I consider several alternatives to a problem before I answer.					
3. I try to use strategies that have worked in the past.					
4. I pace myself while learning in order to have enough time.					
5. I understand my intellectual strengths and weaknesses.					
6. I think about what I really need to learn before I begin a task					
7. I know how well I did once I finish a test.					
8. I set specific goals before I begin a task.					
9. I slow down when I encounter important information.					
10. I know what kind of information is most important to learn.					
11. I ask myself if I have considered all options when solving a problem.					
12. I am good at organizing information.					
13. I consciously focus my attention on important information.					
14. I have a specific purpose for each strategy I use.					
15. I learn best when I know something about the topic.					
16. I know what the teacher expects me to learn.					
17. I am good at remembering information.					
18. I use different learning strategies depending on the situation.					
19. I ask myself if there was an easier way to do things after I finish a task.					
20. I have control over how well I learn.					
21. I periodically review to help me understand important relationships.					
22. I ask myself questions about the material before I begin.					
23. I think of several ways to solve a problem and choose the best one.					
24. I summarize what I've learned after I finish.					
25. I ask others for help when I don't understand something.					
26. I can motivate myself to learn when I need to					
27. I am aware of what strategies I use when I study.					
28. I find myself analyzing the usefulness of strategies while I study.					
29. I use my intellectual strengths to compensate for my weaknesses.					
30. I focus on the meaning and significance of new information.					
31. I create my own examples to make information more meaningful.					
32. I am a good judge of how well I understand something.					
33. I find myself using helpful learning strategies automatically.					
34. I find myself pausing regularly to check my comprehension.					
35. I know when each strategy I use will be most effective.					
36. I ask myself how well I accomplish my goals once I'm finished.					
37. I draw pictures or diagrams to help me understand while learning.					
38. I ask myself if I have considered all options after I solve a problem.					
39. I try to translate new information into my own words.					
40. I change strategies when I fail to understand.					
41. I use the organizational structure of the text to help me learn.					
42. I read instructions carefully before I begin a task.					
43. I ask myself if what I'm reading is related to what I already know.					
44. I reevaluate my assumptions when I get confused.					
45. I organize my time to best accomplish my goals.					
46. I learn more when I am interested in the topic.					
47. I try to break studying down into smaller steps.					
48. I focus on overall meaning rather than specifics.					
49. I ask myself questions about how well I am doing while I am learning something new.					
50. I ask myself if I learned as much as I could have once I finish a task.					
51. I stop and go back over new information that is not clear.					
52. I stop and reread when I get confused.					

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The Emergence and Form of Metacognitive Regulation: Case Study of More and Less Successful Outcome Groups in Solving Geometry Problems Collaboratively

Anis Farida Jamil[1,2], Tatag Yuli Eko Siswono[1], Rini Setianingsih[1]

[1]Universitas Negeri Surabaya, Indonesia,

[2] Universitas Muhammadiyah Malang, Indonesia

anisfarida@umm.ac.id, tatagsiswono@unesa.ac.id, rinisetianingsih@unesa.ac.id*

Abstract: The current research trend of metacognitive regulation has shifted from an individual to a social context. One such social context is collaborative problem-solving. Interaction between group members is an essential factor in completing collaborative problems. Metacognitive regulation is divided into four forms based on the interaction in collaborative problem-solving. This study aims to analyze the emergence and form of metacognitive regulation in groups that are more and less successful at solving geometric problems collaboratively. Each group consists of two undergraduate students who have taken geometry courses. The group tries to solve geometry problems in the form of proof problems. This study examines how metacognitive regulation emerges in group discussion activities. The emergence of metacognitive regulations was identified through student utterances when discussing in groups. In addition, interview data support researchers in exploring metacognitive regulation carried out by groups. This study also identifies forms of metacognitive regulation that occur when groups interact. The study results showed differences in the metacognitive regulation activity in four aspects: orientation, planning, monitoring, and evaluation. The more successful group has a form of co-constructed social metacognitive regulation. In contrast, the less successful group has a form of ignored social metacognitive regulation.

Keywords: Metacognitive Regulation, Collaborative, Problem-Solving

INTRODUCTION

Current curriculum and teaching reforms have focused more on the teaching and assessment of 21st-century skills. Four skills are demanded in the 21st century, namely 4C critical thinking and problem-solving, collaboration, communication, and creativity. Problem-solving is essential to learning (Purnomo et al., 2022). The National Association of College and Employees (NACE) is an American non-profit professional association for college career services, recruiting practitioners, and hiring college graduates. NACE states that there are five primary skills in workforce recruitment: teamwork skills, leadership, communication skills, problem-solving skills, and a strong work ethic. Therefore, collaboration and problem-solving are central skills in the 21st century.

Metacognition is a predictor of problem-solving. Research shows that someone with good metacognitive abilities is associated with good problem-solving (Roick & Ringeisen, 2018; Zan, 2000). Metacognition was introduced by (Flavell, 1979), who mentions "thinking about thinking." Metacognition is divided into metacognitive knowledge and regulation (An & Cao, 2014; Brown, 1987; Schraw & Moshman, 1995). Metacognitive knowledge is individual awareness of their cognitive processes. In comparison, metacognitive regulation describes individual monitoring and control of their cognitive processes.

Metacognitive regulation is considered a more critical aspect than metacognitive knowledge. Awareness of cognitive processes needs to be improved to explain cognitive processing results. It requires examining strategies for monitoring and controlling cognitive processes, which are metacognitive regulations. (Stephanou & Mpiontini, 2017). Thus, this study focuses on the study of metacognitive regulation. In its development, metacognitive regulation has been studied in individuals and groups. Research shows that metacognitive regulation can emerge in group activities, not just individuals (Jin & Kim, 2018; Kim et al., 2013; Magiera & Zawojewski, 2011).

Interaction is essential in student group activities, especially in solving collaborative problems. With successful interaction, it is possible to build a space of shared understanding to solve common problems (Roschelle & Teasley, 1995). However, some interactions were influential in the group activities, and some could have been more effective. Research shows that there are group members who ignore each other's contributions and focus only on their thoughts. Based on the differences in interactions in the collaborative group, Molenaar et al. (2014) create a framework that divides metacognitive activity into four forms that align with the framework of Iiskala et al. (2021). By adopting these two frameworks, this study divides the form of metacognitive regulation into four: ignored social metacognitive regulation, accepted social metacognitive regulation, co-constructed metacognitive regulation, and shared social metacognitive regulation. During the problem-solving process, of course, not all groups can solve it successfully. Some groups are both more successful and less successful.

It would be interesting to study the form of metacognitive regulation in these two groups with different criteria. The research findings are expected to provide information such as the emergence of metacognitive regulation in collaborative problem-solving. Besides, the finding is forms of metacognitive regulation in groups that are more and less successful in solving problems. These findings can be followed up by looking at the differences in metacognitive regulation in collaborative problem-solving in groups with more and less successful results.

Previous research analyzed the form of metacognitive regulation in each aspect based on the coding scheme of Molenaar et al. (2014). However, this research focuses on using collaborative scripts to improve students' metacognitive regulation in task-oriented reading (Kielstra et al., 2022). In this study, we want to see the emergence and dominance of forms of metacognitive regulation in groups that are more successful or less successful in solving geometric problems. Forms of metacognitive regulation are obtained from utterances or conversations when students discuss solving problems. The utterances produced can be classified as verbalized metacognitive regulation and are supported by interview data.

THEORETICAL FRAMEWORK

Metacognitive Regulation in Social Context

In the traditional view, metacognition was studied by individuals. How to see a person's awareness of his cognition is called metacognitive knowledge. How to see someone monitoring and controlling their cognitive process is called metacognitive regulation (Flavell, 1979; Scheiner & Pinto, 2016; Sternberg & Sternberg, 2012). However, metacognition has been studied in a social context in its development. It is because individual learning outcomes cannot be separated from the role of others. Social-based contexts are situations where students interpret multiple perspectives, engage in explanations, and seek mathematical consensus (Magiera & Zawojewski, 2011). It also applies to metacognitive regulation, which is part of metacognition. Jin & Kim (2018) found that elementary school students were metacognitively engaged in collaborative problem-solving activities. Besides that, research by Jin & Kim (2018) challenges the traditional view of metacognitive regulation studied. Metacognitive regulation can emerge at both individual and group levels.

Metacognitive regulation is indicated by orientation, planning, monitoring, and evaluation (Brown, 1987; Veenman et al., 2006). Orientation refers to self-orientation by analyzing the task, realizing the perception of the task by way of task content orientation, generating hypotheses about the task content, and activating prior knowledge. Planning involves selecting and sequencing strategies, allocating resources, and formulating action plans (Jacobs & Paris, 1987; Nelson, 1990; Schraw, 1998). Monitor self-progress by checking the adequacy of solving problems/task solutions and understanding by identifying inconsistencies and modifying problem-solving, including monitoring (Brown, 1987; Veenman et al., 2006).

At the same time, evaluation refers to assessing learning outcomes and learning processes (Brown, 1987; Jacobs & Paris, 1987; Schraw, 1998; Veenman et al., 2006). The indicators of metacognitive regulation above are indicators in a particular context. The meaning of collaborative work must be clearly defined to determine indicators of metacognitive regulation in collaborative problem-solving. Damon & Phelps (1989) distinguish group interaction activities into peer tutoring, cooperative, and collaborative. Peer tutoring occurs in interactions where people with different skills are brought together so that one can instruct the other. In cooperatives, some arrangements allow groups to share tasks and master their separate parts. In collaboration, interaction occurs when students with the same competency level share their ideas to solve challenging problems together (Damon & Phelps, 1989). Collaboration is a mutual process of exploring each other's reasoning and viewpoints to build a shared understanding of the task, generating methods and interpretations of mutually acceptable solutions. Therefore, mutual interaction requires one to propose and defend their ideas and ask their colleagues to clarify and justify ideas they do not understand (Goos et al., 2002). Collaboration is characterized by "togetherness" between group members from the beginning to the end of the problem-solving process. This togetherness occurs in terms of sharing ideas, clarifying each other, and gaining a common understanding in solving problems.

By paying attention to collaborative work, this study offers indicators of metacognitive regulation in solving collaborative problems, presented in Table 1.

Metacognitive Regulation Indicator in Collaborative Problem-Solving	**code**
Orientation	
1. Self-orientation by analyzing tasks aims to prepare the problem-solving process in groups.	Orientation-1
2. Recognizing shared perceptions of the problem to be solved by generating hypotheses about task content and activating previous knowledge	Orientation-2
Planning	
1. Selecting the right strategy from the results of collaborative thinking before and during the problem-solving process	Planning-1
2. Optimizing self and or group resources in solving problems	Planning-2
3. Formulating action plans resulting from collaborative activities	Planning-3
Monitoring	
1. Be aware of self or each other's understanding and cognitive performance	Monitoring-1
2. Monitoring self- or collaborative thinking and actions (participation, interaction, and group cohesion)	Monitoring-2
3. Identifying self or other's cognitive conflicts and inconsistencies and modifying problem-solving if necessary	Monitoring-3
Evaluation	
1. Assessing the quality of self-performance or collaborative performance in problem-solving	Evaluation-1
2. Assessing self or group learning outcomes	Evaluation-2

Table 1: Metacognitive Regulation Indicator in Collaborative Problem-Solving

Form of Metacognitive Regulation in Collaborative Problem-Solving

Interaction is necessary for the process of solving collaborative problems. With successful interaction, it is almost possible to build a space of shared understanding to solve common problems (Roschelle & Teasley, 1995). In collaborative groups, interaction can occur differently (Volet et al., 2009). Interaction in collaborative learning is divided into two: shared interaction and co-constructed interaction (Van Boxtel, 2004). Shared interaction occurs when group members share existing knowledge and mutually acknowledge each other's contributions (mostly without disputes/demands for justification).

Meanwhile, co-constructed interaction occurs when students build their activities to explain and question each other's thoughts and provide feedback. The characteristics of co-constructed are students formulating actions and knowledge that individual group members cannot produce alone. However, not all collaborative activities occur effectively. Studies show that students ignore each other's contributions and concentrate on their thinking (Molenaar et al., 2014).

Following the differences in these interactions, Molenaar et al. (2014) divide four types of interactions into metacognitive activities: ignored social metacognitive activities, accepted social metacognitive activities, co-constructed metacognitive activities, and shared social metacognitive activities. In line with Molenaar et al. (2014), the framework proposed by Iiskala et al. (2021) divides metacognitive regulation into four forms: verbalized metacognitive self-regulation, ignored metacognitive regulation, metacognitive other regulation, and socially shared metacognitive regulation. The forms of metacognitive regulation in this study are defined as four types of interactions in metacognitive regulatory activities in collaborative problem-solving. Metacognitive regulations are manifested verbally. In this study, four forms of metacognitive regulation in collaborative problem-solving are:

1. Ignored Social Metacognitive Regulation
   It occurs when a group member tries to control or monitor group learning activities, but others ignore these efforts. Example: A student evaluating the answers produced by the group commented that the answer was wrong. Other group members did not respond to his comments.
2. Accepted Social Metacognitive Regulation
   It occurs when other group members agree with one group member's metacognitive comments by implementing them in their cognitive activities. Example: A student evaluating the answers produced by the group commented that the answer was wrong. The other group members started rechecking their answers. It shows that evaluation activities are considered and followed up in re-examination. So, group members engage with these metacognitive comments through cognitive contributions.
3. Co-constructed Social Metacognitive Regulation
   Group members build on each other's ideas by collaboratively constructing metacognitive activities to organize collaborative learning. Group members exchange metacognitive

comments that generate new ideas. This new idea emerges when students propose metacognitive comments to one another. Example: a student gives an idea in the form of several strategies that can be used to solve a problem. Another group member commented that he believed one of them was the best. The first student agreed and explained why this strategy was also the best according to him.

4. Shared Social Metacognitive Regulations
   Group members share their metacognitive ideas and respond to each other's contributions, but they do not build each other's ideas toward new ideas. Example: a student evaluating the answers produced by the group comments that the answer is wrong. Another group member commented that he believed his different answers might be wrong too.

METHOD

This research uses a qualitative approach with a multiple-case study. The two cases used in this study have different criteria that the two groups represent. The criteria of one group were more successful in solving geometry problems in collaboration, and the other could have been more successful. The group that has a more successful outcome is the group that can use a logical proof strategy even though the writing has yet to use the proper proof steps. In contrast, the less successful group was the group that failed to prove the questions given. Each group consists of two people. Working in pairs can increase the possibility of group members negotiating, interacting, reaching agreements, and evaluating their assignments (Córdoba Zúñiga et al., 2021). Thus, collaborative work is expected to occur well.

The participants of this study were undergraduate students at the University of Muhammadiyah Malang, Indonesia, who had taken geometry courses. The number of participants is 26 students divided into 13 groups. Three groups are included in the more successful outcome groups, and ten groups are included in the less successful outcome groups. The researcher selected one of the three and ten groups, respectively. The two chosen groups are more interactive than the others enabling researchers to obtain more complete data on metacognitive regulation. Thus, in the result section, two groups are the focus of the study. Henceforth, we will refer to the members of the first group as S1 and S2. While the members of the second group as S3 and S4. Table 2 shows the characteristics of S1, S2, S3, and S4—student characteristics in the form of gender and learning outcomes. Learning outcomes are obtained from the midterm exam score and divided into three criteria: high, medium, and low.

Subjects	Gender	Learning Outcomes
S1	Female	High
S2	Female	Medium
S3	Female	Medium
S4	Female	Low

Table 2: The Characteristic of the Subjects

The instruments used were group assignment sheets and interview guidelines. *Group assignments* are problem-solving tasks that trigger students' metacognitive regulations in conversations or speech. Researchers analyzed the emergence and form of metacognitive regulation based on group student discussions. The interview data were used to examine the students' metacognitive regulation. The interviews were conducted in groups. There are four aspects.

The group was given a geometric problem task with the type of problem to prove. Figure 1 shows the task assigned to the groups. Groups complete tasks collaboratively. While the group discussed completing its task, the recording was done using a video-audio recorder. After completion, the researcher interviewed the group using a task-based interview guide developed based on indicators of metacognitive regulation in collaborative problem-solving. Student conversation and interview transcripts were analyzed to identify the emergence and forms of metacognitive regulation. Four aspects asked during the interview correspond to the four aspects of metacognitive regulation. In the first aspect, orientation, we ask about the nature of the task and what prior knowledge students must have to complete the task. In the second aspect, planning, we ask how many strategies have been discussed to solve the problem. In the third aspect, monitoring, we asked about students' awareness of their understanding and how they understood their colleagues' understanding. In addition, we asked about students' need for more understanding in completing assignments. In the last aspect, evaluation, we asked how they assessed their learning outcomes and their assessments of group solutions and collaboration.

Do the following problems in pairs with your friends!

Quadrilateral $RSTV$ has the vertices $R(a, b)$, $S(c, b)$, $T(c-d, e)$.

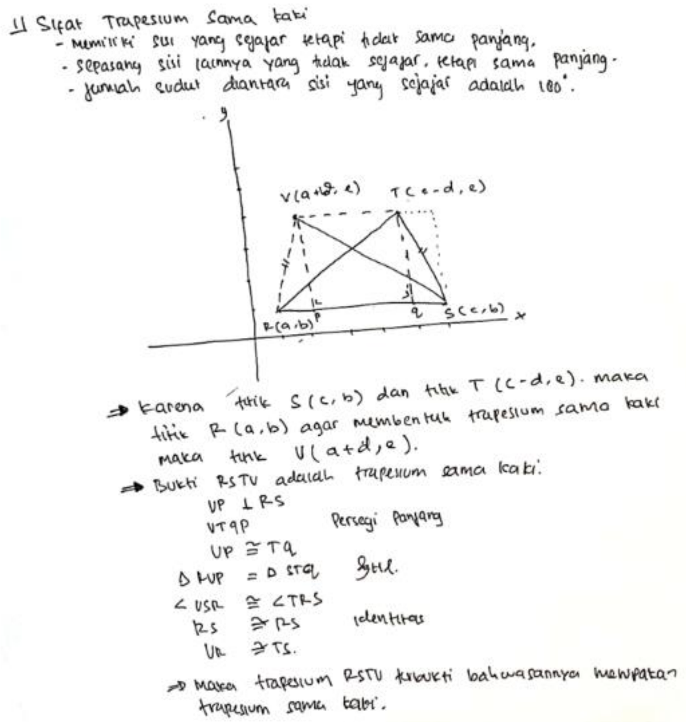
Figure 1: Tasks given to student groups

There are three stages in analyzing data, namely, (1) data condensation, (2) data display, and (3) inclusion drawing/ verification (Miles et al., 2014). The data from group conversations and interview transcripts are coded in the data condensation activity. The coding scheme for metacognitive regulation refers to Table 1 (Iiskala et al., 2021; Molenaar et al., 2014). Then the data is displayed in conversations or interview excerpts that contain examples of the occurrence and form of metacognitive regulation. In addition, the results of data analysis are displayed in a matrix that compares the occurrence of metacognitive regulation in the two groups of subjects. The last stage is drawing conclusions and verification.

RESULT

Metacognitive regulation of more successful outcome groups

Figure 2 is the result of group work that is more successful in solving the given problem. The results of the group's work are still in Indonesian, but we have added a translation in English.



Isosceles Trapezoid Properties:
- Having parallel sides but not the same length
- The other pair of sides are the same length but not parallel
- The sum of angles measured between the parallel sides is 180°

- Since $S(c, b)$ and $T(c - d, e)$ thus $R(a, b)$. To form an isosceles trapezoid then $V(a + d, e)$
- Proof of $RSTV$ is isosceles trapezoid

  $VP \perp RS$

  $VTQP$ is rectangle

  $VP \cong TQ$

  $\Delta RUP \cong \Delta STQ$

  $\angle VSR \cong \angle TRS$

  $RS \cong RS$ (identity)

  $VR \cong TS$
- Thus, proved that $RSTV$ is the isosceles trapezoid

Figure 2: Group work that is more successful at solving problems

In Figure 2, it can be seen that students start their answers by writing down the properties of an isosceles trapezoid. Based on the results of interviews with the student group, they agreed that the first step was to know the properties of an isosceles trapezoid. They drew the known points from the problem, determined the coordinates of the point in question, and drew an isosceles trapezoid. The following is an excerpt from the group interview transcript. Assume that the first group consists of S1 and S2. At the same time, the researcher is written as R. The text in bold, and italics is a code for indicators of metacognitive regulation in collaborative problem-solving.

R: "How did the group agree on a strategy to solve the problem?"

S1: "We first thought that we must first know the properties of an isosceles trapezoid. After that, draw an isosceles trapezoid at Cartesian coordinates" ***(Orientation-2)***

S2:" Then we determine the point. The point is obtained by making the line the same length as the line. Because then" ***(Orientation-2)***

In the interview excerpts, students recognized a shared perception of the problem to be solved by generating hypotheses about task content and activating previous knowledge. It shows one indicator of metacognitive regulation on the orientation aspect. The utterances that appeared when the first student had a discussion showed self-orientation by analyzing tasks. The following is an excerpt of students speech during the discussion.

(After they finish reading the questions)

S1: "do it represents coordinates?" (pointing to the known dot symbol in the problem) ***(Orientation-1)***

S2: "correct. It coordinates."
S1: "means we draw the coordinates first" ***(Orientation-1)***
S2: "Yes, we draw the point first at any initial position."

In the conversation it can be seen that students analyze assignments that aim to prepare for solving group problems. The analysis they did about the coordinates of the points they would later draw to create an isosceles trapezoid shape. From the results of collaborative work, students agreed to prove by showing that RSTV fulfils the properties of an isosceles trapezoid. They mentioned the three properties of an isosceles trapezoid, as shown in Figure 2, although, in the end, they proved it in another way. Another way is that the group proves that an isosceles trapezoid can be constructed from two congruent right triangles and a rectangle ***(planning-3)***. This group is a more successful outcome group, not because of their perfectly correct answers but because of the logic they put into completing the proof. The steps students take in choosing and determining more accessible, appropriate strategies for solving problems are included in the planning aspect of metacognitive regulation.

R: "Why, in the end, did the proof not use the trapezoidal property as written?"
S1: "Initially, we wanted to prove one of the properties, that is, two non-parallel sides are the same length, by using the Pythagorean theorem. However, we cannot continue. Finally, we use the second method. If we can determine that this figure consists of a rectangle and two congruent right triangles, then we can prove an isosceles trapezoid" ***(Planning-1)***
R: "Why is that?"
S2: "in our opinion, the proof will be easier" ***(Planning-1)***
Monitoring activities occur during the problem-solving process. It was also found when researchers conducted interviews with groups. The excerpt of the conversation transcript shows that S1 can identify the lack of understanding of S2. S2's incomprehension is a cognitive conflict that he faces. "Hmmm" was said by S2 and immediately identified by S1 that S2 did not understand, so S1 continued his explanation. Therefore, S1 can identify cognitive conflicts from their group mates, an aspect of monitoring. The following conversation transcript shows the emergence of the monitoring aspect.
S2: "How do we prove an isosceles trapezoid?"
S1: "First, we prove that the non-parallel sides are equal in length" ***(Monitoring-3)***
S2: "Hmmm?"
S1: "means we show the length of the side TS the same as VR. Let us try using the Pythagorean theorem" ***(Monitoring-3)***

The evaluation aspect was not seen when the group discussed it. However, the interview transcripts show that the group evaluated the group's performance and learning outcomes. The following interview transcripts show evaluation aspects of metacognitive regulation.
R: "What do you think about group work?"
S1: "I find it more helpful to work with a group because we can exchange opinions and understandings" ***(Evaluation-1)***
S2: "I agree because we can choose a better way by sharing ideas."

R: "Are you sure about the evidence you have produced?"
S2: "Not sure, we think our strategy is right, but we are not sure about the writing." ***(Evaluation-2)***

Metacognitive regulation of less successful outcome groups

Figure 3 is the result of the group's work which could have been more successful in solving the problem of proving geometric material. As shown in Figure 2, student work is still Indonesian, and we have translated it into English.

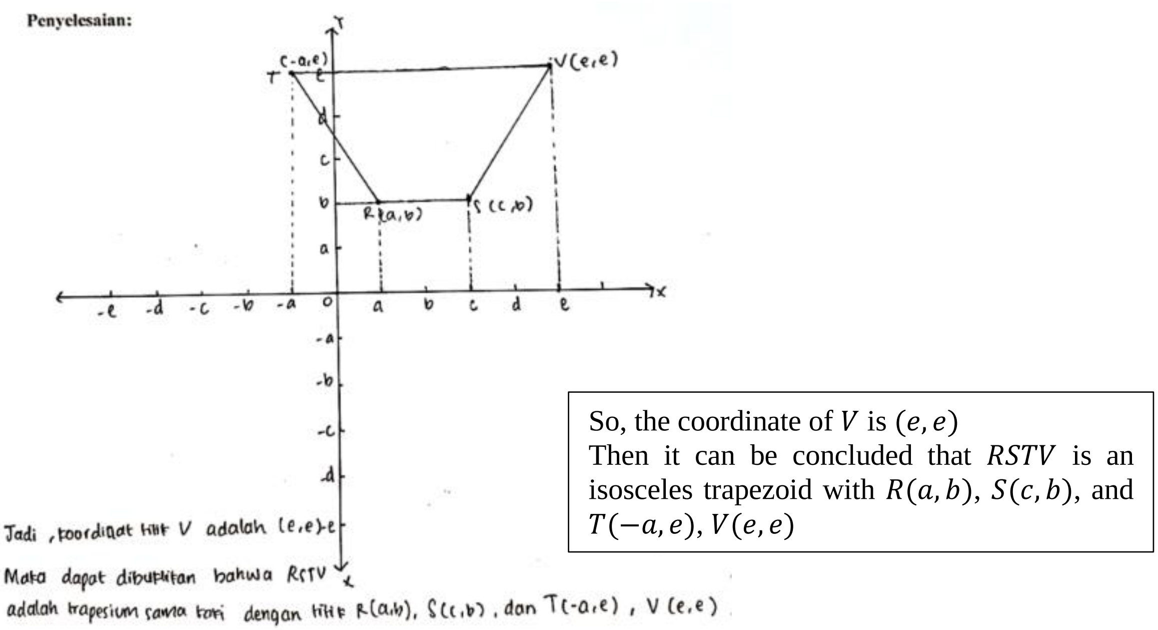


Figure 3: Group work which could have been more successful in solving problems

The group writes down the answers starting with drawing an isosceles trapezoid at Cartesian coordinates. The group needed to write down how to get the point coordinates. There is no basis for providing conclusions on the evidence carried out. The text provided only shows that after the group determined the point coordinates, they concluded that $RSTV$ was an isosceles trapezoid.

We identified that this group could have been more successful in proving an isosceles trapezoid. It is because mathematics conceptual knowledge was not used. Conceptual knowledge is essential to understand the basic concepts in solving mathematical problems (Ho, 2020). Polya (1945) divides problems into "problems to find" and "problems to prove," in which this study used problems to prove mathematics. Students need to be corrected in interpreting symbols a, b, c, d and e, which is known in the problem. They assume that the symbols a, b, c, d and e represent numbers 1,2,3,4 and 5, respectively, on the x-axis and y-axis. At the same time, it should be the symbol a, b, c, d and e and is any number on the x-axis and y-axis. This misrepresentation causes the group to need clarification in drawing points that have coordinates $(c-d,e)$. They change the coordinate of point T to $(-a,e)$, whereas $(c-d,e)$ and $(-a,e)$ are different coordinates.

In group discussions, students orient themselves by analyzing tasks to prepare for problem-solving. However, this orientation indicates that they are unsure of their coordinate knowledge. The following is an excerpt of the group's utterances during the discussion. Assume the group members are S3 and S4 while the researcher is R. The text in bold and italics is a coding scheme for indicators of metacognitive regulation in collaborative problem-solving.

(After they have finished reading the questions)
S3: "Shall we use the coordinates?" ***(Orientation-1)***
S4: "Yes, it is true."
S3: "At $T(c-d,e)$, what point do the coordinates have a minus sign?" ***(Orientation-1)***
"At $T(c-d,e)$, what point do the coordinates have a minus sign?" His sentence shows that S3 has no understanding of point coordinates. S4 also needed help understanding the meaning of the symbol, but in the end, they agreed that the symbol a,b,c,d and e represent the number $1,2,3,4,$ and 5 on the x-axis and y-axis, respectively. The group has a shared perception of the coordinates of the points ***(Orientation-2)***, but the hypothesis about the task content they make needs to be corrected.

There was no choice of strategy discussed by the group in proving an isosceles trapezoid. The group only planned the proof by drawing isosceles trapezoids without using concepts, principles, theorems, properties, corollary, and lemma. It can be seen from the excerpts of the conversation.

S3: "How to prove isosceles trapezoid? You can prove it through pictures alone" ***(Planning-3)***
S4: "I think so" ***(Planning-3)***

Based on the results of interviews with the group, we obtained that the group knew that the steps were incorrect, but they could not find another way to prove that the shape is an isosceles trapezoid.
R: "Do you think the answers that have been written are correct?"
S3: "I do not think it is right, but we have no other way to prove it" ***(Monitoring-1, Evaluation-2)***
S4: "not yet right because we were just guessing" ***(Monitoring-1, Evaluation-2)***
From the results of the interviews above, the group assessed their learning outcomes. They judge their work as not optimal. However, they do not try to improve their work.

Metacognitive regulation form of more successful outcome group in solving geometry problem collaboratively

Based on group conversations in discussing solving problems, two excerpts of conversations are used to identify the dominance of forms of metacognitive regulation.

Conversation Excerpt (1)

S1: "Let us start by drawing the coordinates, shall we?"
S2: "I think so."
S1: "Do point R drawn here?"
S2: "Point R can be drawn in any position."
S1: "True, but we must note that the symbol $a, c, c-d$ are on the x-axis and b, e are on the y-axis."

Conversation Excerpt (2)
S1: "How do we start proving isosceles trapezoids?"
S2: "We should first know the properties of an isosceles trapezoid."
S1: "Okay. It has one pair of parallel sides that are not the same length and another pair of sides that are the same length but not parallel. The sum of the interior angles between parallel sides is 180°."
S2: "So, what next."
S1: "First, we show the side lengths of TS and VR and the same length. Let us just use the Pythagorean theorem" (after trying for a while, they found a problem)
S2: "It seems difficult. How do you prove the other two parallel sides? What if we show that this isosceles trapezoid can be constructed from a rectangle and two congruent right-angled triangles?"
S1: "Okay, let us try. We make an auxiliary line first."

Based on the excerpts of group conversations above, it can be identified that the form of metacognitive regulation is Co-Constructed Social Metacognitive Regulation (CSMR). S1 and S2 share ideas. In excerpt (1), S1 gives the idea that the first step is to draw coordinates, and S2 justifies and emphasizes that points on coordinates can be drawn at any position. S1 agrees and reminds the meaning of the symbols in the problem. In excerpt (1), the group builds on each other's ideas to devise a strategy to solve the problem. A similar explanation can also be identified in excerpt (2).

S2 offers a strategy to prove an isosceles trapezoid. The strategy is to use the properties of an isosceles trapezoid. S1 justifies this method and gives an idea to show one of the isosceles trapezoidal properties by using the Pythagorean theorem. However, after they tried, they ran into problems. Therefore, S2 offers another strategy by giving reasons. S1 supports and continues the strategy by trying to make an auxiliary line. The group exchanges thoughts and generates new ideas to solve problems. Therefore, the dominant form of metacognitive regulation in this group is CMSR. The group exchanges thoughts and generates new ideas to solve problems. Therefore, the dominant form of metacognitive regulation in this group is CMSR. The group exchanges thoughts and generates new ideas to solve problems. Therefore, the dominant form of metacognitive regulation in this group is CMSR.

Metacognitive regulation form of less successful outcome group in solving geometry problems collaboratively

Some excerpts of group conversations are used to identify forms of metacognitive regulation in collaborative problem-solving. The following shows two examples of excerpts of student group conversations.

Conversation Excerpt (1)
S4: "$c - d$ can be interpreted between points c and d on the x-axis. I think it can be easy."
S3: "That time?"

Conversation Excerpt (2)
S3: "I know the point V will be $(\ldots, e)$ or it could even be$(-a + d, e)$"
S4: "That is not true. We calculate from the coordinates, then the point $V\ (e, e)$."

The results of group conversations show that metacognitive regulations that occur in solving dominant collaborative problems in the form of Ignored Social Metacognitive Regulation (ISMR) can be identified. The ISMR form is found in the examples of the two excerpts above conversation, where S4 rejects or ignores S3's ideas and, conversely, S3 doubts S4's opinion. In the collaborative problem-solving process, S4 dominates in providing solutions and often ignores S3's ideas to defend his ideas in finding a solution. The solution provided by S4 was the wrong solution, but S3 seemed forced to accept it despite doubts about the answers they wrote. S4 ignored S3's ideas and concentrated on his thoughts.

DISCUSSION

Metacognitive regulation can emerge well in group activities, not only in individual contexts. This research proves that student metacognitive regulation can emerge in collaboration with other students in problem-solving activities. Shared knowledge construction and joint information problem-solving challenge students to discuss and regulate their and each other's cognitive activities, providing an opportunity to practice with and refine one's metacognition (Raes et al., 2016; Thalemann & Strube, 2004). Iiskala et al. (2011) concluded that metacognitive experience and regulation emerged in collaborative processes. It was not reducible only to the individual level, which he termed Socially Shared Metacognition.

Table 3 compares the metacognitive regulation that emerged in the groups with more and fewer results. The comparison is presented on every aspect of metacognitive regulation. The comparison results showed that every aspect of metacognitive regulation emerged in the more successful and less successful groups. However, not all indicators can be identified during the problem-solving process. The difference in metacognitive regulation in the two groups is the quality of their metacognitive activities. The ability of students to manage their learning is considered necessary for the quality of collaborative learning (Ucan & Webb, 2015). In addition, there are findings that conceptual knowledge has a role in solving mathematical proof problems. The more successful group had better mathematics conceptual knowledge than the less successful group.

Metacognitive Regulation Aspect	More Successful Group	Less Successful Group
Orientation	The group agrees on the task content hypothesis, which shows that their prior knowledge is good enough.	Self-orientation for groups that are less successful shows disbelief in the knowledge they have.
Planning	The group has several settlement strategies planned together. They choose one strategy that they think is easier to solve the problem.	The group has only one agreed strategy. However, the strategy they agreed on could have been more precise.
Monitoring	Students are aware of their understanding of themselves and their group mates. They provide further explanation as a form of their awareness of their partner's cognitive conflict.	The group realizes that their understanding and knowledge could be improved. Nevertheless, they continued to solve the problem approximately.
Evaluation	Evaluation does not appear in speech when the group is discussing. However, the interview data analysis found that the groups reviewed the quality of group performance and learning outcomes.	Evaluation does not appear in speech when the group is discussing. The group realized that the strategy they used needed to be more appropriate. However, there has yet to be an attempt to improve the answer.

Table 3: Comparison of the emergence of metacognitive regulation in the group with more and less results

Based on the results of the conversation analysis that occurred when students discussed solving problems, the researcher found that the more successful group's dominant form of metacognitive regulation was co-constructed social metacognitive regulation. The less successful group has a dominant form of ignored social metacognitive regulation. The results of this analysis are empirical evidence of a form of metacognitive regulation in solving collaborative problems initiated by Molenaar et al. (2014). It also shows that interactions that build ideas between group members give better results than groups that ignore the ideas of other group members.

The difficulty in implementation is that researchers must carefully determine which utterances are included in metacognitive or cognitive activities. Researchers must be directly involved when observing group activities, not just relying on video results. Video is used to reconfirm the researcher's understanding of the activities carried out by students and how their metacognitive regulation appears. This study was limited to only two groups which were case representatives. In future studies, the representative group can be expanded. In addition, we limit the type of problem to prove where problems with that type are complex for students. Further research can develop tasks with the type of problem to find.

CONCLUSIONS

The more successful and the less successful groups demonstrated metacognitive regulatory activity that appeared in verbal form. Both groups carried out every aspect of metacognitive regulation, but the quality differed. The difference in the quality of this metacognitive regulation indicates a level of metacognitive regulation. De Backer et al. (2016) gave the term level of metacognitive regulation, which describes the different qualities of metacognitive regulation called low and deep-level metacognitive regulation. Furthermore, the results of this study indicate that differences in interactions that occur in collaborative groups will provide different forms of metacognitive regulation. In this study, the more successful group had the CSMR form, while the less successful group had the ISMR form. Further research would be exciting to identify other forms of metacognitive regulation in collaborative problem-solving.

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Metacognitive Intervention: Can It Solve Suspension of Sense-Making in Integration Problem-Solving?

Alifiani, Subanji, Hendro Permadi, Santi Irawati

Universitas Negeri Malang, Indonesia

alifiani.2103119@students.um.ac.id, subanji.fmipa@um.ac.id*,

hendro.permadi.fmipa@um.ac.id, santi.irawati.fmipa@um.ac.id

Abstract: This study aims to describe metacognitive interventions for students who experience suspension of sense-making when solving integration problems based on the characteristics of students' thinking. This research applied a qualitative research approach and provided essay tests followed by task-based interviews to classify students who experience suspension of sense-making into two categories based on their thinking characteristics namely pure procedural and mixed conceptual-procedural. The metacognitive intervention in this study was carried out by giving metacognitive questions to the participants based on their thinking characteristics. The metacognitive questions given included comprehension, connection, strategy, and reflection questions. The results of this study indicated that mixed conceptual-procedural students were able to raise their sense-making earlier than pure procedural students. It can be concluded that pure procedural students need metacognitive intervention in the form of complete metacognitive intervention while mixed conceptual-procedural students only need partial metacognitive intervention.

Keywords: integration, sense-making, suspension of sense-making, metacognitive, metacognitive intervention

INTRODUCTION

Calculus has been a subject of study in mathematics education since 1980s as conducted by Orton (1983a) & Orton (1983b). Both of his studies focused on how students understood calculus in terms of differentiation and integration. Subsequently, Davis & Vinner (1986) held a research on understanding the concept of limit in calculus and identifying sources of misconceptions. In recent years, research related to calculus in the field of mathematics education includes cognitive processes in learning calculus, barriers to learning calculus, calculus learning practices, and the transition of calculus from high school to university (Bressoud, 2021; Fung & Poon, 2021; Galanti & Miller, 2021; Ghedamsi & Lecorre, 2021; Kashefi et al., 2012; Rasmussen et al., 2014). Calculus has an important role in secondary and higher education as well as various scientific disciplines (Rasmussen et al., 2014; Yoon et al., 2021). Calculus serves as a prerequisite for further studies in areas such as mathematics, computer science, social sciences, biological sciences, physical sciences, education, statistics, engineering, and medicine (Bressoud, 2021; Ghedamsi & Lecorre, 2021). Calculus notions cover real numbers, infinite sequences, functions, continuity, limits, differentials, basic calculus theorems, differential equations, and integrals (Ghedamsi & Lecorre, 2021).

One of the important calculus notions is integration. Integration is defined as a key component of mathematics education in secondary schools and calculus courses in tertiary institutions (Greefrath et al., 2021). Integration are important for understanding various contextual problems, including various contexts in physics and engineering and also important when studying mathematics (Radmehr & Drake, 2017). However, the integration is often considered difficult by most students at both high school and university levels. Furthermore, according to Greefrath et al. (2021), student difficulties do not always arise from a lack of knowledge, but from the activation of less productive cognitive resources. For example, when solving integration problems, especially in area calculations, students often do not involve sense-making. When solving problems, students often do not realize the irregularities that occur, where the area cannot be 0 when the region exists, and the area cannot be negative either. This can be called a suspension of sense-making. Sense-making in mathematics can be defined as developing an understanding of a context, concept, or situation by relating it to existing knowledge (Keazer & Menon, 2016). Furthermore, the suspension of sense-making describes the sense-making disengagement that students do when trying to solve mathematical problems. For example, it occurs when the students ignore realistic considerations in solving mathematical problems (Carotenuto et al., 2021). When sense-making is not involved, the resulting answers will be inaccurate even though they have been carried out with the right procedures. In line with the opinion of Keazer & Menon (2016), sense-making must be inherent in all mathematical activities.

Suspension of sense-making occurs due to several factors. According to Biccard (2018), the suspension of sense-making comes from the disconnection of procedural and conceptual understanding. This is in line with the opinion of Greefrath et al. (2021) that student knowledge is often limited to procedural knowledge because they are good at integrations but have difficulty connecting the different contexts of definite integrations. The terms conceptual and procedural are used to describe knowledge about a network of interrelated relationships between mathematical concepts and knowledge of mathematical symbols, formulas, algorithms, and procedures respectively (Legesse et al., 2020). The procedure is a series of steps, or actions, which are carried out to achieve a goal (Rittle-Johnson et al., 2015). This is in line with Voutsina (2012) that procedural knowledge is defined as knowledge that allows the application of rules, algorithms and procedures to solve problems that are not necessarily meaningful and can be generalized to new contexts and situations. It is the capacity to follow sequential steps to solve mathematical problems or achieve mathematical goals (Hurrell, 2021). The essence of procedural knowledge is that it involves applying sequential action steps and automated techniques to solve problems (Aydın & Özgeldi, 2019). Meanwhile, conceptual knowledge is characterized as a network of connected knowledge, a network in which the connecting links are as prominent as the separate pieces of information. The relationships established by conceptual knowledge serve as guides for understanding problems, and for generating new solution strategies or for adapting existing strategies to solve unfamiliar problems (Aydın & Özgeldi, 2019).

Furthermore, the interactions between procedural and conceptual knowledge can be mediated by metacognitive processes (Braithwaite & Sprague, 2021). Metacognitive is defined as knowledge or cognitive activity that uses cognitive processes as its object (Lingel et al., 2019). Metacognitive involves the ability to actively control various cognitive processes; it is a mental process used to regulate cognitive processes (Radmehr & Drake, 2017). Schoenfeld (2016) states that sense-making requires a metacognitive process. Jivet et al. (2020) then investigated metacognitive influences on sense-making. There are three latent variables for sense-making: transparency of design, reference frames, and support for action. The results of Jivet et al. (2020) research showed that metacognitive influences these three latent sense-making variables. The results of this study are also supported by Franklin et al. (2018) that metacognitive is related to mindset, reflection, sense-making, and the development of personal judgment and is an integral part of student success in learning. Learning that involves metacognitive is seen as able to trigger individuals to carry out a sense-making process (Shilo & Kramarski, 2019).

Research related to the suspension of sense-making has been carried out. Carotenuto et al. (2021), for example, seek to deepen understanding of the suspension of sense-making by conducting empirical and qualitative studies that focus on the effects of variations in the presentation of story questions (text, images, format) on students' approaches to problems. The results of this study depicted that suspension of sense-making is more precisely a phenomenon of activating alternative types of sense-making: various types of sense-making active seem to be strongly influenced by the presentation of word problems. However, this research is still limited to students' numerical answers. Therefore, Carotenuto et al. (2021) suggested further research on the process of suspension of sense-making cognitively, not just on objective reports.

Additionally, Kirkland & McNeil (2021) investigated the suspension of sense-making experienced when working on word problems and examined the design of word problem questions that can trigger students to involve sense-making, reasoning explicitly about the context described in the problem. The results showed that rewriting story problems into "yes/no" questions affected students' problem-solving performance and sense-making. However, further research is needed to determine the mechanisms involved in these effects due to research. It is a quantitative study, but not a qualitative one involving think-aloud as a retrospective follow-up on participants to better understand their thinking during problem-solving.

Furthermore, Bonotto (2003) suggested remedies that can be given to overcome the suspension of sense-making, including (i) replacing word problem-solving with class activities that are more related to realistic conditions that are close to students and consistent with sense-making dispositions; (ii) changing teachers' conceptions, beliefs and attitudes towards mathematics; (iii) making direct efforts to change the socio-math class norms. However, this suggested remedy has not been related to metacognition. Even metacognitive also plays a role in sense-making as explained by Schoenfeld (2016) who concluded that these components are interrelated with one another. This study is also supported by previous research (Jivet et al., 2020; Shilo & Kramarski, 2019).

Therefore, it can be said that so far the students' suspension of sense-making has not been explained as a cognitive process that refers to the findings of previous studies. Previous studies mainly focused on word problems that cause suspension of sense-making. If the suspension of sense-making is described from a cognitive perspective, it will develop basic knowledge about student thinking. This can then be utilized in developing interventions that are based on the characteristics of student thinking. Furthermore, this study aims to explore further the process of metacognitive intervention in students who experience suspension of sense-making in terms of students' thinking characteristics. This research begins by investigating the characteristics of the thinking of students who experience suspension of sense-making in calculus courses, especially in the process of solving integration problems, in determining the area of a region bounded by a curve. This research is expected to expand knowledge related to metacognitive interventions that can be given to students who experience suspension of sense-making according to their thinking characteristics. Calculus learning practitioners can then take advantage of this research to determine appropriate interventions according to the characteristics of students who experience suspension of sense-making.

LITERATURE REVIEW

Metacognitive Intervention

The term “metacognitive” or “thinking about thinking”, refers to a distinct capacity that allows one to think about one’s cognitive processes (Pennequin et al., 2010). Metacognitive theories are those theories of mind that focus on cognitive aspects of the mind (Schraw & Moshman, 1995). This means that metacognitive involves the ability to control various cognitive processes actively. In other words, metacognitive is a mental process used to regulate cognitive processes (Radmehr & Drake, 2017). Metacognitive involves the ability to assess one's knowledge and cognitive abilities and how one monitors and controls their cognition in completing tasks (Bellon et al., 2019). Therefore, metacognitive is essential in mathematics learning activities to help students learn more effectively and efficiently.

However, metacognitive does not always run smoothly without any barriers, students may experience metacognitive failure. Metacognitive failure is related to the response to red flags (Huda et al., 2018). In the metacognitive process “red flag” indicates the need for someone to stop or re-examine the problem-solving process (Goos, 2002). According to (Goos, 2002), there are three times when “red flags” can occur and can identify metacognitive failures, namely: (1) no progress in the process of finding solutions (lack of progress); (2) error detection in the problem-solving process, (3) ambiguous in the final answer (anomalous result). Metacognitive failure occurs when students are unable to detect red flags (blindness), detect the presence of red flags but the actions taken are inappropriate (vandalism), and assumes there are red flags that are not there (mirage) (Rozak et al., 2018).

In mathematics learning activities, metacognitive can be improved by appropriate instruction that facilitate students to reflect on their own thinking (Lai, 2011). Metacognitive interventions can be given so that students could bring up metacognitive processes. One form of metacognitive intervention is in the form of metacognitive questions. Metacognitive questions consist of comprehension, connection, strategic, and reflection questions (Özcan & Erktin, 2015). According to Faradiba et al., (2019), comprehension questions assist students to understand mathematics problems. Connection questions assist students in connecting the given problem to similar or related problems in the past. The strategic questions assist students in determining the best strategy to solve a problem. Finally, the reflection questions direct students to recheck the process of solving problems and their solutions. The grid of metacognitive questions asked in interviews as part of the metacognitive intervention is in Table 1.

Types of Metacognitive Questions	Metacognitive Questions
Comprehension Question	1) What is the material related to this problem? 2) Can you show the area that you are looking for?
Connection Question	1) Have you ever worked on questions like this before? 2) Did you immediately do this problem in the same way as you did before? 3) What are the differences between this question and the questions you have done before?
Strategic Question	1) What strategies can be used to solve the problem? 2) Why is this considered the right strategy? 3) What is the integration formula?
Reflection Question	1) Is the process correct? 2) Does the solution make sense? 3) Is there any other way to solve this problem?

Table 1: The Metacognitive Questions

Suspension of Sense-Making

Sense-making is a means for learning mathematics and is an important goal of learning mathematics (Biccard, 2018; Keazer & Menon, 2016; Palatnik & Koichu, 2017; Sepeng & Sigola, 2013; Weinberg & Thomas, 2018). Sense-making is an essential cognitive process in all mathematical activities (Keazer & Menon, 2016). Sense-making is involved in mathematical activities, including activities of understanding concepts, representations, reasoning, proving, and problem-solving processes (Keazer & Menon, 2016; Mueller et al., 2011; Palatnik & Koichu, 2017; Rau et al., 2012; Smith, 2006). From a problem-solving perspective, sense-making means forming meaning or giving meaning based on experiences that include the context of everyday life as well as concepts and knowledge possessed, which enables a person to recognize how and when to respond to problems appropriately to solve problems effectively (Xiaofang, 2021).

However, during the problem-solving process, the student's ability to consider real-world information might not be applied, and students tend to ignore this information (Fitzpatrick et al., 2020). Several studies have also described "seemingly absurd things that students do at all levels when they try to solve math problems". An example is when the students respond with numerical answers to nonsensical problems or when they ignore realistic considerations in school math problem-solving (Carotenuto et al., 2021). Schoenfeld (1991) then introduced the phrase "suspension of sense-making" to describe students' disengagement with mathematics.

Palm (2008) explained that the phenomenon of suspension of sense-making occurs when students face a problem, they immediately work on the problem in a stereotyped way, without paying attention to the reality of the 'real' situation described in the task. As a result, the solution they find does not match, and in some cases, it even becomes absurd when it is linked to the 'real' situation. This also happens with students when working on problems. Students have a tendency not to use their real-world knowledge properly and ignore that their solutions must make sense in the 'real' situation. The tendency to provide such 'unrealistic' solutions seems strong and not easily overcome by mere hinting.

METHOD

This study applied a qualitative approach because it aimed to describe metacognitive interventions for students who experience "suspension of sense-making". This type of research can be categorized as exploratory descriptive because it describes the results of exploration related to metacognitive interventions given to students who experience suspension of sense-making when application problems based on the characteristics of students' thinking. The research obtained verbal data in the form of students' expressions when they were solving problems. This research was carried out at the Mathematics Education Study Program, in one of the universities in East Java, Indonesia. The participants were students who experienced a "suspension of sense-making".

To begin the study, researchers concerned with learning of integral calculus by employing 47 students in total as the participants. The learning process was carried out in six meetings, and all activities in each meeting were observed and recorded. The learning process implemented is summarized in the following Table 2.

Meeting 1	Meeting 2	Meeting 3	Meeting 4	Meeting 5	Meeting 6
Discussing the Concept of a Definite Integral with the Concept of Riemann Sums	Discuss the definition and nature of definite integrals	Discussing the Fundamental Theorem of Calculus	Discuss Integration Techniques	Discusses the Mean Value Theorem for Integration and Symmetry	Discuss specific integration applications

Table 2: Learning Activities Implemented

The method used in learning is the IMPROVE self-questioning method because it is one of the well-known methods for growing students' metacognition and improving their mathematical problem-solving skills (Shilo & Kramarski, 2019). This method aims to instil key aspects of sense-making in problem-solving by using general questions directed at understanding, strategy, connection, and reflection across the three metacognitive skills. The IMPROVE method has been trusted in metacognitive research on students at various levels of education as well as in teacher professional development (Shilo & Kramarski, 2019). In every learning process, students were trained to always involve metacognitive.

After finishing the learning on integration, students were then given integration problems. After that, an observation was conducted to see whether the phenomenon of "suspension of sense-making" appeared in the student's answers. The problems given were the following.

(1) Determine the area of the region bounded by $f(x) = \sin x\,, 0 \leq x \leq 2\pi$ and the x −axis

(2) Determine the area of the region bounded by $f(x) = x^2 - 4x, -1 \leq x \leq 2$ and the x −axis

When students were asked to solve the problems, there were 16 students out of 47 students immediately applied definite integration concepts for granted without further activation of cognitive resources. Therefore, the final result obtained was less precise. Furthermore, task-based interviews were conducted with students who indicated they were experiencing suspension of sense-making to find out the students' thinking processes when working on the problem and ensure that the errors on the answer sheets occurred due to suspension of sense-making. Based on the task-based interviews conducted to the sixteen participants related to the error occurred due to suspension of sense-making, two characters of the participants were then found, namely pure procedural and mixed conceptual-procedural students.

From sixteen participants who experienced a suspension of sense-making, eleventh of them were identified to be pure-procedural students. Meanwhile, five students were categorized as mixed conceptual-procedural students. All sixteen participants were then explored in-depth interview in which metacognitive questions were asked as the metacognitive intervention. The interview guide was adapted from Faradiba et al. (2019) which can be seen in Table 1.

RESULTS

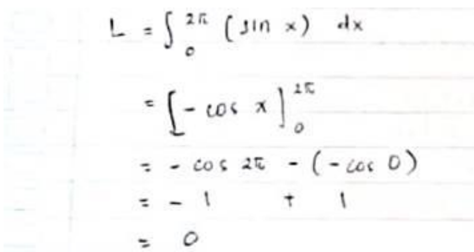
In this study, metacognitive interventions were given in the form of asking metacognitive questions to students who experienced suspension of sense-making. The metacognitive intervention was intended to help students involve their metacognition in problem-solving. Furthermore, this study explored the thinking characteristics of students who experience suspension of sense-making based on the types of conceptual and procedural knowledge involved in the problem-solving process. Next, we described the metacognitive intervention process based on the students' thinking characteristics in the problem-solving process, namely pure procedural and mixed conceptual-procedural. The metacognitive intervention was given during the task-based interview.

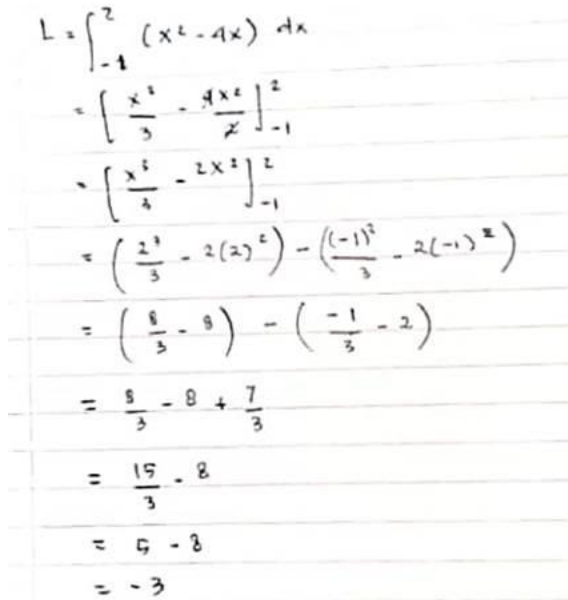
Metacognitive Intervention in Pure Procedural

After being given two integration problems, pure procedural students immediately solved the problems using the usual procedures by applying the integration formula. However, they applied this formula without involving the meaning of the procedure carried out and identified it as experiencing suspension of sense-making. It is because they did not realize the awkwardness that the area cannot be 0 in problem 1 and is unlikely to be negative in problem 2 (Figure 1). Then a metacognitive intervention was carried out on pure procedural students. Initially, a metacognitive intervention was given in problem 1 with a description of the problem, namely: Determine the area of the region bounded by

$$f(x) = \sin x,\ 0 \leq x \leq 2\pi \quad (1)$$

and the x-axis. As shown in Figure 1a, when the students worked on problem 1, they answered that the area of the region is 0. This is seen as unreasonable because the area exists, so the area cannot be 0. One of the pure procedural student's initial answers to problem 1 can be seen in Figure 1a.





(a) (b)

Figure 1: (a) Student's Answers to Problem 1 (b) Student's Answers to Problem 2

Pure procedural students worked on Problem 1 by applying the procedure purely without paying attention to the concept. Even though, pure procedural students had a good understanding of the problems given. This can be seen from the comprehension questions described as follows (P is one of the researchers and S1 is one of the pure procedural students).

P : What material is related to this problem?
S1 : Regarding the application of integrations in area calculations ma'am
P : Try to draw and indicate the area you want to find the area of
S1 :

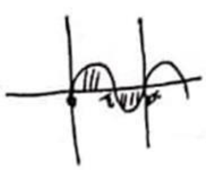


Figure 2: Image of the area sought by S1 (one of the pure procedural students)

Based on the interview with all pure procedural students in the comprehension question section, it is known that students have known how to draw the area they want to find even though they did not describe it during the problem-solving process. This shows that the students have a good understanding of the problem to be worked on.

Next, the metacognitive intervention was continued by providing connection questions. The example of the interview is as follows.

P : Have you ever done something like this before?
S1 : Yes, ma'am.
P : Is there any difference between this problem and the problem you have done before?
S1 : Yes, at least the curve and the boundaries, ma'am.
P : If you look again at the area, you just drew, were there no special conditions that distinguished this problem from the other questions you used to work on before?
S1 : (trying to think)
Erm... What is it, ma'am? Nothing seems to be

Based on the interview, pure procedural students were still not aware of the special conditions that exist in the problem where there should be different treatment between the areas above the x-axis and below the x-axis. Therefore, the metacognitive intervention was continued by providing strategic questions. The example of the interview excerpt with one of the pure procedural students is as follows.

P : Tell me about the strategy you used to solve the problem.
S1 : Yes, ma'am, using integration... all that's left is to enter the graph function formula and its boundaries
P : Okay, now try to remember your answer again (while pointing at the student's answer) and tell me yesterday how come you answered like that. What's the story?
S1 : All that remains is to substitute the curve formula f(x) = sin x and the boundaries, Ma'am (while reading the answers), and the result are 0. (Answers read by students can be seen in Figure 1a)
P : When you found the answer, did you check again or not?
S1 : No, Ma'am.
P : Why not check again
S1 : Yes, because I think it is correct Ma'am. There's nothing strange about my answer, I did it smoothly too.
P : Oh, so there are no obstacles in solving this problem?
S1 : No, Ma'am.

At this strategic question stage, the students were asked about the strategy used, possible strategic choices, and how the strategy was implemented and monitored. From the strategy questions asked, it is known that students directly applied the curve and boundary formulas that they previously knew in integration to determine the area of the area. The students were not aware of the red flag in the form of an anomaly. This indicates a blindness type of metacognitive failure. The last question for Problem 1 is related to the reflection question. Meanwhile, the following is a sample excerpt from an interview related to the reflection question.

P : If you look back now, is there anything strange about your answer?

S1 : *(Checking the answer again) It's correct ma'am. Isn't that right,* Ma'am?

$$\int \sin x \, dx = -\cos x \quad (2)$$

P : *Yes, nothing else is strange, right?*
S1 : *No mom*
P : *Try to look again at the graph that you drew at the beginning. Is it possible that the area is 0?*
S1 : *Oh yes, it can't be 0, ma'am. This is the area.*
P : *What's wrong with it?*
S1 : *(Rethinking the answer)*
S1 : *Oh yes, ma'am, there are curves above the x-axis and below the x-axis. So it seems that you have to calculate it separately, ma'am.*

Based on the interview at the reflection question stage, it is known that the students reflected and re-checked their answers because they were asked by the researcher as part of the intervention. The results of the reflection show that the students initially experienced blindness-type of metacognitive failure because they were not aware of any irregularities in the obtained results. However, after being given metacognitive questions, they could raise metacognitive awareness, engage sense-making, and be aware of existing irregularities. Finally, they could find the right answer by changing the strategy, as shown in Figure 3.

Figure 3: Student's Answer to Problem 1 after Metacognitive Intervention

In Problem 1, the pure procedural students already had a good understanding of the problem, as can be seen from their answers at the comprehension question stage. At the connection stage, the students were not yet aware of the direction of the questions in the connection stage so at the strategic question stage they were still firm on the answer. At the reflection stage, the students could finally raise their awareness, engage sense-making, and realize there were irregularities in the results obtained. After realizing the mistake, the students could make corrections properly even though they experienced confusion because they returned 0 results on L_2. When they got a value of 0, they could get involved in sense-making and realized that the area couldn't be 0, and found something wrong with the positive and negative values in the area of calculation.

Next, in Problem 2, Determine the area of the region bounded by

$$f(x) = x^2 - 4x, -1 \leq x \leq 2 \quad (3)$$

and the x −axes. Pure procedural students answered Problem 2 as can be seen in Figure 1b. The students also had a good understanding of Problem 2. They were able to indicate the area to be searched for when asked even though initially it was not written on the answer sheet as shown in the sample in Figure 4.

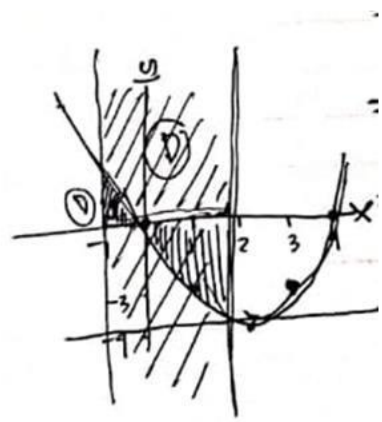


Figure 4: The Results of the S1 Graphic on Problem 2

Based on the results of the interview in the comprehension question stage, it is known that students had a fairly good understanding of Problem 2. They could draw graphs and show the area to be searched for. Furthermore, at the connection question stage, the researcher related it to Problem 1. It can be seen from the sample interview excerpt with S1 as follows.

P : *If you look at the graphics in Problem 2, what do you think is similar to problem 1 that we have just discussed?*

S1 : *This is the same as a curve that lies above the x-axis and below the x-axis. Now, if you look again, the area under the x-axis is bigger, ma'am. This is probably why my answer ended up being negative. If the number 1 was the area above and below the x-axis is the same, then the result is 0.*

P : *Well, that's right. Is it possible that the area is negative?*

S1 : *No mom*

Based on the connection questions, the students looked back at their answers to problem 1, rethought their answers to problem 2, and began to realize the awkwardness of that area was negative. Therefore, it can be said that students' metacognitive awareness appears at this stage spontaneously.

Because students were already aware of the awkwardness that area could not be negative at the connection question stage, then at the strategy question stage the researcher only confirmed a more appropriate strategy to be able to solve Problem 2. An example of the students' answers is in Figure 5. The figure describes S1's answer to Problem 2 after realizing the awkwardness of that area was impossible negative.

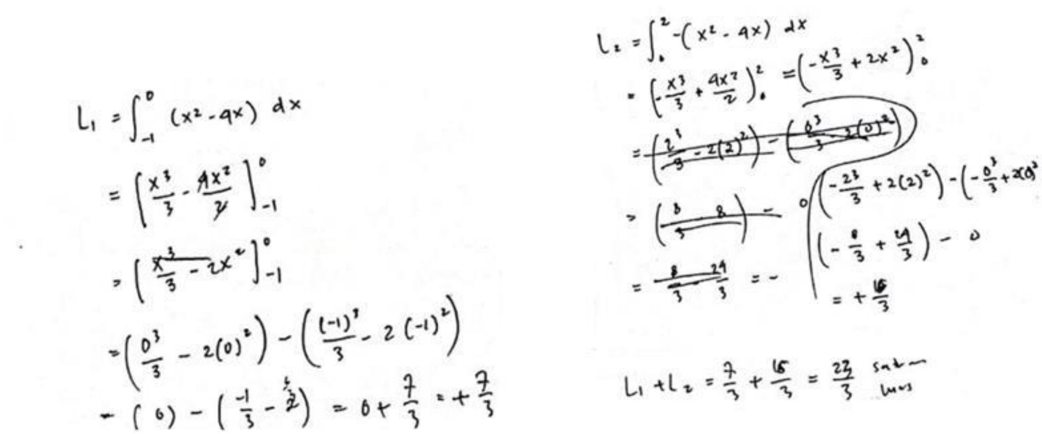


Figure 5: Improvement of S1 Answers on Problem 2 after Metacognitive Intervention

In the process of solving Problem 2, pure procedural students had also started to involve their sense of making and were aware of red flags. They revised their answers that were considered odd and, in the end, they were able to solve Problem 2 properly by finding the right answer.

Finally, after being given a reflection question, pure procedural students could realize that before solving integration problems, they need to know the curve image and the region to be searched for its area first. They cannot directly substitute curve and boundary formulas. Even though the results look right, and it looks like there are no significant obstacles, the results will be awkward.

Metacognitive Intervention on Mixed Conceptual-Procedural

After being given two integration problems, mixed conceptual-procedural students were known to have drawn graphs and understand the purpose of drawing these graphs. Even though they understand the meaning of drawing graphs when solving integration problems, it turns out that drawing graphs are also a routine procedure that students usually do. So, in this study students with these characteristics are hereinafter referred to as mixed conceptual-procedural students. Mixed conceptual-procedural students are identified as experiencing suspension of sense-making because they did not realize the awkwardness that the breadth could not be 0 in

Problem 1 and could not be negative in problem 2. Then a metacognitive intervention was carried out on mixed conceptual-procedural students. First, the process of metacognitive intervention in Problem 1 will be described. Figure 6 presents the example of mixed conceptual-procedural students' answers to Problem 1.

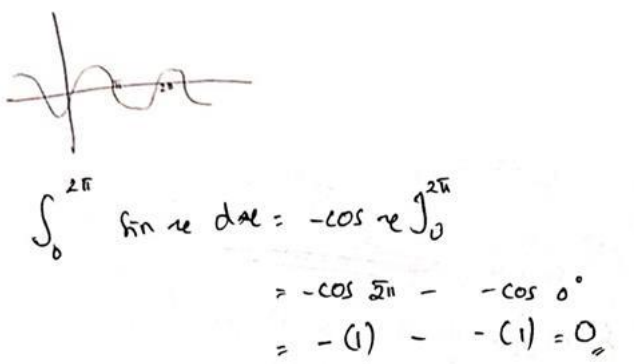


Figure 6: The Answer of Students with Mixed Conceptual-procedural to Problem 1

In contrast to pure procedural students who immediately solved problems by applying certain integration concepts, without drawing a graph first, mixed conceptual-procedural students drew their graphs even though they did not indicate the area to be searched for in the image. However, it is known through the comprehension questions that S2 (one of the mixed conceptual-procedural students) had already understood the given problem and could indicate the area to be searched for.

P	:	*What material is related to this problem?*
S2	:	*Regarding the application of integration ma'am*
P	:	*How do you know that this problem is integration related?*
S2	:	*These are like questions that are usually done in the Calculus course, Ma'am. For questions like this, use the integration.*
P	:	*Okay, here you have drawn the curve. Can you show the area to find the area?*
S2	:	*This or that?*

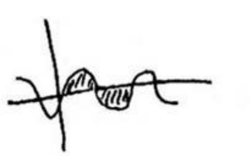


Figure 7: The figure of the area that the mixed conceptual-procedural students are looking for

Based on the interview in the comprehension question section, it is known that students knew the area they wanted to explore even though they did not describe it when solving problems as it is shown by the sample in Figure 7. This shows that they had a good understanding of the problem to be worked on.

Next, the researcher continued the metacognitive intervention by providing connection questions as follows.

P : *Have you ever done something like this before?*
S2 : *Yes ma'am, yesterday in integral calculus class I often worked on problems like this.*
P : *How to solve this kind of problem?*
S2 : *Yes, first draw it ma'am, then when you already know which area we use the integration, we substitute the boundary and curve formula.*
P : *Oh, I see. Then why do we need to draw the curve first before working on it?*
S2 : *Just so you know the area, ma'am, and so you know the boundaries.*
P : *OK about the boundaries. How is that a way of knowing the boundaries?*
S2 : *Yes, seen from the curve ma'am. Sometimes the areas are separated, so to calculate the area you have to divide each area and then add them up. But in question 1, it immediately connects. So yes, the boundaries goes straight from 0 to* 2π
P : *Previously you said that sometimes there are separate areas, what do you mean?*
S2 : *Yes, for example, there is an intersection of 2 curves, ma'am. It's an area that intersects, we don't know where. Sometimes connected, sometimes disconnected.*
P : *Okay, if we look again at the curve image in problem 1. Can't this be said to be separate too? There's one above the x-axis and one below the x-axis. How's that?*
S2 : *(thinking back on the answer while looking at the pictures and making doodles)*

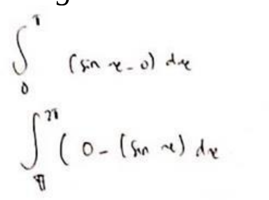


Figure 8: Doodles made by mixed conceptual-procedural students

The one below the x-axis should be negative, right, ma'am?
P : *Naah… That's right, where did you get it?*
S2 : *(explains Figure 8)*
Here, ma'am, I think of this x-axis as f(x) = 0, and then I use the concept of the area bounded by two curves

Based on the interview, mixed conceptual-procedural students assumed that the area to be determined was continuous, so there was no need to divide it per region. They did not realize that there are areas located above and below the x-axis, which should require different treatment when calculating their area. Here the students were again identified as experiencing suspension of sense-making because they did not realize that the areas that were located above the x-axis and below the x-axis should also be said to be separate and to calculate the area, they could not directly apply integration with a boundary of 0 to 2π.

However, when the questions were continued, it was seen that students began to raise their awareness, namely realizing that the area under the x-axis should be negative as shown in Figure 8. Mixed conceptual-procedural students assumed that the x-axis is equal to $f(x) = 0$ and applied the concept of the area between the two curves. Here it can be seen that they did not just think procedurally but started associating with concepts. Therefore, they already realized that the areas located above and below the x-axis need different treatment.

Furthermore, because the students had raised awareness, the researcher only confirmed the strategy used at the strategic question stage. They realized that the answer could not be 0 and that there was an error in the initial strategy used. Mixed conceptual-procedural students immediately tried to implement a new strategy, namely by calculating the area of each area that lies above and below the x-axis

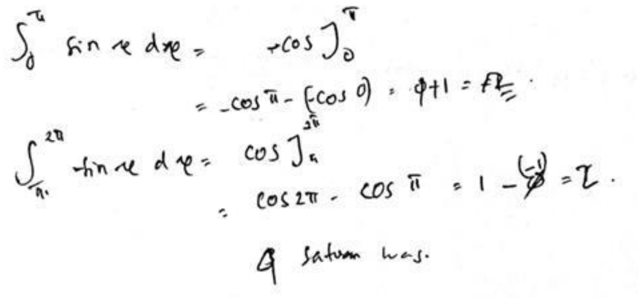


Figure 9: Answers to Problem 1 by S2 after the Metacognitive Intervention

Based on the example of the student's answers in Figure 9, it is known that students reflected and re-checked their answers when they found results that were deemed inappropriate and, in the end, get the right solution.

In Problem 1, mixed conceptual-procedural students already had a good understanding of the problem, as seen from the students' answers at the comprehension question stage. At the connection stage, they could also answer questions well, they understood the importance of drawing curves before determining the area. They had also begun to realize that there were special conditions in Problem 1 at the connection question stage. Furthermore, at the strategic question stage, the students mixed conceptual-procedural started implementing a new strategy, having realized the need for a different treatment between the areas located above and below the x-axis. In the process of work, they always reflected when they found inappropriate results. In the end, the students could revise correctly and find the right results.

Next, in Problem 2, figure 10 shows the initial answer to Problem 2 of one of the mixed conceptual-procedural students.

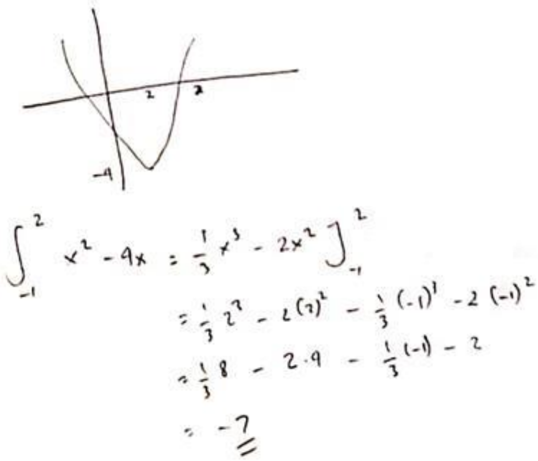


Figure 10: The Answer to Problem 2 by the student with mixed conceptual-procedural

Before being asked further regarding the answer to Problem 2, the mixed conceptual-procedural students immediately realized that area could not be negative. This is because mixed conceptual-procedural students had received metacognitive intervention in Problem 1, so students' metacognitive spontaneously appeared in Question 2. Mixed conceptual-procedural students could bring up metacognitive awareness spontaneously and were able to realize the irregularities that existed even though they had not been given metacognitive questions related to Problem 2. After carrying out the process of re-checking the answers, S2, for example, realized that there was something wrong with the graphic image and the integration formula. Following are the improvements made by mixed conceptual-procedural students on Problem 2.

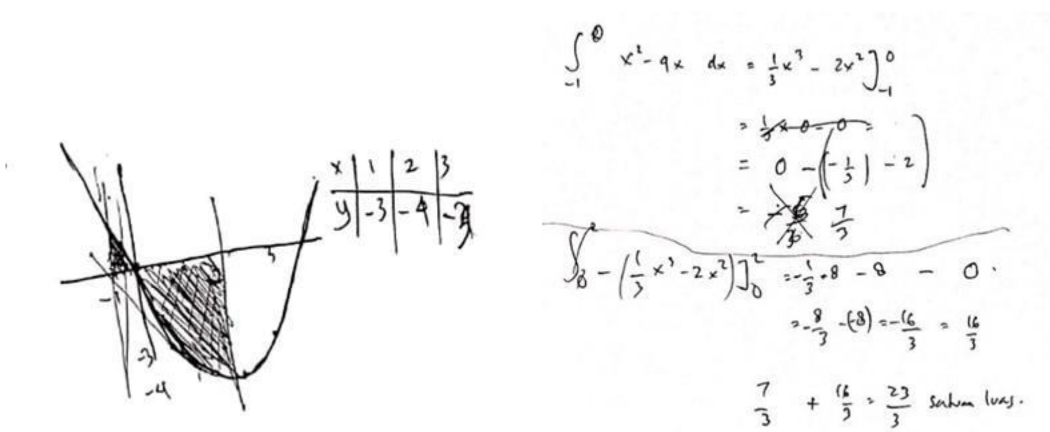


Figure 11: Improvements of mixed conceptual-procedural student answers to Problem 2

The research results presented can be summarized in the following table.

Thinking Characteristics	Types of Suspension of Sense-making	Metacognitive Intervention
Pure Procedural	Do not realize the awkwardness that the area cannot be 0 or negative	Comprehension, Connection, Strategic, Reflection Question
Mixed Conceptual-Procedural	Do not realize that the graphs above and below the x-axis require different treatment	Comprehension, Connection Question
	Do not realize the awkwardness that the area cannot be 0 or negative	Comprehension, Connection Question

Table 3: Metacognitive Interventions based on Thinking Characteristics

DISCUSSION

Pure Procedural Students

Pure procedural students initially experienced suspension of sense-making when solving Problems 1 and 2. This happened because of three factors. (1) Pure procedural students did not activate cognitive resources in the form of sense-making during the problem-solving process. This is in conjunction with Greefrath et al. (2021) that errors do not always arise from a lack of knowledge but from the activation of less productive cognitive resources. (2) Pure procedural students are identified as experiencing metacognitive failure of the metacognitive blindness type because they are not aware of the red flag in the form of an anomaly (Goos, 2002). (3) Pure procedural students do not connect conceptual and procedural understanding, so they experience suspension of sense-making according to opinion (Biccard, 2018).

Furthermore, the metacognitive interventions given to pure procedural students are summarized in Table 2. In the end, after being given metacognitive interventions in the form of metacognitive questions: comprehension questions, connection questions, strategic questions, and reflection questions. Pure procedural students can raise metacognitive awareness at the reflection question stage. Meanwhile, metacognitive awareness is a process of using reflective thinking in developing one's awareness of personal knowledge, tasks, and strategies in a context (Kesici et al., 2011). Pure procedural students could overcome the problem of suspension of sense-making experienced and be able to find solutions to Problems 1 and 2 appropriately. Based on the results of the metacognitive intervention given in the form of metacognitive questions, pure procedural students were able to raise metacognitive awareness and realize existing irregularities. In this case, it can be noted that pure procedural students need complete metacognitive intervention.

Mixed Conceptual-Procedural Students

Like pure procedural students, mixed conceptual-procedural students also did not involve cognitive resources or sense-making in the beginning process of solving a problem. As a result, mixed conceptual-procedural students experienced a suspension of sense-making where students did not realize the awkwardness that the area cannot be 0 in problem 1 or even negative as in problem 2. This is in line with Greefrath et al. (2021) that errors do not always arise from a lack of knowledge but from the activation of less productive cognitive resources.

At first, mixed conceptual-procedural students had drawn graphs and knew the area to be searched for. Graphics are drawn to help define boundaries. This indicates that students have started associating with the concept and not just doing it procedurally. However, at first, the mixed conceptual-procedural students did not realize that a different treatment was needed between graphs located above the x-axis and below the x-axis. Students only focused on finding the overall area without distinguishing the areas below and above the x-axis first. This indicates that students are again experiencing suspension of sense-making because it fails to recognize the existence of different contexts and leads to the need for different treatment between the regions located above and below the x-axis. This is further associated with visual metacognitive concepts.

Visual metacognition is the ability to evaluate one's performance on visual perception tasks (Rahnev, 2021). Visual metacognition is concerned with how people give judgments of confidence in perceptual tasks. According to Sternberg and Sternberg (2012), perception is a set of processes which include recognizing, organizing, and making sense of the sensations we receive from environmental stimuli, which in this case are graphics. Visual metacognitive and awareness are considered to be closely interrelated, with knowledge of the correctness of perceptual choices depending on the level of awareness of the stimulus (Jachs et al., 2015). Visual metacognition is an important skill in our daily lives that enables us, for example, to recognize our poor ability to see in foggy conditions and thus, drive more slowly (Rahnev, 2021). Furthermore, in this study visual metacognitive is associated with how students perceive graphs.

In the visual metacognitive process, failures in visual awareness can occur, such as intentional blindness and change blindness (Ortega et al., 2018). Inattentional blindness is defined as a failure to pay attention to the unexpected but is fully visible when attention is diverted to another aspect of the display being seen (Jensen et al., 2011). The lack of attention of people who experience inattentional blindness can be caused by a demanding main task (Redlich et al., 2021). Change blindness is a staggering failure to detect substantial visual changes. Change blindness, for example, occurs when someone divides attention. When an individual divides attention, the individual is often not aware of some cognitive limitations, such as failure to pay attention to unexpected important changes that occur, and there is an inability to accurately record the events we see (Ortega et al., 2018). Both types of visual metacognitive failure reveal a startling discrepancy between what we believe we see and what we see.

The failure experienced by mixed conceptual-procedural students in viewing graphs was identified as inattentional blindness because these pure procedural students failed to notice that there are areas located below and above the x-axis. Mixed conceptual-procedural students only focus on finding the overall area without first dividing the areas below and above the x-axis.

Then after being given a metacognitive intervention to mixed conceptual-procedural students, they could raise their metacognitive awareness at the connection question stage in problem 1. Students began to realize that there are different contexts and the need for different treatment between the areas located above the x-axis and below the x-axis. After the mixed conceptual-procedural student realizes his mistake, the student can correct his answer and get the right solution from problem 1. Then in Problem 2, the mixed conceptual-procedural student could raise his metacognitive awareness spontaneously. In the end, students with mixed conceptual-procedural could fix the error in Problem 2 and could determine the right solution. The metacognitive interventions provided are summarized in Table 2.

In the end, based on the results of the metacognitive intervention given in the form of metacognitive questions, students only need questions up to the connection question stage. In other words, mixed conceptual-procedural students do not need complete metacognitive questions, or only need partial metacognitive intervention up to the connection question stage.

CONCLUSION

Students often do not involve sense-making when solving problems. This is hereinafter referred to as suspension of sense-making. In this study, students who experienced suspension of sense-making were grouped into pure procedural and mixed conceptual-procedural students. The suspension of sense-making that occurs is caused by several factors: (1) activation of students' cognitive resources is less productive; (2) there is a gap between conceptual and procedural understanding; (3) there is a type of metacognitive failure of metacognitive blindness because they are not aware of the red flag in the form of an anomaly; (4) it is found the presence of inattentional blindness visual metacognitive failure in the process of interpreting graphs.

Furthermore, students who experience suspension of sense-making are given intervention in the form of a metacognitive intervention. Students with pure procedural characteristics can raise metacognitive awareness and engages sense-making at the reflection question stage, so pure procedural students can be said to need complete metacognitive intervention starting from comprehension, connection, and strategic questions to reflection. Meanwhile, students with mixed conceptual-procedural characteristics can raise metacognitive awareness and engages sense-making at the questioning stage connection question or it can be said that it does not require complete intervention, hereinafter referred to as partial metacognitive intervention.

Consequently, metacognitive intervention can solve suspension of sense-making in integration problem-solving. Both pure-procedural and mixed conceptual-procedural students who experiencing a suspension of sense-making in solving integration problem may eventually engage in sense-making after accepting metacognitive intervention. Therefore, metacognitive interventions can accommodate students who experience a suspension of sense making in accordance with their individual thinking characteristics. Metacognitive intervention can activate students' metacognitive awareness and make students engage sense-making in solving integration problem. Metacognitive intervention can complement the learning strategies used in calculus learning, especially in integration notion.

This research has not yet found subjects with pure conceptual characteristics. Therefore, further research is suggested to be able to reveal whether students who solve problems with pure conceptual knowledge characteristics may also experience suspension of sense-making. It is also recommended for further research to investigate further related to metacognitive intervention in students with pure conceptual characteristics in the problem-solving process concerning the suspension of sense-making.

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High School Students' Beliefs about Mathematical Problem Solving: A Cluster Analysis

Edgar J. Sintema*, Mogege Mosimege

Department of Mathematics, Natural Sciences and Technology Education, University of the Free State, Bloemfontein, South Africa

sintema.ej@ufs.ac.za

*Abstract***:** *This paper investigates profiles of high school students' mathematical problem-solving beliefs following the introduction of a new mathematics curriculum. In this quantitative study, 490 high school students responded to a mathematical problem-solving beliefs questionnaire to provide data about their beliefs related to mathematical problem solving. A cluster random sampling technique was used to select participants for the study. To accomplish the major goals of the study, a K-means clustering technique was conducted to analyze patterns that were discernible from their beliefs. In addition, a one-way ANOVA conducted to examine mean differences of their beliefs between clusters. Results revealed that, in general, students strongly believe that conceptual understanding is important in mathematics. In one of the clusters students hold strong beliefs about the usefulness of mathematics in their da-to-day lives while in another cluster it was strongly believed that effort was key for one to increase their mathematical ability. Results are important for students' confidence to solve mathematical problems and for implementation of a problem-solving approach in the new mathematics curriculum.*

Keywords: Cluster analysis; High school; Mathematical problem-solving; Students' beliefs

Introduction

Problem solving is essential for mathematics conceptual development at all levels of education. It is the major reason for learning mathematics in all classrooms (Purnomo et al., 2022). Problem solving supports development students' critical thinking due to high order mathematical problems they encounter and enables them to think systematically and mathematically during the process of problematizing (Goutlet-Lyle et al., 2020; Rott et al., 2021). This view motivated curriculum reform in Zambia.

Zambia embarked on reforming the mathematics curriculum whose implementation was launched by the Curriculum Development Centre (CDC) in 2019, and in which problem solving was adopted as one of the key teaching approaches (CDC, 2019). The reformed curriculum was named Science, Technology, Engineering, Mathematics (STEM) syllabus because it placed science, technology, engineering, and mathematics at the center of all classroom activities. Teaching and learning approaches were to integrate science, technology, and mathematics. CDC guided that problem-solving was important for fostering critical analysis of mathematical problems for learners and collaboration among teachers through sharing ideas about improvement of their classroom practice. CDC further guided that the new curriculum would provoke creativity, innovativeness, and critical thinking among students, and schools would train students to be problem solvers who would effectively apply mathematics to solving problems facing their society.

This view is anchored on the premise that mathematical problem solving remains the "*most important skill*" that students should acquire from school (Erlina & Purnomo, 2020) and must essentially be a core aspect of the mathematics curriculum in Zambia and beyond (Purnomo et al., 2022). However, Zambian high school students continue to exhibit lack of mathematical skills required for solving high order problems and they demonstrate low problem-solving abilities (Karimah et al., 2018; Piñeiro et al., 2021; Yeni et al., 2020). This has been attributed to the lack of mathematical creativity, poor choice of strategies, and inability to fully understand mathematical tasks especially word problems (Carotenuto et al., 2021; Payadnya et al., 2021; Purnomo et al., 2021; Supandi et al., 2021). Resolving these issues requires the intervention of teachers. Teachers are expected to intervene through the curriculum because curriculum implementation is a sole responsibility of teachers (Rafiepour & Farsani, 2021; Wang et al., 2022) and accounts for their key performance indicators in Zambia.

Teachers have faced challenges implementing the new STEM curriculum because it was not published together with revised or new mathematics textbooks that teachers would use to teach, and learners to learn. This is despite evidence that students and teachers heavily depend on materials like textbooks and worksheets for the learning and teaching process, and for development of students' problem-solving skills (Jäder et al., 2020; Ulandari et al., 2019). Textbooks serve as the major reference materials for teachers' lesson planning activities and for students exercises and notes. Sahendra et al. (2018) emphasized that textbooks, for example, can improve students' mathematical word problem-solving abilities. This is supported by Yuanita et al. (2018) who posit that mathematical representations can enhance problem solving skills by acting as mediators between students' beliefs and mathematical problem-solving.

As indicated earlier, teachers play a central role in students mathematics development. Several studies have reported about mathematics teachers' beliefs in mathematical problem solving and the role teacher beliefs play in fostering students' problem-solving abilities (Fatmanissa & Qomaria, 2021; Muhtarom, 2019; Siswono et al., 2019; Van Dooren et al., 2019). It has been argued that teachers should always consider and include students' context and incorporate real life situations into mathematical problems they prepare as tasks for students (Fatmanissa & Qomaria, 2021). Such problems should be of high quality and demanding of deep intuition from students (Agustina et al., 2021). They may be time-consuming to solve but teachers should not confuse them with long procedural problems that may also take long to be solved. The quality of a mathematical problem should be emphasized when selecting tasks for students.

Students' mathematical beliefs are known to be the most central and powerful characteristics that enhance their learning and account for their performance in mathematics (Habók et al., 2020; Hidayatullah & Csíkos, 2022; Yin et al., 2020; Wang et al., 2019). When students exhibit positive beliefs in their mathematics problem solving abilities, they are motivated to work hard towards improving their mathematics competences. Evidence from the literature indicates that previous studies inquired on high school students' mathematical problem-solving beliefs focusing on different aspects of the topic, with some exploring gender differences (E.g., Awofala, 2017), while Rojo Robas et al. (2020) studied the role of beliefs on students' motivation to engage in problem solving. Others concluded that students' problem-solving abilities can be enhanced by their positive beliefs (NoprianiLubis et al., 2017; Özcan & Eren Gümüş, 2019; Prendergast et al., 2018; Surya et al., 2017; Zulkarnain et al., 2021).

In a study aimed at examining the effect of STEM project-based learning on secondary school students' mathematical problem solving beliefs, Kwon et al. (2020) found that mathematical problem solving beliefs were the best predictor of students' STEM career perception. The same study further found that mathematical problem solving beliefs and perceptions towards mathematics increased. In a related study, Hidayatullah & Csíkos (2022) found that students' beliefs in mathematics play a very important role in predicting their performance when solving word problems in mathematics.

We acknowledge that mathematical problem solving and student beliefs are among the widely researched areas of mathematics education in the world. However, little research related to these areas has been conducted in Zambia (E.g. Banda, & Mwansa, 2020; Chidyaka, & Nkhata, 2019). While these studies partly focused on problem solving, they did not address any aspect of mathematical problem-solving beliefs of learners. For example, Chidyaka, & Nkhata (2019) conducted a qualitative case study aimed at exploring metacognitive strategies employed by secondary school students in their mathematical problem solving. Results from their study showed that while students were involved in problem solving, they were not using metacognitive strategies because they were not aware of them.

In another study, Banda and Mwansa (2020) investigated factors that influenced female high school students' development of negative self-concept. They found that poor mathematics background and negative perception towards mathematics were the most prominent factors that contributed to girls' negative self-concept about mathematics. Thus, to contribute to reducing the gap in high studies in mathematics, the purpose of the current study was to explore profiles of high school students' mathematical problem-solving beliefs by clustering them into relatively homogeneous groups based on their belief characteristics. In addition, the study aimed at highlighting implications of their beliefs for curriculum implementation. This was accomplished via answers to the following research questions:

1. What beliefs do high school students report about mathematical problem-solving?
2. What patterns can be discerned in high school students' beliefs about mathematical problem solving?

3. What differences exist between clusters of high school students' beliefs about mathematical problem solving?

Methodology

The current study follows a quantitative research design with a cross-sectional survey method. Precisely, a single, one-time-only, cross-sectional survey was administered to a sample of 490 high school students. The survey focused on high school students' beliefs about mathematical problem-solving.

Participants

Participants consisted of 490 Grade 12 students aged between 15 and 18 years who were selected from three public secondary schools in Chipata District located in the Eastern province of Zambia. Considering schools in the district are geographically spread, we used cluster random sampling, a probabilistic sampling method, to select the schools and participants. Within the established clusters, we requested all students to participate in responding to the survey upon explaining the purpose of the study. Only 490 students from the clusters volunteered to participate in the study. At the time of data collection, these students were following the newly introduced STEM Mathematics Curriculum and were expected to be examined based on the new curriculum. Convenience sampling, a non-probabilistic sampling technique, was used to select 490 students (288 boys and 202 girls) from three high schools. Grade 12 students were selected because they had followed the newly introduced mathematics Curriculum for more than one year. It was thus assumed that their mathematical problem-solving beliefs would be more representative of all senior secondary school students in the Eastern province.

Data Collection Instrument

The data collection instrument used in this study was a survey adapted from Kloosterman and Stage (1992, p. 115) which was developed to measure students' mathematical problem-solving beliefs. The survey consisted of 36 items on a 5-point Likert scale (1 = *strongly disagree*, 2 = *disagree*, 3 = *not sure*, 4 = *agree*, and 5 = *strongly agree*). The items were evenly sub-divided into six beliefs. The beliefs and sample items are represented in Table 1.

Table 1. Beliefs about mathematical problem solving with sample items (Kloosterman & Stage, 1992, p. 115)

Belief	Sample items
Belief 1: I can solve time-consuming mathematics problems	Math problems that take a long time don't bother me I feel I can do math problems that take a long time to complete
Belief 2: There are word problems that cannot be solved with simple, step-by-step procedures	There are word problems that just can't be solved by following a predetermined sequence of steps Memorizing steps is not that useful for learning to solve word problems
Belief 3: Understanding concepts is important in mathematics	A person who doesn't understand why an answer to a math problem is correct hasn't really solved the problem

Belief 4: Word problems are important in mathematics	In addition to getting a right answer in mathematics, it is important to understand why the answer is correct A person who can't solve word problems really can't do math Computational skills are of little value if you can't use them to solve word problems
Belief 5: Effort can increase mathematical ability	By trying hard, one can become smarter in math Working can improve one's ability in mathematics
Belief 6: Mathematics is useful in daily life	I study mathematics because I know how useful it is Mathematics is a worthwhile and necessary subject

Data Analysis

Data were analyzed using a quantitative data analysis tool, Statistical Package for Social Sciences version 23 (SPSS 23). Considering that some items in the questionnaire were reverse coded (negatively worded), we started our analysis by reverse scoring all negatively worded items so that high values of the scale indicated responses of the same type. We then proceeded with the data analysis process by first calculating descriptive statistics (mean scores and standard deviations) to gain insight about participants' mathematical problem-solving beliefs. Descriptive statistics enabled us to adequately answer the first research question. Then, to analyze the existence of participants' mathematical problem-solving belief profiles, we performed a K-means cluster analysis procedure. The technique enabled us to generate homogeneous subgroups of participants based on the mean scores of their mathematical problem-solving beliefs. Finally, to examine differences in participants' mathematical problem-solving beliefs across the three clusters, a one-way analysis of variance (ANOVA) was conducted. In addition, a Bonferroni test for multiple comparison was performed for all significant results to determine the clusters that were significantly different.

Results

This section presents results following the order of the research questions. We begin by presenting results related to participants' mathematical problem-solving beliefs before delving into findings about the clustering of participants' mathematical problem-solving beliefs. We end the section by presenting results related to cluster differences of participants' mathematical problem-solving beliefs.

High school students' beliefs about mathematical problem solving

To explore students' mathematical problem-solving beliefs, we based our analysis on mean scores and standard deviations of their beliefs (Table 2). Students were not sure about their beliefs related to conceptual understanding in mathematics is essential (M = 3.79, SD = .58) and that they can solve mathematics problems which are time-consuming (M= 3.32, SD = .68). However, they negatively believed that effort was key for the increase of one's mathematical ability (M = 2.97, SD = .32). Further, they were also uncertain about their beliefs concerned with the importance of word problems in mathematics (M = 3.14, SD = .54) although they exhibited negative beliefs about word problems in mathematics that can be solved using simple step-by-step procedures (M = 2.55, SD = .54). In addition, students exhibited negative beliefs about the usefulness of mathematics to their daily lives (M = 2.97, SD = .46) which potentially limits their application of mathematical concepts to solving problems in their immediate environment.

Table 2. Mean scores and standard deviations of students' mathematical problem-solving beliefs

Belief	N	M	SD
Belief 1: I can solve time-consuming mathematics problems	490	3.32	.68
Belief 2: There are word problems that cannot be solved with simple, step-by-step procedures	490	2.55	.54
Belief 3: Understanding concepts is important in mathematics	490	3.79	.58
Belief 4: Word problems are important in mathematics	490	3.14	.54
Belief 5: Effort can increase mathematical ability	490	2.97	.32
Belief 6: Mathematics is useful in daily life	490	2.97	.46

High school students' belief profiles about mathematical problem solving

To determine profiles of students' mathematical problem-solving beliefs we used a K-means clustering technique which revealed three subgroups of relatively homogeneous beliefs (Table 3). We then derived mean scores and standard deviations of students' beliefs from each cluster (see Table 3) for assessment. As we present results, we would like to clarify the contextual understanding on time-consuming mathematical problems referred to in beliefs 1. In the context of this study, time-consuming mathematical problems are high quality problems that require deep intuition from students. They are different from long procedural problems that may also be time-consuming. Sometimes such high-quality problems may require less time than these long procedural ones. Thus, in this study we emphasize the quality of a mathematical problem.

Table 3. Cluster profiles of students' beliefs about mathematical problem solving

Belief	Cluster 1 (N = 203)		Cluster 2 (N = 137)		Cluster 3 (N = 150)	
	M	SD	M	SD	M	SD
Belief 1: I can solve time-consuming mathematics problems	3.89	.43	3.24	.41	2.64	.46
Belief 2: There are word problems that cannot be solved with simple, step-by-step procedures	2.43	.53	2.55	.55	2.71	.51
Belief 3: Understanding concepts is important in mathematics	4.12	.42	3.14	.34	3.92	.45
Belief 4: Word problems are important in mathematics	3.29	.49	3.08	.54	2.99	.54
Belief 5: Effort can increase mathematical ability	3.00	.29	2.87	.35	3.02	.32
Belief 6: Mathematics is useful in daily life	2.90	.39	3.01	.52	2.99	.48

Analyzing Table 3, students from Cluster 1 hold positive beliefs about the importance of conceptual understanding in mathematics (M = 4.12, Sd = .42), while those in cluster 2 (M = 3.14, SD = .34) and Cluster 3 (M = 3.92, Sd = .45) were not sure about their beliefs. Results also show that students in Cluster 2 (M = 3.01, SD = 52) were not sure about their beliefs related to the usefulness of mathematics in their daily lives as opposed to their counterparts in Cluster 1 (M = 2.90, SD = .39) and those in Cluster 3 (M = 2.99, SD = .48) who exhibit negative beliefs. Compared to those in Cluster 2 (M = 2.87, SD = .35), Cluster 1 (M = 3.00, SD = .29) and Cluster 3 (M = 3.02, SD = .32) were dominated by students who were not sure about their beliefs that one can increase his or her mathematical ability by applying effort to studying the subject. Students in clusters 1 and 2 were uncertain about their beliefs that word problems are important in mathematics while exhibiting negative beliefs that some mathematical word problems can be solved by simple, step-by-step procedures. It can also be deduced from Table 3 that students in clusters 1 and 2 were not sure about their belief that they can solve mathematics problems that are time consuming while those in cluster 3 clearly report negative beliefs. We further generated a bar chart to show a visual representation of the final cluster centers (Figure 1) of the clustering procedure.

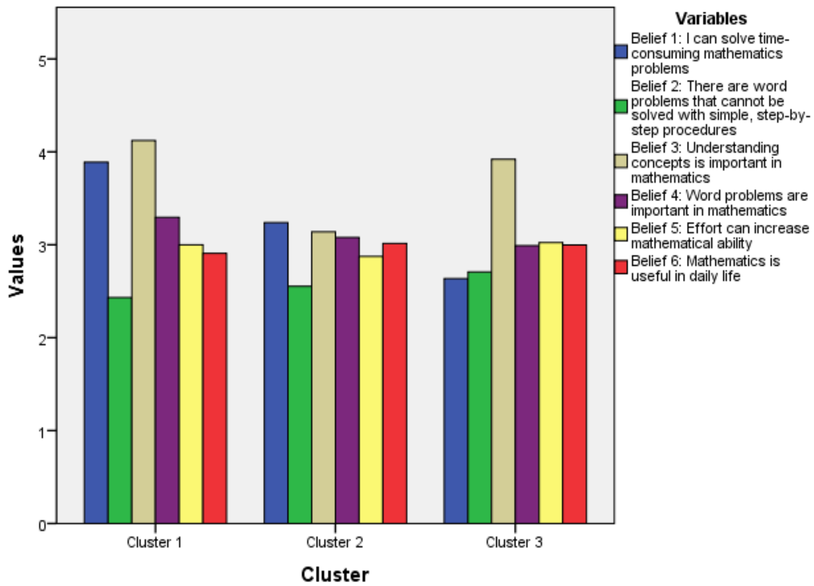

Figure 1. Visual representation of final cluster centers of students mathematical problem-solving beliefs

Cluster differences of high school students' belief profiles about mathematical problem solving

We also sought to examine differences in students' mathematical problem-solving beliefs across the three clusters. Except for the belief about the usefulness of mathematics in one's day-to-day life which was not statistically significant across clusters [F(2.76), p = .065], all other beliefs held by students statistically differed across clusters (see Table 4). For the beliefs that were statistically different across clusters we went on to determine where the actual differences were using the Bonferroni post hoc test for multiple comparison (table 5).

Table 4. ANOVA for differences in students' beliefs about mathematical problem-solving beliefs across clusters

Belief		SS	df	MS	F	Sig.	$\eta^2_{Partial}$
Belief 1: I can solve time-consuming mathematics problems	Between Groups	136.52	2	68.26	363.23	.000	.599
	Within Groups	91.52	487	.19			
Belief 2: There are word problems that cannot be solved with simple, step-by-step procedures	Between Groups	6.61	2	3.30	11.85	.000	.046
	Within Groups	135.75	487	.28			
Belief 3: Understanding concepts is important in mathematics	Between Groups	83.06	2	41.53	246.30	.000	.503
	Within Groups	82.12	487	.17			
Belief 4: Word problems are important in mathematics	Between Groups	8.69	2	4.35	15.83	.000	.061
	Within Groups	133.72	487	.28			
Belief 5: Effort can increase mathematical ability	Between Groups	1.85	2	.93	9.06	.000	.036
	Within Groups	49.77	487	.10			
Belief 6: Mathematics is useful in daily life	Between Groups	1.16	2	.58	2.76	.065	
	Within Groups	102.19	487	.21			

In addition to the post hoc test on all significantly different results, we calculated the effect size of these differences to determine the practical significance of the ANOVA test results. Effect size adds value to statistical results by providing insight about the practical value of the said result for real world use. To accomplish this task, we used partial Eta squared value which reduces the bias of the Eta squared. Table 4 shows that the effect size ranges from $\eta^2_{Partial}$ = .036 to $\eta^2_{Partial}$ = .599. Students' belief that they can solve mathematics problems that are time consuming accounted for the highest effect size ($\eta^2_{Partial}$ = .599) followed by their belief that conceptual understanding is important in mathematics ($\eta^2_{Partial}$ = .503). this implies that 59.9% of the differences across clusters can be explained by students' belief that they can solve time consuming mathematics problems and their belief about the importance of mathematics accounts for 50.3 of the variability. The rest of the beliefs accounted for less than 10% of the variability in the differences across clusters. Thus, these beliefs had very low practical significance for the differences across factors.

Table 5. Bonferroni Post Hoc test for multiple comparison of significant cluster differences

Dependent Variable	Cluster (I)	Cluster (J)	Mean Difference (I-J)	Std. Error	Sig.
Belief 1: I can solve time-consuming mathematics problems	1	2	.65*	.048	.000
		3	1.25*	.05	.000
	2	1	-.65*	.05	.000
		3	.60*	.05	.000
	3	1	-1.25*	.05	.000
		2	-.60*	.05	.000
Belief 2: There are word problems that cannot be solved with simple, step-by-step procedures	1	2	-.12	.06	.109
		3	-.28*	.06	.000
	2	1	.12	.06	.109
		3	-.15*	.06	.041
	3	1	.28*	.06	.000
		2	.15*	.06	.041
Belief 3: Understanding concepts is important in mathematics	1	2	.98*	.05	.000
		3	.20*	.04	.000
	2	1	-.98*	.05	.000
		3	-.78*	.05	.000
	3	1	-.20*	.04	.000
		2	.78*	.05	.000
Belief 4: Word problems are important in mathematics	1	2	.22*	.06	.001
		3	.30*	.06	.000
	2	1	-.22*	.06	.001
		3	.09	.06	.468
	3	1	-.30*	.06	.000
		2	-.09	.06	.468
Belief 5: Effort can increase mathematical ability	1	2	.13*	.04	.001
		3	-.02	.03	1.000
	2	1	-.13*	.04	.001
		3	-.15*	.04	.000
	3	1	.02	.03	1.000
		2	.15*	.04	.000

Examining Table 5, while the ANOVA test revealed statistically significant differences across all clusters about students' belief that there are word problems that cannot be solved with simple, step-by-step procedures, the post hoc test shows that clusters 1 and 2 did not significantly differ. Similarly, there was no statistically significant difference between students in clusters 2 and 3 about their beliefs related to the importance of word problems in mathematics. Table 5 further shows that students' beliefs about the need for effort to increase one's mathematical ability did not statistically differ between students in clusters 1 and 3.

Discussion and conclusion

With regards to students' mathematical problem-solving beliefs' profiles, we identified three distinct profiles, each of which exhibited characteristics that were relatively different from the other. Cluster 1, the largest of the three with 41.4% membership, is comprised of students that exhibit overall high motivation to engage in mathematical problem-solving than students in the other two clusters. Cluster 1 is characterized by students who demonstrated high positive beliefs that conceptual understanding in mathematics is important in mathematics than their colleagues in the other two clusters. This means that these students are motivated to engage with high quality mathematics problems and would persevere to understand the concept behind every mathematical problem they encounter. The result is consistent with findings of Mason (2003) and Surya et al. (2017) who also found that understanding mathematical concepts was important for students' development of mathematical ability.

This result is particularly crucial for the problem-solving approach to teaching and learning on which the new mathematics curriculum is anchored. This is because the aim of mathematical problem solving is to enhance understanding of mathematical concepts as opposed to memorizing facts and procedures for solving problems. This is coupled with their strong belief that they can engage with mathematical problems that demand a lot of time to solve. The belief characteristics exhibited by these students are important as they imply that students in cluster 1 would embrace problem solving in their classroom activities that their teachers would plan for them. The other positive thing about students in cluster 1 is that they recognize the importance of word problems in mathematics and believe that simple steps would not be sufficient for solving word problems. This agrees with the findings of Mason (2003). The only worrying thing about cluster 1 students is that they do not strongly believe that mathematics is useful in their daily lives. This implies that they would not see the value of applying mathematical concepts in solving real life problems. It is expected that students who value conceptual understanding and believe that they can solve time-consuming problems. This result is in tandem with the findings of Ozturk & Guven (2016) who reported that students who did not find mathematics useful also found the subject to be difficult.

Pertaining to beliefs profiles of students in cluster 2, unlike their colleagues in clusters 1 and 2, they strongly believe that mathematics is useful in the daily life of individuals. This is especially important when it comes to application of mathematics concepts to solving real life problems. Thus, by recognizing the usefulness of mathematics in their daily lives, this group of students show potential to find solutions to problems facing their society using mathematics. This would also be supported by their strong beliefs about the importance of conceptual understanding in mathematics and that they can solve time-consuming mathematical problems. however, students in this cluster do not strongly believe that effort is important for enhancement of one's mathematical ability. This seems to contradict their belief that they can solve time-consuming problems because such problems would demand considerable amount of effort to be put in. in fact, these students would require effort as they face the process of mathematical problem solving because tasks that are likely to be given won't be easy and straightforward. They would demand a lot of effort from students.

Cluster 3 was dominated by students with weak characteristics about majority beliefs. Generally, they hold strong beliefs that conceptual understanding is important in mathematics, like their counterparts in the other two clusters, and that effort is a fundamental requirement for one to improve their mathematical abilities. These two beliefs are important for successful mathematical problem solving and this would potentially help them to engage with problem solving tasks that they would face (Simamora & Saragih, 2019; Son et al., 2020; Surya et al., 2017). However, like their colleagues in cluster 1 but unlike those in cluster 2, they do not strongly believe that mathematics is useful to one's daily life. This poses a challenge on their ability to apply mathematics to solving real life problems. In addition, they do not strongly believe that they can solve time-consuming mathematical problems.

When cluster differences were analyzed, we found that there were significant differences between clusters for five beliefs. However, further analysis revealed small effect size for majority of the beliefs. We concluded that sample size may have caused significant some of these differences between clusters. Large sample sizes tend to make small differences present themselves as significant when their practical differences (effect size) may not be large enough. Thus, we observed from the effect sizes that apart from students' beliefs about solving time-consuming mathematical problems and that conceptual understanding being important in mathematics, the differences were practically insignificant (small). In conclusion we argue that it is important to consider psychological factors like students' beliefs, attitudes and perception about mathematics and mathematical problem solving and to prepare teachers to consider these factors when planning work for their students. The results of this study are important to Zambia's Ministry of Education as feedback on the newly introduced curriculum. The findings are also a contribution to mathematics classroom practice in general and contribute empirical evidence to the literature in mathematics education about high school students' beliefs about mathematical problem solving.

Limitations and future research

The limitation we note is about the data collection tool that was used to gather information upon which the analysis and inferences were based. The instrument was limited to collecting only self-reported beliefs which could not be probed by the researcher. This also limited the extent of the analysis to only statistical testing. We recommend the use of multiple instruments for future research which would enable the collection of data like observations and interviews that would allow for both qualitative and quantitative analysis. This would ensure rich and depth of findings. In future we also hope to conduct a study that includes participants from Grade 10 to 12 to assess student beliefs based on grade level and to examine whether their beliefs change as they advance in grade level. Finally, we recommend similar studies that would investigate teachers' beliefs about problem solving approach to teaching as this also has a bearing on how well they prepare tasks for their students.

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The Problem Corner

Ivan Retamoso, PhD, *The Problem Corner* Editor

Borough of Manhattan Community College

iretamoso@bmcc.cuny.edu

The Purpose of **The Problem Corner** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Greetings, fellow problem solvers!

I am delighted to announce that I have received solutions to both Problem 10 and Problem 11, and I am pleased to report that they were all correct, as well as fascinating and innovative. By showcasing what I deemed to be the most outstanding solutions, I aim to enrich and elevate the mathematical understanding of our global community.

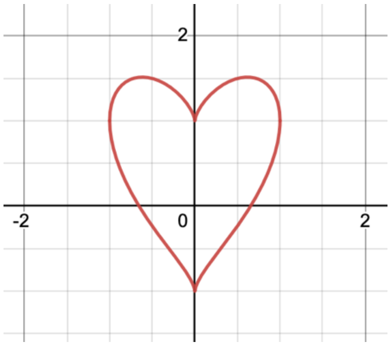
Solutions to **Problems** from the Previous Issue

"Heart" problem.

Problem 10

Proposed by Ivan Retamoso, BMCC, USA.

The Graph of the equation $\left(y - x^{\frac{2}{3}}\right)^2 = 1 - x^2$ is a "heart" and is shown below:



Find the slope of the tangent line of the “heart” at the point $\left(\frac{1}{8}, \frac{2+3\sqrt{7}}{8}\right)$ in exact form.

Solution to problem 10

By Masayuki Kirsch, Borough of Manhattan Community College, Japan.

This minimalist and efficient solution takes full advantage of the local nature of the slope of the tangent line, rather than computing $\frac{dy}{dx}$ *in general the substitution of the values of the coordinates* $\left(\frac{1}{8}, \frac{2+3\sqrt{7}}{8}\right)$ *is done immediately after taking derivatives of both sides of the equation, which quickly leads to the computation of the slope in exact form, kudos to this great solution.*

$$\frac{d}{dx}\left(y - x^{\frac{2}{3}}\right)^2 = \frac{d}{dx}\left(1 - x^2\right)$$

$$\to 2\left(y - x^{\frac{2}{3}}\right)\left(\frac{dy}{dx} - \frac{2}{3x^{\frac{1}{3}}}\right) = -2x$$

Substitution =

$$2\left(\frac{2+3\sqrt{7}}{8} - \frac{1}{8}^{\frac{2}{3}}\right)\left(m - \frac{2}{3\left(\frac{1}{8}\right)^{\frac{1}{3}}}\right) = -2\left(\frac{1}{8}\right)$$

$$= \frac{3\sqrt{7}}{4}\left(m - \frac{4}{3}\right)$$

$$= \frac{3\sqrt{7}\,m}{4} - \sqrt{7} = -\frac{1}{4}$$

$$\left[m = -\frac{\sqrt{7} - 28}{21}\right]$$

Second Solution to problem 10

By Aradhana Kumari, Borough of Manhattan Community College, USA.

This second solution follows a different approach, first $\frac{dy}{dx}$ *is symbolically obtained as a general formula, then the final solution if found by applying the general formula for a particular case.*

Consider the equation

$$(y-x^{\frac{2}{3}})^2 = 1-x^2$$

differentiate both side with respect to x we get

$$\frac{d}{dx}\left(y-x^{\frac{2}{3}}\right)^2 = \frac{d}{dx}(1-x^2)$$

$$2\left(y-x^{2/3}\right)\left\{\frac{d}{dx}\left[y-x^{2/3}\right]\right\} = -2x$$

$$2\left(y-x^{2/3}\right)\left\{\left[\frac{dy}{dx}-\frac{2}{3}x^{-\frac{1}{3}}\right]\right\} = -2x$$

$$\frac{dy}{dx}-\frac{2}{3}x^{-\frac{1}{3}} = \frac{-2x}{2\left(y-x^{\frac{2}{3}}\right)}$$

$$\therefore \frac{dy}{dx} = \frac{-2x}{2\left(y-x^{2/3}\right)}+\frac{2}{3}\frac{1}{x^{1/3}}$$

$$\left.\frac{dy}{dx}\right|_{\left(\frac{1}{8},\frac{2+3\sqrt{7}}{8}\right)} = \frac{-2\times\frac{1}{8}}{2\left[\frac{2+3\sqrt{7}}{8}-\left(\frac{1}{8}\right)^{2/3}\right]}+\frac{2}{3}\frac{1}{\left(\frac{1}{8}\right)^{1/3}}$$

$$\left.\frac{dy}{dx}\right|_{\left(\frac{1}{8},\frac{2+3\sqrt{7}}{8}\right)} = \frac{-\frac{1}{4}}{\frac{2+3\sqrt{7}}{4}-2\times\frac{1}{4}}+\frac{2}{3}\frac{1}{\left(\frac{1}{2}\right)}$$

$$= \frac{-\frac{1}{4}}{\frac{2+3\sqrt{7}-2}{4}}+\frac{4}{3}$$

$$= \left(\frac{-1}{4}\times\frac{4}{3\sqrt{7}}\right)+\frac{4}{3}$$

$$= -\frac{1}{3\sqrt{7}}+\frac{4}{3} = \frac{-1}{3}\left[\frac{1}{\sqrt{7}}-4\right]$$

$$= \frac{-1}{3}\left[\frac{\sqrt{7}}{7}-4\right] = \frac{-1}{3}\left[\frac{\sqrt{7}-28}{7}\right]$$

$$= \frac{28-\sqrt{7}}{21}$$

$\therefore$ The slope of the tangent line to the graph of the equation $(y-x^{\frac{2}{3}})^2 = 1-x^2$ at the point $\left(\frac{1}{8},\frac{2+3\sqrt{7}}{8}\right)$ is $\frac{28-\sqrt{7}}{21}$

Interesting "Parallel lines" problem.

Problem 11

Proposed by Ivan Retamoso, BMCC, USA.

Find the coordinates of the point (x, y) that belongs to the graph of $x^2 + 18xy + 81y^2 = 144$ that is closest to the origin $(0,0)$ and lies on the third quadrant.

Solution to problem 11

By Dr. Michael W. Ecker, (retired) Pennsylvania State University, USA.

We are truly privileged to have the distinguished Dr. Michael W. Ecker, a prominent figure in the world of problem-solving, provide his outstanding solution to problem 11. Dr. Ecker's remarkable achievements include a PhD in mathematics from the City University of New York in 1978, and an illustrious 45-year career as a mathematics professor, with the last 30 years spent at Pennsylvania State University's Wilkes-Barre campus. In addition to his numerous academic accolades, Dr. Ecker also serves as the Problem Section Editor of the MathAMATYC Educator journal. It is an honor to receive his valuable contribution.

Solution: The left side is a perfect square so the equation is equivalent to (*) $(x+9y)^2 = 12^2$. Note that replacing x by $-x$ *and also* y by $-y$ in (*) results in an equivalent equation. That means that the graph is symmetric with respect to the origin, which we will call O.

Next, by taking square roots, the graph of (*) is equivalent to the point (x, y) lying on the graph of one of the two lines with equations $x+9y=12$ and $x+9y=-12$. The second of these is in the third quadrant. Let's call the closest point on this line (to the origin) A. Then the line segment OA with this shortest length is found by dropping a perpendicular from the origin to the line.

Rewrite $x+9y=-12$ as $y=-\frac{x}{9}-\frac{4}{3}$ to identify our original line as having slope $= -\frac{1}{9}$. Hence, our perpendicular line has slope equal to the negative reciprocal of this, or +9. It follows that the equation of our perpendicular, by the Point-Slope formula, is $y=9x$. So, our nearest point lies on both the lines $y=-\frac{x}{9}-\frac{4}{3}$ and $y=9x$. To find the intersection, we set $9x=-\frac{x}{9}-\frac{4}{3}$ and solve in usual fashion, such as multiplying each side by 9 and isolating x. So, we get $x=-\frac{6}{41}$, and substituting this back into either of the two equations gives $y=-\frac{54}{41}$. It is straightforward to check, at least, that $\left(-\frac{6}{41}, -\frac{54}{41}\right)$ does satisfy the linear equation $x+9y=-12$.

Second Solution to problem 11

By Aradhana Kumari, Borough of Manhattan Community College, USA.

This great second solution is detailed oriented and includes a graph to support the steps that were followed to find the solution.

Consider the equation

$x^2+18xy+81y^2=144$

$(x+9y)^2=12^2$

$(x+9y)^2-12^2=0$

$(x+9y-12)(x+9y+12)=0$

$\Rightarrow \quad x+9y-12=0 \quad \text{or} \quad x+9y+12=0$

$\Rightarrow \quad x+9y=12 \quad \text{or} \quad x+9y=-12$

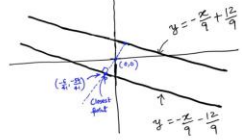
$\Rightarrow \quad y=\frac{-x}{9}+\frac{12}{9} \quad \text{or} \quad y=\frac{-x}{9}-\frac{12}{9}$

These represents set of parallel lines with slope $-\frac{1}{9}$.

In order to find the minimum distance between a point (x,y) lying on the equation $x^2+18xy+81y^2=144$ (which represents a set of parallel lines) and the point $(0,0)$. First we need to find the equation of perpendicular line to the line $y=\frac{-x}{9}+\frac{12}{9}$ & which passes through $(0,0)$.

$\therefore$ The equation of perpendicular line to $y=\frac{-x}{9}+\frac{12}{9}$ which passes through $(0,0)$ is

$y=9x$ (shown as dotted blue)



Point $P(x,y)$ is the intersection point between the line $y=\frac{-x}{9}-\frac{12}{9}$ & $y=9x$.

We need to find the coordinate of the point $P(x,y)$. For this consider

$y=\frac{-x}{9}-\frac{12}{9}$

$9x=\frac{-x}{9}-\frac{12}{9}$ (for intersection point)

$9x+\frac{x}{9}=-\frac{12}{9}$

$\frac{82x}{9}=\frac{-12}{9} \Rightarrow 82x=-12$

$x=\frac{-12}{82}=\frac{-6}{41}$

$\therefore y=-9\times\frac{6}{41}=\frac{-54}{41}$

Hence the coordinate of P is $\left(\frac{-6}{41},\frac{-54}{41}\right)$

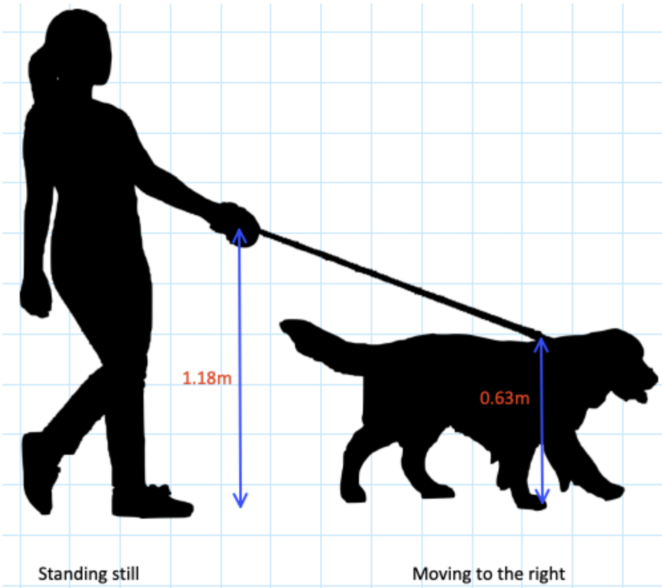
Dear fellow problem solvers,

I trust that you not only enjoyed but also gained valuable insights from solving problems 10 and 11. It's time to move on to the next challenges, and I am excited to present you with the following two problems.

Problem 12

Proposed by Ivan Retamoso, BMCC, USA.

Eva is standing still holding her dog via an extendable leash which she keeps at the height of $1.18\ m$ above the ground as shown in the figure below, suddenly her dog walks to the right at a constant speed of $0.9\frac{m}{s}$, at what rate is the leash extending when the end of the leash is $3m$ horizontally away from Eva?



Problem 13

Proposed by Ivan Retamoso, BMCC, USA.

Prove that the equation $x^3 - 14x + k = 0$ where k is any real number, has at most one real number solution in the interval $[-2,2]$.

Students' Mathematical Thinking Process in Algebraic Verification Based on Crystalline Concept

Nihayatus Sa'adah[1], Siti Faizah[1*], Cholis Sa'dijah[2], Siti Khabibah[3], Dian Kurniati[4]

[1]Faculty of Education at Universitas Hasyim Asy'ari, Indonesia,

[2]Faculty of Mathematics and Natural Sciences at Universitas Negeri Malang, Indonesia,

[3]Faculty of Mathematics and Natural Sciences at Universitas Negeri Surabaya,

[4]Faculty of Teacher Training and Education at Universitas Jember, Indonesia

nihayatussaadah@unhasy.ac.id, faizah.siti91@gmail.com, cholis.sadijah.fmipa@um.ac.id, sitikhabibah@unesa.ac.id, dian.kurniati@unej.ac,

Abstract: Crystalline concept is the main concept used as the reference by students in algebraic verification. This concept divided the way of solving algebraic verification into two types: symbolic and embodied compression. This research aimed to explore the students' mathematical thinking process in solving algebraic verification based on the Crystalline concept types. The subjects of research were 15 students who took abstract algebra course. Those subjects were asked to solve algebraic verification and were divided based on their types. To get a deeper data, one student was randomly chosen from each type to be interviewed. The verification and interview data were analyzed by using the steps of mathematical thinking process. Those steps are abstracting, representation, and verification. Abstracting is the step to find the ideas: definition of group and abelian group. Representation is the way to communicate the suitable ideas with the conditions. The last step is verification in which students performed the process based on the results of two previous steps. The symbolic student tends to solve the verification symbolically while the embodied one solved the verification arithmetically. Based on the findings, it is essential to design a learning that can accustom students to solve algebraic verification symbolically as the verification should be done deductively.

Keywords: Algebraic, Crystalline Concept, Mathematical Thinking, Verification

INTRODUCTION

Mathematical thinking and mathematical verification are built based on representation, reasoning, and students' mistakes during verification and abstractings process (Lo & McCrory, 2009). Abstracting is a learning process from previous experiences as the formation base in forming new experiences (Khasanah et al., 2019). New experiences from time to time can be built through conceptual embodiment by combining perceptions and actions that develop through the mental world (Tall, 2005).

In the theory of the three worlds of mathematics, there are three components that need to be considered, namely: the conceptual-embodied world, the proceptual-symbolic world, and the axiomatic-formal world (Tall, 2008). The conceptual embodied does not only discuss about the embodiment of thinking, but is also perception and reflection as a result of the representation of a mathematical concept. The proceptual symbolism arises when students perform calculations and when they use symbols to derive schemas. Both components work as a process that is conducted and a concept that is thought about, so it can be called a procept (as the abbreviation of process and concept). The process of constructing symbols, processes, and concepts is called the basic procept. While axiomatic-formal is the student knowledge based on axioms, theorems, and definitions of a mathematical object.

The development of mathematics theory begins with introduction of the definition formulated as axioms to prove a theorem or a statement in order to obtain formal verification. Tall (2013; 2014) formulated definition in mathematics as "crystalline concept" which is a mathematics concept with internal structure that can cause the emergence of properties from a certain definition. Crystalline concept can also be called as the main concept in the form of properties or axiom based on context of the mathematical problem. The use of concept in solving mathematical problems can be in the form of: (1) geometrical object which consist of dots, lines, triangles, circle, congruent triangles, parallel lines which has the property of Euclid verification, (2) operation of symbols as flexible process and concept (procept) in arithmetic, algebra, and symbolic calculus needed in calculation and manipulation, (3) a group of theories from mathematical concept as properties of a certain axiom to obtain formal conclusion. Crystalline concept is often found in mathematics especially in algebra and number operation, algebraic expressions, as well as process and concepts using symbol operation with various methods. So, based on this concept, students can be divided into two groups seen the way to complete the verification. Those groups are symbolic compression and embodied compression. The students of symbolic compression complete the verification through symbol operations. Meanwhile, students with embodied compression complete the verification through number operations.

Literature Review

Algebra contains concepts that introduce symbol manipulation through theory system (Pedemonte, 2008). One of the concepts in abstract algebra course is a group and an abelian group. Students are asked to solve the verification based on the definition of group and abelian group. The set with binary operation is called group if it fulfills the closure with respect to the group operation and associative properties, the set has an identity element, and every element of the set has an inverse element; while in abelian group, it is necessary to add commutative properties (Gilbert & Gilbert, 2015).

Mathematical thinking and verification are the parts of mathematics that is useful to train students' reasoning abilities (Varghese, 2009; Faizah et al., 2022). A person's cognitive development in doing mathematical thinking and verification is based on the mathematical language contained in the sensory-motor capabilities by combining perception, operation, and reasoning (Tall et al., 2013). All three aspects can be communicated through enactive gestures, iconic images, written and spoken language as well as arithmetic symbol operations and axiomatic formal symbols based on logical deduction (Bruner, 1966).

There are many researchers who studied algebraic verification and mathematical thinking (Hannah et al., 2014; Onal et al., 2017; Hidayah et al., 2020; Reyhani et al., 2012; Aristidou, 2020; Noto et al., 2019). Students perform mathematical thinking to achieve new knowledge or concepts through abstracting, estimation, generalization, testing hypotheses, and verification processes using the definitions obtained from previously studied concepts (Yorulmaz, 2017; Bukova, 2006). Mathematical thinking is defined as mathematical techniques, concepts, and methods that are used directly or indirectly in the problem-solving process (Henderson et al., 2002).

Mathematical thinking contains the following components: abstracting, synthesis, generalization, modeling, problem solving, and verification (Tall, 2002). Furthermore, Mason (2010) defines the components of mathematical thinking as: specializing, generalizing, making conjectures, justifying, and convincing. Mathematical thinking also includes estimation, induction, deduction, sampling, generalization, analogy, formal and informal reasoning, assertion and equations of processes (Uyangör, 2019). Therefore, this research uses three steps mathematical thinking processes: abstracting, representation, and verification. Based on these reasons, it is clear that in doing algebraic verification all students' mathematical abilities related to the previously known concepts will be involved.

According to the observations in abstract algebra class, there are students who solve the verification problem without using algebraic symbols. They tend to prove it by using specific numbers which are the elements of the appropriate set. They think that it is easier to be solved by using number rather than symbol (variable) because they just need to perform the simple calculation in each property of the verification problem. They keep doing it even though the lecture has explained that the proof has to be able to be applied to all of the elements of the set. Instead of calculations t, the proof can be simplified by using algebraic symbols. Therefore, the aim of this research is exploring students' mathematical thinking process in completing algebraic verification seen from Crystalline concept being used.

METHOD

This research is a qualitative research that uses the purposive sampling technique. The researchers divided the students into two groups based on their types in completing the verification. The first type is the students who complete the verification by using the rules of group and abelian group definitions symbolically and the second type are students who do it arithmetically. Then, the researchers choose one student randomly from each type to be interviewed to obtain the deeper data about their mathematical thinking process.

The subjects of research are the third semester students of Mathematics Education Department at Hasyim Ay'ari University. They are chosen because they have received the lesson about the definition of group and abelian group. The research instruments are the written test sheet and the interview guideline. The researchers also conducted the validity test to the expert to find out the validity of the written test instrument before it was given to the students. During the interview, the researchers recorded the process using video tape recorder to simplify the transcription process. Students were requested to explain the concepts that they used in the verification process through think-aloud technique or by explaining the procedure of the verification process in detail. The written test can be seen in Figure 1.

"Let Q^+ be a set of positive rational number, with binary operation which is defined as $p * q = \frac{pq}{3}$ for all $p, q \in Q^+$, please prove that Q^+ with the stated binary operation is included as abelian group"

Figure 1: Test instrument

The data were analyzed by using three steps: data reduction, data interpretation, and conclusion (Creswell, 2014). Data reduction is selecting and reducing the data based on the relatedness to research aims. The unrelated data can be considered as the findings. Data interpretation is conducted by describing the data from reduction step. The last step is making a conclusion based on the data from interpretation step. There are indicators that be used to analyze the students' mathematical thinking process (Yorulmaz, 2017; Lo & McCrory, 2009; Tall, 2013; 2014). Those indicators are explained in Table 1.

Table 1: Indicators of students' mathematical thinking process

Steps in mathematical thinking process	**Description**
Abstracting	• Determining the idea from the basic concept that will be used to complete the algebraic proof. • The basic concepts used are the definition of group and abelian group.

Steps in mathematical thinking process	Description
Representation	• Communicating the information that is contained in the written test sheet in the form of numbers or algebraic symbols. • Write the information based on the concepts from the abstracting level.
Verification	Completing the algebraic verification by using the definition of group that contained a number of conditions: fulfill the closure and associative properties, the set has an identity element, every element of the set has the inverse element. The definition of abelian group that contained a number of conditions: fulfill the closure, associative, and commutative properties, the set has an identity element, every element of the set has an inverse element.

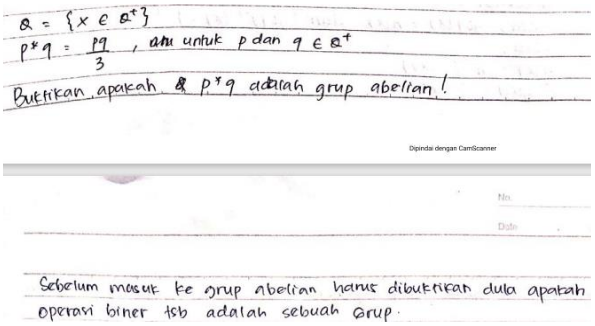
RESULTS AND DISCUSSION

The result of the written test divides the 15 students into two types: 6 students with symbolic compression type and 9 students with embodied compression type. The researchers chose one student randomly from each type to be interviewed. The researcher gave an SS code for subjects with Symbolic Compression and SE for Embodied Compression student. The result shows that they performed algebraic verifications through mathematical thinking processes consisting of abstracting, representation, and verification as explained below:

Abstracting

Abstracting is a mental activity formulated to determine the basic concept to be used in solving a mathematical problem (Lo & McCrory, 2009; Witheley, 2009). Abstracting can also be called as the initial idea to be used in solving the problem based on what was previously experienced in solving similar problems (Skemp, 2012). Abstracting can happen through construction process of a certain knowledge obtained through experience and past event (Nurhasanah et al., 2017).

SS performed the abstracting by mentioning that to prove whether the binary operation $*$ in positive rational number set (Q^+) is an abelian group, we should prove the group first. If the set Q^+ with $*$ operation fulfills the closure and associative properties, has an identity element, and every element of the set Q^+ has an inverse, then the set Q^+ with $*$ operation forms a group. Furthermore, a commutative property should be added to make the set Q^+ with $*$ operation be abelian group.


$Q = \{x \in Q^+\}$
$p * q = \frac{pq}{3}$, atau untuk p dan $q \in Q^+$
Buktikan apakah & $p*q$ adalah grup abelian!
Dipindai dengan CamScanner
No.
Date
Sebelum masuk ke grup abelian harus dibuktikan dulu apakah operasi biner tsb adalah sebuah grup.

Translate version
$Q = \{x \in Q^+\}$
$p * q = \frac{pq}{3}$ for p and $q \in Q^+$
Prove that $p * q$ is an abelian group!
Before begin to prove the abelian group, it is important to prove that the binary operation above is a group.

Figure 2: Idea which will be used by SS

Interviewer : To prove an abelian group, do you need to prove the group first?
SS : Of course, ma'am. Because in abelian group definitions, there is a rule stated that if we want to prove an abelian group, the first thing we have to do is making sure that a non-empty set is a group first. Therefore, I prove that Q^+ which is a rational number with binary operation is a group.
Interviewer : What is the next step?
SS : If it can fulfill the 4 rules in group definition, I will continue to prove the commutative properties. But if any of the rules not met the requirements then I will not continue.

The idea that SS used is group definition to solve problem related to abelian group. In the interview process, SS mentioned that the definition of abelian group is a non-empty set with * binary operation which fulfill 4 rules of group definition and the commutative property. If it is found that any one of the 4 rules are not met during the verification, the process will not be continued.

Meanwhile, SE performed the abstracting by using the idea of an abelian group definition: fulfill the closure, associative, and commutative properties. As shown in the following interview transcript:

Interviewer : What is your idea in solving the problem?
SE : I used the idea based on my previous experience about the similar problem that I have solved. The problem contained commutative property and binary operation.
Interviewer : May I know the previous problem that you mentioned before?
SE : Ehmm… if I am not mistaken, the problem was "given a non-empty set of Z^+ with binary operation $x * y = |x - y|$ if $x \neq y$ and $x * x = x$ for every $x, y \in Z^+$. Please determine whether the operation fulfills closure, commutative, and associative properties!"
Because as far as I remember, the abelian group can be called a commutative group, so I used the closure, commutative, and associative properties similar to the previous problem that I have solved.

From the result of the interview, it is known that SE proved the abelian group based on her memory about similar problem that has been done previously. SE did not realize that the current problem was different from the previous problem. Therefore, the idea used by SE in solving the problem is inappropriate because she missed the conditions about the identity and inverse elements. Even though the definition of the abelian group contains these conditions: fulfill the closure and associative properties, the set has an identity element, every element of the set has an inverse element, and fulfill the commutative property (Gilbert & Gilbert, 2015; Gallian, 2010). SE's mistake in selecting the idea can be called the fact error because the student chose the inappropriate facts with the problem they worked on (Hidayah et al., 2020). From the abstracting process performed by SE, it is known that she used her previous experience to complete the abelian group verification. This is in line with Tall (2008) which states that previous experiences form connections in the brain that can affect the way how students understand new situations.

Representation

Representation is a tool to communicate ideas or answers including graphs, numbers, diagrams, geometry, algebraic symbols, or others (Ernaningsih & Wicasari, 2017; Bannister, 2014). Representation and symbolization are the core of mathematics related to cognitive (Mainali, 2021). Furthermore, NCTM (2000) also mentions that representation is an important element in supporting students' understanding of a mathematical concept by communicating understanding to themselves and to others. Representation can help students understand the abstract mathematical concepts (Samsuddin & Retnawati, 2018).

This study found that SS represented the given information by using algebraic symbols. The symbols used by SS are $p \in Q^+, q \in Q^+$ and $pq \in Q^+$ as shown in Figure 3. He decided to use the variable as the representation of the set because he knew that the verification must apply to all of the elements of the set as shown in the following interview transcript:

Interviewer : What do you mean by $p \in Q^+, q \in Q^+$?

SS : To solve this problem, we have to mention the elements of the set. I choose elements p and q because both of them are the elements of positive rational number set.

Interviewer : Why do you choose p and q as the elements of the set?

SS : Because the verification have to apply to all of the elements of the set. If I don't use the variable, then the verification don't apply generally to all of the elements of the set.

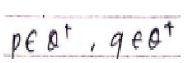


Figure 3: Idea representation of SS

While SE represented the given information by using number such as $p = 2$ and $q = 3$ with $pq \in Q^+$ as shown in Figure 4.he used the specific number as the representation of the set as shown in the following interview transcript:

Interviewer : What is your idea to solve the problem?

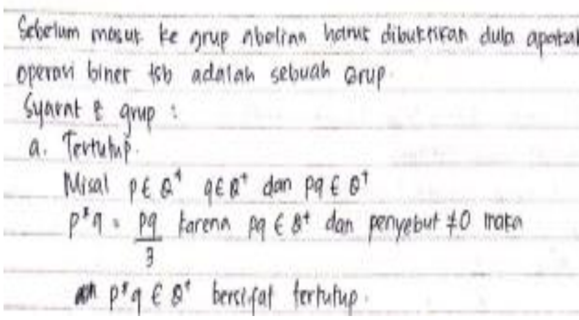
SE	:	I used the idea based on my previous experience about the similar problem that I have solved. The problem contained of commutative property and binary operation.
Interviewer	:	Why do you mean by $p = 2$ and $q = 3$?
SE	:	2 and 3 are the elements of the positive rational number set.
Interviewer	:	Are the elements of the positive rational number just 2 and 3?
SE	:	No. But I think it's enough if I verify to 2 and 3 as those numbers really are the element of the positive rational number set.

Figure 4: Idea representation of SE

Verification

Verification and mathematical thinking are the most important aspect in mathematics learning (Knuth, 2002). The subjects of this research have different ways in completing abelian group verification. SS proved by using all the properties in the definition of abelian group. First, SS proved the closure property by using the symbols contained in the problem. SS stated that $p * q \in Q^+$ fulfills the closure property because $\frac{pq}{3}$ is also a positive rational number. The second property, SS proved the associative property by $(p * q) * r = \left(\frac{pq}{3}\right) * r$ and $p * (q * r) = p * \left(\frac{qr}{3}\right)$. SS stated that the results of both operations are the same, to confirm that it fulfills the associative property. SS proved the associative property through symbol manipulation using algebraic operations. Algebraic operations are stages of mathematical thinking in solving verification problems (Faizah et al., 2020). The third property is about identity element. SS proved the identity element by $p * e = p$ with $p \in Q^+$ so it is obtained that $e = 3$. Therefore, Q^+ has the identity element. The fourth property is about the inverse element. SS determined the inverse by $p * p^{-1} = e$ so it is obtained that $\frac{3e}{p} \in Q^+$.

SS mentioned that to prove whether a non-empty set with binary operation is an abelian group or not, it is necessary to prove the group first. If all the conditions in the group are met, then it can be added the commutative property to complete the verification of abelian group. However, if any condition of the group does not met, then it is automatically not abelian group. Figure 5 show that all the properties of group are met, so it is necessary to add the commutative property to complete the abelian group verification (Figure 6).



Sebelum masuk ke grup abelian harus dibuktikan dulu apakah operasi biner tsb adalah sebuah grup

Syarat 2 grup :

a. Tertutup

Misal $p \in Q^+$ $q \in Q^+$ dan $pq \in Q^+$

$p*q = \frac{pq}{3}$ karena $pq \in Q^+$ dan penyebut $\neq 0$ maka

$p*q \in Q^+$ bersifat tertutup.

Translate version

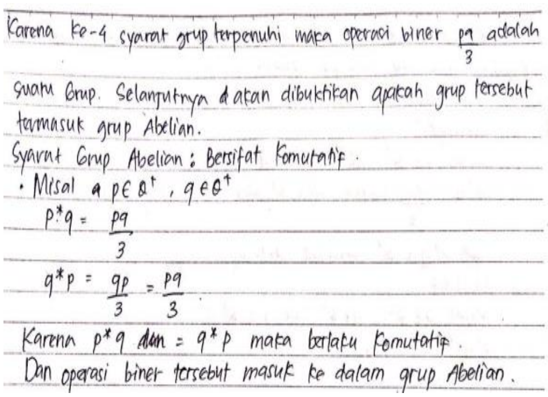
It needs to prove whether binary operation is group or not before proving the abelian group. The conditions of group:

a. Closure

Let $p \in Q^+, q \in Q^+$ and $pq \in Q^+$

$p * q = \frac{pq}{3}$ because $pq \in Q^+$ and the denominator $\neq 0$ therefore, $p * q \in Q^+$ is closure.

Figure 5: Group verification by SS



Karena ke-4 syarat grup terpenuhi maka operasi biner $\frac{pq}{3}$ adalah suatu Grup. Selanjutnya d akan dibuktikan apakah grup tersebut termasuk grup Abelian.

Syarat Grup Abelian : Bersifat Komutatif.

· Misal a $p \in Q^+$, $q \in Q^+$

$p*q = \frac{pq}{3}$

$q*p = \frac{qp}{3} = \frac{pq}{3}$.

Karena $p*q$ dan $= q*p$ maka berlaku Komutatif.

Dan operasi biner tersebut masuk ke dalam grup Abelian.

Translate version

Since all conditions of the group is fulfilled, then the binary operation $\frac{pq}{3}$ is a group. Next, it will be proven whether or not the group is an abelian group.

The condition for abelian group: fulfills the commutative property

$$p * q = \frac{pq}{3}$$

$$q * p = \frac{qp}{3} = \frac{pq}{3}$$

Since $p * q = q * p$ then it fulfills the commutative property.

And that binary operation is an abelian group

Figure 6: Abelian group verification by SS

SS mentioned that Q^+ with binary operation $*$ is an abelian group as it fulfills the 4 conditions of group and it applies the commutative property. The operation which is translated in calculation, addition, division is symbolized as a concept that can be manipulated in the form of arithmetic and symbol of algebra (Tall et al., 2013).

Based on the verification results performed by SS, it can be seen that he proved the problem by using general algebraic symbols. These symbols are used to prove through algebraic calculation and manipulation based on the properties of group and abelian group. This is in line with Tall (2002) which states that symbol manipulation is one step in proving algebra. Thus, the verification performed by SS is general because he used algebraic symbols based on the conditions in the definition. Therefore, the verification done by SS is formal (Pedemonte, 2008). SS's mathematical thinking process in completing algebraic verification can be seen in Figure 7.

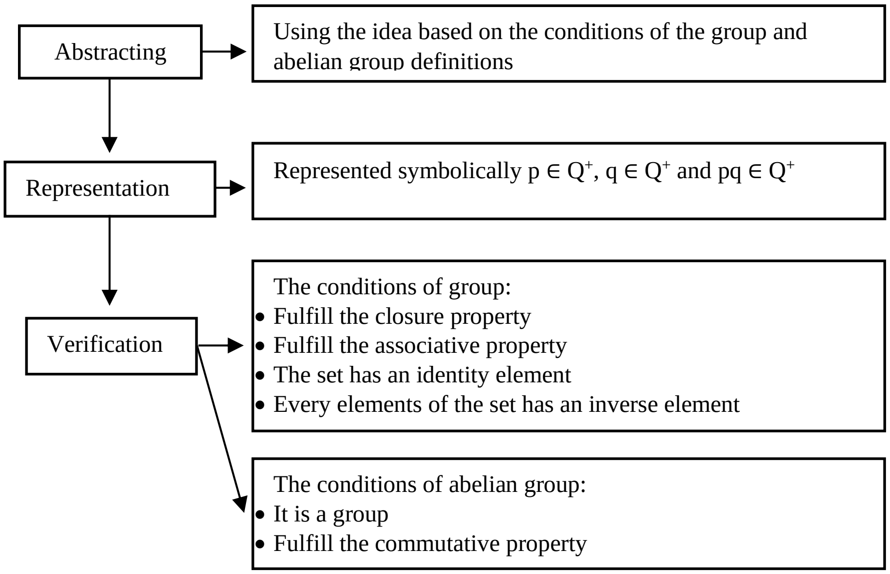
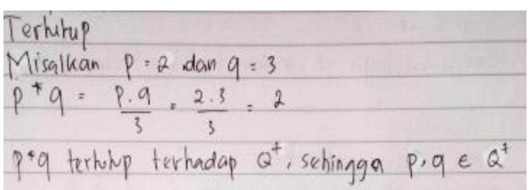


Figure 7: SS's Mathematical Thinking Process

SE performed an abelian group verification by using number as the representation of the set Q^+ with $p = 2$ and $q = 3$ to prove the commutative and closure property. Then, she added $r = 3$ to prove the associative property (Figure 8). Based on the verification of those three properties, SE concluded that it is an abelian group.



Translate version
Closure property:
Let $p = 2$ and $q = 3$

$$p * q = \frac{p.q}{3} = \frac{2.3}{3} = 2$$

$p * q$ is closure to Q^+, therefore $p, q \in Q^+$

Figure 8: SE's Verification Result

SE did not complete the verification because the subject only wrote three conditions even though she knows that abelian group has five properties. Therefore, she asked the permission to add the answer when the interview process was conducted as shown in the following interview transcript:

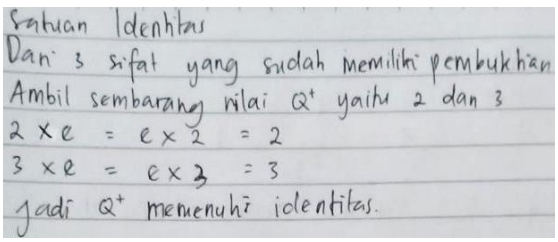
Interviewer	:	Do you agree that the closure, associative, and commutative properties are the conditions of abelian group?
SE	:	Emm… I think, actually there are two more properties that I have to prove but I did not have enough time. Therefore, based on the three conditions that I have proven, I concluded that $p * q = \frac{p.q}{3}$ for $p, q \in Q^+$ is an abelian group.
Interviewer	:	Did you not mention earlier that the three properties are based on your previous experience in solving the similar verification?

SE : Yes, what I mean was I did the test based on my memory when I solved the similar problem. But an abelian group had 5 conditions that need to be proven.

Interviewer : You said that there are 5 conditions, but you did only 3 conditions in the past. What about the other two conditions?

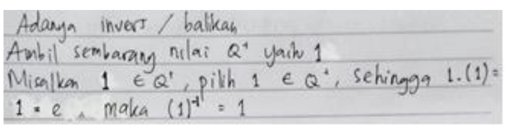
SE : May I add the answers now? I have not finished it because of the lack of time.

SE's additional answers are depicted in Figure 9 and Figure 10 below. SE did the multiplication of the identity element (e) and any element of Q^+ (she chose the number of 2 and 3), but in the end she did not succeeded in finding the identity element. She just mentioned that Q^+ fulfills the identity element. SE proved the inverse element by taking any element of Q^+ which has multiplication results equal to 1.


Satuan Identitas
Dari 3 sifat yang sudah memiliki pembuktian
Ambil sembarang nilai Q^+ yaitu 2 dan 3
$2 \times e = e \times 2 = 2$
$3 \times e = e \times 3 = 3$
Jadi Q^+ memenuhi identitas.

Translate version
Identity element:
From the 3 conditions that have been proven, take any element of Q^+ such as 2 and 3:
$2 \times e = e \times 2 = 2$
$3 \times e = e \times 3 = 3$
Therefore, Q^+ fulfil the identity

Figure 9: Additional data about the identity element of SE's verification


Adanya invers / balikan
Ambil sembarang nilai Q^+ yaitu 1
Misalkan $1 \in Q^+$, pilih $1 \in Q^+$, sehingga $1.(1) =$
$1 = e$, maka $(1)^{-1} = 1$

Translate version
Inverse element:
Take any element of Q^+ such as 1
Let $1 \in Q^+$, choose $1 \in Q^+$ so $1.(1) = 1 = e$, then $(1)^{-1} = 1$

Figure 10: Additional data about the inverse element of SE's verification

Based on the results of verification done by SE, it is visible that she has difficulty in using algebraic symbols, which resulted in her using a number as the representation of the set. The verification using numbers can be categorized as arithmetic verification (Uyangör, 2019). Arithmetic verifications that based on formal axioms are included as informal verifications. The arithmetic verification scheme is based on the use of number, while the analytical verification scheme is based on reasoning and logical deduction to obtain valid arguments (Mukuka & Shumba, 2016). SE's mathematical thinking process in solving algebraic verifications can be seen in Figure 11.

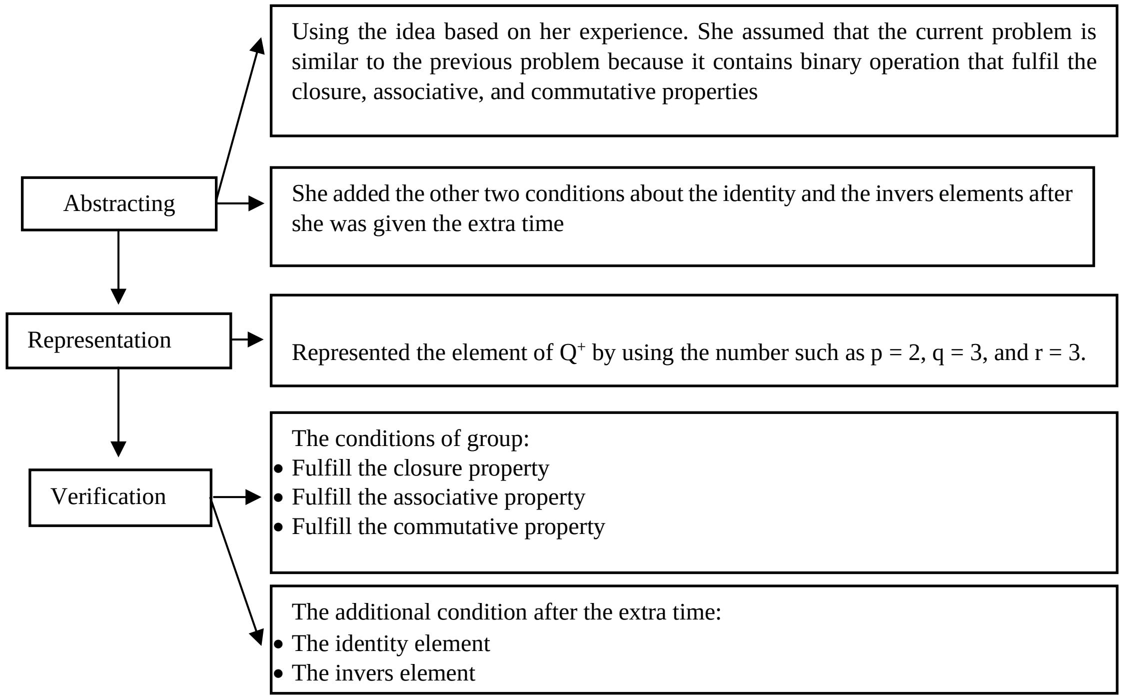


Figure 11: SE's mathematical thinking process

The two types of student thinking indicate that processes and concepts are inseparable. Processes and concepts are two important aspects in mathematics because they contain the meaningful symbols based on definition or axioms (Mukuka & Shumba, 2016). On the other hand, mathematical thinking is a mathematics technique which contain concepts and processes in solving problem directly or indirectly (Çelik & Özdemir et al., 2020). Based on the interpretation above, the difference between the two types based on the process and concept that they used can be simplified as seen in Table 2.

Table 2: The differences of students' thinking process types in completing the verification

Types of Students	**Crystalline Concept**	
	Operational Process	**Set-Theoretic Concept**
Symbolic compression	Calculation and manipulation processes are conducted symbolically	Known concepts about axiomatic formal mathematics are deduced by formal verification.
Embodied compression	Calculation and manipulation processes are conducted arithmetically	Known concepts about axiomatic formal mathematics are deduced by informal verification.

It is seen from Table 2 above that the students with different type of Crystalline concept complete the algebraic verification in different ways. The symbolic compression student used the variable that represent the member of the set, meanwhile, student with the embodied compression used the specific number as the representation of the set. This result is in line with Tall (2014) which states that the symbolic compression student thinks generally because he/she does not use the specific numbers as it is used by embodied compression student. Embodied student does not use the algebraic symbol as she assumes that the verification with specific number is enough. If she succeeds to prove the verification with specific number then automatically the verification is proven for all of set members. One of the reasons this can happen is because the transition process from embodiment to symbolism has not been completed. Tall (2008) states that there are two types of thinking throughout school mathematics: embodiment and symbolism. The first type of thinking is used to give specific meanings in abstract context, while the second type is used to build the computational mental.?? Then, further research needs to be conducted for high school students about the type of student thinking in solving the verification problem.

To support the embodied students to be able to think symbolically, the lecture has to design a learning that extends the context so the students accustom to solve the algebraic verification symbolically. Tall (2008) states that the transition to the formal axiomatic can be built through experiences about embodiment and symbolism. Then the lecture has to give the opportunity for students to solve problems on various kinds of the verification. Those various kinds of problem can give students experiences that can be used as the transition medium of thinking type.

CONCLUSIONS

The results showed that both students with symbolic compression and embodied compression types completed algebraic verifications through mathematical thinking in the form of: abstracting, representation, and verification. Students performed abstractings to determine the ideas that would be used in the verification. Student with symbolic compression type chooses to determine whether or not the positive rational number set and binary operation $p * q = \frac{pq}{3}$, for all $p, q \in \mathrm{Q}^+$ form a group. If it has been proven that it forms a group, he continues by showing whether or not the group is abelian. Meanwhile, student with embodied compression type works for some properties of abelian group: fulfill the closure, associative, and commutative properties. Although she know that there are still two properties left, but he cannot work for them because of the limited time. She acknowledges it through the interview process. Next, students conduct representations to communicate their ideas. Student with symbolic compression type represents the given information through algebraic symbols such as $p \in Q^+, q \in Q^+$ and $pq \in Q^+$. Meanwhile student with embodied compression type represents the given information through the number as the representation of the positive rational number set such as $p = 2, q = 3$ and $pq \in Q^+$. The last step is verification. Both students perform the verification step based on their idea in the abstracting step and complete the verification by using the result of the representation step. Student with symbolic compression type completes the verification process by proving that the positive rational number set and binary operation $p * q = \frac{pq}{3}$, for all $p, q \in \mathrm{Q}^{+}$ forms a group. Then, he continues the verification by showing that it is an abelian group. Meanwhile, student with embodied compression type works the verification process by proving that the positive rational number set and binary operation $p * q = \frac{pq}{3}$, for all $p, q \in \mathrm{Q}^{+}$ fulfills the closure, associative, and commutative properties. Then, she adds the other properties such as the identity element and inverse element in the interview process.

The description above shows that both of embodied and symbolic compression type students complete the algebraic verification problem through mathematical thinking by using different style. The differences are embodied compression type student completes the verification arithmetically while symbolic compression type students complete the verification symbolically. Therefore, it is essential to design a learning that can accustom students to solve the algebraic verification symbolically as the verification should be done deductively by giving various kind of verification problem.

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Creative Conjecture: Abductive Reasoning to Generate Some Ideas in Algebra

Indriati Nurul Hidayah*, Santi Irawati, Mohammad Agung, Cholis Sa'dijah, Subanji, Sudirman

Universitas Negeri Malang, Jl. Semarang no 5, Malang, Indonesia

indriati.nurul.fmipa@um.ac.id

Abstract: Most students practice abductive reasoning in solving mathematical problems that encourage creativity. This study analyses the process of making conjectures based on abductive reasoning. This study used a qualitative design and involved 106 undergraduate mathematics students enrolled in the mathematics course Introduction to Ring. We analyzed the students' conjectures on two mathematics problems. The study was completed by grouping the types of conjectures made by students and then investigating each student's explanation of each conjecture. The results suggested two types of conjectures practised by students, namely creativity in investigating the converse of the proposition and creativity in dividing into cases.

INTRODUCTION

Creativity represents the required character for students in solving mathematical problems. Creative thinking is closely related to problem-posing and project-based learning (Ayllon et al., 2016; Wijayati et al., 2019). During this learning, students are frequently provided with problems and learning methods to build up comfortable learning that enhances students' creative thinking (Ngiamsunthorn, 2020). A previous study has reported a positive relationship between students' facts finding and problem finding with their number and originality of ideas (van Hooijdonk et al., 2020). Moreover, problem-solving correlates positively with image completion, whereas fact-finding does not (Dewijani, 2015).

Problem-solving is also a powerful evaluation tool for a person's mathematical reasoning and creativity (Ayllon et al., 2016). Students' reasoning in solving mathematical problems is divided into several types. Some researchers argued that students' reasoning in solving math problems is deductive (Leighton, 2006; Niu et al., 2007) or inductive reasoning (Haverty et al., 2000; Hozzov & Kov, 2020; Moguel et al., 2019). Meanwhile, Molnár et al. (2013) reported a link between inductive reasoning and complex mathematical problem solvency. Meanwhile, several researchers analyzed the use of deductive and inductive reasoning in solving mathematical problems (Arslan et al., 2009; Stephens et al., 2020). Previous research by Arslan et al. (2009) found that educational students prefer inductive rather than deductive reasoning. Similarly, Stephens et al. (2020) investigated the way students in the USA concluded when given a series of premises.

In the initial stage of solving problems, abductive reasoning plays a role in discovering new knowledge. Students actively construct new knowledge in problem-solving situations that may contradict their current knowledge, resulting in astonishment and obstacles (Radford, 2008). Other studies supporting abductive reasoning in forming discoveries include the studies carried out by Tschaepe (2014) and Walton (2005). Abductive reasoning also carries a role in the discovery of new rules through conjectures at the beginning of a discovery (Abe, 2003; Levin-Rozalis, 2010; Magnani, 2001; Paavola, 2006; Prendinger & Ishizuka, 2005; Woosuk, 2017). Niiniluoto (2018) also adds that abductive reasoning is required in the discovery of new concepts because abductive reasoning may produce various results that still need to be proven deductively.

Many studies have investigated abductive reasoning in solving mathematical problems. Several studies have also addressed the relationship between abductive, deductive, and inductive in solving algebraic problems (Moscoso, 2019). Additionally, previous studies have discussed the relationship between abductive reasoning and creativity (Hidayah et al., 2021; Moscoso, 2019; Tomiyama et al., 2010).

Literature Review

Creativity

Creativity in this research is defined as the process of generating ideas. In mathematics learning, ideas can be in the form of a theorem, a new solution to a problem, or further examples of a concept. Students generate ideas or connect with them through mathematical writing, mainly in solving problems. To generate ideas, students can make guesses or formulate a hypothesis as a part of their creativity (Torrance, 1965). Creativity can also be defined as a process of being sensitive to the problems at hand, deficiencies and knowledge gaps, missing parts, friction, and so forth. Creativity also includes identifying difficulties, seeking solutions, making guesses, formulating hypotheses about deficiencies, testing and retesting these hypotheses, possibly modifying and retesting the hypothesis, and finally communicating the results (Torrance, 1965).

In addition, four components of creativity have been proposed, including fluency, flexibility, elaboration, and originality (Torrance, 1965) Fluency is the number of relevant responses from the subject, while flexibility is thinking of different questions, causes, or consequences. Meanwhile, originality represents the statistical infrequency of these questions, reasons, or effects, as well as the extent to which the response represents a mental leap departure from the apparent and commonplace. Lastly, elaborate is the detail and specificity incorporated into the questions.

In mathematics learning, creativity has particular criteria. Mathematical creativity is the ability to formulate mathematical objectives and find inherent relationships among them (Ervynck, 1991 in Tall, 2002). Mathematics learning focusing on creativity will improve students' representational ability, strategic fluency, and flexibility, as well as an appreciation for new problems or solutions (Silver, 1997). Therefore, dimensions of creativity, namely fluency, flexibility, and novelty, are a core part of mathematics.

Silver (1997) details the indicators of creativity dimensions in solving mathematics problems, as presented in Table 1.

Table 1. The Indicators of Creativity Dimensions

Dimensions of Creativity	Indicators
Fluency	• generating multiple mathematical ideas, • generating various answers to a mathematical problem (if it exists), • exploring mathematical situations.
Flexibility	• generating new mathematical solutions when at least one has already been produced.
Novelty	• exploring many solutions to a mathematical problem and generating a new one.

Another aspect of creativity is divergent and convergent thinking. Divergent thinking includes finding patterns, breaking fixed mindsets, formulating mathematical conjectures, evaluating original mathematical ideas, identifying missing components, and moving from general to specific concepts. Meanwhile, convergent thinking enables students to answer without requiring significant creativity (Mann, 2006). In terms of supporting a statement, there is little difference between divergent and convergent topics (Pamungkas et al., 2018).

Student creativity can be enhanced through open-ended problems since solving mathematical problems in multiple ways is closely related to personal mathematical creativity and suggests evaluating mathematical creativity (Yaftian et al., 2011). Open-ended tasks also promote students' mathematical creativity (Fatah et al., 2016; Molad et al., 2020; Rahayuningsih et al., 2019). Besides, lecturers may adopt schemes from other lessons with a sufficient challenge to increase students' creativity (Diezmann & Watters, 2002).

Abductive Reasoning

In solving mathematical problems, students practice reasoning. Reasoning is a thinking process that connects known facts or realities to conclusions (Krawczyk, 2017), leading to problem solvency. It is also regarded as an incredibly logical or analytical thinking process (Subanji & Supratman, 2015). Thus, reasoning can be interpreted as a logical process for reaching conclusions based on the available information.

Several studies reported that students' reasoning in solving problems is deductive (Lachmy & Koichu, 2014; Niu et al., 2007; Stephens et al., 2020) and inductive (Bisanz et al., 2013; Haverty et al., 2000; Hozzov & Kov, 2020). Other researchers have also analyzed deductive and inductive reasoning in their studies (Arslan et al., 2009; Nickerson, 2010; Rivera, 2008; Stephens et al., 2020).

The reasoning that facilitates creativity is abductive reasoning. It is classified as the first phase of new ideas generation. Abductive reasoning contributes to improving creative knowledge in the learning process (Moscoso, 2019; O'Reilly, 2016). Abductive reasoning consists of examining and using facts to propose a conjecture (Peirce, 1960) with unproven validity. Conjecture can also be referred to as the origin of a new idea in mathematics learning. Further, once the conjecture is verified by deductive reasoning, then it is determined as a theorem.

People are frequently perplexed by the many styles of reasoning, particularly the distinctions between deductive, inductive, and abductive reasoning. To describe abductive reasoning, the examples of deductive, inductive, and abductive reasoning proposed by Peirce (1960) are presented in the following.

Examples of deductive reasoning

Rule: All marbles in this bag are white.
Case: These marbles come from inside this bag.
∴Result: These marbles are white.

Examples of inductive reasoning

Case: These marbles come from inside this bag.
Result: These marbles are white.
∴Rule: All the marbles in this bag are white.

Examples of abductive reasoning

Rule: All marbles in this bag are white.
Result: These marbles are white.
∴Case: These marbles come from inside this bag.

In addition, the conclusion made based on abductive reasoning in mathematics learning is presented in the following.

Every two even numbers, when added together, will produce an even number
Two numbers add to an even number
So, maybe the two numbers are both even

By using abductive reasoning, we can conclude many conclusions based on known rules and results.

Abductive reasoning is closely related to the form of logic. In logic, the sentence which has a truth value, true or false, is called a statement, symbolized by p, q, r, and so forth. A statement can contain universal quantifiers or existential quantifiers with symbols $\forall x$ and $\exists x$, respectively. The example of the symbol for a statement with a quantifier is $(\forall x)\, p(x)$. The example of symbol creation for a statement is shown in the following.

Let $p(x)$: x be an even number

$q(x)$: $5x$ is an even number

Then the statement "For every x, if x is an even number, then 5x is also an even number" can be denoted by $\forall x, p(x) \rightarrow q(x)$

Using the abductive form proposed by Peirce (Niiniluoto, 2018), the statement can be written as

$$(\forall x)F(x) \rightarrow G(x)$$
$$(\exists x = a)G(a)$$
$$\therefore F(a)$$

in a simple form,

$$A \rightarrow B$$
$$B$$
$$\therefore A$$

When compared with the modus ponens and the modus tolens in deductive reasoning, it becomes

$$A \rightarrow B$$
$$A$$
$$\therefore B$$

and

$$A \rightarrow B$$
$$-B$$
$$\therefore -A$$

As observed in the example, the modus ponens and modus tolens rules do not encompass abductive reasoning. In abductive reasoning, the inference attempts to modify the modus ponens and the modus tolens. The validity of deductive reasoning has been confirmed by the arguments $'(A \rightarrow B) \wedge A \rightarrow B'$ and $'(A \rightarrow B) \wedge -B \rightarrow A'$ form a tautology. Although, abductive reasoning is not completely valid, it opens opportunities to come up with creative conclusions (Niiniluoto, 2018).

Several studies have investigated types of students' abductive reasoning in problem-solving. Four abductive reasoning based on fact were reported, including creative conjecture, fact optimization, factual error, and mistaken facts (Hidayah et al., 2020). In the creative conjecture, to solve the problem, students have to use every piece of information inside the problem, understand the questions, and use "actual" facts from outside the problem. Students create conjectures from facts by writing, describing, or drawing problem-solving designs and composing a new conjecture on a problem related to the question (Hidayah et al., 2020).

In this study, we investigated students' abductive reasoning in solving two mathematics problems. Specifically, we analyzed the students’ answers to two mathematics problems. The analysis was carried out by grouping the types of conjectures made by students, followed by an examination of each student’s explanation based on their reasoning. For the answers using abductive reasoning, the explanation was observed based on the use of facts and the form of logic.

RESEARCH METHOD

The participants of this study were 106 undergraduate mathematics students consisting of 86 female and 20 male students in the fourth semester of the mathematics course introduction to ring at the Department of Mathematics, Faculty of Science, Universitas Negeri Malang, Indonesia. In the pandemic era, learning was carried out online. In this study, the researcher acted as a lecturer in the classes. During the learning, the lecturers often gave questions and assignments that enhanced students' curiosity or creativity. Besides, the learning also facilitated students to discuss material they had not understood or convey new ideas to lecturers and friends. Sometimes lecturers also asked students to form small discussion groups to discuss assignments.

The instruments in this research were two mathematical problems facilitating students to make some conjecture. Moreover, the mathematical problems also improved students' creativity because they aided students in developing many conjectures along with explanations.

The instrument validity was checked through discussion with an expert group. In detail, two mathematics experts were involved in the content validity test on the mathematics questions and interview sheets. The validity test included the eligibility of the item test, the concept’s authenticity, multiple interpretations, and appropriate instructions for abductive reasoning. Additionally, the questions provided several facts, and the participants were asked to check whether the provided statement was true or not based on their existing knowledge, as shown in Figure 1.

Let R be a commutative ring with unity, $x, y, z \in R$.

1. If x is a unit, does $y \mid z$ imply $xy \mid z$?
2. Did your answer in part (1) still hold if x is not a unit? Justify your answer!

Figure 1. Open-ended Problem to Observe Students' Abductive Reasoning

The problem is about a commutative ring in Algebra. The commutative ring R is a non-empty set with two binary operations, addition (denoted by $a+b$) and multiplication (denoted by ab), such that for all a, b, c in R (Gallian, 2016), as presented in the following.

1. $a+b=b+a$
2. $(a+b)+c=a+(b+c)$
3. There is an additive identity 0 in R such as $a+0=a$ for all a in R.
4. There is an element $-a$ in R such as $a+(-a)=0$
5. $a(bc)=(ab)c$
6. $a(b+c)=ab+ac$ and $((b+c)a=ba+ca$
7. $ab=ba$

In the first question, if the student answers 'yes,' then their statement became 'If x unit and $y|z$, then $xy|z$.' The students who answer 'no' produced a statement '$\exists x,y,z \in R,\ x$ units or $y|z$, but $xy|z$.' Additionally, the students could also present other answers to generate many conjectures. By answering this problem, students' creativity is expected to increase.

From the first question, we constructed the second question by modifying the first question of "If x is not a unit, does $y \mid z$ implies $xy \mid z$?". Through students' answer to this question, we investigated the conjectures developed by students and their explanations. Further, we analyzed the students' answers that contained new ideas. The analysis was performed by investigating the students' responses, relating them to the logical form of the conjecture's proposition, and discussing them based on creativity criteria.

RESULTS AND DISCUSSION

Through data reduction, only 83 out of 106 students' answers could be analyzed. Besides, nine of 83 students did not answer the second question. The analysis was completed by grouping the types of conjectures proposed by students and examining each student's conjecture based on the reasoning. For the students who use abductive reasoning, the explanation was observed based on facts and logic.

In solving the first question, three different conjectures were made by students. Fifty-three students created the conjecture 'If $y \mid z$ and x unit then $xy \mid z'$ and used deductive reasoning to their answers' validity and authenticity. The students' valid conjecture and justification represent their proper reasoning enhancing their discovery (Folger & Stein, 2017; Niiniluoto, 2018; Peirce,1960). In mathematics, deductive reasoning is used to justify some theorem (Ayalon & Even, 2010; Ellis, 2007; Leighton, 2006).

Our obtained data suggested that many students presented the same conjecture with different explanations. Meanwhile, twenty students proposed incomplete facts, and they took $x = 1$ and $x = -1$ as a unit element in R. These facts allow the production of a conjecture, but they remain insufficient for the justification of a statement in mathematics. Further, other students present different conjectures with an explanation.

Two students created two creative conjectures, namely 'If $y \mid z$ and x unit then $xy \mid z'$ and 'If $xy \mid z$ then $y \mid z$.' Then, they attempted to converse with the true proposition. The student's creative conjecture is listed in Table 2.

Table 2. Conjecture Proposed by the Student for the First Question

Conjecture Proposed by the Student	Description	Number of Students
Let R be a commutative ring with unity, $x, y, z \in R$. • If $y \mid z$ and x unit, then $xy \mid z$	• Using deductive reasoning • Using incomplete facts • Using the wrong form of logic • Assuming the questionable concept as a given fact	53 20 4 2
Let R be a commutative ring with unity, $x, y, z \in R$. • If $y \mid z$ and x unit, then $xy \nmid z$	• Using incomplete facts • Using the wrong form of logic	1 1
Let R be a commutative ring with unity, $x, y, z \in R$. • If $y \mid z$ and x unit, then $xy \mid z$ • If $xy \mid z$, then $y \mid z$	• Using deductive reasoning	2

For the second question, we observed three different conjectures and their explanation. Nine students answered, 'If $y \mid z$ and x non-unit then $xy \mid z$ is false' and used deductive reasoning, shown from $\exists x, y, z \in R$, $y \mid z$, and x non-unit, but $xy \nmid z$. Students use abductive reasoning to create conjectures and justify them with deductive reasoning (Nandasena et al., 2018; Peirce, 1960.). As reported in a previous study, many students provide one counterexample to refute a false conjecture, but others give some counterexamples (Zeybek, 2017). Besides, numerous students create the same conjecture with different explanations.

Our data also showed that forty-two students misunderstood the negation of the statement. They assumed that 'If $y \mid z$ and x non-unit then $xy \nmid z$' is a negation form of 'If $y \mid z$ and x unit then $xy \mid z'$. From the first question, the students comprehended 'If $y \mid z$ and x unit, then $xy \mid z'$ is true. Therefore, they assumed 'If $y \mid z$ and x non-unit, then $xy \mid z'$ was false. As reported in a previous study, merely denying the meaning of the statements was less successful than employing symbolic principles of negation in a recursive style (Piatek-Jimenez, 2010). Besides, nine students provided no answer to the second question.

In addition, one student presented two conjectures creatively. They divided case by case for conjecture, from 'Let R be a commutative ring with unity, $x, y, z \in R, y \mid z \leftrightarrow z = yc$ with the first case, If c prime and x non-unit then $xy \nmid z$ and the second case, If c is non-prime and x non-unit then $xy|z$, as summarized in Table 3.

The students who practiced abductive reasoning concluded based on facts and provided reasoning. Students' conclusions are referred to as conjectures since they have not been proven true. However, in their explanation, 53 and nine students used deductive reasoning in justifying their conjectures for the first and second questions, respectively. Meanwhile, the remaining students presented other explanations. As described by Peirce (1960), the construction of conjectures and justification in abductive reasoning is part of the inquiry process.

Table 3. Conjecture Made by Students for Second Question

Conjectures Presented by Students	Explanation	Number of Students
Let R be a commutative ring with unity, • $\exists x, y, z \in R$, $y \mid z$ or x non-unit but $xy \nmid z$	• Using deductive reasoning • Using incomplete fact	9 12
Let R be a commutative ring with unity, $x, y, z \in R$. • If $y \mid z$ and x are non-unit, then $xy \nmid z$	• Using incomplete facts • Using the wrong form of negation • Assuming the questionable thing as a given fact	8 42 2
Let R be a commutative ring with unity, $x, y, z \in R, y \mid z \leftrightarrow z = yc$ • If c prime and x non-unit, then $xy \nmid z$ • If c is non-prime and x non-unit, then $xy|z$	• Dividing into cases	1

Open-ended problem facilitates students to make different conjectures (Fatah et al., 2016; Molad et al., 2020; Rahayuningsih et al., 2019; Suyitno et al., 2018). The open-ended problem can improve students' mathematical reasoning (Bernard & Chotimah, 2018). Generating conjectures is an essential mathematical habit helping students develop their mathematic skills (Meagher et al., 2020)

Although the students' conjecture is not necessarily true, it is a candidate for the new theorem that requires some corrections. Making conjectures is the initial stage in developing a new mathematical theory. Further, those conjectures can be a theorem if equipped with valid proof steps, known as deductive reasoning. However, sometimes students cannot practice these valid steps, as shown in Tables 2 and 3.

Two students (coded as S1 and S2) answered the first question, and one student (coded as S3) answered the second question by generating a creative conjecture because they presented a new conjecture correlated to the problem in the question (Hidayah et al., 2020). The students practising creative conjectures generate different conjectures from the existing questions, as illustrated in Figures 2 and 3.

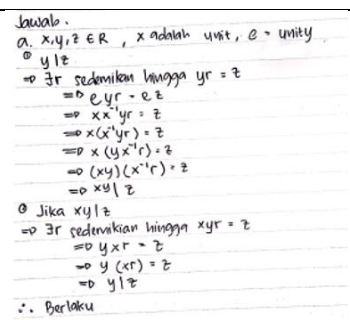
Creativity in Investigating the Converse of Proposition

Students S1 and S2 produced creative conjectures (Hidayah et al., 2020) because, in the first question, they presented "If $y \mid z$ and x unit then $xy \mid z$" and used deductive reasoning to justify the proposed conjecture. Besides, these students also made another conjecture. S1 and S2 attempted to generate the converse of the previously made propositions, as shown below.

Proposition: If $y \mid z$ and x unit, then $xy \mid z$

Converse: If $xy \mid z$ then $y \mid z$ and x unit

In the converse form, S1 and S2 eliminated or ignored the x-unit condition so that for any x, the converse still applies. This conjecture is excellent as it extends the sufficient condition of a theorem. The student's creative answer in investigating the converse of the proposition is illustrated in Figure 2.

 Jawab. a. x,y,z ∈ R , x adalah unit, e = unity ① y|z ⇒ ∃r sedemikian hingga yr = z ⇒ eyr = ez ⇒ xx⁻¹yr = z ⇒ x(x⁻¹yr) = z ⇒ x(yx⁻¹r) = z ⇒ (xy)(x⁻¹r) = z ⇒ xy|z ② Jika xy|z ⇒ ∃r sedemikian hingga xyr = z ⇒ yxr = z ⇒ y(xr) = z ⇒ y|z ∴ Berlaku	Translate: a. $x, y, z \in R$, x is a unit, e unity If $y\|z \rightarrow \exists r$ such that $yr = z$ $\rightarrow eyr = ez$ $\rightarrow xx^{-1}yr = z \rightarrow x(x^{-1}yr) = z$ $\rightarrow x(yx^{-1}r) = z$ $\rightarrow (xy)(x^{-1}r) = z$ $\rightarrow xy\|z$ If $xy\|z \rightarrow r$ such that $xyr = z$

	$\rightarrow yxr = z \rightarrow y(xr) = z$ $\rightarrow y\|z$

Figure 2. Students' Creative Conjecture in Investigate Converse of Proposition

In the first question, S1 and S2 made conjectures. Then, they used deductive reasoning to prove their conjectures, as shown in Figure 2. Students S1 and S2 were not satisfied with the answer, then they tried to investigate the converse of the true proposition in part (i). Through deductive proof, S1 found the true proposition "If $xy \mid z$ then $y \mid z$." They modified this form of converse, such as S1 omitted x as a unit in the ring R. So x applies to any element in the ring R.

Students' ability to construct different conjectures is essential. Although the justification for the conjecture is still lacking, creativity in making new propositions should be appreciated. However, the justification provided by these two students is still lengthy, and there are more effective justification steps, as written in the following.

Given that $y \mid z$, so $z = ky$ for some $k \in R$. Since x is a unit, we have

$$z = ky$$
$$z = k.1y$$
$$z = k(x^{-1}x)y$$
$$z = (kx^{-1})xy$$

S1 and S2 present excellent creativity because they explore mathematical situations within the problems to generate multiple conjectures. Their answers represent their high level of flexibility. From its first proposition, S1 generates conversion of implication for flexibility criteria. The conversion of the implication becomes a mathematic rule used by S1 and S2 in their abductive reasoning.

In addition to making two new conjectures, their implications, and conversations, S1 and S2 also provided explanations using deductive reasoning. As their explanation is valid, the conjectures from S1 and S2 are included in the novelty criteria.

Creativity in Dividing into Cases

The second creativity type in solving a mathematical problem is creativity in dividing into cases. One of our respondents, coded as S3, presented a conjecture, 'If $y \mid z$ and x unit then $xy \mid z$' and practised deductive reasoning to explain the conjecture. So, this student has a true proposition. Further, S3 discovered a piece of new knowledge through abductive and deductive reasoning in solving the problem (Folger & Stein, 2017; Nandasena et al., 2018; Żelechowska et al., 2020)

Student S3 answered the second question by dividing two cases based on primary property. S3 explored the mathematical situation in the problem and obtained the possible value c divided into two cases. The first case was c is non-primary; x is not a factor of c, while the second case was c is non-primary; x is a factor of c, as presented in Figure 3.

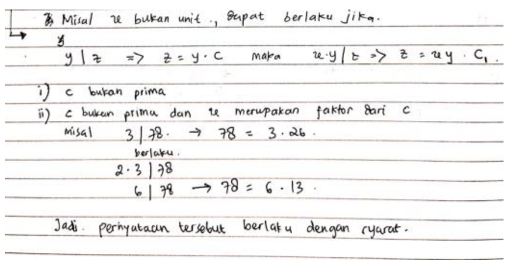
<table>
<tr>
<td>
2. Misal x bukan unit, dapat berlaku jika.
y | z => z = y · c maka x·y | z => z = xy · c1 .
i) c bukan prima
ii) c bukan prima dan x merupakan faktor dari c
Misal 3 | 78 → 78 = 3 · 26 .
berlaku.
2 · 3 | 78
6 | 78 → 78 = 6 · 13 .
Jadi. pernyataan tersebut berlaku dengan syarat.</td>
<td>Translate:
(2) if x is not a unit, is the conclusion in (1) hold?
Let x is not a unit; the statement will behold, if
$y|z \to z = yc$ then $xy|z \to z = xy$
i) c is not prime
ii) c is not prime, and x is a factor of c
For the example
$3|78 \to 78 = 3.26$
So
$2.3|78$
$6|78 \to 78 = 6.13$
So, the proposition is held if we add the sufficient condition</td>
</tr>
</table>

Figure 3. S3 Exploration of the Mathematical Situation

In S3's answer, he added the sufficient condition in the proposition, namely, c was not prime, x was not a factor of c, c was not prime, and x was a factor of c. So, S3 divided sufficient condition in two cases, namely, c was not prime, while x was not a factor of c, and c was not prime, while x was a factor of c.

So, the conjectures made by S3 are presented below,

(i) if $y|z \to z = yc$, and c is not prime, x is not a factor of c, then $xy \nmid z$;

(ii) If $y|z \to z = yc$, and c is not prime, x is a factor of c, then $xy|z$.

S3 justified the conjecture by using an example, as presented in the following.

In the example of $y|z \to z = yc$, and c is not prime, x is not a factor of c

$$y = 3, z = 78, c = 26$$

$$3|78 \to 78 = 3.26$$

There is $x = 7$ is not a factor of 26, so that $7.3 \nmid 78$

In the example of $y|z \to z = yc$, and c is not prime, x is a factor of c

$$y = 3, z = 78, c = 26$$

$$3|78 \to 78 = 3.26$$

If $x = 2$, is a factor of 26, then $2.3|78 \to 6|78 \to 78 = 6.13$

S3 justified his conjecture by using an example. In mathematics, the example is just one case from a proposition, so it can not be proof of some theorem or proposition. Further, S3 must prove his conjecture by deductive reasoning to preserve it. However, S3 creativity in generating conjectures deserves to be appreciated, though it cannot be proven.

One method for proving a theorem in mathematics is to divide it into cases. One commonly used method for verifying a statement of the implication form is breaking up the proof into several cases(Bloch, 2011; Hammack, 2013). By dividing it into cases, the process of finding proof is simplified. This method is often advantageous in splitting the problem into many minor problems (Stefanowicz et al., 2014).

From the mathematical situation presented in Figure 3, S3 obtained $z = yc$, and then S3 constructed some conjectures by dividing c into two cases. S3 generated new ideas when at least one has already been produced, so the conjecture remains to be in creative criteria. Based on the answer of S3, novelty criteria are held because S3 generates a new idea, different from other students. Creativity leads to a novel and useful outcome idea, product, or expression (Schubert, 2021).

CONCLUSIONS

One sort of abductive reasoning connected to the use of facts while solving algebraic problems is creative conjecture. In the creative conjecture, students use all the facts inside the problem to solve it. Besides, the students must know the question's meaning and use authentic facts outside the problem to solve the problem. Therefore, students develop conjectures based on facts by writing, describing, or drawing problem-solving designs and writing a new conjecture outside the question but still related to the problem in the question (Hidayah et al., 2020). In this research, three students presented creative conjecture.

In the data collection process, we provided two interrelated problems that facilitate the making of conjectures. Our obtained data showed two types of creativity in constructing conjectures. The first type is creativity in investigating the converse of the proposition, and the second type is creativity in dividing into cases. For the first type, the students made two conjectures in implication form and converse from the implication. The conjectures were 'If $y \mid z$ and x unit, then $xy \mid z$ and If $xy \mid z$ then $y \mid z'$. Further, they completed their answer with deductive reasoning to justify the conjecture. The student also divided into two cases based on the primary property for the second type. He explored the mathematical situation in the problem and obtained the possible value of c divided into two cases. The cases are c primary, and c is non-primary.

In addition, those students also presented a high level of creativity because they have fluency, flexibility, and novelty in solving open-ended problems. They explored the mathematical situation in the problem and used abductive reasoning to make some conjectures. Some students even used deductive reasoning to justify conjectures, but one student did not. Even so, they found a piece of knowledge that would be useful for increasing their creativity.

Therefore, as lecturers are obligated to enable students to think creatively, they have to provide open-ended problems that stimulate students to create conjectures. The type of reasoning that increases students' creativity is abductive reasoning. Abductive reasoning, completed by deductive reasoning, is essential to discovery learning.

This study of abductive reasoning was carried out on mathematics students who learned pure mathematics. The results would be slightly different from the studies involving junior high students with cognitive abilities that require the usage of natural objects. Therefore, future investigation is encouraged to examine the abductive reasoning of students in junior high school.

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Building on Students' Prior Mathematical Thinking: Exploring Students' Reasoning Interpretation of Preconceptions in Learning Mathematics

Robert Wakhata[1], Sudi Balimuttajjo[2], Védaste Mutarutinya[3]

[1]African Centre of Excellence for Innovative Teaching and Learning Mathematics and Science (ACEITLMS) University of Rwanda, College of Education (UR-CE), Rwamagana, Rwanda,

[2]Mbarara University of Science and Technology; Mbarara, Uganda,

[3]University of Rwanda, College of Education (UR-CE), Rwamagana, Rwanda.

rwakhata@gmail.com*, balimuttajjos@gmail.com, mutavedik@gmail.com

Abstract: *The present study explored 285 11th-grade students' preconceptions, misconceptions, and errors in solving mathematics tasks by graphical method. A descriptive-explorative study design was adopted. Cluster sampling was used to select students from sampled secondary schools in eastern and central Uganda. Students' paper and pen solution sketches together with the task-based interviews were used to identify students' preconceptions, misconceptions, and errors in linear programming (LP). Students' responses were analyzed thematically and interpreted as students' learning gaps. The results indicated that students lacked proficiency in relating basic algebraic concepts and procedures to the mathematical language with which LP is conveyed. Generally, most students could not adequately use their previous knowledge and connect it to the learning and solving of LP tasks. Besides applying wrong mathematical algebraic concepts, students had difficulties interpreting and writing correct models (inequalities) from LP word problems. Misconceptions and errors were common and peculiar to and in individual student's solution sketches especially in applying the concepts of equations and inequalities to graphically solve and optimize LP tasks. Students held extremely weak concept images of graphing equations, and inequalities and their linkage to optimizing feasible regions. This research provides insight into the learning of mathematics word problems (LP) and recommends that mathematics educators should effectively apply students' preconceptions, misconceptions, and errors as opportunities for enhancing students' LP conceptual changes. For mathematical proficiency, suitable learning approaches, methods, and strategies should be adapted to address specific individual student's flawed conceptual and procedural knowledge and understanding. These approaches may guide educators in helping students to construct the connections between the old and the new knowledge.*

INTRODUCTION

Research in the 21st-century education system on learning science and mathematics has taken a differential trend. Sometimes and most often pre-existing knowledge may aid or hinder acquisition of students' subsequent conceptual knowledge and understanding. The learning of mathematics is still more challenging for both students and for some teachers as it requires both parties to have a deeper and broader mathematical conceptual understanding of basic algebraic concepts and principles (NCTM, 2014). There exists a relationship between students' prior knowledge and understanding, and the learning of new concepts. Indeed, how teachers use students' prior knowledge and understanding to improve their pedagogy is what may define effective learning. This scenario is called teachers' pedagogical content knowledge. Research shows that assessing students' prior scientific and mathematical knowledge and understanding allows educators to plan, focus, adopt, and adapt new learning strategies (Dong et al., 2020; Otero & Nathan, 2008). To students, prior mathematical knowledge helps them to construct connections between the old and new knowledge. Thus, students' understanding of mathematical concepts can be improved by teachers reviewing prior knowledge and understanding before introducing new concepts. The implication is that students may understand better when educators review their prior knowledge and understanding, and effectively link it to subsequent learning. Students' understanding of new concepts may explicitly influence knowledge acquisition and capacity to apply higher-order cognitive problem-solving skills.

Educators can identify students' proficiency/learning gaps by reviewing their prior mathematical knowledge and understanding (Celik & Guzel, 2017). These may include pointing out previous and subsequent topics to be covered, providing lesson roadmaps, inviting reflective problem-solving tasks, and application of active learning activities like concept maps or case studies. These strategies can be implemented in small groups or for individual students taking into account their learning gaps and differences. Research (e.g., Celik & Guzel, 2017; Rach & Ufer, 2020) shows that using students' prior knowledge, understanding and experiences, may help in generating practical examples through scaffolding learning for making connections to increase knowledge acquisition and retention. Thus, educators can use students' prior knowledge and understanding to identify the learning gaps (preconceptions, misconceptions, and errors), and topical learning difficulties, justify why students are struggling and consequently correct the flawed concepts.

According to the cognitive load theory, the information learned must be held in the working memory until it has been processed sufficiently to pass into the long-term memory to acquire highly complex knowledge and skills (Kalyuga & Singh, 2016; Paas & Ayres, 2014). Therefore, the teachers' pedagogical content knowledge is potentially key in identifying students' prior conceptions and effectively using preconceptions to improve pedagogy. According to Shulman (1986), pedagogical content knowledge (PCK) refers to "the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations in a word, the ways of representing and formulating the subject that make it comprehensible to others" (p.9). Shulman argued that the teachers' PCK dimension is unique and combines pedagogical knowledge (PK) and content knowledge (CK). Thus, application of prior knowledge schemata is significant in restructuring the learning process (Galili & Goldberg, 2001).

The teachers' PCK relates to teachers' effectiveness in enhancing students' achievement and proficiency in a typical classroom learning environment. Generally, PCK involves the teachers' interpretations and transformations of the subject-matter knowledge (SMK) in the context of facilitating effective learning. Indeed, what differentiates mathematics teachers from other scientists is how the knowledge of teaching is organized and used to foster students' learning. Thus, the learning practices should be used effectively in supporting specific content (NCTM, 2014). This may involve teachers' competencies in delivering the conceptual approach, relational understanding and adaptive reasoning of the subject matter (Kathirveloo & Marzita, 2014). This knowledge component is what Hill et al. (2008) referred to as mathematical knowledge for teaching (MKT) and the mathematical quality of instruction (MQI); the unique knowledge that intersects with the specific subject teacher characteristics to produce effective and meaningful instruction. According to Hill, "teachers with weak MKT would have teaching characterized by few affordances and many deficits". Hill further noted elements for MQI as those that involve dealing with students' mathematical errors, responding to students appropriately, connecting classroom practice to mathematics in real-life, mathematical language and richness of the mathematics (p. 437).

Some empirical studies conducted in different settings and contexts (e.g., Baumert et al., 2010; Kleickmann et al., 2015; Cankoy, 2010; Halim et al., 2013; Jong, 2018) have demonstrated the significance of the above knowledge dimensions (PCK, MKT and MQI) in enhancing students' understanding of science and mathematics. In understanding the learning of mathematics and LP in particular, some studies (e.g., Kenney et al., 2020; Shikuku, 2017) show that the topic of LP is challenging and that students have limited conceptual understanding of equations and inequalities. These factors impede students' understanding of basic concepts (e.g., gradient, equations and inequalities) and its application in graphing (Dolores-Flores et al., 2021). The causes of students' learning challenges in LP are enormous and mainly stem from preconceptions of equations and inequalities. Students, on one hand, fail to understand the relationship between equations, inequalities, and LP while teachers on the other hand have not adequately applied suitable learning approaches to address the causes and sources of students' flawed conceptions (misconceptions and errors). The annual UNEB reports on students' performance support the above claim (UNEB., 2020, 2019, 2018, 2016). The above learning challenges (and other related factors) have limited students' conceptualization of LP and related concepts.

Yet, LP is applied in vast areas of science, technology, engineering, and mathematics (e.g., Dhal, 2016; Parlesak et al., 2016; Romeijn et al., 2006; van Dooren, 2018; Von Gadow & Walker, 1982; Wilen & Fadel, 2012). For instance, LP concepts are necessary for finding solutions to everyday non-routine mathematics problems and in making relevant decisions in management. Indeed, LP is a subset of operations research that aims to optimize scarce resources as a result of opportunity cost in different sectors of any country's economy (e.g., business, engineering, manufacturing, medical, etc.). In learning LP, many students come to class with previously learnt and developed concepts, ideas, principles and particular flawed ways of thinking and reasoning. Thus, the purpose of this study was to use LP mathematics tasks to explore students' preconceptions, misconceptions and errors in LP. It is expected that this research may provide insight to mathematics educators in using students' preconceptions as a springboard for devising suitable pedagogies. This can be achieved by providing varied and meaningful instruction to enhance students' conceptual and procedural knowledge and understanding. This is because students may persistently retain flawed prior knowledge even after undergoing an effective instruction.

The Conceptual Framework

This research is situated on the PCK conceptual framework based on Shulman (1986). Shulman conceptualized that "pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons" (p. 9). According to Shulman, effective learning strategies involve teachers' integration of students' preconceptions and misconceptions held previously and how these preconceptions relate to subsequent learning. In supporting students' mathematical thinking and understanding, Taşdan & Çelik (2016) developed a framework for examining mathematics teachers' PKC. The framework is related to Shulman's conceptualization of PCK, and is important in enhancing teachers' PCK (e.g., the use of graphics, manipulatives) with the main objective of understanding students' mathematical thinking.

Indeed, the above theoretical framework aligns with the five strands of mathematical proficiency. Kilpatrick, Swafford and Findell (2001) proposed a multidimensional five interwoven and interdependent strands of mathematical proficiency teachers should target during the classroom instruction. These strands are conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (NCTM, 2014). Kilpatrick, Swafford and Findell have argued "that proficiency should enable them to cope with the mathematical challenges of daily life and enable them to continue their study of mathematics in high school and beyond" (p. 116). The above five interrelated strands are inevitable for learning mathematics in the sense that they foster, support and promote students' identification and acquisition of conceptual knowledge, procedural knowledge, problem-solving skills, abilities, and beliefs. These skills are all supported by the cognitive load theory. However, educators should ask themselves the effective ways of motivating learners to represent and connect prior knowledge and understanding and effectively use it deeply and broadly during problem solving. According to NCTM (2014), students' effective learning "depends fundamentally on what happens inside the classroom as teachers and learners interact over the curriculum" (p. 8). To support the learning process, and enhance pedagogy, Stein et al. (1996) resonated that students' proficiency and competency is determined by mathematical tasks they are given. Tasks at the lower cognitive stage (memorization level), for example, must be different from those at the highest cognitive level (doing mathematics).

The Study Context

In this paper, students were engaged by prospective teachers with classroom scenarios grounded on the notion that learning and teaching are inseparable, and that students' previous knowledge and understanding are inevitable for subsequent learning. We drew research experiences from a cohort of 285 students to highlight and identify students' preconceptions, misconceptions, and errors in learning LP. The tasks provided to students exhibited model-eliciting activities (Hamilton et al., 2008; Lesh et al., 2000). In a typical classroom scenario, these tasks are perceived as those having suitable principles to adequately measure students' knowledge and understanding in LP. In particular, the tasks measured students' abilities in model formulation from word problem statements, problem-solving, and critical thinking. Thus, the tasks were engaging and challenging. Students' paper and pen responses provided feedback on their preconceptions with the main objective of enhancing pedagogy through variation of instructional practices.

The tasks were suitable for engaging the 11th grade (locally known as senior four class) students within the Ugandan lower secondary school curriculum (see Appendix 1). It was predicted that when prospective teachers provide suitable tasks to students based on their cognitive level, their critical thinking, conceptual understanding, procedural fluency and problem-solving abilities may be enhanced. This may thus address students' learning gaps. This can be achieved by teachers varying their classroom learning approaches, methods and strategies. Thus, teachers are engaged to help students work collaboratively in small groups or individually. The purpose of engagement is to help students make sense of, and apply previous mathematical ideas, principles, rules to guide them reason mathematically when learning new concepts.

This research mainly focused on the learning of mathematics and linear programming (LP) in particular. Linear programming is one of the topics taught to the 11th grade Ugandan lower secondary school students (NCDC, 2008, 2018). At this level, the graphical solution of LP problems and in two dimensions (x, y) is mostly emphasized (other methods e.g., simplex method is outside the scope of this study). Students' prior conceptual knowledge and understanding of equations and inequalities is a prerequisite for learning LP word problems. The Ugandan lower secondary school syllabus content (8th grade to the 11th grade) outlines the application of equations and inequalities in finding optimal solutions to LP problems. The 8th grade is locally referred to as senior one class while the 11th grade is senior four class, the terminal class where students sit for national examinations (UNEB). Moreover, the Ugandan lower secondary school curriculum materials are more procedural than conceptual. This partly aligns with the conceptual framework of this study. An example of a LP problem at this level adapted for this study is shown in question 1 below. This particular LP task was adapted from Ugandan mathematics curriculum materials (textbooks, teachers' reference books and past paper national examinations).

Question 1.

A Geography club in a certain school wishes to go for the field work excursion to a national park. The club hired a **mini-bus** and a **bus** to take students. Each trip for the bus had to cost Shs.**500,000** and that of a mini-bus, Shs.**300,000**. Due to Covid-19 pandemic, the bus transported **36** students and the mini–bus, **9** students. The maximum number of students allowed to go for the excursion was **216**. The number of trips the bus made did not exceed those made by the mini-bus. The club had mobilized Shs.**3,000,000** for the transportation of students.

(i) Write down five inequalities representing the above information.

(ii) Plot a suitable graph for the inequalities in (i) shading out the unwanted regions.

(iii) How many journeys should the bus and mini-bus make so as to minimize transport?

We recognize that question 1 above was challenging to students. Students spend at least 30 minutes thinking about this problem. Most of them wrote and represented different models (both wrong and some correct) on separate Cartesian coordinates, which could not yield optimal solutions. While others plotted wrong models. The teachers' responses and engagement was aimed at supporting, fostering and improving students' understanding. This paper aims to demonstrate the effect of students' preconceptions in understanding and learning LP. This task (and others) were applied to help teachers identify their PCK necessary for engaging learners for deeper and broader understanding. We drew our conclusions from teachers' responses for effective pedagogy aimed at enhancing the learning of LP. It is expected that this study will improve teachers' pedagogical strategies for learning and solving LP (and related) tasks. The above LP problem in question 1 can be symbolically written in the form:

$$\begin{aligned} &\text{min./max. } x_1 \pm x_2 \\ &s.t\ ax_1 \pm bx_2 \leq\geq c \\ &\quad dx_1 \pm ex_2 \leq\geq f \\ &\quad gx_1 \pm hx_2 \leq\geq i \\ &\quad x_1 \geq 0 \\ &\quad x_2 \geq 0 \end{aligned}$$

To adequately solve the above LP problem, the non-negative constraints $(x_1 \geq 0)$ and $x_2 \geq 0$ and the set of points satisfying the main constraints ($ax_1 \pm bx_2 \leq\geq c, dx_1 \pm ex_2 \leq\geq f$ and $gx_1 \pm hx_2 \leq\geq i$) are plotted on the same coordinate axes, the feasible region is identified at the region of intersection of the lines representing inequalities (after shading unwanted regions). The corner points of the bounded half-plane constraint set are substituted into (min./max. $x_1 \pm x_2$), called the objective function, and used for finding optimal solutions.

Methodology

The Research Subjects and their Classroom Experiences

This study involved 285 students (126 male and 159 female). The subjects were engaged in solving LP tasks (by graphical method). It was an activity-based three months mathematics interventional research study designed to investigate how students' prior knowledge and understanding could aid subsequent mathematical learning, reasoning, and problem-solving. Whereas LP may appear to be an independent topic, equations and inequalities form prerequisite conceptual knowledge for understanding this topic. This means students need to develop a better conceptual understanding of equations and inequalities before learning LP. All sampled students participated in the study. All the subjects were the 11th grade students who were preparing to write national examinations for the academic year 2020/2021. Students had varied mathematical conceptual background. To correlate paper and pen responses with students' learning experiences, focus group discussions and interviews were conducted with students and also with their respective teachers. Students were interviewed as a group and where possible individually after completing a unit on LP which lasted for four weeks with each week covering 4 hours of class periods (three times a week, 80 minutes per lesson). The sub-topics discussed in class during the learning of LP included formation of equations and inequalities from mathematics word problems (appendix 1), plotting equations and inequalities on the same coordinate axes and optimization of the feasible region.

The Research Design

This study adopted an exploratory-descriptive study design. Exploratory-descriptive designs are used to collect data in natural settings to explain phenomena from the perspective of the persons being studied (Creswell, 2014). This design was useful in summarizing and understanding student's previous mathematical thinking. It was appropriate for this study in describing and exploring students' preconceptions and their understanding of LP. This was done with the view of restructuring students' thinking by constructing meaning from equations and inequalities to adequately help students graphically solve LP and related tasks. The goal was to help students to have a complete understanding of LP and to have a critical and creative understanding of solving non-routine LP word problems by graphical method. The research subjects were all the 11th grade students from eastern and central Uganda. It was predicted that the sampled students had had prior knowledge and understanding of LP and related concepts at the time of data collection.

FINDINGS

The aim of this study was to investigate student's prior mathematical thinking and explore how their reasoning and interpretation of preconceptions can enhance the learning of mathematics. Data was coded and analyzed and discussed qualitatively following clearly defined steps (Miles et al. 2014). The mathematical content of question 1 above involving the solution of LP tasks (by graphical method) for optimizing the objective function subject to the constraints is central as outlined in the 11th grade Ugandan mathematics syllabus, developed by the national curriculum development centre (NCDC, 2018).

The notion of graphical solutions of LP problems encompasses several concepts and mathematical principles (Appendix 1) which are sequentially learned from 8th grade to 11th grade. For students' conceptual understanding, the stated LP prior concepts (Appendix 1) should be reviewed by mathematics teachers before introducing and/or learning LP. The Ugandan mathematics textbook and related reference materials have been designed procedurally. In particular, this research focused on students' prerequisite knowledge in solving LP tasks by graphical method and how teachers applied students' preconceptions knowledge component to enhance the learning of LP.

Analysis of LP Task(s)

Students were interviewed by the principal researcher assisted by four research assistants. The focus group interview (Appendix 2) with selected mathematics teachers consisted of a set of tasks arranged in a sequence, each aiding the learning and solving of LP tasks by graphical method. All audio recordings of interviews were transcribed verbatim. In each case, the selected students were also asked related tasks to write models from the given mathematics word problem, represent models (inequalities) on the same coordinate axes, shed out the unwanted regions and finally optimizing the objective region. To obtain supportive information, students were also asked to solve a series of additional tasks related to the learning of equations and inequalities and their relationship with LP. Students were finally asked to describe their conceptual understanding of LP concepts as indicated in question 1.

Students' failed to comprehend mathematics word problems, and also wrote incorrect inequalities (please see vignettes in Appendix 3). . This was the first step and perhaps the greatest hindrance to attaining optimal solutions. Most students could not mathematize LP word tasks by transforming a mathematics word problem into correct models (inequalities). There were conceptual flaws and misinterpretations across the group of students. Moreover, the preconceptions, misconceptions and errors were relatively consistent amongst the majority of individual students. This suggests that most students shared a common conceptual challenge generally attributed to the linkage between equations, inequalities, and the learning of LP. This was heavily attributed to students limited or poor command of English language vocabulary and its relationship to mathematics (exceed, at least, at most, greater than, less than, minimize, maximize, etc.). Yet, on the beliefs about access and equity in learning mathematics, the NCTM (2014) asserts that "Students who are not fluent in the English language are less able to learn mathematics and, therefore, must be in a separate track for English language learners" (p. 63). Thus, teachers are argued to promote students' meaningful learning by formulating effective approaches, methods, and techniques. This research may help educators to identify a community of resources to understand how students use contexts, culture, conditions, and language to support mathematical learning.

The coding of students' interaction was performed deductively (Kooloos et al. 2020) while the coding of interpretation and decisions was inductive. This helped to explore the teacher's interpretation of their students' thinking, and the decisions they made in solving LP tasks. Over 90 percent of students were unable to interpret, link mathematical symbols to mathematical meaning. Consequently, some students failed to write correct equations and inequalities, used wrong equations to obtain coordinates, and plotted incorrect graphs by separating instead of plotting them simultaneously on the same coordinate axes. This meant that they could neither identify the correct feasible region nor get corresponding coordinates for optimization. Some students could not decide on whether or not to shade wanted or unwanted regions. Therefore, they were unable to solve and optimize the given LP problem.

Table 1. Rubric for Classifying Strategies used in Learning Linear Programming

Correct use of the Strategy	Partial use of the Strategy	Limited or no use of the Strategy
5 points	3 points	1 point
Evidence of students' insightful thinking in LP question interpretation and problem exploration.	Limited clarity on question interpretation and writing some wrong models.	There is completely limited students' understanding of LP word statements from which models are written.
Learner's procedures for solving LP problems are all clear and focused.	Learners apply some correct strategies which yield some partial correct answers.	The procedures have no relationship with the questions asked
Appropriate strategies are applied and demonstrate a students' understanding and thinking of basic concepts.	The learner starts the problem appropriately but fails to apply some of the basic algebraic concepts for solving LP problems, and loses focus.	The learner does not completely understand the basic LP concepts to be used to fully explore the problem at hand linking to the strategies.
The learner portrays a clear understanding and provides extensions or generalizations of procedures and possible strategies to the solution of the given problem.	Learners may recognize procedures, and/or patterns but fail to correctly identify relationships hence applying concepts incorrectly.	The learner portrays incorrect or no understanding of concepts and steps, hence providing no extensions or generalizations of procedures and possible strategies to the solution of the given problem.

From Table 1 above, 156 students mixed up the concepts by shading unwanted regions on separate graphs. At least 80 percent of students plotted graphs of inequalities instead of equations while others interchanged the axes (x and y respectively) and could not adequately interpret the feasible region. Students' inability to identify the feasible (accessible) region, identify integral values, and use the objective function to optimize the stated LP problem was the major hindrance. Thirty-five students completely failed to identify the feasible region even after writing the other correct procedures. Fifty-six students had correctly represented equations (transformed from inequalities) from word problem statements but failed to plot them and/or identify the points of intersection, x-intercepts and y-intercepts from graphs. Thus, these particular students wrote wrong coordinates and hence wrong optimal solutions. All these challenges partly stemmed from students' English language deficiency. Below is an extract of focus group interview transcripts conducted with some students.

Students' Argumentation on the Learning of Linear Programming

During the interview process, students expressed their conceptual knowledge and understanding, elaborating the prior knowledge they held. The teachers together with the research team engaged students to understand their underlying concepts. In these transcriptions, the selected students struggled to apply prior knowledge in writing correct models to represent word statements. The students wrote wrong or incomplete models since they could not mathematize statements related to the learning of LP. This provided an opportunity to explore students' understanding and effectively use it to build on their thinking. In some instances, however, the teachers confirmed and where necessary, corrected wrong models to help students plot correct graphs. The following is an excerpt of selected students' preconceptions.

Interviewer: Read for me question 1.

Student: She reads fluently.

Interviewer: interpret and summarize what the question is all about?

Student: Sir, some words are confusing.

Interviewer: Which words are confusing?

Student: Number of trips the bus make did not exceed those made by the mini-bus.

Interviewer: Okay! Which symbol correctly represents this statement?

Student: Sir, it is $<$

Interviewer: Why not $>, \leq or \ \geq$?

Student: Because they should not exceed.

Interviewer: Okay! Now write all the five inequalities.

Student: $x \geq 0;\ y > 0;\ 500{,}000x + 300{,}000y = 3{,}000{,}000; 36x + 9y \geq 216\ and\ x < y$

Interviewer: Are all the above inequalities, correct?

Student: Yes sir.

Interviewer: May you please tell me the meaning of using the inequality symbols$<, \leq, >, \geq$?

Student: Sir, they confuse me. They mean more than and less than when you read and interpret tasks.

Interviewer: How will you get coordinates from this inequality $500{,}000x + 300{,}000y = 3{,}000{,}000$?

Student: Sir, you just remove all zeros.

Interviewer: Okay, write the final inequality

Student: $5x + 3y = 3$

Interviewer: Using the above inequality, get your coordinates to be plotted.

Student:

x	0	0
y	1	0.6

Interviewer: How do you plot the pair of coordinates (0, 0.6)?

Student: Sir, you get the scale.

Interviewer: How do you plot coordinate for all the five lines?

Student: You plot each line separately on the graph and shade unwanted regions

Interviewer: now, how do you get the feasible region?

Student: Okay, after plotting separate lines, you now combine and plot all lines on the same graph, shade outside and leave the middle part as the feasible region.

Interviewer: Is the feasible part always in the middle after shading the outer side of each equation?

Student: Of course, sir. Our teacher told us to do so.

Interviewer: Now, how do you minimize transport?

Student: You get some coordinates from the middle of the feasible region and substitute them in the equation.

Interviewer: Which equation?

Student: Sir, it is not given in this question.

From the above verbal transcription, we can infer some characteristics of students' common conceptions and misconceptions arising from the interview transcripts (see vignettes in Appendix 3). Students were asked to write the procedure for optimizing a LP word problem (see question 1) and explain the process of optimization. It was evident that most students had flaws in answering LP problem 1 and other related LP tasks. However, the students' responses to the above problem provided evidence for exploring the best instructional practices for enhancing the learning of LP and related concepts. Thus, teachers must be aware of students' prior preconceptions, misconceptions and errors in terms of their beliefs and incomplete understandings that may directly or indirectly conflict with subsequent learning. Teachers should also create an enabling platform and circumstances where students' LP learning gaps can be externalized, expressed and discussed explicitly.

DISCUSSIONS

This study applied a descriptive-explorative study design to explore 285 11th-grade students' preconceptions in solving LP tasks by graphical method. From the above findings, it is evident that students faced learning challenges in LP. This perhaps explains the practicability of LP in our daily lives outside the classroom environment. It reveals students' conceptual and learning gaps in applying mathematics in solving societal problems. This research acknowledges the fact that LP is one of the challenging mathematics word problems topics. Thus, educators should teach LP sequentially for students' conceptualization. From the above research findings, and to effectively teach this topic, teachers need to teach from simple to complex and from concrete to abstract to build on the pre-existing mathematical ideas and principles. Students may be guided to develop and understand challenging and more complex mathematical concepts. There is need for prospective teachers to include students' preconceptions during lesson preparation and instruction to minimize misconceptions and consequently mathematical errors. Lastly, the competence-based curriculum (CBC) recently introduced in Ugandan lower secondary school curriculum emphasizes that teachers should innovatively by providing students with suitable learning materials so that they use them to observe and make practical experiments. The use of learning materials develops and improves students' generic skills. Inclusion of students' preconceptions, misconceptions and errors exposes their potential, weaknesses and learning gaps which can be corrected during instruction (Kooloos et al., 2022). According to NCTM (2014), scientific explanations backed with clarity and, with possible, examples lead to better conceptual understanding. Teachers should, therefore, allot adequate time with individual commitment to engage, explore, explain, elaborate and evaluate their learning objectives. This is a restructured and blended mode of instruction to counter students' prior misconceptions and errors. To achieve this, learning should be designed in such a way that mathematics topics are taught in a sequence and that the content should be taught deeply and broadly than covering many topics in a superficial manner.

Evidence from this study indicates that students failed to link prior conceptual knowledge and understanding to the learning of LP concepts which consequently affected subsequent learning. It can further be noted that some students had not adapted the learning environment at the time of data collection. Some students who had joined the sampled schools during their 10th and 11th grades could not fully solve LP and related tasks. Thus, to such students, the knowledge they had acquired cannot be represented as a well-defined intermediate state. This research adds to other empirical findings (e.g., Connell, 2015; Kooloos et al., 2022; Nelson, 1992), and suggests that teachers should develop a strong restructuring between pre-instruction, instruction and evaluation of students' learning outcomes. The teachers' conceptions of the relevant mathematical concepts may play a significant role in interpreting and correcting students' flawed concepts. Prospective teachers should inculcate these learning principles by enhancing their mathematical knowledge for teaching and mathematical quality of instruction. This may help to overcome students' challenges arising from preconceptions, misconceptions and errors. The research conducted by Kooloos et al. (2022) on teachers' orientations toward using student mathematical thinking as a resource during whole-class discussion shows that teachers might be supported in their novice attempts at whole-class discourse by explicitly discussing students' conceptions and the learning gaps.

Comprehension of LP word problems accounted for students' misconceptions and errors. Fifty-four percent of students made these errors (see vignettes in Appendix 3). Comprehension errors varied based on students' previous academic abilities, background and preconceptions. At least 70% of students had difficulty understanding LP questions. Due to students' inability to interpret mathematical word problems, wrong inequalities were written. Consequently, erroneous models were utilized to represent and solve the LP task (question 1), and this resulted in inaccurate solution sets. This was mainly attributed to students' limited understanding and proficiency in English language and its relationship to the interpretation and application of basic mathematical principles. The interviews conducted demonstrate a link between students' English language development, mathematics background, and the learning environment, all of which hampered students' knowledge acquisition.

Students failed to translate LP mathematical word statements from English to their individual "mother tongue" languages and vice versa, as well as the connection to symbolic representations. This contributed to most of the misconceptions and errors in LP. The findings are in agreement with Pongsakdi et al. (2020), and, Makonye and Fakude (2016) findings since students are constantly confronted with linguistic challenges, relational challenges, and interactions between linguistic and symbolic representations. Generally, students had limited mathematical vocabulary as evidenced by their incorrect use of mathematical symbolism ($<$, $>$, $\leq$, and $\geq$) representing statements conveyed in the question. Other related phrases wrongly interpreted included "at least, at most, greater than, "less than," "minimize," "maximize, "feasible" etc." In contrast to their peers in urban settings, these errors were seen more often in students' paper and pen works from those studying from rural secondary schools.

At least half (50%) of students were unable to link mathematical symbols, variables, constraints, mathematical operations, algebraic expressions to mathematical language, and the solution of inequalities and LP in particular (see students' vignettes in Appendix 3). Students' inability to consistently use symbols ($=$, $<$, $>$, $\leq$ and $\geq$) to represent numbers rather than objects and expressing them in mathematical sense contributed to most misconceptions and errors. Many students who were interviewed could not distinguish the mathematical interpretation of the symbols $\leq$ from $\geq$, relating to "at least" and "at most," respectively. Similarly, the traditional sense and usage of $<$and $>$ was perplexing, and this led to wrong interpretations and solution sets. Students repeatedly failed to make clear distinction between the symbols ($<$, $>$), and ($\leq$, $\geq$), and their contextual applications. Students' understanding of the graphical solution of LP problems was hampered by this mystery. Many of them could not write correct equations from inequalities or could not correctly plot equations on the x-y plane.

Over 80% of students were unable to appropriately represent the specified inequalities (albeit some inequalities were incorrect) on the Cartesian plane. In addition, inequalities that were incorrectly written could not yield optimal solutions (see vignettes in Appendix 3). The majority of students (99% of low achievers) failed to use the graphical approach appropriately in finding the feasible region. To some students, it was debatable whether or not to draw graphs equations or inequalities. The graphical solution to LP problems became more complicated in the case where two or more inequalities were to be plotted on the same coordinate axes. Students instead plotted graphs on different coordinate axes without taking into account their intersection in this scenario. Thus, they could not identify the feasible region. This was due to the repetitious procedures that some students found difficult to comprehend. Students' inability to shade the correct side of the graph to obtain the feasible area resulted in a mix of graphical and arithmetic difficulties. Furthermore, some students were unable to transform inequalities to equations and vice versa prior to obtaining the appropriate coordinates for plotting on coordinate axes. Students plotted the graphs of inequalities instead of equations, illustrating a mismatch between inequalities and equations. To illustrate this concept, students plotted the graph of the inequality $y \geq x\text{-}a$, instead of $y = x\text{-}a$. This means the Cartesian coordinates could then be obtained using the equation $y = x\text{-}a$ before sketching the graph of $y = x\text{-}a$.

Procedural difficulties were the main cause of skill manipulation. The failure by some students to understand LP word problems, as well as their inability to reason analytically and insightfully, had an impact on the entire process. At least 60% of students struggled with the option of whether or not to draw dotted and solid lines to express inequalities with mathematical symbols $<$ or $>$, and $\leq$ or $\geq$ respectively. For question 1, 72% of students failed to correctly shade unwanted regions and isolate the feasible region (see vignettes in Appendix 3). Between numeric and symbolic representations, there were many misunderstandings and overgeneralizations. It is possible that this occurred owing to a lack of prior knowledge of equations and inequalities and their relationship to the learning of LP. When it came to identifying the feasible area, which is defined as an intersection of at least two constraints with relational symbols of $<$, $>$, $\leq$ or $\geq$, there were numerous observable inconsistencies, all of which led to wrong optimal solutions.

Even after defining the feasible region correctly, some students (42%) were unable to use the objective function to find the numerical optimized solution to question 1. As a result, the stated numerical answers were wrong. It is possible that students found the reversal of this procedure more baffling or intimidating. Research by Botty et al., (2015), Kenney et al., (2020) and Tsamir and Almog (2001) supports the research findings. Furthermore, some students who correctly identified the feasible region were unable to write correct integral and/or "corner coordinates" from the feasible region which consequently led to writing wrong models (equations and inequalities). Yet, the correct objective function was to be written for students to optimize the LP problem 1. Correct coordinates were to be extracted and appropriately used in writing models. It was, however, observed that erroneous graphs were plotted in some cases resulting in inaccurate feasible regions.

Students also made careless mistakes (Appendix 3). As mentioned earlier, this could have stemmed from poor command of English language. This mostly affected low and average achievers who were unable to write correct inequalities from word problem statements. Although high achievers also made these types of errors, they were not common. Approximately 18% of high achievers were unable to correctly use the scale and align the axes. Consequently, some students failed to write the necessary corner coordinates (integral coordinates) for optimization or derive suitable coordinates from equations for plotting graphs. Inconsistencies in shading wanted regions instead of unwanted regions, interchanging the axes (x and y, respectively), failure to distinguish points of intersection, x-intercepts, and y-intercepts from graphs, and failure to obtain coordinates from equations and/or inequalities and their inability to map them were major challenges. Students' failed to interpret the graph and accurately answer the tasks as portrayed in the question. During face-to-face interviews, some students couldn't interpret and connect their cognitive schema to the given LP problems. When representing dotted and solid lines for the graphs of $y \geq x-4$ and $y>-3x$, some students couldn't distinguish the difference between the processes and relational inequality symbols. Specifically, some students found it difficult to graphically represent equations of the form $y=a$ and $x=a$, hence failing to plot the graph of $y>-3x$. To the majority of students, it was difficult to relate $y=a$ and $x=a$ in representing a horizontal line or a vertical line.

Conclusion

The purpose of this study was to use LP mathematical tasks to explore 285 grade 11 students' preconceptions, misconceptions and errors in learning the topic of LP. The findings show that students generally held weak and flawed algebraic concepts (equations and inequalities). Students failed to link prior conceptual knowledge and understanding of equations and inequalities to the learning of LP concepts. Comprehension of LP word problems accounted for most of the students' misconceptions and errors. Students failed to translate LP mathematical word problem statements. They could not link mathematical symbols, variables, constraints, and mathematical operations, algebraic expressions to mathematical language, and the solution of inequalities and LP (see students' vignettes in Appendix 3). Students struggled to distinguish equations from inequalities and their relationship to the learning of LP. Some students failed to represent inequalities graphically. Thus, they could not identify the feasible region. To some students, even after defining the feasible region correctly, they failed to use the objective function to find the numerical solution. These (and related factors) at large limited students' understanding of and finding suitable solutions to sampled LP problems. The findings provide practical insight in using students' preconceptions to aid subsequent learning. This was an exploratory study with inherent limitations. Future researchers in different settings and contexts may apply quantitative and mixed methods approached to compare and contrast the stated research findings. In conclusion, adopting and consistently applying students' preconceptions in learning mathematics can be a key ingredient and factor in identifying students' flawed learning gaps which is aimed at achieving students' mathematical proficiency. It is likely that using suitable LP and related mathematics tasks in the typical classroom and teacher training education contexts can be employed to:

1. Identify and explore mathematical, didactical, and pedagogical issues. There is need for student engagement that may trigger students' reflection of preconceptions about the learning of mathematics and LP in particular.
2. Apply LP tasks as opportunities for preparing and addressing students' learning challenges.
3. Apply LP tasks to develop teachers' pedagogical content knowledge and their awareness in addressing students' flawed concepts.
4. Use students' preconceptions, misconceptions and errors as opportunities for addressing and developing teachers' continuous professional development programs.
5. Applying mathematics in real-life scenarios beyond the compulsory level at the 11th grade.

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Appendix 1

Eleventh Grade Students' Content Understanding and Key Mathematical Ideas for Learning the Graphical Method of Solving Linear Programming Problems

Content (with Related Examples) for Solving LP Tasks	Key Mathematical Ideas for Learning LP in Secondary Schools
1. Representation of inequalities on a number line and writing down the solution set. e.g., Using a number line, find the integral values of x which satisfy	▪ Correct use of a number line and related symbols. ▪ Knowledge of equations, i.e., the gradient of a straight line, negative gradient, positive gradient, finding the equation of the straight line in the form $y = \pm mx \pm c$, $y = \pm mx, y = \pm m, x = \pm m$, etc.

the sets: $\{3x > 2x + 5\} \cap \{3x < 32 - x\}$ 2. Solving linear inequalities algebraically (non-graphical). e.g., Solve: $\frac{x-2}{4}$ - $\frac{x-2}{3}$ < 1	▪ Solving linear equations correctly including the use of lowest common multiples, simplifying fractions, etc. ▪ Correct use of set notations e.g., $\cup, \cap, \emptyset, \in$, etc. ▪ The distinction between solving equations and inequalities. ▪ Correct use of symbols in solving linear equations.
3. Solving quadratic inequalities algebraically (non-graphical). e.g., Solve: $3x^2 + 7x - 20 < 0$	▪ Identification of correct factors (critical points). ▪ Distinguishing critical values from solutions to the quadratic inequality. ▪ Review of the methods of solving quadratic equations. Showing students differences between the following graphs $y = \pm ax^2, y = \pm x^2$, $y = \pm ax^2 \pm bx, y = \pm ax^2 \pm bx \pm c$, $y = (x \pm p)^2$, $y = (x \pm p)^2 + k$, $y = \pm a(x \pm p)(x \pm p), = \pm(x \pm p)(x \pm p)$ etc. substitute = with symbols like $>, <, \geq$ and $\leq$. Students' knowledge of $b^2 - 4ac = 0$, $b^2 - 4ac \leq 0$ and $b^2 - 4ac \geq 0$ ▪ Consistent use of mathematical symbols ▪ The distinction between inequalities with equations. ▪ Review of representation of equations and inequalities on the coordinate axes.
4. Solving linear inequalities graphically e.g., Show by shading unwanted regions the region satisfying the inequalities: $x + y \leq 3$, $y > x - 4$, and $y \geq -3x$	▪ Obtaining correct coordinates from the given equations. ▪ Distinguishing dotted from solid lines based on mathematical symbols $<, >, \leq, \geq$. ▪ Identification of the correct feasible region. ▪ Correct use of scales on the coordinate axes. ▪ Decision on whether to shade wanted/unwanted regions. ▪ Identification of correct coordinates for plotting and testing the objective function to find the feasible region. ▪ Plotting graphs of equations instead of inequalities.

5. Solving simultaneous inequalities graphically. e.g., On the same coordinate axes, draw the curve y= $4x^2$ and the line y = 1. Show by shading unwanted regions, the region represented by: y > 1 and y < $4x^2$. Hence, state the integral coordinates of the points which lie in the region $\{y > 1 \cap y < 4x^2\}$	▪ The distinction between graphs of curves and straight lines. ▪ Correct use of mathematical symbolism when representing the coordinate axes. ▪ Failure to obtain coordinates from equations. ▪ Obtaining correct coordinates from the feasible region. ▪ Obtaining correct integral values and convex points.
6. The graphical solution of a LP problem. e.g., A school has organized a geography study tour for 90 students. Two types of vehicles are needed; Taxis and Costa buses. The maximum capacity of the taxi is 15 passengers while that of the Costa bus is 30 passengers. The number of taxis will be greater than the number of Costa buses. The number of taxis will be less than five. The cost of hiring a taxi is Shs.60,000 while that of the Costa bus is Shs. 100,000. There is only Shs. 600,000 available. (a) If *x* represents the number of taxis and *y* the number of Costa Buses, write inequalities for the given information. (b) Represent the inequalities on the graph paper by shading the unwanted regions. (c) Find from your graph the number of taxis and Costa Buses which are full that must be ordered so that all the students are transported?	▪ Students' comprehension of mathematical word problems, translating and writing correct inequalities from mathematics word statements. ▪ Students' ability to link mathematical symbols, variables, constraints, operations, algebraic expressions into mathematical language. ▪ Correct use of scales when representing equations on the coordinate axes. ▪ Writing the correct models and the objective function for optimization. ▪ Students' ability in solving equations simultaneously by graphical means. ▪ Review of basic mathematical vocabulary (at least, at most, greater than, less than, minimize, maximize, etc. ▪ Emphasize finding the correct feasible region. ▪ Identification of correct coordinates for optimization. ▪ Decision on shading unwanted regions and leaving out wanted regions. ▪ Adequate knowledge of the graphical solution of linear and quadratic equations simultaneously. ▪ Plotting graphs of inequalities, not equations. ▪ Correct labeling and use of axes (x and y respectively). ▪ Identification of correct points of intersection, x-intercepts, and y-intercepts from the graph.

(d) Find the minimum and maximum cost of transporting 90 students?	▪ Ability to obtain coordinates from equations, and correctly plotting them on the same coordinate axes. ▪ Identifying the correct feasible region, obtaining integral values, and optimizing the coordinates. ▪ Ability to write correct equations, inequalities from the given feasible region. ▪ Interpretation of optimization terms (maximum or minimum), this leads to correct or incorrect substitutions and numerical values. ▪ Ability to use symbols to represent numbers, not objects.

Appendix 2

Dear participant,

You are requested to answer the following questions related to the effective learning of linear programming. This interview is likely to take approximately 45 minutes. It is expected that the answers arising from our interaction are aimed at improving the learning of linear programming in secondary schools.

1. What is linear programming within the context of secondary school mathematics?
2. What specific students' prior knowledge is necessary for the learning of linear programming?
3. Differentiate between equations, inequalities, and linear programming.
4. What are the challenges of teaching linear programming as compared to other topics?
5. What are the causes of students' challenges in learning linear programming?
6. When can the learning of linear programming concepts be introduced?
7. What techniques and procedures do you apply when teaching linear programming?
8. How do you make students understand these techniques and procedures?
9. Do these techniques and procedures become more complex as students' progress?
10. What learning materials are used to help students grasp linear programming concepts?
11. What steps do you consider inevitable for students to understand linear programming?
12. What procedures are followed when introducing (teaching) linear programming concepts?
13. Which aspects of linear programming are most problematic for students to comprehend?

14. How do you help your students to overcome the challenges of learning linear programming?
15. How do you identify students with learning challenges in linear programming?
16. What else are you doing to help students understand linear programming concepts?
17. In your own opinion, what should be done to improve the learning of linear programming?

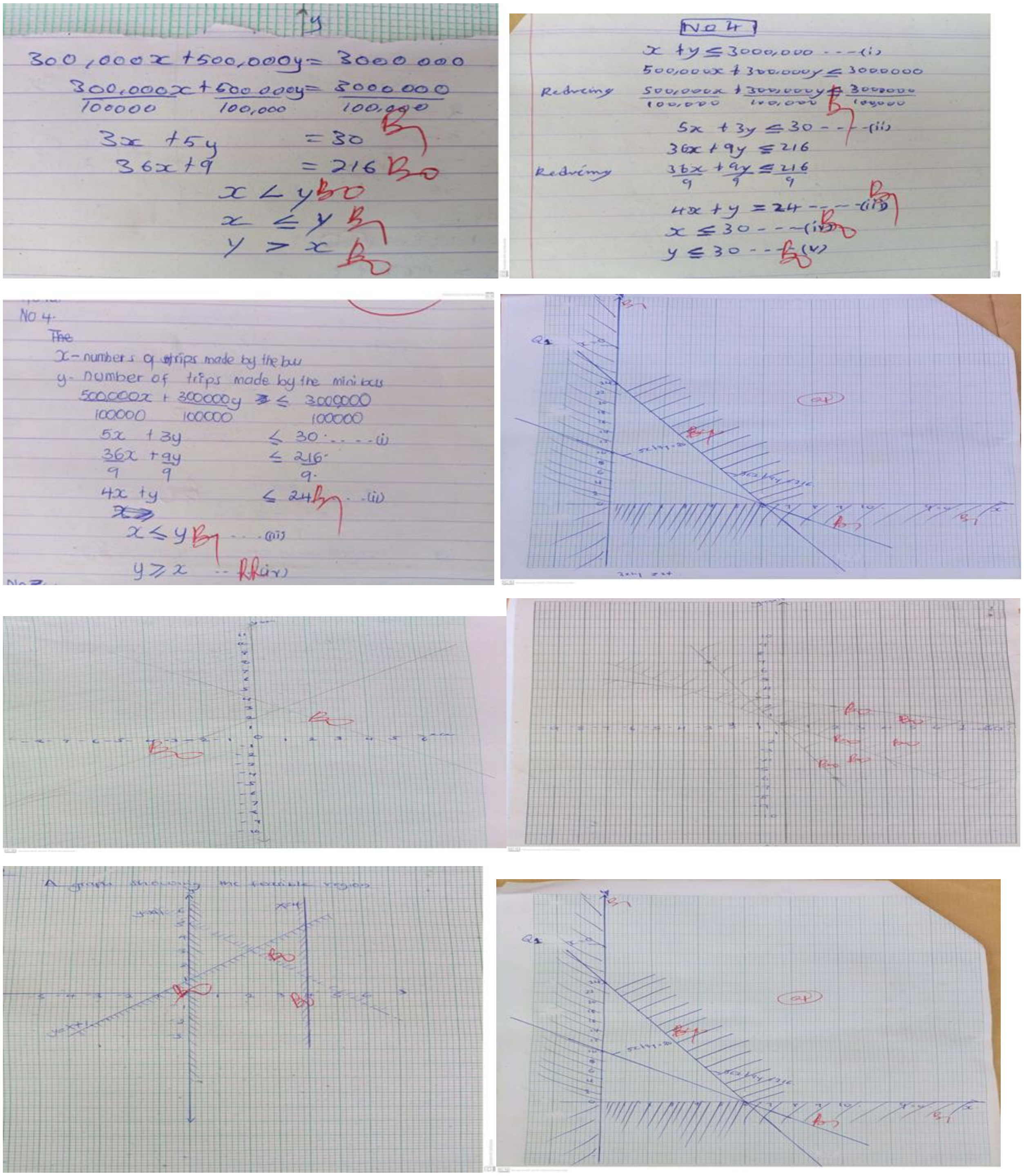
Looking at question 6 in Appendix 1:

a. What concepts are necessary for illustrating and solving this question?
b. How many standard marks can be allocated for this question?
c. How can the concepts be broken down to help students answer the question effectively?
d. Briefly explain how you will allocate marks for this particular question.
e. Solve this question the way you would expect students to solve it.
f. What concepts are easily grasped by students in this question?
g. What concepts limit subsequent learning of this question?
h. Some teachers say this question is usually hard for most students. What do you say?
i. Some teachers say this topic is challenging to introduce and teach. What is your view?
j. Do students elude LP questions during national examinations? If yes, why?

Thank you for your feedback.

Appendix 3

Students' Vignettes on LP Model Formulation and Graphing Abilities



Primary School Students' Understanding of Four Operation Symbols (+, −, x, ÷, =) and Using Them in Arithmetic Operations and Word Problems

Halil Önal

Burdur Mehmet Akif Ersoy University, Türkiye

halilonal@mehmetakif.edu.tr

Abstract: The aim of this study is to determine the primary school fourth-grade students' understanding of four operation symbols, using them in arithmetic operations and word problems. Phenomenology research design, one of the qualitative research methods, was used in the research. The participants of the research consist of 123 students, attending the fourth grade of primary school (9 years old). The participants of the study were determined by using convenience sampling, The "four operation symbol study form" developed by the researcher was used as a data collection tool. Content analysis method was used in the analysis of the data. According to the results of the research, it was determined that the students formed meanings in different categories for the meanings of symbols, they were more successful in writing arithmetic operations by using symbols than in solving arithmetic operations. It was concluded that the students showed lower success in writing and solving word problems by using symbols compared to writing and solving arithmetic operations. It was determined that the lowest success in the use of symbols in arithmetic operations and word problems was in the use of the "=" symbol.

INTRODUCTION

Mathematics can be considered as a way of thinking or a language, but the challenge for many people is what it usually looks like when it is written. As mathematics develops, communication tools become more concise through words, symbols and diagrams (Cooke, 2007). Making sense of mathematics is possible by structuring language symbols and concepts in students' minds (Yalvaç, 2019). The use and interpretation of mathematical symbols begin very early in school life, with arithmetic symbols forming the basis of most of the mathematics learned today (Angliheri, 2005). Mathematical signs and symbols have a decisive role in the coding, constructing and communicating of mathematical knowledge. Mathematical concepts are constructed as symbolic relational structures and encoded through symbols that can be logically combined in mathematical operations (Steinbring, 2006; Goldin, 2020). A symbol is needed to make it easier for students to recognize concepts and to express a concept.

Mathematical symbols (0, 1, 2, 3, +, –, <,>, %, etc.) are important tools used in the transmission of mathematical knowledge (Olkun & Toluk Uçar, 2018). It is impossible for us to manipulate concepts visually without symbols. The symbols of mathematics make it possible for us to discover and express relationships between various concepts. For example, when we write down the statement 4 +2 = 6 with symbols, we actually refer to the relationship between the concepts of four, two, six addition and equality (Haylock & Cockburn, 2014). Symbols form the basis of mathematical communication by conveying meanings and messages (Angliheri, 2005; Bardini & Pierce, 2015). Symbols also help to mediate thinking about mathematical concepts, enabling operations on concepts (Gray & Tall, 1994). According to Angliheri (2005), it is very important for children to be able to understand what the teacher says and how it relates to the symbols they use for calculations they see on a page, to understand the relationships that exist between numbers and the operations we use on numbers.

Since symbols may have different meanings depending on the mathematical context, the use and interpretation of symbols may not be easy for students. Students often have difficulties in attributing meaning to mathematical symbols (Powell, 2015; Powell & Driver, 2015). Although young children can identify and write symbols, this does not reflect an understanding of the mathematical meaning of symbols or their relationship to numbers (Ilany & Hassidov, 2018). In mathematics, there is common use of keywords denoting the four operations (addition, subtraction, multiplication, and division). Since mathematics includes symbols and signs, the inability to distinguish between them for visual reasons may hinder learning. For example; the symbols "<, >, +, –, x, ÷, =" may cause confusion between the numbers 2 and 4, 6 and 8 (Patkın, 2011). One of the main challenges at this stage of learning mathematics concerns children's misconception of the symbols used (+, −, x, ÷, =) as a result of their erroneous experiences (Baroody & Standifer, 1993). It is very important to teach the correct mathematical use of symbols from the pre-school period (Angliheri, 2005; Hassidov & Ilany, 2017).

As students progress from primary school to university in mathematics education, as a result of the discontinuity in the use of symbols, unlimited expansion, increased complexity in symbol load and not being familiar with the meanings of symbols, students lose their self-confidence in mathematics and may choose a way of study that minimizes their need for mathematics (Bardini & Pierce, 2015). Students must be able to decipher the meanings of symbols in order to effectively read mathematics (Adams, 2003). Students construct their algebraic understanding by associating them with their understanding in arithmetic (Akkan, Baki & Çakıroğlu, 2012; Yıldız & Atay, 2019). It is important to determine how the primary school senior students, who will encounter the field of learning algebra in the secondary school period and where arithmetic is mainly at the centre of mathematics teaching, make sense of the four operation symbols, their use in arithmetic operations and their use in word problems. It is important to determine the pre-knowledge of students regarding the meaning and use of the four operation symbols. For these reasons, the aim of this study is to determine the primary school fourth-grade students' understanding of four operation symbols, using them in arithmetic operations and word problems.

METHODS

The phenomenology design, one of the qualitative research methods, was used in this study, which aimed to get the thoughts of the fourth-grade students about their experiences, how the symbols of the four operations were interpreted by the students, how to write and solve arithmetic operations by using symbols and how to write and solve word problems by using symbols. The research was guided by the following questions.

Research questions

1. What are the opinions of the 4th-grade primary school students regarding the meanings of the four operation symbols (+, −, x, ÷, =)?
2. How are primary school 4th-grade students in writing and solving arithmetic operations by using the four operation symbols (+, −, x, ÷, =)?
3. How are primary school 4th-grade students in writing and solving word problems by using the four operation symbols (+, −, x, ÷, =)?

The participants of the research consist of 123 students (65 girls and 58 boys), attending the fourth grade (9 years old) in two different public primary schools in Ankara, Türkiye, in May and June of the 2021-2022 academic year. The schools of the participants are public schools located in the settlement where the families live at the middle socio-economic level. The participants of the study were determined by using convenience sampling, which is one of the purposive sampling methods.

In the research, the "four operation symbol study form" developed by the researcher was used as a data collection tool. The study form prepared to determine the level of understanding of the four operation symbols (+, −, x, ÷, =) of primary school fourth-grade students consists of three parts. In the first part of the data collection tool, there are questions to determine how the four operation symbols are interpreted by the students, in the second part, how to write and solve arithmetic operations by using four-operation symbols, and in the third part, there are questions to determine how to write and solve word problems by using four-operation symbols. The study form prepared by the researcher was sent to 1 mathematics education specialist and two classroom teachers, and expert opinion was applied to ensure the content validity. In line with the expert opinions received, the question sentences were rearranged by making arrangements regarding the suitability of the student level and supporting them with appropriate visuals. The reorganized study form was applied to 26 fourth-grade students studying in another primary school, which were not among the participants of the research, and a pre-application was made. As a result of the pre-application, it was determined that the questions were understood by the students, could be answered and were suitable for their level. The final form of the study form is given in Figure 1.

Symbol	Questions	Answers	Symbol	Questions	Answers	Symbol	Questions	Answers	Symbol	Questions	Answers	Symbol	Questions	Answers
	1.What does this symbol mean?			1.What does this symbol mean?			1.What does this symbol mean?			1.What does this symbol mean?			1.What does this symbol mean?	
+	2.Write an arithmetic operation by using this symbol and find your result.		−	2.Write an arithmetic operation by using this symbol and find your result.		X	2.Write an arithmetic operation by using this symbol and find your result.		÷	2.Write an arithmetic operation by using this symbol and find your result.		=	2.Write an arithmetic operation by using this symbol and find your result.	
	3.Write a word problem by using this symbol and find your result.			3.Write a word problem by using this symbol and find your result.			3.Write a word problem by using this symbol and find your result.			3.Write a word problem by using this symbol and find your result.			3.Write a word problem by using this symbol and find your result.	

Figure 1: Four operation symbol study form

The administrators and teachers in the schools where the application was carried out were informed by the researcher about the purpose of the study and necessary permissions were obtained. The necessary consent was obtained from the participants and their families to participate in the study. The data collection tool was applied to the fourth-grade primary school students during the mathematics lesson in their classrooms, and the students were informed about how to answer the questions in the data collection tool and what should be considered. Students were given an average of 30-35 minutes to answer the questions in the study form individually.

The data obtained through the four operation symbol study forms were analyzed with the content analysis method. Content analysis is an analysis method that focuses on how many times measurement units such as a particular speech pattern or phrase are used, summarizing some words of a text with smaller content categories with coding based on certain rules (Merriam & Tisdell, 2015).

The written answers given by students to the study form were arranged by labelling as S1, S2,…,S123 (S: students). The edited data (the answers) were coded by applying content analysis, then the similar codes were arranged in line with the emerging concepts, and the concepts were brought together under categories based on frequency and significance analysis. Using the four operation symbols of the students, arithmetic and word problems were digitized as the numbers of students who were successful, were not successful and were partially successful. Randomly selected samples from student worksheets were analyzed at different times and similar results were obtained to ensure reliability. In the study, the data were analyzed independently by the researcher and a mathematician and the researcher diversification was used, thus reliability was tried to be ensured. The list made by the researcher and the list prepared by the specialist were compared and the consistency of the results was calculated by applying the kappa statistics. Kappa statistic value (κ) was found to be 0.843. The fact that the Kappa statistic value (κ) is between 0,81 - 1,00 indicates that the level of agreement is very high (Landis & Koch, 1977 as cited in Bilgen & Doğan, 2017). In order to ensure the credibility of the analysis results of the data, the photographs from the raw data sources were included in the findings section.

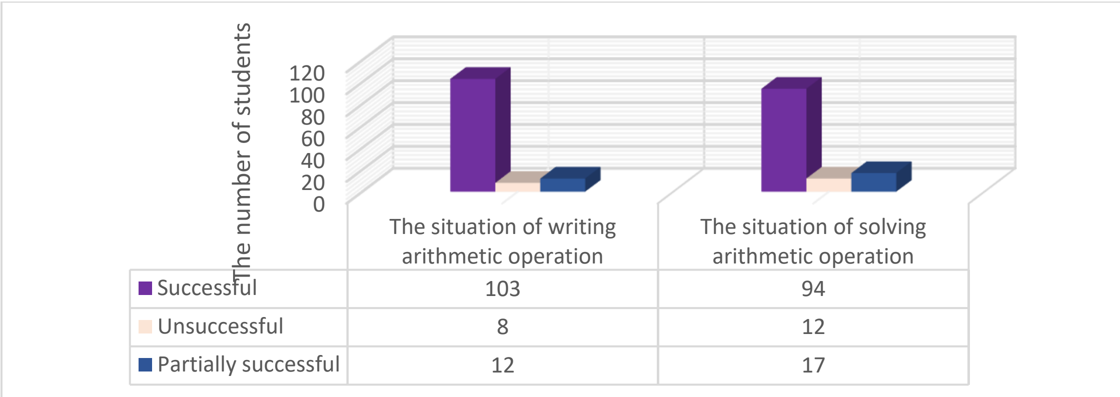
RESULTS

In this part of the study, the findings that emerged in line with the data obtained were presented in the form of tables, graphs and student response examples. The views of the fourth-grade primary school students on the meaning of the "+" symbol were given in Table 1.

Category	Code	f
Addition sign (43)	It's an addition sign	39
	It is the sign that we use to collect things	3
	It is the addition sign used when adding in problems	1
Addition (36)	It is used in addition	31
	It is the addition symbol used in the four operations	1
	It is used in mathematical operations	3
	To make an addition subject	1
To increase (27)	It is the symbol that makes the numbers increase	22
	It allows to increase the objects	5
Plus sign (21)	It's a plus sign	19
	Plus sign, that is (+)	2
To add (13)	It means to add	9
	Addition, that is, adding	3
	If a number is two, we add	1
To multiply (9)	It multiplies the numbers	7
	This sign serves to add, to multiply an object	2
To excess (6)	It means excessing	5
	It means an excess of something	1
To combine (4)	It is the symbol of combining	3
	This sign is a symbol for combining a number with a number	1
Correct sign (3)	It is the correct sign	2
	It is used if our answers to the problems are correct	1
Other (7)	It is similar to the multiplication sign	1
	Teachers want it, we do it	1
	It's the first operation we learn	1
	The most used symbol	1
	The symbol we learned in first grade	1
	It is forward rhythmic counting	1
	It is the sign of balancing	1

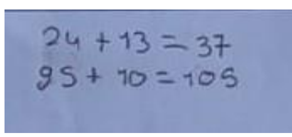
Table 1: The views of fourth-grade primary school students on the meaning of the "+" symbol.

When table 1 is examined, it is seen that primary school fourth-grade students stated the "addition sign" category most regarding the meaning of the "+" symbol, and the opinion of "it is an addition sign" was the most frequent expression within this category. The situations of writing and solving arithmetic operations by using the "+" symbol of primary school fourth-grade students were given in Graph 1.



Graph 1: The situations of students' writing and solving arithmetic operations by using the “+” symbol

When writing and solving arithmetic operations of the primary school fourth-grade students by using the “+” symbol were examined, it was determined that the majority of the students were able to write arithmetic operations and solve arithmetic operations. The example showing that the student with code S45 could write and solve an arithmetic operation using the “+” symbol is in Figure 2. The example, in which the student with code S68 could write an arithmetic operation but could not solve it correctly, is in Figure 3.



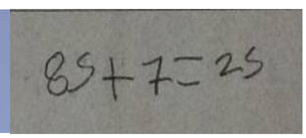
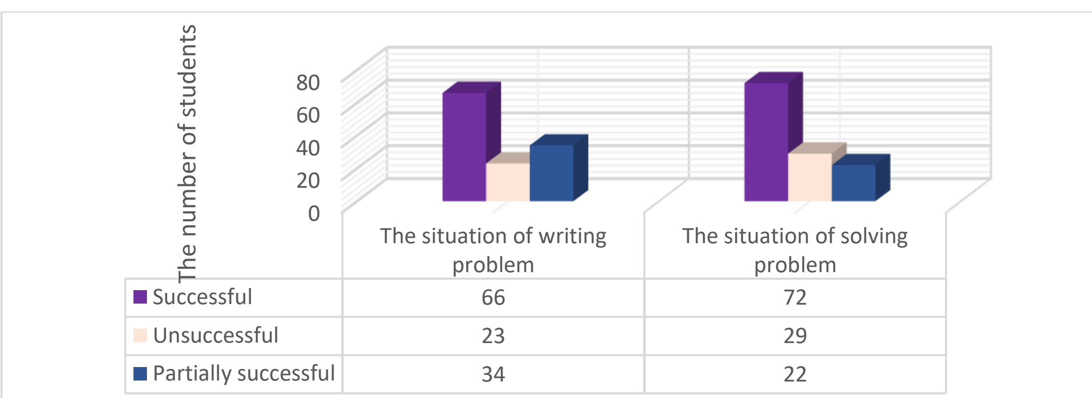


Figure 2: S45, an example of response

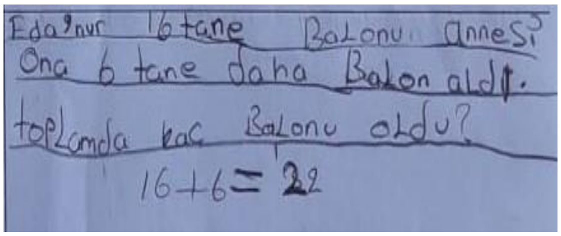
Figure 3: S68, an example of response

The findings on the writing and solving of word problems of primary school fourth-grade students using the “+” symbol were given in Graph 2.



Graph 2: The situations of students” writing and solving word problems by using the “+” symbol

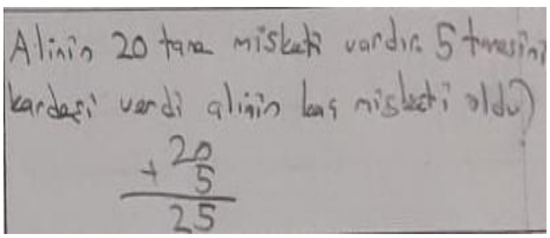
When the word problem writing and solving of primary school fourth-grade students by using the "+" symbol were examined, it was revealed that students are more successful in solving problems by using the "+" symbol than in writing. The example of the correct usage of the "+" symbol by the student coded S34 in problem writing and solving was included in Figure 4. The example of the incorrect usage of the symbol "+" in writing of the word problem and the correct usage in solving it by the student coded S96 was included in Figure 5.



Edanur had 16 balloons. His mother bought him 6 more balloons. How many balloons did he have in total?

16 + 6 = 22

Figure 4: S34, an example of response



Ali had 20 marbles. He gave 5 of them to his brother. How many marbles did Ali have?

$$\begin{array}{r} 20 \\ +5 \\ \hline 25 \end{array}$$

Figure 5: S96, an example of response

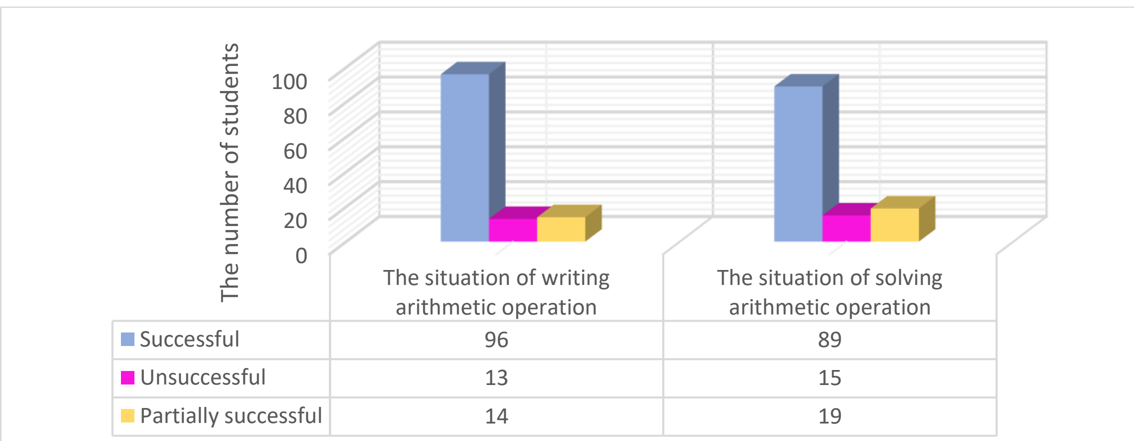
The views of the fourth-grade primary students on the meaning of the "–" symbol were given in Table 2.

Category	Code	f
Subtraction sign (41)	It's a subtraction sign.	36
	It is the subtraction sign that indicates the decrease.	2
	It's a subtraction sign in math.	3
Subtraction (34)	It is used in subtraction.	26
	It subtracts a number from a number.	4
	It means to subtract a number.	3
	Having a subject of subtraction	1
To reduce (26)	It helps to reduce the numbers.	18
	It reduces the objects.	4
	It means to reduce.	4
To decrease (23)	It means to decrease.	16
	It is used to decrease something.	4
	It's useful to decrease.	3
Minus sign (14)	It means minus sign.	13
	Minus sign, so small	1
Wrong sign (5)	It's the wrong sign.	4

	Wrong refers to controlling.	1
To separate (3)	It is the separation of one number and another number.	2
	Writing separately.	1
Other (6)	It means negative.	1
	It is used after addition.	1
	The sign next to words that have no endings.	1
	It means subtrahend.	1
	It means to delete.	1
	Getting a minus lowers points.	1

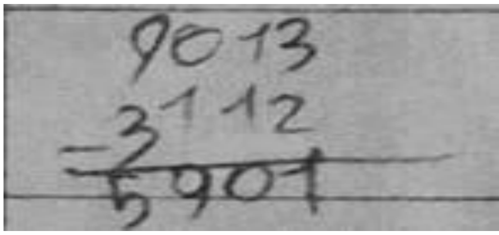
Table 2: The views of fourth-grade primary school students on the meaning of the "−" symbol

When table 2 is examined, it is seen that primary school fourth grade students stated the "subtraction sign" category most regarding the meaning of the "−" symbol, and the opinion of "it's a subtraction sign" was the most frequent expression within this category. The situations of writing and solving arithmetic operations by using the "−" symbol of primary school fourth grade students were given in Graph 3.



Graph 3: The situations of students' writing and solving arithmetic operations by using the "−" symbol

When the primary school fourth grade students' writing and solving arithmetic operations by using the "−" symbol were examined, it was determined that the majority of the students were able to write arithmetic operations and solve arithmetic operations. The example showing that the student with the code S26 could write and solve arithmetic operation using the symbol "−" was included in Figure 6. The example of the student with the code S51, which shows that he could write arithmetic operation but could not solve the solution of the arithmetic operation correctly, was included in Figure 7.



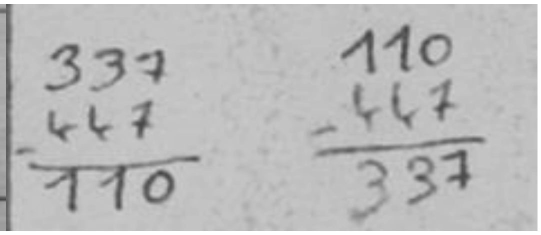
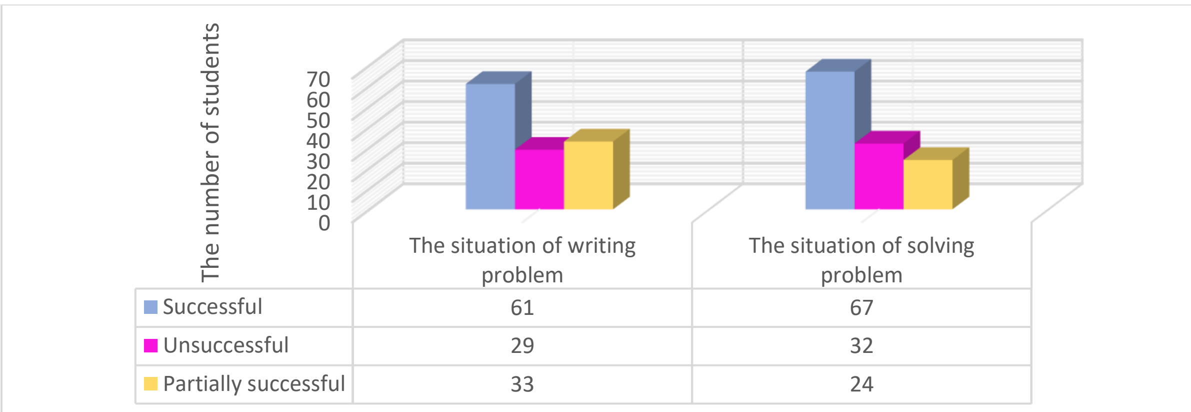


Figure 6: S26, an example of response Figure 7: S51, an example of response

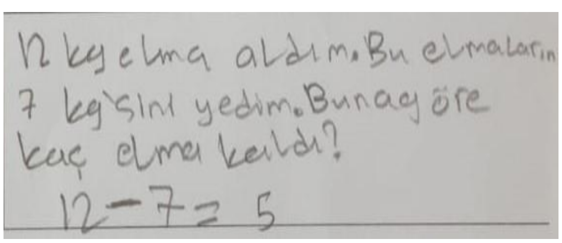
The findings regarding the writing and solving situations of word problems of primary school fourth grade students by using the "–" symbol were given in Graph 4.



	The situation of writing problem	The situation of solving problem
Successful	61	67
Unsuccessful	29	32
Partially successful	33	24

Graph 4: The situations of students' writing and solving word problems by using the "–" symbol

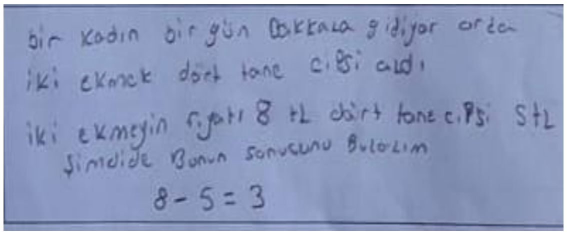
When the primary school fourth grade students' writing and solving word problems by using the "–" symbol were examined, it was determined that students are more successful in solving problems using the "–" symbol than in writing. The example of the correct usage of the symbol "–" of the student with the code S40 in writing and solving word problems was included in Figure 8, and the example of the incorrect usage of the symbol "–" in writing of the word problem but the correct usage in solving it by the student coded S03 was included in Figure 9.



I bought 12 kg of apples. I ate 7 kg of these apples. How many apples are left?

12 – 7 = 5

Figure 8. S40, an example of response


bir kadin bir gün Bakkala gidiyor orda
iki ekmek dört tane cipsi aldı
iki ekmeyin fiyatı 8 tL dört tane cipsi 5tL
şimdide Bunun sonucunu Bulalım
8-5=3

A woman is going to the grocery store one day. There he bought two breads and four chips. The price of two breads is 8 TL, the price of four chips is 5 TL. Now let's find the result of this.
$8 - 5 = 3$

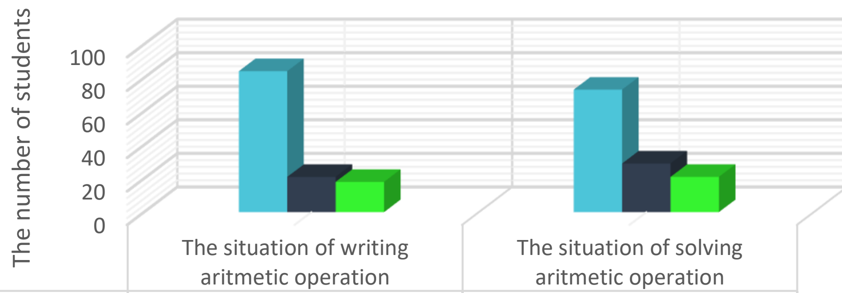
Figure 9. S03, an example of response

The views of the fourth-grade students on the meaning of the “x” symbol were given in Table 3.

Category	Code	f
Multiplication sign (32)	It's a multiplication sign.	27
	It is the multiplication sign that allows us to find the result in a short way instead of adding two numbers in a long way.	3
	It indicates the multiplication sign we use to multiply a number.	2
Multiplication (30)	It is used in multiplication.	20
	The short path for addition is multiplication.	8
	It only works for multiplication.	1
	Making a multiplication.	1
Cross (16)	It indicates a cross.	14
	It is the cross that increases the numbers.	2
Multiply (14)	It means multiply.	8
	This symbol means multiply two numbers.	6
Times (11)	It means times.	7
	It indicates operation of times.	4
Coefficient (8)	It tells how many times one number will be another number.	7
	I understand that it will be multiplied by the given coefficient number.	1
Wrong(4)	It means wrong.	3
	It is used if the problems we do are wrong.	1
Plus sign (3)	This sign means plus sign.	2
	It is the plus sign that increases the numbers.	1
Other (9)	We use it in important places.	1
	Obtaining different numbers by collision of numbers	1
	It means that something will multiply	1
	It's a no sign	1
	It means factor	1
	It does not change if the place of the numbers in the operation changes	1
	It teaches us multiplication	1
	It is forbidden to enter	1
	It is dangerous	1

Table 3: The views of fourth-grade primary school students on the meaning of the “x” symbol

When table 3 is examined, it is seen that primary school fourth grade students stated the "multiplication sign" category most regarding the meaning of the "x" symbol, and the opinion of "it is a multiplication sign" were the most frequent expressions in this category. The situations of writing and solving arithmetic operations by using the "x" symbol of primary school fourth grade students were given in Graph 5.



	The situation of writing aritmetic operation	The situation of solving aritmetic operation
Successful	84	73
Unsuccessful	21	29
Partially successful	18	21

Graph 5: The situations of students' writing and solving arithmetic operations by using the "x" symbol

When the primary school fourth grade students' writing and solving arithmetic operations by using the "x" symbol were examined, it was determined that students are more successful in writing arithmetic operations using the "x" symbol than in solving them. The example showing that the student coded S103 can write and solve arithmetic operations using the "x" symbol is in Figure 10, and the example that the student with the code S23 can write arithmetic operation but cannot solve the arithmetic operation, is in Figure 11.

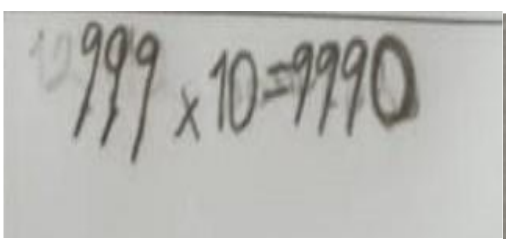


Figure 10: S23, an example of response

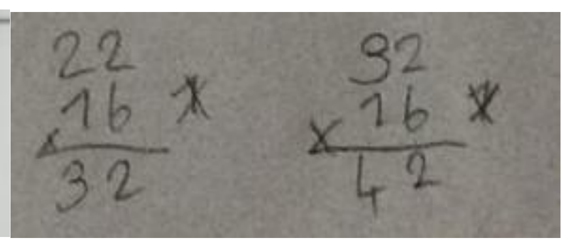
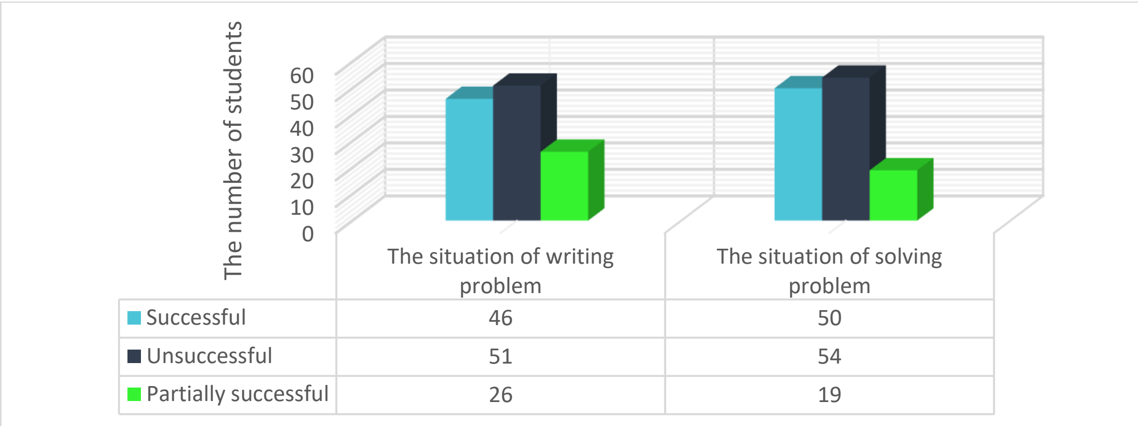


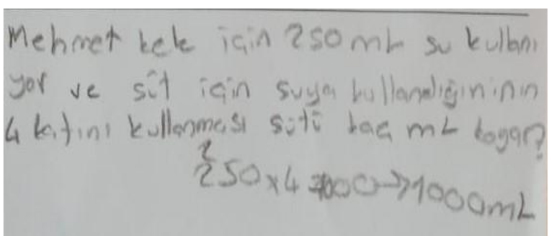
Figure 11: S103, an example of response

The findings regarding the writing and solving situations of word problems of primary school fourth grade students by using the "x" symbol were given in Graph 6.



Graph 6: The situations of students' writing and solving word problems by using the “x” symbol

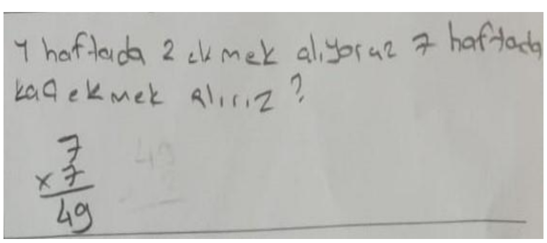
When the primary school fourth grade students' writing and solving word problems by using the “x” symbol were examined, it was determined that students are more successful in solving problems by using the “x” symbol than in writing. The example of the correct usage of the “x” symbol of the student with the code S71 in writing and solving word problem is in Figure 12, and the example of the correct usage of the symbol “x” in writing of the word problem but the incorrect usage in solving it by the student coded S42 is in Figure 13.



Mehmet uses 250 ml of water for cake and uses 4 times more water for milk than he uses water. How many ml does he put in the milk?

250 x 4 = 1000 ml

Figure 12: S71, an example of response



We buy 2 breads in 1 week. How many breads do we buy in 7 weeks?

$$\begin{array}{r} 7 \\ \underline{x7} \\ 49 \end{array}$$

Figure 13: S42, an example of response

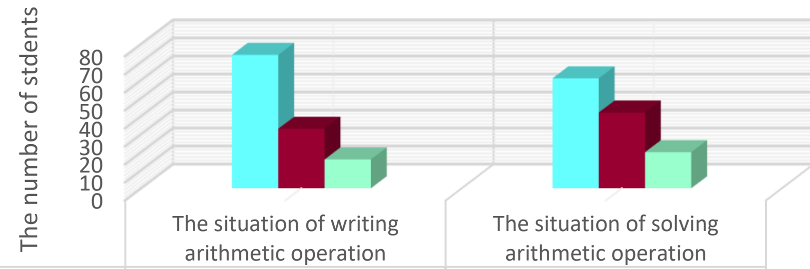
The views of the fourth grade students on the meaning of the symbol “÷” were given in Table 4.

Category	Code	f
To share (43)	It means to share	30
	It ensures that something is shared equally.	9
	This sign is the sign of the sharing of two numbers.	3

	It's to share.	1
Division sign (24)	It is a division sign	21
	We make a division with division sign	2
	The sign we use when we solve a division problem	1
Division (22)	It is used in division	12
	It indicates by how many times we can divide a number	6
	Division is a shortcut for subtraction	3
	Making a division	1
To group (17)	It means to group.	12
	It indicates how many groups a number will have.	3
	It means to divide into groups.	2
Slash sign (11)	It is a slash sign.	8
	It is a sign that means to divide one number by another number.	3
To divide (7)	It means to divide.	5
	It divides two things between each other.	2
Minus sign (4)	It means minus sign.	3
	Abbreviation for subtraction	1
To decrease (3)	It means a decrease.	2
	This symbol means decrease.	1
Other (6)	We use it in important places.	1
	We use it in numbers	1
	It cannot be found in the quotient of any number.	1
	It helps to distribute the numbers equally.	1
	It is the most difficult operation.	1
	It means to separate.	1

Table 4: The views of fourth-grade primary school students on the meaning of the symbol "÷"

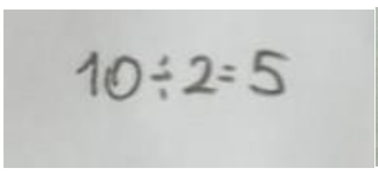
When table 4 is examined, it is seen that primary school fourth grade students stated the category of "to share" the most regarding the meaning of the "÷" symbol, and the most frequent expressions were the opinion of "it means to share" in this category. The situations of writing and solving arithmetic operations by using the symbol "÷" of primary school fourth grade students were given in Graph 7.



	The situation of writing arithmetic operation	The situation of solving arithmetic operation
Successful	74	61
Unsuccessful	33	42
Partialy successful	16	20

Graph 7: The situations of students' writing and solving arithmetic operations by using the “÷” symbol

When the writing and solving arithmetic operations of the primary school fourth grade students by using the “÷” symbol was examined, it was determined that students are more successful in writing arithmetic operations using the symbol “÷” than in solving. The example showing the student coded S98 can write and solve arithmetic operations using the symbol “÷” is in Figure 14, and the example that the student coded S39 can write arithmetic operations but cannot solve the arithmetic operation correctly, is in Figure 15.



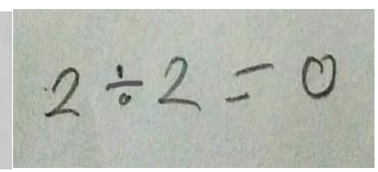
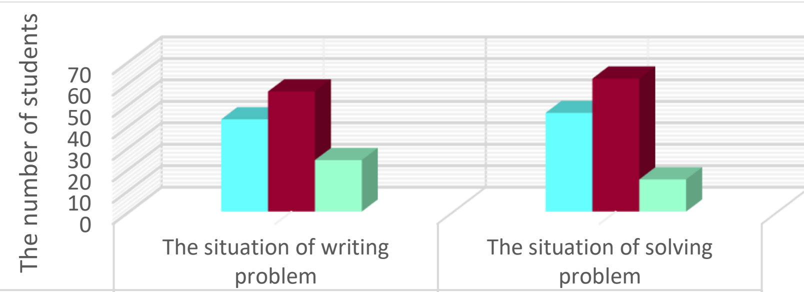


Figure 14: S98 an example of response

Figure 15: S39 an example of response

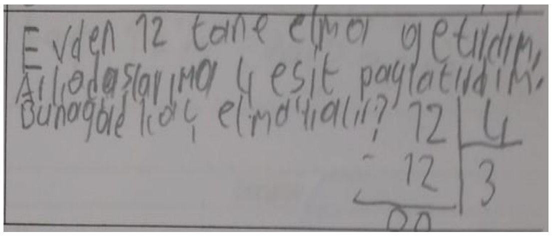
The findings regarding the writing and solving situations of word problems of primary school fourth grade students by using the symbol “÷” were given in Graph 8.



	The situation of writing problem	The situation of solving problem
Successful	43	46
Unsuccessful	56	62
Partialy successful	24	15

Graph 8: The situations of students' writing and solving word problems by using the symbol “÷”

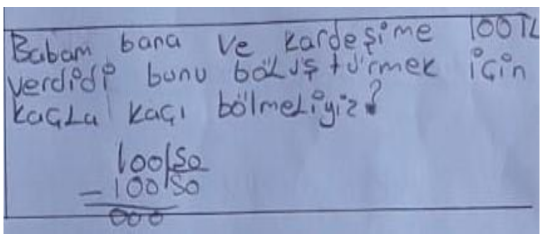
When the writing and solving word problems of the primary school fourth grade students by using the "÷" symbol was examined, it was determined students are more successful in solving problems by using the symbol "÷" than in writing. The example of the partially correct usage of the "÷" symbol of the student with the code S05 in writing and the correct usage in solving word problem is in Figure 16, and the example of the correct usage of the symbol "÷" in writing of the word problem but the incorrect usage in solving it by the student coded S102 is in Figure 17.



I brought 12 apples from home. I shared 4 equal parts with my friends. How many apples are left?

12 | 4
-12 | 3
00

Figure 16: S05 an example of response



My father gave me and my brother 100 TL. How many should we divide by how many?

100 | 50
-100 | 50
000

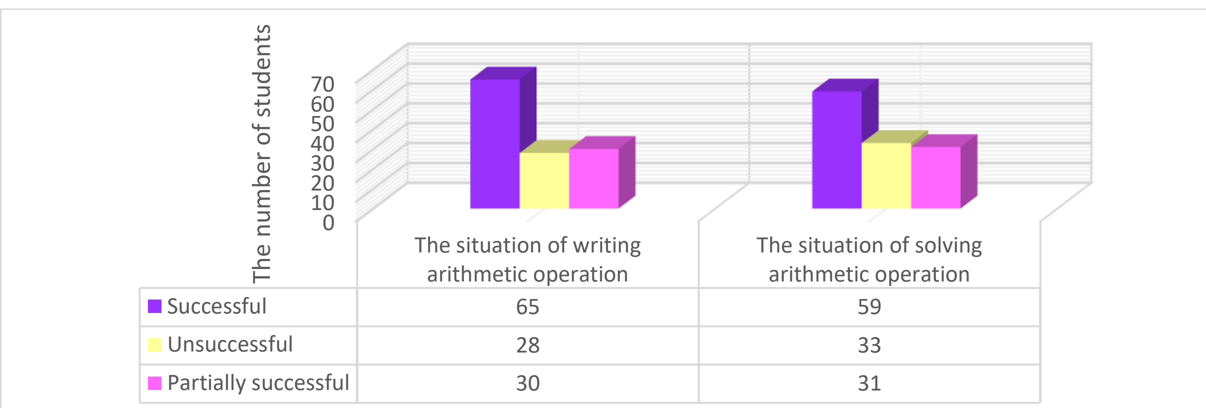
Figure 17: S102 an example of response

Category	**Code**	**f**
Result (37)	It is used to find the result.	21
	It shows the result of each problem.	10
	It is equal to the result.	6
	When we add something with something, we write it to the result.	1
Equal sign (23)	It's an equal sign.	19
	It is the equal sign found in all operations.	3
	It is an equal sign that indicates the equality.	1
Answer (22)	It means to find the answer	14
	It shows the answer in addition, subtraction, multiplication, division.	8
Same (19)	It means the same.	13
	It is said that if all numbers are the same, they are equal.	4
	It indicates that two numbers are the same.	2
	It is put between the same numbers.	2
Equivalent (13)	It means equivalence.	8
	If two things are equal, It says they are equivalent.	4
	It means equivalent to something.	1
Across (10)	The sign written across the question.	6

	The across side of the solution.	4
The end (8)	We use it at the end of the numbers	5
	It is used at the end of the operation	2
	It is put at the end of +, -, x, and ÷	1
Next to (7)	It is written next to the operations.	4
	It is placed next to the numbers.	2
	The sign that precedes the result of adjacent operations	1
To start (4)	It is the signal to start the operation.	3
	It means to start.	1
Other (6)	It is found everywhere.	1
	It is used in tests and books.	1
	It evens everything out.	1
	It means common.	1
	I think of this symbol as the sign between “< and >”.	1
	It provides the crosscheck of an operation.	1

Table 5: The views of fourth grade primary school students on the meaning of the “=” symbol

When table 5 is examined, it is seen that primary school fourth grade students stated the category of “result” the most regarding the meaning of the “=” symbol, and the most frequent expressions were “it is used to find the result” in this category. The situations of writing and solving arithmetic operations by using the symbol “=” of primary school fourth grade students were given in Graph 9.



Graph 9: The situations of students’ writing and solving arithmetic operations by using the “=” symbol

When the writing and solving arithmetic operations of the primary school fourth-grade students by using the “=” symbol was examined, it was determined that students are more successful in writing arithmetic operations using the “=” symbol than in solving. The example showing the student coded S63 can write and solve arithmetic operations using the symbol “=” is in Figure 18, and the example that the student coded S84 tried to write an equation using the symbol “=” but made an incorrect solution is in Figure 19.

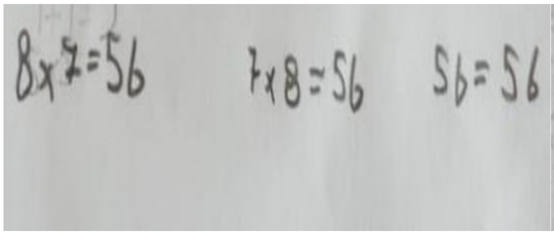


Figure 18: S63 an example of response

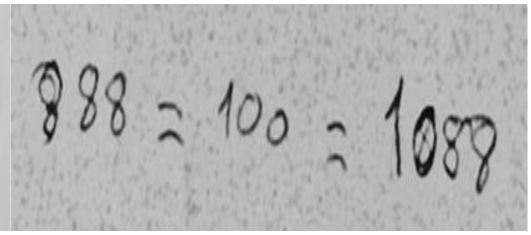
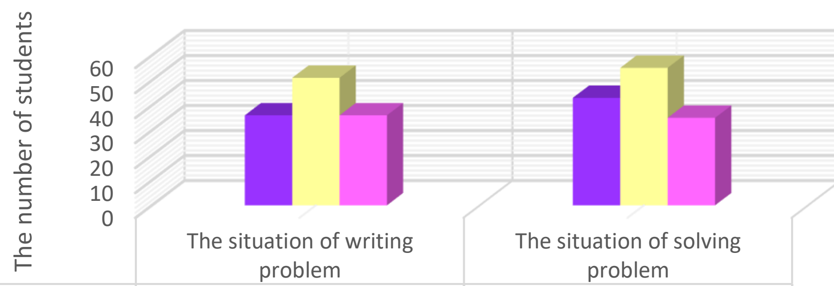


Figure 19: S84 an example of response

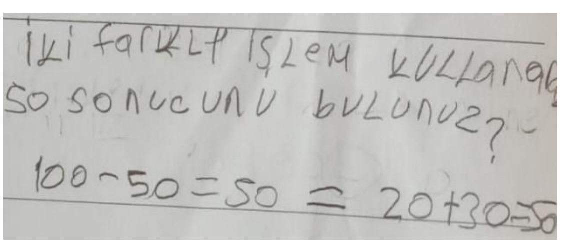
The findings regarding the writing and solving situations of word problems of primary school fourth grade students by using the "=" symbol were given in Graph 10.



	The situation of writing problem	The situation of solving problem
Successful	36	43
Unsuccessful	51	55
Partially successful	36	35

Graph 10: The situations of students' writing and solving word problems by using the "=" symbol

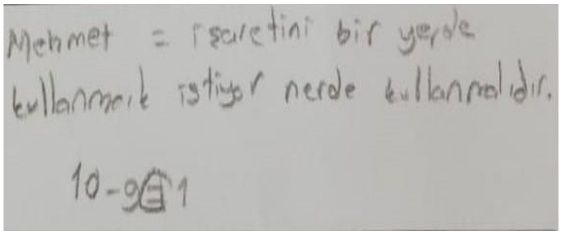
When the writing and solving word problems of the primary school fourth-grade students by using the "=" symbol was examined, it was determined students are more successful in solving problems by using the "=" symbol than in writing. The example of the correct usage of the "=" symbol of the student with the code S79 in writing and solving a word problem is in Figure 20, and the example of the correct usage of the symbol "=" in writing the word problem but the incorrect usage in solving it by the student coded S58 is in Figure 21.



Find the result 50 using two different operations?

100 – 50 = 50 = 20 + 30 = 50

Figure 20: S79 an example of response



Mehmet wants to use the = sign somewhere. Where should use.

10 – 9 = 1

Figure 21: S58 an example of response

CONCLUSIONS AND DISCUSSIONS

In this study, which was conducted to determine how the four operation symbols are interpreted by the students, how to write and solve arithmetic operations using symbols, and how to write and solve word problems using symbols, it was concluded that the fourth graders of the primary school made sense of the "+" symbol in connection with the categories of "addition sign, addition, to increase, plus sign, to add, to multiply, to combine and correct sign". According to Anghileri (2005), children should learn that the "+" symbol can be read as "and", "add" or "plus" and that they need to put meaning into the symbol to solve written sums. According to Altun (2018), in order to gain the knowledge of the addition operation at the primary school level, the expressions such as "and, more, total, plus, added, multiplied" should be included in the problem sentences given and the students should be provided with meaning to the addition. As another result of the research, it was determined that the fourth-grade students of primary school put forward the categories of "subtraction sign, subtraction, to reduce, to decrease, minus sign, wrong sign and to separate" regarding the meaning of the "–" symbol.

It was concluded that primary school fourth-grade students put forward the categories of "multiplication sign, cross, multiply, times, coefficient, wrong and plus sign" regarding the meaning of the "x" symbol. According to Pesen (2020), it should be emphasized that 4 x 2 =? can be expressed as "4 times 2 equals 8, 4 by 2 is 8", the words "by", "times" are related to multiplication, the symbol of multiplication is "x" and it is read as "cross", and it should be explained to students that this expression is written as 4 x 2 = 8. The fact that multiplication also means repeated addition explains the coefficient meaning of multiplication. It was determined that the fourth-grade students of primary school put forward the categories of "to share, division sign, division, to group, slash sign, to divide, minus sign and to decrease" regarding the meaning of the symbol "÷". Angliheri (2005) states that the "÷" symbol used in the expression "12 ÷ 3" can be read in many ways. The word "sharing" is often associated with division. Although the word "divide" may be less familiar to most young children, there is an increasing acceptance and use of this interpretation as children grow up. It was determined that the fourth-grade students of primary school put forward the categories of "result, equal sign, answer, same, equivalent, across, the end, next to and to start" regarding the meaning of the "=" symbol. Under these categories, it was found that the symbol "=" was used the most in the "result" category in the sense of "it is used to find the result". Students often interpret the equal sign as an operational symbol. They do not see the equal sign as a relational symbol that requires the use of numerical relations between the two sides of the equation (Carpenter & Levi, 2000; Knuth, Alibali, McNeil, Weinberg & Stephens, 2005; Mirin, 2020; Powell, 2015; Yıldız & Atay, 2019).

When the primary school fourth-grade students' writing and solving arithmetic operations using four operation symbols were examined, it was concluded that students were more successful in writing arithmetic operations using symbols than in solving arithmetic operations. In addition, it was determined that students were more successful in writing and solving arithmetic operations using "+" and "−" symbols than in writing and solving arithmetic operations using "x", "÷" and "=" symbols. Önal (2017), in his research concluded that misinterpretation of symbols and texts is one of the important factors that cause students to make mistakes in the four operations. According to the research findings, it was concluded that the students had the most difficulty in using the "=" symbol in writing and solving arithmetic operations. It may not seem important for teachers to focus on the relational meaning of the equal sign. However, when students move to secondary school and high school and begin to solve mathematical (or., 9 = 81 ÷ x) and algebraic equations (or., 3a = b + 12), the sign of equality needs to be understood relationally (Powell, 2015). Misconceptions developed in primary school, especially for concepts that form the basis of algebraic thinking such as equality prevent students from understanding algebra issues correctly (Byrd, McNeil, Chesney, & Matthews, 2015; Knuth et al., 2005; Mirin, 2020).

When the writing and solving word problems of primary school fourth-grade students using four-operation symbols was examined, it was concluded that students were less successful in writing and solving arithmetic operations using symbols, and they were more successful in problem-solving than in problem-writing. It can be said that students are not able to fully convey their thoughts into words as a reason why they are more successful at problem solving than writing problems. Angliheri (2006) states that the use of the subtraction symbol "−" to represent the words "take away" in some problems and "difference between" in others causes difficulties for children. Where a child knows subtraction only as 'take away', confusion may arise in representing the "difference between" using exactly the same symbol. Children should learn that there are many meanings associated with arithmetic symbols, the certain interpretations will better match the different contexts and the different solution procedures that are appropriate.

RECOMMENDATIONS

Students do not always attribute the correct meaning or any meaning to their mathematical symbols. Teachers need to provide opportunities for their students to learn the correct definitions of symbols and to apply symbol understanding in a variety of contexts (Powell, 2015). In the study of Kabael and Ata Baran (2016), the participants suggested that a section called "the corner of symbols and their meanings" should be created on the blackboard or on the school math boards during the course process in order to gain the meanings of symbols.

Misconceptions that may occur in the four operation symbols can reveal conceptual problems in mathematical studies. In line with the findings of the study, different examples and exercises can be included in mathematics lessons, textbooks and workbooks regarding the meanings of the four operation symbols (+, −, x, ÷, =), their use in arithmetic operations, and their use in word problems. In the teaching of symbols, speeches and concrete objects containing daily life situations can be used. Research can be conducted on the meanings attributed to other symbols (<, ≤, >, ≥, %, etc.) used in primary school mathematics lessons, the determination of their use in arithmetic operations and word problems, and the difficulties experienced in the use of symbols.

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Secondary School Students' Construction Processes of Square Root Concept with Realistic Problems: An APOS Perspective

Elif Nur Akgul[1], Rezan Yilmaz[2*],

[1]Ministry of National Education, Samsun, Turkey,

[2]Ondokuz Mayis University, Samsun, Turkey

rezzany@omu.edu.tr

Abstract: *This study aims to examine the construction process of the square root concept of secondary school students in Realistic Mathematics Education (RME) based environment. The study was carried out in an 8th-grade classroom with 16 students from a secondary school in a city in the Black Sea Region. The research was designed qualitatively as a case study. To measure the preliminary knowledge of students about the square root concept, a test was prepared. The students were grouped heterogeneously according to the results obtained from the test and observations. The students were asked to solve two contextual problems in their groups. The teaching process was supported by in-group and inter-group discussions. Interviews were conducted two times with each of the three participants who were selected from the groups by purposeful sampling and whose readiness levels were advanced, intermediate, and lower-intermediate. The data from the transcripts of the teaching process, in-group discussions, interviews, and also individual and group worksheets and observations, were evaluated with content analysis within the framework of APOS theory. As a result of the study, the concept of the square root of square numbers was constructed as an object by all of the participants. Also, the participants could determine the location of the square roots of non-square positive integers between two natural numbers. It has been determined that having strong coordination is very important, and exponential numbers, area and perimeter measurement, unit, and rational-decimal numbers are used in this coordination.*

INTRODUCTION

Numbers, their properties, and relations cover the main and important part of the mathematics curriculums taught in schools. Natural numbers and their operations on them which are started to be taught in primary school expand with integers and rational numbers in secondary school. At the end of secondary school, irrational numbers are started to construct, and real numbers and complex numbers are taught in high school. During this period, high school students understand the number system and compare its characteristics with others (MNE, 2018; NCTM, 2000).

It is essential to understand irrational numbers for extending and reconstructing rational numbers to real numbers (Sirotic and Zazkis, 2007a). However, the case of this extension is particularly dramatic. The students have difficulties with the strict hierarchy among number sets and the abstract structure of irrational numbers, and also they have misconceptions about them (Arcavi, Bruckheimer, and Ben-Zvi, 1987; Fischbein, Jehiam and Cohen, 1995; Guven, Çekmez, and Karataş, 2011; Sirotic and Zazkis, 2007a). It has been assumed that the concept of irrational number is intuitively difficult because of their incommensurability. After the surprising discovery of early Greek mathematicians about the existence of incommensurable segments, the rigorous theory of irrational numbers has been fully established with the contribution of Dedekind, Cantor, and Weierstrass in the nineteenth century with a long, historical delay (Fischbein, Jehiam, and Cohen, 1995). So, it would not be plausible to expect students to overcome its epistemological obstacle and understand the concept of irrationality easily.

The advantage of using real-world models and pictorial representations can be considered to improve the understanding of natural or rational numbers, but this cannot look easy for irrational numbers due to their more abstractness. On the other hand, considering that the mathematical concepts arise from real-life needs of knowledge and relations, it is thought that teaching these concepts should be realized in a more informal and real-life context (Campbell and Zazkis, 2002; Gravemeijer, 1999). Thus, when students learn by relating to their own lives and performing their mathematization processes, mathematics will become meaningful for them (Gravemeijer and Terwel, 2000). However, attention to informal meanings and familiar contexts in mathematics education should not be considered separately from the development of conceptual foundations (Campbell and Zazkis, 2002). In this sense, realistic problems in the environments in which the student has actively re-invented the mathematical structures by using his/her paths and the models he/she has developed might serve this purpose (Gravemeijer, 1999). The steps which are firstly the development of operations-relations in some ordinary real-life contexts, then the realizing the same structure in other contexts, and finally, the formulation of the common structure by symbolizing it (Treffers; 1991), will constitute generalization and abstraction (Mitchelmore, 2002). If we consider that the concept is a cognitive structure formed as a result of abstraction (Von Glaserfeld, 1991; Yilmaz and Argun, 2018), the environment in which abstraction will take place will be important in the concept formation process.

Although the necessity and importance of understanding irrational numbers in the transition from the set of rational numbers to the set of real numbers are emphasized by the studies conducted, it is seen that the researches on the subject are still quite insufficient. Also, studies examining how these concepts are abstracted are almost non-existent. When the related studies are examined, understanding of irrational numbers generally is conducted and these studies are carried out on teachers, prospective teachers, and undergraduate students. For example, Sirotic and Zazkis examined prospective secondary school teachers' understandings of the representation of irrational numbers as points on a number line (2007a) and their knowledge regarding the relationship between the two sets, rational and irrational (2007b). Also, Zazkis and Sirotic focused on how different representations influenced their responses concerning for to irrationality (2010). Guven, Cekmez, and Karatas (2011) researched prospective elementary school teachers' understandings of defining rational and irrational numbers, placing them on the number line, and operations with them. Patel and Varma's study (2018) is on undergraduate students from different departments and they examined the performance on a magnitude comparison task about irrationals denoted by radical expressions and on a number line estimation task about these irrational numbers. Fischbein, Jehiam, and Cohen (1995) investigated the presence and the effects of obstacles for 9^{th} and 10^{th}-grade students and prospective teachers: the difficulty to accept that two magnitudes (line segments) may be incommensurable and the difficulty to accept that the set of rational numbers does not cover all the points in an interval. Arbour (2012) studied college science students' understanding and concept images of real, rational, and irrational numbers, and Voskoglou and Kosyvas (2013), designed a general plan in terms of the APOS/ACE treatment for teaching real numbers at high school and college level. Kidron (2018) analyzed 10^{th}-12^{th} graders' conceptions of irrational numbers by using their representation as non-repeating infinite decimals and their conceptions of irrational numbers on the number line.

In the below-mentioned theoretical framework, RME and the role of realistic problems, and APOS theory have been explained.

THEORETICAL FRAMEWORK

Realistic Mathematics Education (RME) and Realistic Problems

The idea of RME is based on that mathematics is a human activity. This approach put forwards that the learner re-invents the formal mathematics knowledge by transforming his/her informal knowledge in real life with the help of contextual problems that are meaningful and experientially real to them (De Lange, 1996; Gravemeijer, 1999; Treffers, 1991). Thus, they can experience these mathematization processes similar to the processes of real mathematicians (Gravemeijer and Terwel, 2000) and this provides learners to construct meaningful knowledge. Here, realistic problems are mathematical problems that students can experience or imagine in real life and are based on meaningful contexts. Besides the importance of contexts, the most crucial criterion for a problem within RME is its offering opportunities for modeling and mathematization (van den Heuvel-Panhuizen, 2005).

'*Guided reinvention through progressive mathematization, didactical phenomenology*, and *self-developed models*' are three key principles of RME for implementing and planning the teaching and learning process. During guided reinvention through progressive mathematization, students configure and organize the realistic problem by finding mathematical aspects of it (Fauzan, 2002) and this strong intuitional component prompts the students to reinvent the mathematical concept (Yilmaz, 2020). Progressive mathematization involves a two-stage process: horizontal and vertical. In the first stage, students transform the realistic problem into a mathematical problem by using their informal strategies, and in the second stage, they produce a new algorithm and move the mathematization process to a higher level in the light of informal strategies and abstract the conception (Gravemeijer and Terwel, 2000; van den Panhuizen and Drijvers, 2014). Didactic phenomenology means that the students make sense of the events as a phenomenon and thereby, they can produce their phenomenon like real mathematicians (Freudenthal, 1983). The self-developed models mean that students' models that they develop with the help of the realistic problem have a crucial role to pose as a bridge between informal and formal knowledge (Fauzan, 2002; van den Heuvel-Panhuizen and Drijvers, 2014) and these models (model for) that are dependent on the context transforms into the models (model of) independent from the problem situation (Zandieh and Rasmussen, 2010).

The design of instruction to be prepared according to the principles of RME requires: starting to learn with a concrete foundation with rich content problems that support mathematical organization and following informal solution processes of the students in context (constructing and concretizing); using models and schemes that may arise during the solution process of the problem to complete the gaps between students' abstraction levels and to facilitate their cognitive transitions (levels and models); giving special assignments to the students to reveal their free products and thoughts (reflection and special assignment); working with groups (social context and interaction) and providing students to learn the concept in a spiral relation with related prior knowledge (structuring and interviewing) (Treffers, 1991; Gravemeijer and Terwel, 2000; van den Heuvel-Panhuizen and Drijvers, 2014).

APOS Theory

Piaget's one major idea is reflective abstraction (1980) which is the main mechanism for the mental constructions and logico-mathematical structures in the development of the thought of individuals. APOS Theory is based on his idea and developed on the terms Action-Process- Object-Schema. As a constructivist theory, it gives a theoretical framework about how mathematical concepts can be learned and examined in the process of their formation.

According to this theory, mental structures called genetic decomposition are built in the mind while learning a concept. This construction can vary from one individual to other and can also be different from the definition/formulation of the concept. With the possible genetic decompositions, suggestions can be made as a model about how to design instruction for students and, what differences and difficulties in acquiring the concept might reveal. The individual makes sense of the concept with certain mental structures and these structures (*Action, Process, Object,* and *Schema*) include mental mechanisms (*interiorization, coordination, (de)encapsulation, reversal, generalization,* and *thematization*) (Asiala et al., 1997; Arnon et al., 2014). During the construction, mental objects that have been previously constructed begin to be transformed in the action stage where external stimulus is needed, and this stage is static (Arnon et al., 2014). As it is controlled and reflected, the action no longer needs external stimulus and is consciously interiorized, the process stage is passed, and then a dynamic structure is formed. These internal processes allow individuals to make sense of perceived phenomena (Dubinsky, 2002). A new process can be achieved by the coordination of different processes or the reversal of the process, as well as the process stage by interiorizing the actions. If the individual perceives this dynamic structure as a whole and realizes that he/she can apply different actions and processes to it, the process is encapsulated as an object and thus a static structure is obtained (Asiala, et al., 1997). Sometimes the object(s) can be de-encapsulated by returning to the process stage from which it was acquired, thus objectifying new processes by coordination between these processes (Arnon et al., 2014). A coherent collection of all these actions, processes, and objects constructs the schema. At the end of the construction process, if the learner is successful, the problem or new object has been assimilated by the schema. And also, the schema includes other schemas related to the concept. When not successful, his/her existing schema might be accommodated to handle the new phenomenon (Dubinsky, 2002).

Remembering that the concepts of irrational and real numbers were first learned in secondary school, it is thought that the examination of the formation of these concepts at this level is very important and necessary due to the lack of studies conducted at this level. Also, it is thought that examining the formation of these concepts by providing a transition environment from informal situations to formal situations through problems based on real-life contexts will give important ideas about how these concepts can be learned as well as how they can be taught.

In this study, 8^{th}-grade learners' construction process of the irrational number concept is examined with APOS theory by trying to define its genetic decomposition. The study is focused on the square root concept and the learning environment is grounded in the theory of RME. Therefore, the research question of the study is considered as *'how is the 8^{th}-grade students' construction process of square root concept in RME based teaching environment?'*.

METHOD

Research Design

In this study, we qualitatively researched the 8^{th}-grade students' construction processes of the square root concept with realistic problems. The research was conducted in a case-study pattern to investigate their constructions in depth (Yin; 2003). The study teaching process has in accordance with the RME approach and focuses to construct with realistic problems. The mental structures and mechanisms of the students were interpreted within the context of the APOS theory.

Participants

This study was conducted with three participants selected from a 16-student class in a state secondary school in the Black Sea region where one of the researchers was a mathematics teacher. A purposeful sampling method was used to reveal the mathematical lattice in students' minds regarding their conceptual understanding of the square root concept. The participants were selected voluntarily according to the following criteria:

Since the learning and teaching environment of the study would take place with heterogeneous groups, a test (explained in data collection tools) including the subjects related to the square root concept was applied to all the students in the class to determine the groups before conducting the activities. After this test was applied, 16 students were divided into heterogeneous groups of four, according to this test results, the observations, and the opinions of the other teachers about their academic achievement and personal characteristics. In the formation of heterogeneous groups, the difference in the readiness of the students in the same group about the concept of square root, their interaction within the group, and the mathematical skills of the students in the same group was considered. Three participants, one from each group and each level group (advanced, intermediate, lower-intermediate), were selected for interviews as in Table 1.

Groups	Students in Groups	Participants	Gender	Levels
Group 1	S1, S7, S8, S9	S1	Male	Advanced
Group 2	S2, S3, S10, S11	S3	Male	Intermediate
Group 3	S4, S6, S12, S13	-	-	-
Group 4	S5, S14, S15, S16	S5	Female	Lower-Intermediate

Table 1: Participants in groups and their levels

Teaching Process and Data Collection

The teaching and learning process was planned according to the principles of RME. It was tried to design a natural environment where students could think freely in groups. In the learning environment, the mathematization processes of the students were tried to be supported by guiding. Contextual problems provided students with a structured beginning that helps the formation of the concept, and students could relate the given problem with their existing knowledge and skills. In addition, informal solution strategies, didactic phenomena, and the models they create could be revealed in the process of solving problems and abstracting the concept. After the problems were solved in the groups, interviews were conducted.

Data collection tools were readiness test, observations, interviews, and worksheets which were used by the participants during activities in groups and during the interviews. The test consisted of a total of 14 questions, one of which was true/false, and the others were short-answered and open-ended, including the concepts that form the basis of the square root and its properties. The test included units of length and area and their transformations, perimeter and area of square and rectangle, rational numbers and decimals, prime numbers, factors and multiples, and exponential expressions and their properties. The test was finalized by taking the opinions of four mathematics educators and applying it to the 8^{th}-grade students of a different secondary school.

From the beginning of the semester, the researcher (who is the teacher) regularly got information about the students from the course teachers and the class mentor. She had the opportunity to observe the students in the lessons, thus getting to know the classroom environment, having information about both the academic and personal situations of the students, and getting an idea about which students to choose as participants. Observations were continued during the teaching and learning process. What the students thought, what kind of conversations they had with their group friends, and how they reacted when faced with contextual problems during the lessons were observed. During the lessons, each group was recorded with video cameras, and the whole class was recorded with an extra video camera. Subsequently, interviews were also recorded.

To reveal what the students thought in the groups, each student used different colored pencils in the group and personal papers, and thus, it tried to follow them. They were asked to write down all their thoughts on their papers and it was collected at the end of each lesson.

Two interviews were conducted with each participant at the end of each lesson and before moving on to the next lesson. The interviews were held in an environment where the participants feel comfortable and the students were asked to think aloud. During the interviews, as in the classroom environment, materials (such as chess, rulers, squared objects and calculators) were kept ready, and semi-structured interview questions which were finalized after expert opinions were used. In the first interview, reflections of the participants in *'determining the relationship between the square numbers and the square roots of these numbers'* were examined. Participants were asked to describe how they developed a solution for the relevant problem (Chess Problem 1 in Appendix), what prior information they used while solving it, and how they used it. Also, another problem (Parkour Problem in Appendix) was presented to reveal the formation of the process. In the second interview, reflections of the participants in *'determining the location of the square root of a positive integer between two natural numbers even if this positive integer is not a square number'* were similarly examined after the relevant problem (Chess Problem 2 in Appendix) and other problems (I know! Problem 1 & 2 in Appendix) was presented to reveal the formation of the process. The participants were asked to write down all they thought during the interviews and the solutions to the questions and problems they reached on their papers.

Data Analysis

The data obtained from the transcription of the video recordings of the teaching process and interviews were analyzed via content analysis. After the transcribed data were coded according to APOS theoretical framework, categories were identified by linking different codes (Creswell, 1998). A full consensus has been reached about the formation of data, codes, and categories by researchers to provide the reliability of the study. Also, to ensure that the results can be conveyed in similar media, the obtained findings have been supported with quotations and detailed descriptions have been made. In addition to the transcripts, the written documents collected from the students and the observations were also included in the analysis process. In the study, a genetic decomposition (in Figure 1) was prepared and then, finalized by taking the opinions of two experts. This decomposition gave an idea before the teaching process about how the participants could construct the concept and was useful in designing the teaching process.

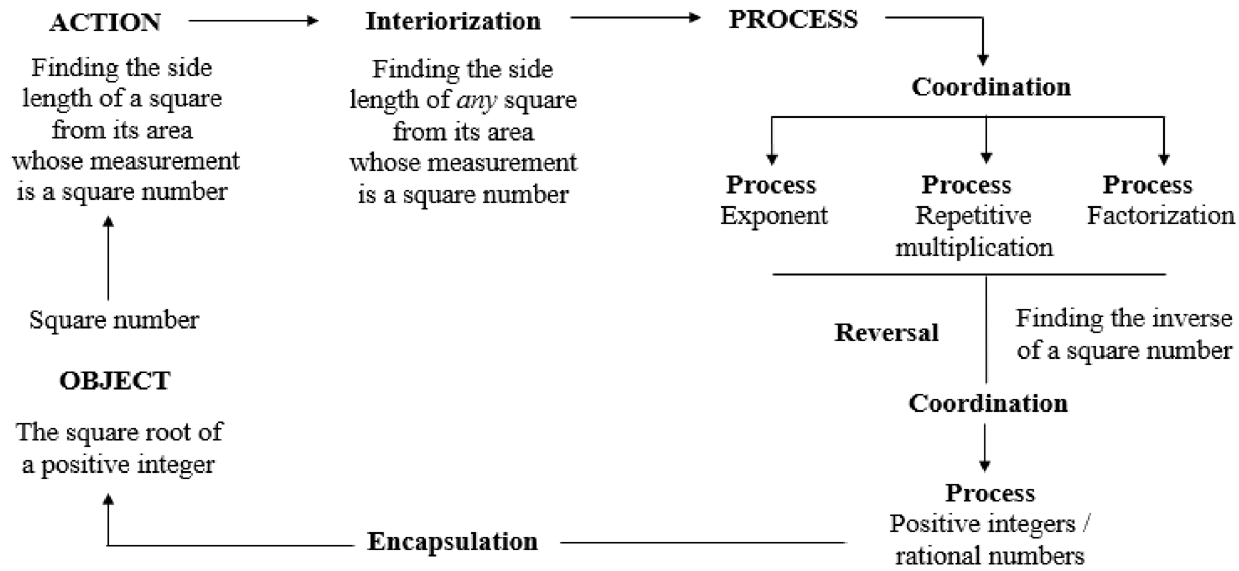

Figure 1. Genetic decomposition of the square root concept

According to genetic decomposition in Figure 1, students start their constructions with a square number in the context of Problem 1. In action, they try to find the side length of the square from the area of it whose measurement is a square number. After interiorizing the process, they could systematically find the side length of any square from the area of it whose measurement is a square number. Also, in the process, students may coordinate the way(s) of finding these numbers with processes of concepts such as exponent, repetitive multiplication, or factorization. Thus, they could systematically find the square roots of square numbers by making sense. The students are asked about the sides of the table in the context of Problem 2, they find the inverse of the square numbers and then they can coordinate the process of the square root of square numbers with the process of positive rational numbers/integers. After determining the location of the square root of a positive integer between two natural numbers even if this number is not a square number and finding the square roots of positive integers in the context of I know! Problems 1 & 2, they can encapsulate the process by using the square root symbol out of the context and can reach the square root of positive integers as an object. The genetic decomposition of the square root concept is given in Figure 1.

RESULTS

The findings related to square root were organized within the APOS theoretical framework. Differences in the construction processes constitute the main criteria of this research. The findings of the study are tried to be given under two sub-titles related to the square root of square numbers and the square root of positive integers by synthesizing constructions of the participants in their group studies and interviews.

Construction Process of the Square Root of a Square Number

S1 and his group mates had hesitations about the solution to the problem because the numerical values were not given. However, after in-group discussions, S1 expressed his thought, *"There are 64 squares in its field. Its side is 8. Then its perimeter would be 32"* and represented it as in the first part of Figure 2. When the researcher asked the reason for his thought about the length of squares, he explained *"It can be 16 meters too, teacher. One of them is 2 meters. Then it will be 16… or we can decrease. It's half a meter. Then it will be 4 meters"*. But he was limited by the thought that there should be only one answer to this problem. S3 said that he and his group mates could not solve the problem through numbers and that they would find it by using x and y variables. He thought about the relation between the number of the squares and the length and explained *"If its lengths were x, then its area would be x times x of* x^2*, and its perimeter would be 4x"* and visualized as in the second part of Figure 2. Similarly, S5 and her group mates thought the number of squares, expressed the area measurement as 64 (square units), and found the perimeter as 32 (units). However, they did not think that the area could take different values and this is considered that there was not enough sign that would make them feel their internalization.

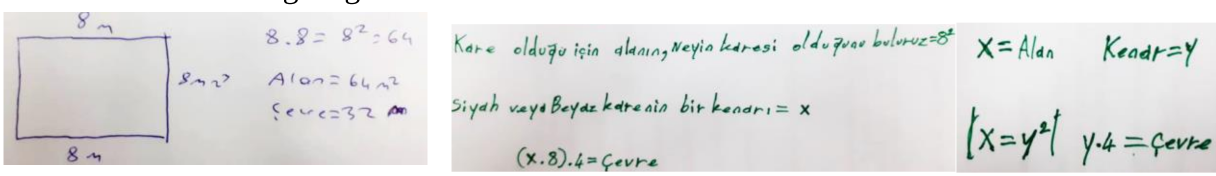

Figure 2. Representations of S1 and S3 in their Group Works on Chess Problem 1

The interviews started with discussions about the problems and the participants' thoughts on them by remembering the activity, and they were asked to explain their thoughts. Then Parkour Problem asked to search for their constructions.

During the interview, when the researcher asked S1 whether the area of the chessboard could take another value, he gave examples for the areas like '16 times 16'. He explained these values as "*It could have been a multiple of 8"* because of the 8 tiles. So, it is considered that his interiorization was limited to multiples of 8. Then when it is asked *"What can the area be if it was not 64 …, but something else?"* S1 replied that *"x could happen*" and explained *x* as *"anything could happen… for example 50"*. Then, when the researcher asked what the edge length can be when the area is 50, his thought was *"…the multiplication of two same numbers does not equal 50... if it is 25, the edge can be 5"*. Here, it is considered that the participant realized which (square) numbers can be as the area value. He explained the strategy he used while trying to find the edge length of the square as *"searching for numbers that are the product of two same numbers"*. Here, it is thought that the participant started to interiorize that there may be different area values, progressed to the process stage, and coordinated with repeated multiplication while finding the square numbers. When he was asked how he could symbolize it, he was able to use the information he learned in the classroom and wrote the notation correctly. S1 answered the researcher's question about the relationship between square root and square concepts *"In the square, we multiply the number by itself, ..., in the square root, ...Hmmm… it says the number that which two equal numbers we can find by multiplying them"*. Thus it is considered that he did a generalization in the process stage.

S3 said that he started solving the problem by thinking about the number of square tiles. Since no length was given in the problem, he said that he thought to set up an equation. He mentioned finding the 'square of what' for the area of the chessboard and gave 8^2 as an example. When the researcher asked what values the area of the square could take, firstly S3 said 12 and when asked to explain, he added that *"the multiplication of the same two numbers cannot be equal to 12"*. He explained his thoughts through reflection upon the variable-algebraic expression he had constructed. He then exemplified as 25 and 100. Thus, while finding the square numbers, he researched which was the multiplication of two same numbers, and coordinated square numbers with repeated multiplication. When he was asked how they could symbolize it, he wrote the notation correctly as in Figure 3.

$\sqrt{25}=5$ $\sqrt{16}=4$

Figure 3. Representation of S3 on Square Root Symbol

Then, when asked about the relationship between the square of a number and its square root, S3 said, *"Square multiplies, that is,* (gives example) *5 times 5. The square root divides...* (thinks) *to the same number..."* and related the concepts of square and square root. When S3 was asked what the square root was for, his answer was *"finding the edge from the area*". When asked how he could represent the number 25, which is his example, he answered, *"square of 5"* and wrote it as $\sqrt{5^2}$. He also explained, *"First, we write the exponential number as a natural number, it becomes 25, then we take its square root, so it will be the same again".* Thus, it is considered that S3 interiorized the action and progressed to the process stage.

S5 had the idea that by considering the square tiles on the floor as a unit square and its area would be equal to the total number of squares in it. So, the researcher (R) continued to interview giving samples from the square objects around them to make her aware of the units she could think of. Using the square tiles on the floor, she was asked to think about the side lengths of these tiles and was expected to interpret the distance between the door and the trash can in the interview environment through these tiles.

R: *…What can you say about the distance between the door and the trash can?*
S5: *It is about 1 meter.*
R: *What about the square tiles?*
S5: *... 4.5 squares... but.*
R: *So the length of one side of a square is 1 (m)?*
S5: *No.*
R: *What can it be?*
S5: *We divide 100 cm by 4. It is about 20…*

When the researcher asked her whether the area in the problem had values different from 64, S5 thought about the numbers 32 and 128 as related to 64 in terms of taking different values. But when finding the side length, she thought of factoring it and said *"We will find the factors of 32, but I can't find it right now. what and what… If I say 32 is divided by 2, it doesn't work either, I guess it would be wrong. 32 must-have factors…"*. She thought the same for 128 and 56 and could not reach the side length from these numbers. When the researcher continued on the squares whose side length she knew, she said that the value of the area would be 36 by taking the side length 6, 49 by taking the side length 7, and 81 by taking the side length 9, and showed that she interiorized the examples in different situations. When asked about the property that did not change when the area changed, she said *"it becomes its square"*. If the area was found this way when the side was known, it was asked how to find the side when the area was known, and she said *"The number of the square itself… how can I say… the square root is already this operation, we find the number 49 itself, that is, we find 7. Its square root"*. She also correctly symbolized it by remembering the notation in the lesson. Thus, from her expressions that "*the area is always equal to the square of the side length",* it is considered that she interiorized the action and was able to coordinate with the exponential numbers in the process stage.

Finally, the researcher asked the Parkour Problem to the participants and they discussed their solution. S1 paid attention to the segments given as examples in the problem, found the square numbers in these segments, and then correctly determined the other square numbers for parkour 1-3. In the 4th part, he found the number of flags taken by the three athletes competing in both teams separately and calculated the points he collected from these flags. He correctly determined 106 total points for team A and 119 total points for team B (Figure 4).

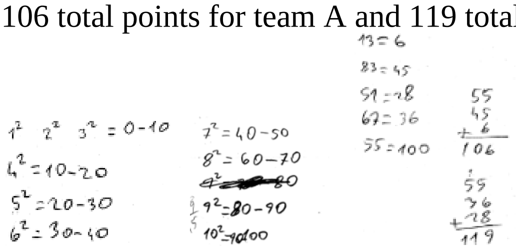

Figure 4. Solution of S1 on Parkour Problem

Since S3 first understood the problem as a flag being planted for every 10 m, he commented that *"10 has no square, 20 has no square… none of them has a square"* and when asked the reason, he said, *"4 times 4 is 16, 5 times 5 is 25... No"*. But he realized that he had misunderstood after he re-read the problem. He said, "*The square of 1 is 1. Then there is the square of ... There is a square of 4…"*. S3 then correctly thought about the intervals between the other square numbers and found the places where all the flags could be planted, and then correctly answered the problem for parkour 1-3. When he read the last parkour (4), he could not understand the problem at first, and then said *"There is no square of 1"* in parkour 1, and also added *"There is no square root of 1"* in this parkour. S3 ended his hesitation about number 1 while finding the places where the flags would be planted (Figure 5).

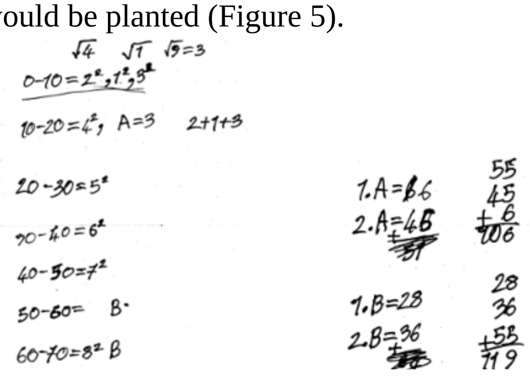

Figure 5. Solutions of S3 on Parkour Problem

On the other hand, while calculating the scores to be taken from the places where the flags are planted, he noticed that the obtained scores were always equal to the base, and questioned whether this was a coincidence.

> **S3:** *...these are (1^2,2^2,3^2,4^2...) we always have the base (1,2,3,4...). Is it a coincidence?... First, we take the square of this, then its square root, and then we find this again. It's not a coincidence…*
> **R:** *Then, what is the relation between square and square root?*
> **S3:** *It's like inverse operation. For example, it's like adding and subtracting 1. These are inverse operations. So here (he's talking about the square root of 4^2) is 4, here's 5, here's 6, here's 7, here's 8, here's 9.*

It was thought that S3 progressed in the process stage by discovering the relationship between square numbers and the square roots of these numbers as inverses of each other by doing a reversal.

S5, like the other participants, focused on every 10 meters distance and thought about the solution over the numbers 10, 20, and 30… and could not find an answer. Then the researcher asked, *"What… if it is 5?"* and she answered as *"25" and* added *"between 20 and 30"* to the question of where to plant it and stated that she understood the problem. S5, like the other participants, did not consider 1 as a square number because the square of the number 1 is equal to itself, but assimilated it to an *"identity element".* Later in the process, she accepted the number 1 as a square number, found the square numbers up to 100, and answered correctly. She also thought that she would get the square root of the sum of the distances where the flags were planted, and therefore added these values. Later, when she was asked to read the problem again, she understood it correctly and solved it (Figure 6).

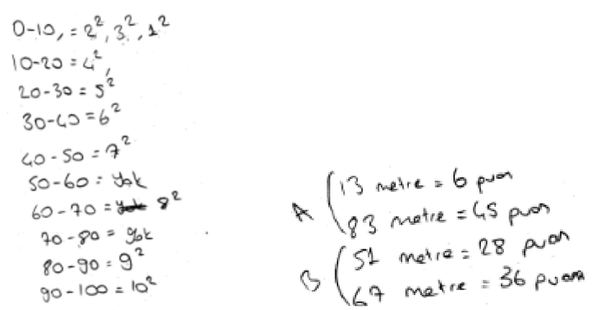

Figure 6. Solution of S5 on Parkour Problem

She realized that the square root of the square of a number (as an exponent) would be equal to this number in the base by saying *"...the square of 2 is 4, square root of 4 is 2. The square of 3 is 9, the square root of 9 is 3 …. The square of 4 is 16, and the square root of 16 is 4* (that's how it gets to 9). *As directly, we take the bases…".*

It was observed that S5 had no difficulty in finding what the values of the square roots of square numbers were equal to. Although she couldn't define it exactly, she interiorized that the number in the square root was the square of a number and that the base number was equal to the square root expression. It is thought that the reason why the participant has difficulty in the chess problem is the lack of knowledge in concepts such as area and square unit. In this sense, it is considered that S5 progressed to the process of the square root of square numbers.

Construction Process of the Square Root of a Positive Integer

To encourage the students to conceptualize the square root of positive integers, Chess Problem 2 was presented to them and the solution was discussed by the groups. The participants and their group mates generally started to think that the area of the sticker was 64 (square units) and the area of the table was 81 (square units). Their reasoning was based on the number of squares on the chessboard. Also, they thought of the units as cm. It is realized that their constructions on the unit concept were limited and they had a lack of knowledge about it. For example, S1 told his group about the table: *"…then this will also be a square since the margin will be equal here. If it's square, it will be 9. One side is 9 cm, then according to the previous problem,".* His reasoning was, *"There are 8 squares. 8 times 8 is 64. Which is the smallest number greater than 8, it is 9. Then 9 times 9 equals 81, the area of the table"* (Figure 7). He also expressed the lengths in units *"It must be centimeter. If it is meter, it will be very big…".*

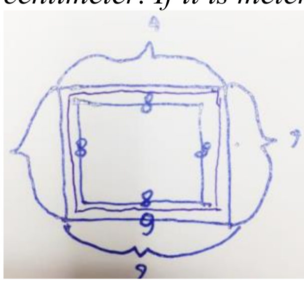

Figure 7. Visual from Group Paper of S1 on Chess Problem 2

The researcher asked if there were other values. The groups of S1 and S3 offered different values related to the area, but S5 and her group mates could not. For example, S1 said *"it must be greater than 64"* and then his group mate S7 added, *"it can be 65, ... or 68, then it can be 72".* When the researcher asked what the edge of the table would be if the area was 72 (square unit), S8 stated that it would be between the sticker and the edge of the table. Then S1, stating that it could not be an integer in this case, attributed his thought to 72 being between 64 and 81. When the researcher asked what the side length of the table would be if the area of the sticker was 50 (square unit), S1 said, *"Let's find one side if its area is 50. 7 times 7 is 49. Then it could be 7,5 units. 7,5 times 7,5. Let's find this result"* (Figure 8).

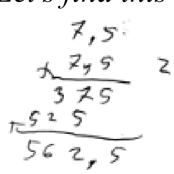

Figure 8. S1'calculation of the square of 7,5

S1 calculated the multiplication wrongly in the decimal notation and the value obtained according to this result caused the participant to hesitate. When the researcher asked them to explain the value of 7.5, S1 said, *"...you asked the side length of a square whose area is 50 square units…"* and for the side length, he said *"Something between 7 and 8…".* When the researcher asked which one was closer, S1 replied, *"It is closer to 7"* and explained the reason as *"7 times 7 is 49. 50,* *because it is closer to 49"*. He also stated, *"If it is closer to 7, then it is less than 7.5"*. Similarly, S3 expressed his thoughts *"There are numbers with commas between 8 and 9"*. The students in his group first thought that a table could be a square if its side lengths were integers. The reason was that when they wanted to divide this square table into unit squares, they thought that the remaining part would be a decimal number and it couldn't be written as a square number. Then, S3 stated that the closest number to 8 was 8.1 which was between 8 and 9. He said *"At first, I thought 81. Then its side was 9… For example, if it was 8.1, its area was 8.1 times 8.1."* When the researcher asked him to answer approximately, he said, *"It is greater than 64 and it is less than 81"* and commented on the side *"Then I can say it is between 8 and 9"*. He continued, *"…there can be a lot of tables… All of them have areas between 64 and 81, and side lengths between 8 and 9"* (Figure 9).

Figure 9. Visual of S3's comment on Table of Chess

It is thought that these participants tended to coordinate the process of the square root of square numbers with the process of positive rational numbers/integers. To see their construction, second interviews were conducted.

In the interview, S1 expressed his solution similar to what he thought in the classroom. When he was asked whether the length of the table to be built could be smaller than 9, he replied, *"Yes, it will be a decimal number"* and gave an example *"It will be 8.5"*. To calculate the measure of the area of the table, he multiplied 8,50 and 8,50 and incorrectly calculated it as 7225,00. S1, who is deficient in operations of decimals, said, *"It is very long. But the side is short… 8.5 is a small number…It should be between 64 and 81"*. He also added to the researcher's question *"Is there only 8.5 between 8 and 9?"* by saying *"No,… 8.4; 8.3; 8.2; 8.1… their areas are also between 64 and 81"*. It is thought that S1 coordinated the square root of square numbers with rational numbers in decimals.

S3 explained his thoughts similarly to S1. When the researcher asked whether the area could have other values, he answered *"It can be 25. If it was 25, one side will be 5. It can be 36… Its side length will be 6 cm."* When the researcher asked, *"Could it be something other than 6?"*, he explained, *"Yes, it can be... Then there can be numbers between 5 and 6 on one side, and the area will be between 25 and 36 at that time"*. When he was asked *"What can you think if you have your area between 25 and 36?"*, he said *"30"* and thought *"the side length of the square will be between 5 and 5,5"* and he explained by saying *"30 is closer to 25"*. He also determined approximately the location for 20 which is between 16 and 25 on the number line he represented by saying *"Since this is (square root of)16, it is between 4 and 5, between 4 and 4,5… it's closer to 4 because it's closer to 16… it can be 4,4."* (Figure 10).

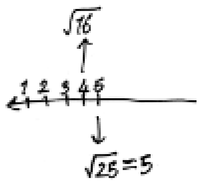

Figure 10. S3' Representation of square roots of numbers on line

S5 also spoke of the areas as 64 and 81 and the lengths as 8 and 9. When the researcher asked whether the table could be smaller, she said, *"Then, it must be between 64 and 81"* and first added, *"I'll take half of 8^2 and 9^2, because I want to get a number between them".* Then she thought of a value for the area by saying *"64 and 81… can it be 75"* and tried to explain the length of the table by saying *"We need to find the square root of 75… a decimal number"*. But since she could not comment on this side length, she chose 81 as the area of the table again. When the researcher asked for numbers between 8 and 9, she answered *"8.1; 8.2; 8.3;…"* and then the researcher wanted her to draw a square with a side length of 8,2 units. She drew the square and expressed its area as 64,4 by calculating incorrectly. She also compared this area with other areas (64 and 81) and said *"It must be somewhere around"* by showing any point on the edges of the table which she accepted its measurement as 9 units. When asked to visualize the table with a length of 8.2 units, she added rectangles with lengths of 0.2×1 square units at the end of each row. (Figure 11).

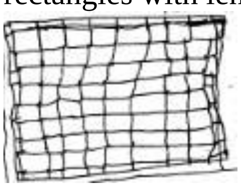

Figure 11. Representation of S5' Table of Chess.

When the researcher asked about the side length of a square with an area of 30 cm^2, she said, *"We find the square root of 30"* and explained as *"5 times 5 is 25. 6 times 6 is 36. Then we need to find something between them… It may be 5,5".* When asked about 43, she said, *"6 times 6 is 36. Since the next one is 7 times 7, it is 49… It can be 6,5... Because it will be greater than 6 and less than 7".* When asked whether there were any other numbers between them or not, she said, *"It can be 6,6… whichever is closer to 43…43 is closer to 49".*

It is considered that participants S1 and S3 could coordinate the square root of square numbers with rational numbers in decimal and interiorized the location of the square root of a positive integer between two positive integers. But it is thought that the coordination of S5 was not strong enough like others.

With the evidence for encapsulation and progress to the object stage, the researcher asked S1 for the value of $2+\sqrt{14}$. S1 said, *"14 doesn't have things,… teacher, in other words, the same multiplier"* and when the researcher asked how he could calculate it, he said *"It must be a decimal number"* and added, *"This number is between 9 and 16".* He explained the reason as *"Because the least square root of this is 9, not the square root, teacher. The previous 9, that is,… teacher. The next one is 16".* When the researcher asked what the answer would be if it was 9 instead of

14, he said *"5"* and explained as *"the square root of 9 is... 3",* if it was 16, he said *"6"* and explained it as *"the square root of 16 is... 4",* and if it was 14, he said *"a number between 3 and 4".* When asked about his estimated location, he said, *"It should be closer to 4"* and explained *"because 14 is closer to 16, …it can be 3,8"* and got a result as *"it is 5,8".*

When the researcher asked S3 for the value of $2+\sqrt{17}$, S3 said, *"First we have to find the square root. The square root is a decimal number, so let's find out which two numbers it is between. Between 4^2 and 5^2"* and thought its approximate value was *"it is closer to 4 to the power of 2,… it can be 4,2, approximately 4,2"*. Then she had the result as *"approximately 6.2"* (Figure 12).

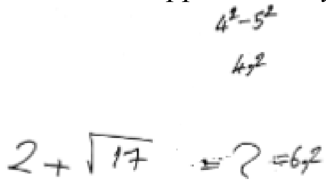

Figure 12. Answer of S3 for $2+\sqrt{17}$

S5 thought the result of $2+\sqrt{14}$ as *"First, we find the square root of 14. 4 times 4 is 16. 3 times 3 is 9. It is a decimal number again… In other words, it is between 3 and 4".* When she was asked about the closer one, she said, *"3 times 3 is 9. The difference is 5. 4 times 4 is 16. The difference is 2. It is closer to 4… it can be 3.8… the result will be 5.8".*

The researcher continued the interview by asking I know! Problem. S1 evaluated the answer 'between 6 and 7' of Merve's question 'square root of 46' as *"correct"*. He first said, *"the square root of 36 is 6, the square root of 49 is 7"* and then he tried to correct his expressions, *"the square of 6 is 36… 4 and 9… between 2 and 3"*. Then, he examined the answers given in the problem one by one and found which one gave the correct or incorrect answer. He hesitated about the square root of 2 in the problem, but later when the researcher questioned him, *"No, teacher. It is a decimal number so… a number between 1 and 2"* and explained as *"1 time 1 equals 1. 2 times 2 is 4. It is a number between 1 and 2".* He thought of the square root of 195 and stated its location between two positive integers and said the closer one. His explanations were *"square of 12 is 144. Square of 13 is 169. 195 is bigger than these numbers"*, and he accepted Merve's answer in the problem as correct and said, *"Square of 10 is 100, square of 20 is 400. 195 is between these numbers"*. When the researcher asked the location of the square root of 195, she answered *"between 14 and 13"* and when asked the closer one, he replied *"14".* Thus, it has been considered that S1 could estimate the location of the square root of a number between square numbers, state the closer one, and also calculate its value approximately.

S3 thought of the problem as *"…which two integers are the square root of 46? 46… which two numbers I multiply make 46…, 6 times 6 is 36, 7 times 7 is 49,… between 6 and 7. True. …between which two positive numbers is the square root of 87,… the square root of 87,… 9 times 9 is 81, 10 times 10 is 100, this is wrong".* He said *"1 time 1 is 1. 2 times 2 is 4. Then it will be between 1 and 2. This is wrong…"* and completed the solution correctly. Regarding the square root of 195, he said *"I don't know the square root of 13"* and then calculated the square of 13 as 169, and quickly said that the answer had to be between the squares of 13 and 14.

S5 also examined the answers given in the problem and then explained which one was right or wrong. When asked about the square root of 2, she interpreted it correctly with an uncertain tone and explained as *"it is between 1 and 2,… square of 1 is 1, square of 2 is 4. 2 is between them, therefore… it is something decimal".* She was also able to find the square root of 195.

It is considered that the participants used the square root symbol outside the original context of geometry and square numbers, extended it to non-square numbers, estimated decimals between integers, and then used it in a new situation with the addition of integers. So, they could construct the square root of natural numbers as an object.

DISCUSSION AND CONCLUSIONS

In the study, it was determined that the genetic decomposition proposed before the RME activities and the schemas of the students seen as a result of these activities were consistent with each other. It has been seen that the environment designed with the problem situations within the framework of this context by associating the concepts with real-life situations generally supports the concept formation. The reason for the difficulties encountered by the participants in constructing the concept of the square root is that they encountered this concept for the first time in both formal and informal life, its irrationality, and therefore it is thought to be more abstract. At the same time, it has been seen that the deficiencies and misconceptions in the preliminary information, which are thought to be necessary for the formation of the square root concept, make the construction of the square root concept difficult and therefore affect it. However, it was observed that the participants were generally able to conceptualize the square root of square numbers and to determine the location of the square root of positive integers between which two positive integers. In this respect, it can be said that the results of this study support the studies (e.g., Bray, and Tangney (2016), Juandi, Kusumah, and Tamur, (2022), Ozdemir and Uzel (2011), Sitorus and Masrayati (2016)) that advocate the positive effect of RME on the teaching of the concepts. In addition, it was determined that the participants knew the square root of a square number and the location of the square root of a (non-square) positive integer between two natural numbers. But sometimes they had difficulties in estimating their locations or calculating approximate values of these numbers when performing operations on square root values or displaying them on the number line. It can be said that this situation is partially similar to the results of studies (e.g. Arbour (2012), Güven, Çekmez ve Karataş, (2011), Ercire, Narlı, and Aksoy, (2016)) which state that students learn irrational and rational numbers at the knowledge level, but have problems in understanding whether a number is irrational or rational.

While constructing the concept of the square root of square numbers, all three participants adhered to the visual of the contextual problem given in the action stage. In this stage, the advanced participant set out with the idea that the area measurement can take value in the square unit and thought that the area of the square can be found by dividing it into unit squares. However, his idea remained attached to the visual in the context and he obtained a single result by considering only one of the different values that can be considered in this modeling problem, where no numerical data was given and the result can take different values. Likewise, the intermediate participant had similar thoughts, and the fact that there was no numerical data in the problem and that he had taken a value for the solution bothered him, and he thought that using variables would be more meaningful. The lower-intermediate participant, who had similar thoughts to other participants, had difficulties due to the inability to construct the area measurement units exactly. It was seen that the main reasons underlying the three participants' adherence to the visual in solving the contextual problem and difficulty in transitioning from the action phase to the process phase were that they could not consider that there could be different numbers of unit squares due to the area-edge relationship while calculating the area of the square, that they could not interiorize the unit concept, and that they have insufficient modeling skills in solving the contextual problem used in the teaching environment.

It was observed that the participants had different thoughts while trying to find the side length of the square given as the chessboard in the contextual problem. In the action stage, the lower-intermediate participant, who has incomplete/incorrect information on the area and perimeter measurement, exponential expression, performed operations such as dividing the area measure of the square by 2 while trying to find the side length of the square whose area is known. It was determined that this participant had thought that the values of the exponential numbers would be found by multiplying the base and the exponent so that the reverse operation would be achieved by dividing by 2. Therefore, he had difficulty in establishing a relationship between the number of unit squares he deals with and the edge area and in making this situation independent from the problem. The advanced participant, who first thought that the area of the given square could be equal to only a single value, later interiorized that the multiples of the side length obtained from this value could also be considered as different values and passed to the process stage. The lower-intermediate participant thought about the area measure instead of the side length and thought that there could be values that are multiples of this area measure value. However, since the numbers he received were not square numbers, he could not find the side length from the area measurement. The intermediate participant established an equation on the area measurement that changes depending on the side length of the given square. In addition, although the participants did not express that the number they should receive has to be a square number, they realized that they had to choose the numbers that are the product of two same numbers or the square of a number as the area value, and they found the square roots of numbers by coordinating this information with factoring, exponential expressions or repeated multiplication.

In the process stage, it was seen that the participants could generally relate the number of unit squares in the given area of the square with the side length of the square, recognize that the meanings of these concepts were different, state that the side lengths could change according to the size of the unit squares taken, but they could not use this information actively in the context of the given problem. It has been observed that the participant who can interiorize this information was the intermediate participant who could say different values for the side length of the given area of the square and could also calculate the area measurement that changes according to these values.

Interestingly, it was observed that the advanced participant had difficulty in interpreting the square root of the square number in exponential form and first thought that the square root of this exponential expression could not be found. On the other hand, the intermediate participant found the square root value of the square number in exponent form without any difficulty and interiorized this knowledge. Also, this participant stated that there is no need to perform operations in finding the square root of the square of a number and that these operations indicate opposite situations. It was observed that the lower-intermediate participant did not make any comments. In addition, it was determined that all three participants stated that 1 cannot be thought of as a square number because it is equal to itself, and therefore, it will not have a square root. Afterwards, it was seen that the participants interiorized the concept of the square root as a situation of finding a number whose square is the given number, and they were able to generalize the knowledge that the square roots of numbers with a square exponent are equal to the number in their base, thanks to the coordination they established. As a result of the interviews, all three participants with different readiness levels conceptualized the square root of square numbers as an object, but the coordination with the unit, area-side relationship, perimeter-side relationship, exponential expressions, and factorization were very important in conceptualization and that the participants expressed 'deleting the square, finding the side from the area, the opposite of squaring' for the concept of the square root as phenomena.

While thinking about the location of the square root of (non-square) positive integers between two natural numbers, the advanced participant, who constructed the decimal representations of rational numbers stronger than the others, carried out coordination by considering other rational numbers between these numbers. During the coordination, this participant made mistakes in multiplying the rational numbers in decimal forms, but these errors did not complicate his thoughts so much. Similarly, it was observed that the intermediate participant established coordination between square numbers and the square roots of these numbers and decimal notations. It can be said that these participants were able to find the location of the square root of (non-square) positive integers between two natural numbers, they were able to determine the closer natural number, and also they could estimate their approximate values. In this sense, it is thought that they could conceptualize the square root of positive integers.

It was observed that the lower-intermediate participant had difficulty in performing operations on non-integer numbers and made mistakes while expressing decimal notations and performing operations in this notation. However, he was able to recognize the value of the square root of (non-square) positive integers between two natural numbers with the guidance of the researcher.

While determining the location of the square root of positive integers between two natural numbers, it was observed that the participants were able to encapsulate that there are infinitely many numbers between two natural numbers and that the square roots of (non-square) positive integers will always locate between these two natural numbers. Also, it has been observed that they could correctly represent the approximate values on the number line. As a result, it can be said that they were able to generalize and construct as an object, and coordination of rational numbers, their decimal notations, and their operations were very important. It has been seen that the concepts that form the basis of the square root concept greatly affect the students' formation of this concept. For this reason, concepts such as area and perimeter measurement, exponential numbers, unit, rational numbers, and decimal representations and their operations should be formed strongly before the teaching process of the concept.

In the study, it was observed that the students were not familiar with the RME-based environment, modeling situations, and had difficulty in reasoning and making comments. For this reason, it will be beneficial to design contextual situations, to associate them with real life that students be familiar with, to present them in an environment where students are allowed to make sense of them, and to use similar teaching designs where each student takes responsibility for their learning. In addition, the students' in-group discussions helped them realize and correct each other's mistakes, and support their learning by sharing their knowledge.

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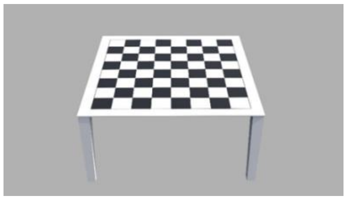
APPENDIX

Chess Problem

1. The head of the school is considering preparing a place in the school garden where the students can play chess to increase their interest in the game of chess. To do this, he tiled a chess board with black and white tiles in a suitable place in the garden. He wants to surround this area with a security strip so that it will not be stepped on until it dries. Since he knows the measurement of the area from the laid tiles, what minimum length of the strip should he use for this?

2. The head of the school was pleased with the increasing interest in chess and decided to build a chess table for each classroom so that the students could continue playing in cold weather. First, he bought sticky chessboard pictures to apply on the surfaces of the tables. Then he asked a carpenter to make tables that would have equal margins on the edges when these pictures were pasted. If the carpenter wants to reduce the cost, what minimum edge length can the table have?



Parkour Problem

For the parkour race, flags will be planted on the 100-meter linear line at the points where the squares of the positive integers coincide. Let's consider 10-meter segments such as 0-10, 10-20, 20-30, 30-40... starting from the beginning.

Parkour 1: *In how many of the 10-meter segments will multiple flags be planted?*
Parkour 2: *In how many of the 10-meter segments will the flag not be planted?*
Parkour 3: *How many flags will someone pass if leaves the race at 57 meters?*
Parkour 4: *Teams A and B each of them have three athletes racing. Two athletes from team A leave the race at the 13^{th} and 83^{rd} meters. Two athletes from team B leave the race at the 51^{st} and 67^{th} meters. The other two athletes complete the race. For each flag passed, points are scored as the square root of the number on which it is planted.*

Which team wins the race?

I know! Problem

1. *Merve (M) and Elif (E) compete in the TV play "I Know!". The questions come out separately from the envelopes of the two competitors. The moderator announces the scores of the competitors when the questions in the envelopes are finished. The questions (Q) and the answers (A) of the competitors given are as follows:*

Q of M: *Between which two positive integers is the square root of 46?*
A of M: *6 and 7*
Q of E: *Between which two positive integers is the square root of 87?*
A of E: *10 and 11*
Q of M: *Between which two positive integers is the square root of 2?*
A of M: *2 and 4.*
Q of E: *Between which two positive integers is the square root of 101?*
A of E: *10 and 11.*
Q of M: *Between which two positive integers is the square root of 91?*
A of M: *8 and 9.*
Q of E: *Between which two positive integers is the square root of 75?*
A of E: *8 and 9.*

Which competitor will the moderator announces as the winner of the competition?

2. *In the competition, the moderator explains the new rule as "Both competitors will answer the question by pressing the button first. The correct answer will be evaluated and the competitor who gives the correct answer will score points". The competition continues:*

Moderator: *Between which two positive integers is the square root of 195? (Elif presses the button at first.)*
Elif: *12 and 13 (Merve presses the button)*
Merve: *10 and 20. (Elif presses the button)*
Elif: *11 and 12.*

When the moderator confirms that the answer of Elif was correct, Merve objects and claims that her answer is correct. Do you think Merve is right? Please explain.

Teaching of Topology and its Applications in Learning: A Bibliometric Meta-Analysis of the Last Years from the Scopus Database

Diego Vizcaíno[1], Victor Vargas[2], Adriana Huertas[3]

[1]Facultad de Educación, Universidad Antonio Nariño, Bogotá, Colombia,

[2]Institute of Mathematics and Statistics Federal University of Rio Grande do Sul,

[3]Universidad Militar Nueva Granada, Bogotá, Colombia

diegofabvizcaino@uan.edu.co *, vavargascu@gmail.com, adripahb@gmail.com

Abstract: In this work, a bibliometric analysis of the investigations of the last 54 years focused on the teaching of topology and its applications in the learning of other areas of knowledge was carried out. The articles that appear in the SCOPUS database were taken into account under the search criteria of the words topology and teaching, connected with the Boolean expression AND in the search field ABS. As a result, 329 articles were obtained which, based on the PRISMA methodology, were reduced to 74 papers. In them publication trends, impact of publications, citation frequencies, among others, were compared. In addition, its use was identified for learning topology at different levels of training, areas of knowledge where this discipline is most applied and strategies used to teach these applications.

INTRODUCTION

According to (Wagner et al., 2011) and (Machado et al., 2015), bibliometric analysis is a research technique that identifies trends in scientific knowledge, for which it is possible to know about the current state of research in a specific area. This technique has been used in politics, science, technology, education, etc., in order to characterize scientific production in these fields (Cui, 2018; Koseoglu, Rahimi, Okumus, & Liu, 2016). For the development of a bibliometric study, it is necessary to select a database according to the scope of the research, identify keywords, establish search criteria and set categories that allow the interpretation of the information collected (Vallejo, Huertas, & Baracaldo, 2014).

In the area of mathematics, few bibliometric analyzes have been developed, in this sense, the study proposed by Behrens & Luksch (2011) stands out, which aimed to establish and characterize scientific productivity in the mathematics field between the years 1868 and 2008, and the bibliometric analysis of the Relime journal (Machado, Fanjul, López & Povedano, 2015). In the first, the results allowed to conclude that the publications and researchers in this area presented an exponential growth, which allowed to improve the cohesion index of the publications and in the second a behavior of the publications in educational mathematics is shown.

In this way, we believe that bibliometric studies in the field of mathematics can contribute to significantly improve knowledge about the applications of some of its branches, in this case, topology from an educational approach. Therefore, this study presents a bibliometric analysis where authors, quartiles, keywords, journals, among others, are identified using the PRISMA methodology (Moher, 2010). Additionally, a qualitative analysis is presented that determines the research trends, strategies and methods implemented in the teaching of topology, the areas of knowledge in which topological concepts are applied, and types of applications that have been developed.

WHAT IS TOPOLOGY?

Topology is a branch of mathematics originating in the 18th century from the need of resolving some issues that could not be approached from Euclidean geometry, reason why, some authors identified the first findings in this area as a type of abstraction from metric geometry, in which a broader concept on the proximity between space points was held (Callender & Weingard, 1996), this more universal concept associated to the element's position in space allowed relating certain objects that were completely different from a geometrical approach, for example, finding properties that remain unchanged in convex polyhedrons and that characterize non-convex polyhedrons based on their number of holes.

A mathematics topic that evidences the importance of topology is the study of sequences and their accumulation points (Schulte & Juhl, 1996), the foregoing, because a great variety of problems in the real world and different theories of some important knowledge areas like Ecology in the case of Biology, the study of the noble gasses in Physics, the central limit theorem in Statistics, etc., depend on the concept of accumulation point to guarantee that the estimations conducted are a close approximation to reality.

This discipline originated in Euler (1741), Where the solution to a famous puzzle proposed by the citizens of Königsberg-Prusia was found, which asked if it were possible to cross all of that city's bridges without having to cross the same bridge twice, this study gave rise to the so-called geometry of position, which later came to be known as topology.

Because of these new invariants, from a topological approach, geometrical concepts that depend on some type of distance lost importance, for example, angles and areas. From a ludic mathematical perspective, this way of understanding space allowed finding a solution to some problems that would be impossible to approach from a merely geometrical perspective, for example, the problem of the 5 Yens and the puzzle of the topological entanglement (Nishiyama, 2011).

The first study considered relevant in the topology area is Euler (1758), where the first topological invariant known of in the mathematical literature was found, this invariant tells us that any convex polyhedron fulfills the equation $V\text{-}E+F=2$, where V is the number of vertices, E is the number of edges, and F is the number of faces. After these studies, some contributions arise, like the classification of non-convex polyhedrons through the Euler characteristic, the classification of closed surfaces through its curvature, and the relationship between the Euler characteristic of a closed surface and its genus g.

A few years later, in 1857, the physicist Frederick Guthrie proposed the four-color problem, which assured that every map on a plane or on a 2-sphere could be colored with a maximum of four colors. Some attempts of solving this problem were proposed in the following years, among them, the most famous published in Kempe (1879), proof that was refuted one year later by the mathematician Percy Heawood. After that, nearly a century had to go by for a solution to this conjecture to be found, which is disclosed in Appel and Haken (1977) and Appel, Haken and Koch (1977), studies that used computational tools to classify all the possible types of maps that can be found in these surfaces and thus solve this famous problem.

The following contributions that are considered relevant in the area were realized in Poincaré (1892), which introduces the connectivity of surfaces, subsequently he published Poincaré (1895), denominated Analysis Situs and considered the paper that gave rise to modern topology, followed by its five supplements (Poincaré, 1899, 1900, 1902-1, 1902-2, 1904).

In these studies, Poincaré developed important contributions in the topology area, like the generalization of the Euler characteristic to higher dimensions, the topological invariance of the fundamental group, the triangulation of closed surfaces, etc. However, the most important problem proposed by Poincaré in this conglomerate of articles, appears in the fifth and last supplement (Poincaré, 1904), and was known until the beginning of the 21st century as the Poincaré conjecture, furthermore, this conjecture was part of the seven millennium problems proposed by the Clay Institute.

This conjecture assures that every 3-manifold with a trivial fundamental group is isomorphic at a 3-sphere, in the 2-dimension case, this problem is known as the characterization of the closed 2-manifolds, and it is associated to the equivalence, in topological terms, of the surface of a coffee mug to the surface of a donut. The analogue to this conjecture in the 4-dimension case, was resolved in Freedman (1982), but the conjecture proposed in Poincaré (1904) was only resolved until the beginning of the 21st century as a particular case of a problem related to Ricci flows, this solution was found by Perelman and appears in three articles uploaded to the arXiv platform between the years 2002 and 2003.

TEACHING OF TOPOLOGY

The teaching of school mathematics faces different challenges, from academia, proposals have been made on how to take on this challenge. for Freudenthal (2002), it is important that school mathematics always include axiomatization, formalization, and schematization elements. According to this author, axiomatizing relates to organizing knowledge; formalizing is adapting and transforming the language based on symbolization; and schematizing requires generalizing the language in the form of laws and rules that are adapted to reality through abstraction. Freudenthal emphasizes that the problem is due to the fact that only formalizations appear in school mathematics.

Polya (2004) proposes a work route based on problem solving. For him it is important to follow four steps: First, start by comprehending the problem. Second, imagine a resolution plan that is related to the type of relationships described between the variables. Third, execute the plan. Fourth, check and interpret. Check the result, the reasoning used and the possible alternatives to obtain the result. Evaluate whether the result is obvious and whether the result or method can be used to solve the other problems.

This work proposal from a problem-solving perspective allows motivating the student, which is an important element in their learning, however, it is easy to find students that desist and dismiss their ability to learn mathematics. This, according to Wallrabenstein (1973), is due to the fact that the conceptual part is often rigorously worked on, rather than the problems that make the subject interesting, creating monotonous educational environments, far from motivation, leaving the educational process subject only to the formalization.

Faced with this problem, the teaching of topology appears as one of the possible solutions, since from the study of this area it is possible to start from the interests and experiences of the students (Hilton, 1971), allowing them to broaden their base of experiences and, for example, build more elaborate notions such as geometric space (Wallrabenstein, 1973). According to the school level, the experiences that students acquire in the study of topology can become significant, becoming the basis of more sophisticated knowledge that becomes indispensable for more abstract constructions that the student could encounter (Ke, Monk, & Duschl, 2002).

Another advantage, according to Piaget and García (1998), occurs because the usual notion of tridimensional space is constructed starting from topological space, followed by forecast space, and ending with Euclidean space, following an order contrary to that of the historical order in which geometry was formalized.

According to Wallrabenstein (1973) children are more easily interested in topological problems than in analytical geometry problems. Topology allows crossed connections with other mathematics disciplines, like set theory, logic, and combinatory. For Kawauchi and Yanagimoto

(2012), the study of topology allows the development of spatial thinking and occurs at all levels of schooling.

Hence, if the topology is included in early school years, we will achieve an important degree of familiarity with it, making it possible for it to be used as a common thread in school mathematics, allowing students to relate different parts of mathematics from an early. According to Whitney (2017), topology has penetrated almost all other branches of mathematics. Basically, the study of topological notions in school mathematics courses opens the door to create an important symbolic conceptual base that the student uses for their own benefit.

METHODOLOGY

This research aimed to determine by means of a bibliographic study what the literature indicates about the teaching of topology, to identify methods used in the teaching of topology, and on the other hand, to detect the areas of knowledge where this is applied. To do this, the publications that relate topology and teaching are consulted in the Scopus database, the results of this search are applied the PRISMA methodology for systematic reviews, and thus obtain a selection of documents to which a Bibliometric analysis of the articles selected between 1966 and 2020.

The Scopus database, from the Elsevier publishing house, was selected as it is considered one of the most recognized bases for abstracts and citations of the scientific literature in the world. Within this database, searches can be carried out by using Boolean operators and field codes that connect the author's name, title, keywords, publication date, area of interest, among others. The words that guided the search were topology and teaching, connected with the Boolean expression AND in the search field ABS, that is, the query was focused on the articles that contained these two terms in the abstract.

The search performed identified 329 publications, these were analyzed using the PRISMA (Moher, 2010). methodology and 70 articles were selected from this first stage, which were subsequently reviewed based on the criteria described in Table 1.

Criterion	Description
Article Title	Article title
Keywords	Keyword 1, Keyword 2, Keyword 3, ...
Number of authors	Quantity of authors (1, 2, 3, …)
Authors	Name(s) of the author(s)

Institutions of the Authors	Universities or research centers of the authors
Project or grant related to the project	Type of relationship of the article with any project
Year	Year of publication of the article (1974, 1975 , 1976, …, 2018).
Country	Place of origin of the journal where the article was published.
Journal	Name of the Journal.
Journal's SJR Quartile 2017	It is an indicator that evaluates the importance that a journal has within its area.
Journal's H Index	It is a numeric value that measures the quality of the scientific production of a journal.
Number of Citations	Number of times that the article has been cited by other authors.

Table 1. Criteria for article analysis.

The analysis of the articles was conducted through a database that contains the criteria of table 1, the systematization of the information was developed from the detailed reading of each article, and the quantitative analysis of the information was performed with the Statistical Package for the Social Sciences (SPSS).

In a second part of the study, a qualitative analysis of the selected articles was carried out by classifying them into two categories: 1- teaching of topology, and 2- applications of topology in learning other areas of knowledge. These categories arose a priori since our objective was to identify methods used in the teaching of topology, and areas of knowledge where this discipline is applied. In table 2, the criteria used in each one of the categories are described.

Analysis Criterion	**Teaching of Topology**	**Application of topology in learning**
Level of Education	High school students, university students, university professors, or workers	
Area of education	Career being studied by the population in which the study was developed	
Conceptions addressed	Topological concept that appears in the article	NA
Description of the method	Method used for teaching the concept	NA
Topic of application	NA	Subject matter on which the study is developed
Type of application	NA	Software, course, theoretical work

Table 2. Criteria of the qualitative analysis. *(NA: Not Applicable*).

RESULTS

Using the steps indicated by the PRISMA methodology for the systematic analysis of documents, 70 works were selected that in some way mentioned the teaching of topology and the applications of topology in the learning of different areas of knowledge.

In the descriptive analysis, we found that 27% of the articles were written by a single author, 24.2% were written by two authors, 18.5% were written by three authors, and the remaining 30% were written by more than three authors (Figure 1), consequently, it is possible to conclude that this type of research has a low index of cohesion. In addition to this, among the 208 authors of the articles analyzed, we found that only two of them, Debao Chen and Feng Zou, possess authorship in two studies, which evidences a potential research current because of the small number of authors that conduct studies in this direction.

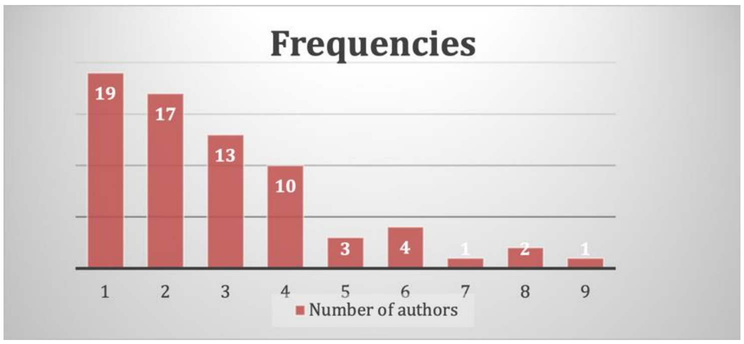


Figure 1: Number of authors per article.

On the other hand, in the articles analyzed, we found 240 different keywords, of which, only 13 are repeated in more than one article, while the remaining appear only once in any of the articles. The keywords appearing with a frequency of greater than one are: Education, Power Electronics, Project-Based-Learning (PBL), Teaching-Learning-Based-Optimization (TLBO), Topology and Virtualization (Figure 2).

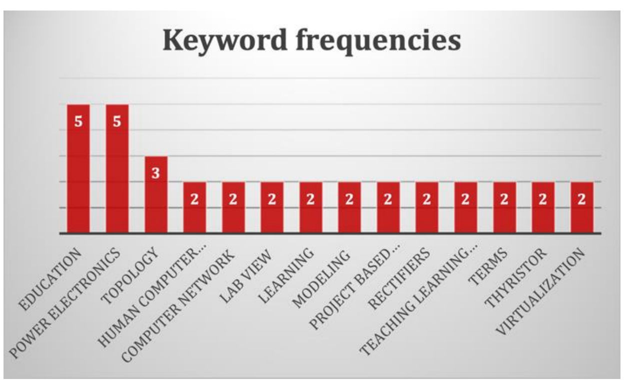


Figure 2: Keywords with Highest Frequency.

Table 3 presents the list of the 46 journals that were found in the search, as well as the country, quartile and journal's H index. The list allows establishing that there are four journals that have published the most on the topic: International Journal of Mathematical Education in Science and Technology (United States), IEEE Transactions on Education (United States), International Journal of Electrical Engineering Education (United Kingdom), and IFIP International Federation for Information Processing (Germany). The first, is a journal specialized in mathematical education at all levels of education. The second, focuses on educational methods and technology related to the theory and practice of electrical engineering and computer science. The third, publishes on learning in engineering addressing topics such as: teaching methods, curriculum design, assessment, among others. The fourth, is a journal on information systems that contributes to the knowledge on the use of computers, free information access, right to privacy, and protection of confidential data.

Journal Title	Country	Quartile SJR 2020	H index	Items
International Journal of Mathematical Education in Science and Technology	United States	Q2	33	4
IEEE Transactions on Education	United States	Q1	68	3
International Journal of Electrical Engineering Education	United Kingdom	Q3	23	3
IFIP International Federation for Information Processing	Germany	Q4	37	2
Acta Scientiarum Naturalium Universitatis Pekinensis	China	Q3	18	1
2008 IEEE Vehicle Power and Propulsion Conference	United States	NYAQ	26	1
2017 Australasian Universities Power Engineering Conference, AUPEC 2017	Australia	NYAQ	4	1
2019 5th International Conference on Control, Automation and Robotics, ICCAR 2019	United States	NYAQ	3	1
2019 Electric Power Quality and Supply Reliability Conference and 2019 Symposium on Electrical Engineering and Mechatronics, PQ and SEEM 2019	United States	NYAQ	5	1
2019 IEEE 15th Brazilian Power Electronics Conference and 5th IEEE Southern Power Electronics Conference, COBEP/SPEC 2019	Brazil	NYAQ	NYAQ	1
32nd AAAI Conference on Artificial Intelligence, AAAI 2018	United States	NYAQ	72	1
ACM International Conference Proceeding Series	United States	NYAQ	123	1
ACM Sigcse Bulletin	United States	NYAQ	47	1
Advanced Science Letters	United States	NYAQ	27	1

American Biology Teacher	United States	Q2	30	1
American Journal of Physics	United States	Q2	99	1
Applied Categorical Structures	Netherlands	Q2	31	1
ASEE Annual Conference and Exposition, Conference Proceedings	United States	NYAQ	34	1
CEUR Workshop Proceedings	United States	NYAQ	52	1
Communication Quarterly	United Kingdom	Q2	45	1
COMPEL - The International Journal for Computation and Mathematics in Electrical and Electronic Engineering	United Kingdom	Q3	31	1
Computer Applications in Engineering Education,	United States	Q2	29	1
Computer Communications	Netherlands	Q!	105	1
Computers & Geosciences	United Kingdom	Q1	123	1
Computers and Structures	United Kingdom	Q1	138	1
EDUNINE 2018 - 2nd IEEE World Engineering Education Conference: The Role of Professional Associations in Contemporaneous Engineer Careers, Proceedings	United States	NYAQ	5	1
Energies	Switzerland	Q2	93	1
ICIC Express Letters, Part B: Applications	United States	Q4	12	1
IECON Proceedings (Industrial Electronics Conference)	United States	NYAQ	71	1
IEEE International Conference on Electro Information Technology	United States	NYAQ	20	1
IEEE LATIN AMERICA TRANSACTIONS	United States	Q3	26	1
IEEE Transactions on Power Apparatus and Systems	United States	NYAQ	0	1
IEEE TRANSACTIONS ON SYSTEMS, MAN, AND CYBERNETICS: SYSTEMS,	United States	Q1	64	1
IET Circuits, Devices and Systems	United Kingdom	Q3	49	1
International Journal on Digital Libraries	Germany	Q2	32	1
Int. J. Advanced Media and Communication	United Kingdom	Q2	12	1
International Journal of Emerging Technologies in Learning (iJET)	United States	Q3	11	1
International Journal of Engineering Education	Ireland	Q1	50	1
International Journal of Pure and Applied Mathematics	Bulgaria	Q4	24	1
International Journal of Recent Technology and Engineering	India	NYAQ	20	1
International Review of Research in Open and Distance Learning	Canada	Q1	50	1

Journal of Molecular Modeling	Germany	Q3	69	1
Journal of Computational Chemistry	United States	Q1	188	1
Journal of Higher Education Policy and Management,	United Kingdom	Q1	42	1
Journal of Mathematical Psychology	United States	Q1	69	1
Journal of Physics: Conference Series	United Kingdom	Q4	85	1
Journal of Systems and Software	Netherlands	Q1	109	1
JournalofNortheasternUniversity	China	Q4	20	1
Mathematical Modelling of Natural Phenomena	France	Q2	36	1
Neurocomputing	Netherlands	Q1	143	1
Open Cybernetics and Systemics Journal	Netherlands	Q4	8	1
Plos One	United States	Q1	332	1
POLLACK PERIODICA	Hungary	Q3	11	1
Power System Technology	China	Q1	68	1
Proceedings - 9th International Conference on Information Technology in Medicine and Education, ITME 2018	United States	NYAQ	6	1
Proceedings of 2019 8th International Conference on Modern Power Systems, MPS 2019	United States	NYAQ	3	1
Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences	United Kingdom	Q1	135	1
Revista Facultad de Ingeniería, Universidad de Antioquia.	Colombia	Q4	12	1
Scientific Reports	United Kingdom	Q1	213	1
SIGCSE Bulletin (Association for Computing Machinery, Special Interest Group on Computer Science Education),	United States	NYAQ	47	1
Structural and Multidisciplinary Optimization	Germany	Q1	117	1
Studies in Computational Intelligence	Germany	Q4	68	1

Table 3. General Journal List. (*No yet assigned quartile 2020. NYAQ)*

Figure 3 presents the journals' quartile in which the research found in this analysis were published, which represents the importance of each publication in their corresponding area. We found that 17 journals belong to the first quartile, 10 to the second, 8 to the third, and 8 to the fourth, in addition to that, we found 19 journals that do not report a quartile since they are Conference Proceedings.

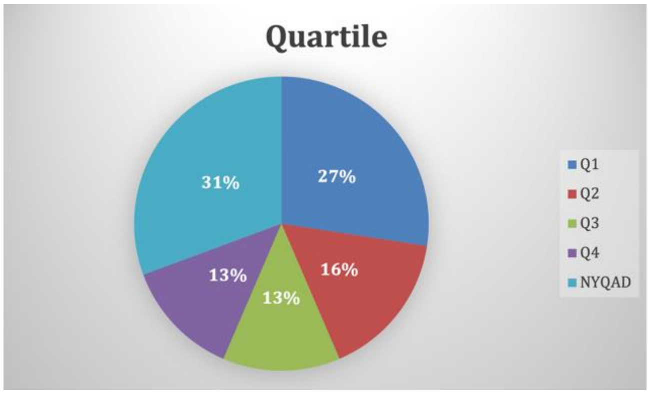


Figure 3. Journals' Quartile

Figure 4 shows each journal's country of origin, together with the number of publications. As can be observed, the country with the highest number of articles is the United States (45%), followed by the United Kingdom (17.7%) and, the latter in turn, by Germany and the Netherlands with (8%). The foregoing allows concluding that the United States is the largest generator of knowledge on teaching of topology and applications of this area in learning of other branches of knowledge.

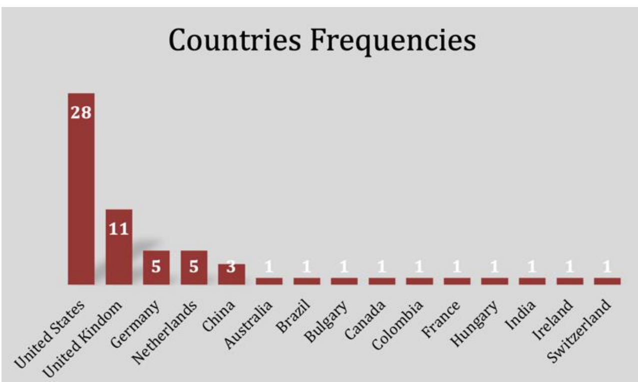


Figure 4. Journals' countries of origin.

It was possible to determine the number of citations of each article according to the Scopus database and establish that the range of citation of the articles, except for one, is between 0 and 62, table 4 shows the list of most cited articles that were found in the analysis. The atypical value corresponds to the article published by Lu and Chen (2012), cited 8321 times as of June 2021. This article describes the development of a software named Multiwfn, whose objective is to analyze wavefunctions, visualization of molecular and orbital structures, analysis of energy and density variations, generation of initial optimum conditions for wavefunctions, and other types of tasks that are performed in the area of quantum chemistry.

Article Title	Authors	Year	Journal Name	Citations According to Scopus
Multiwfn: A multifunctional wavefunction analyzer	Tian Lu and Feiwu Chen	2012	Journal of Computational Chemistry	2,684
Experiences in the application of project-based learning in a switching-mode power supplies course	Diego Lamar, Pablo Miaja, Manuel Arias, Alberto Rodríguez, Miguel Rodríguez, Aitor Vázquez, Marta Hernando y Javier Sebastián	2012	IEEE Transactions on Education	33
A new approach for teaching power electronics converter experiments	Jacinto Jiménez, Fulgencio Soto, Esther de Jódar, José Villarejo y Joaquín Roca	2005	IEEE Transactions on Education	32
An improved teaching-learning-based optimization with neighborhood search for applications of ANN	Lei Wang, Feng Zou, Xinhong Hei, Dongdong Yang, Debao Chen y Qiaoyong Jiang	2014	Neurocomputing	27
On some fundamental properties of structural topology optimization problems	Mathias Stolpe	2010	Structural and Multidisciplinary Optimization	20

Table 4. List of most cited articles.

From the number of publications, we can observe that between the years 1971 and 2005, a 34-year period, 12 articles were published, while between the years 2006 and 2020, a 14-year period, 58 articles were published (Figure 5), which shows a significant increase in the amount of research related to this topic in recent decades. Of the 70 articles analyzed, only 21 mentioned having received support from an institution or research funds, and among these institutions, the National Natural Science Foundation of China (NSFC) is noteworthy because it is the one that supported the greatest number of articles in this field of knowledge.

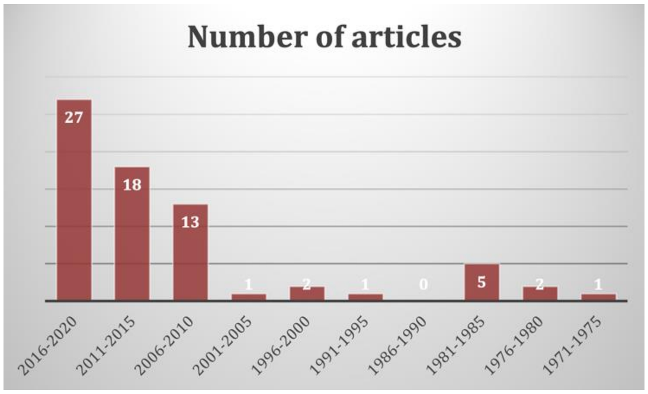


Figure 5. Number of articles in periods of five years.

QUALITATIVE ANALYSIS

The second part of the results corresponds to a qualitative analysis of the chosen articles by classifying them into two categories: the first, articles that refer to the teaching of topology, and the second, applications of topology in the learning of other areas of the knowledge. These categories arose a priori since our objective was to identify methods used in the teaching of topology, and areas of knowledge where this discipline is applied.

Based on the rigorous reading of the chosen articles, we found 7 belonging to the first category, among these, some focused in the teaching of topology for mathematics students, others seeking to bring students with basic and mid-levels of education closer to topological conceptions, and, finally, articles that seek to get students, belonging to careers in knowledge areas with no evident relationship with mathematics, to make interpretations of topological notions from the perspective of their field of expertise.

Table 5 shows the characteristics that were considered for the analysis of each one of the articles.

Article	Conceptions addressed	Level of Education	Area of specialization	Method Description
(Vukmirovic, Devetakovic, & Petrusevski, 2012)	Möbius strip; Genus of closed surfaces.	University students.	Architecture.	Building materials; Paper; Software for Image Design;
(Nishiyama, 2011)	Space continuity; Topological puzzles.	University students.	Teaching of mathematics.	Riddles; Drawings; Flexible Materials.
(Lawvere, 1996)	Continuity in space proposed by Walter Noll.	NM	NM	Theoretical approach.
(Garay, 1993)	Genus of closed surfaces; Homeomorphism; Fundamental Group.	University students.	Teaching of mathematics.	Theoretical approach; Drawings.

(Kanellopoulou, 1982)	Euler Characteristic; Nodes; Regions.	High school students.	NM	Curriculum and learning primers.
(Neubrand, 1981)	Set theory; Group theory; Topology; Category theory.	University students.	Mathematics.	Spiral curriculum.
(Baylis, 1977)	Continuity on sets.	NM	NM	Theoretical approach.

Table 5. Research on teaching of topology. (*NM: No Mention*).

In the articles focused on teaching of topology for mathematics students, the authors associate some basic topological concepts to other mathematics areas like group theory, category theory, and even unidimensional calculus, aiming for students to find the connection between these branches and understand the importance of topology in learning of mathematics in general (Neubrand, 1981).

In the articles focused on teaching topological notions to students with basic and mid-levels of education, the authors show the properties from approaches that do not use rigorous language corresponding to mathematical writing, for example, in Nishiyama (2011) the notion of continuity, probably the most important conception studied in topology, is approached performing an intuitive approximation through the use of flexible materials, drawings, and riddles, putting mathematical writing aside, that in part, because the author intends to explain this conception to a population that does not possess the necessary mathematical knowledge to understand it intricately.

In Garay (1993), using intuitive approaches and some formal definitions, the aim is to bring students closer to more advanced concepts like homeomorphism, fundamental group, Euler characteristic, genus of closed surfaces, among others. In this case, literary language and some drawings are used, with the intent of bringing the student closer to these conceptions unconsciously, subsequently the topological notion is introduced, followed by a formal definition. With this strategy, the authors seek the theoretical concepts addressed to be understood in a simple manner by students.

In Vukmirovic, Devetakovic, and Petrusevski (2012), students with careers having no evident relationship with mathematics are brought closer to some topological notions, more specifically, the objective is for some students from the career of architecture, to appropriate the conceptions that can be approached intuitively like the Möbius strip and the genus of closed surfaces, and by making use of the materials and software specialized for the design, an interpretation of these concepts is made from the perspective of their area of expertise through the design and construction of sculptures inspired in these topological surfaces.

It is possible to establish that the areas that most incorporate applications of topological concepts in their curricula are Systems Engineering, Electronic Engineering, and Electrical Engineering.

In the case of Systems Engineering, the topological application that most appears in the chosen articles is the use of networks for data management, currently this topic is of great importance because of the growing need of exchanging information between computer servers throughout the world. From a theoretical perspective, the servers' interconnection networks and their software are based on discrete topology constructions, where each connection point and each connection line are modeled through graphs (Maheswaran et al., 2009; Montero & Manzano, 2017). The schemes used in these networks are called topologies, being the most common the mesh, star, tree, bus, and ring topologies, that applied to computer networks present characteristics that allow certain benefits and limitations in information exchange, these characteristics are taken advantage of by engineers when designing networks (Wang et al., 2014).

In Electronic and Electrical Engineering, it is also possible to find the property of having elements interconnected between each other, whether to transport information or electricity. In this case, each topology allows different constructions that are taken advantage of according to the need of optimizing the performance of the circuit to be designed (Mohammed, Kawar, & Abugharbieh, 2013; Kazimierczuk, & Murthy-Bellur, 2012). Each circuit has its own topological structure, and in the cases of the power networks also a geometrical structure, which allows the construction of mathematical models that describe its behavior (Altintas, 2011; Xu et al., 2012). Since the industry and technology require more effective and reliable systems every day, new applications of topology arise in this area, showing us that for our case, a high frequency of articles in this direction.

To a lesser extent, we find articles in diverse areas like: Management, Architecture, Arts, Librarianship, Biology, Economics, Physics, Geology, Chemistry, and Environmental Engineering. In arts, an article was found for the design of a piano course (Wei, 2018), another in which Chinese characters are studied using their topological structure (Sun et al., 2014), and one of artistic creation where the students performed organized tasks from social interaction networks (Kawka, Larkin, & Danaher, 2011). In psychology, there were articles that analyzed interpersonal relationships from the topology of networks (Dudley, 1976), and development of critical thinking and analysis of learning in school environments (Hamizan, Zaid, & Noor, 2016). In most of these articles, we find that topology becomes a tool for the analysis of a structure, a class, or a method. Other applications of topology are related to the visualization of complex structures like DNA or molecular structures that can be understood from their topological properties (Halverson, 2010; Lu & Chen 2012; Robic & Jungck, 2011).

It is noteworthy to mention, how most of the applications are addressed toward the university level of education, this could be because the management of technological concept is currently carried out to a greater degree in higher education, so its application does not easily reach research in basic and middle education.

PROJECTION

This work allows us to think about the convenience of including or not including notions of topology in school. When reviewing the data, we noticed the slightly diffuse idea that it is possible to do it. This, instead of providing us with answers, introduces us to other questions about how topology should be introduced in our schools. In this regard, different works appear in the literature. Steven Greenstein (2014) mentions how studies based on Piaget show that children are able to work and understand conceptual geometry, a fact that for many means that geometric work can be oriented towards topological work.

Gavilan et al (2022) carry out work on graph theory that, although they do so with university students, work on an interesting proposal in the field of analysis of student learning processes and possible cognitive conflicts that appear, and that can be overcome through the use of different types of representation systems.

In another study, Whitney (2017) mentions how it is possible to work on topology in preschool level courses. For this author, the first objective is to show that introducing topological terms and teaching topological concepts in our schools could be improved. On the one hand, students' geometric intuition could be strengthened if we routinely encourage them to compare and contrast topological and geometric ideas. Furthermore, if topology is introduced at school stages, students can become familiar with an area of mathematics that in modern times has penetrated almost every other branch of mathematics.

This author calls on the research community to continue studying how basic education students understand topological conceptions in relation to geometric concepts. He ultimately proposes that: If the K-12 mathematics curriculum begins to incorporate some topological ideas, we might possibly observe many beneficial consequences (Whitney, 2017).

In this sense, Genevieve and Eunsook (2005) mention how the teaching of geometry in elementary school goes hand in hand with the discovery of the world, its forms and its regularities, to conceptualize a vision of space. *Overall, while not totally disproven, the topological primacy theory is not supported. It may be that children do not construct first topological and later projective and Euclidean ideas. Rather, it may be that ideas of all types develop over time, becoming increasingly integrated and synthesized. These ideas are originally intuitions, grounded in action...* (Clements & Battista, 1992,pp. 425-426). These authors do not affirm that, however, the ideas of topology will have to appear first in children and then Euclid's projectivist geometry, but rather that the two are gradually being built, interrelated and consolidated.

For Genevieve and Eunsook, when working with children in the construction of maps, it is possible to find topological relations involving proximity and separation, spatial order, enclosure and continuity; projective relationships involving perspective; and Euclidian relationships including proportion, distance, and relative size. (Genevieve & Eunsook, 2005 p 80)

Finally, Sugarman (2014) presents us in her thesis a work proposal in topology for school grades supported by the study of mathematics education reforms. Here, the author prepared a topology teaching work in grades 4, 5 and 6, applying it only in grade 5, where he concludes that it is possible to teach topology at the school level. For the Discussing Unknowns in Topology presents math as a subject in which we do not have all the answers, a viewpoint which is not commonly seen in early mathematics education.

As a complement to this bibliometric analysis work, some teaching designs of topology didactics were made to a group of future mathematics teachers. There, the theoretical part was worked on, complementing it with practical exercises on each of the topology topics covered. This type of work can be extrapolated to apply in secondary education. The sequence of proposed topics is divided into chapters, which can be worked on at different levels of the secondary education.

Unit 1: Introduction to graph theory

1. Definition of graph
2. Concepts and terminology
3. Eulerian paths and cycles
4. Fleury algorithm
5. Hamiltonian paths and cycles

Unit 2: Polyhedra and Euler characteristics

1. Regular Polyhedra
2. Graph, plane
3. Faces of a flat graph
4. Euler's formula for polyhedra
5. Triangulation of surfaces
6. Euler characteristic, topological invariant

Unit 3: The 4-color theorem

1. History of the 4 Color Theorem
2. Activity: Map painting
3. Graphs, plans
4. Double a map
5. Coloring graphics
6. The error in Kempe's proof

Unit 4: The Mobius strip and the Klein bottle

1. The Mobius Strip.
2. The Klein Bottle.
3. The projective plane as conscious spaces.
4. Activity (Triangulate and color).

Unit 5: Introduction to knot theory

1. History of knot theory.
2. Knots in topology and their classification
3. Prime knots
4. Activity (Knotplot Software)

Unit 6: Basic topological notions.

1. Topology definition

2. Open, closed sets.
3. Interior, exterior, adherence, limit, isolated points.
4. Topology on finite sets
5. Usual topology in the line and the plane.
6. Continuous functions. Homeomorphism.

Each of the proposed themes has the possibility of being introduced from the students' own conceptions and taken to active work, for example, playing at following paths in graphs, using polyhedrons of different materials to check the Euler characteristic, drawing and coloring maps, building the Mobius strip and describing characteristics of the knots that the students know how to make. Also from set theory, and using different types of representations, it is possible to understand the basic topological notions. Each experience lived by the students enables the creation of new experiences that enhance learning not only of topology but of other fields of mathematics, by requiring in each activity processes such as: serialization, hierarchy, problem solving, among others.

CONCLUSIONS

At the beginning of this study, we saw that topology originated in consequence of the need of identifying objects by their position in space, putting aside the notion of distance between thereof (Euler, 1741), this way of conceiving space began to gain importance in recent years, mainly by the methods used to exchange information. In this bibliometric study, publications focused on learning topology and the use of topology in teaching other areas were identified.

The criteria of the bibliometric analysis allowed concluding that the country where most of the articles of this type were published was the United States, followed by the two main European academic powers, United Kingdom and Germany, which can be interpreted as a consequence of tradition and strength of the educational institutions in these countries. On the other hand, the articles that were found in this search are in their majority published in Q2 and Q3 quartile journals, which demonstrates that these publications achieve a medium impact on the academic environment, this generates certain confusion if the relevance that these areas have to the development of technologies are considered, in addition to their importance in the construction of the theoretical framework of mathematics.

Regarding the number of authors per article, it was found that in most cases, they do not exceed more than four per article, this shows a low index of cohesion in these researches, which implies a potential research current because of the small number of authors that conduct studies in this direction. On the citations, it was possible to establish a low range of citation, with the exception of an atypical datum appearing in the Lu and Chen (2012) article, which is cited 8321 times by other authors. Finally, with respect to the H index, it is possible to conclude that the journals with the highest index, are in their majority, focused on applications of this branch of mathematics, while the journals specialized in teaching have slightly lower indexes.

Through the qualitative analysis, it was possible to identify some methods used by mathematics professors to encourage the learning of topology in university students, furthermore, articles were found focused toward the teaching of this area to basic and middle school students. Among the most common methodologies, is the use of intuitive approximations to topological concepts through drawings, puzzles and examples (Garay, 1993), in addition, it was possible to identify curricula constructions in which attention is given to the connection of topology with other branches of mathematics (Neubrand, 1981). From this we can conclude that topology allows intuitive approaches to its basic conceptions, which can facilitate its learning in students with elementary levels of training in mathematics (Gavilan et all 2022). The number of publications on the teaching of topology suggests that it is still a branch of work that does not receive much attention from the academic community of researchers in mathematics and mathematics teaching.

On the other hand, it was possible to find some topological applications in learning different branches of the knowledge, which were classified in area, topic, and type of application. Within these categories, there are designs of courses, software use, among others. One of the most interesting articles in design of courses is MacArthur and Anderson (1981) (2017), in which a learning program for power plant operators is designed, in these cases the adequate operation of the plant depends mainly on the interconnection with other similar plants, consequently, it becomes relevant for operators to understand the topological structure of these networks.

Among the articles that use some type of software to teach applications of any topological notion, one of the most important ones is Lu and Chen (2012), which uses, like one of its multiple functions, the topological structure of the molecules to study their properties from a quantum chemistry approach, which implies a potential learning tool in this area of knowledge. Another type of software appearing in some of the articles analyzed, for example in Altintas (2011) and Xu et al. (2012), consists in simulation programs of power plant processes, which are used to train operators and university students with less resources than those used in traditional teaching methods. Finally, it is noteworthy to name the application in the study of computer networks, which use the topological structure of the hardware and software used in the network to optimize information Exchange processes (Montero & Manzano, 2017; Restrepo et al., 2016; Xu et al., 2012).

The above verifies the relevance of topology in the evolution of the modern society, since these articles use applications of topological properties in the teaching of optimization processes and in the generation of other types of useful knowledge, mainly in the case of technological development, therefore the learning of this area in basic and middle education curricula should be encouraged, in a similar way than in other branches of mathematics like algebra, calculus, and geometry (Vizcaino & Terrazzan 2020), (Castiblanco & Nardi, 2018).

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