# The Use of Variation Theory of Learning in Teaching Solving Right Triangles

Joel B. Mendoza[1], Minie Rose C. Lapinid[2]

[1]South View High School, Hope Mills, North Carolina, USA

[2] De La Salle University, Manila, Philippines

joel.balat.mendoza@gmail.com, minie.lapinid@dlsu.edu.ph

*Abstract: Teachers are often perplexed realizing students ending up with different understandings of the same lesson after attending the same class. This study investigates the use of Variation Theory as a pedagogical design tool in improving students' problem-solving skills in trigonometry. This action research utilizing the 'Learning Study' approach was conducted in a Filipino-Chinese private school in a highly urbanized city in the Philippines. Two Grade 10 intact classes consisting of a total of 41 students and three mathematics teachers participated in the learning study. Selected students were interviewed to validate students' intended and lived objects of learning. The video-recorded lessons were examined to determine the alignment of the intended and enacted objects of learning. The analysis of the pretest and posttest showed the use of different patterns of variation and invariance in teaching was able to address students' misconceptions and difficulties in solving right triangles and helped the students understand problems better. Thus, the Variation Theory of learning as a pedagogical design tool is deemed effective in improving the students' knowledge, procedural, and problem-solving skills, and in bridging the gap between the intended objects of learning and the lived objects of learning.*

Keywords: Variation Theory, Learning Study, Object of learning, Problem-solving, Trigonometric ratios

## INTRODUCTION

Over the years, different studies have been conducted in order to determine how students learn (Felder & Brent, 2005; Garfield, 1995). Findings from these studies have suggested different approaches on how teaching can be conducted and how learning can be facilitated (Prince, 2004; Dunlosky et al., 2013). Despite these efforts to make teaching and learning more effective, "educators, researchers, and policy-makers worldwide continue to struggle to understand the needed changes to improve educational outcomes and educational attainment for students, particularly in the content areas such as science and mathematics" (O'Dwyer et al., 2015, p. 1). There still seems to have no consensus reached as to which kind of teaching is the best that would ensure effective learning (Lo, 2012). Educators are still puzzled as to how two students sitting in the same class, given the same instruction and using the same materials, end up with a different understanding of the concept taught (Bussey et al., 2013). The Philippines is no exception to this problem. The performance of the Filipino students in international large-scale assessments such as the Programme for International Student Assessment (PISA) is a glaring indicator the country has consistently performed poorly in mathematics (OECD, 2019).

One interesting thing to ponder is how our neighboring East Asian countries such as Singapore, Hong Kong SAR, South Korea, Chinese Taipei, and Japan performed very well in the said study. The results published by the International Association of Educational Assessment (IAEA) showed that these countries continue to dominate the rankings and outperform other participating countries notwithstanding a pronounced gap of 48 points is observed between these top performing countries and the next highest performers (Mullis et al., 2016). Several studies have been conducted as to how these East Asian countries outperformed other countries in international benchmarking tests despite the unfavorable classroom image such as large classes, teacher-dominated classroom, among others (Mok, 2006; Wong, 2013; Lim, 2007). The East Asian paradox intrigued many educators which led to conducting studies among Asian classrooms. Findings of the studies showed that contrary to the notion of the outsiders that Asian classrooms are teacher-dominated and students seemingly practice rote memory learning, the teachers presented the lessons with variations (Lim, 2007; Mok, 2006). These findings support Ference Marton's study and his Theory of Variation. The Variation Theory (VT) provides a framework that the learners must experience variation in the critical feature of a concept, within limited space and time, in order for the concept to be learnable. VT started to gain popularity in Hong Kong, Mainland China, and Sweden. Studies were also conducted in Brunei, Japan, and Malaysia with results revealing positive effects of using VT as a pedagogical tool in designing lessons.

According to Marton and Booth (1997), one possible reason for students' difference in understanding of the concept is the difference in their perspective of the lesson taught. One common mistake that teachers make is to assume that their students comprehend the lesson the way they expect them to. This assumption becomes a barrier to facilitating learning. As pointed out by Pang and Lo (2012), students experience a phenomenon differently; therefore, teachers must craft and deploy a pedagogy that suits the varying learning styles and other differences that impact comprehension.

From the theoretical point of view of phenomenography, every individual experience a certain phenomenon in a unique way. Hence, variation occurs among individuals who experienced the same phenomenon which leads to difference in conceptions (Samuelsson & Pramling, 2016). In order for understanding of the concept to happen, students' perspective should be drawn to the intended similar aspect. Discernment of this perspective allows students to learn what the teachers ought them to know. "The aspects of the phenomenon and the relationships between them that are discerned and simultaneously present in the individual's focal awareness define the individual's way of experiencing the phenomenon" (Marton & Booth, 1997, p. 101).

The first researcher has been teaching at the secondary school level for more than ten years when the study was conducted. Based on his teaching experience, a common problem among students is difficulty in solving problems. It is in this context that the researcher conducted this study using

VT in the hope that it can improve the problem-solving skills of the students in Trigonometry. As Lo (2012) pointed out, improvement in teaching can be done by changing the mindset through determining students' views on the concepts being taught since these are primarily the reasons for the variation in the attainment of the learning outcomes.

**Variation Theory**

Variation theory is a theory of learning and experience which explains "how a learner might come to see, understand, or experience a given phenomenon in a certain way" (Orgill, 2012, p. 3391). Moreover, this theory claims that an object may be interpreted by people differently, which results in different understanding (Lo, 2012). In view of this, the theory states that for learning to happen, discernment of the critical aspects of learning must take place. Discernment only happens when the students are directed at the object of learning.

According to Marton and Pang (2006), there are four ways wherein discernment of variation can happen: contrast, separation, generalization, and fusion.

> a. "The principle of contrast. To discern quality X, a mutually exclusive quality ~X needs to be experienced simultaneously. For instance, to understand what a fraction is, students need to be presented with non-examples, such as a whole number or a decimal.
> b. The principle of separation. To discern a dimension of variation that can take on different values, the other dimensions of variation need to be kept invariant or varying at a different rate. For instance, if teachers want students to understand the relationship of a numerator to the value of a fraction, then they may keep the denominator invariant but vary the numerator. In this way, students' attention will be drawn to the numerator, which has been separated from the other critical aspects that affect the value of the fraction.
> c. The principle of generalization. To discern a certain value, $X_1$, in one of the dimensions of variation X from other values in other dimensions of the variation, $X_1$ needs to remain invariant while the other dimensions vary. For instance, to help students to generalize the concept of 1/2, teachers may give all kinds of examples that involve 1/2, say half of a pizza, half of an apple, half of an hour, etc.
> d. The principle of fusion. To experience the simultaneity of two dimensions of variation, these two dimensions need to vary simultaneously and be experienced by the learner. For instance, to enable students to understand the two critical aspects of numerator and denominator in determining the value of a fraction, teachers may vary both the numerator and the denominator at the same time, systematically, such as 1/2, 2/3, 3/4, 4/5, etc." (Pang, 2008, pp. 5-6)

Several research studies have already been carried out to explore the use of VT as a guiding principle for pedagogical design in teaching. The findings of the study conducted by Lam (2012) in examining the effectiveness of VT as a pedagogical tool to help students learn chemical reaction rates by using the group-based scientific investigation methodology proved VT to have helped academically-challenged students understanding their lessons.

According to Lo (2012), the tenets of VT complement the other teaching principles that are deemed effective by the academic community. Specifically, the VT principles guide the teachers in sharpening the "focus of the object of learning, which resulted in the students acquiring a better understanding of the role of characteristics and interaction with a storyline" (Tong, 2012). The theory also serves as a reference for educators in designing proper pedagogical approaches to assist students in determining the object of learning. However, a particular pattern of variation and invariance must be in place to cater to different objects of learning (Marton & Pang, 2006). Given this context, VT can be considered as a "theoretical grounding to understand some of the necessary conditions of learning so that wise pedagogical decisions can be made. The principles of VT imply what features of the object of learning have to be invariant and what should vary in the students' experience" (Lo & Marton, 2012, p. 7).

According to Kullberg et al. (2017), studies on the use of systematic variation in teaching mathematics within a VT framework prove that the theory, as a design principle, can help students notice specific contents of the lesson, which eventually result in better understanding and learning. Hence, it is essential to carefully select the factors that are critical to learning and to keep the unimportant invariant.

The findings of Cheng (2016) are in parallel with the conclusions of other studies about the use of VT in teaching mathematics (e.g., Al-Murani, 2007; Mhlolo, 2013). The latter studies assert that using the variation framework as a pedagogical design helps students discern certain aspects of the lessons and therefore, increases their ability to comprehend the concepts being discussed. Lo (2012) stressed further that using VT as a guiding principle for pedagogical design ensures that teachers employ effective teaching strategy and learning activities that are focused on the object of learning and critical aspects. This prevents the lesson from deviating from its objective and wasting valuable teaching time, and avoids students discerning other objects of learning that are inappropriate and not worth learning.

In a study conducted by Pang & Lo (2012), teachers who intentionally used the tools of VT in teaching were better compared to those who unknowingly used variation in teaching. The former group was able to effectively manage the class discussion and to inject changes in the lesson plan which resulted in enhanced student learning. This highlights the need for teachers and course designers to have a full grasp of the critical aspects of learning and the patterns of variation and invariance that help students notice them. Moreover, teachers should have a clear plan on how they can draw their students toward these patterns of variation and invariance. As Marton & Pang (2013) noted, "if an aspect that we want our students to notice is varied against an invariant background, it is more likely that students will discern it."

## Object of Learning

An *object of learning* refers to the "specific insight, skill, or capability that the students are expected to develop during a lesson or during a limited sequence of lessons" (Marton & Pang, 2006). It is used to denote "the 'what' aspect of teaching and learning" (Häggström, 2008). In the context of the classroom, the object of learning encompasses everything that students are supposed to learn from what the teachers are teaching.

There are three perspectives that are used in VT to study the object of learning, namely the (1) lived object of learning, (2) intended object of learning, and (3) enacted object of learning. In these three perspectives, VT observes and evaluates the object of learning. The *lived object of learning* is the object of learning from the students' perspective. It describes what students actually learn which is influenced by what they perceive as important or valuable (Marton et al., 2004 as cited in

Häaggström, 2008). Their experience within the learning environment provides the basis for how they will make sense of the object of learning presented to them. Hence, it requires a close observation of how a particular object of learning is being developed and implemented vis-à-vis the direct and indirect aspects of learning during class discussions. On the other hand, when the lens by which classroom scenarios are viewed come from the perspective of the teachers, it is referred to as the *intended object of learning.* It denotes the intention of the teacher to help students acquire specific skills and capabilities by using his or her sphere of knowledge and experience. This is manifested in the end-to-end process of creating instructional materials (Marton & Tsui, 2004).

Lastly, the *enacted object of learning* deals with the perspective of a researcher with regard to the kind of learning experience that transpires in a particular situation. This is "co-constituted in the interaction between learners and the teacher or between the learners themselves. It is described by the researcher from the point of view of what was afforded to the learners." (Runesson, 2005, p. 7). The researcher examines the direct and indirect objects of learning that both impact learning in positive and negative manners (Marton & Pang, 2006). According to Runesson (2005), there are factors other than the teachers' intentions that dictate the possibility of learning; the classroom, books, and other instructional materials as well as the interactions among the teachers and students comprising the learning environment that influence the enacted object of learning.

The interrelationship of the three perspectives on the objects of learning is illustrated in Figure 1 adapted from Häaggström (2008). A Venn diagram is used to present the interplay of the different perspectives in particular instances. The partial overlap between the intended and enacted objects of learning indicate that not all intended objects of learning may be enacted in the classroom and/or not all objects enacted in the classroom are part of the intended object of learning. The same is true with the partial overlap between the enacted and lived objects, and the partial overlap between the intended and the lived objects. The central part which is common to all three circles indicates the intended object of learning that was enacted by the teacher and lived by the students. In an ideal scenario, the intended, enacted and lived objects of learning must totally overlap.

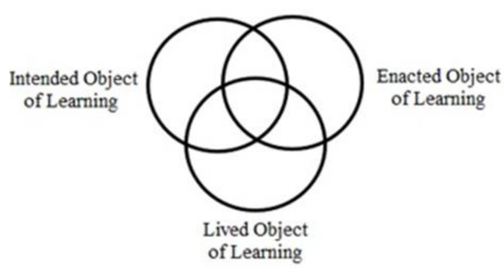



Figure 1: The object of learning.
Source: Häggström (2008)

**Learning Study**

The Learning Study is an approach where a group of teachers, ranging from two to six members, collaborate to determine effective ways to help students absorb a particular object of learning (Cheng & Lo, 2013). According to Marton and Pang (2006), Learning Study is inspired by the *design experiment* (Brown, 1992; Collins, 1992) where a lesson plan is taught to a control group and an experimental group, with the latter's lesson subjected to the VT (Kirkman, 2014) and the *Japanese lesson study* (Elipane, 2012; Stigler & Hiebert, 1999) where the lesson plan is subjected to the teach-review-teach-review cycle (Kirkman, 2014). To be specific, the Learning Study approach follows a systematic process, beginning with identifying the specific object of learning that students have difficulty comprehending. Once the need has been determined, the group proceeds with planning the lessons. The main goal of the research lesson plan is to help students absorb the object of learning. To achieve this, the teachers share their knowledge and experiences related to the particular subject. Kirkman (2014) added that it is also essential that assessment of the depth and breadth of knowledge of the students is taken into consideration when planning the lesson. From the insights gained in this step, the group then identifies which method helps them facilitate learning effectively. One of the teachers from the group then uses the lesson plan to teach the students while the rest observe so that they would know what needs to be enhanced, if necessary. Should there be revisions, another teacher can facilitate the discussion to a different group of students; this step illustrates the use of VT which Stigler and Hiebert (1999, as cited by Kirkman, 2014) noted as the border line between the Learning Study and the Lesson Study. Aside from this, Cheng and Lo (2013) added that the Learning Study focuses on the object of learning and how it can be taught to students in the easiest way possible while the Lesson Study looks at the different factors that affect the lesson such as management style and methodology of teaching. There are also two main features that distinguishes the Learning Study from its origins (Marton & Pang, 2006):

1. Focus. The Learning Study has a narrower focus compared to the design experiment and is only concerned about identifying how a particular object of learning can be effectively taught to the students.

2. Teacher's Role. In the Learning Study, the teachers are mainly responsible in determining how the framework can be utilized in the lesson plan design and implementation.

The Learning Study has two aspects as described by Marton and Pang (2006). Aside from pooling teachers' valuable experiences to improve teaching and learning, it aims to build cutting-edge learning environments for theoretically grounded research studies.

**Problem Statement**

Since Trigonometry has been identified as one of the difficult areas in mathematics due to its abstract nature (Dhungana et al., 2023), this study has chosen to apply VT in one of the lessons in Trigonometry. The purpose of this study is to improve the problem-solving skills of students in solving right triangles in Trigonometry using the VT. Specifically, it sought to answer the following questions:

1. What are the intended objects of learning based on students' pretest results?
2. What are the patterns of variations and invariance in the learning study plan and the enacted objects of learning?
3. What are the lived objects of learning?

# METHOD

## Research Design

The research follows an action research design which uses the Learning Study approach. Using this approach, the researcher focused on the object of learning and sought strategies that aim to help improve the facilitation of learning problem-solving skills. The study adapted the learning study procedure by Lo (2012) which uses the VT as a guiding principle throughout its entire process following a "systematic process of inquiry which involves planning, implementation and evaluating a research lesson" (Cheng & Lo, 2013, p. 5).

## Research Participants

The first researcher who was also the lesson implementer, was joined by two teacher-observers in designing the research lesson plan. The teacher-observers were given a briefing on how to conduct a learning study. The teacher-observers had at least 10 years of teaching experience.

Two Grade 10 intact classes from a private school in a highly urbanized metropolitan area in the Philippines were chosen as the research lesson participants for the two cycles of the learning study. The age of the students ranges from 14 to 16. The classes were composed of Filipino and foreign students, mostly of Chinese descent. The students already had a background on right triangles which was discussed when they were in Grade 9. The first class where the first cycle of the learning study was conducted had 17 students and this cycle is referred to as the pilot study. The second class where the second cycle of the learning study was conducted had 24 students and this cycle is referred to as the main study.

Purposive sampling was used in selecting participants to be interviewed in order to have representatives for different levels of students' performance. In this connection, the researcher selected six (6) students from each learning study cycle. The sample was composed of two (2) above average, two (2) average and two (2) below average performers in the written tests.

## Research Instruments

The pre- and posttest were written tasks composed of eight short response questions on trigonometric ratios and its application in real life, particularly the angle of elevation and angle of depression. These were constructed by the researcher and validated by three seasoned Mathematics teachers. The posttest utilized the same questions given to the students during the pretest. The reason for using the same set of questions is to help chart the progress of their understanding. The tests were written in English since this is the primary medium of instruction in the school.

The interviews enabled the researcher to dig deeper on the thoughts of the students and validate students' responses in the pre- and posttests. General and open-ended questions were used to probe and uncover students' thought processes with regard to understanding the object of learning. These served as reference for mapping out students' cognitive processes when solving problems.

The research lessons for the two (2) classes were video-recorded and transcribed verbatim. The researcher also took note of nonverbal cues such as facial expressions used by the students during the research lesson implementation. In addition, the researcher used the video-recording to examine whether the intended object of learning is aligned with the enacted object of learning. Data gathered from these research instruments were triangulated to validate or to cross-check the results of each instrument.

**Data Gathering Procedure**

Briefing sessions were conducted to provide participants an overview of the study and informed consents to participate in the study were secured from them. The pretest was administered for 60 minutes to determine the objects of learning and the patterns of variations and invariance for the research lesson. By administering the written tasks, the researcher was able to have an initial assessment of the capability of the students to understand the concept, as well as determine different methods of solving. Moreover, the tasks enabled the researcher to see the critical aspects of learning that aid in solving these types of problems. The results of the pretest written tasks were then used in the next step: the preparatory meetings. This step aimed to produce essential materials for the study such as research lesson plan, activity worksheets, presentation slides, and other teaching aids. The researcher set preparatory meetings with the teacher-observers to design the research lesson and to plan how to teach students the trigonometric ratios and its application in solving real-life problems. The following guide questions from Marton and Pang (2006) helped the teachers in this step:

1. What are the important points of teaching this topic?
2. What common errors and confusions do students have when learning this topic?
3. How do students make sense of the topic?

Moreover, the teachers were also asked how they facilitated learning in terms of the same object of learning in their previous teaching engagement. They were guided by the following questions as suggested by Marton and Pang (2006):

1. How did you handle the same object of learning in the past?
2. What do you think are the critical aspects of understanding this topic?
3. What were the difficult points of teaching this topic in the past?
4. How could we help students from the phenomenon?

The outputs from the preparatory meetings were used in preparing the research lesson. The research lesson implementation lasted for four days with 80-minute time allotment for each day. The lessons were videotaped and were used to analyze the enacted object of learning in terms of variance and invariance in the actual classroom contexts as the researcher implemented the lesson plan in the classes in line with his own personal style and with any modifications that he considered necessary. The two other teachers involved observed the research lessons and gave comments in the post-lesson meeting for evaluation and modification of the lessons. For example, the number of review questions on Day 1 was reduced and the illustrations used for angle of elevation and depression were improved.

The posttest was conducted to determine how much of the intended object of learning was experienced (lived) by the students. Just like the pretest, students were given 60 minutes to answer the questions in the posttest. Two sets of semi-structured interviews were conducted for each cycle of the research lesson. The first set of interviews were conducted after the pretest and the second set of interviews after the learning study. These were recorded and transcribed verbatim. Students were asked to elaborate their answers in the pre- and posttest and to expound on what they have learned about the lessons, the misconceptions that have been cleared, and what they still need to understand. During the posttest interview, the students were asked to give feedback on the teaching approach, activities, and examples given. The research lesson plan was again slightly modified based on students' suggestions and comments from the interviews.

**Data Analysis**

A scoring rubric was used to reliably evaluate the students' performance in the written tasks. A set of descriptors was developed to help the researcher categorize students' answers into different levels of understanding. The following points are assigned for each correct response corresponding to knowledge of concept questions: 1 point for a correct answer and another 1 point for correct explanation. As for the problem-solving questions, the following points are given for each correct response: illustration (1 point), equation (1 point), solution (2 points), final answer (1 point).

To check whether there is a significant difference between the pretest and posttest results related to students' knowledge in the two cycles of the research lesson, paired t-test was used. To check whether there is a significant difference between the posttest results related to students' knowledge in the two cycles of the research lesson, Welch t-test for unequal sample size and unequal variance was conducted. Descriptive summaries of the mean scores for each question in the posttest of the two learning study cycles were computed to assess whether the students have acquired (lived) the intended object of learning and to determine whether statistically significant differences exist between the posttest of the two cycles by means of independent samples t-tests. Statistical tests of the hypothesis using the differences in gain scores for each question and total gain scores posttest in the two cycles were also analyzed. These descriptive summaries provided an understanding of how students in the first and second cycles differed in relation to the lived object of learning.

This study used the framework of variation to analyze the research lessons in a qualitative way, that is, how the teacher handled the object of learning and implemented the lesson plan in the two cycles. The analysis does not describe the details of the teaching processes. Rather, it focuses on the patterns of variation and invariance in the lessons. In this study, the teacher used the initial research lesson plan, teaching and learning materials designed in the teachers' preparation meeting in the first cycle and used a modified lesson plan in the second cycle based on feedback from the learning study group.

## RESULTS

### Intended Objects of Learning

The analysis of the pretest revealed the following: (1) the students have limited knowledge in solving right triangles, (2) the students have no prior knowledge of angle of elevation and angle of depression, (3) the students have no prior knowledge of trigonometric ratios, and (4) the students have no idea on how to solve problems involving right triangles.

Some of the students were able to determine which of the triangles is solvable. However, the reasons they provided showed they have limited understanding on how to determine whether the given right triangle is solvable or not. Nonetheless, they tried to recall their previous lessons on right triangles as reflected in some of the reasons they gave: (a) the measure of the length of the three sides can be solved, (b) the measures of the three angles can be solved, (c) Pythagorean Theorem can be used, (d) 30°-60°-90° Triangle Theorem can be used to find the measure of the missing side, (e) The sum of the angles in a triangle is 180°, and (c) “I don’t know/I have no idea”. Some students provided no reason at all.

On questions that require students to determine which trigonometric ratio can be used to solve the unknown part of the triangle, to identify the angle of elevation and angle of depression, and solving real-life problems on right triangles, no one among the students were able to answer any of the questions correctly. However, some attempts were made to answer the given questions. This includes (a) drawing a figure based on the given problem, (b) using Triangle Angle Sum Theorem to find the measure of the angles, (c) using Pythagorean Theorem to find the measure of the sides despite having only one given side measurement, and (d) putting random numbers. Some students got correct answers in the multiple-choice type questions by guessing as they were not able to provide a valid explanation to their answers. The reasons provided by the students include the following: (a) “It looks elevated”, (b) “I don’t have any idea yet”, (c) “I just guessed”, (d) “The line of sight is going up”, and (e) no answer at all. The results showed that the students lack the necessary knowledge of the concepts involved in the questions such as trigonometric ratios, and angle of elevation and angle of depression.

Based on the above results, the learning study group decided to have the following intended objects of learning, that is, to develop students’ capability to discern that: (OL1) a right triangle can be solved if either of the following are given: (a) two sides, or (b) an acute angle and a side; (OL2) For OL1-b, the choice of trigonometric ratio to use will vary depending on which pair of side and angle measures are given; and (OL3) trigonometric ratios can be used to model and solve real-life problems that involve right triangles.

**Patterns of Variations in the Lesson and the Enacted Objects of Learning**

In order to help the students 'live' the intended objects of learning, several patterns of variations were created by the researcher through the inputs of the teacher-observers. The principles of *contrast*, *separation*, and *generalization* were used.

**Activity 1: Finding the Unknown Measures of a Right Triangle**

**Materials:** ruler, protractor, calculator

**A.** Which of the following sets of given would make it possible to solve $\Delta ABC$, where $\angle C = 90°$? Draw and label the triangle based on each given. Solve the triangle. Show your solution.

1. $\angle BAC = 34°, \angle ABC = 56°$
2. $\angle BAC = 34°, AC = 4\ cm$
3. $\angle ABC = 56°, AC = 4\ cm$
4. $AC = 4\ cm, AB = 4.8\ cm$
5. $AC = 4.8\ cm$

**B.** Based on part A, what are the conditions (given) needed to solve a right triangle?

Figure 2: Activity for OL1

For OL1, *separation* was used. The right triangle remained invariant while the given conditions were variant. In this activity, students were given a right triangle with five different combinations of measurements given. These are as follows: (1) two acute angles, (2) an acute angle and its adjacent leg, (3) an acute angle and its opposite leg, (4) two sides, and (5) only one side. Please see Figure 2. Here is an excerpt of the class discussion:

*T: So, based on the activity, when do we say that a right triangle is solvable?*
*S: When you can form a unique right triangle. That is when the given is at least the measurement of an angle and a measurement of a side.*
*T: What kind of angle?*
*S: Acute angle.*
*T: Aside from that, are there any other conditions that would allow us to solve a right triangle?*
*S: If measurements of two sides are given.*

For OL2, *separation* and *generalization* were used. The students were shown a right triangle and were then asked to give the trigonometric ratio that can be used to find the value of the labeled part. The teacher kept the triangle, the measure of the acute angle, and the missing side invariant while making the side with known measure variant. The purpose of the activity is for the students to discern that the trigonometric ratio to be used is largely dependent on which parts of the triangle the measurements are given albeit they can use either the primary or its secondary trigonometric ratio counterpart (e.g. cosine and secant in solving for b given hypotenuse equals 17 units and acute angle equals 28°).

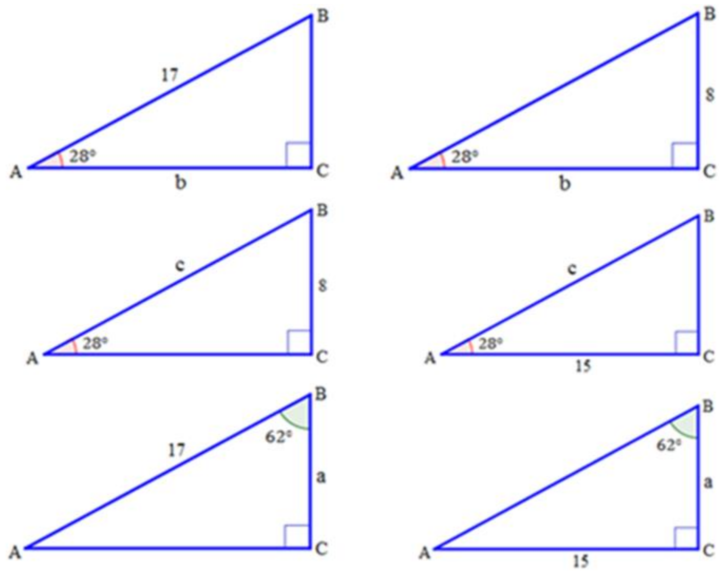



Figure 3: In each pair of triangles, the measure of the acute angle and the unknown were kept invariant while the given measure of the side varied

*T: What can you say about the examples?*
*S: We can use two different trigonometric ratios to find the unknown measure.*
*T: Is that true to all the examples that we had?*
*S: Yes, sir!*
*T: Do you have any generalization or conclusion?*
*S: In certain given measurements, for example, when an angle and an adjacent leg are given, we can either find the hypotenuse using cosine or secant.*
*T: Correct! Anything else?*
*S: If an acute angle is given, you can find the measure of the other acute angle and find the unknown using that angle and the given sides.*
*T: Correct! Any other answers?*
*S: More than one trigonometric ratio can be used to find the measure of the unknown sides.*

**Activity 3: Solving Right Triangles**

**A.** Solve $\Delta ABC$ where $\angle C = 90°$. Round your final answer to the nearest hundredths. Show your solution.

1. $\angle B = 40°, AC = 5\ cm$

2. $\angle A = 50°, AB = 7.8\ cm$

3. $\angle A = 25°, BC = 10\ cm$

Figure 4: Solving right triangles using different trigonometric ratios

Furthermore, in order to reinforce this object of learning, the students solve right triangles presented in Figure 4. Variations were given thereby the use of trigonometric ratios in the initial step also vary. This pattern of variation helps the students discern that the choice of trigonometric ratios depends largely on the pair of given measures. This is validated by the student's response:

*T: What can you conclude based on the activity?*
*S: The activity showed that the choice of trigonometric ratio to use will be based on the given.*

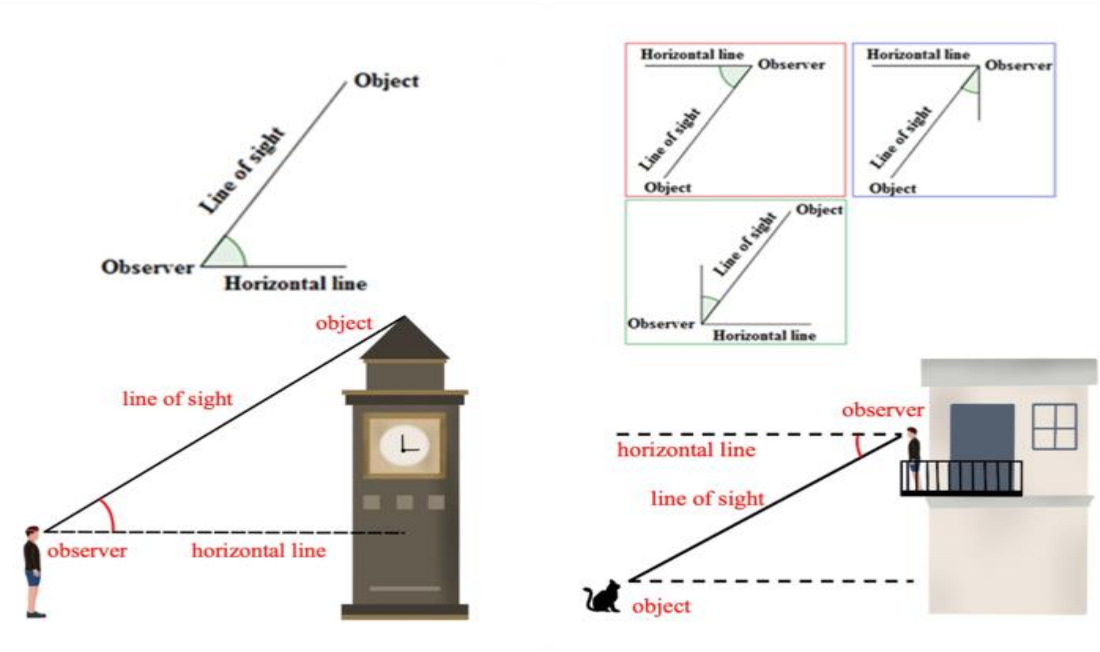
Understanding the concept of angle of elevation and angle of depression is paramount in solving problems involving trigonometric ratios. In order for the students to discern the concept of angle of elevation and angle of depression, the teacher used *contrast* as a pattern of variation. See Figure 5 for the examples and non-examples of angle of elevation.

Through *contrast*, the students' attention was focused on the critical aspects of the angle of elevation and the angle of depression – that is the location of both the observer and the object, the line of sight, and the horizontal line. The students were able to discern the definition of both the angle of elevation and angle of depression by identifying the similarities among examples and contrasting these with the non-examples. The following is an excerpt of the class discourse.

*T: The figures on the board show examples and non-examples of angle of elevation. What do you observe in the figures shown?*
*S: In the examples, the observer is somewhat below the object.*
*T: That's right. What else?*
*S: The line of sight is above the observer.*
*T: Okay. What else?*
*S: The observer is always looking above.*
*T: Any other observations?*
*S: The angle is formed by the horizontal line and the line of sight.*
*T: That's correct! So, what separates the examples and the non-examples of the angle of elevation?*
*S: In the non-example, the object is above the observer and so is the line of sight. But if we look at the angle, the angle is formed by the vertical line and the line sight. In the non-example, the angle is formed by the horizontal line and the line of sight but the object is below the horizontal line.*
*T: So based on your observations, what is an angle of elevation?*
*S: The angle of elevation is an angle formed by the horizontal line and the line of sight.*
*T: Hmmm… Do you think the definition you gave is complete? Look at this non-example (pointing at the red-boxed figure at the right). The angle is also formed by the horizontal line and the line of sight. But why is that figure a non-example?*
*S: Ahhh… The object should be above the observer.*
*T: So what is an angle of elevation?*
*S: The angle of elevation is an angle formed by the horizontal line and the line of sight where the object is above the horizontal line.*
*T: Very good! Now take a look at the next set of figures. We have examples and non-examples of angles of depression. What are your observations?*
*S: The observer is always looking downward.*
*T: What else?*

*S: Just like in angle of elevation, the angle is formed by the horizontal line and the line of sight.*
*T: So, based on the examples and non-examples, what is an angle of depression?*
*S: The angle of depression is an angle formed by the horizontal line and the line of sight where the object is below the horizontal line.*

Angle of Elevation




Examples Non-examples

Figure 5: Examples and non-examples of angle of elevation

The designed plan was carried out and the teacher-observers noted that the teacher was able to enact the intended object of learning. Some of the remarks from the observers during the post-lesson discussion are the following:

> *Observer 1: The teacher is commended for directing the students' attention to the objects of learning… The teacher has facilitated well in directing the students' attention to the important things to consider/focus in order to come up with the definition."*
> *Observer 2: The activities are well-thought out. It has enacted the intended object of learning.*

**Lived Objects of Learning in the Two Cycles of Research Lesson**

The analysis of students' answers in the posttest in the two cycles of the research lesson in problem-solving revealed the usual mistakes some of the students still make. These are: (a) incorrect manipulation of the equation (Figure 6), (b) carelessness (Figures 7), (c) use of incorrect trigonometric ratio, and (d) failure to understand the problem presented (Figures 9). Nonetheless, the majority of the students were able to correctly draw the correct illustration to represent the given problem, write the correct equation and solve the given measure in question.

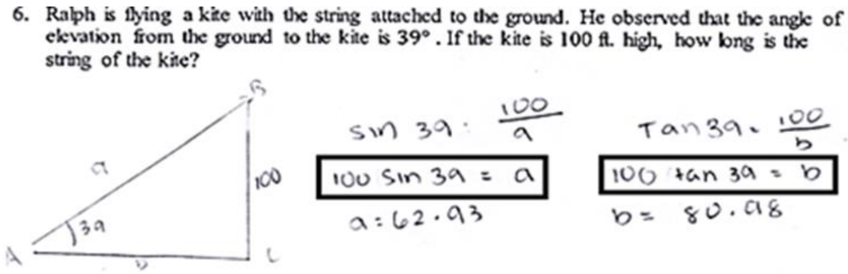



Figure 6: Incorrect manipulation of the working equation

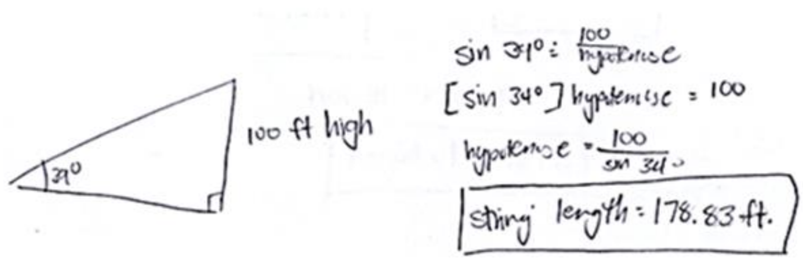



Figure 7: Carelessness in copying the angle measure

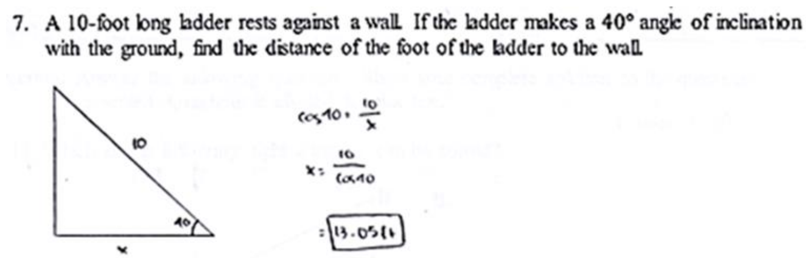



Figure 8: The use of the incorrect ratio

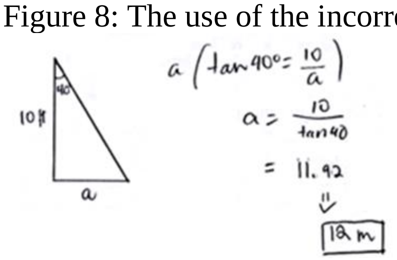


Figure 9: Failure to draw a correct figure by a student

**Pilot Study**

Table 1 shows the summary of the pre- and post-test results in the pilot study. The computed p-values are less than 0.05 significance level for OL1, OL2, and OL3 show that there is a statistically significant difference in the pre- and post-test mean scores. It can be concluded that the change in scores in each object of learning is due to the intervention done by the teacher – the use of VT in teaching the lesson. The posttest results further show a significant improvement among the students in terms of the three objects of learning. Thus, the pronounced gap between the intended object of learning and the lived object of learning before the research lesson was also addressed.

Table 1: Summary of the Pre-Test and Post-Test Results of the Pilot Study

| Object of Learning | Test Items | Full Score | Pretest | | Posttest | | Pretest vs. Posttest | Level of Significance |
|---|---|---|---|---|---|---|---|---|
| | | | Mean | SD | Mean | SD | *t* | (p = 0.05) |
| OL1 | 1a to 1f | 12 | 4.647 | 1.281 | 11.235 | 1.059 | 22.125 | 0.000 |
| OL2 | 2a to 2c & 3 | 11 | 1.294 | 0.824 | 9.118 | 2.928 | 13.137 | 0.000 |
| OL3 | 4 to 8 | 20 | 1.059 | 0.998 | 16.941 | 3.638 | 19.015 | 0.000 |
| Overall | 1a to 8 | 43 | 7 | 2.169 | 37.294 | 6.711 | 23.256 | 0.000 |

**Main Study**

Table 2 shows the summary of the pre- and posttest results of the main study. The table shows the computed p-values for each object of learning. Since all p-values < 0.05, it implies that there is a significant difference between the performance of the students before and after the research lesson. The post-test mean score for each object of learning revealed a significant improvement in the students' lived object of learning. The overall mean score of 39.21 which is 91.19% of the total score 43 points is considered high indicating students' lived objects of learning are very close to the intended objects of learning.

Table 2: Comparison of Pre-Test and Post-Test Results of the Main Study

| Object of Learning | Test Items | Full Score | Pretest | | Posttest | | Pretest vs. Posttest | Level of Significance |
|---|---|---|---|---|---|---|---|---|
| | | | Mean | SD | Mean | SD | *t* | (p = 0.05) |
| OL1 | 1a to 1f | 12 | 4.083 | 1.412 | 10.833 | 1.374 | 15.087 | 0.000 |
| OL2 | 7 to 10 | 11 | 0.542 | 0.498 | 10.167 | 1.700 | 27.915 | 0.000 |
| OL3 | 11 to 15 | 20 | 1.000 | 1.555 | 18.167 | 2.734 | 32.361 | 0.000 |
| Overall | 1 to 15 | 43 | 5.625 | 2.563 | 39.208 | 4.153 | 38.364 | 0.000 |

The average score of the students in the pilot and main study are 37.294 and 39.208 out of a possible score of 43, translating to 86.73% and 91.18%, respectively. To check whether there is a significant difference between the performance of the students in terms of understanding and ability to solve problems involving trigonometric ratios, the posttest results in the two cycles of the research lesson were subjected to Welch t-test for unequal sample size and unequal variance was conducted. The 4.45% difference between the post-test results in the two cycles of the research lesson is not significant (p-value > 0.05). This implies that the minimal changes in the lesson are not significant to greatly affect the manner in which the lessons were taught between the two classes leading to small and non-significant differences in their lived objects of learning.

The use of different patterns of variations enabled the students to see the critical aspects of the lesson. Thus, discernment of the object of learning occurred. This is likewise validated by students' responses in the interview when asked how the use of patterns of variations helped them. The answers are summarized below.

a. It enabled the students to see deeper meaning of the topics.

*"The different examples and explanations used showed me the deeper meaning by showing critical aspects in trigonometry … and how useful trigonometry is…"*

b. It helped the students in relating one problem with another problem.

*"It helped me in solving different problems, such as figures where I have to solve different measurements and relating problems to each other so I can solve the problem in a similar way and at the same time, knowing that I can use solutions for specific types of problems."*

c. It guided the students on how to approach the lesson.

*"The teachers' varying examples give out a distinct guide on how to tackle a problem… especially the examples that use the same unknown but have different parts that were given measures—it just really makes my mind work a little more to process the said problem…"*

## DISCUSSION AND CONCLUSIONS

Marton et al. (2004) underscored the objects of learning to be clearly stated and identified and Kullberg et al. (2017) advised administering a test in order to identify students' prior knowledge. Particularly, we noted that VT has also been conducted in solving right triangles by Peng et al. (2017) but the focus had been in the use of VT in the six typical phases of the national pattern of teaching of problem solving in China. In this study, the learning study group teachers, based on the context of the test results, deemed OL1 to be essential in teaching solving right triangles using Trigonometric ratios to Grade 10 students as teaching through progressive variation problems and core connections (Gu et al., 2017). Students previously learned solving right triangles using Pythagorean theorem and solving the missing angle measure using the Triangle Angle Sum Theorem. This is being connected to the current lesson on solving right triangles using Trigonometric ratios. Moreover, we included the concept of angle of elevation and angle of depression as most of the word problems would make use of these terminologies.

In this study, we illustrated the use of variation patterns of *contrast*, *separation* and *generalization* to help students discern that not all triangles are solvable especially when there is lack of given information. When a triangle is solvable, students were made to discern when to use the Pythagorean Theorem, the Triangle Angle Sum Theorem and the Trigonometric Ratios, all of which are dependent on which part (side and/or angle) the measures are given and their relation to the unknown measure one wish to solve. Furthermore, students were taught to correctly solve a problem through proper illustration of the model representation of the problem by discerning between examples and non-examples of angle of elevation and angle of depression, neither of these angles are formed by a vertical line with the observer's line of sight. Watson (2017) argues that the use of variation in mathematics teaching should draw out students' attention to "dependency relationships" that are invariant in mathematics and how such careful use of variation can lead to abstraction of new ideas. Likewise, categorization of the different parts of a triangle as hypotenuse, a given acute angle, its opposite side and its adjacent side had played an important role as Gu et al. (2017) states that categorization is an important mathematical thinking method in VT.

The intended objects of learning were drawn out from students' pretest results and discussed by the teachers in the learning study. Students' lived objects of learning as reflected in their posttests were very close to the intended objects of learning by way of the patterns of variations in the enacted objects of learning. While there are students who still exhibit mistakes, these are minimal as most students were able to draw a correct illustration based on the given problem, write a correct equation and answer the given question. The students who do not usually perform well in class have made significant improvements after the research lesson. Some students who have below average academic ability before the research lesson were able to answer all the problem-solving questions correctly and were able to get a perfect score in the posttest despite performing poorly in the pretest. The consistent high scores of the students in the two cycles of the research lesson are indications that the use of VT as a pedagogical design tool is effective in bridging the gap between the intended object of learning and the lived object of learning of the students. Careful selection and proper use of different variation patterns such as *contrast*, *separation*, and *generalization* for the enactment of objects of learning, proved to be effective in helping the students keep their focus and discern the critical aspects of the lesson to achieve the intended objects of learning, thereby narrowing the gap between the intended and lived objects of learning. The results of the study confirmed the productive applicability of VT (Clarke, 2017). Peng et al. (2017) explains that VT allows learners to develop their capability to discern which aspects must be considered in the process of achieving one's goal even in an unfamiliar situation. Exposing students to discerning important aspects of mathematical concepts and procedures help them assimilate the skills even beyond the classroom lessons.

The following pedagogical implications based on the findings of the study are thus recommended. The researchers posit that in order to ensure the intended object of learning will be lived by the students, the teacher should plan teaching and structure learning activities that will keep the students' focus on the critical aspects of the lesson which are the intended objects of learning for which VT has been of general utility. For the students to live the intended object of learning, the enacted object of learning has to be as close to the intended object of learning. As with any skill, problem solving demands a considerable level of motivation, effort and devotion from students to imbibe a good understanding of the underlying concepts involved in the problems and mastery of the procedures. A learning study as a form of professional learning community, is considered to be a good opportunity for teachers to collaborate to better their teaching practice as there is more than 1 pair of critical eyes looking at the lesson and at least two heads in sharing their expertise, content and pedagogical knowledge, and experiences.

Future research may consider a long-term study to further examine the long-term effects of the use of VT as a pedagogical design tool in the problem-solving skills of the students most especially those of low academic ability. Likewise, a study on using variation theory in online and blended learning modes of delivery may be conducted.

## References

[1] Al-Murani, T. (2007). The deliberate use of variation to teach algebra: A realistic variation study. A Doctoral thesis, Linacre College, University of Oxford. Retrieved from http://ethos.bl.uk/OrderDetails.do?uin=uk.bl.ethos.486965

[2] Brown, A. (1992). Design experiments: Theoretical and methodological challenges in creating complex interventions in classroom settings. *The Journal of the Learning Sciences*, 15(2), 141–178.

[3] Bussey, T., Orgill, M., & Crippen, K. (2013) Variation theory: A theory of learning and a useful theoretical framework for chemical education research. *Chemistry Education Research and Practice, 14*, 9–22.

[4] Cheng, E., & Lo, M. (2013). The approach of learning study: Its origin and implications. Retrieved from https://www.semanticscholar.org/paper/The-Approach-of-Learning-Study%3A-Its-Origin-and-Imp-Cheng-Ling/0041f101d306341f5a2209f3cf4de91ad9e337c6

[5] Cheng, E. (2016). Learning through the variation theory: A case study. *International Journal of Teaching and Learning in Higher Education, 28*(2), 283–292.

[6] Clarke, D. (2017). Introduction: Putting variation theory to work. In R. Huang & Y. Li (Eds.), *Teaching and learning mathematics through variation*. Sense Publishers, p. 297-300.

[7] Collins, A. (1992). Toward a design science of education. In E. Scandlon & T. D. Shea (Eds.), *New directions in educational technology***,** Berlin: Springer (pp. 15–22).

[8] Dhungana, S., Pant, B. P., Dahal, N. (2023). Students' experience in learning trigonometry in high school mathematics: A phenomenological study. *Mathematics Teaching-Research Journal, 15*(4), 184-201.

[9] Dunlosky, J., Rawson, K., Marsh, E., Nathan M., Willingham, D. (2013) Improving students' learning with effective learning techniques: Promising directions from cognitive and educational psychology. *Psychological Science in the Public Interest*, 14(1), 4–58.

[10] Elipane, L. (2012). Integrating the essential elements of lesson study in pre-service mathematics teacher education. *IND skriftserie***.** Copenhagen: Department of Science Education, v. 27.

[11] Felder, R., & Brent, R. (2005) Understanding student differences. *Journal of Engineering Education*, *94*(1), 57–72.

[12] Garfield, J. (1995). How students learn Statistics. *International Statistical Review, 63*(1), p. 25–34.

[13] Gu, F., Huang, R., & Gu, L. (2017). Theory and development of teaching through variation in mathematics in China. In R. Huang & Y. Li (Eds.), *Teaching and learning mathematics through variation* (pp. 13-42).

[14] Häggström, J. (2008). Teaching systems of linear equations in Sweden and China: What is made possible to learn? *Göteborg Studies of Educational Sciences*. Göteborg: Acta Universitatis Gothoburgensis.

[15] Kirkman, J. (2014). The potential of learning study to enhance initial teacher education and continued professional development at the University of Birmingham. Retrieved from: https://www.birmingham.ac.uk/Documents/college-social-sciences/education/research-centres/U21-learning-study-report.pdf

[16] Kullberg, A., Runesson, U., & Marton, F. (2017). What is made possible to learn when using the variation theory of learning in teaching mathematics? *ZDM Mathematics Education*, *49*(4), 559-569.

[17] Lam, S. (2012). *The use of variation theory to improve student understanding of reaction rate through scientific investigation*. A Thesis, University of Hong Kong, Pokfulam, Hong Kong SAR. Retrieved from http://dx.doi.org/10.5353/th_b4832995

[18] Lim, C. (2007). Characteristics of mathematics teaching in Shanghai, China: Through the lens of a Malaysian. *Mathematics Education Research Journal*, *19*(1), 77–89.

[19] Lo, M., Marton, F. (2012). Towards a science of the art of teaching: Using variation theory as a guiding principle of pedagogical design. *International Journal for Lesson and Learning Studies*, *1*(1), 7–22.

[20] Lo, M. (2012). *Variation theory and the improvement of teaching and learning*. Gothenburg Studies in Educational Sciences, Göteborgs Universitet.

[21] Marton, F., & Booth, S. (1997). *Learning and awareness*. Hillsdale, NJ: Lawrence Erlbaum Associates.

[22] Marton, F. (2009). *Sameness and difference in learning*. Lecture at the Swedish Research Links Symposium on Phenomenography and Variation Theory, The University of Hong Kong, Hong Kong SAR, December 1–3, 2009.

[23] Marton, F., & Pang, M. (2006). On some necessary conditions of learning. *The Journal of the Learning Sciences*, *15*(2), 193–220.

[24] Marton, F., & Pang, M. (2013). Meanings are acquired from experiencing differences against a background of sameness, rather than from experiencing sameness against a background of difference: Putting a conjecture to test by embedding it into a pedagogical tool. *Frontline Learning Research*, *1*(1), 24–41.

[25] Marton, F., Runesson, U., & Tsui, A.B. (2004). The space of learning. In F. Marton, & A.B.Tsui (Eds.), *Classroom discourse and the space of learning*. New Jersey: Lawrence Erlbaum, pp. 3-62.

[26] Marton, F., & Tsui, A. B. M. (2004). *Classroom Discourse and the Space of Learning*. Mahwah, NJ: Lawrence Erlbaum Associates.

[27] Mhlolo, M. (2013). The merits of teaching mathematics with variation. *Pythagoras*, *34*(2), Art. #233.

[28] Mok, I. (2006). Shedding Light on the East Asian Learner Paradox: Reconstructing student-centeredness in a Shanghai classroom. *Asia Pacific Journal of Education*, *26*(2), 131-142.

[29] Mullis, I., Martin, M., Foy, P., & Hooper, M. (2016). *TIMSS 2015 International Results in Mathematics*. Retrieved from Boston College, TIMSS & PIRLS International Study Center website: http://timssandpirls.bc.edu/timss2015/international-results/

[30] O'Dwyer, L.M., Wang, Y., & Shields, K.A. (2015). Teaching for conceptual understanding: A cross-national comparison of the relationship between teachers' instructional practices and student achievement in mathematics. *Large-scale Assessments in Education*, *3*(1), 1-30.

[31] Organization for Economic Co-operation and Development (OECD). (2019). *Philippines – Country Note – PISA 2018 Results*. Retrieved from https://www.oecd.org/pisa/publications/PISA2018_CN_PHL.pdf.

[32] Orgill, M. (2012). Variation theory. In N. M. Seel (Ed.), *Encyclopedia of the sciences of learning*, pp. 3391–3393. Heidelberg, Germany: Springer-Verlag GmbH.

[33] Pang, M. (2008). *Using the learning study grounded on the variation theory to improve students' mathematical understanding*. Presented at the 11th International Congress on Mathematical Education (ICME) 2008, Monterrey, Mexico, 6-13 July 2008.

[34] Pang, M. F., & Lo, M. L. (2012). Learning study: Helping teachers to use theory, develop professionally, and produce new knowledge to be shared. *Instructional Science*, *40*(3), 589–606.

[35] Peng, A., Li, J., Nie, B., & Li, Y. (2017). Characteristics of teaching mathematical problem solving in China. In R. Huang & Y. Li (Eds.), *Teaching and learning mathematics through variation*. Sense Publishers, pp. 111-126.

[36] Prince, M. (2004). Does active learning work? A review of the research. *Journal of Engineering Education*, *93*(3), 223–231.

[37] Runesson, U. (2005). Beyond discourse and interaction. Variation: A critical aspect for teaching and learning mathematics. *Cambridge Journal of Education*, *35*(1), 69–87.

[38] Samuelsson, I., & Pramling, N. (2016). Variation theory of learning and developmental pedagogy: Two context-related models of learning grounded in phenomenography, *Scandinavian Journal of Educational Research*, *60*(3), 286–295.

[39] Stigler, J., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom.* New York: Free Press.

[40] Tong, S. (2012). Applying the theory of variation in teaching reading. *Australian Journal of Teacher Education, 37*(10), 1-19.

[41] Watson, A. (2017). Pedagogy of variation. In Huang, R., & Li, Y. (Eds.), *Teaching and learning mathematics through variation: Confucian heritage meets Western theories* (pp. 85-103). Rotterdam, NL: Sense Publishers.

[42] Wong, N. (2013). The Chinese learner, the Japanese learner, the Asian learner: Inspiration for the (mathematics) learner. *Scientiae Mathematicae Japonicae, 76*(2), 376–384.