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The Theory on Loops and Spaces. Part 1.

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Abstract: We are all fascinated by loops and their formation in space. When a line cuts itself, it forms an intersection point and creates a space. This is an experimental study done by analyzing several loops, forming a concrete formulation by visualizing the patterns observed, and then proving the formulations proposed using the known standard mathematical methods. This piece of mathematics is studied under graph theory and forms the basis for understanding and developing thinking of the graph theory at the elementary level of mathematics. This article develops the thinking behind how to analyze patterns in nature and write them in the form of mathematical statements or formulas This article has been inspired by a YouTube video posted by the mathematician Dr. James Tanton on 27th Sept. 2021 on his YouTube Channel.

1. INTRODUCTION

We have all been fascinated by loops and curves in space. "A loop is a path whose initial and terminal point is the same." It may or may not cut itself. We do not need to lift the pencil while drawing a loop. Now when two lines intersect at a point is called an Intersection Point. So, while drawing a loop, it is possible that a line can cut itself at one or more than one point. It is equally probable that a line may cut the exact intersection point multiple times. This leads to a few interesting questions in our mind!

2. QUESTIONS/ PROPOSALS:

1. Is the theory being, Pieces + Intersection = Spaces?
2. Can we prove that if we draw a group of several loops that intersect at least once such that no loop can be isolated, then we can draw the entire picture of loops without lifting our pencil?
3. Can we prove that every picture we draw can be two colorable such that no two regions can share a section of boundaries of the same color?
4. Can we find the relation that while tracing a loop, we pass through an intersection point 'P,' then the number of intersections passed before reaching 'P'?
5. Can we find the relation to find the sum of intersection points we have passed starting from P and reaching P again?
6. Can the above results work in Higher Dimensions also?

Before we begin, let us introduce "The value of a point" and a "Piece."

3. VALUE OF A POINT AND A PIECE:

If we consider a point as a source of two or more rays emanating from it in opposite directions, like if we take a point on a line, then we have two rays emanating from that point. Similarly, a point with two intersecting lines will have four rays emanating from it. Now the Value of a Point, 'P,' will be evaluated as:

$$V(P) = 1 + \frac{n-4}{2} = \frac{n-2}{2} \quad (1)$$

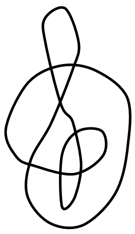
Where n is the number of rays emanating from the point 'P.' The Value of a point is non-zero only at intersection points; the rest everywhere is zero.

We define a "Piece" as a single loop or collection of loops such that no loop can be isolated from the group.

We will always restrict ourselves to only "one-piece," as calculating spaces for one piece and then adding to get the final number of spaces is more effective than calculating spaces for multiple pieces simultaneously.

4. TESTING THE THEORY: PIECES + INTERSECTION = SPACES:

Whenever we draw a loop, we either cut a line to form an intersection point or form a closed curve without any intersection point (e.g., Circle). Let us take the following examples:

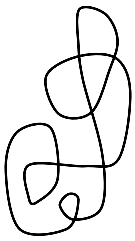


No. of pieces: 01

No. of Intersection points: 07

No. of Spaces: 7+1 = 8

Figure 1: Example 1



No. of pieces: 01

No. of Intersection points: 06

No. of Spaces: 6+1 = 7

Figure 2: Example 2

4.1 HYPOTHESIS:

From the above example, we can find a relation as:
Given a loop, let the value of intersection points be V_1, V_2, V_3 ... V_n, also let No. of intersection points with values V_1, V_2, V_3, ... V_n be N_1, N_2, N_3, ... N_n respectively, then:

Number of Spaces (S) = No. of Pieces (1) + $N_1V_1 + N_2V_2 + N_3V_3 + ... + N_nV_n$

$$S = 1 + \sum_{k=1}^{n} N_k V_k \quad (2)$$

Before we begin the proof, let us look at the following postulate:

4.1.1 POSTULATE

The Value 'V' of an Intersection Point 'P' must be a Natural Number.

$$V(P) \in \mathbb{N} \tag{3}$$

4.1.2 PROOF OF HYPOTHESIS 4.1, PROOF BY CONSTRUCTION:

Suppose we draw a closed loop with no intersection point (E.g., a Circle), then we have created one space inside the Loop (Figure 3).

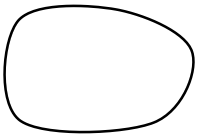


Figure 3: A closed loop with no intersection point having one space within it

If we draw another loop with one intersection point with value V_1, we will observe that we have created $(1 + V_1)$ Spaces (Figure 4).

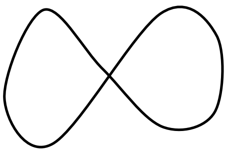


Figure 4: Example of a closed loop having 2 Spaces, ($N_1 = 1$; $V_1 = 1$)

Similarly, if we draw another loop with N_1 intersection points, each having value V_1, we will observe that we have created $(1 + N_1V_1)$ Spaces (Figure 5).

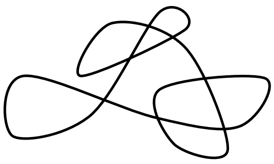


Figure 5: Example of a closed loop having 8 Spaces, ($N_1 = 7$; $V_1 = 1$)

If we generalize it more by drawing another loop having (N_1+N_2) Number of Intersection points with N_1 points having value V_1 and N_2 points having value V_2, we will observe that we have created $(1 + N_1V_1 + N_2V_2)$ Spaces (Figure 6).

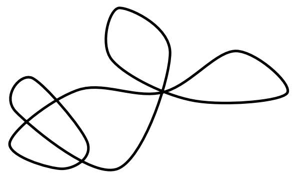


Figure 6: Example of a closed loop having 6 Spaces, ($N_1 = 3$; $V_1 = 1$; $N_2 = 1$; $V_2 = 2$)

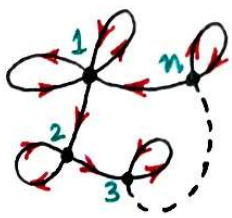


Figure: 7

Now, let us simplify things radically further. On a Blank Piece of paper, we slowly start drawing a self-intersecting loop. Every time the loop intersects itself, we write +1 in the tally of intersection points whether it has created a new intersection point or crossed the same intersection point again, +1 in the tally of value of point, and +1 for the tally of regions, as one intersection point will create at least one region. When we complete one loop, we write +1 for loops and +1 for regions, as when we connect the loop to the starting point, one more region is created, and we have completed one loop. We will also consider the value of the intersection point in the following table that corresponds to the Loop in Figure 7. Here, the dotted line represents that the loop continues further, but the initial few intersection points are shown out of total 'n' intersection points. Let us see it in the following table:

Loop/Piece	Intersection Point	Region/ Space	The number assigned to that Intersection Point	Value
-	+1	+1	1	+1
-	+1	+1	1	+1
-	+1	+1	2	+1
-	+1	+1	3	+1
-	...	...	...	...
-	+1	+1	n	+1
+1	0	+1	-	-

Table 1: Tally of Number of Regions, Intersection Points, Loops and Value of Intersection Points

Here, we observe that, for each intersection point, the value is determined by the sum of all the values corresponding to that point. E.g., the total Value of Intersection Point (1) is equal to 2 (+1+1), the total Value of Intersection Point (2) is +1, and so on. We get +1 when we connect the loop to the initial point, which completes one loop, so +1 for Loop/Piece. Therefore, the total number of regions produced equals 1+ (sum of all the values in the last column). The sum of the values in the last column can be simplified to a total sum of (1 × Number of times loop passes an intersection point). e.g., from the above table, the number of spaces can be given as 1 +

[(1×2)+(1×1)+(1×1)+...+(1×1)]. Here, we have assumed that the intersection point of a particular value, V occurs only once. Now, let us generalize the above observations.

4.2 GENERALIZATION:

"If we have given a loop or a collection of connected loops such that no loop can be isolated from the group, having intersection points of values V_1, V_2, V_3, ... V_n also let No. of intersection points with values V_1, V_2, V_3, ... V_n be N_1, N_2, N_3, ... N_n respectively, then the number of Spaces (S) created will be given as:"

$$No. of\ Spaces\ (S) = 1 + \sum_{k=1}^{n} N_k V_k \tag{4}$$

5. CLASSIFICATION OF INTERSECTION POINTS:

5.1 PURE INTERSECTION POINTS:

These points are inherent to the original loop.

5.2 MIXED INTERSECTION POINTS:

These are the intersection points created/ formed when one loop intersects with the other at a minimum of 2 points. These intersection points are formed by the Intersection of one Loop with one or more than one loop.

6. CONNECTING THE LOOPS:

Let us prove that if we draw several loops that intersect, we can draw the entire picture without lifting our pencil from the page (as though it were one loop).

To begin with, let us consider the loops P_1, P_2, P_3, … P_k having p_1, p_2, p_3, … p_k 'pure' Intersection points, respectively. Now let all these loops from P_1 to P_k intersect such that no loop can be isolated from the rest.

Let us suppose that $P_i\ \{1 \leq i \leq k\ ; i\ \neq j\}$ intersects with $P_j\ \{1 \leq j \leq k\ ; j \neq i\}$ at K_m Number of intersection points. Where $K_m = \{K_1, K_2, K_3, \dots K_n;\ 1 \leq m \leq n;\ K_m \epsilon\ \mathbb{Z}^+\}$. Then, in this case, the total number of intersection points will be given as:

Total No. of Intersection Points: {No. of pure Intersection Points} + {No. of Mixed Intersection Points}

Hence,

Total number of Intersection points when all loops from P_1 to P_k intersect will be given as:

$$\{p_1 + p_2 + p_3 + \dots + p_k\} + \{K_1 + K_2 + K_3 + \dots + K_n\} \tag{5}$$

Therefore,

Number of intersection points:= $\{p_1 + p_2 + p_3 + K_1 + K_2 + K_3 + \dots + p_k + K_n\}$

Now, it is possible to draw a new loop (L) with $\{p_1 + p_2 + p_3 + K_1 + K_2 + K_3 + \ldots + p_k + K_n\}$ number of intersection points, and there is only one way in which these intersection points can be placed in the same orientation as they were in original small loops (P_1, P_2 … and so on.). So, from the definition of loops (Introduction), if we go in reverse as we already have intersection points and then trace the bigger loop through those intersection points, it is possible to draw the entire picture without lifting our pencil.

6.1 A SPECIAL CASE:

It might be possible that pure and mixed intersection points overlap. Then, in that case, we will only increase the value of that intersection point and eventually increase the number of spaces, but the relative orientation of intersection points will remain the same, and there is only one way these intersection points can be fixed in space. Therefore, it is still possible to define a new loop L with the same orientation of intersection points in space, and hence it is possible to draw the entire picture without lifting our pencil.
Hence, the above statement is proved.

7. TWO COLORABLE:

Let us now prove that every picture we draw "can be" two-colorable, meaning that we can color the spaces blue and yellow (for example) so that no two regions that share a section of the boundary are of the same color.
To build this, if we look closely at any arbitrary intersection point then, we can "separate" any intersection point in the following ways:

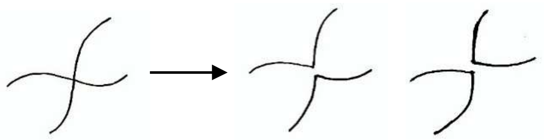


Figure 8: Separating Intersection Points

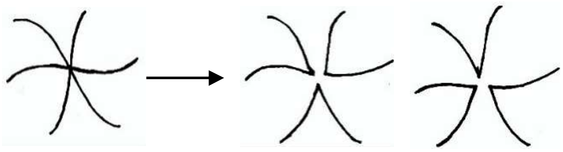


Figure 9: Separating Intersection Points

Now, from the above observation, we can form the following postulate:

7.1 POSTULATE:

For any given intersection point in a loop, we can separate in "only two" possible ways. Figures 8 and 9 can be considered as visual proof for this postulate.

Now, we will extend this idea to the bigger picture of loops as:

For any given loop 'L,' we can separate every intersection point to create disjoint loops with no intersection points and do not share any common boundary. Then from postulate 7.1, there are only two ways to separate them, which will result in 'only two' distinct figures and can be colored with only two distinct colors. When we combine both the distinct figures to get the original loop, we will get the loop colored so that none of the space shares the boundary of the same color.

7.2 ILLUSTRATION:

The above argument can be illustrated as follows:

Two ways to separate intersection points

Figure 10: Illustration that two coloring schemes do exists!

Hence, from the above discussion, it is established that "It is possible that every picture we draw "can be" two-colorable so that no two regions that share a section of the boundary are of the same color.

8. REFERENCES:

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