Students' Mathematical Thinking Process in Algebraic Verification Based on Crystalline Concept

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Abstract: Crystalline concept is the main concept used as the reference by students in algebraic verification. This concept divided the way of solving algebraic verification into two types: symbolic and embodied compression. This research aimed to explore the students' mathematical thinking process in solving algebraic verification based on the Crystalline concept types. The subjects of research were 15 students who took abstract algebra course. Those subjects were asked to solve algebraic verification and were divided based on their types. To get a deeper data, one student was randomly chosen from each type to be interviewed. The verification and interview data were analyzed by using the steps of mathematical thinking process. Those steps are abstracting, representation, and verification. Abstracting is the step to find the ideas: definition of group and abelian group. Representation is the way to communicate the suitable ideas with the conditions. The last step is verification in which students performed the process based on the results of two previous steps. The symbolic student tends to solve the verification symbolically while the embodied one solved the verification arithmetically. Based on the findings, it is essential to design a learning that can accustom students to solve algebraic verification symbolically as the verification should be done deductively.

Keywords: Algebraic, Crystalline Concept, Mathematical Thinking, Verification

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INTRODUCTION

Mathematical thinking and mathematical verification are built based on representation, reasoning, and students' mistakes during verification and abstractings process (Lo & McCrory, 2009). Abstracting is a learning process from previous experiences as the formation base in forming new experiences (Khasanah et al., 2019). New experiences from time to time can be built through conceptual embodiment by combining perceptions and actions that develop through the mental world (Tall, 2005).

In the theory of the three worlds of mathematics, there are three components that need to be considered, namely: the conceptual-embodied world, the proceptual-symbolic world, and the axiomatic-formal world (Tall, 2008). The conceptual embodied does not only discuss about the embodiment of thinking, but is also perception and reflection as a result of the representation of a mathematical concept. The proceptual symbolism arises when students perform calculations and when they use symbols to derive schemas. Both components work as a process that is conducted and a concept that is thought about, so it can be called a procept (as the abbreviation of process and concept). The process of constructing symbols, processes, and concepts is called the basic procept. While axiomatic-formal is the student knowledge based on axioms, theorems, and definitions of a mathematical object.

The development of mathematics theory begins with introduction of the definition formulated as axioms to prove a theorem or a statement in order to obtain formal verification. Tall (2013; 2014) formulated definition in mathematics as "crystalline concept" which is a mathematics concept with internal structure that can cause the emergence of properties from a certain definition. Crystalline concept can also be called as the main concept in the form of properties or axiom based on context of the mathematical problem. The use of concept in solving mathematical problems can be in the form of: (1) geometrical object which consist of dots, lines, triangles, circle, congruent triangles, parallel lines which has the property of Euclid verification, (2) operation of symbols as flexible process and concept (procept) in arithmetic, algebra, and symbolic calculus needed in calculation and manipulation, (3) a group of theories from mathematical concept as properties of a certain axiom to obtain formal conclusion. Crystalline concept is often found in mathematics especially in algebra and number operation, algebraic expressions, as well as process and concepts using symbol operation with various methods. So, based on this concept, students can be divided into two groups seen the way to complete the verification. Those groups are symbolic compression and embodied compression. The students of symbolic compression complete the verification through symbol operations. Meanwhile, students with embodied compression complete the verification through number operations.

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Literature Review

Algebra contains of new concepts to introduce symbol manipulation through theory system (Pedemonte, 2008). One of the concepts in abstract algebra course isa group and an abelian group. Students are asked to solve the verification based on the definition of group and abelian group. The set with binary operation is called group if it fulfills the closure with respect to the group operation and associative properties, the set has an identity element, and every element of the set has an inverse element; while in abelian group, it is necessary to add commutative properties (Gilbert & Gilbert, 2015).

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Mathematical thinking and verification are the parts of mathematics that is useful to train students' reasoning abilities (Varghese, 2009; Faizah et al., 2022). A person's cognitive development in doing mathematical thinking and verification is based on the mathematical language contained in the sensory-motor capabilities by combining perception, operation, and reasoning (Tall et al., 2013). All three aspects can be communicated through enactive gestures, iconic images, written and spoken language as well as arithmetic symbol operations and axiomatic formal symbols based on logical deduction (Bruner, 1966).

There are many researchers who studied algebraic verification and mathematical thinking (Hannah et al., 2014; Onal et al., 2017; Hidayah et al., 2020; Reyhani et al., 2012; Aristidou, 2020; Noto et al., 2019). Students perform mathematical thinking to achieve new knowledge or concepts through abstracting, estimation, generalization, testing hypotheses, and verification processes using the definitions obtained from previously studied concepts (Yorulmaz, 2017; Bukova, 2006). Mathematical thinking is defined as mathematical techniques, concepts, and methods that are used directly or indirectly in the problem-solving process (Henderson et al., 2002).

Mathematical thinking contains the following components: abstracting, synthesis, generalization, modeling, problem solving, and verification (Tall, 2002). Furthermore, Mason (2010) defines the components of mathematical thinking as: specializing, generalizing, making conjectures, justifying, and convincing. Mathematical thinking also includes estimation, induction, deduction, sampling, generalization, analogy, formal and informal reasoning, assertion and equations of processes (Uyangör, 2019). Therefore, this research uses three steps mathematical thinking processes: abstracting, representation, and verification. Based on these reasons, it is clear that in doing algebraic verification all students' mathematical abilities related to the previously known concepts will be involved.

According to the observations in abstract algebra class, there are students who solve the verification problem without using algebraic symbols. They tend to prove it by using specific numbers which are the elements of the appropriate set. They think that it is easier to be solved by using number rather than symbol (variable) because they just need to perform the simple

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calculation in each property of the verification problem. They keep doing it even though the lecture has explained that the proof has to be able to be applied to all of the elements of the set. Instead of calculations t, the proof can be simplified by using algebraic symbols. Therefore, the aim of this research is exploring students' mathematical thinking process in completing algebraic verification seen from Crystalline concept being used.

METHOD

This research is a qualitative research that uses the purposive sampling technique. The researchers divided the students into two groups based on their types in completing the verification. The first type is the students who complete the verification by using the rules of group and abelian group definitions symbolically and the second type are students who do it arithmetically. Then, the researchers choose one student randomly from each type to be interviewed to obtain the deeper data about their mathematical thinking process.

The subjects of research are the third semester students of Mathematics Education Department at Hasyim Ay'ari University. They are chosen because they have received the lesson about the definition of group and abelian group. The research instruments are the written test sheet and the interview guideline. The researchers also conducted the validity test to the expert to find out the validity of the written test instrument before it was given to the students. During the interview, the researchers recorded the process using video tape recorder to simplify the transcription process. Students were requested to explain the concepts that they used in the verification process through think-aloud technique or by explaining the procedure of the verification process in detail. The written test can be seen in Figure 1.

"Let Q^+ be a set of positive rational number, with binary operation which is defined as $p * q = \frac{pq}{3}$ for all $p, q \in Q^+$, please prove that Q^+ with the stated binary operation is included as abelian group"

Figure 1: Test instrument

The data were analyzed by using three steps: data reduction, data interpretation, and conclusion (Creswell, 2014). Data reduction is selecting and reducing the data based on the relatedness to research aims. The unrelated data can be considered as the findings. Data interpretation is conducted by describing the data from reduction step. The last step is making a conclusion based on the data from interpretation step. There are indicators that be used to analyze the students' mathematical thinking process (Yorulmaz, 2017; Lo & McCrory, 2009; Tall, 2013; 2014). Those indicators are explained in Table 1.

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Table 1: Indicators of students' mathematical thinking process

Steps in mathematical thinking process	Description
Abstracting	• Determining the idea from the basic concept that will be used to complete the algebraic proof. • The basic concepts used are the definition of group and abelian group.
Representation	• Communicating the information that is contained in the written test sheet in the form of numbers or algebraic symbols. • Write the information based on the concepts from the abstracting level.
Verification	Completing the algebraic verification by using the definition of group that contained a number of conditions: fulfill the closure and associative properties, the set has an identity element, every element of the set has the inverse element. The definition of abelian group that contained a number of conditions: fulfill the closure, associative, and commutative properties, the set has an identity element, every element of the set has an inverse element.

RESULTS AND DISCUSSION

The result of the written test divides the 15 students into two types: 6 students with symbolic compression type and 9 students with embodied compression type. The researchers choose one student randomly from each type to be interviewed. The researcher gave an SS code for subjects with Symbolic Compression and SE for Embodied Compression student. The result shows that they performed algebraic verifications through mathematical thinking processes consisting of abstracting, representation, and verification as explained below:

Abstracting

Abstracting is a mental activity formulated to determine the basic concept to be used in solving a mathematical problem (Lo & McCrory, 2009; Witheley, 2009). Abstracting can also be called as the initial idea to be used in solving the problem based on what was previously experienced in solving similar problems (Skemp, 2012). Abstracting can happen through construction process of a certain knowledge obtained through experience and past event (Nurhasanah et al., 2017).

SS performed the abstracting by mentioning that to prove whether the binary operation * in positive rational number set (Q^+) is an abelian group, we should prove the group first. If the set

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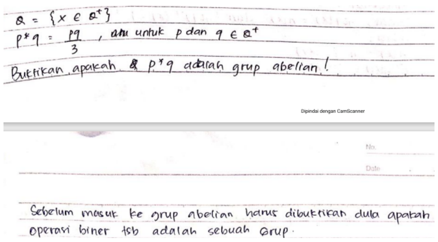
Q^+ with * operation fulfills the closure and associative properties, has an identity element, and every element of the set Q^+ has an inverse, then the set Q^+ with * operation forms a group. Furthermore, a commutative property should be added to make the set Q^+ with * operation be abelian group.



$Q = \{x \in Q^+\}$

$p * q = \frac{pq}{3}$, ~~ah~~ untuk p dan $q \in Q^+$

Buktikan apakah ~~Q~~ $p*q$ adalah grup abelian!

Dipindai dengan CamScanner

No.

Date

Sebelum masuk ke grup abelian harus dibuktikan dulu apakah operasi biner tsb adalah sebuah grup.

Translate version
$Q = \{x \in Q^+\}$
$p * q = \frac{pq}{3}$ for p and $q \in Q^+$
Prove that $p * q$ is an abelian group!
Before begin to prove the abelian group, it is important to prove that the binary operation above is a group.

Figure 2: Idea which will be used by SS

Interviewer : To prove an abelian group, do you need to prove the group first?

SS : Of course, ma'am. Because in abelian group definitions, there is a rule stated that if we want to prove an abelian group, the first thing we have to do is making sure that a non-empty set is a group first. Therefore, I prove that Q^+ which is a rational number with binary operation is a group.

Interviewer : What is the next step?

SS : If it can fulfill the 4 rules in group definition, I will continue to prove the commutative properties. But if any of the rules not met the requirements then I will not continue.

The idea that SS used is group definition to solve problem related to abelian group. In the interview process, SS mentioned that the definition of abelian group is a non-empty set with * binary operation which fulfill 4 rules of group definition and the commutative property. If it is found that any one of the 4 rules are not met during the verification, the process will not be continued.

Meanwhile, SE performed the abstracting by using the idea of an abelian group definition: fulfill the closure, associative, and commutative properties. As shown in the following interview transcript:

Interviewer : What is your idea in solving the problem?

SE : I used the idea based on my previous experience about the similar problem that I have solved. The problem contained commutative property and binary operation.

Interviewer : May I know the previous problem that you mentioned before?

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SE	:	Ehmm… if I am not mistaken, the problem was "given a non-empty set of Z^+ with binary operation $\mathrm{x} * \mathrm{y} = \vert\mathrm{x} - \mathrm{y}\vert$ if $\mathrm{x} \neq \mathrm{y}$ and $\mathrm{x} * \mathrm{x} = \mathrm{x}$ for every $\mathrm{x}, \mathrm{y} \in \mathrm{Z}^+$. Please determine whether the operation fulfills closure, commutative, and associative properties!" Because as far as I remember, the abelian group can be called a commutative group, so I used the closure, commutative, and associative properties similar to the previous problem that I have solved.

From the result of the interview, it is known that SE proved the abelian group based on her memory about similar problem that has been done previously. SE did not realize that the current problem was different from the previous problem. Therefore, the idea used by SE in solving the problem is inappropriate because she missed the conditions about the identity and inverse elements. Even though the definition of the abelian group contains these conditions: fulfill the closure and associative properties, the set has an identity element, every element of the set has an inverse element, and fulfill the commutative property (Gilbert & Gilbert, 2015; Gallian, 2010). SE's mistake in selecting the idea can be called the fact error because the student chose the inappropriate facts with the problem they worked on (Hidayah et al., 2020). From the abstracting process performed by SE, it is known that she used her previous experience to complete the abelian group verification. This is in line with Tall (2008) which states that previous experiences form connections in the brain that can affect the way how students understand new situations.

Representation

Representation is a tool to communicate ideas or answers including graphs, numbers, diagrams, geometry, algebraic symbols, or others (Ernaningsih & Wicasari, 2017; Bannister, 2014). Representation and symbolization are the core of mathematics related to cognitive (Mainali, 2021). Furthermore, NCTM (2000) also mentions that representation is an important element in supporting students' understanding of a mathematical concept by communicating understanding to themselves and to others. Representation can help students understand the abstract mathematical concepts (Samsuddin & Retnawati, 2018).

This study found that SS represented the given information by using algebraic symbols. The symbols used by SS are $p \in Q^+, q \in Q^+$ and $pq \in Q^+$ as shown in Figure 3. He decided to use the variable as the representation of the set because he knew that the verification must apply to all of the elements of the set as shown in the following interview transcript:

Interviewer	:	What do you mean by $p \in Q^+, q \in Q^+$?
SS	:	To solve this problem, we have to mention the elements of the set. I choose elements p and q because both of them are the elements of positive rational number set.
Interviewer	:	Why do you choose p and q as the elements of the set?
SS	:	Because the verification have to apply to all of the elements of the set. If I don't use the variable, then the verification don't apply generally to all of the elements of the set.

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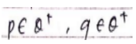

Figure 3: Idea representation of SS

While SE represented the given information by using number such as $p = 2$ and $q = 3$ with $pq \in Q^+$ as shown in Figure 4.he used the specific number as the representation of the set as shown in the following interview transcript:

Interviewer	:	What is your idea to solve the problem?
SE	:	I used the idea based on my previous experience about the similar problem that I have solved. The problem contained of commutative property and binary operation.
Interviewer	:	Why do you mean by $p = 2$ and $q = 3$?
SE	:	2 and 3 are the elements of the positive rational number set.
Interviewer	:	Are the elements of the positive rational number just 2 and 3?
SE	:	No. But I think it's enough if I verify to 2 and 3 as those numbers really are the element of the positive rational number set.

Figure 4: Idea representation of SE

Verification

Verification and mathematical thinking are the most important aspect in mathematics learning (Knuth, 2002). The subjects of this research have different ways in completing abelian group verification. SS proved by using all the properties in the definition of abelian group. First, SS proved the closure property by using the symbols contained in the problem. SS stated that $p * q \in Q^+$ fulfills the closure property because $\frac{pq}{3}$ is also a positive rational number. The second property, SS proved the associative property by $(p * q) * r = \left(\frac{pq}{3}\right) * r$ and $p * (q * r) = p * \left(\frac{qr}{3}\right)$. SS stated that the results of both operations are the same, to confirm that it fulfills the associative property. SS proved the associative property through symbol manipulation using algebraic operations. Algebraic operations are stages of mathematical thinking in solving verification problems (Faizah et al., 2020). The third property is about identity element. SS proved the identity element by $p * e = p$ with $p \in Q^+$ so it is obtained that $e = 3$. Therefore, Q^+ has the identity element. The fourth property is about the inverse element. SS determined the inverse by $p * p^{-1} = e$ so it is obtained that $\frac{3e}{p} \in Q^+$.

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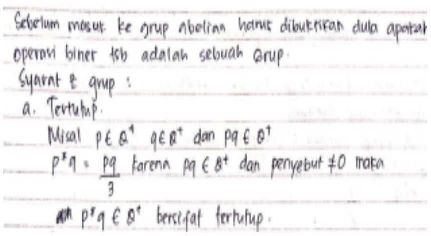
SS mentioned that to prove whether a non-empty set with binary operation is an abelian group or not, it is necessary to prove the group first. If all the conditions in the group are met, then it can be added the commutative property to complete the verification of abelian group. However, if any condition of the group does not met, then it is automatically not abelian group. Figure 5 show that all the properties of group are met, so it is necessary to add the commutative property to complete the abelian group verification (Figure 6).



Sebelum masuk ke grup abelian harus dibuktikan dulu apakah operasi biner tsb adalah sebuah grup.
Syarat 2 grup :
a. Tertutup.
Misal $p \in Q^+$ $q \in Q^+$ dan $pq \in Q^+$
$p*q = \frac{pq}{3}$ karena $pq \in Q^+$ dan penyebut $\neq 0$ maka
$p*q \in Q^+$ bersifat tertutup.

Translate version

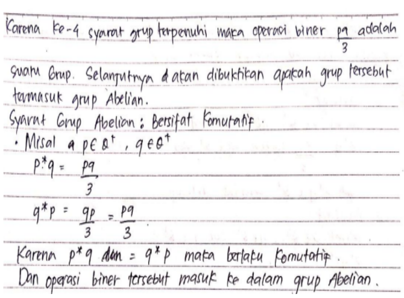
It needs to prove whether binary operation is group or not before proving the abelian group. The conditions of group:

a. Closure

Let $p \in Q^+, q \in Q^+$ and $pq \in Q^+$

$p * q = \frac{pq}{3}$ because $pq \in Q^+$ and the denominator $\neq 0$ therefore, $p * q \in Q^+$ is closure.

Figure 5: Group verification by SS



Karena ke-4 syarat grup terpenuhi maka operasi biner $\frac{pq}{3}$ adalah suatu Grup. Selanjutnya akan dibuktikan apakah grup tersebut termasuk grup Abelian.
Syarat Grup Abelian : Bersifat Komutatif.
• Misal $p \in Q^+$, $q \in Q^+$
$p*q = \frac{pq}{3}$
$q*p = \frac{qp}{3} = \frac{pq}{3}$
Karena $p*q$ = $q*p$ maka berlaku Komutatif.
Dan operasi biner tersebut masuk ke dalam grup Abelian.

Translate version

Since all conditions of the group is fulfilled, then the binary operation $\frac{pq}{3}$ is a group. Next, it will be proven whether or not the group is an abelian group.

The condition for abelian group: fulfills the commutative property

$$p * q = \frac{pq}{3}$$
$$q * p = \frac{qp}{3} = \frac{pq}{3}$$

Since $p * q = q * p$ then it fulfills the commutative property.

And that binary operation is an abelian group

Figure 6: Abelian group verification by SS

SS mentioned that Q^+ with binary operation * is an abelian group as it fulfills the 4 conditions of group and it applies the commutative property. The operation which is translated in calculation, addition, division is symbolized as a concept that can be manipulated in the form of arithmetic and symbol of algebra (Tall et al., 2013).

Based on the verification results performed by SS, it can be seen that he proved the problem by using general algebraic symbols. These symbols are used to prove through algebraic calculation

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and manipulation based on the properties of group and abelian group. This is in line with Tall (2002) which states that symbol manipulation is one step in proving algebra. Thus, the verification performed by SS is general because he used algebraic symbols based on the conditions in the definition. Therefore, the verification done by SS is formal (Pedemonte, 2008). SS's mathematical thinking process in completing algebraic verification can be seen in Figure 7.

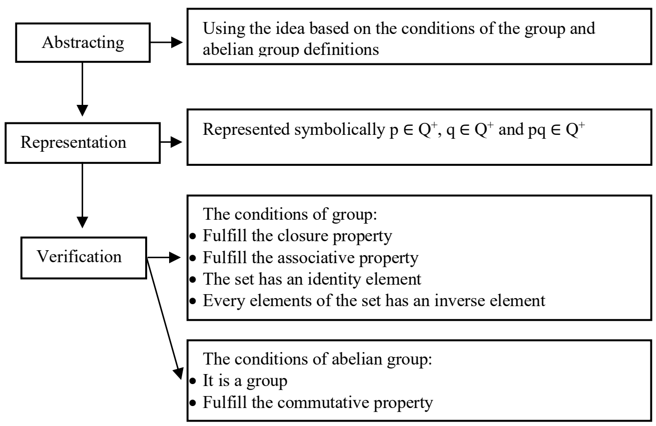
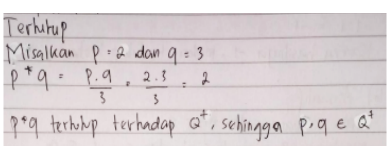


Figure 7: SS's Mathematical Thinking Process

SE performed an abelian group verification by using number as the representation of the set Q^+ with $p = 2$ and $q = 3$ to prove the commutative and closure property. Then, she added $r = 3$ to prove the associative property (Figure 8). Based on the verification of those three properties, SE concluded that it is an abelian group.



Translate version
Closure property:
Let $p = 2$ and $q = 3$

$$p * q = \frac{p.q}{3} = \frac{2.3}{3} = 2$$

$p * q$ is closure to Q^+, therefore $p, q \in Q^+$

Figure 8: SE's Verification Result

SE did not complete the verification because the subject only wrote three conditions even though she knows that abelian group has five properties. Therefore, she asked the permission to

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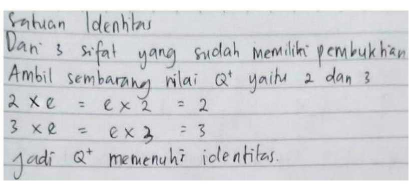
add the answer when the interview process was conducted as shown in the following interview transcript:

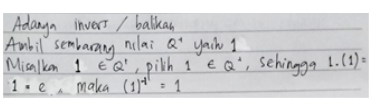
Interviewer	:	Do you agree that the closure, associative, and commutative properties are the conditions of abelian group?
SE	:	Emm… I think, actually there are two more properties that I have to prove but I did not have enough time. Therefore, based on the three conditions that I have proven, I concluded that $p * q = \frac{p.q}{3}$ for $p, q \in Q^+$ is an abelian group.
Interviewer	:	Did you not mention earlier that the three properties are based on your previous experience in solving the similar verification?
SE	:	Yes, what I mean was I did the test based on my memory when I solved the similar problem. But an abelian group had 5 conditions that need to be proven.
Interviewer	:	You said that there are 5 conditions, but you did only 3 conditions in the past. What about the other two conditions?
SE	:	May I add the answers now? I have not finished it because of the lack of time.

SE's additional answers are depicted in Figure 9 and Figure 10 below. SE did the multiplication of the identity element (e) and any element of Q^+ (she chose the number of 2 and 3), but in the end she did not succeeded in finding the identity element. She just mentioned that Q^+ fulfills the identity element. SE proved the inverse element by taking any element of Q^+ which has multiplication results equal to 1.


Satuan Identitas
Dari 3 sifat yang sudah memiliki pembuktian
Ambil sembarang nilai Q⁺ yaitu 2 dan 3
2 × e = e × 2 = 2
3 × e = e × 3 = 3
Jadi Q⁺ memenuhi identitas.

Translate version
Identity element:
From the 3 conditions that have been proven, take any element of Q^+ such as 2 and 3:
$2 \times e = e \times 2 = 2$
$3 \times e = e \times 3 = 3$
Therefore, Q^+ fulfil the identity

Figure 9: Additional data about the identity element of SE's verification


Adanya invers / balikan
Ambil sembarang nilai Q⁺ yaitu 1
Misalkan 1 ∈ Q⁺, pilih 1 ∈ Q⁺, sehingga 1.(1) =
1 = e, maka (1)⁻¹ = 1

Translate version
Inverse element:
Take any element of Q^+ such as 1
Let $1 \in Q^+$, choose $1 \in Q^+$ so $1.(1) = 1 = e$,
then $(1)^{-1} = 1$

Figure 10: Additional data about the inverse element of SE's verification

Based on the results of verification done by SE, it is visible that she has difficulty in using algebraic symbols, which resulted in her using a number as the representation of the set. The

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verification using numbers can be categorized as arithmetic verification (Uyangör, 2019). Arithmetic verifications that based on formal axioms are included as informal verifications. The arithmetic verification scheme is based on the use of number, while the analytical verification scheme is based on reasoning and logical deduction to obtain valid arguments (Mukuka & Shumba, 2016). SE's mathematical thinking process in solving algebraic verifications can be seen in Figure 11.

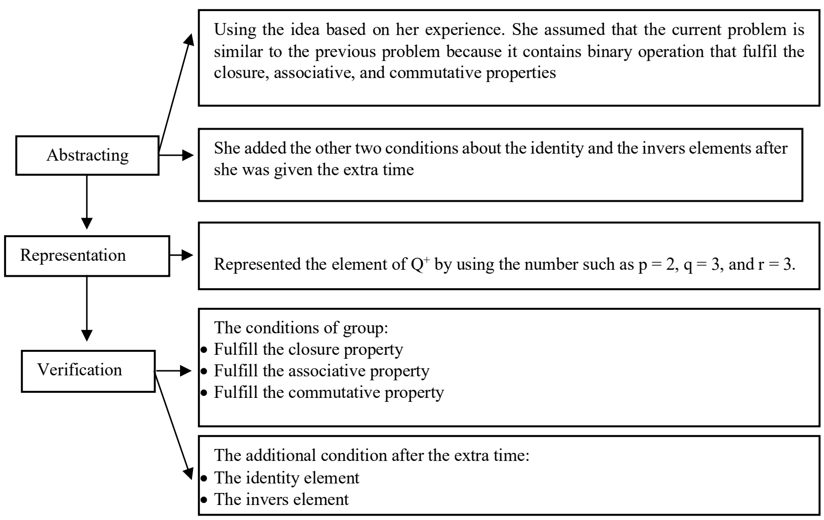

Figure 11: SE's mathematical thinking process

The two types of student thinking indicate that processes and concepts are inseparable. Processes and concepts are two important aspects in mathematics because they contain the meaningful symbols based on definition or axioms (Mukuka & Shumba, 2016). On the other hand, mathematical thinking is a mathematics technique which contain concepts and processes in solving problem directly or indirectly (Çelik & Özdemir et al., 2020). Based on the interpretation above, the difference between the two types based on the process and concept that they used can be simplified as seen in Table 2.

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Table 2: The differences of students' thinking process types in completing the verification

Types of Students	Crystalline Concept	
	Operational Process	**Set-Theoretic Concept**
Symbolic compression	Calculation and manipulation processes are conducted symbolically	Known concepts about axiomatic formal mathematics are deduced by formal verification.
Embodied compression	Calculation and manipulation processes are conducted arithmetically	Known concepts about axiomatic formal mathematics are deduced by informal verification.

It is seen from Table 2 above that the students with different type of Crystalline concept complete the algebraic verification in different ways. The symbolic compression student used the variable that represent the member of the set, meanwhile, student with the embodied compression used the specific number as the representation of the set. This result is in line with Tall (2014) which states that the symbolic compression student thinks generally because he/she does not use the specific numbers as it is used by embodied compression student. Embodied student does not use the algebraic symbol as she assumes that the verification with specific number is enough. If she succeeds to prove the verification with specific number then automatically the verification is proven for all of set members. One of the reasons this can happen is because the transition process from embodiment to symbolism has not been completed. Tall (2008) states that there are two types of thinking throughout school mathematics: embodiment and symbolism. The first type of thinking is used to give specific meanings in abstract context, while the second type is used to build the computational mental.?? Then, further research needs to be conducted for high school students about the type of student thinking in solving the verification problem.

To support the embodied students to be able to think symbolically, the lecture has to design a learning that extends the context so the students accustom to solve the algebraic verification symbolically. Tall (2008) states that the transition to the formal axiomatic can be built through experiences about embodiment and symbolism. Then the lecture has to give the opportunity for students to solve problems on various kinds of the verification. Those various kinds of problem can give students experiences that can be used as the transition medium of thinking type.

CONCLUSIONS

The results showed that both students with symbolic compression and embodied compression types completed algebraic verifications through mathematical thinking in the form of: abstracting, representation, and verification. Students performed abstractings to determine the ideas that would be used in the verification. Student with symbolic compression type chooses to

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determine whether or not the positive rational number set and binary operation $p * q = \frac{pq}{3}$, for all $p, q \in \mathrm{Q}^{+}$ form a group. If it has been proven that it forms a group, he continues by showing whether or not the group is abelian. Meanwhile, student with embodied compression type works for some properties of abelian group: fulfill the closure, associative, and commutative properties. Although she know that there are still two properties left, but he cannot work for them because of the limited time. She acknowledges it through the interview process. Next, students conduct representations to communicate their ideas. Student with symbolic compression type represents the given information through algebraic symbols such as $p \in Q^{+}, q \in Q^{+}$ and $pq \in Q^{+}$. Meanwhile student with embodied compression type represents the given information through the number as the representation of the positive rational number set such as $p = 2, q = 3$ and $pq \in Q^{+}$. The last step is verification. Both students perform the verification step based on their idea in the abstracting step and complete the verification by using the result of the representation step. Student with symbolic compression type completes the verification process by proving that the positive rational number set and binary operation $p * q = \frac{pq}{3}$, for all $p, q \in \mathrm{Q}^{+}$ forms a group. Then, he continues the verification by showing that it is an abelian group. Meanwhile, student with embodied compression type works the verification process by proving that the positive rational number set and binary operation $p * q = \frac{pq}{3}$, for all $p, q \in \mathrm{Q}^{+}$ fulfills the closure, associative, and commutative properties. Then, she adds the other properties such as the identity element and inverse element in the interview process.

The description above shows that both of embodied and symbolic compression type students complete the algebraic verification problem through mathematical thinking by using different style. The differences are embodied compression type student completes the verification arithmetically while symbolic compression type students complete the verification symbolically. Therefore, it is essential to design a learning that can accustom students to solve the algebraic verification symbolically as the verification should be done deductively by giving various kind of verification problem.

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