

# Improving the Performance of Mathematics Teachers through Preparing a Research Lesson on the Topic of Maximum and Minimum Values of a Trigonometric Function

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Abstract: In each educational curriculum, trigonometry is an important subject at upper secondary schools and pre university level that apply in many other subjects such as algebra, calculus, geometry and physics. Many of students have serious problem in learning the trigonometric materials because usually mathematics educators transfer the trigonometric concepts to students through traditional methods that encourage students to memorize the trigonometric concepts. The purpose of this qualitative case study is to introduce the Lesson Study as a new teaching method based on problem solving approach in order to increase the performance of teachers in teaching trigonometry. In this study, a group of three mathematics teachers from an international pre-university centre in Malaysia and the researcher contributed in preparing a research lesson on the topic of the maximum and minimum values of a trigonometric function. Also, this research lesson improved in an existing class containing 10 students. Data collected through observations of discussion meetings and analyzed descriptively. In this research lesson, the researcher discussed teaching the maximum and minimum values of a trigonometric function through variety of solutions and likely misconceptions among students. Maybe this article helps mathematics educators to have better performance in teaching trigonometry through Lesson Study based on problem solving approach. eachers from an international pre-university centre in Malaysia and the researcher<br>eachers from an international pre-university centre in Malaysia and the researcher<br>of a trigonometric function. Also, this research lesson

Keywords: Lesson Study, Misconception, Problem solving, Trigonometry

# INTRODUCTION

Trigonometry is a complex part of mathematics that plays an important role in our daily life. Trigonometric concepts are difficult to understand by learners and usually this subject is challenging for teaching (Martin-Fernandez et al., 2019). For example, a study by Gholami et al. favourite subject because of the complexity of this subject. They preferred to teach courses related to the calculus, algebra, statistics and probability. In teaching trigonometry, the challenge resides in the fact that many of traditional methods of teaching, primarily emphasizes superficial skills





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and such methods do not allow learners to understand the trigonometric topics conceptually (Altman & Kidron, 2016). Therefore, students face difficulties in learning trigonometry through problem solving due to misconceptions about trigonometric contents (Weber, 2005). In learning trigonometry, students experience numerous obstacles due to misconceptions, for instance, students' misunderstandings with the concepts angle and angle measure at the starting point for learning trigonometry are the most basic problem among students in depth understanding of trigonometric concepts (J. Nabie et al., 2018). Discussing different methods of solving trigonometric problems and common students 'misunderstandings about them will help improve teachers' performance in the classroom. Based on Xenofontos and Andrews (2014) a mathematics task or a goal-directed activity is considered as a problem for students if this task is new and challenging to them. They further added that a mathematical exercise is not a problem because learners solve mathematics exercises by following steps they have learned. Teaching mathematics materials straight from textbooks is common in our educational institutions including schools and universities (Dhakal et al., 2020). Therefore, most of mathematics teachers still prefer to teach mathematical concepts through traditional methods by emphasizes on solving routine exercises in teaching (Voskoglou, 2019). It seems new methods of teaching such as Lesson Study requires a high level of mathematical knowledge and pedagogy to prepare suitable mathematical materials based on the ability of students. Considering appropriate activities and mathematics problems in the prepared lessons help learners to have better performance in the classes and enhance their abilities in problem solving (Gholami, Ayub, & Yunus, 2021).

The Japanese Lesson study approach, not only focuses on a team-oriented educational design and shared responsibility for the educational processes and outcomes but also clearly focuses on students' experience of learning process and not simply on the methods of teaching (Elliott, 2019; Hanfstingl et al., 2019). For example, familiarity with students' misunderstandings about different mathematical concepts provides a good opportunity for the Lesson Study team to provide appropriate research lessons. In this educational method, Lesson Study group members prepare lessons that are called research lessons in a participatory manner and after teaching in real classes, they constantly improve them (Coenders & Verhoef, 2019; Lewis et al., 2006). Therefore, Lesson Study as a kind of professional development programs improves the teaching knowledge of educators especially their pedagogical content knowledge through discussions among them regarding the students' learning (Coenders & Verhoef, 2019). This educational approach helps mathematics teachers to overcome difficulties facing students such as their misconceptions about mathematical concepts and to improve student learning (Leavy & Hourigan, 2018). Japanese Lesson Study has various models and is now spreading to educational systems of other countries in order to increases students' learning through supports teachers in improving their skills and teaching practices (Grimsaeth & Hallas, 2015). Research lesson is the most important part of Lesson Study and the procedure of preparing a research lesson is as follows (Lewis, 2002).

- 1) The Lesson Study group members set the goals for students' learning based on their abilities and skills
- 2) The members of the Lesson Study group collaborate to improve a plan for a teaching session to provide better learning situation for students

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- 3) One of the Lesson Study group members teaches the research lesson, while the others collect data by observation.
- 4) In a post-lesson discussion, the members of Lesson Study group analyze their observations in order to improve the quality of research lesson.
- 5) If necessary, the members of Lesson Study group plan to improve their teaching practice for a new research lesson.

Discussion about the misunderstandings of students about mathematical concepts is very beneficial for educators to have effective teaching. In other words, knowing the nature of misconceptions and misunderstandings and their sources regarding various contents of mathematics that are common among students of all educational levels, helps educators to plan suitable instructions for students' learning. Understanding mathematics concepts depend on linking from the prior knowledge and new topics, which may help or hinder the process of learning. Incorrect prior knowledge regarding mathematical concepts are called misconceptions, that cause a disability to learn the new contents (Alkhateeb, 2020). Mathematics educators can change the misconceptions after brief instructions and provide an appropriate situation for conceptually learning (Durkin & Rittle-Johnson, 2015). In fact, Suitable teaching methods eliminate the mistakes and misconceptions that students have about mathematical concepts (Yilmaz et al., 2018). In teaching trigonometry, mostly misconceptions arise from the teaching method. For instance, as identified by Tuna (2013) about 90% of the novice mathematics teachers had misconceptions regarding the definition of the trigonometric concept of radian. They explained incorrect definitions for this concept such as, "the expression of degree in terms of  $\pi$ ", "the unit of length of degree", and "I just know the formula of  $\frac{D}{180} = \frac{R}{\pi}$  and "I do not know what radian is". Therefore, mathematics educators, beside the appropriate subject matter knowledge, require knowing the common misunderstandings and misconceptions that students face in a specific topic. Regarding this issue, knowing the variety of creatively solutions for trigonometric problems and students' misconceptions about them provide a good opportunity for educators to improve their teaching knowledge (Dundar, 2015). The purpose of this study is to prepare an effective research lesson for teaching a given trigonometry problem and investigate the misconceptions emerging in students learning regarding this problem in order to provide better teaching and learning situation.

# METHODOLOGY

# Research Design and Sample

This study was conducted during the academic year 2020. In this study, a group of three mathematics teachers (Two males and a female) from an international school in Malaysia and the researcher collaboratively planned, discussed and designed a research lesson on the topic of the maximum and minimum values of a trigonometric function. Meanwhile, all the members of the Lesson Study group were experienced teachers with at least 15 years experiences in teaching mathematics. Furthermore, 10 students (4 male and 6 female) from an existing class were participating in this study.





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# Data Collection

Finding the maximum and minimum values of a trigonometric function is an important part of trigonometry subject that apply in many trigonometric concepts such as the range of functions, drawing the graph of functions and optimization the real world problems. For example, the following problem shows an application of the maximum and minimum values of a trigonometric function in the real world. HEMATICS TEACHING RESEARCH JOURNAL 29<br>
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Problem: In a four season country, the length of each day of a year calculated based on the following trigonometric function

$$
L(t) = 12 + 2.4 \sin \frac{2\pi}{365} (t - 1)
$$

year) and  $L(t)$  is the length of the day in hours. Determine the length of the longest and shortest day of the year.

Based on the importance of maximum and minimum values of the trigonometric functions, the researcher studied regarding the topic "Maximum and minimum values of the function  $f(x) =$ members suggested three solution methods for this problem and discussed the likely misconceptions of students about these solutions.

Two weeks before starting this study, the researcher introduced the topic of this research lesson to the members of Lesson Study group and asked them to prepare suitable material for a rich research lesson. In a meeting, they planned, discussed and designed a research lesson and a teacher of Lesson Study group taught this research lesson in a class and the others observed and collected data. In a post-discussion meeting, they tried to improve the quality of this research lesson.

# Analyzing the Data

All members of the Lesson Study group were familiar with the Lesson Study approach because they participated in an in-service program related to the Lesson Study a few months before starting this study. The researcher introduced the topic of this research lesson to teachers and asked them to share their knowledge and experience to produce a research lesson on the topic entitled day of the year.<br>
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Based on the importance of maximum and minimum values of the function  $f(x) =$ <br>
ersience the studied regarding the During three sessions, they planned, discussed and designed a research lesson for pre-university level students. The materials in this research lesson gathered through observations of discussion meetings and teaching this research lesson for students in a real class by a member of Lesson Study group. The researcher analyzed the methods and suggestions of the Lesson Study group members descriptively to prepare a research lesson.





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### FINDINGS ON THE PREPARED RESEARCH LESSON

The members of the Lesson Study group prepared a research lesson entitled "Maximum and **MATHEMATICS TEACHING RESEARCH JOURNAL** 30<br>
THE PREPARED RESEARCH LESSON<br>
The members of the Lesson Study group prepared a research lesson entitled "Maximum and<br>
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# First solution

We know the formula

$$
\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y.
$$

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Since we do not know whether  $a^2 + b^2$  is<br>  $m$  such that  $(ma)^2 + (mb)^2 = 1$ .<br>  $c^2 = 1$ <br>  $c^2 = 1$ <br>  $c^2 = 1$ <br>  $\Rightarrow c^2 = 1$ <br>  $\frac{1}{\sqrt{a^2 + b^2}}$ <br>  $\Rightarrow c^2 + (\frac{b}{\sqrt{a^2 + b^2}})^2 = 1$ .<br>  $\Rightarrow c \cos x$ <br>

1 1 = sin ± cos ⇒ y = sin

.

Therefore, we have

$$
(ma)^2 + (mb)^2 = 1 \Rightarrow \left(\frac{a}{\sqrt{a^2 + b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2 + b^2}}\right)^2 = 1.
$$

Now, we can write

$$
y = a \sin x \pm b \cos x
$$
  
\n
$$
\Rightarrow y = \frac{ma \sin x}{m} \pm \frac{mb \cos x}{m}
$$





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\n
$$
= y = \frac{1}{m} (ma \sin x \pm mb \cos x)
$$
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$$
\Rightarrow y = \frac{1}{\sqrt{a^2 + b^2}} (\frac{a}{\sqrt{a^2 + b^2}} \sin x \pm \frac{b}{\sqrt{a^2 + b^2}} \cos x)
$$
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$$
\Rightarrow y = \sqrt{a^2 + b^2} (\frac{a}{\sqrt{a^2 + b^2}} \sin x \pm \frac{b}{\sqrt{a^2 + b^2}} \cos x)
$$
\nAssume that  $\frac{a}{\sqrt{a^2 + b^2}} = \cos \theta$ , then,  $\sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \frac{a^2}{a^2 + b^2}} = \frac{b}{\sqrt{a^2 + b^2}}$ .

\nBased on the above calculations we obtain the following trigonometric function  $y = \sqrt{a^2 + b^2} (\cos \theta \sin x \pm \sin \theta \cos x)$ 

\n
$$
\Rightarrow y = \sqrt{a^2 + b^2} \sin(x \pm \theta).
$$
\nSince,  $-1 \le \sin(x \pm \theta) \le 1$ , then

\n
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-\sqrt{a^2 + b^2} \le \sqrt{a^2 + b^2} \sin(x \pm \theta) \le \sqrt{a^2 + b^2}.
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\nIt means that

\n
$$
-\sqrt{a^2 + b^2} \le a \sin x \pm b \cos x \le \sqrt{a^2 + b^2}.
$$
\nSecond Solution:

\nWe convert this function into an expression consisting of only sine function as follows

\n
$$
a \sin x + b \cos x = c \sin(k + x)
$$

 $b$ Assume that  $\frac{a}{\sqrt{a^2+b^2}} = \cos \theta$ , then,  $\sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \frac{a}{a^2+b^2}} = \frac{b}{\sqrt{a^2+b^2}}$ .<br>Based on the above calculations we obtain the following trigonometric function en, sin  $\theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \frac{a^2}{a^2 + b^2}} = \frac{b}{\sqrt{a^2 + b^2}}$ <br>
we obtain the following trigonometric function<br>  $\sqrt{a^2 + b^2} (\cos \theta \sin x \pm \sin \theta \cos x)$ <br>  $\Rightarrow y = \sqrt{a^2 + b^2} \sin(x \pm \theta)$ .<br>
en<br>  $\overline{b^2} \le \sqrt{a^2 + b^2} \sin(x \pm \theta) \le \sqrt{a^2 + b^2}$ 

$$
y = \sqrt{a^2 + b^2} (\cos \theta \sin x \pm \sin \theta \cos x
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$$
\Rightarrow y = \sqrt{a^2 + b^2} \sin(x \pm \theta).
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-\sqrt{a^2+b^2} \le \sqrt{a^2+b^2} \sin(x \pm \theta) \le \sqrt{a^2+b^2}.
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$$
-\sqrt{a^2 + b^2} \le a \sin x \pm b \cos x \le \sqrt{a^2 + b^2}.
$$

We convert this function into an expression consisting of only sine function as follows

$$
a\sin x + b\cos x = c\sin(k + x)
$$
  
\n
$$
\Rightarrow a\sin x + b\cos x = c\sin k \cos x + c\cos k \sin x
$$

A sin  $x + b$  cos *o*, then, sin  $b - \sqrt{1 - \cos^2 b} - \sqrt{1 - \frac{2 + b^2}{a^2 + b^2}} - \frac{2}{\sqrt{a^2 + b^2}}$ <br>
Based on the above calculations we obtain the following trigonometric function<br>  $y = \sqrt{a^2 + b^2} \cos \theta \sin x \pm \sin \theta \cos x$ <br>
Since,  $-1 \le \sin(x \pm \theta$ the left and right sides. Therefore,  $\frac{1}{a^2 + b^2} \sin(x \pm \theta)$ .<br>  $\frac{1}{a^2 + b^2} \sin(x \pm \theta) \le \sqrt{a^2 + b^2}$ .<br>  $\sin x \pm b \cos x \le \sqrt{a^2 + b^2}$ .<br>
a consisting of only sine function as follows<br>  $\phi \cos x = c \sin(k + x)$ <br>  $= c \sin k \cos x + c \cos k \sin x$ .<br>
are coefficients of  $\sin x$  and  $\cos x$  should b Since,  $-1 \le \sin(x \pm \theta) \le 1$ , then<br>  $-\sqrt{a^2 + b^2} \le \sqrt{a^2 + b^2} \sin(x \pm \theta) \le \sqrt{a^2 + b^2}$ .<br>
It means that<br>  $-\sqrt{a^2 + b^2} \le a \sin x \pm b \cos x \le \sqrt{a^2 + b^2}$ .<br>
Second Solution:<br>
We convert this function into an expression consisting of only s

$$
a = c \cos k
$$

$$
b = c \sin k.
$$

$$
\frac{b^2}{b^2} \le a \sin x \pm b \cos x \le \sqrt{a^2 + b^2}.
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pression consisting of only sine function as follows  
in  $x + b \cos x = c \sin(k + x)$   
 $b \cos x = c \sin k \cos x + c \cos k \sin x$ .  
of x, the coefficients of sin x and cos x should be equal on  
 $a = c \cos k$   
 $b = c \sin k$ .  
ations we have,  
 $a = c \cos k \Rightarrow \cos k = \frac{a}{c}$   
 $b = c \sin k \Rightarrow \sin k = \frac{b}{c}$ 

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ression consisting of only sine function as follows<br>  $n x + b \cos x = c \sin(k + x)$ <br>  $\cos x = c \sin k \cos x + c \cos k \sin x$ .<br>
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Vol 14 no 2<br>  $k + cos^2 k = 1 \Rightarrow (\frac{b}{c})^2 + (\frac{a}{c})^2 = 1 \Rightarrow c = \pm \sqrt{a^2 + b^2}$ .<br>
relations  $a = c \cos k$  and  $b = c \sin k$  we obtain<br>  $\frac{c \sin}{c \cos k} = \frac{b}{a} \Rightarrow \tan k = \frac{b}{a} \Rightarrow k = \tan^{-1} \frac{b}{a}$ . HEMATICS TEACHING RESEARCH JOURNAL<br>
1ER 2022<br>  $k = 1 \Rightarrow (\frac{b}{c})^2 + (\frac{a}{c})^2 = 1 \Rightarrow c = \pm \sqrt{a^2 + b^2}$ .<br>  $a = c \cos k$  and  $b = c \sin k$  we obtain<br>  $\frac{1}{b \cos k} = \frac{b}{a} \Rightarrow \tan k = \frac{b}{a} \Rightarrow k = \tan^{-1} \frac{b}{a}$ . S TEACHING RESEARCH JOURNAL<br>  $y^2 + (\frac{a}{c})^2 = 1 \Rightarrow c = \pm \sqrt{a^2 + b^2}.$ <br>
and  $b = c \sin k$  we obtain<br>  $n k = \frac{b}{a} \Rightarrow k = \tan^{-1} \frac{b}{a}.$ MATHEMATICS TEACHING RESEARCH JOURNAL<br>
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SUMMER 2022<br>  $\left(\frac{e^{3n}}{2} + \cos^2 k = 1 \Rightarrow \$ ATHEMATICS TEACHING RESEARCH JOURNAL<br>
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os  $a = c \cos k$  and  $b = c \sin k$  we obtain<br>  $\frac{c \sin}{c \cos k} = \frac{b}{a} \Rightarrow \tan k = \frac{b}{a} \Rightarrow k = \tan^{-1} \frac{b}{a}$ .<br>  $x + b \cos x = \pm \sqrt{a^2 + b^2} \sin(x + \tan$ ATHEMATICS TEACHING RESEARCH JOURNAL<br>
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1 ⇒  $\left(\frac{b}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1 \Rightarrow c = \pm \sqrt{a^2 + b^2}$ .<br>
c cos k and b = c sin k we obtain<br>  $\frac{b}{a} \Rightarrow \tan k = \frac{b}{a} \Rightarrow k = \tan^{-1} \frac{b}{a}$ .<br>
s x =  $\pm \sqrt{a^2 + b^2} \sin(x + \tan^{-1} \frac{b}{a})$ . MATHEMATICS TEACHING RESEARCH JOURNAL<br>
SUMMER 2022<br>
Vol 14 no 2<br>  $k + cos^2 k = 1 \Rightarrow (\frac{b}{c})^2 + (\frac{a}{c})^2 = 1 \Rightarrow c = \pm \sqrt{a^2 + b^2}$ .<br>
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+  $cos^2 k = 1 \Rightarrow {p \choose c}^2 + {p \choose c}^2 = 1 \Rightarrow c = \pm \sqrt{a^2 + b^2}$ .<br>
lations  $a = c \cos k$  and  $b = c \sin k$  we obtain<br>  $\frac{e \sin}{c \cos k} = \frac{b}{a} \Rightarrow \tan k = \frac{b}{a} \Rightarrow k = \tan^{-1} \frac{b}{a}$ .<br>
sin x + b cos x =  $\pm \sqrt{a^2 + b^2} \sin(x + \tan^{-1} \frac{b}{$

$$
\sin^2 k + \cos^2 k = 1 \Rightarrow (\frac{b}{c})^2 + (\frac{a}{c})^2 = 1 \Rightarrow c = \pm \sqrt{a^2 + b^2}.
$$

$$
\frac{c \sin}{c \cos k} = \frac{b}{a} \Rightarrow \tan k = \frac{b}{a} \Rightarrow k = \tan^{-1} \frac{b}{a}.
$$
 Therefore, we have

$$
a \sin x + b \cos x = \pm \sqrt{a^2 + b^2} \sin(x + \tan^{-1} \frac{b}{a}).
$$

). If we limit the arctan to be within

$$
-\frac{\pi}{2} < \tan^{-1}\frac{b}{a} < \frac{\pi}{2}.
$$

. Then we obtain the following relation

$$
a\sin x + b\cos x = \sqrt{a^2 + b^2}\sin(x + \tan^{-1}\frac{b}{a}).
$$

$$
sin^{2}k + cos^{2}k = 1 \Rightarrow \left(\frac{b}{c}\right)^{2} + \left(\frac{a}{c}\right)^{2} = 1 \Rightarrow c = \pm \sqrt{a^{2} + b^{2}}.
$$
  
Now, according to the relations  $a = c \cos k$  and  $b = c \sin k$  we obtain  

$$
\frac{c \sin k}{c \cos k} = \frac{b}{a} \Rightarrow \tan k = \frac{b}{a} \Rightarrow k = \tan^{-1} \frac{b}{a}.
$$
  
Therefore, we have  

$$
a \sin x + b \cos x = \pm \sqrt{a^{2} + b^{2}} \sin(x + \tan^{-1} \frac{b}{a}).
$$
  
If we limit the arctan to be within  

$$
-\frac{\pi}{2} < \tan^{-1} \frac{b}{a} < \frac{\pi}{2}.
$$
  
Then we obtain the following relation  

$$
a \sin x + b \cos x = \sqrt{a^{2} + b^{2}} \sin(x + \tan^{-1} \frac{b}{a}).
$$
  
Since  $-1 \leq \sin(x + \tan^{-1} \frac{b}{a}) \leq 1$ , we see that  

$$
-\sqrt{a^{2} + b^{2}} \leq \sqrt{a^{2} + b^{2}} \sin(x + \tan^{-1} \frac{b}{a}) \leq \sqrt{a^{2} + b^{2}}
$$

$$
\Rightarrow -\sqrt{a^{2} + b^{2}} \leq f(x) = a \sin x + b \cos x \leq \sqrt{a^{2} + b^{2}}.
$$
  
Similarly, according to this method we can find the maximum and minimum values of the  
trigonometric function  $y = a \sin x - b \cos x$ .  
Third solution:  
Definition 1:  
Definition 1:  
The dot product of two vectors  $\vec{u} = (a, b)$  and  $\vec{v} = (x, y)$ , written  $\vec{u} \cdot \vec{v}$  is given by the definition  
 $\vec{u} \cdot \vec{v} = (a, b) \cdot (x, y) = ax + by$ .  
Definition 2:  
Assume that the angle between two vectors  $\vec{u} = (a, b)$  and  $\vec{v} = (x, y)$  is  $\theta$  then the dot product of these vectors is defined as  

$$
\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta
$$
  
where  $|\vec{u}| = \sqrt{a^{2} + b^{2$ 

e obtain<br>  $n^{-1} \frac{b}{a}$ .<br>  $+ \tan^{-1} \frac{b}{a}$ .<br>  $\tan^{-1} \frac{b}{a}$ .<br>  $\le \sqrt{a^2 + b^2}$ .<br>  $c \le \sqrt{a^2 + b^2}$ .<br>  $\tan \frac{b}{a}$ .  $-\sqrt{a^2 + b^2} \le f(x) = a \sin x + b \cos x \le \sqrt{a^2 + b^2}$ .<br>
Similarly, according to this method we can find the maximum and minimum values of the<br>
dirigonometric function  $y = a \sin x - b \cos x$ .<br>
Third solution:<br>
Definition 1:<br>
Definition 1:<br>
The

Definition 1:

of these vectors is defined as

$$
\vec{u}.\vec{v} = |\vec{u}| |\vec{v}| \cos \theta
$$

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Now for solution of this given problem, we consider two vectors  $\overrightarrow{u_1} = (a, b)$  and  $\overrightarrow{u_2} = (\sin x, \cos x)$  and find the dot product of them through two different methods based on the EARCH JOURNAL 33<br>  $\overrightarrow{u_1} = (a, b)$  and  $\overrightarrow{u_2} =$ <br>
rent methods based on the<br>  $+ b \cos x$ <br>  $\cos \theta = \sqrt{a^2 + b^2} \cos \theta$  $\overrightarrow{u_2}$  = MATHEMATICS TEACHING RESEARCH JOURNAL<br>
SUMMER 2022<br>
Now for solution of this given problem, we consider two vectors  $\overrightarrow{u_1} = (a, b)$  and  $\overrightarrow{u_2} =$ <br>
(sin x, cos x) and find the dot product of them through two different m definitions 1 and 2. MATHEMATICS TEACHING RESEARCH JOURNAL<br>
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ven problem, we consider two vectors  $\overrightarrow{u_1} = (a, b)$  and  $\overrightarrow{u_2} =$ <br>
dot product of them through two different methods based on the<br>  $\overrightarrow{u_2} = (a, b)$ . (sin MATHEMATICS TEACHING RESEARCH JOURNAL<br>
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given problem, we consider two vectors  $\overline{u_1} = (a, b)$  and  $\overline{u_2} =$ <br>
the dot product of them through two different methods based on the<br>  $\overline{u_1} \cdot \overline{$ MATHEMATICS TEACHING RESEARCH JOURNAL<br>
SUMMER 2022<br>
Now for solution of this given problem, we consider two vectors  $\overline{u_1} = (a, b)$  and  $\overline{u_2} =$ <br>
(sin x, cos x) and find the dot product of them through two different m EACHING RESEARCH JOURNAL<br>
sider two vectors  $\overline{u_1} = (a, b)$  and  $\overline{u_2} =$ <br>
hrough two different methods based on the<br>  $\cos x$ ) =  $a \sin x + b \cos x$ <br>  $\sin^2 \theta + \cos^2 \theta \cos \theta = \sqrt{a^2 + b^2} \cos \theta$ <br>  $-\overline{b^2} \cos \theta \le \sqrt{a^2 + b^2}$ <br>  $\overline{u_1} \cdot \$ CHING RESEARCH JOURNAL<br>
or two vectors  $\overrightarrow{u_1} = (a, b)$  and  $\overrightarrow{u_2} =$ <br>
ugh two different methods based on the<br>  $(x) = a \sin x + b \cos x$ <br>  $\overrightarrow{e\theta} + \cos^2 \theta \cos \theta = \sqrt{a^2 + b^2} \cos \theta$ <br>  $\overrightarrow{c\theta} = \sqrt{a^2 + b^2}$ <br>  $\overrightarrow{u_2} \le \sqrt{a^2 + b^2}$ <br>  $b \cos$ MATHEMATICS TEACHING RESEARCH JOURNAL<br>
SUMMER 2022<br>
Now for solution of this given problem, we consider two vectors  $\overline{u_1} = (a, b)$  and  $\overline{u_2} =$ <br>
(Sin *x*, cos *x*) and find the dot product of them through two differ

$$
\overrightarrow{u_1} \cdot \overrightarrow{u_2} = (a, b) \cdot (\sin x, \cos x) = a \sin x + b \cos x
$$

 $\vec{u_1} \cdot \vec{u_2} = |\vec{u_1}||\vec{u_2}| \cos \theta = \sqrt{a^2 + b^2} \sqrt{\sin^2 \theta + \cos^2 \theta \cos \theta}$ 

**MATHEMATICS TEACHING RESEARCH JOURNAL** 33  
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\n**INOM OF solution of this given problem, we consider two vectors** 
$$
\overrightarrow{u_1} = (a, b)
$$
 and  $\overrightarrow{u_2} =$   
\n(sin *x*, cos *x*) and find the dot product of them through two different methods based on the  
\ndefinitions 1 and 2.  
\n
$$
\overrightarrow{u_1} \cdot \overrightarrow{u_2} = |\overrightarrow{u_1}||\overrightarrow{u_2}| \cos \theta = \sqrt{a^2 + b^2} \sqrt{\sin^2 \theta + \cos^2 \theta} \cos \theta = \sqrt{a^2 + b^2} \cos \theta
$$
\nSince  $-1 \le \cos \theta \le 1$ ,  
\n
$$
-\sqrt{a^2 + b^2} \le \sqrt{a^2 + b^2} \cos \theta \le \sqrt{a^2 + b^2}
$$
\n
$$
\Rightarrow -\sqrt{a^2 + b^2} \le \overrightarrow{u_1} \cdot \overrightarrow{u_2} \le \sqrt{a^2 + b^2}
$$
\n
$$
\Rightarrow -\sqrt{a^2 + b^2} \le \overrightarrow{u_1} \cdot \overrightarrow{u_2} \le \sqrt{a^2 + b^2}
$$
\n
$$
\Rightarrow -\sqrt{a^2 + b^2} \le a \sin x + b \cos x \le \sqrt{a^2 + b^2}.
$$
\nTherefore, for the trigonometric function  $y = a \sin x + b \cos x$  we have  
\n
$$
-\sqrt{a^2 + b^2} \le y \le \sqrt{a^2 + b^2}.
$$
\nThrough similar process, we can show that  $-\sqrt{a^2 + b^2} \le a \sin x - b \cos x \le \sqrt{a^2 + b^2}.$ 

\nExample 1:

\nFind the range of the function  $f(x) = 2 + 3 \sin x - 4 \cos x$ .

\nSolution:

\nWe know that

\n
$$
-\sqrt{3^2 + 4^2} \le 3 \sin x - 4 \cos x \le \sqrt{3^2 + 4^2} \Rightarrow -5 \le 3 \sin x - 4 \cos x \le 5.
$$
\nTherefore,  $-3 \le 2 + 3 \sin x - 4 \cos x \le 7$  and the range of this function is  $R_f = [-3, 7]$ .

\

$$
-\sqrt{a^2 + b^2} \le y \le \sqrt{a^2 + b^2}.
$$

Therefore, for the trigonometric function  $y = a \sin x + b \cos x$  we have<br>  $-\sqrt{a^2 + b^2} \le y \le \sqrt{a^2 + b^2}$ .<br>
Through similar process, we can show that  $-\sqrt{a^2 + b^2} \le a \sin x - b \cos x \le \sqrt{a^2 + b^2}$ .<br>
Example 1:<br>
Find the range of the function

We know that

$$
-\sqrt{3^2 + 4^2} \le 3\sin x - 4\cos x \le \sqrt{3^2 + 4^2} \Rightarrow -5 \le 3\sin x - 4\cos x \le 5.
$$

Determine the maximum and minimum values of the following two variables function

$$
f(x, y) = 3\sin x + 4\cos x - 3\cos y + 4\sin y - 4.
$$

We know,

$$
-\sqrt{3^2 + 4^2} \le 3\sin x + 4\cos x \le \sqrt{3^2 + 4^2}
$$

$$
-\sqrt{3^2 + 4^2} \le 4\sin y - 3\cos y \le \sqrt{3^2 + 4^2}.
$$

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Find the range of the function  $f(x) = 2 + 3 \sin x - 4 \cos x$ .<br>
Solution:<br>
We know that<br>  $-\sqrt{3^2 + 4^2} \le 3 \sin x - 4 \cos x \le \sqrt{3^2 + 4^2} \Rightarrow -5 \le 3 \sin x - 4 \cos x \le 5$ .<br>
Therefore,  $-3 \le 2 + 3 \sin x - 4 \cos x \le 7$  and the range of this functi





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Vol 14 no 2<br>  $-10 \le 3 \sin x + 4 \cos x + 4 \sin y - 3 \cos y \le 10.$ <br>
Now, we have<br>  $-14 \le 3 \sin x + 4 \cos x + 4 \sin y - 3 \cos y - 4 \le 6.$ <br>
It means  $-14 \le f(x, y) \le 6.$

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$$
-10 \le 3 \sin x + 4 \cos x + 4 \sin y - 3 \cos y \le 10.
$$

$$
-14 \le 3 \sin x + 4 \cos x + 4 \sin y - 3 \cos y - 4 \le 6
$$

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THEORY SUMMER 2022<br>
Vol 14 no 2<br>  $-10 \le 3 \sin x + 4 \cos x + 4 \sin y - 3 \cos y \le 10$ .<br>
Now, we have<br>  $-14 \le 3 \sin x + 4 \cos x + 4 \sin y - 3 \cos y - 4 \le 6$ .<br>
It means  $-14 \le f(x, y) \le 6$ .<br> *Generalization of this Trigo* MATHEMATICS TEACHING RESEARCH JOURNAL<br>
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TRACHING<br>  $-10 \le 3 \sin x + 4 \cos x + 4 \sin y - 3 \cos y \le 10$ .<br>
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It means  $-14 \le f(x, y) \le 6$ .<br> *Deneralization of this Trigonometric P* MATHEMATICS TEACHING RESEARCH JOURNAL<br>
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SUMMER 2022<br>  $-10 \le 3 \sin x + 4 \cos x + 4 \sin y - 3 \cos y \le 10.$ <br>
Now, we have<br>  $-14 \le 3 \sin x + 4 \cos x + 4 \sin y - 3 \cos y - 4 \le 6.$ <br>
It means  $-14 \le f(x, y) \le 6.$ <br>
Generalization of this Trig  $t - 3 \cos y \le 10.$ <br>  $- 3 \cos y - 4 \le 6.$ <br>  $\cos kx + c$  is calculated based on the<br>  $\ln kx + b \cos kx \le \sqrt{a^2 + b^2}$ . The<br>  $kx \pm b \cos kx$  discussed as follows:<br>
therefore,  $\sin \alpha = \frac{\pm b}{\sqrt{a^2 + b^2}}$ .<br>  $kx$ <br>  $\frac{b}{\sqrt{a^2 + b^2}} \cos kx$  $-4 \le 6$ .<br>
is calculated based on the<br>
os  $kx \le \sqrt{a^2 + b^2}$ . The<br>
sin  $\alpha = \frac{\pm b}{\sqrt{a^2 + b^2}}$ .<br>
cos  $kx$ <br>  $kx \pm \alpha$ .

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SUMMER 2022<br>
Vol 14 no 2<br>  $-10 \le 3 \sin x + 4 \cos x + 4 \sin y - 3 \cos y \le 10$ .<br>
Now, we have<br>  $-14 \le 3 \sin x + 4 \cos x + 4 \sin y - 3 \cos y - 4 \le 6$ .<br>
It means  $-14 \le f(x, y) \le 6$ .<br>
It means  $-14 \le f(x, y) \le 6$ .<br> MATHEMATICS TEACHING RESEARCH JOURNAL<br>
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Vol 14 no 2<br>  $-10 \le 3 \sin x + 4 \cos x + 4 \sin y - 3 \cos y \le 10$ .<br>
Now, we have<br>  $-14 \le 3 \sin x + 4 \cos x + 4 \sin y - 3 \cos y - 4 \le 6$ .<br>
It means  $-14 \le f(x, y) \le 6$ .<br>
It means  $-14 \le f(x, y) \le 6$ .<br>
One in t SIUMMER 2022<br>
TRACHING SUITS TO UP 14 no 2<br>  $-10 \le 3 \sin x + 4 \cos x + 4 \sin y - 3 \cos y \le 10$ .<br>
Now, we have<br>  $-14 \le 3 \sin x + 4 \cos x + 4 \sin y - 3 \cos y - 4 \le 6$ .<br>
It means  $-14 \le f(x, y) \le 6$ .<br>
It means  $-14 \le f(x, y) \le 6$ .<br>
It means  $-14 \le f(x, y) \le 6$ .<br>
O  $a \t b$  $\sqrt{a^2+b^2}$  and  $\sqrt{a^2+b^2}$ .

$$
-10 \le 3 \sin x + 4 \cos x + 4 \sin y - 3 \cos y \le 10.
$$
  
\nNow, we have  
\n
$$
-14 \le 3 \sin x + 4 \cos x + 4 \sin y - 3 \cos y - 4 \le 6.
$$
  
\nIt means  $-14 \le f(x, y) \le 6$ .  
\n*Generalization of this Trigonometric Problem*  
\nProve that the range of the function  $g(x) = a \sin kx + b \cos kx + c$  is calculated based on the  
\nProve that  $\theta_n = [c - \sqrt{a^2 + b^2}, c + \sqrt{a^2 + b^2}]$ .  
\nProof:  
\nFirstly, we show that,  $\forall a, b, k, x \in R, -\sqrt{a^2 + b^2} \le a \sin kx + b \cos kx \le \sqrt{a^2 + b^2}$ . The  
\nmaximum and minimum values of the function  $y = a \sin kx \pm b \cos kx$  discussed as follows:  
\nSince,  $-1 \le \frac{a}{\sqrt{a^2 + b^2}} \le 1$ , we assume that  $\cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}$  therefore,  $\sin \alpha = \frac{\pm b}{\sqrt{a^2 + b^2}}$ .  
\n $y = a \sin kx \pm b \cos kx$   
\n $\Rightarrow \frac{y}{\sqrt{a^2 + b^2}} = \frac{a}{\sqrt{a^2 + b^2}} \sin kx \pm \frac{b}{\sqrt{a^2 + b^2}} \cos kx$   
\n $\Rightarrow \frac{y}{\sqrt{a^2 + b^2}} = \cos a \sin kx \pm \sin a \cos kx = \sin(kx \pm a)$ .  
\nWe know,  $-1 \le \sin(kx \pm a) \le 1$ , thus  
\n $-1 \le \frac{y}{\sqrt{a^2 + b^2}} \le 1 \Rightarrow -\sqrt{a^2 + b^2} \le y \le \sqrt{a^2 + b^2}$ .  
\nTherefore,  
\n $c - \sqrt{a^2 + b^2} \le a \sin kx \pm b \cos kx + c \le c + \sqrt{a^2 + b^2}$ .  
\nH means  
\n $c - \sqrt{a^2 + b^2} \le g(x) \le c + \sqrt{a^2 + b^2}$   
\n $\Rightarrow R_g = [c - \sqrt{$ 

$$
\frac{y}{\sqrt{a^2 + b^2}} = \cos \alpha \sin kx \pm \sin \alpha \cos kx = \sin(kx \pm \alpha).
$$
  
We know,  $-1 \le \sin(kx \pm \alpha) \le 1$ , thus  

$$
-1 \le \frac{y}{\sqrt{a^2 + b^2}} \le 1 \Rightarrow -\sqrt{a^2 + b^2} \le y \le \sqrt{a^2 + b^2}
$$

$$
\Rightarrow -\sqrt{a^2 + b^2} \le a \sin kx \pm b \cos kx \le \sqrt{a^2 + b^2}.
$$
Therefore,  

$$
c - \sqrt{a^2 + b^2} \le a \sin kx \pm b \cos kx + c \le c + \sqrt{a^2 + b^2}
$$
It means  

$$
c - \sqrt{a^2 + b^2} \le g(x) \le c + \sqrt{a^2 + b^2}
$$

$$
\Rightarrow R_g = [c - \sqrt{a^2 + b^2}, c + \sqrt{a^2 + b^2}].
$$
Example 3:  
Find the range of the function  $h(x) = [2 \sin 4x - 3 \cos 4x]$  where [] is the symbol of partial integer.  
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$$
c - \sqrt{a^2 + b^2} \le a \sin kx \pm b \cos kx + c \le c + \sqrt{a^2 + b^2}
$$

It means

$$
c - \sqrt{a^2 + b^2} \le g(x) \le c + \sqrt{a^2 + b^2}
$$
  
\n
$$
\Rightarrow R_g = [c - \sqrt{a^2 + b^2}, \ c + \sqrt{a^2 + b^2}].
$$

integer.





# MATHEMATICS TEACHING RESEARCH JOURNAL 35 SUMMER 2022 Vol 14 no 2 THEMATICS TEACHING RESEARCH JOURNAL<br>
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the real numbers. As respect to the above theorem we have<br>  $-\sqrt{13} \le 2 \sin 4x - 3 \cos 4x \le \sqrt{13}$ <br>  $R_h = \{-4, -3, -2, -1, 0, 1, 2, 3\}.$

Solution:

The function  $h$  is continues on the real numbers. As respect to the above theorem we have

$$
-\sqrt{13} \le 2\sin 4x - 3\cos 4x \le \sqrt{13}
$$

Therefore, we obtain

$$
R_h = \{-4, -3, -2, -1, 0, 1, 2, 3\}.
$$

MATHEMATICS TEACHING RESEARCH JOURNAL<br>
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Solution:<br>
The function *h* is continues on the real numbers. As respect to the above theorem we have<br>  $-\sqrt{13} \le 2 \sin 4x - 3 \cos 4x \le \sqrt{13}$ <br>
Therefore, we obtain<br>  $R_h = \{-4,$ improve the ability of students in problem solving. MATHEMATICS TEACHING RESEARCH JOURNAL<br>
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Solution:<br>
The function *h* is continues on the real numbers. As respect to the above theorem we have<br>  $-\sqrt{13} \le 2 \sin 4x - 3 \cos 4x \le \sqrt{13}$ <br>
Therefore, we obtain<br>  $R_h = \{-4, -$ ଶା√ଷ ୱ୧୬ ௫ିସ ୡ୭ୱ ௫ ଵା√ସ ୱ୧୬ ௫ାଷ ୡ୭ୱ ௫  $R_h = \{-4, -3, -2, -1, 0, 1, 2, 3\}.$ <br>
blems are suitable to discuss in the classroom, because such problems<br>
udents in problem solving.<br>
sin  $x + 5 \cos x$ , write the linear combination of sine and cosine as<br>
function.<br>
minimum val

Problem 1:

consisting of only cosine function.

Problem 2:

Find the maximum and minimum values of the function  $f(x) = \frac{2 + \sqrt{3} \sin x - 4 \cos x}{1 + \sqrt{4 \sin x + 3 \cos x}}$ .<br>Problem 3:

Prove that

$$
(\sin x + a \cos x)(\sin x + b \cos x) \le 1 + \left(\frac{a+b}{2}\right)^2.
$$
Problem 4:

Prove that

$$
(\sin 3x + a \cos 3x)(\sin 3x + b \cos 3x) \le \frac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2b^2 + 1}).
$$

(sin 3x + 5 cos 3x) (sin 3x + b cos 3x)  $\leq \frac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2b^2 + 1})$ <br>
(sin 3x + a cos 3x)(sine function.<br>
2.<br>
4.<br>
(sin x + a cos x)(sin x + b cos x)  $\leq 1 + (\frac{a+b}{2})^2$ .<br>
4.<br>
4.<br>
(sin 3x + a cos 3x)(sin 3x + b cos improve the ability of students in problem solving.<br>
Problem 1:<br>
For the function  $y = 12 \sin x + 5 \cos x$ , write the linear combination of sine and cosine as<br>
consisting of only cosine function.<br>
Problem 2:<br>
Find the maximum and Based on a research by Gholami et al. (2021), some common misconceptions that students encounter regarding the problem "find the maximum and minimum values of the trigonometric

Problem 2:<br>
Find the maximum and minimum values of the function  $f(x) = \frac{2+\sqrt{3\sin x - 4\cos x}}{1+\sqrt{4\sin x + 3\cos x}}$ <br>
Problem 3:<br>
Prove that<br>  $(\sin x + a \cos x)(\sin x + b \cos x) \le 1 + (\frac{a+b}{2})^2$ .<br>
Problem 4:<br> **Prove that**<br>  $(\sin 3x + a \cos 3x)(\sin 3x + b \cos 3x)$ Find the maximum and minimum values of the function  $f(x) = \frac{2+\sqrt{3 \sin x - 4 \cos x}}{1+\sqrt{4 \sin x + 3 \cos x}}$ <br>
Problem 3:<br>
Problem 4:<br>
(sin  $x + a \cos x$ )(sin  $x + b \cos x$ )  $\leq 1 + (\frac{a+b}{2})^2$ .<br>
Problem 4:<br>
(sin  $3x + a \cos 3x$ )(sin  $3x + b \cos 3x$ )  $\leq \frac$ the maximum and minimum values of the function  $f(x) = \frac{1}{1 + \sqrt{4 \sin x + 3 \cos x}}$ <br>
ce that<br>
(sin  $x + a \cos x (\sin x + b \cos x) \le 1 + (\frac{a+b}{2})^2$ .<br>
blem 4:<br>
ce that<br>
(sin  $3x + a \cos 3x (\sin 3x + b \cos 3x) \le \frac{1}{2} (1 + ab + \sqrt{a^2 + b^2 + a^2b^2 + 1})$ .<br> **CUSSION**<br> For a state of the two states are allowed by a Gensive Common steeds. All photometric of the two functions  $\left( \sin 3x + a \cos 3x \right) \leq \frac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2b^2 + 1})$ .<br>
CUSSION<br>
CUSSION<br>
CUSSION<br>
The two first argument of a is entired to the value of sin  $x + a \cos x$  (sin  $x + b \cos x$ )  $\leq 1 + (\frac{a+b}{2})^2$ .<br>
belom 4:<br>
that<br>
(sin  $3x + a \cos 3x$ )  $(\sin 3x + b \cos 3x) \leq \frac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2b^2 + 1})$ .<br>
CUSSION<br>
ed on a research by Gholami et al. (2021), some c formula  $sin^2x + cos^2x = 1$ . For instance, for the angles in the first quartile of the unit circle, (sin  $x + a \cos x$ ) (sin  $x + b \cos x$ )  $\le 1 + (\frac{a+b}{2})^2$ .<br>  $+ a \cos 3x$ ) (sin  $3x + b \cos 3x$ )  $\le \frac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2b^2 + 1})$ .<br>
arch by Gholami et al. (2021), some common misconceptions that stt or the problem "find the maximum and in  $x + a \cos x$ ) (sin  $x + b \cos x$ ) ≤ 1 +  $\left(\frac{a+b}{2}\right)^2$ .<br>  $x$ ) (sin  $3x + b \cos 3x$ ) ≤  $\frac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2b^2 + 1})$ .<br>
Gholami et al. (2021), some common misconceptions that students<br>
roblem "find the maximum and minimum valu (sin  $x + a \cos x$ )(sin  $x + b \cos x$ )  $\le 1 + (\frac{n+b}{2})^2$ .<br>
ve that<br>
(sin  $3x + a \cos 3x$ )(sin  $3x + b \cos 3x$ )  $\le \frac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2b^2 + 1})$ .<br>
CUSSION<br>
con a research by Gholami et al. (2021), some common misconceptions that students





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**CALCHING RESEARCH JOURNAL**<br> **CALCHING**<br>  $x +$ <br>  $x +$ <br>  $\ln \theta$ <br>
and  $cos^{2}x = 1$  it is clear that  $cos x = 0$  when  $sin x = 1$ . The above argument is correct to find the MATHEMATICS TEACHING RESEARCH JOURNAL<br>
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ot consider sin  $x = 1$  and  $\cos x = 1$  simultanously. By using the formula  $\sin^2 x +$ <br>  $x = 1$  it is clear that  $\cos x = 0$  when  $\sin x = 1$ . The above argument is corr MATHEMATICS TEACHING RESEARCH JOURNAL<br>
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cannot consider sin  $x = 1$  and  $\cos x = 1$  simultanously. By using the formula  $\sin^2 x + \cos^2 x = 1$  it is clear that  $\cos x$ **MATHEMATICS TEACHING RESEARCH JO<br>
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cannot consider sin**  $x = 1$  **and**  $\cos x = 1$  **simultanously. By using the<br>**  $\cos^2 x = 1$  **it is clear that**  $\cos x = 0$  **when**  $\sin x = 1$ **. The above Both sides of the function**<br> **BUMMER 2022**<br> **BUMMER 2022**<br> **BUMMER 2022**<br> **CONTAGE SEARCH SUMMER 2022**<br> **CONTAGE SEARCH SIDES**<br> **CONTAGE SEARCH SIDES**<br> **CONTAGE SEARCH SIDES**<br> **CONTAGE SEARCH SIDES OF THE SIDE SIDES OF T** MATHEMATICS TEACHING RESEARCH JOURNAL 36<br>
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sider sin  $x = 1$  and  $\cos x = 1$  simultanously. By using the formula  $\sin^2 x +$ <br>
it is clear that  $\cos x = 0$  when  $\sin x = 1$ . The above argument is correct to fin FEACHING RESEARCH JOURNAL 36<br>
simultanously. By using the formula  $sin^2x +$ <br>  $sin x = 1$ . The above argument is correct to find the<br>  $sin x + cos y$  because the two values of  $sin x$  and<br>  $y = sin x + cos x$  we have<br>  $x + cos^2x + 2 sin x cos x = 1 + sin 2x$ .<br>  $\le h$ MATHEMATICS TEACHING RESEARCH JOURNAL 36<br>
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sin x = 1 and cos x = 1 simultanously. By using the formula  $sin^2x$  +<br>
car that cos x = 0 when  $sin x$  = 1. The above argument is correct to find the<br>
inimum MATHEMATICS TEACHING RESEARCH JOURNAL<br>
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COS<sup>2</sup>X = 1 it is clear that cos x = 1 simultanously. By using the formula  $sin^2 x$  +<br>
maximum and minimum values of  $A = sin x + cos y$  because the two values of sin x an URNAL 36<br>
: formula  $sin^2x +$ <br>
s correct to find the<br>
values of sin x and<br>
+ sin 2x.<br>  $\overline{2}$ .<br>
(x)  $\leq 2 \Rightarrow 0 \leq$ <br>  $\leq$  2 and hence<br>
sconception MATHEMATICS TEACHING RESEARCH JOURNAL<br>
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cannot consider sin x = 1 simulanously. By using the formula  $sin^2 x +$ <br>
cannot consider sin x = 1 and cos x = 1 simulanously. By using the formul TEACHING RESEARCH JOURNAL 36<br>
1 simultanously. By using the formula  $sin^2x + sin x = 1$ . The above argument is correct to find the<br>  $sin x + cos y$  because the two values of  $sin x$  and<br>  $x$ ) =  $sin x + cos x$  we have<br>  $x^2x + cos^2x + 2 sin x cos x = 1 + sin 2x$ .<br> MATHEMATICS TEACHING RESEARCH JOURNAL 36<br>
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cannot cosider sin x = 1 and cos x = 1 simultanously. By using the formula  $sin^2 x +$ <br>  $cos^2 x = 1$  it is clear that  $cos x = 0$  when  $sin x = 1$ . The abov MATHEMATICS TEACHING RESEARCH JOURNAL<br>
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cannot consider sin  $x = 1$  and cos  $x = 1$  simultanously. By using the formula stin<sup>2</sup>x +<br>
maximum and minimum values of  $A = \sin x + \cos y$  because the

 $h^{2}(x) = (\sin x + \cos x)^{2} = \sin^{2} x + \cos^{2} x + 2 \sin x \cos x = 1 + \sin 2x$ .

A logical argument for this problem was shown as follows

cannot consider 
$$
\sin x = 1
$$
 and  $\cos x = 1$  simultaneously. By using the formula  $\sin^2 x + \cos^2 x = 1$  it is clear that  $\cos x = 0$  when  $\sin x = 1$ . The above argument is correct to find the maximum and minimum values of  $A = \sin x + \cos y$  because the two values of  $\sin x$  and  $\cos y$  are independent.  
By squaring both sides of the function  $h(x) = \sin x + \cos x$  we have  $h^2(x) = (\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2 \sin x \cos x = 1 + \sin 2x$ .  
 $0 \le 1 + \sin 2x \le 2 \Rightarrow 0 \le h^2(x) \le 2 \Rightarrow 0 \le h(x) \le \sqrt{2}$ .  
In the above argument, the misconception is related to the concept  $0 \le h^2(x) \le 2 \Rightarrow 0 \le h(x) \le \sqrt{2}$  because the inequality  $0 \le h^2(x) \le 2$  is equivalent to  $h^2(x) \le 2$  and hence  $-\sqrt{2} \le h(x) \le \sqrt{2}$ .  
 $h(x) = \sin x + \cos x \Rightarrow h(x) - \cos x = \sin x \Rightarrow -1 \le h(x) - \cos x \le 1$   
 $\Rightarrow -1 + \cos x \le h(x) \le 1 + \cos x \Rightarrow -2 \le h(x) \le 2$ .  
In this argument, obtaining the inequality  $-1 \le h(x) - \cos x \le 1$  is a misconception because the values  $\sin x$  and  $\cos x$  are dependent.  
gical argument for this problem was shown as follows  

$$
h(x) = \sin x + \cos x = \sqrt{2} \left( \frac{\sqrt{2}}{2} \sin x + \frac{\sqrt{2}}{2} \cos x \right) = \sqrt{2} (\cos 45 \sin x + \sin 45 \cos x)
$$
  
 $\Rightarrow h(x) = \sqrt{2} \sin(x + 45)$ .  
 $\sec, -1 \le \sin(x + 45) \le 1$  we have  $-\sqrt{2} \le \sin(x + 45) \le \sqrt{2}$ , thus  $-\sqrt{2} \le h(x) \le \sqrt{2}$ .

maximum and minimum values of  $A = \sin x + \cos y$  because the two values of  $\sin x$  and<br>
b) By squaring both sides of the function  $h(x) = \sin x + \cos x$  we have<br>  $h^2(x) = (\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2 \sin x \cos x = 1 + \sin 2x$ .<br>  $0 \le 1 + \sin 2x \le 2 \Rightarrow 0 \le h^2(x$ If  $\ln(x) \le x^2 \le \ln(x) \le x^2$ ,  $\ln(x) \le x^2 \le 2$  is equivalent to  $h^2(x) \le 2$  and hence  $-\sqrt{2} \le h(x) \le \sqrt{2}$ .<br>  $\ln(x) = \sin x + \cos x \Rightarrow h(x) - \cos x = \sin x \Rightarrow -1 \le h(x) - \cos x \le 1$ <br>  $\Rightarrow -1 + \cos x \le h(x) \le 1 + \cos x \Rightarrow -2 \le h(x) \le 2$ .<br>
In this argument, obtaining the vi(x) = y 2 occasses the mediantly 0 = n (x) = 2 as equivalent to n (x) = 2 and netice<br>
c)  $h(x) = \sin x + \cos x \le h(x) = \cos x = \sin x \Rightarrow -1 \le h(x) - \cos x \le 1$ <br>  $\Rightarrow -1 + \cos x \le h(x) \le 1 + \cos x \Rightarrow -2 \le h(x) \le 2$ .<br>
In this argument, obtaining the inequality - $-\sqrt{2} \le \ln(x) \le \sqrt{x}$ .<br>
c)  $\frac{h(x)}{\sqrt{x}} = \sin x + \cos x \Rightarrow h(x) - \cos x = \sin x \Rightarrow -1 \le h(x) - \cos x \le 1$ <br>
⇒  $-1 + \cos x \le h(x) \le 1 + \cos x \Rightarrow -2 \le h(x) \le 2$ .<br>
In this argument, obtaining the inequality  $-1 \le h(x) - \cos x \le 1$  is a misconception<br>
because the values sin c)  $n(x) = 8m x + \cos x \Rightarrow n(x) - \cos x \le \pi$  sin  $x \Rightarrow -1 \le n(x) - \cos x \le 1$ <br>  $\Rightarrow -1 + \cos x \le h(x) \le 1 + \cos x \Rightarrow -2 \le h(x) \le 2$ .<br>
In this argument, obtaining the inequality  $-1 \le h(x) - \cos x \le 1$  is a misconception<br>
because the values sin x and  $\cos x$  are depen these variables can be negative. Therefore, based on this incorrect argument, the maximum and In this argument, obtaining the inequality  $-1 \le \frac{\pi}{2}$  is a misconception<br>
because the values sin x and cos x are dependent.<br>
A logical argument for this problem was shown as follows<br>  $h(x) = \sin x + \cos x = \sqrt{2} \left( \frac{\sqrt{2}}{2} \sin x$ respectively. Students understood that the above argument is correct for the two variables function A logical argument for this problem was shown as follows<br>  $h(x) = \sin x + \cos x = \sqrt{2} \left( \frac{\sqrt{2}}{2} \sin x + \frac{\sqrt{2}}{2} \cos x \right) = \sqrt{2} (\cos 45 \sin x + \sin 45 \cos x)$ <br>  $\Rightarrow h(x) = \sqrt{2} \sin(x + 45) \le 1$  we have  $-\sqrt{2} \le \sin(x + 45) \le \sqrt{2}$ , thus  $-\sqrt{2} \le h(x) \le \sqrt{2}$ . simultaneously. The simultaneously. Therefore, the maximum since  $\sqrt{2}$ <br>
Since,  $-1 \le \sin(x + 45) \le 1$  we have  $-\sqrt{2} \le \sin(x + 45) \le \sqrt{2}$ , thus  $-\sqrt{2} \le h(x) \le \sqrt{2}$ .<br>
In this study, three students considered  $\sin x = 1$  and  $\cos x =$  $h(x) = \sin x + \cos x = \sqrt{2} \left( \frac{\sqrt{2}}{2} \sin x + \frac{\sqrt{2}}{2} \cos x \right) = \sqrt{2} (\cos 45 \sin x + \sin 45 \cos x)$ <br>
⇒ h(x) =  $\sqrt{2} \sin(x + 45)$ .<br>
Since, -1 ≤ sin(x + 45) ≤ 1 we have  $-\sqrt{2}$  ≤ sin(x + 45) ≤  $\sqrt{2}$ , thus  $-\sqrt{2}$  ≤ h(x) ≤  $\sqrt{2}$ .<br>
In this study, considered sin  $x = 1$  and  $\cos x = 1$  to find  $a(1) + b(1) = a + b$  as<br>
motion  $f(x) = a \sin x + b \cos x$ . Also, they argued that the minimum<br>  $-1) + b(-1) = -(a + b)$ . In this argument, teachers can refer to the<br>
ans that students face. Firstly, st

$$
-|b| \le f(x) - a\sin x \le |b|
$$





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$$
\Rightarrow -|b| - |a| \le f(x) \le |b| + |a|.
$$

MATHEMATICS TEACHING RESEARCH JOURNAL<br>
THEATING SUMMER 2022<br>
Vol 14 no 2<br>  $\Rightarrow -|b| - |a| \le f(x) \le |b| + |a|$ .<br>
In the above method of solution, students considered sin x and cos x, as two independent variables<br>
wrongly, whereas wrongly, whereas these two variables are dependent. Teachers explained this misconception to **STRAGE STEACHING RESEARCH JOURNAL**<br> **SUMMER 2022**<br> **SUM** MATHEMATICS TEACHING RESEARCH JOURNAL<br>
SUMMER 2022<br>
Yol 14 no 2<br>  $\Rightarrow -|b| - |a| \le f(x) \le |b| + |a|$ .<br>
In the above method of solution, students considered sin x and cos x, as two independent variables<br>
wrongly, whereas these two MATHEMATICS TEACHING RESEARCH JOURNAL<br>
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Wol 14 no 2<br>  $\Rightarrow -|b| - |a| \le f(x) \le |b| + |a|$ .<br>
In the above method of solution, students considered sin x and cos x, as two independent wrongly, whereas these two v MATHEMATICS TEACHING RESEARCH JOURNAL<br>
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SUMMER 2022<br>  $\Rightarrow -|b| - |a| \le f(x) \le |b| + |a|$ .<br>
In the above method of solution, students considered sin x and cos x, as two independent varial<br>
surveyly, whereas these two va MATHEMATICS TEACHING RESEARCH JOURNAL ST<br>
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Vol 14 no 2<br>  $\Rightarrow -|b| - |a| \le f(x) \le |b| + |a|$ .<br>
In the above method of solution, students considered sin x and cos x, as two independent variables<br>
wrongly, whereas these t MATHEMATICS TEACHING RESEARCH JOURNAL 37<br>
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FREEFING CONDITER 2022<br>  $\Rightarrow -|b| - |a| \le f(x) \le |b| + |a|$ .<br>
In the above method of solution, students considered sin x and cos x, as two independent variables<br>
surgerly, whe HERENE INTERTING TEACHING RESEARCH JOURNAL. 37<br>
TERGENING TERM MITER ADI2<br>
Vol 14 no 2<br>  $\Rightarrow -|b| - |a| \le f(x) \le |b| + |a|$ .<br>
In the above method of solution, students considered sin x and cos x, as two independent variables<br>
sw  $\Rightarrow -|b| - |a| \le f(x) \le |b| + |a|$ .<br>
In the above method of solution, students considered sin x and cos x, as two indep<br>
wrongly, whereas these two variables are dependent. Teachers explained this n<br>
students using a specific fu  $-9 - |u| = |u| \ge f(x) \le |v| + |u|$ .<br>
In the above method of solution, students considered sin x and cos x, as two indep<br>
wrongly, whereas these two variables are dependent. Teachers explained this n<br>
students using a specific fun  $\Rightarrow$  - |b| - |a| ≤ f(x) ≤ |b| + |a|.<br>
thdo of solution, students considered sin x and cos x, as two independent variables<br>
as these two variables are dependent. Teachers explained this misconception to<br>
a specific functi In the above method of solution, students considered sin x and cos x, as two independent variables<br>wromgly, whereas these two variables are dependent. Teachers explained this misconception to<br>students using a specific fun

The rest of the students in the class got the right answer by considering specific situations for this problem. For example, one of the students considered equal value for both coefficients  $a$  and  $b$ and he found the values  $|a|\sqrt{2}$  and  $-|a|\sqrt{2}$  as the maximum and minimum values of the function

concept of linear combination of two variables.

For the problem 2, "Find the maximum and minimum values of the function  $f(x) =$ 

obtan 7 as the maximum value of this function because when we put sin 
$$
x =
$$
  
hould be only zero.  
is in the class got the right answer by considering specific situations for this  
; one of the students considered equal value for both coefficients *a* and *b*  
is  $|a|\sqrt{2}$  and  $-|a|\sqrt{2}$  as the maximum and minimum values of the function  
 $s x$  respectively.  
or the function  $y = 12 \sin x + 5 \cos x$ , write the linear combination of sine  
ing of only cosine function" some students by using the formula  $\sin x =$   
changed it as  $y = \pm 12\sqrt{1 - \cos^2 x} + 5 \cos x$ . They didn't understand the  
bination of two variables.  
and the maximum and minimum values of the function  $f(x) =$   
e students found the maximum and minimum values of numerator 2 +  
follows  
 $-\sqrt{3^2 + (-4)^2} \le 3 \sin x - 4 \cos x \le \sqrt{3^2 + (-4)^2}$   
 $\Rightarrow -5 \le 3 \sin x - 4 \cos x \le 5$   
 $\Rightarrow 0 \le \sqrt{3 \sin x - 4 \cos x} \le \sqrt{5}$   
 $\Rightarrow 2 \le 2 + \sqrt{3 \sin x - 4 \cos x} \le \sqrt{5}$   
 $\Rightarrow 2 \le 2 + \sqrt{3 \sin x - 4 \cos x} \le \sqrt{5}$   
 $\Rightarrow 2 \le 2 + \sqrt{3 \sin x - 4 \cos x} \le \sqrt{5}$   
 $\Rightarrow 1 + \sqrt{4 \sin x + 3 \cos x}$  respectively. They argued in order to find the  
equation, they found the numbers 1 and 1 +  $\sqrt{5}$  as the maximum and minimum  
or 1 +  $\sqrt{4 \sin x + 3 \cos x}$  respectively. They argued in order to find the  
function  $f(x)$  we require to divide the maximum value of numerator to the

problem. For example, one of the students considered equal value for both coefficients a and b<br>and he found the values  $|a| \sqrt{2}$  and  $-|a| \sqrt{2}$  as the maximum and minimum values of the function<br>In the Problem 1, "For th in the Problem 1, "For the function  $y = 12 \sin x + 5 \cos x$ , write the linear combination of sine<br>and cosine as consisting of only cosine function" some students by using the formula sin  $x = \pm \sqrt{1 - \cos^2 x} + 5 \cos x$ . They didn't unde maximum value of the function  $f(x)$  we require to divide the maximum value of numerator to the minimum value of denominator. Therefore, the maximum value of the function  $f(x)$  is  $\frac{2+\sqrt{5}}{1}$  $\frac{1}{1}$ concept of linear combination of two variables.<br>
For the problem 2, "Find the maximum and minimum values of the function  $f(x) = \frac{2+\sqrt{3}\sin x - 4\cos x}{1+\sqrt{4}\sin x + 3\cos x}$ ," some students found the maximum and minimum values of nu  $rac{2}{1+\sqrt{5}}$ . This argument is not correct For the problem 2, "Find the maximum and minimum values of the function  $f(x) = \frac{2 + \sqrt{3} \sin x - 4 \cos x}{1 + \sqrt{4} \sin x + \cos x}$ ," some students found the maximum and minimum values of numerator 2 +  $\sqrt{3} \sin x - 4 \cos x$  as follows<br>  $-\sqrt{3^2$  $\frac{2+\sqrt{3}\sin x - 4\cos x}{1+\sqrt{4}\sin x + 3\cos x}$ , some students found the maximum and minimum values of numerator 2 +<br> $-\sqrt{3^2 + (-4)^2} \le 3\sin x - 4\cos x \le \sqrt{3^2 + (-4)^2}$ <br> $\Rightarrow -5 \le 3\sin x - 4\cos x \le 5$ <br> $\Rightarrow 0 \le \sqrt{3}\sin x - 4\cos x \le 5$ <br> $\Rightarrow 0 \le \sqrt{3}\sin x - 4\cos x \le 2$ 1+ $\sqrt{4 \sin x + 3 \cos^2 x}$  some students found the maximum and minimum values of numeral  $\sqrt{3 \sin x} - 4 \cos x$  as follows<br>  $-\sqrt{3^2 + (-4)^2} \le 3 \sin x - 4 \cos x \le \sqrt{3^2 + (-4)^2}$ <br>  $\Rightarrow -5 \le 3 \sin x - 4 \cos x \le 5$ <br>  $\Rightarrow 0 \le \sqrt{3 \sin x - 4 \cos x} \le 2 + \sqrt{5}$ .<br>
Thro  $\sqrt{3} \sin x - 4 \cos x$  as follows<br>  $-\sqrt{3^2 + (-4)^2} \le 3 \sin x - 4 \cos x \le \sqrt{3^2 + (-4)^2}$ <br>  $\Rightarrow -5 \le 3 \sin x - 4 \cos x \le 5$ <br>  $\Rightarrow 0 \le \sqrt{3 \sin x - 4 \cos x} \le \sqrt{5}$ <br>  $\Rightarrow 2 \le 2 + \sqrt{3 \sin x - 4 \cos x} \le 2 + \sqrt{5}$ .<br>
Through similar calculation, they found the numbers 1  $\frac{2+\sqrt{3}\sin x-4\cos x}{1+\sqrt{4\sin y+3\cos}}$ . because of independency of nominator and denominator of the statement C.<br>There is a common misconception among students regarding this problem that is so important for teachers to know. We have the following inequalities





# **EMATICS TEACHING RESEARCH JOURNAL 38**  SUMMER 2022 Vol 14 no 2 **EXECUTE ANTICS TEACHING RESEARCH JOURNAL**<br>
2 ≤ 2 + √3 sin x − 4 cos x ≤ 2 + √5<br>
2 ≤ 1 + √4 sin x + 3 cos x ≤ 1 + √5.<br>
2 ≤  $\frac{2+\sqrt{3} \sin x - 4 \cos x}{2}$  ≤  $\frac{2+\sqrt{5}}{\sqrt{5}}$ <br>
2 ≤  $\frac{2+\sqrt{3} \sin x - 4 \cos x}{2}$  ≤  $\frac{2+\sqrt{5}}{\sqrt{5}}$ . MATHEMATICS TEACHING RESEARCH JOURNAL<br>
TEACHING<br>
SUMMER 2022<br>
Vol 14 no 2<br>  $2 \le 2 + \sqrt{3} \sin x - 4 \cos x \le 2 + \sqrt{5}$ <br>  $1 \le 1 + \sqrt{4} \sin x + 3 \cos x \le 1 + \sqrt{5}$ .<br>
By dividing all sides of the above inequalities we obtain<br>  $2 \le \frac{2 + \sqrt{3} \sin x -$ EMATICS TEACHING RESEARCH JOURNAL<br>
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2 +  $\sqrt{3} \sin x - 4 \cos x \le 2 + \sqrt{5}$ <br>
1 +  $\sqrt{4} \sin x + 3 \cos x \le 1 + \sqrt{5}$ .<br>
inequalities we obtain<br>  $2 \le \frac{2 + \sqrt{3} \sin x - 4 \cos x}{1 + \sqrt{4} \sin y + 3 \cos y} \le \frac{2 + \sqrt{5}}{1 + \sqrt{5}}$ <br>
eptable because  $2 > \frac{2 +$ ATICS TEACHING RESEARCH JOURNAL<br>  $\sqrt{3} \sin x - 4 \cos x \le 2 + \sqrt{5}$ <br>  $\sqrt{4} \sin x + 3 \cos x \le 1 + \sqrt{5}$ .<br>
analities we obtain<br>  $\frac{2 + \sqrt{3} \sin x - 4 \cos x}{1 + \sqrt{4} \sin y + 3 \cos y} \le \frac{2 + \sqrt{5}}{1 + \sqrt{5}}$ <br>
ble because  $2 > \frac{2 + \sqrt{5}}{1 + \sqrt{5}}$ ;<br>
must be as f

$$
2 \le 2 + \sqrt{3 \sin x - 4 \cos x} \le 2 + \sqrt{5}
$$
  

$$
1 \le 1 + \sqrt{4 \sin x + 3 \cos x} \le 1 + \sqrt{5}
$$

$$
2 \le \frac{2 + \sqrt{3\sin x - 4\cos x}}{1 + \sqrt{4\sin y + 3\cos y}} \le \frac{2 + \sqrt{5}}{1 + \sqrt{5}}.
$$

 $1+\sqrt{5}$ It is clear that this result is not acceptable because  $2 > \frac{2+\sqrt{5}}{1+\sqrt{5}}$ .<br>The logical argument for this problem must be as follows

MATHEMATICS TEACHING RESEARCH JOURNAL  
\n**38**  
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\n2 ≤ 2 + 
$$
\sqrt{3} \sin x - 4 \cos x \le 2 + \sqrt{5}
$$
  
\n1 ≤ 1 +  $\sqrt{4} \sin x + 3 \cos x \le 1 + \sqrt{5}$ .  
\nBy dividing all sides of the above inequalities we obtain  
\n
$$
2 \le \frac{2 + \sqrt{3} \sin x - 4 \cos x}{1 + \sqrt{4} \sin x + 3 \cos x} \le \frac{2 + \sqrt{5}}{1 + \sqrt{5}}
$$
\nIt is clear that this result is not acceptable because  $2 > \frac{2 + \sqrt{5}}{1 + \sqrt{5}}$ .  
\nThe logical argument for this problem must be as follows  
\n
$$
2 \le 2 + \sqrt{3} \sin x - 4 \cos x \le 2 + \sqrt{5}
$$
\n
$$
1 \le 1 + \sqrt{4} \sin x + 3 \cos x \le 1 + \sqrt{5} \Rightarrow \frac{1}{1 + \sqrt{5}} \le \frac{1}{1 + \sqrt{4} \sin x + 3 \cos x} \le 1
$$
\nBy multiplying all sides of these inequalities we can see  
\n
$$
\frac{2}{1 + \sqrt{5}} \le \frac{2 + \sqrt{3} \sin x - 4 \cos x}{1 + \sqrt{3} \sin x + 3 \cos x} \le 2 + \sqrt{5}.
$$
\nTherefore, the maximum and minimum values of  $C = \frac{2 + \sqrt{3} \sin x - 4 \cos x}{1 + \sqrt{4} \sin y + 3 \cos y}$  are  $2 + \sqrt{5}$  and  $\frac{2}{1 + \sqrt{5}}$   
\nrespectively.  
\nAnother misconception is about the problem 3, "Prove that (sin *x* + *a* cos *x*)(sin *x* + *b* cos *x*) ≤  
\n $1 + \left(\frac{a+b}{2}\right)^2$ ."

$$
\frac{2}{1+\sqrt{5}} \le \frac{2+\sqrt{3\sin x - 4\cos x}}{1+\sqrt{4\sin y + 3\cos x}} \le 2+\sqrt{5}.
$$

 $1+\sqrt{5}$ respectively.

 $\frac{1}{x+3\cos} \le 1.$ <br>are  $2 + \sqrt{5}$  and  $\frac{2}{1+\sqrt{5}}$ <br>cos  $x)(\sin x + b \cos x) \le$ 2  $\sqrt{1 + \frac{1}{2}}$  $)^2$ .".

For this problem, the members of Lesson Study group members were faced with the following argument

$$
(\sin x + a \cos x)(\sin x + b \cos x) \le \sqrt{1 + a^2}\sqrt{1 + b^2}.
$$

The logical argument of this problem indicate  $\cos x \le 1 + \sqrt{5}$ <br>  $1 \le 1 + \sqrt{4 \sin x + 3 \cos x} \le 1 + \sqrt{5} \Rightarrow \frac{1}{1 + \sqrt{5}} \le \frac{1}{1 + \sqrt{4 \sin x + 3 \cos x}} \le 1$ .<br>
By multiplying all sides of these inequalities we can see<br>  $\frac{2}{1 + \sqrt{5}} \le \frac{2 + \sqrt{$  $\frac{+b}{2}$ )<sup>2</sup> but this inequality is not correct i  $1 \le 1 + \sqrt{4} \sin x + 3 \cos x \le 1 + \sqrt{5} \Rightarrow \frac{1}{1+\sqrt{5}} \le \frac{1}{1+\sqrt{4} \sin x + 3 \cos x} \le 1$ .<br>
By multiplying all sides of these inequalities we can sec<br>  $\frac{2}{1+\sqrt{3} \sin x - 4 \cos x} \le 2 + \sqrt{5}$ .<br>
Therefore, the maximum and minimum values of  $C$  $\left(\frac{1+(-1)}{2}\right)^2 \Rightarrow 2 \le 1$ . This misconception is tiplying all sides of these inequalities we can see<br>  $\frac{2}{1+\sqrt{5}} \le \frac{2+\sqrt{3}\sin x - 4\cos x}{1+\sqrt{4}\sin y + 3\cos x} \le 2 + \sqrt{5}$ .<br>
orc, the maximum and minimum values of  $C = \frac{2+\sqrt{3}\sin x - 4\cos x}{1+\sqrt{4}\sin y + 3\cos y}$  arc  $2 + \sqrt{5}$  and  $\frac{2}{1+\sqrt{5}}$  $\frac{2}{1+\sqrt{5}} \leq \frac{2+\sqrt{3 \sin x - 4 \cos x}}{1+\sqrt{4 \sin y + 3 \cos x}} \leq 2 + \sqrt{5}$ .<br>
Therefore, the maximum and minimum values of  $C = \frac{2+\sqrt{3 \sin x - 4 \cos x}}{1+\sqrt{4 \sin y + 1 \cos y}}$  are  $2 + \sqrt{5}$  and  $\frac{2}{1+\sqrt{5}}$ <br>
Another missonception is about the probl Therefore, the maximum and minimum values of  $C = \frac{2+\sqrt{3 \sin x - 4 \cos x}}{1+\sqrt{4 \sin y + 3 \cos y}}$  are  $2 + \sqrt{5}$  and  $\frac{2}{1+\sqrt{8}}$ <br>respectively.<br>Another misconception is about the problem 3, "Prove that  $(\sin x + a \cos x)(\sin x + b \cos x) \le$ <br> $1 + (\frac{a+b}{2$ Therefore, the maximum and minimum values of  $C = \frac{2+\sqrt{3}\sin x - t \cos x}{1+\sqrt{4}\sin y + 3\cos y}$  are  $2 + \sqrt{5}$  and  $\frac{2}{1+\sqrt{8}}$ <br>
Another missconception is about the problem 3, "Prove that  $(\sin x + a \cos x)(\sin x + b \cos x) \le 1 + (\frac{a+b}{2})^2$ ."<br>
A Anothe Fraction is about the problem 3, "Prove that  $(\sin x + a \cos x)(\sin x + b \cos x)$ <br>
Transpectively.<br>
Another misconception is about the problem 3, "Prove that  $(\sin x + a \cos x)(\sin x + b \cos x) + (1 + \frac{a+b}{2})^2$ ."<br>
For this problem, the members of Lesson S  $\lambda = \frac{1}{1 + \sqrt{4 \sin y + 3 \cos y}}$  are 2 + √5 and  $\frac{1}{1 + \sqrt{5}}$ <br>
sove that  $(\sin x + a \cos x)(\sin x + b \cos x) \le$ <br>
soup members were faced with the following<br>  $\cos x$ )  $\leq \sqrt{1 + a^2} \sqrt{1 + b^2}$ .<br>  $\leq 1 + (\frac{a+b}{2})^2$  but this inequality is not corre respectively.<br>
Another misconception is about the problem 3, "Prove that  $(\sin x + a \cos x)(\sin x + b \cos x) \le$ <br>  $1 + (\frac{a+b}{2})^2$ .".<br>
After that students tried to prove  $\sqrt{1 + a^2}\sqrt{1 + b^2} \le 1 + (\frac{a+b}{2})^2$  but this inequality is not correct<br>

 $\frac{(n+1)(n+2)}{2}$ , which is obviously true. We assume that  $\cos x \neq 0$ , thus by dividing both sides of the given inequality by  $\cos^2 x$  gives

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(tan  $x + a$ )(tan  $x + b$ )  $\leq [1 + (\frac{a+b}{2})^2] \sec^2 x$ .<br>
then  $\sec^2 x = 1 + m^2$ . Therefore, the above inequality changes to<br>  $(a + b)m + ab \leq (\frac{a+b}{2})^2 m^2 + m^2 + (\frac{a+b}{2})^2 + 1$

$$
(\tan x + a)(\tan x + b) \le \left[1 + \left(\frac{a+b}{2}\right)^2\right] \sec^2 x.
$$

Now, we set tan  $x = m$ , then  $\sec^2 x = 1 + m^2$ . Therefore, the above inequality changes to

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\n(tan x + a)(tan x + b) ≤ 
$$
\left[1 + \frac{(a+b)}{2}\right]sec^2x
$$
.  
\nNow, we set  $\tan x = m$ , then  $sec^2x = 1 + m^2$ . Therefore, the above inequality changes to  
\n $m^2 + (a + b)m + ab \le \left(\frac{a+b}{2}\right)^2 m^2 + m^2 + \left(\frac{a+b}{2}\right)^2 + 1$   
\n $\Rightarrow \left(\frac{a+b}{2}\right)^2 m^2 + 1 - (a + b)m + \left(\frac{a+b}{2}\right)^2 - ab \ge 0$   
\n $\Rightarrow \left[\left(\frac{a+b}{2}\right)^2 m^2 + 1 - (a + b)m\right] + \left[\frac{a^2 + b^2 - 2ab}{4}\right] \ge 0$   
\n $\Rightarrow \left(\frac{(a+b)m}{2} - 1)^2 + \left(\frac{a+b}{2}\right)^2 \ge 0$ .  
\nThe proof is complete because all of the above statements are reversible.  
\nThe same misconception is discussed by Problem 4, "prove that (sin 3x + a cos 3x)(sin 3x +  
\nb cos 3x) ≤  $\frac{1}{2}$ (1 + ab +  $\sqrt{a^2 + b^2 + a^2b^2 + 1}$ )." In this problem, we can write  
\n(sin 3x + a cos 3x)(sin 3x + b cos 3x) ≤  $\sqrt{1 + a^2}\sqrt{1 + b^2}$ .  
\nNow, we should prove  
\n $\sqrt{1 + a^2}\sqrt{1 + b^2} \le \frac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2b^2 + 1})$ .  
\nThe above inequality is a misunderstanding, because by setting  $a = 2$  and  $b = -2$  we obtain  
\n $\sqrt{1 + (2)^2}\sqrt{1 + (-2)^2} \le \frac{1}{2}(1 + (2)(-2) + \sqrt{(2)^2 + (-2)^2 + (2)^2(-2)^2 + 1}) \Rightarrow 5 \le 1$ .  
\nA member of the Lesson Study group suggested a logical solution for this problem as follows  
\n(sin 3x + a cos 3x)(sin 3x + b cos 3x  
\n= sin<sup>2</sup>3x + b sin 3x cos 3x + a cos 3x sin 3x + abcos

 $b \cos 3x \leq \frac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2b^2 + 1})$ ." In this problem, we can write

$$
\sqrt{1+a^2}\sqrt{1+b^2} \le \frac{1}{2}(1+ab+\sqrt{a^2+b^2+a^2b^2+1}).
$$

$$
\sqrt{1+(2)^2}\sqrt{1+(-2)^2} \le \frac{1}{2}(1+(2)(-2)+\sqrt{(2)^2+(-2)^2+(2)^2(-2)^2+1}) \Rightarrow 5 \le 1.
$$

$$
\Rightarrow (\frac{(a+b)m}{2} - 1)^2 + (\frac{a-b}{2})^2 \ge 0.
$$
  
\nso complete because all of the above statements are reversible.  
\nnisconception is discussed for problem 4, "prove that (sin 3x + a cos 3x)(sin 3x +  
\n $\frac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2b^2 + 1})$ ." In this problem, we can write  
\n(sin 3x + a cos 3x)(sin 3x + b cos 3x)  $\le \sqrt{1 + a^2}\sqrt{1 + b^2}$ .  
\nwould prove  
\n
$$
\sqrt{1 + a^2}\sqrt{1 + b^2} \le \frac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2b^2 + 1})
$$
\nline  
\ninequality is a misunderstanding, because by setting  $a = 2$  and  $b = -2$  we obtain  
\n $2\sqrt{1 + (-2)^2} \le \frac{1}{2}(1 + (2)(-2) + \sqrt{(2)^2 + (-2)^2 + (2)^2(-2)^2 + 1}) \Rightarrow 5 \le 1$ .  
\nof the Lesson Study group suggested a logical solution for this problem as follows  
\n(sin 3x + a cos 3x)(sin 3x + b cos 3x)  
\n $= sin^2 3x + b sin 3x cos 3x + a cos 3x sin 3x + abcos^2 3x$   
\n $= 1 + (ab - 1)cos^2 3x + (\frac{a+b}{2})sin 6x$   
\n $= 1 + (ab - 1)(\frac{1 + cos 6x}{2}) + (\frac{a+b}{2})sin 6x$   
\n $= \frac{ab + 1}{2} + (\frac{a+b}{2})sin 6x + (\frac{ab - 1}{2})cos 6x$   
\nso'veved by a Creative Commons license. Atirbution-NonCommerical-Srae-Alike 4.0 International (CG BY-NC-SA





# MATHEMATICS TEACHING RESEARCH JOURNAL 40 SUMMER 2022 Vol 14 no 2 TEACHING RESEARCH JOURNAL<br>  $\frac{(a+b)}{2}$ <br>  $\frac{(a+b)}{2}$ <br>  $\frac{(ab-1)}{2}$ <br>  $\frac{(a+b)(a+b)}{2}$ <br>  $\frac{(a+b)(a+b)(a+b)}{2}$ <br>  $\frac{(a+b)(a+b)(a+b)}{2}$ <br>  $\frac{(a+b)(a+b)(a+b)}{2}$ <br>  $\frac{(a+b)(a+b)(a+b)}{2}$ <br>  $\frac{(a+b)(a+b)(a+b)}{2}$ <br>  $\frac{(a+b)(a+b)(a+b)}{2}$ <br>  $\frac{(a+b)(a+b)(a+b)(a+b)}{2}$ CHING RESEARCH JOURNAL<br>  $y^2 + (\frac{ab-1}{2})^2$ <br>  $\frac{b^2 + a^2b^2 + 1}{b^2}$ G RESEARCH JOURNAL 40<br>  $\frac{ab-1}{2}$ <br>  $\frac{2}{a^2b^2+1}$ .

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\n
$$
\leq \frac{ab+1}{2} + \sqrt{\frac{a+b}{2}}^2 + (\frac{ab-1}{2})^2
$$
\n
$$
= \frac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2b^2 + 1}).
$$

MATHEMATICS TEACHING RESEARCH JOURNAL<br>
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TEACHING<br>
SUMMER 2022<br>
Vol 14 no 2<br>  $\leq \frac{ab+1}{2} + \sqrt{\frac{a+b}{2} + \frac{ab-1}{2}}$ <br>  $= \frac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2b^2 + 1}).$ <br>
CONCLUSIONS<br>
Improving the mathematical knowledge of mathem Improving the mathematical knowledge of mathematics teachers affects the quality of lesson design, teaching methods and classroom atmosphere (Copur-Gencturk, 2015). Therefore, teachers require enhancing their mathematical knowledge continually. One of the best ways regarding this issue is sharing their knowledge and experiences through Lesson Study approach. In teaching mathematics concepts through problem solving, teachers need to understand mathematical ideas regarding a problem in a deep and connected way, and further they should be familiar with different methods of solution (O. Masingila et al., 2018). In this study, the Lesson Study group members suggested three different solution methods to find the maximum and minimum values of TRESERVIES SUMMER 2022<br>
TRESERVIENT Vol 14 no 2<br>  $\frac{ab+1}{2} + \sqrt{\frac{(a+b)}{2} + \frac{(ab-1)}{2}}$ <br>  $= \frac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2b^2 + 1}).$ <br>
CONCLUSIONS<br>
Improving the mathematical knowledge of mathematics teachers affects the quality of le maximum and minimum values of the function  $y = a \sin x + b \cos x$  are the function by function to function the function  $\frac{ab + 1}{2} + \sqrt{\frac{(a+b)^2 + a^2b^2 + 1}{2}}$ .<br>
CONCLUSIONS<br>
Improving the mathematical knowledge of mathematics teacher  $\leq \frac{ab+1}{2} + \sqrt{\frac{(a+b)^2}{2}} + \frac{(b-b)^2}{2}}$ <br>=  $\frac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2b^2 + 1})$ .<br>CONCLUSIONS<br>Himproving the mathematical knowledge of mathematics teachers affects the quality of lesson<br>design, taching methods and classroom at helps learners to improve their abilities in problem solving. Also, teachers generalized this given  $\leq \frac{2^{n-1}}{2} + \sqrt{(\frac{n-2}{2})^2 + (\frac{n-2}{2})^2}$ <br>  $= \frac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2b^2 + 1}).$ <br> **CONCLUSIONS**<br>
Improving the mathematical knowledge of mathematics teachers affects the quality of lesson<br>
design, teaching methods and cla  $\frac{2}{3}(1 + ab + \sqrt{a^2 + b^2 + a^2b^2 + 1}).$ <br> **CONCLUSIONS**<br>  $= \frac{1}{2}(1 + ab + \sqrt{a^2 + b^2 + a^2b^2 + 1}).$ <br>
(the sumplimetholds and classroom atmosphere (Copur-Genolutive, 2015). Therefore, teachers<br>
require enhancing their mathematical know concepts. They enhanced the quality of this research lesson by considering some suitable problems related to this given trigonometric problem and discussing regarding the variety of students' misconceptions in order to improve the performance of educators in their teaching. In this study, the members of the Lesson Study group found that students had serious problems to solve a general Improving the mathematical knowledge of mathematics teachers affects the quality of lesson equite change, teaching methods and classroom atmosphere (Copur-Gencturk, 2015). Therefore, teachers require enhancing their mathe design, teaching methods and classroom atmosphere (Copur-Geneturk, 2015). Therefore, teachers<br>require enhancing their mathematical knowledge contrainally. One of the best ways regarding this<br>issue is sharing their knowled because the coefficients  $a, b, c$  and  $k$  is not clear for students. They can solve the problems with issue is sharing their knowledge and experiences through Lesson Study approach. In teaching mathematics concepts through problem solving, teachers need to understand mathematics ideas regarding a problem in a deep and con mathematics concepts through problem solving, teachers need to understand mathematical ideas<br>regarding a problem in a deep and connected way, and thritten they should be familiar with<br>different methods of solution (O. Mas 5" easily because of the clearance coefficients in these functions.

In this article, the researcher discussed about the variety of solutions for a given trigonometric problem and related misconceptions to improve the mathematical knowledge of educators. Although mathematical misunderstandings are common among students, some novice teachers are also involved. Therefore, this research lesson may help teachers to provide a better learning environment for students regarding this trigonometric problem. Meanwhile, experienced teachers can improve this research lesson based on their students' abilities to reduce the trigonometric misconceptions among students.





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