A New Tool for the Teaching of Graph Theory: Identification of Commognitive Conflicts

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Abstract: *In this exploratory work, the discourse of first-year computer engineering undergraduate students of graph theory was analyzed with the aim of improving the teaching of this branch of mathematics. The theoretical framework used is the theory of commognition, specifically, we focus on commognitive conflicts because they are learning opportunities since they foster the learning process when resolved by students, and so teachers should consider them in their practice. A qualitative analysis of the written responses to a questionnaire dealing with definitions and the concepts of path and cycle graphs was performed. Thus, several commognitive conflicts were found, coming from the confluence of discourses governed by different discursive rules. Furthermore, the conflicts encountered were classified according to their origin, into object-level and metalevel conflicts. Concretely, the object-level conflicts had to do with the school discourse of geometry or sequences, and with the discourse of directed graphs; the metalevel commognitive conflicts were associated with the school discourse of the mathematical practice of defining. Finally, our findings are contrasted with related works in the literature, and also a series of implications for the teaching of graph theory are presented.*

Keywords: Graph theory, written discourse, commognitive conflict, object-level, metalevel, teaching

INTRODUCTION

Discrete mathematics is dedicated to the study of discrete structures, i.e., finite, or numerable sets. It is of growing interest in society because although it is a relatively new branch of mathematics, its recent development is "linked to the evolution of the society and also other disciplines such as computer science, engineering, business, chemistry, biology, and economics, where discrete mathematics appears as a tool as well as an object" (Ouvrier-Buffet, 2020, p.182). Discrete mathematics comprises several areas, among which graph theory stands out. In few countries, graph theory is studied in secondary education, but in most countries, it appears for the first time in undergraduate courses, mainly in mathematics or engineering (González et al., 2019; Milková, 2009; Vidermanová & Melušová, 2011), as it is a powerful tool for modeling reality. Authors like

Rosenstein (2018) advocate for their inclusion in the school curriculum. Apart from their usefulness in modeling, other reasons they present are that many of their concepts and problems can be understood without having great knowledge in mathematics and that they allow the development of mathematical practices like those of professional mathematicians (Balsim & Feder, 2008; Ouvrier-Buffet, 2020).

Regarding the teaching of graph theory, there are research works that describe didactic sequences or resources to be used in the classroom (Cartier, 2008; Costa et al., 2014; Geschke et al., 2005; Hart & Martin, 2018; Smithers, 2005; Wasserman, 2017). However, all these works do not deepen into students' reasoning. In fact, there are only a few studies considering this issue such as the work by Hazzan and Hadar (2005), who analyzed understanding from the perspective of the reduction of abstraction. Also, Medová et al. (2019) studied the errors made by students when using algorithms, and more recently, González et al. (2021) extended the Van Hiele model to characterize students' reasoning in graph theory.

Gavilán-Izquierdo et al. (2021) propose a new perspective to consider students' reasoning during the teaching process of graph theory: the sociocultural theory of commognition (Sfard, 2008). This framework has as its focus the study of discourse and its change. This discourse can be spoken or written, and learning is perceived as a change in discourse. Thus, learning can occur when commognitive conflicts appear, that is, when participants in the discourse act according to different discursive rules. The resolution of these conflicts results in learning.

This paper aims to investigate the written discourse of first-year undergraduate students of graph theory, thus identifying possible commognitive conflicts that teachers may use in their classrooms. In addition, based on our findings, we intend to provide a series of guidelines for the teaching of this area.

Literature Review

There are several recent works on commognitive conflicts. We present a synthesis of these works in chronological order, focusing on those studying the secondary-tertiary transition (Gueudet & Thomas, 2020) since they will be helpful in the discussion of our results.

Research has revealed different commognitive conflicts in a variety of concepts. Jayakody (2015) identified several commognitive conflicts in university students when they approached the concept of continuous functions. In his data collection instrument, she included a questionnaire in which they had to first describe and then define what a continuous function is. Two types of commognitive conflicts were identified: A first one, interpersonal, in different uses of the word "domain" and a second one, intrapersonal, arising from different inconsistent definitions extracted from textbooks. This author concludes that the identified conflicts provide information about the thinking of functions and thus have implications for teaching. Also, Ioannou (2018) identified commognitive conflicts in first-year university students when studying group theory. The first conflict has its origin in that these students at the university tend to consider that all sets have an internal composition law defined (e.g., the set of integers with addition). This author points out that "in the secondary education mathematics discourse, mathematical sets have algebraic structure, and, in particular, a binary operation with some properties. The notion of a set without an operation is new for novice undergraduate students" (Ioannou, 2018, p. 140). He also identified a second commognitive conflict in relation to the characteristic that a set can be empty, which does not occur in high school since all sets that appear in this stage have at least one element. Moreover, in high school sets are usually described in terms of their elements and not axiomatically, as often happens at university. Then both conflicts are closely related to the transition between school mathematics and university mathematics. The author concluded that his findings have some teaching implications, such as the need for these teachers to guide their students when using old ideas of school discourse in a new tertiary discourse.

Thoma and Nardi (2018) focused on manifestations of unresolved commognitive conflicts in first-year university students, such as the absence of a specification of the appropriate numerical context in tasks; the confusion between the symbols and the rules of school algebra and set theory discourses; the symbols of probability and set theory discourses; and between the symbols and rules of the probability theory discourse. They deduced that teachers should make a more explicit and systematic presentation of the differences between discourses and facilitate flexible movements between them. Regarding applied mathematics, Viirman and Nardi (2019) detected some commognitive conflicts in undergraduate biology students when performing mathematical modeling tasks. Some conflicts were classified as intramathematical (relating to what is understood as a math task) and others as extramathematical (relating to what constitutes a valid solution to the tasks in the biological sense). These authors believe that considering these conflicts in the teaching of this subject would be beneficial for learning.

Schüler-Meyer (2019) investigated conflicts in upper secondary students on the topic of elementary number theory and stated that "difficulties in learning processes in transition can be conceptualized as the students' attempts to communicate in secondary discourses while being engaged in tertiary discourses." (Schüler-Meyer, 2019, p. 165). His results illustrate the convenience of dealing with and explaining metanarratives when teaching. Fernández-León et al. (2021), agreeing with Sánchez and García (2014), also indicated that it is possible to consider conflicts between the students' discourse and the academic discourse of mathematicians. Specifically, they identified conflicts in the discourse of undergraduate students when constructing definitions, due to differences between the metarules of the students' activities and the sociomathematical norms.

González-Regaña et al. (2021) identified different commognitive conflicts in first-year undergraduate students (of the bachelor's degree in primary school education) when describing and defining geometric solids. They classified them as object-level and metalevel commognitive conflicts. An example of a commognitive conflict is described as follows: "The passage from a mathematical discourse that allows describing geometric objects [...] to one that is capable of elaborating formal definitions of these objects entails a development at the metalevel of the first discourse" (González-Regaña et al., 2021, p. 93). On the other hand, an example of an object-level conflict they found is the use of 2D geometry metarules to solve 3D situations. Thus, in the first case, there is a metalevel conflict between discourses about the mathematical practice of defining, and in the second there is an object-level conflict between the concepts of 2D and 3D geometry.

Finally, Kontorovich (2021), using the construct of precedent pockets as prior experiences in learning, analyzed the intrapersonal commognitive conflicts that appear in preuniversity students when using the concept of square root. He recommends that teachers provide students with tasks that are sufficiently varied and well thought out to be able to detect these commognitive conflicts.

Theoretical Framework

Fundamentals of Graph Theory

We first state the concepts of graph theory (Rosen, 2019) that will appear in this paper to ensure that it is self-contained. We say that a *graph* G is a pair (V, E), where V is any set (called the set of *vertices*), and E (the set of *edges*) is a set of unordered pairs of vertices. Two vertices are *adjacent* if they form an edge, and the *degree* of a vertex, $d(v)$, is defined as the number of vertices adjacent to it, i.e., $d(v) = |\{u \in V \ s.t. \{u, v\} \in E\}|$. Although there are several systems of graph representation, one of the most common consists of associating points in the plane to the vertices that are joined by lines, provided that the corresponding vertices are adjacent (*pictorial representation*). This type of representation is not unique, as shown in Figure 1 for the graph G_1 = ({a, b, c, d, e}, {{a,b}, {b,c}, {c,d}, {d,e}}).

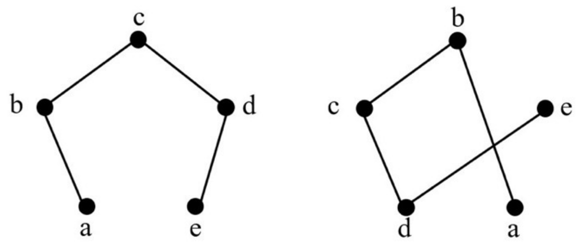


Figure 1: Two pictorial representations of graph G_1

There are other graph representation systems besides the set representation and the pictorial representation, such as matrices, degree sequences, intersections of objects, etc.

Formally, two graphs are said to be *isomorphic* if there exists a bijection between their vertex sets such that it preserves the edges. In Figure 2, two isomorphic graphs with bijection a$\leftrightarrow w, b \leftrightarrow y, c \leftrightarrow x, d \leftrightarrow z$ can be seen.

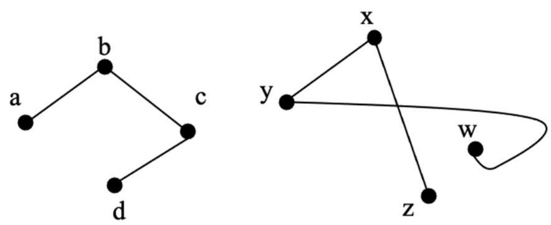


Figure 2: Two isomorphic graphs

A graph G = (V, E) *contains* a graph G' = (V', E') as a *subgraph* if V' ⊆ V and E' ⊆ E. On the other hand, a graph is said to be *connected* if any pair of vertices can be joined by a sequence of adjacent vertices. This property allows us to define classical families such as *paths*, which are connected graphs having two vertices of degree one and the rest of degree two, and *cycles*, which are connected graphs with all their vertices of degree two. Finally, a graph is said to be *directed* if its set of edges is a set of ordered pairs of vertices.

The theory of commognition

We have selected the sociocultural theory of commognition (Sfard, 2007, 2008) to analyze our data. As Presmeg (2016) stated, this is a framework with a high potential to consider issues of teaching and learning of mathematics. The word *commognition* derives from the words "communication" and "cognition". This theory holds that thinking does not exist without discourse, and reciprocally. Therefore, changes in mathematical discourse produce changes in mathematical thinking and, likewise, changes in thinking about mathematics produce changes in discourse, that is, in the way students communicate mathematically. According to this theory, mathematical *learning* occurs when the discourse is modified, extended, and enriched academically, in short, when the discourse changes, and teaching involves facilitating these changes. This theory offers a holistic view of mathematical learning since it considers the types of change that result from learning, the process followed by the participants (students and teachers) to achieve the change, and the expected results of the change (Sfard, 2007).

According to the theory of commognition (Sfard, 2007, 2008), mathematics is a type of discourse, and discourse is mathematical when it does not refer to material, tangible objects but to mathematical objects, which are abstract discursive objects, constructed in discourse, with signifiers considered mathematical (numbers, geometric figures, etc.). Mathematical discourse is characterized by these four properties:

-*Word use.* It encompasses the use of terms that are specifically mathematical (e.g., subgraph) and the use of common language words that may have mathematical meanings (e.g., degree).

-*Visual mediators.* These are the means through which discourse participants identify the objects to which they are referring and coordinate their communication. While in colloquial discourse visual mediators are concrete material objects (which may be present or mentally visualized), more specialized discourses often involve symbolic artifacts, created specially for this form of communication. Some examples are mathematical formulae or diagrams.

-*Narratives.* These statements report the characteristics of mathematical objects, their properties, or relationships between them, and are subject to acceptance or rejection by the community. If they are considered true because they have been substantiated by the community, they are called endorsed narratives. An example in mathematical discourse is the statement "all vertices in a cycle have degree 2".

-*Routines.* These are delimited and identifiable recurrent patterns in the discourse. They can be inferred from the discourse by observing word use, visual mediators, and analyzing how narratives are created and endorsed. Examples of routines specific to mathematics are defining, conjecturing, proving, executing an algorithm, etc.

As we have pointed out above, in the theory of commognition, learning mathematics means changing the discourse (i.e., its properties). Two types of mathematical learning can be distinguished. On the one hand, at the object level (object-level learning), which is expressed in the enrichment of the discourse through vocabulary expansion, construction of new routines, and production of new endorsed narratives, and, on the other hand, at the metalevel (metalevel learning), which implies changes in the metarules of the discourse. That is, some usual tasks, such as defining an object or identifying geometric figures, are done in a different way than usual, and some words used so far may change their meaning (Sfard, 2008).

Learning occurs mainly when the learner encounters a new discourse. If this new discourse is governed by different rules than the ones he/she knows so far, what Sfard (2008) called a *commognitive conflict* arises in the learner. The resolution of a commognitive conflict results in learning. We will adopt here the classification of commognitive conflicts used by González-Regaña et al. (2021), who distinguish between *object-level* and *metalevel* commognitive conflicts. Object-level commognitive conflicts are those that, when resolved, produce object-level learning, that is, those related to mathematical concepts. Similarly, metalevel conflicts are those that, when resolved, produce metalevel learning, that is, those related to mathematical practices, such as defining, conjecturing, or proving.

Thus, in this work, of an exploratory nature, we will focus on the construct of commognitive conflict. As mentioned above, it plays a relevant role in discourse change. Specifically, the research objectives addressed are as follows:

- Identify evidence of commognitive conflicts in the written discourse about graph theory of first-year computer engineering students.

- Classify these commognitive conflicts, and, if possible, determine their origin.

- Present some general recommendations for the teaching of graph theory based on the identified commognitive conflicts.

METHOD

Considering the nature of our data and the research objectives, an interpretive qualitative methodological approach is used. The participants and the context of the study, the instrument used for data collection, the data collection process, and how the data analysis was carried out are described below.

Participants and context

The study involved thirty-nine students (numbered from 1 to 39) from a group of the basic first-year course "Logic and Discrete Mathematics", taught in the first semester, with a duration of 60 hours. This group included students from different computer engineering degrees from the Polytechnic University of Madrid in Spain. The previous training required to take this subject is the one that any student who has taken a technology or health sciences baccalaureate is supposed to have acquired. No previous knowledge of the subject is required since most of the syllabus consists of topics that are developed in a self-contained manner. Specifically, the subject deals with 6 topics: 1. (Introduction) Sets, applications, and relationships; 2. Propositional and predicate logic; 3. Induction and recursion; 4. Combinatorics; 5. Binary relationships; 6. Graphs and digraphs. The most extensive is the one dedicated to logic since it is intended to be an instrument that facilitates reasoning and formalization in all subjects of the degree. The rest of the subjects are presented more briefly, emphasizing the formal aspects, since in later subjects the aspects more related to computer science (such as algorithm programming) are taken up and dealt with.

Data collection instrument

We used a written questionnaire to collect data, as other questionnaires designed to determine commognitive conflicts (Kontorovich, 2021). Indeed, according to the commognitive approach (Sfard, 2008), communication about mathematics in written or verbal responses is not a window to thinking, but an inseparable part of it. Thus, in this study, students' responses to this questionnaire are considered acts of communication and, therefore, part of their meaning making

(Biza, 2017, p. 1994). Besides, as Dimitrić (2018) stated, students' written work can be used to discover how they understood mathematical concepts and to suggest several methods to improve teaching.

Specifically, our questionnaire consisted of open-ended questions instead of multiple-choice items, like in the work by Manero and Arnal-Bailera (2021), because in this way more of the discourse and reasoning of the students will be shown.

The tasks of the questionnaire are presented below. All of them were related to the formulation of definitions of the concepts of cycle and path graphs.

1. Define a 6-vertex cycle. Are there any properties that you can eliminate from your definition of a 6-vertex cycle so that it remains equally valid? If so, indicate which one(s).
2. Define any path. Are there any properties that you can eliminate from your definition of a path so that it remains equally valid? If so, indicate which one(s).
3. Could you give a definition of a path equivalent to the one you have already given but containing the concept of a cycle?

Among these items, we find questions that can be classified as object-level tasks, i.e., they refer mainly to the mathematical objects of discourse, and it is not necessary that metalevel learning has been produced or need not be produced to solve them correctly; for example, when a definition of a 6-vertex cycle or path is requested. On the other hand, we find questions in the metalevel, i.e., associated with mathematical practices, specifically referring not to mathematical objects but to the definition of these objects. In our case they ask to find other definitions equivalent to those given before. Solving these metalevel questions correctly requires that the student either knows or learns the metarules that dominate the construction of equivalent definitions. These metalevel questions may cause the students to go back and ask themselves if the definition given at the beginning is correct and equivalent to the one given later, so they have the potential to produce metalevel learning.

Data collection

The participants answered the questionnaire individually and in writing during the last class of the course. The written responses are the data of our study, and the students gave their consent for them to be used for this purpose. These assignments were also part of the course assessment, and no notes or bibliographic material was allowed to be used to answer them.

Data analysis

Data analysis consisted of two phases. In the first phase, we focused on identifying the properties of the discourse: word use, visual mediators, narratives, and routines. In the second phase, based on the results of the first phase, we identified evidence of unresolved commognitive conflicts. In both phases, each researcher first analyzed the written responses individually, followed by sharing sessions with all members of the research team in which discrepant cases were discussed until consensus was reached. It should be noted that all the responses of the 39 students were analyzed, but for reasons of space, we only show evidence of the most representative ones in the findings section.

FINDINGS

The results are presented using vignettes. Each vignette is characterized by the identification of a particular type of commognitive conflict. We have classified them as object-level or metalevel commognitive conflicts. Within the object-level type, we have distinguished between the subtypes *intra* (within graph theory) or *inter* (in relation to other areas of mathematics, such as geometry, or other mathematical concepts, such as sequences). The metalevel commognitive conflicts found are mainly related to the mathematical practice of defining. In each vignette, the empirical data are shown together with the item being answered and the analysis performed. Below the students' answers, we show its literal translation into English.

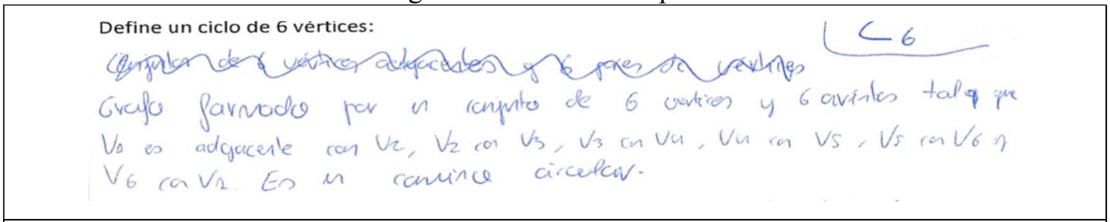
Object-level commognitive conflicts

Vignette 1. Object-level inter commognitive conflicts, between the discourse of graph theory and the discourse of Euclidean geometry.

This subtype of commognitive conflicts has its origin in the knowledge that students have of plane Euclidean geometry when they begin their study of graph theory at university. In fact, there is a certain analogy between plane geometric figures and graphs, since many graphs resemble geometric figures in their pictorial representation, and also possess vertices and edges. Moreover, rigid movements, which preserve the shape and size of geometric figures, are a particular case of topological transformations, which do not alter the sets of vertices and edges of the graph (González et al., 2021).

In Figure 3, student 12 is asked to define a six-vertex cycle. The first sentence that appears is a correct definition. In the second sentence, he/she intends to give an alternative definition using the word “circular”. This definition is incorrect since a correct mathematical definition, according to Zaslavsky and Shir (2005), should not depend on the chosen representation, and a cycle does not necessarily have a circular shape. It would suffice to consider a pictorial representation with edges that intersect at points that are not vertices. Thus, this definition is clearly dependent on the chosen representation and is not useful to recognize cycles in other representation systems. In addition, the use of this definition may lead to conflict when the student encounters pictorial representations of six-vertex cycles that have no circular shape. This could lead him to conclude, for example, that both graphs are not the same graph (or isomorphic), because he/she takes into account only the shape instead of the combinatorial information that the graph possesses, which is precisely what defines it.

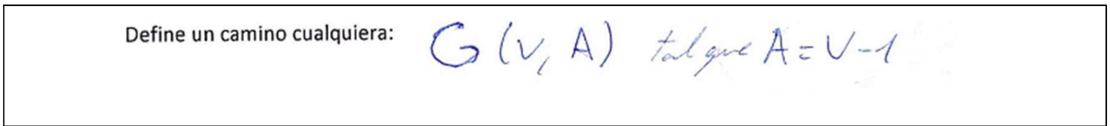
Figure 3: Student 12 response



Define a 6-vertex cycle:

"Graph formed by a set of 6 vertices and 6 edges such that v_1 is adjacent to v_2, v_2 to v_3, v_3 to v_4, v_4 to v_5, v_5 to v_6 and v_6 to v_1. It is a circular path."

In Figure 4, the student is asked to define any path. It can be understood that this student answers that the number of edges must be equal to the number of vertices minus one, although he/she uses incorrect visual mediators (symbols) (since it is not specified that it refers to the cardinality of both sets). This student did not give any information about how the connections between vertices must be in this type of graph, although there exist many different graphs given a specific number of edges and vertices. We think that the error could have its origin in a conflict with the discourse of geometry, with some definitions of polygons. For example, a pentagon is defined as a shape with five angles and five sides, and only with these two data (number of angles and sides) this type of shape is univocally determined, unlike what happens in graphs.

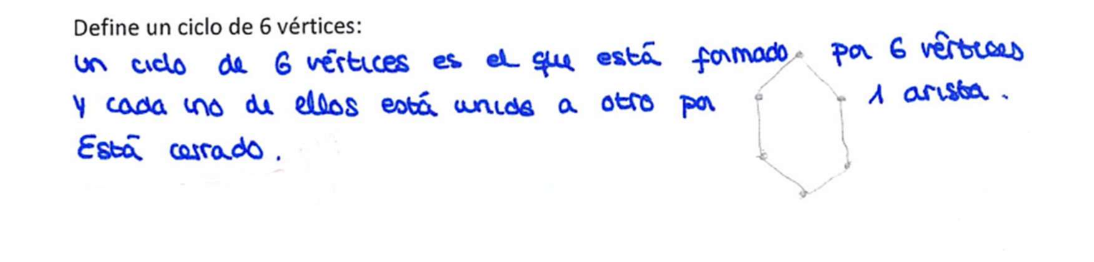


Define any path:

"G(V, A) such that A=V-1"

Figure 4: Student 10 response

Figure 5 shows a different conflict from the previous one, also with its origin in plane geometry. As in Figure 3, the student is asked to define a six-vertex cycle. The first sentence is redundant, and the set of edges is missing. The second sentence may be ambiguous, but it can be understood, according to the example drawn, that he/she refers to the fact that each vertex is joined *(at least)* to another by an edge, that is, there are no isolated vertices. In the third sentence, he/she indicates that this type of graph is closed. We believe that this use of words *("closed")* together with the pictorial representation given may indicate a conflict with the discourse of geometry, since the word closed is used in the discourse of geometry to indicate a closed polygonal chain or a closed curve, but it is not used in the discourse of graph theory to mention a property of a graph. Note that the given definition is incorrect because it includes graphs that are not six-vertex cycles (e.g., the disjoint union of two 3-vertex cycles or the graph in Figure 6, left). We also think that some very different pictorial representations of six-vertex cycles might not be identified as such by this student (e.g., see Figure 6, right). This is because there is some evidence that he/she thinks in a specific geometric shape (hexagon); even from the position of the representation and the text in the answer (Figure 5), he/she seems to have drawn this pictorial representation first and then described it.



Define a 6-vertex cycle:

"A 6-vertex cycle is the one that is formed by 6 vertices and each of them is connected to another by an edge. It is closed."

Figure 5: Student 19 response

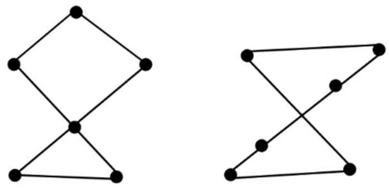
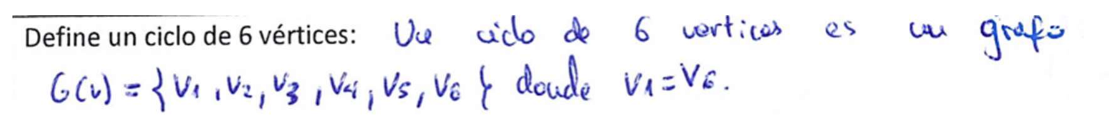


Figure 6: Six-vertex graphs

Vignette 2. Object-level inter commognitive conflicts, between the discourse of graph theory and the discourse of sequences.

This conflict occurs mainly when graphs appear in their set representation. Thus, in Figure 7, the student defines a six-vertex cycle by giving only the set of vertices as a set of elements where the subscripts of the elements indicate order. He/she also states that the last element that appears is equal to the first. This definition has two fundamental errors: first, if one vertex is equal to another, then by convention in set theory this set has five elements instead of six. Furthermore, it does not mention the set of edges, that is, the connections between these vertices. We believe therefore that there is a conflict with the discourse of sequences, since a sequence is defined precisely as an ordered set of elements and these elements are not related.

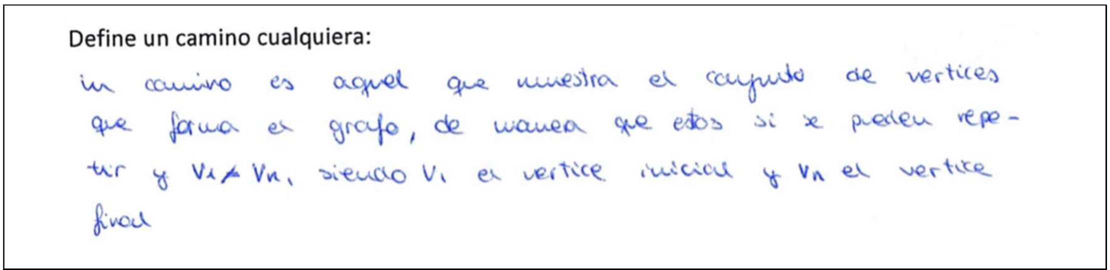


Define a 6-vertex cycle:

"A 6-vertex cycle is a graph $G(v)=\{v_1,v_2,v_3,v_4,v_5,v_6\}$ where $v_1=v_6$."

Figure 7: Student 30 response

Figure 8 shows a similar conflict between the discourse of graph theory and the discourse of sequences. Here the student defines a path as an ordered set of vertices, writes that there is an initial and a final vertex and does not mention the edges.



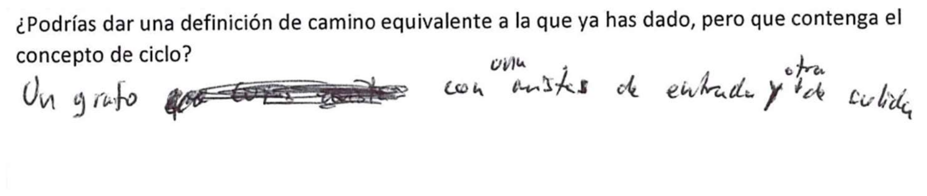
Define any path:

"A path is one that shows the set of vertices that form the graph in such a way that they can be repeated, and $v_1 \neq v_n$, where v_1 is the initial vertex and v_n is the final vertex."

Figure 8: Student 25 response

Vignette 3. Object-level intra commognitive conflicts, within the discourse of graph theory itself.

Figure 9 shows a student's answer in which he/she uses the words "one in-edge in and one out-edge" (sic). These concepts do not apply for graphs, but there are others that have a similar name and are specific to the discourse of directed graphs, such as "in-degree / out-degree" of a vertex. In addition, we have found more evidence of this conflict, for example, in student 37, but in that case, he/she did not use the words mentioned but the associated visual mediators, $g^+(v)$ and $g^-(v)$ (see Figure 10).

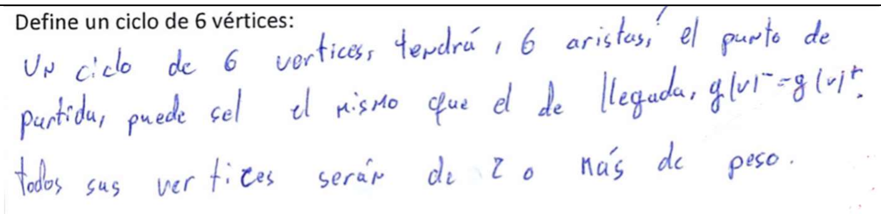

¿Podrías dar una definición de camino equivalente a la que ya has dado, pero que contenga el concepto de ciclo?

Un grafo ~~[struck out]~~ con una arista de entrada y otra de salida

Could you give a definition of a path equivalent to the one you have already given but containing the concept of a cycle?

"A graph with one in-edge and another out-edge" (sic)

Figure 9: Student 24 response


Define un ciclo de 6 vértices:

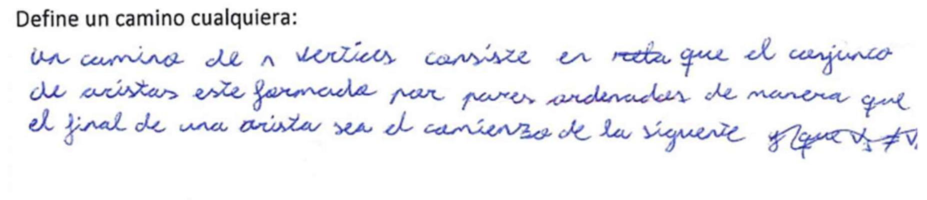
Un ciclo de 6 vertices, tendrá, 6 aristas; el punto de partida, puede ser el mismo que el de llegada, $g(v)^- = g(v)^+$. todos sus vertices serán de 2 o más de peso.

Define a 6-vertex cycle:

"A 6-vertex cycle will have 6 edges; the starting point can be the same as the end point. All its vertices will have weights greater than or equal to 2."

Figure 10: Student 37 response

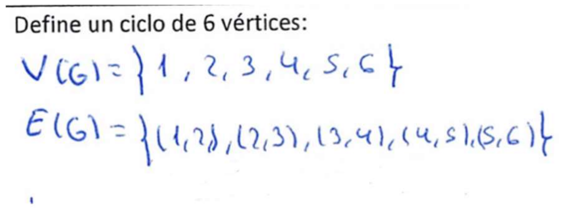
Figure 11 shows another conflict of type intra in the use of words between the discourse of graphs and the discourse of directed graphs. Note that, by general convention in graph theory, unless otherwise explicitly stated, when we say "graph" we refer to an undirected graph. The student here used the words "ordered pairs" to refer to the edges, as happens in directed graphs and contrary to undirected graphs, where the pairs are considered unordered. Several more students showed this conflict in their use of visual mediators: they denoted edges in the form (a,b) instead of {a,b} (see Figure 12).


Define un camino cualquiera:
Un camino de n vértices consiste en recta que el conjunto de aristas este formado por pares ordenados de manera que el final de una arista sea el comienzo de la siguiente

Define any path:

"A path of n vertices consists of the set of edges being made up of ordered pairs such that the end of one edge is the beginning of the next"

Figure 11: Student 3 response


Define un ciclo de 6 vértices:
V(G)= {1, 2, 3, 4, 5, 6}
E(G)= {(1,2),(2,3),(3,4),(4,5),(5,6)}

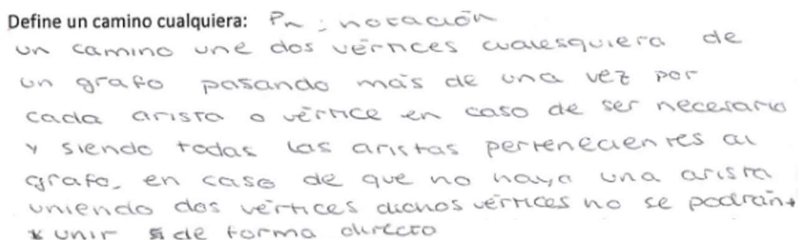
Define a 6-vertex cycle:

V(G)= {1,2,3,4,5,6}

E(G)= {(1,2),(2,3),(3,4),(4,5),(5,6)}

Figure 12: Student 6 response

Figure 13 shows a conflict of type intra related to word use. Thus, some of the students, such as student 15, understood that the item asked to define a path as a subgraph within a graph instead of a path graph.


Define un camino cualquiera: P_n : notación
un camino une dos vértices cualesquiera de un grafo pasando más de una vez por cada arista o vértice en caso de ser necesario y siendo todas las aristas pertenecientes al grafo, en caso de que no haya una arista uniendo dos vértices dichos vértices no se podrán unir de forma directa

Define any path:

"P_n: notation

A path joins any two vertices of a graph passing more than once through each edge or vertex if necessary and being all the edges belonging to the graph, in case there is no edge joining two vertices, these vertices cannot be joined directly."

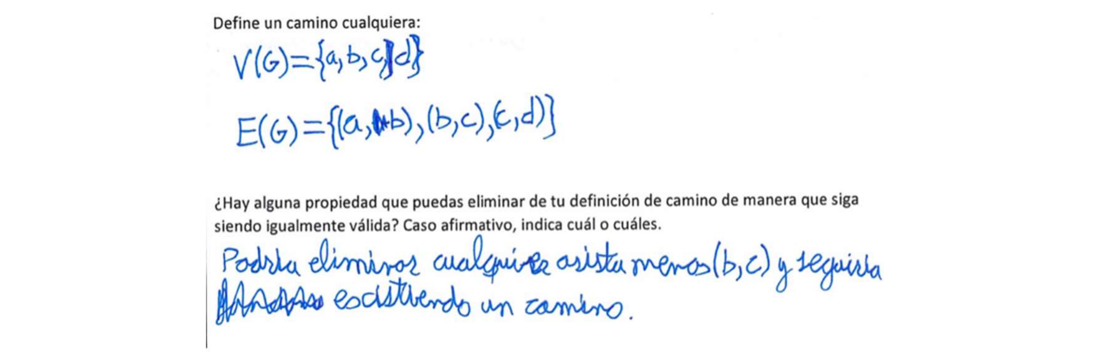
Figure 13: Student 15 response

It should be noted that several students in the sample, such as student 15 (Figure 13), defined a path or a six-vertex cycle referring to the idea of physically traversing the graph, which was evidenced by their use of expressions such as "passing through each vertex/edge" or "starting to traverse it from one vertex, it ends at another/the same vertex".

Metalevel commognitive conflicts

Vignette 4.- Metalevel commognitive conflicts, related to the mathematical practice of defining

Several responses from student 16 are shown below. When asked to define any path, he/she responds by giving an example of a (directed) path with four vertices in its set representation (Figure 14). The metalevel conflict may have its origin in the differences between the discourses of the practices of defining and proving, since in the latter, examples are usually given to prove the existence or counterexamples to prove that something is not true. We think that a conflict in the use of the word "any" may also have contributed to this error. The student may have interpreted the question as follows: "choose any path and define it", which corresponds to a non-formal or everyday use of the word "any", as opposed to its use in formal mathematics, where the proposed statement means: "give a valid definition for any path". Apart from this, we can observe in the second question that the student thinks of an edge as a property of the graph (instead of as an element of the graph), thus trying to "avoid" the task as it is presented, which is a metalevel task, and therefore more complicated. The purpose of this task was for the student to define the concept of a path graph using a set of necessary and sufficient properties; however, he/she answered by moving the task to the object level, referring to the concept "subgraphs of paths that are also paths". Finally, in the last task (Figure 15), the student did the same as in the previous one, he/she interpreted a metalevel task, in which he/she was asked to provide an equivalent definition of a path including the concept of cycle, as an object-level task using this time the concepts of cycle, path and subgraph, Specifically, he/she might have interpreted the sentence "contain the concept of cycle", as "a cycle is a subgraph of a path", as we can infer from his answer.



Define un camino cualquiera:

V(G)={a,b,c,d}

E(G)={(a,b),(b,c),(c,d)}

¿Hay alguna propiedad que puedas eliminar de tu definición de camino de manera que siga siendo igualmente válida? Caso afirmativo, indica cuál o cuáles.

Podría eliminar cualquier arista menos (b,c) y seguiría existiendo un camino.

Define any path:

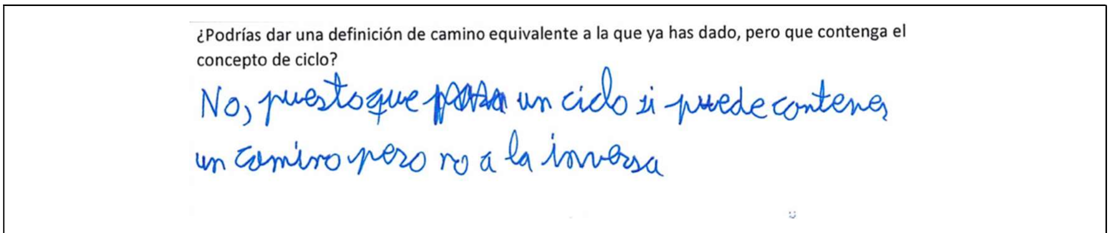
"V(G)={a,b,c,d}

E(G)={(a,b), (b,c), (c,d)}"

Are there any properties that you can remove from your definition of a path so that it is still valid? If so, indicate which one(s).

"I could remove any edge except (b,c) and a path would still exist."

Figure 14: Student 16 responses



¿Podrías dar una definición de camino equivalente a la que ya has dado, pero que contenga el concepto de ciclo?

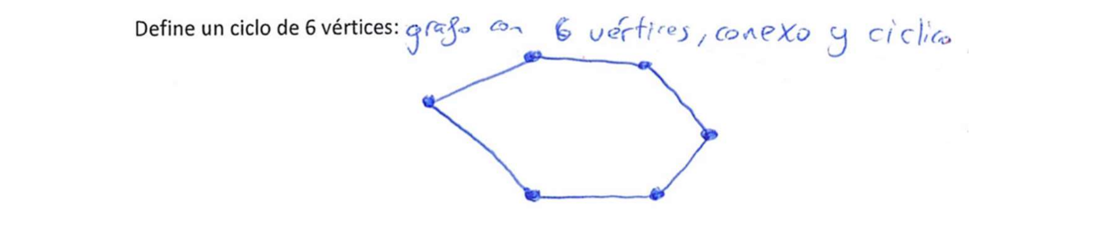
No, puesto que un ciclo sí puede contener un camino pero no a la inversa

Could you give a definition of a path equivalent to the one you have already given but containing the concept of a cycle?

"No, since a cycle can contain a path, but not vice versa."

Figure 15: Another student 16 response

Figure 16 shows a response of student 11 to the task of defining a six-vertex cycle. He/she presents an example of this graph in its pictorial representation and gives a series of characteristics: "it has 6 vertices, it is connected, and cyclic". We can identify a conflict in the mathematical practice of defining because he/she uses in his definition a word derived (*cyclic*) from the one it is defining (*cycle*), i.e., there is circularity in the definition (Zaslavsky & Shir, 2005). Furthermore, if we omitted this word, there would not be a set of necessary and sufficient properties, thus skipping another condition that a correct mathematical definition must have (Zaslavsky & Shir, 2005).



Define a 6-vertex cycle:

"Graph with 6 vertices, connected and cyclic"

Figure 16: Student 11 response

Finally, in Figure 17, in the same task, student 28 presents another conflict different from the previous ones in this vignette, also in the discursive activity of the mathematical practice of defining. This student shows only a concrete pictorial representation of a six-vertex cycle and does not refer to mathematical properties. He/she also did the same when asked to define the concept of a path. Moreover, in this case he/she provided a pictorial representation of a path with a concrete number of vertices, instead of somehow indicating that it can have any number of vertices, this is, in addition to not having understood the task of defining, he/she did not understand the generalization that was asked for. This may be due to a misinterpretation of the word "any", just as it happened to student 16 in Figure 14.

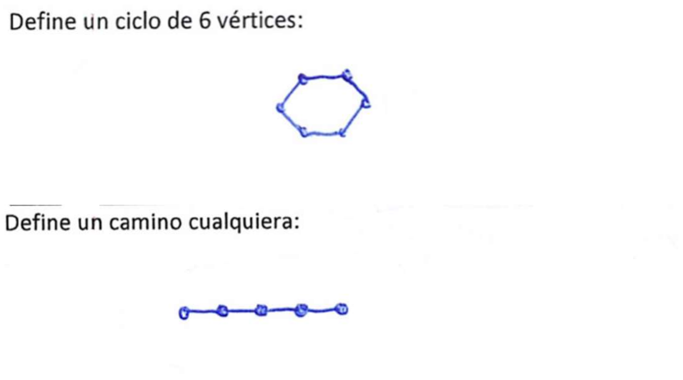


Figure 17: Student 28 responses

Now we present a quantitative summary of our study. If we consider the variable “number of conflicts per student”, its mean is 0.9487 and its standard deviation is 0.5969. Specifically, there were 8 students who did not show evidence of any type of conflict, 25 students showed evidence of only one type of conflict, and 6 students showed evidence of two types of conflict.

Table 1 shows the number and the percentage of students who presented each type of commognitive conflict. We have identified the vignette number in the results with the type of conflict, i.e., type 1 corresponds to the conflicts of vignette 1, and so on. The percentages do not add up to 100% because there are students who presented more than one type of conflict and students in whom no evidence of any type of conflict was found. It can be observed that the most frequent conflicts are those of types 3 and 4, that is, those related to the graph theory itself and the metalevel ones related to the characteristics of a mathematical definition. The number of students who presented type 4 conflicts is not so large, which does not indicate that all those who did not present this type of conflict knew the characteristics of a mathematical definition, since many students left blank items that could bring up this type of conflict.

	Number of students (percentage)
Type 1	5 (12.82 %)
Type 2	2 (5.13 %)
Type 3	22 (56.41 %)
Type 4	8 (20.51 %)

Table 1: Percentage of students presenting each type of commognitive conflict

DISCUSSION AND CONCLUSION

This study provides findings applicable to improving the teaching and learning of graph theory. Responding to the need pointed out by Hazzan and Hadar (2005) and Ouvrier-Buffet et al. (2018) to employ appropriate theoretical frameworks to investigate the teaching and learning of graph theory, we selected the commognitive theoretical framework (Sfard, 2008). The use of this framework has allowed us to identify commognitive conflicts, which is relevant for teaching and learning because when they occur, they generate learning opportunities, and their resolution results in learning. Specifically, in this article we have analyzed the written responses of first-year university students about basic concepts of graph theory. We have found commognitive conflicts that arose from the confluence of discourses governed by different rules. These conflicts have been classified using the same classification as González-Regaña et al. (2021). That is, they have been classified as object-level commognitive conflicts (i.e., related to the mathematical content) and metalevel commognitive conflicts (i.e., related to mathematical practices). Within the object-level ones, we have distinguished the subtypes inter, in our case between the discourse of graph theory and other mathematical areas or concepts, such as geometry or sequences; or intra, which appear within the discourse of graph theory itself. The identified metalevel conflicts are related to the mathematical practice of defining.

Regarding the conflicts found between the discourse of graph theory and that of Euclidean geometry, the data show the erroneous use or extrapolation of properties of the discourse of Euclidean geometry in the context of graph theory, due to the analogy between these two areas (González et al., 2021). It is influenced by the common use of words in the two areas; for example, the word “vertices” is used in geometry and graphs, and the word “edges” is also used in both graphs and geometry. As for visual mediators, there are similarities between the pictorial representation of graphs and 2D and 3D geometry; that is, similar visual mediators are used, but they have different meanings. Thus, this analogy between mathematical objects, which sometimes proves to be beneficial, for example, in solving problems, at other times leads to errors when the differences between the objects are not considered.

We have also presented evidence of conflicts between the discourse of graph theory and that of sequences. Several students defined cycle or path graphs as ordered sets of vertices, which resembles the definition of sequences as ordered sets of elements. Some of them also omitted the set of edges in their definition, which can be related to the work by Ioannou (2018). Indeed, this author states that, in university discourse, unlike what happens in secondary school, the sets may or may not have a binary relationship defined, which in some cases can confuse students who assume that they always have it defined or, on the contrary, that they never have it.

We have also observed that students frequently refer to the idea of physically traversing the cycle or path graph, which does not usually happen with other families of graphs. The origin of this type of reasoning can be found in the name of these graphs, since the words “cycle” or “path” are used in common language associated with the meaning of a route. We can also think that this emphasis on the traversal process may have its origin in real problems that are commonly used to introduce graph theory. For instance, the famous problem of the Königsberg bridges, which consists of finding whether a certain traversal is possible in a city represented by a graph. Thus, many students do not see these types of graphs as static objects that have a series of properties, but as elements that are associated with an action or process (traverse them). If we take into account that the students analyzed are taking their first steps in graph theory, this agrees with process-object theories such as Sfard’s (1991) reification theory and Dubinsky’s (1991) APOS theory, which state that procedural conceptions (processes) precede structural ones (objects). This may also be an indicator that students use the reduction of abstraction in the sense of Hazzan and Hadar (2005), which states that when they find it difficult to solve a mathematical problem due to its degree of abstraction, they unconsciously reduce it. Note that the association of cycles or paths with routes can also be related to the conflicts found in relation to the use of words and visual mediators of directed graphs in undirected graphs, since a route can be identified in some way with a certain direction on the edges of the graph.

Regarding now the commognitive conflicts in relation to the mathematical practice of defining, other works have also dealt with students' difficulties in this practice (Fernández-León et al., 2021; Zaslavsky & Shir, 2005), although in relation to other areas of mathematics, such as analysis or geometry. However, the conflicts found in our case are similar to those of these works, since many of them are related to the lack of knowledge of the characteristics that a correct mathematical definition should have: noncontradicting, unambiguous, invariant (under change of representation), hierarchical, noncircular, and minimal (Borasi, 1992; Zaslavsky and Shir, 2005). Therefore, there is evidence to deduce that several students in the sample do not understand the concept of definition given by a set of necessary and sufficient conditions and its usefulness in mathematics, which in the framework proposed by González et al. (2021) would be similar to stating that they have not reached Van Hiele's level 3 in graph theory. There were also conflicts related to the misinterpretation of metalevel tasks as object-level ones. Specifically, in these tasks they were asked about the definition of a type of graph, and some students gave answers that only referred to the type of graph defined. This again presents an analogy with the theoretical framework of the reduction of abstraction used by Hazzan and Hadar (2005).

To sum up, all these conflicts can be related, on the one hand, to the difference of discourses in the transition from high school to university education (Gueudet & Thomas, 2020; Ioannou, 2018; Schuler-Meyer, 2019), which is a critical point in teaching and learning because the rules of discourse change, and often the new rules of discourse are tacit, that is, not endorsed explicitly by the teachers (Sfard, 2008). On the other hand, these difficulties have to do with the passage from elementary mathematical thinking to advanced mathematical thinking (Tall, 1991); in our context, it is the passage from describing objects to providing mathematical definitions that meet all the characteristics indicated above.

Finally, Sfard (2007) stated that Van Hiele's cognitive theory and her sociocultural theory of commognition can complement each other. In her words "Van Hiele's levels can be interpreted as a hierarchy of mutually incommensurable geometric discourses" (p. 597), that is, they differ in the use of their properties. This complementarity was addressed by Wang and Kinzel (2014), who proposed that at each Van Hiele level (in the geometric case) there can be different types of discourse. The proposal by González et al. (2021) of Van Hiele levels of reasoning for graph theory suggests that object-level inter commognitive conflicts related to geometry may be linked to Van Hiele level 1 (recognition) since they come from descriptions of graphs supported by visual referents and the use of words that refer to their shape rather than to their topological or combinatorial properties, the latter being more typical of a higher level (González et al., 2021).

Implications for teaching

The relevance of our results is explained by the fact that the commognitive conflicts found give an idea of the way of thinking and the difficulties of the students when they begin the study of graph theory, and therefore they can be used by teachers for the design of hypothetical learning trajectories in the sense of Simon (1995). According to this author, hypothetical learning trajectories have three components: the learning objectives, the learning activities, and the hypothetical learning process (i.e., a prediction of how students' thinking and understanding will evolve). We can then examine this evolution in practice by analyzing their discourse using the commognitive framework, and based on this analysis, modify or design a new hypothetical learning trajectory. After all, hypothetical learning trajectories are not static but may undergo continuous modifications, that are mainly caused by the results of the assessment of student learning and the changes that occur in teachers' knowledge about the teaching-learning process.

All this leads us to suggest that in the early stages of the teaching of graph theory, sequences of tasks could be proposed in order to bring to the surface the identified commognitive conflicts since their resolution will facilitate progression through the Van Hiele levels. Furthermore, as Thoma and Nardi (2018) stated, during teaching it is important to make explicit reference to the change in the rules of discourse, in our case both the differences between the discourse of graph theory with other areas of mathematics such as geometry, and the difference between the rules of school and university discourse, or elementary and advanced mathematics discourse. However, before making the differences between discourses explicit, we believe that it is essential to promote in class situations in which these conflicts arise, so that students experience the need for discourse change. As an example, when students are asked to provide a definition, we can present examples of several graphs that meet their definition but are not of the class of the defined object, or vice versa, we can present examples of graphs that do not meet their definition but are of the class that was asked to be defined. This triggers a metalevel commognitive conflict, which, when resolved, leads to learning.

It is also important to try to avoid the excessive use of visualization or visual-based reasoning by learners. Several authors (e.g., Hershkowitz (1989)) pointed out the limitations of purely visual reasoning in the context of geometry, which is favored because often only prototypical examples are presented to students when they try to learn a concept. Similarly, as we have said above, Hazzan and Hadar (2005) interpret visualization as a mechanism of reduction of the level of abstraction in graph theory that leads students to erroneous reasoning or poor understanding of concepts. To prevent this, we recommend teachers to present, when introducing concepts and proposing tasks, both prototypical and non-prototypical examples of graphs in their pictorial representation, as well as examples in other systems of representation: intersection of objects, set representation, matrices, degree sequences, etc. This is also relevant because various authors (e.g., Dagan et al., 2018) found that the use of different representations enhances cognitive interest, creative thinking, and a deeper understanding of the concepts involved. To conclude, it is important to remark that the findings of this study can be used to guide teachers in the development of learning materials for graph theory.

Examples of application of the tool in the classroom

As a first example, we describe a situation that could arise in the class when the teacher asks to define a 6-vertex cycle. We suppose that a student replies that it is a graph with 6 vertices and 6 edges. This response provides evidence that the student presents a type 1 conflict, such as the one in Figure 4. The teacher could then provide pictorial representations of various ad hoc selected graphs (like the ones in Figure 18) selected ad hoc to collect further evidence of this conflict. Thus, the teacher would ask if these representations corresponded to six-vertex cycles. We assume that the student says that the graph in Figure 18(a) is a 6-vertex cycle because it has 6 vertices, six edges, and it is hexagon-shaped. Regarding the graph in Figure 18(b), the student could say that it is not a six-vertex cycle because although it has 6 vertices and 6 edges, it is not hexagon-shaped. The teacher would thus confirm that there is a type 1 conflict. We assume that this student also states that the graphs of Figures 18(c) and Figure 18(d) are not six-vertex cycles. This would also reveal signs of a type 4 conflict related to the mathematical practice of defining. The teacher could tell the student that according to his/her definition, Figure 18(c) and Figure 18(d) are six-vertex cycles and could then explain to him/her the function of mathematical definitions to classify objects. The teacher would also tell the student that indeed the graphs in Figure 18(c) and Figure 18(d) are not six-vertex cycles, but they meet his/her definition and therefore his/her definition is wrong. The teacher would state the correct definition for a six-vertex cycle and would remark that it makes no reference to the shape of the graph and that graph in Figure 18(b) meets it, therefore, it is a six-vertex cycle, thus resolving the type 1 conflict. The teacher could further explain to the students the characteristics that correct mathematical definitions must have, for example, by focusing on invariance by change of representation, using graphs in Figure 18(a) and Figure 18(b). They could also deal with minimality using the correct definition of a six-vertex cycle by adding or removing properties from the definition and asking for or presenting examples that meet the definitions resulting from these modifications. Here again, more type 4 conflicts could arise, which could be resolved with the help of examples and counterexamples of the modified definition.

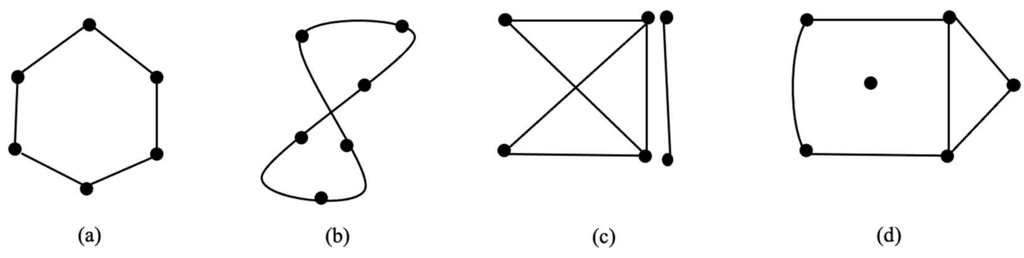


Figure 18: Some graphs having 6 vertices and 6 edges

As a second example, the teacher could ask the students to define and represent a path graph with four vertices and we assume that one of them gives this expression $G = \{v_1, v_2, v_3, v_4\}$ and the representation in Figure 19(a). The teacher then asks this student if the expression $G = \{v_2, v_1, v_4, v_3\}$ also represents a path graph and, in such case to represent it. We assume that the student draws a path graph like the one in Figure 19(b). The teacher would therefore detect a conflict of type 2, as the student considers the order of the vertices when representing and does not detect that his/her "definition" is wrong because the set of edges is missing. The teacher would ask him/her if the graphs he/she has represented in Figure 19(a) and Figure 19(b) are the same graph, and we assume that the student says they are not. The teacher could then ask the student to represent the graph in Figure 19(c) in a similar way. We assume that he/she represents it like this: $G = \{v_1, v_2, v_3, v_4\}$. The teacher could ask if the latter graph is the same as graphs in Figure 19(a) or Figure 19(b). If the student says no, the teacher could answer that indeed, they are not the same graph and could point out that the set representation he/she has given for the three graphs is the same because in graph theory the vertex sets are not ordered, and therefore the given expressions do not univocally determine those graphs. The teacher might add that, in fact, graphs in Figure 19(a) and Figure 19(b) are not the same path graph, although they are isomorphic, and Figure 19(c) is not even a path graph. In conclusion, students should learn that in the set representation of graphs it is essential to give the set of edges, since there is an infinite number of graphs with the same set of vertices, thus resolving the conflict of type 2.

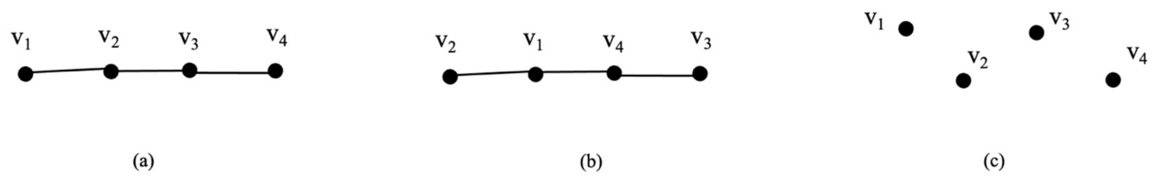


Figure 19: Some graphs having 4 vertices

Finally, this exploratory study is limited by the size of the sample, but it has allowed us to make a first approximation to the written discourse of graph theory students. In future works, we intend to complete the study by expanding the sample and taking data also from spoken discourse, both in class and in individual interviews. We also would like to identify commognitive conflicts relating to other concepts and other mathematical practices within graph theory.

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