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Constructing Students' Thinking Process through Assimilation and Accommodation Framework

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Abstract: Thinking is a tool to construct knowledge in learning mathematics. However, some college students have not been fully aware of the importance of constructing their knowledge. Therefore, this study aims to explore students' thinking processes in completing mathematical proofs through assimilation and accommodation schemes. This research was conducted on students majoring in mathematics from three different universities in East Java as research subjects. The data was collected through a mathematical proof test instrument and interviews which is then qualitatively analyzed. The results of the study show that there were students who completed the test through the assimilation scheme only, and there were students who completed the test using both assimilation and accommodation schemes. Students construct their thinking processes through 5 stages, namely: identifying, determining rules to be used, proving with symbol manipulation, reviewing, and justifying. Students use the five stages of thinking to construct knowledge. However, students who use assimilation schemes made some errors in proving the mathematics problem due to their carelessness in doing the proving with symbol manipulation and reviewing stages.

INTRODUCTION

Thinking process is an important component to know someone's abilities and talents in learning mathematics (Polly et al., 2007; Uyangör, 2019). Thinking can be said as a tool for learning mathematics and a tool to construct one's knowledge (As'ari et al., 2019; Fisher, 2005). Thinking process includes reasoning that occurs through a mental activity in the students' brain. This reasoning can occur when the students are performing algebraic operations, problem solving, decision making, critical thinking, reflective thinking, or analytical thinking. This process is not only to produce abstract mathematical numbers and concepts but also as an important skill in thinking analytically and logically, as well as reasoning quantitatively (Onal et al., 2017).

Thinking analytically is a highly necessary thinking process used to solve mathematical proof problems. At the university level, these problems are formal which require analytical thinking capability. However, some students tend to complete the mathematical proving problems related to abstract algebra intuitively (Korolova & Zeidmane, 2016). Intuitive mathematical proving is not necessarily wrong but students could possibly use the wrong concept in solving the problems. Some students solved subgroup problems in abstract algebra by using the Lagrange Theorem because they understand only the Lagrange Theorem concept and unfortunately do not understand subgroups concept very well (Leron & Hazzan, 2009; Leron, 2014). Therefore, students who complete an abstract algebra proving by using only the existing knowledge, need to construct their thinking process in order to accept a new knowledge scheme. The new knowledge scheme can be built by assimilation and accommodation. Thus, the question of this research is "how does the students' thinking process in solving the algebra proving problem based on the assimilation and accommodation framework?"

This research focuses on how do the students build their knowledge in solving algebra proofing problems through constructive thinking. Piaget said that the thinking process could be done through a construction process that occurs based on the previous knowledge to gain a new one. This construction could have occurred through five components, namely activating previous knowledge, owning and understanding a new knowledge, using the knowledge, then reflecting (Aseeri, 2020). Construction was the process of student's interaction related to previously owned ideas with new ideas to understand a concept being studied. Construction could be combined with interaction due to the existence of knowledge that were being used to perform a mental activity (Guler & Gurbuz, 2018).

Assimilation and Accommodation Framework

Piaget's theory states that there are two kinds of adaptation process of each individual to their environment; assimilation and accommodation (Kaasila, et al., 2014). Piaget divided the intellectual growth that occurs through one's mental activity into the following six steps: reflexively, obtained through a fundamental adaptation, interest on a new situation, relation to new discoveries, and combining the discoveries in mental activities (Piaget, 1965). A new scheme obtained by the students could be included in the assimilation object by organizing a new definition. The scheme on Piaget's theory contained assimilation and accommodation as a process of knowledge translation. Both were influenced by the development of Piaget's theory in mathematics learning (Ernest, 2003).

Assimilation is a process conducted by students in inserting a new stimulus into the existing scheme. The assimilation was a positive influence of the environment that occurs on one's mental activity. At the time a new object is being assimilated into the existing scheme. While the accommodation is a process of adjusting the schemes conducted by students to build a new scheme based on the existing scheme. Accommodation indicates that the process which is conducted by the student is influenced by the object being transformed. In other words, assimilation and accommodation could be represented as an interaction between the subject and the object which makes assimilation and accommodation closely related (Zhiqing, 2015). At the time when assimilation is dominated by a new scheme, then the scheme is a part of the accommodation. Therefore, assimilation can occur even though there is no accommodation, but accommodation will not occur without the existence of assimilation. For instance: students who have learnt about addition operation of natural numbers but have never learnt about the addition operation of fractions will solve the mathematics problem as follow $\frac{2}{3}+\frac{1}{2}=\frac{3}{5}$. In this process, students only perform assimilation as they only use the previous knowledge without reconstructing to gain new knowledge about addition operation of fractions. If students are able to operate addition in the form of "$\frac{2}{3}+\frac{1}{2}=\frac{4+3}{6}=\frac{7}{6}$", these students already performed accommodation as they equalized the denominators into 6 before adding the numerators into $4+3=7$.

Students can construct their knowledge when doing the assimilation to form a new scheme. Assimilation and accommodation are the adaptation process to the environment based on cognitive structures. While assimilation is the process of interpreting an event by using the existing cognitive structures, accommodation on the other hand is the process of increasing knowledge by modifying the existing knowledge or cognitive structures to gain a new experience (Kaasila et al., 2014; Netti et al., 2016). Therefore, in the process of assimilation, a new stimulus is directly absorbed and integrated into the existing knowledge schemes. Meanwhile the process of accommodation on the existing knowledge structures cannot directly absorb the new stimulus; it needs a phase to integrate the stimulus. The process of assimilation and accommodation can be illustrated into a diagram in order to help us understand the process or procedure of those two adaptation process (Subanji & Nusantara, 2016).

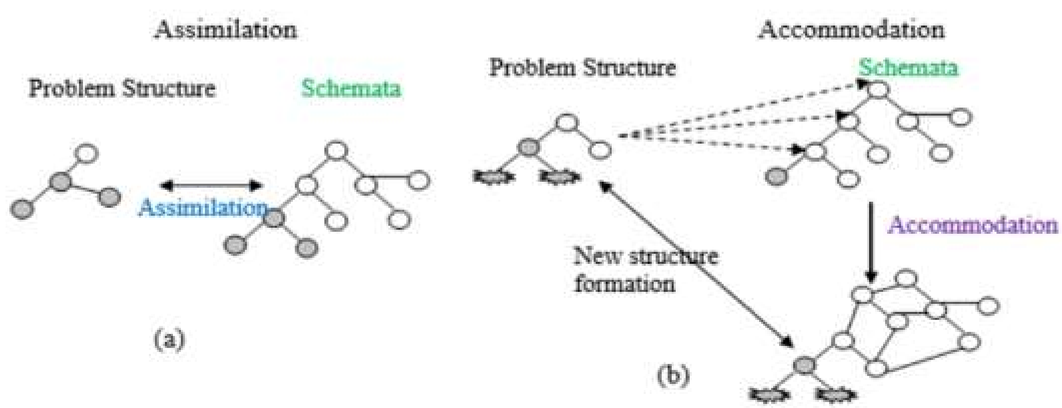


Figure 1: Assimilation and accommodation process

Figure 1 (a), shows that assimilation occurs when the structure of the problem is in accordance with the existing scheme. It will be interpreted directly into the correct way in order to form new structures. Figure 1 (b), shows that the structure of the thinking scheme does not match with the structure of the problem. The students need to convert the new scheme with the existing schemes in order to create a new thinking structure related to the problem when they constructing correctly. Therefore, the thinking activity through the assimilation and accommodation framework in this research can be seen in Table 1.

Table 1: Thinking process based on assimilation and accomodation schemes.

Thinking Process	Mental Activity
Assimilation	Employing an existing scheme to solve the mathematical proof problem
Accommodation	Employing both the existing scheme and a new scheme in order to solve mathematical proof problems

METHOD

This research is a qualitative research with descriptive explorative. Data for this research is collected through written test and interview. The researcher is the main instrument in collecting and analyzing the data obtained from written test results and interview. This research was conducted to college students in mathematics department from three different universities. The three universities are in Jombang, Mojokerto and Malang city. The subjects were chosen based on the students' abilities in constructing their knowledge through assimilation and accommodation scheme as shown in Table 1. The participant are those mathematics students who have passed the abstract algebra course. From 78 students in three different universities in East Java, Indonesia, 9 students were able to do assimilation without accommodation, and 13 students were able to do assimilation and accommodation. The students who were chosen as the research subjects were those who were able to reveal their thinking process verbally. Table 2 shows the number of students who could construct their idea through thinking process.

Table 2: The construction of students' thinking process

University	Number of Students	Assimilation	Assimilation and Accommodation
A	27	2	4
B	31	4	5
C	20	3	4
Total	78	9	13

From Table 2, we can see that 9 students were able to do the thinking process of assimilation and 13 students were able to do the thinking process of assimilation and accommodation. In general, one out of nine students could express their mind verbally in solving problem through assimilation. Two out of 13 students could express their mind in assimilation and accommodation. In short, three students were chosen as the research subjects. Table 3 showed the number of research subjects in this study.

Table 3: Selection of subjects

Thinking Process	Number of Students	Research Subjects
Assimilation	9	1
Assimilation and Accommodation	13	2
Total number	22	3

Table 3 shows that there were 3 students who were selected as research subjects. The 3 subjects are Dwi as subject 1, Alex as subject 2, and Dita as subject 3 (pseudonym). They were selected as research subjects as they were able to do verbal and written communication related to the thinking process that have been conducted in completing the abstract algebra proving test. This is due to the fact that thinking process is a form of communication between individuals and themselves based on cognitive activities they have conducted (Sfard & Kieren, 2001; Sfard, 2012).

The main instrument in this qualitative research is the researcher assisted with research instruments in the form of mathematical proving problem test and interview. The mathematical proving test instrument used in this research was adapted from Hungerford (2000). as follows:

"Let $p * q = p + q - pq$ with p, q elements of natural numbers in binary operations. Determine whether $p * q = p + q - pq$ is semigroup or not!"

The proving test consisted of semigroup material in abstract algebra. Semigroup is non-empty set G together with a binary operation * on G that is associative $a(bc) = (ab)c$ for all $a, b, c \in G$ (Hungerford, 2000).

The definition of semigroup:

a. A binary operation * on a non-empty set G is a function $\mu: G \times G \to G$.
b. An operation * on a set G is associative if $(a * b) * c = a * (b * c)$ for every $a, b, c \in G$

The data analysis used in this research is a qualitative with the following details:

Data analysis was conducted by observing the results of written tests and semi-structured interviews. In this research, interviews were used as a triangulation to obtain valid data. Creswell (2012) stated that the validity and reliability test of qualitative research can be done through triangulation. The researcher conducted task-based interviews on subjects with the help of a tape recorder and field notes containing important points from the subjects' expressions. The results of the interviews were transcribed exactly to the subjects' answers and expressions and then reduced based on assimilation and accommodation presented in Table 1. The data is presented in matrix form as one of the methods of qualitative research data analysis (Miles et al., 2014). This matrix is a table containing the relationship between variables obtained from the results of written tests and interviews. In this research, the researcher was actively involved in designing research, collecting data, and analyzing data.

RESULT AND DISCUSSION

The results of this research showed that there were three students selected as the research subjects. The selection of three subjects was based on their oral and written communication skills in constructing knowledge to complete a mathematical proving test. The three subjects were able to complete the test by integrating the previously owned knowledge scheme with a new scheme.

Subject 1

Dwi completed the test by assimilation because she performed the procedural proving by using the existing knowledge scheme. She identified the problem by reading the information that would be proven and then she wrote that the natural numbers with binary operations at $p * q = p + q - pq$ is closed. The claim was given spontaneously because she did not think of any element of the natural numbers by symbol of N. Dwi used the knowledge scheme about real numbers to complete the test. She considered that natural numbers were real numbers that are closed to all types of operations of numbers in the form of addition, subtraction, multiplication, and division operation. This can be seen from Figure 2.

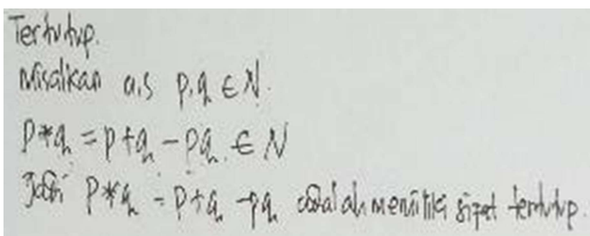
	Translation: Closed For example, take any $p, q \in N$ $p * q = p + q - pq \in N$ So, $p * q = p + q - pq$ has a closed nature.

Figure 2: Answer of subject 1 on the first stage

On the second stage, subject 1 used the associative nature to prove the semigroup. She proved the associative nature by using the existing knowledge scheme about real numbers to prove the semigroup of the natural numbers. She used the symbols $p, q, r \in R$ to prove the associative nature. Dwi performed algebraic operations by manipulating symbols. Firstly, she assumed that on $(p * q) * r = p * (q * r)$ the associative nature was not applicable because the results of algebraic operations between the left-hand and right-hand side of the equation were not the same as in the circle sign in Figure 3.

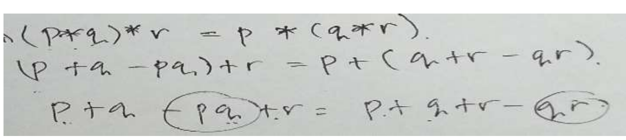
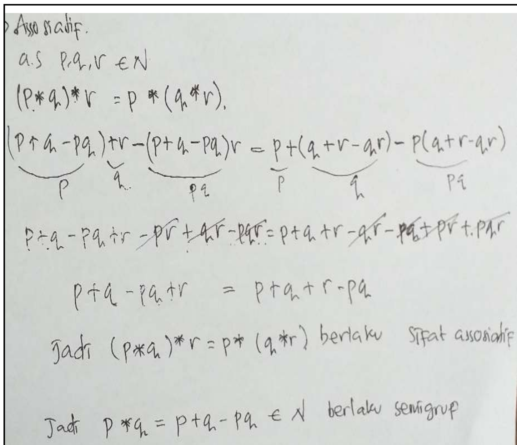


Figure 3: The initial answer of the subject before the construction process

The subject then constructed the knowledge by reproving to make sure that she got the right answer of associative proofing. She used cancellation characteristics by crossing out the same elements between the right-hand and left-hand side, so that it is obtained $(p * q) * r = p * (q * r)$ as in Figure 4.



Translation:
Take any $p, q, r \in N$

$$(p * q) * r = p * (q * r)$$
$$(p + q - pq) + r - (p + q - pq)r = p + (q + r - qr) - p(q + r - qr)$$
$$p + q - pq + r - \cancel{pr} + \cancel{qr} - \cancel{pqr} = p + q + r - \cancel{qr} - \cancel{pq} + \cancel{pr} + \cancel{pqr}$$
$$p + q - pq + r = p + q + r - pq$$

So, $(p * q) * r = p * (q * r)$ applies semigroup

Figure 4: The answer of subject 1 on the second stage

The subject's proving process shows that the closed and associative nature were applicable. At first, she assumed that $(p * q) * r$ was not associative because she obtained $(p * q) * r \neq p * (q * r)$. Furthermore, the subject constructed the knowledge that she had in order to obtain $(p * q) * r = p * (q * r)$ as in Figure 4. From the proving result of the closed and associative nature, Dwi concluded that $(p * q) = p + q - pq$ is a semigroup on binary operations of natural numbers. This is indicated from the interview transcript as follows:

R : *Why?*
D : *Because in the beginning I did an algebraic and the result was* $(p * q) * r \neq p * (q * r)$. *However, after I carefully observed by decomposing it one-by-one, the result showed that* $(p * q) * r = p * (q * r)$

Dwi as subject 1 completed the mathematical proving test related to the semigroup by first identifying the problem. Identification of the problem is done spontaneously by mentioning the semigroup conditions in the form of closed and associative nature. Then she gave a claim that $p * q = p + q - pq$ is closed on the binary operation of natural number (N). Then she proved the associative nature by using the assimilation scheme to obtain $(p * q) * r = p * (q * r)$. She proved it through symbol manipulation in algebraic operations and obtained $(p * q) * r \neq p * (q * r)$. After that she claimed that $p, q \in N$ with respect to binary operations on natural numbers is not associative. However, Dwi conducted a review on her result by re-checking it again. From the review, she found that the associative nature that she previously concluded was incorrect. Then she re-constructed her knowledge to perform algebraic operations again and obtained $(p * q) * r = p * (q * r)$. Therefore, Dwi justified that $p * q = p + q - pq$ for all $p, q \in N$ is semigroup of binary operations. The justification was done analytically based on the thinking construction, but the final conclusion that she gave was incorrect. Dwi performed procedural proving as she only explained the proving of semigroup in natural number as in the proving procedure for real number. Although she had written N in her proving, she did not realise that N is a natural number. Thus, she only performed assimilation without accommodation as she did not reconstruct her previous knowledge to conduct the proving of N as natural number.

Subject 2

Alex used his previous knowledge scheme about semigroups proving on real numbers to prove the semigroups on natural numbers. It can be seen from the mental activity performed by Alex in identifying the problems. He mentioned that the semigroup requiring the closed and associative nature. Then he proved and concluded that $p * q = p + q - pq$ for all $p, q \in N$ is not semigroup of binary operations because it did not fulfill the associative nature. The conclusion was correct but the steps taken in reaching the conclusion were not correct. From his proving of the closed nature, an error was seen. The right answer should: $p * q = p + q - pq$ for all $p, q \in N$ is not closed on binary operations.

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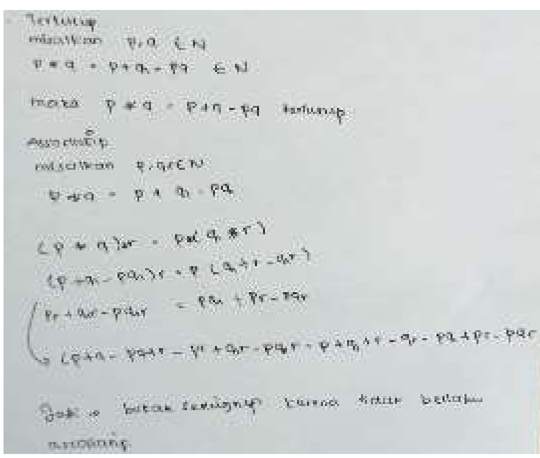
<td></td>
<td>Translation:
• Closed
For example, $p, q \in N$
$p * q = p + q - pq \in N$
Then
$p * q = p + q - pq \in N$
Are closed
• Associative
For example, $p, q \in N$
$p * q = p + q - pq$
$(p * q) * r = p * (q * r)$
$(p + q - pq)r = p(q + r - qr)$
$pr + qr - pqr = pq + pr - pqr$
$(p + q - pq + r - pr + qr - pqr$
$= p + q + r - qr - pq + pr - pqr$
So, it is not a semigroup because it does not apply associative nature</td>
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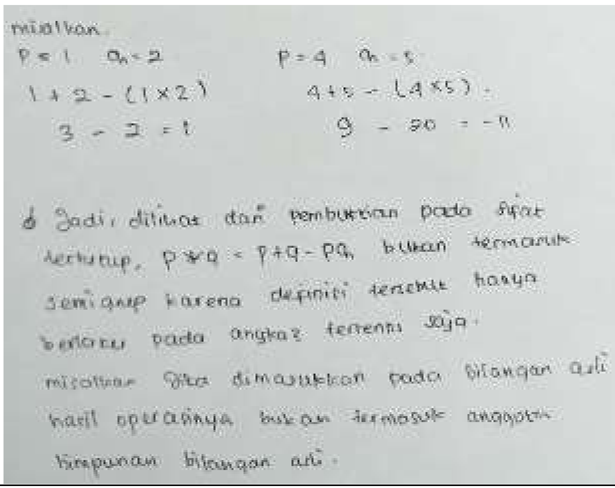
Figure 5: The result of subject 2

Figure 5, shows that Alex described his proving through the closed nature and the associative nature was performed spontaneously as he only described it procedurally. The proving result showed that the claim was correct that the problem was not a semigroup, but the steps taken by Alex to prove it were incorrect. Alex was doubtful about the result of the associative nature proving, so that he constructed the existing knowledge to be re-prove by taking any element of the set of natural numbers in the form of $N = \{1,2,4,5\}$ as in Figure 6.



Translation:
For example,

$$p = 1; q = 2$$
$$p + q - pq = 1 + 2 - (1 \times 2) = 3 - 2 = 1$$

For example,

$$p = 4; q = 5$$
$$p + q - pq = 4 + 5 - (4 \times 5) = 9 - 20 = -19$$

So, it is not closed nature, then $p * q = p + q - pq$ is not a semigroup.

Figure 6: The results of Alex's thinking construction

Alex did the proving twice by substituted elements of the set of Natural Numbers (N). First, Alex considered $p = 1$ and $q = 2$ to obtain $p + q - pq = 1 + 2 - 2.1 = 1$, because 1 is an element of N as a set of natural numbers, then he claimed that $p * q = p + q - pq$ for all $p, q \in N$ is closed. Second, he did the proving by assuming that p = 4 and q = 5 and then obtained $p + q - pq = 4 + 5 - 4.5 = -11$. Since the result is -11, he changed his claim into $p * q = p + q - pq$ for all $p, q \in N$ was not semigroup because it is not closed on binary operations in natural number.

A : *Means that for example something like this, p = 1 and q = 2, then* $p + q - pq = 1 + 2 - 2.1 = 1$ *is obtained. The result is the natural numbers, ma'am.*
For example, p = 4 and q = 5 and then $p + q - pq = 4 + 5 - 4.5 = -11$
Oh, right…. It's not, ma'am.
So that the closed nature is not applicable, isn't it?

The thinking process conducted by Alex in completing the test was by identifying the problem first. The information in the problem stated that N is a set of natural numbers, but Alex used a real number scheme to prove it. This is because he only knew the semigroup proving in real numbers in which it can be said that he did an assimilation process. The proving of closed nature was done only by looking at the information in the problem then made a claim that $p * q = p + q - pq \in N$ is closed on binary operations. After that he performed algebraic operations by manipulating symbols to prove the associative nature. From the proving of associative nature, it was obtained that $p * q = p + q - pq \in N$ was not associative on binary operations because $(p * q) * r \neq p *$

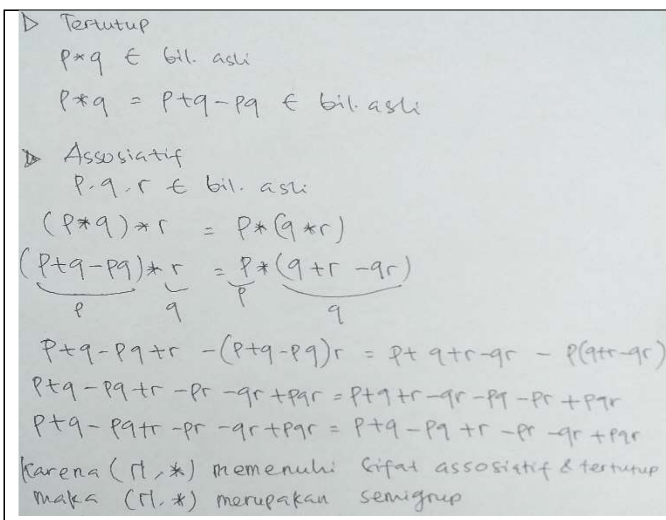
$(q * r)$. Then Alex made a new claim that $p * q = p + q - pq \in N$ was not semigroup on the binary operations because it did not fulfill the associative nature.

Alex conducted a review on his own result. He checked the correctness of the claim by proving the problem using the elements of natural numbers. Then he obtained the result in the form of a negative number (-11) so that he changed the claim in which $p * q = p + q - pq \in N$ was not closed on binary operations because -11 is not an element of the natural numbers. From the claim, he said that proving associative nature was not needed because the first condition of the semigroup was not fulfilled. After that, he made a justification that $p * q = p + q - pq \in N$ was not semigroup on binary operations.

Based on the result of exploration to subject Alex, it is known that he conducted procedural proving as he used real numbers to proof close property of natural numbers which resulted in incorrect conclusion. However, Alex tried to re-examine the statement in the test and presupposed the element of natural numbers in the form of N = {1,2,4,5} to perform the close property proving. Thus, Alex actually performed assimilation but obtained the incorrect conclusion. He then reconstructed his knowledge by presupposing any element of natural numbers so as to say that he performed accommodation.

Subject 3

Dita identified the problem that would be proven in almost the same way as what subject 2 did. First, Dita identified the problem by using semigroup proving on real numbers and integers. Dita claimed that $p * q = p + q - pq \in N$ with $p, q \in N$ is a semigroup on binary operations because it fulfilled the closed nature and the associative nature.



Translation:

- closed

$p * q \in natural\ numbers$

$p * q = p + q - pq \in natural\ numbers$

- Associative

$p, q, r \in real\ numbers$

$(p * q) * r = p * (q * r)$

$(p + q - pq) * r = p * (q + r - qr)$

$p + q - pq + r - (p + q - pq)r = p + q + r - qr - p(q + r - qr)$

$p + q - pq + r - pr - qr + pqr = p + q + r - qr - pq - pr + pqr$

	$p+q-pq+r-pr-qr+pqr$ $=p+q-pq+r-pr$ $-qr+pqr$ Because the natural number satisfies the associative and closed nature, so that natural number is a semigroup on the binary operations

Figure 7: The result of subject 3

From Figure 7, it can be seen that Dita did the closed nature proving only by writing $p*q=p+q-pq \in N$. She only paid attention to the shape of the symbol without paying attention to the element of the set of natural numbers, it is to say that the proving was done spontaneously. Then the associative nature proving was done through manipulation of symbols by assuming that on the left-hand side $p+q-pq=p;\ r=q$ and on the right-hand side $p=p;\ q+r-qr=q$. From this assumption, she performed algebraic operations and obtained the result of $(p*q)*r=p*(q*r)$. Her mistakes in deciphering the associative nature proving resulted in errors in her claim. Dita claimed that $p*q=p+q-pq$ is a semigroup on binary operations of natural numbers. Then she reconstructed her knowledge by saying that the proving of semigroup of natural numbers needed not only algebraic symbols but also needed to be proven by using numbers which were elements of natural numbers. She said that it was based on the thinking process so that it was not written on the answer paper. This was revealed in the interview transcript as follows:

R : *From the claim you have obtained, are you sure that $(p*q)*r=p*(q*r)$ included in semigroups on binary operations of natural numbers?*

Di : *Actually, I'm not sure about that ma'am ...*
Because the proving of the semigroup on original numbers will be more valid if it is to be done by using the algebraic symbols and also the numbers

R : *What do you mean by that?*

Di : *Let me explain this ma'am ... suppose I take p = 10 and q = 12 so we get* $p*q=p+q-pq=10+12-10.12=32-120=-98$

Dita identified the semigroup problem by assimilation based on the known semigroup definition. She said that the semigroup contained a closed and associative nature. She did the proving of closed nature just by looking at $p*q=p+q-pq \in N$. Then she did the proving of the associative nature by performing algebraic operations through symbol manipulation. Dita said that $p*q=p+q-pq \in N$ is associative because $(p*q)*r=p*(q*r)$. Then she claimed that $p*q=p+q-pq \in N$ is a semigroup on binary operations of natural numbers because it satisfied the closed and associative nature. This claim existed because she performed procedural proving without paying attention to the element of natural numbers.

However, Dita conducted a review of her proving that has been done. She conducted accommodation by reconstructing the existing knowledge schemes. She assumed that the natural numbers are $p = 10$ and $q = 12$ which are then substituted into $p * q = p + q - pq$. He obtained -98 as the result of the substitution while -98 is not a natural number. Therefore, she changed her claim by saying that $p * q = p + q - pq \in N$ was not semigroup on binary operation of natural numbers as it doesn't apply the close property.

Based on the exploration process done to all subjects, it is obtained that the construction of students' thinking in completing mathematical proving tests related to abstract algebra can be simplify as shown in Table 4. The following Table 4 will explain the constructing activities conducted by students based on the assimilation and accommodation framework.

Table 4: The thinking process of the subjects.

Subjects	Schemes	Mental Activities	Students' Construction
Dwi	Assimilation	Performed algebraic operations to get $(p * q) * r \neq p * (q * r)$ so that $p * q = p + q - pq$ with binary operations in natural number is not associative but it is closed.	Reviewed the results of the associative nature proving then performed algebraic operations through symbol manipulation. The review was conducted to change the claims that stated $(p * q) * r \neq p * (q * r)$ into $(p * q) * r = p * (q * r)$. Justified that $p * q = p + q - pq$ is semigroup to respect binary operations of natural numbers.
Alex	Assimilation and accommodation	Identified the problem by using semigroup proving of real numbers based on his previously understood scheme. Then did the proving by manipulating the symbols.	Determined mathematical rules in the form of semigroup definitions in the set of real numbers. He then performed the reconstruction by presupposing the element of natural number in the form of p=4 and q=5.
			Gave justification by changing the claim which initially was in the form of $p * q = p + q - pq$ is semigroup to respect with binary operations in natural numbers, and then the claim became $p * q = p + q - pq$ with binary operations in natural numbers is not semigroup because it doesn't fulfill the closed nature.
Dita	Assimilation and accommodation	Identified the problem that would be proven by using previous knowledge related to the proving of semigroups of real numbers and integers.	Gave a statement that to prove the semigroup does not only need the mathematical symbols, but also the numbers which are elements of natural numbers. Then substituted the natural numbers element into $p * q = p + q - pq$.

			Changed the claim with a new claim in the form of $p * q = p + q - pq$ with a binary operation in N is not a semigroup because it didn't meet the requirement of closed nature of the natural numbers.

Students first constructed their knowledge from assimilation and then performed accommodation. The students' thinking construction began by conducting problem identification to solve the mathematical proving problem. The identification was done as a first step in understanding the problem to be proven (Öztürk & Kaplan, 2019). Then separated the object with its context (Sternberg et al., 2008). In this research, the students tried to understand the problem to be proven by identifying the information presented in the problem. The students mentioned that in the proving of semigroup in a non-empty set G with binary operations, the semigroup requirements in the definition need to be understood first.

The semigroup definition includes the closed and associative nature (Hungerford, 2000). The students mentioned that non-empty set of G could be real numbers or integers. Then the students used the scheme of knowledge about real numbers to prove the semigroup of natural numbers. The proving of closed nature happened quickly by looking at the symbols $p, q \in N$ and $p * q = p + q - pq$ without considering about the members of natural numbers set. The proving that has been done through thinking quickly and automatically is called thinking intuitively (Leron & Hazzan, 2009; Leron, 2014). After that, the students performed algebraic operations by manipulating symbols to prove the associative nature. Symbol manipulation is an activity conducted by students to solve mathematical problems related to algebra (Bleiler et al., 2014). The students performed algebraic operations to prove the associative nature of $(p * q) * r = p * (q * r)$. From the results of the closed and associative nature proving, the students gave a claim that $p * q = p + q - pq \in N$ was a semigroup on binary operations of natural numbers. Claims are statements that are often used in solving mathematical proving problems that need to be verified (Panza, 2014).

Furthermore, the students reviewed or re-checked the claims they made (Mason, 2010). The students constructed their knowledge to check the correctness of the claims (Quansah et al., 2018). The students did the thinking construction by re-considering the statements that would be proven by assimilation and accommodation. The students said that the proving of semigroup of natural numbers was not only by using algebraic symbols but also by using numbers that are elements of a set of natural numbers. Then the students did the proving again by using numbers to get new claims: $p * q = p + q - pq \in N$ was not semigroup because it didn't meet the requirement of closed nature on binary operations of natural numbers. The students can make justification from the proving activity twice. Justification shows the confidence level of the students on the conclusions made based on scheme (Mason, 2010).

The result found that intuitive and analytical thinking are not two separate things because students can construct intuitive and analytical thinking processes using assimilation and then accommodation schemes (Rusou et al., 2013; Iannello & Antonietti, 2008). Some of the students solved the problems of semigroup intuitively because they only used the assimilation scheme to construct their existing knowledge. Whereas the other students who were able to solve the problem intuitively and analytically because they did the process of assimilation and accommodation. At the students accepted the problems, they intuitively solved it based on their existing knowledge, even though the context of the problem was different. Therefore, the students' thinking process in constructing their knowledge to complete the proving of abstract algebra can be described as follows (see Table 5):

Table 5: students' thinking process based on assimilation and accommodation schemes

Steps	Activity
Identifying	Mentioning information in the question or problem
Determining rules	Using definition concept.
Proving with symbol manipulation	Performing algebraic operations to prove
Reviewing	Re-checking the claim that has been made. If they are not sure yet, then the proving activity needs to be done again
Justifying	Make a conclusive conclusion based on the result of the review

Students are able to combine intuitive and analytical thinking to make reasoning in solving mathematical problems (Macchi & Bagassi, 2012). Intuitive and analytical thinking are two different things (Rusou et al., 2013). Intuitive thinking is a model of thinking that occurs quickly, spontaneously, automatically (Leron, 2014). Meanwhile, analytical thinking is a model of thinking which is conducted through a slow process related to mathematical rules. Analytical thinking is related to situations, practices, statements, ideas, theories, and arguments (Thaneerananon et al., 2016). The process of analytical thinking starts from observation, determining the supporting rules, and checking or rejecting intuitive responses (Sternberg et al., 2008). The supporting rules act as a guarantor for the students in giving reason for each step of the mathematical proof (Faizah et al., 2020a).

CONCLUSION

Based on the result of this research, it can be concluded that accommodation happens when students re-construct their knowledge based on the assimilation scheme through 5 steps of thinking process. The five steps are the identification; determining the mathematical rules to be used; carrying out the mathematical proving by means of symbol manipulation, review, and justification.

Therefore, the finding of this research can be used as a tool to develop students' knowledge in solving the mathematical proving problems through assimilation scheme and accommodation scheme to ensure that the proving is not conducted spontaneously. Students should understand the meaning of each symbol presents in the question to avoid misconception to the result of the proving that have been performed.

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