Heuristic Method for Minimizing Distance without using Calculus and Its Significance

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Abstract: A very common Applied Optimization Problem in Calculus deals with minimizing a distance given certain constraints, using Calculus, the general method for solving these problems is to find a function formula for the distance that we need to minimize, take the derivative of the distance function, set it equal to zero, and solve for the input value, that should most of the time, lead to the optimal solution. In this article we provide alternatives for solving some Applied Optimization Problems related to minimizing a distance, without the use of the Derivative from Calculus, and instead, using a "Reflection Principle" based on symmetry, Geometric properties, and heuristic methods.

INTRODUCTION

Let's be clear, it is not our intention to diminish the importance of Calculus as a fundamental tool for solving optimization problems, our purpose is to show that using our alternatives will facilitate uncovering Algebraic-Geometric properties and patterns, that are often overseen by students when using Derivatives to solve applied optimization problems. This properties and patterns when put together can be associated to real-life events, this is pedagogically important since it can help instructors better explain fundamental ideas and principles when teaching Calculus.

Over years teaching Calculus, I realized that students understand the principles of Calculus better when they can visualize the abstract fundamental ideas given in our Lectures as real-life events, this is because, as instructors, building upon something that our students already understand is a pedagogical advantage, based on this, I decided to investigate the relationship between the way light travels as an electromagnetic wave in our universe (see [2]) and the minimization of a distance function given some constraints, which is a very common problem when teaching Applied Optimization Problems in Calculus. Additionally, I investigated the relationship between a billiard ball's path before and after it hits an edge (crease) on an idealized frictionless billiard table (see [1] and [4]), it turns out the two real-life events mentioned before are strongly related to the "shortest path" problem in Calculus and related to each other as well.

METHOD

Since our intention is not to ignore Calculus as a fundamental tool for solving applied optimization problems, let's start by solving a classical basic problem about minimizing a distance using Calculus, then we will solve the same problem using our method, and gradually we will cover more challenging cases.

Problem 1

Paul's house is located at point A, the farm of his grandmother is located at point B, there is a river as shown in the figure below, every morning, Paul needs to go to the river, get water and bring it to his grandmother's farm, what is the length of the shortest path Paul should follow? See figure 1 below.

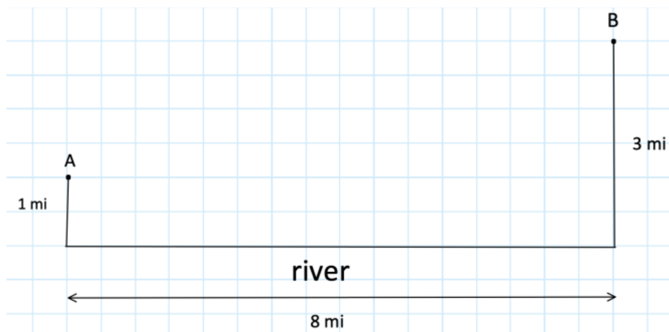

Figure 1: Classical problem in Calculus

A common solution to this classical problem using Calculus goes like this, let C be the point where Paul will reach the river, let x be the distance between O and C, see figure 2 below.

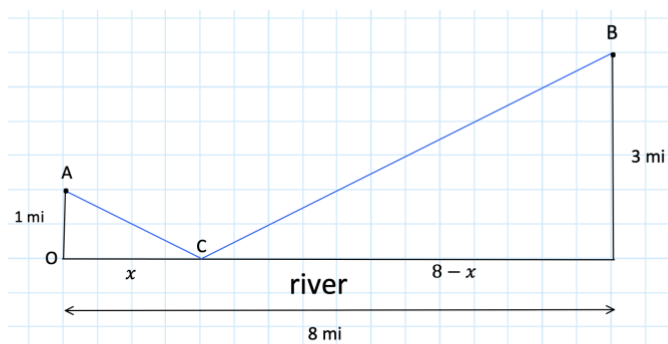

Figure 2: Common solution using Calculus

then we must minimize the distance $AC + CB$, equivalently, we must minimize the function:

$$f(x) = \sqrt{1+x^2} + \sqrt{(8-x)^2+9} \quad (1)$$

Taking the Derivative of $f(x)$, setting it equal to zero, and solving for x we obtain:

$$\frac{1}{2\sqrt{1+x^2}}\,2x+\frac{1}{2\sqrt{(8-x)^2+9}}\,2(8-x)(-1)=0 \tag{2}$$

$$\frac{x}{\sqrt{1+x^2}}=\frac{8-x}{\sqrt{(8-x)^2+9}} \tag{3}$$

$$\frac{x^2}{1+x^2}=\frac{(8-x)^2}{(8-x)^2+9} \tag{4}$$

$$x^2(8-x)^2+9x^2=(8-x)^2+x^2(8-x)^2 \tag{5}$$

$$9x^2=(8-x)^2 \tag{6}$$

$$3x=8-x \tag{7}$$

$$x=2 \tag{8}$$

For $0 \le x \le 8$, $f(x)$ is a continuous function.

The extreme values are $x=0$ and $x=8$, evaluating $f(x)$ at the extreme values and the critical point we obtain:

$$f(0)=\sqrt{1+0^2}+\sqrt{(8-0)^2+9}=9.54 miles. \tag{9}$$

$$f(2)=\sqrt{1+2^2}+\sqrt{(8-2)^2+9}=8.94\, miles. \tag{10}$$

$$f(8)=\sqrt{1+8^2}+\sqrt{(8-8)^2+9}=11.06\, miles. \tag{11}$$

The evaluations above show that $x=2$ is a critical point associated to a minimum value of $f(x)$, then we conclude that shortest path is $A \to C \to B$ having as its length $8.94\, miles$.

A graphical verification using OER DESMOS graphing calculator is shown below in Figure 3.

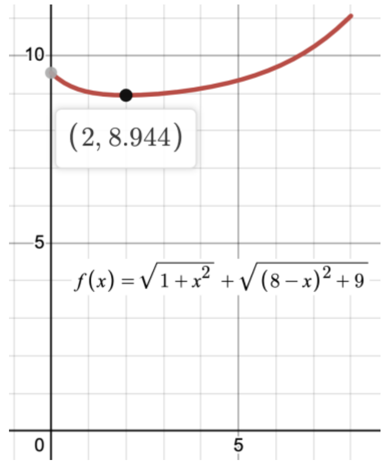

Figure 3: Graphical verification of the shortest path

The Reflection Principle

The minimum distance between two points is the length of a straight path that connects them, if we want to go from a point A to a point B via a sequence of straight paths touching once a line L as shown below, then ideally we may start at point A' the reflection of point A about the line L, this assumption can be made because moving towards line L from A is equivalent, in terms of distance covered, to move towards line L from A' given that the triangles formed above and below L are always congruent, this in turn leads to the equality of the reference angles formed by the paths and the line L, which makes the connection with the principle of reflection in physics, which shows that if a beam of light is aimed at a mirror positioned at line L, it will generate equal angles of incidence an reflection.

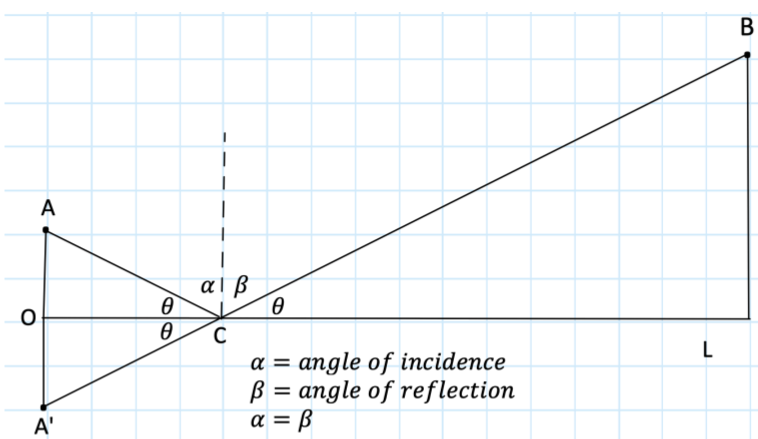

Figure 4: Explanation of The Reflection Principle

Now, we present an alternative solution to problem 1 that does not use Calculus, for this purpose we will use "The Reflection Principle".

Since Paul must reach the river anyway, following "the reflection principle" ideally we may assume that he can start his path at point A' the reflection of point A over the river (over the horizontal line that represents the river), now the shortest path to go from point A' to B is clearly the segment $A'B$, so where $A'B$ intersects the river that must be the point C, also since Triangle AOC is congruent to Triangle $A'OC$ then the path $A \to C \to B$ is the shortest path Paul should follow, see figure 5 below:

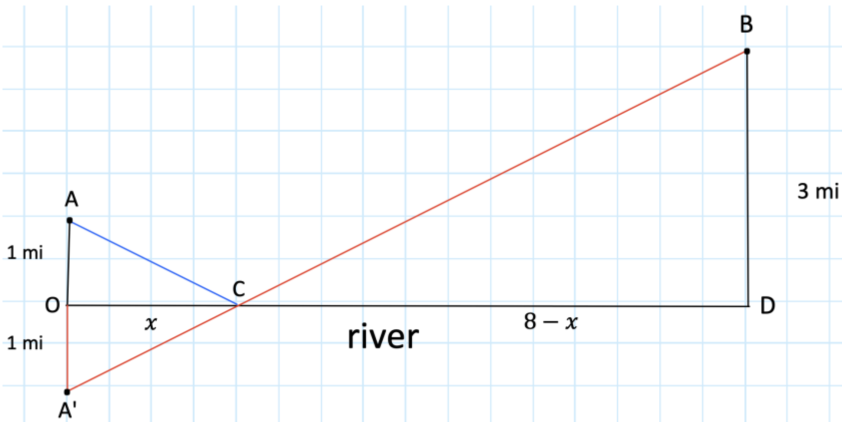

Figure 5: First application of The Reflection Principle.

Now, to determine how far is C from O, since triangle OCA' is similar to triangle BCD then

$$\frac{1}{x} = \frac{3}{8-x} \tag{12}$$

$$3x = 8 - x \tag{13}$$

$$x = 2 \tag{14}$$

Which leads to the shortest path $A \to C \to B$ having as its length:

$$\sqrt{1+2^2} + \sqrt{(8-2)^2+9} = 8.94\, miles. \tag{15}$$

Our alternative solution using "The Reflection Principle" has the following Educational and Computational advantages:

- It is Geometrically constructive; one can draw the shortest path using only straight edge and compass, this is particularly important because as Calculus instructors we can ask our students to build the "shortest path" as an in-class activity.
- It is algebraically simple, basically, we need to solve a rational equation which reduces itself to a linear equation, this means that this exercise could be given to students in a Precalculus class as a special group project.
- It shows that the reference angles formed by the path components with the horizontal line that represents the river are the same.
- It can be extended via "The Reflection Principle" to the solution of more challenging scenarios as we will show later.

Proposed Educational Activity

1. Give students problem 1 (or a similar problem) for them to solve individually or in groups.
2. Ask students to solve it using Calculus.
3. Explain the idea behind "The Reflection Principle".
4. Ask students to solve the same problem they solved in part 1 using "The Reflection Principle".
5. Ask students to draw "The minimum path" solution using only straight edge and compass.
6. Ask students to verify that the reference angles formed by consecutive paths with the horizontal line of reference are the same.
7. Ask students to measure the total length of the minimum path and compare it to the result they got in part 2.
8. Ask students to compare both methods and share comments about the activity.

Let's consider now a more challenging scenario where in order to go from point A to point B it is needed to touch once two perpendicular lines.

Problem 2

Suppose you have points $A(1,6)$ and $B(5,2)$ on an xy coordinate system, find the length of the shortest path to go from A to B touching first y-axis once and then x-axis once, see figure 6 below.

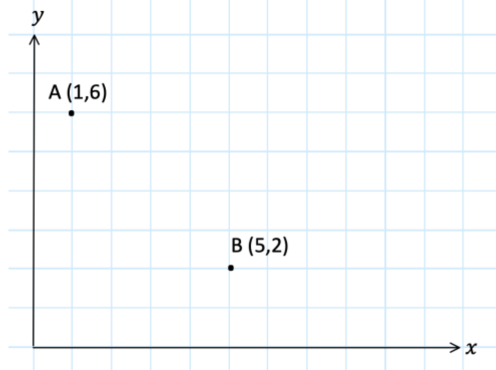

Figure 6: First extension of our basic problem.

Using our "Reflection Principle", ideally we may start at A' and end at B', which are the reflections of points A and B about y-axis and x-axis, respectively, clearly the shortest path between A' and B' is the segment $A'B'$, where $A'B'$ intersects the y-axis let's call that point C and where $A'B'$ intersects x-axis let's call that point D, then $A \to C \to D \to B$ is the shortest path, see figure 7 below.

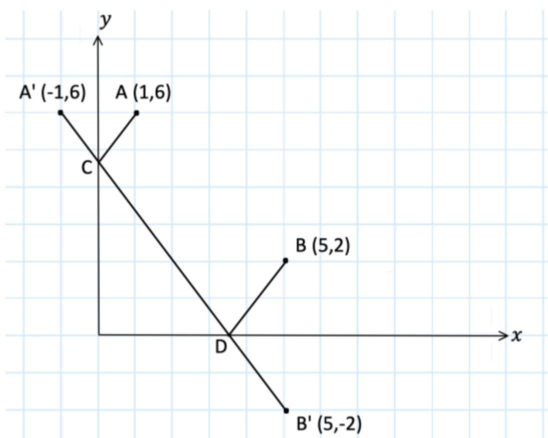

Figure 7: Second application of The Reflection Principle.

The equation of the line passing through the points $A'(-1,6)$ and $B'(5,-2)$ is $y = -\frac{4}{3}x + \frac{14}{3}$ with y-intercept $C = \left(0, \frac{14}{3}\right)$ and x-intercept $D = \left(\frac{7}{2}, 0\right)$ this determines the shortest path, then the length of the shortest path $A \to C \to D \to B$ can be calculated using the distance formula for the points A' and B' as shown below:

$$\sqrt{(5-(-1))^2 + (-2-6)^2} = 10\, units. \tag{16}$$

Remarks

- Solving Problem 2 using Calculus, may in general, require the minimization of a distance objective function that after being differentiated, could lead to a complicated algebraic equation to solve involving radicals, since we would have to deal with the sum of distance formulas as shown in the first solution of Problem 1.
- Our solution of problem 2 only used knowledge about equations of lines in an xy coordinate system and location of x-intercepts and y-intercepts, which makes problem 2 suitable for a group project once "The Reflection Principle" is explained.

How our solutions can be associated to real-life events

After solving problem 1 and problem 2 using "The Reflection Principle" a pattern emerges and a natural question comes to our minds, are these "shortest paths" related to any events in real life? and the answer to this question is yes, when we consider the path of light as an electromagnetic wave traveling in our universe, it turns out since light travels minimizing time at a constant speed (see [3]) then if we consider the pairwise perpendicular lines in problems 1 and 2 as mirrors, a beam of light from a flash light using the angles we found in the solutions of problems 1 and 2 will follow the same paths as we found for our solutions in problems 1 and 2, this can be confirm experimentally and also using the free OER applet in [2] https://phet.colorado.edu/sims/html/bending-light/latest/bending-light_en.html

Additionally, if we think of the game of billiards (see [1] and [4]) If we hit a billiard ball with the stick, using angles found in our solutions to problems 1 and 2, the paths followed by the billiard ball before and after hitting the edges of the billiard table will be the same as the "shortest paths" that we found in our solutions to problems 1 and 2, this of course, assuming a frictionless billiard table.

There is a way to link how light travels a as wave and the way a billiard ball hits and bounces off a crease:

A ball hitting a crease and bouncing off, acts as a wave of light reflecting off a mirror as seen in figure 8 below:

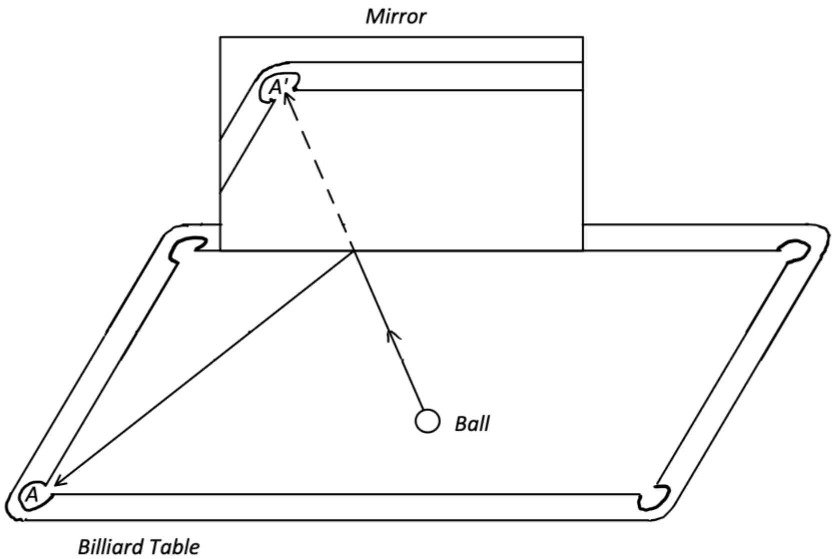

Figure 8: Explanation of the link between the paths of a ray of light and a billiard ball.

Let's consider now a step further challenging scenario where in order to go from point A to point B it is needed to touch once a sequence of pairwise perpendicular lines.

Problem 3

Suppose you have points $A(2{,}4)$ and $B(6{,}3)$ on an xy coordinate system, find the length of the shortest path to go from A to B touching first y-axis once, then x-axis once, and finally the vertical line to x-axis y'-axis once, see figure 9 below.

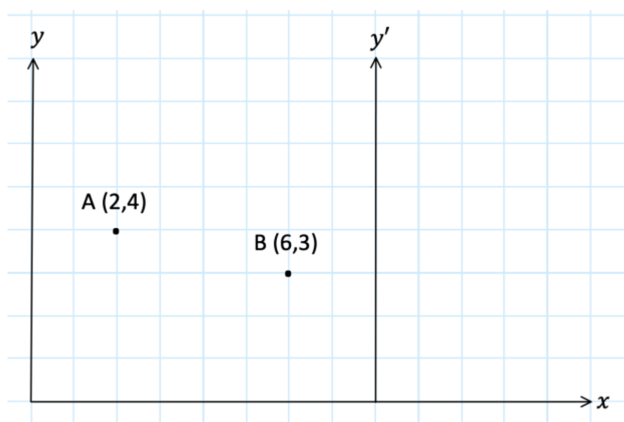

Figure 9: Second extension of our basic problem.

Using our "Reflection Principle", we may ideally start at A' and end at B', which are the reflections of points A and B about y-axis and y'-axis respectively, this in turn is equivalent to ideally starting at A' and ending at B'', where B'' is the reflection of B' about the x-axis. The point where $A'B''$

intersects y-axis let's call it C, and the point where $A'B''$ intersects x-axis let's call it D, the point where DB' intersects y'-axis let's call it E, then the path $A \to C \to D \to E \to B$ is the shortest path, to go from A to B touching once y-axis, x-axis and y'-axis respectively see figure 10 below:

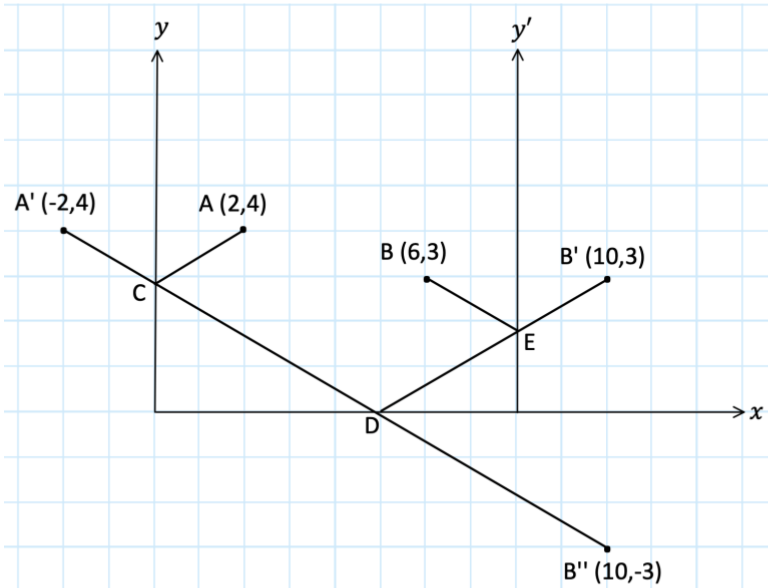

Figure 10: Third application of The Reflection Principle

The equation of the line passing through the points $A'(-2,4)$ and $B''(10,-3)$ is $y = -\frac{7}{12}x + \frac{17}{6}$ with y-intercept $C = \left(0, \frac{17}{6}\right)$ and x-intercept $D = \left(\frac{34}{7}, 0\right)$, the intersection of the line containing the points D and B' $y = \frac{7}{12}x - \frac{17}{6}$ with the vertical line $x = 8$ is the point $E = \left(8, \frac{11}{6}\right)$ this determines the shortest path, the length of the shortest path $A \to C \to D \to E \to B$ can be calculated using the distance formula for the points A' and B'' as shown below:

$$\sqrt{(10-(-2))^2 + (-3-4)^2} = 13.89\ units. \tag{17}$$

CONCLUSION

The methods shown to solve the problems presented without the use of Calculus, using "The Reflection Principle" have the following Educational and Computational overall advantages:

All our solutions for problems 1, 2, and 3 using "The Reflection Principle" are constructive, which means that we can geometrically draw the "shortest paths" using only straight edge and compass, this could be the basis for in-class activities, which ultimately, would give our students a "real-world sense" of what the solutions to the "shortest path" problems should be.

The lengths of the "shortest paths" shown in problems 1, 2, and 3 are similar to the paths that a beam of light from a flashlight would follow if we considered the horizontal and vertical lines as mirrors. This would allow students to confirm a principle that comes from physics, which states that light, in our universe, travels following "the shortest path" (see [2] and [3]).

A confirmation (an experimental test) for the "shortest paths" can be performed using a billiard table and a billiard ball, considering the angles found in our constructions of the "shortest paths", to hit a billiard ball towards the edges (creases) of the table and trace how they bounce off, as seen in [1] and [4].

Our method, based on "The Reflection Principle", admits a natural extension to solve more challenging problems related to finding "shortest paths" subject to given constraints, this was shown going gradually through problems 1, 2, and 3.

When finding the "shortest paths", the path components that are not consecutives are parallel, this is a consequence of the angles of "incidence" and "reflection" (see [1] and [4]) being the same when finding the path of the shortest length associated to light traveling as a wave or a billiard ball being hit by a stick on a billiard table towards the edges.

When solving Applied Optimization Problems like problem 1, 2 and 3 our method based on "The Reflection Principle" uncovers properties, patterns, and associations with real-life events, which are overseen by students when they "blindly" follow the algorithm: "take derivative, set it equal to zero, and solve for the variable".

All the above gives Calculus Instructors material to make up activities and projects for students to go over, individually or in groups, so they can see how theoretical principles in Mathematics can be tested in real life which serves as an instructional motivation, especially for undergraduate students who often ask for real-life examples associated to the abstract knowledge given to them in the lectures. For online classes in [2] there is an free OER applet https://phet.colorado.edu/sims/html/bending-light/latest/bending-light_en.html that students can use virtually, to verify the angles and "shortest paths" that we have shown as solutions to problems 1, 2, and 3.

Note: As of now because of the pandemic-related teaching limitations, it is not possible for me to fully implement the activities suggested in this article. Once the pandemic is completely over, I will use my Calculus class for an implementation of the suggested activities. The results could be the basis for a new article.

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