Editorial from Mónica Arnal-Palacián, Didactics Editor of MTRJ

Research in Mathematics Education is a field in continuous evolution. New, and not so new, perspectives have been appearing in recent years: interdisciplinary work at different educational levels, studies that identify the emotions of teachers and students in the mathematics classroom, mathematics teaching approached from an inclusive, innovative and reflective society, among others. Nor can we forget how the use of new technologies in the teaching and learning of mathematics enhances the dynamic management of some mathematical notions, or how some new contexts require adaptation through new methodologies. In the Mathematics Teaching-Research Journal (MTRJ) we aim to address all these types of studies, and to promote research in mathematics education that is of interest to experienced teachers and researchers as well as to novice and trainee teachers. Among these papers, we are particularly interested in studies that describe, analyse and integrate teaching and research in the classroom, and in which the teacher's voice is central.

In this issue, **Vol. 14 No. 5**, Winter 2022, MTRJ publishes 10 new manuscripts from 7 different countries: USA, Nepal, Indonesia, United Arab Emirates, Morocco and Malasya.

In the first one, Brandt and Columba (USA) explore students' motivation and the learning strategies they put in place in a blended learning environment with a paper entitled "**Choice in Blended Learning: Effects on Student Motivation and Mathematics Achievement**".

Subsequently, we find three articles that deal with mathematical notions linked to geometry. Thapa, Dahal and Pan (Nepal) use GeoGebra to help secondary school students remember and understand the notion of a circle, through group work and student motivation, "**GeoGebra Integration in High School Mathematics: An Experiential Exploration on Concepts of Circle"**. Herawati, Herman, Also, the paper entitled "**Concept of Polygon: Case Study of Elementary Students' Difficulties**" written by Suryadi and Prabawanto (Indonesia) determine from qualitative research the difficulties pupils have with the properties of polygons. Relating geometry to multiplication, we find the study "**A visualization approach to multiplicative reasoning and geometric measurement for primary-school students: a pilot study**". Jain, Leung y Kamalov (UAE) provide tasks that encourage the application of multiplicative reasoning when students are asked to measure the areas of geometric figures.

The next paper was written by El Guenyari, Chergui and El Wahbi (Marocco) with the title "**A study on the effectiveness of some cognitive activities in teaching integrals in secondary school"**. They analysed three levels of the Moroccan educational curriculum on the mathematical notion of the integral: the design, the implementation and what is supposed to be learned. A mathematical notion of calculus is also addressed by Sulastri, Sulastri, Prabawanto and Cahya (Indonesia). In their manuscript, "**Epistemological Obstacles on Limit and Functions Concepts: A Phenomenological Study in Online Learning"**, they explore the barriers to learning various concepts related to function limits that are misunderstood by secondary school students,

From Indonesia, Jabar, Gazali, Ningrum, Atsnan and Prahmana, present an article entitled "**Ethnomathematical Exploration on Traditional Game Bahasinan in Gunung Makmur Village the Regency of Tanah Laut"**. This research analyse the traditional game Bahasinan, which is to be played in a group, and with which some mathematical concepts, e.g., rectangles, can be developed.

Furthermore, Yazgan-Sağ (Turkey), with the title "**Views on Mathematical Giftedness and Characteristics of Mathematically Gifted Students: The Case of Prospective Primary Mathematics Teachers",** interview with prospective teachers linking mathematical giftedness, social environment and effort.

The ninth article, entitled **"Children's Errors in Written Mathematics",** was written by Liew, Leong, Julaihi, Lai, Ting, Chen y Hamdan (Malasya). The authors consider the errors in mathematics are essential to learning, design and contextualise new instruction based on written errors in a primary school classroom on numbers, operations and statistics. After that, we find the article by Pacheco-Muñoz, Nava-Lobato, Juárez-López and de León-Palacios (México), who in their article entitled **"Division problems with remainder: A study on strategies and interpretations with fourth grade Mexican students"** analyse the responses of primary school students in relation to the resolution and interpretation of non-routine problems, precisely measured division and division-partition with remainder.

Finally, in the **Problem Corner section**, our Problem Corner Editor Ivan Retamoso presents the best solutions received for Problems 8 and 9, hoping to enrich and improve the mathematical knowledge of our international community. Two new problems have been proposed.

Mónica Arnal-Palacián
Didactics Editor of MTRJ

Contents

Choice in Blended Learning: Effects on Student Motivation and Mathematics Achievement

Edward C. Brandt, Lynn Columba

Lehigh University (USA)

ecb415@lehigh.edu, hlc0@lehigh.edu

Abstract: Over the past decade, there has been an urgent calling for the improvement of teaching and learning of mathematics. At that same time, research has provided promising findings in the value of self-regulation and choice on learning for student achievement. With the ever-increasing availability of resources for educators to support student learning, this study explores student motivation, learning strategies and learning associated with student choice in a blended learning environment. A quantitative quasi-experimental designed study revealed no significant difference in student motivation, learning strategies, or mastery of geometry content between groups of students.

INTRODUCTION

The effective learning of mathematics is crucial for students to acquire valuable life skills. Critical assessments point to the lack of student success in acquiring such skills, thus providing evidence of a need for improved mathematics education. Results of the 2019 National Assessment of Education Progress (NAEP) posted that only 34% of eighth grade students tested scored at or above proficiency (National Center for Education Statistics, 2019). In a study focused on the success and failure of community college students working towards associate's degrees, mathematics was identified as "a major stumbling block and gatekeeper area" (Hudesman et al., 2013, p. 2). As further support in the value of mathematics education, research conducted by Evans, Kochalka, Ngoon, Wu, Qin, Battista, and Menon (2015) revealed a relation between specific areas of brain structure and the numerical competency of children and adolescents. The researchers extend the study's significance in pointing out mathematical literacy as a key component in the foundation of "future academic and professional success in an increasingly technological society." (2015, p.11743)

Educators have the challenge and duty to meet every child's learning needs. In order to best meet those needs, teachers must serve as designers of the learning experiences rather than simply the main conduit of information. Effective lesson design not only helps in supporting student learning of mathematics, but it aligns with Iwuanyanwu's (2021) challenge that mathematics education must support the "avoidance of leading the learner to frustration, aversion, and … hatred not only of mathematics as a subject but also … of the teacher who handles mathematics" (Iwuanyanwu, 2021). Blended learning can equip educators with the most effective mix of tools to meet those needs as it can provide a variety of elements in a given learning experience. Reporting on a study of the effect blended learning had with secondary level science education, Longo (2016, p.35) pointed out that "by providing various approaches to learning, students are not confined to one … style", which is "particularly true when teachers allow for choice" (p. 35). Blended learning promises to be an approach that provides students with the opportunity of choice in their classroom learning experience, rather than being restricted to following a fixed path designed by an educator. Following fixed learning paths is an element of what Glasser (1998, p.540) would consider as obedient schooling as opposed to a more effective "useful education" aligned with the principles of Choice Theory. For this study, blended learning is defined as a purposeful mix of learning modalities, including but not limited to online tools, flipped model of lesson design, cooperative learning groups, and direct instruction.

LITERATURE REVIEW

A void exists in the literature regarding the effects of blended learning designs and strategies on student learning in secondary mathematics. While there are studies that provide support in the use of blended learning and choice as effective practices for students learning, not one is associated directly with blended learning and choice in secondary mathematics education. A study was conducted of college science courses in which students learned within a blended environment of videotaped lectures and in-class workshop time. Despite no significant difference found in summative assessment scores, students in the blended learning classes demonstrated higher levels of engagement compared to students in a more traditional learning environment (Baum, 2013). Staying with the theme of increased student engagement, in a study of secondary mathematics students, specifically two high school classes and one seventh-grade class, the blended learning experience followed a rigid flipped model that afforded students no choice of learning modality. Specifically, each student was assigned videotaped lectures to be viewed prior to class so that in-class time would be focused on students working and the teacher providing support and assistance as needed through monitoring (Hodgson et al., 2017). Overall, the research revealed a possible link between the flipped-model blend and student engagement. However, some of the results supported no difference between flipped and traditional classrooms, leading to a need for further study. Absent from this study was any type of student choice in the specific blend in the design of the learning experience.

There have been promising reports in the literature focused on student learning in college level courses as well. A study on the effects of blended learning and the flipped model used in the design of lessons in numerical methods courses at three engineering programs yielded noteworthy results. Specifically with respect to the identification of an enhanced understanding and a motivation for learning among students interviewed in focus groups (Clark et al., 2018, p.12). In analyzing interventions geared towards student success in mathematics, researchers pointed out that "the majority of …interventions have emphasized content competency to the exclusion of … self-regulatory strategies" (Hudesman et al., 2013, p.3). This identifies not only a need for self-regulatory strategies such as choice in learning interventions, but further research in its effectiveness as well. Findings from another study on college level students point to the conclusion that self-regulatory strategies purposely woven into various interventions contributed to successful learning of the objectives in a chemistry course (Zhao et al., 2014). The authors encourage a model of embedding self-regulatory strategies, with choice being among them, that are incorporated into the learning design for any content. Research of self-regulating strategies across many disciplines at the high school level, including reading, writing, science and mathematics, point out some positive effects in the overlap of aspects of metacognition and self-regulation, but concludes that more studies are warranted for content-specific research (Greene et al., 2015). Similar conclusions were found in a study of secondary physics students who demonstrated a positive efficacy and stronger confidence in solving mathematics problems if they followed metacognitive strategies throughout their learning and problem solving process (Di Camillo et al., 2020).

There is promising research regarding blended learning as an effective design for a wide range of learners. Research also supports the value of choice in the learning experiences of students and its relationship to motivation. When each of the middle school students was granted more autonomy through the offering of choice in unstructured physical education, a maximization of student motivation to engage in physical activities on a long-term basis was revealed (Kinder et al., 2020). Similarly, secondary chemistry students afforded the freedom to develop their own learning plans as well as choose their strategies to perform had increased levels of motivation and enthusiasm (Vogelzang et al., 2019). However, there is a significant lack of literature when it comes to student choice in the blended learning design. A similar need for research is in the area of secondary mathematics. This study provides a necessary step to fill that void in a way that would inform educators associated with such a critical area of learning.

The purpose of this study was to analyze the data associated with secondary mathematics students' ability to make effective choices when given agency in a blended learning environment. In doing so, this research addresses the following questions:

1. What differences exist in attitudes about learning mathematics between students who have choice in blended learning paths compared to those without choice, as measured by a motivation and learning strategies questionnaire?
2. What differences in achievement in learning mathematics exist between students who have choice in learning paths compared to those without choice, as measured by a post-treatment assessment?

METHOD

Blended learning offers a wide range of tools and strategies to educators as they design effective learning experiences for students. Teachers have numerous choices to make in creating lessons to support the success of as many students as possible. One strategy educators can employ in this process is to put the choice of the modality of learning experience into the hands of the students themselves. Providing students with choice in their learning modality supports student agency, which "provide rich learning spaces for students and teachers" (Vaughn, 2020). Glasser's choice theory (1998), a foundational piece to this study, provides learners with the opportunity to select from a menu of learning modalities. This research supports the notion that "with more choices and much less schooling, [students] may do very well" (Glasser, 1998, p.475). Bray & McClaskey (2013) support the assertion that student choice and voice are key elements to successful personalized learning models. By measuring the effect of the combination of blended learning tools and student choice in learning modality has on learning, educators gain valuable information to capitalize on the opportunity of providing students with the wide variety of learning modalities present among the online and more traditional tools.

Research Design

A quantitative, quasi-experimental design model, with two groups selected through convenient sampling, was used to address the research questions. Through a random selection process, one class was selected as a treatment group, while two other classes collectively served as the control group. A post-experiment questionnaire was provided to all assenting students, along with a common mathematics content assessment. Quantitative analyses were conducted on the results of the questionnaire and the assessment in alignment with each research question.

Setting and Participants

This study took place at a large suburban high school in northeastern United States. Participants were tenth-grade geometry students enrolled in the same course. To reduce variability, a convenience sample of three classes of geometry was selected, with the students placed in the classes through partially random assignment as part of the school's scheduling program algorithm. The classes were similar in profiles in that they had the same teacher, they had comparable class sizes, 22, 22, and 23 respectively, with students from similar mathematics backgrounds, and were physically in the same classroom with identical resources. Through a random number generator, one class was selected to be the treatment group, while the other two served collectively as the control group.

Instruments

All students and parents in the three classes were provided with appropriate assent and consent forms, respectively. All assenting students were issued an adapted version of the Motivated Strategies for Learning Questionnaire (MSLQ) (see Appendix A for the adapted versions) that relates to the purpose of this study (Pintrich et al., 1991). The MSLQ consists of Likert scale items with a range of indicators from 1 for "not at all true of me" through 7 for "very true of me." Two specific sections of the MSLQ aligned to the research question were used in this study. The two sub-sections were on Motivation and Learning Strategies. A check for data integrity was supported through Cronbach's alpha values ranging from .52 to .93, providing acceptable measures of reliability of the tool. Appendix B provides the specific MSLQ subsections that were used along with corresponding data. Additionally, students were administered a common departmental assessment to measure students' learning of the content focused on during the timeframe of the study. This summative assessment was designed by certified mathematics teachers in this school to measure the level of understanding of content associated with right triangles identified by the Common Core State Standards for Mathematics (2010). Specifically, the standard of G-SRT: Similarity, Right Triangles, and Trigonometry was assessed by the tool.

Procedure

Assent was requested of all students in the three identified class sections for the study (see Appendices B and C for sample forms), along with consent from their parents or guardians. Anonymity was secured through the use of Google Forms to collect the questionnaire responses without collecting student identifiable email addresses. Over the course of the school year, students in all of the geometry classes at this school gained experience with blended learning modalities through a number of units of study developed as part of routine practice by the teacher. The lessons, specifically the learning tools, modality, and sequence of experiences, were designed by the classroom teacher. Each consisted of online, interactive programs for instruction, review, and/or assessment, along with in-person experiences through methods such as cooperative group learning, individual focused sessions, teacher-led mini-lectures (approximately 20 minutes in length) and modeling sessions. In previous lessons leading up to the study, students experienced each of the methods of learning that were ultimately among the choices for the experimental group or assigned as part of the control group.

To begin the unit of focus for the study, namely on the content standard of Similarity, Right Triangles, and Trigonometry, the students in the treatment group were provided the opportunity to choose their own learning path from a menu of offerings developed by the teacher. The menu consisted of the following choices: student works individually with personal control of resources that include online tutorials, interactive applets, paper-based instruction and practice with feedback from peers or teacher; student works in a small group with similar control of the afforded resources; or student works in small group that is led by the classroom teacher as far as which resources are used and when. In order to facilitate the students' selections, the classroom teacher arranged the classroom based on the menu choices. At the conclusion of the four days, all participating students completed the questionnaire via Google Forms and the post-assessment of their mastery of the content through the departmental paper-based assessment. The post-assessment, designed by a professional learning community of Geometry teachers at the school, was administered to all students in the class, with results for the study participants extracted for analysis.

Meanwhile, the students in the control group experienced learning of the same concepts in the focus unit in a similar manner in which they had all school year leading up to this study, which was teacher-designed with a variety of teacher-led lecture, workshop, online self-directed, flipped, and cooperative group modalities. The learning paths for the control group were designed and maintained throughout the study by the teacher, without any student choice. At the conclusion of the first four days of the focal unit of this study, all participating students in the control group completed the questionnaire via Google Forms and were assessed of their mastery of the content through the departmental paper-based assessment in the same manner as the treatment group.

Data Analysis

At the conclusion of the first four days of the unit, when the treatments were applied, both the experimental and the control groups were administered the appropriate portions of the MSLQ.Results were analyzed for significant differences in student perceptions pertaining to each subsection of the questionnaire, namely motivation and learning strategies. Two separate two-tailed *t*-tests were conducted to check for significant differences among groups. The first *t*-test compared the difference in levels of motivation and satisfaction of learning styles from students who had a choice in their learning paths against those who did not. The second *t*-test compared the difference in content and skill assessment results between the students who had choice in their learning paths and those who did not. All data were screened for missing values, revealing that there were two students in the control group who responded to the questionnaire, but did not take the content assessment during the time of the study.

RESULTS

Research Question 1: What differences exist in learning mathematics between students who have choice in blended learning paths compared to those without choice, as measured by a motivation and learning strategies questionnaire?

Analysis was conducted on the data to address each of the research questions via a two-tailed *t*-test for difference in means between the students who had choice in their learning paths and the students who did not. The mean score for each student was determined by taking the mean value of the 19 questionnaire prompts that measured motivation. The scores from the questionnaire associated with students' motivation in learning are as follows: The students in the treatment group, who had a choice of learning paths ($n = 7$, $M = 4.94$, $SD = 0.84$), were comparable to students in control group who had no choice in their learning path ($n = 21$, $M = 5.03$, $SD = 0.75$).

When the focus in the questionnaire was on learning strategies, the scores with the treatment students who had choice were as follows ($n = 7$, $M = 4.26$, $SD = 1.06$). The scores for the students in the control group are ($n = 21$, $M = 3.98$, $SD = 1.00$). The mean score for each student was determined by taking the mean value of the 17 questionnaire prompts that focused on students' learning strategies. Table 1 provides a concise listing of the results.

	Treatment				Control			
	n	*M*	*SD*	*SE*	*n*	*M*	*SD*	*SE*
Motivation	7	4.94	0.84	0.32	21	5.03	0.75	0.16
Learning Strategies	7	4.26	1.06	0.40	21	3.98	1.00	0.22

Table 1: Descriptive Statistics of Questionnaire

Trends in the data revealed that students who had a choice in their learning path scored lower on the motivation section of the survey (M = 4.94, SE = 0.32) than the students who were directed to follow the path designed by the teacher (M = 5.03, SE = 0.16). The difference, -0.09, BCa 95% CI [-0.889, 0.714], was not significant $t(26)$ = -0.25, p = .810; additionally, it represented a small-sized effect, d = 0.12. Conversely, students who had choice in their learning path scored higher on the learning strategies portion of the survey (M = 4.26, SE = 0.40) than the students who followed the teacher-designed experience (M = 3.98, SE = 0.22). The difference, 0.28, BCa 95% CI [-0.737, 1.303], was not significant $t(26)$ = 0.62 , p = .550; however, it represented a larger effect than in motivation, d = 0.28.

Research Question 2: What differences in achievement in learning mathematics exist between students who have choice in learning paths compared to those without choice, as measured by a post-treatment assessment?

Results of the mathematics assessment are as follows: Students with choice in their learning path (n = 7, M = 40.50, SD = 6.10); Students in the control group who did not have choice as part of their learning experience (n = 19, M = 41.16, SD = 7.29).

	Treatment				Control			
	n	*M*	*SD*	*SE*	*n*	*M*	*SD*	*SE*
Content Assessment	7	40.50	6.10	2.31	19	41.00	6.69	1.67

Table 2: Descriptive Statistics of Content Assessment

Students who had choice in the design of their learning paths scored lower on the mathematics assessment (M = 40.50, SE = 2.31) than the students who were directed to follow the path designed by the teacher (M = 41.16, SE = 1.67). The difference, -0.66, BCa 95% CI [-6.824, 5.508], was not significant $t(24)$ = -0.23, p = .820; additionally, it represented a small-sized effect, d = 0.09.

DISCUSSION AND CONCLUSION

There is insufficient evidence of a significant difference in the students' attitudes towards learning mathematics given a choice in their learning modalities based on the results of the questionnaire. However, close analysis of the data provided valuable information. For example, the data showed that the students in the experimental group, which received both metacognitive experiences as well as the ability to choose their learning paths in a new unit, had the highest mean score in the learning strategies questionnaire. Findings contribute to the research that a blended learning model has a positive effect on students' motivation for learning (Yagci, 2016) and with a meta-analysis that revealed a positive correlation between students' metacognitive processes and their performance on academic tasks (Dent & Koenka, 2015). The lack of any significant difference between the groups may be attributed to the amount of choice afforded the students, supporting research that too many choices may contribute to added stress to the students in the learning process (Aiken et al., 2017). A more "collaborative" approach, as described in their research, offers choice in the design of the learning paths without overwhelming the learners in the process. The findings of this study and previous research strengthen the call to provide students with a quality education that includes choice as opposed to "coerced schooling" (Glasser, 1998).

For the second research question, the data demonstrated a lack of significant effect on the students' learning of mathematical content. This aligns with Cabi's (2018) research on students who used a specific blended learning version, namely the flipped classroom model, as compared to students who learned more directly from the classroom teacher. Conversely, Alsancak Sirakaya and Ozdemir (2018) revealed that students in a flipped-model learning environment achieved significantly greater than students in a more traditional blended learning environment. In this study, blended learning is defined as a purposeful mix of learning modalities. Perhaps the right mix is critical as seen when students, who learned the mathematics because they were required to prepare to teach it to others, demonstrated higher achievement in problem solving in mathematics than students who learned solely for themselves (Muis et al., 2015).

The research questions associated with this study focused on student motivation and learning strategies, as well as learning of content. Although neither question yielded a substantial effect, the study may have value in line with literature associated with student choice and its positive effect on adolescents' perceptions of autonomy (Williams et al., 2016).

The data indicated no significant differences between groups in measuring levels of motivation, learning strategies, and content knowledge. However, results are promising in that students who had a choice in their modality of learning did not suffer any setbacks with respect to their scores on summative assessments. Although this research strived to analyze the effect student choice in blended learning design has on student learning of mathematics, it does still recognize the value an effective mathematics teacher has on designing learning experiences for students. In this case, students who did not have a choice in their learning paths were not merely assigned a set of lessons in a random fashion. Instead, they experienced learning modules as designed by a certificated, experienced mathematics teacher who carefully matched the best methods they thought would help these specific students with the district curriculum. In either case, it would be wise to be prudent in the overall number of modalities within each choice offered so as to not overwhelm the learner. As indicated in research on the effects of the amount of modalities and online tools infused in lesson designs, Wang (2021) found that students with a lesser number of learning technologies performed better on assessments than those with too many tools (Wang, 2021).

Limitations of the study included the small number of students involved in the study, which may be a reason for the large margin of error in the statistical analyses. While this researcher was fortunate to find one teacher who had three classes of the same course, only a fraction of each of the classes assented to participate in the study. Students who did not provide appropriate assent were still included in the learning and assessment design, but did not complete the questionnaire. Furthermore, the duration of the study may have been too brief for any significant learning to take hold. Lastly, another limitation of the study is that it is only quantitative in structure for both research questions, therefore leaving a void in any informative anecdotal data and themes that may typically come from qualitative interview techniques.

Future Considerations

The limitations identified provide insights for considerations in any future research. Previous research, along with findings from this study, provide direction for future analysis regarding the value of student choice of blended learning design as a powerful, effective combination in the learning of mathematics. Expanding the duration of treatments designed to build students' self-regulatory development, along with the time for students to absorb and thoroughly learn the concepts, arise as key elements for future research designs.

Additionally, the inclusion of qualitative methods such as interviews of students will widen the breadth of information available to educators working to meet the needs of learners. The questionnaire used in this study did not gain access into the reasoning students had for their choices made in the learning paths, specifically what motivations they had for their selections. Similarly, students in the groups who had no choice were not asked about how they felt about being directed to follow a strict path designed by their teacher. Such insight from the learners would provide valuable depth to the current research available as well as constructive feedback for the educators involved in this study.

REFERENCES

[1] Aiken, K. D., Heinze, T. C., Meurer, M. L., & Chapman, K. J. (2017). The impact of collaboration, empowerment, and choice: An empirical examination of the collaborative course development method. *Marketing Education Review*, 27(1), 39-50.

[2] Alsancak Sirakaya, D. & Ozdemir, S. (2018). The effect of a flipped classroom model on academic achievement, self-directed learning readiness, motivation and retention. *Malaysian Online Journal of Educational Technology*, *6*(1), 76-91.

[3] Baum, E. J. (2013). Augmenting guided-inquiry learning with a blended classroom

approach. *Journal of College Science Teaching, 42*(6), 27-33.

[4] Bray, B., & McClaskey, K. (2013). A step-by-step guide to personalize learning. *Learning and Leading with Technology.*

[5] Cabi, E. (2018). The impact of the flipped classroom model on students' academic achievement. *International Review of Research in Open and Distributed Learning, 19*(3), 202-221.

[6] Clark, R., Kaw, R., Lou, Y., Scott, A., & Besterfield-Sacre, M. (2018). Evaluating blended and flipped instruction in numerical methods at multiple engineering schools. *International Journal for the Scholarship of Teaching & Learning, 12*(1), 1-16.

[7] Coles, A. (2013). On metacognition. *For the Learning of Mathematics, 33*(1), 21-26.

[8] Daher, W., Anabousy, A., & Jabarin, R. (2018). Metacognition, positioning and emotions in mathematical activities. *International Journal of Research in Education and Science, 4*(1), 292-303. doi:10:21890/ijres.383184.

[9] Di Camillo, K. & Dawson, V. 2020. Using metacognitive strategies to support the application of mathematics skills of three Year 11 physics students. *Teaching Science, 6*(2), 43-54.

[10] Dent, A. L., & Koenka, A. C. (2016). The relationship between self-regulated learning and academic achievement across childhood and adolescence: A meta-analysis. *Educational Psychology Review, 28,* 425-474. doi:10.1007.

[11] Evans, T. M., Kochalka, J., Ngoon, T. J., Wu, S. S., Qin, S., Battista, C., & Menon, V. (2015). Brain structural integrity and intrinsic functional connectivity forecast 6 year longitudinal growth in children's numerical abilities. *The Journal of Neuroscience, 35*(33), 11743-11750. doi:10.1523/jneurosci.0216-15.2015.

[12] Field, A. (2013). *Discovering Statistics Using IBM SPSS Statistics, 4th Edition*. Los Angles: Sage.

[13] Glasser, W. (1998). *Change theory: A new psychology of personal freedom* [HarperCollins ebooks version]

[14] Greene, J. A., Bolick, C. M, Caprino, A. M., Deekens, V. M., McVea, M., Yu, S., & Jackson, W. P. (2015). Fostering high school students' self-regulated learning online and across academic domains. *The High School Journal, 99*(1), 88-106.

[15] Hodgson, T. R., Cunningham, A., McGee, D., Kinne, L. J., & Murphy, T. J. (2017). Assessing behavioral engagement in flipped and non-flipped mathematics classrooms: Teacher abilities and other potential factors. *International Journal of Education in Mathematics, Science and Technology, 5*(4), 248-261.

[16] Hudesman, J., Crosby, S., Flugman, B., Everson, H., & Clay, D. B. (2013). Using formative assessment and metacognition to improve student achievement. *Journal of Developmental Education, 37*(1), 2-13.

[17] Inglis, M., & Foster, C. (2018). Five decades of mathematics education research. *Journal for Research in Mathematics Education, 49*(4), 462-500.

[18] Iwuanyanwu, P. (2021). Contemporary problems of teaching and learning in mathematics education. *Mathematics Teaching Research Journal, 13*(2), 23-35.

[19] Kinder, C. J., Gaudreault, K. L., & Simonton, K. (2020). Structured and unstructured contexts in physical education: Promoting activity, learning and motivation. *Journal of Physical Education, Recreation & Dance, 91*(6), 30-35. doi:10.1080/07303084.

[20] Longo, C. M. (2016). Changing the instructional model: Utilizing blended learning as a tool of inquiry instruction in middle school science. *Middle School Journal, 47*(3), 33-40.

[21] Muis, K. R., Psaradellis, C., Chevrier, M., Di Leo, I., & Lajoie, S. P. (2015). Learning by preparing to teach: Fostering self-regulatory processes and achievement during complex mathematics problem solving. *Journal of Educational Psychology, 1*(1), 1-19. doi:10.1037.

[22] National Center for Education Statistics. (2019). *NAEP mathematics & reading assessments: Highlighted results at grades 4 and 8 for the nation, states, and districts.* Retrieved from https://www.nationsreportcard.gov/highlights/mathematics/2019/

[23] National Governors Association Center for Best Practices & Council of Chief State School

[24] Officers. (2010). *Common Core State Standards for Mathematics.* Washington, DC: Authors.

[25] Pintrich, P. R., Smith, D. A., Garcia, T., & McKeachie, W. J. (1991). *A Manual for the Use of the Motivated Strategies for Learning Questionnaire (MSLQ)*. Ann Arbor, MI: University of Michigan, National Center for Research to Improve Postsecondary Teaching and Learning,

[26] Purnomo, D., Nusantara, T., Subanji, & Rahardjo, S. (2017). The characteristic of the process of students' metacognition in solving calculus problems. *International Education Studies, 10*(5), 13-25. doi:10.5539.

[27] Vaughn, M. (2020). What is student agency and why is it needed now more than ever? *Theory into Practice, 59*(2), 109-118. doi:10.1080/00405841.

[28] Williams, J. D., Wallace, T. L., & Sung, H. C. (2016). Providing choice in middle grade classrooms: An exploratory study of enactment variability and student reflection. *Journal of Early Adolescence,* 36(4), 527-550.

[29] Vogelzang, J., Admiraal, W. F., & van Driel, J. H. (2019). Scrum methodology as an

effective scaffold to promote students' learning and motivation in context-based secondary chemistry education. *EURASIA Journal of Mathematics, Science and Technology Education, 15*(12), 1-13. doi:10.29333.

[30] Yagci, M. (2016). Blended learning experience in a programming language course and the effect of the thinking styles of the students on success and motivation. *The Turkish Online Journal of Educational Technology, 15*(4), 32-45.

[31] Wang, H. (2021). What works and what does not: A reflective practice on an online mathematics class. *Mathematics Teaching Research Journal, 13*(1), 16-30.

[32] Zhao, N., Wardeska, J. G., McGuire, S. Y., & Cook, E. (2014). Metacognition: An effective tool to promote success in college science learning. *Journal of College Science Teaching, 43*(4), 48-54.

GeoGebra Integration in High School Mathematics: An Experiential Exploration on Concepts of Circle

Raju Thapa[1], Niroj Dahal[2*], Binod Prasad Pant[2]

[1]Kaasthamandap Vidhyalaya, Mandhikhatar, Kathmandu, Nepal

[2]Kathmandu University School of Education, Department of STEAM Education, Hattiban, Lalitpur, Nepal

*Corresponding Author: niroj@kusoed.edu.np

Abstract: This paper is an outcome of using GeoGebra in Nepal's grade ten math curriculum (CDC, 2015). Based on a teaching experiment as one of the research methodologies (Dahal, 2019; Dahal et al., 2019; Dahal et al., 2022a), this paper provides students with active learning environments and the possibility of integrating GeoGebra as an ICT application by documenting the experiences, assessment procedures, emotions, and behaviors, as well as the learning process of eighteen secondary level students (ten boys and eight girls). In this regard, GeoGebra is a computer and online-based application that teaches geometry, algebra, and statistics. GeoGebra's features could help students visualize abstract geometric concepts quickly, correctly, and effectively (Tamam & Dasari, 2021, Dahal et al., 2022a). Theoretically grounded on social constructivism, this paper informs that GeoGebra helps students recall and understand circle terminology, encourages engaged learning through group work, and promotes meaningful learning, conceptual learning, learner-centered teaching, and student motivation. The benefits of using GeoGebra to teach and learn the circle concepts are demonstrated in the experimental classes and reported in this paper. Students become more active builders of the mathematical knowledge of the circle while teaching/learning the concepts of the circle using GeoGebra. It is a crucial ICT tool for supporting innovative approaches to teaching and learning mathematics in the twenty-first century.

Keywords: GeoGebra, circle, teaching experiment, constructivism, innovate approaches

INTRODUCTION

Mathematics teachers and learners present challenges are visualizing abstract geometrical concepts and ideas quickly, correctly, and effectively (Dahal et al., 2022a). This challenge creates an ample opportunity for mathematics teachers and students to learn mathematics using available ICTs tools. Integrating the technological tools to the mainstream of the mathematics classes is one of the ways to connect the conceptual and procedural understanding in the concepts of the circle (Maharjan et al., 2022). Technology integration is likely to be the relevant and innovative way of interactive teaching and learning (Agyei et al., 2021), as technologies are aligned with daily actions. In Nepal, some mathematics teachers are somewhat limited to the chalk-and-talk method of teaching mathematics. For instance, "technology has become an essential component of sustaining civilization, and its incorporation into education is consequently unavoidable" (Ghory & Ghafory, 2021, p. 168).

Likewise, in the professional career of the first author, he realized that students lack the visualizing abilities to understand the concept of the circle. He provides the necessary materials to help students comprehend the ideas of circles, but he notices that they struggle to use those concepts to solve problems and explain the meaning of such concepts while administering various assessments (e.g., quizzes, tests and projects). This might be because of teaching the concepts of the circle through a traditional method, which shall hinder students from manipulating and conceptualizing the circle's properties and visualizing those concepts with animation features. Battista (1999) found that students faced challenges in studying geometrical concepts and struggled to grasp the concepts and required knowledge because of the traditional ways of teaching and learning. Explanation, fear of punishment, textbook-based instruction, and homework and classwork rituals are all part of the traditional teaching methods. Likewise, the first author's students have always struggled to understand geometrical concepts, especially circles. Hence, we have acknowledged the necessity of incorporating innovative pedagogies by incorporating some form of technology. So, there is a need to modify or challenge the existing traditional teaching practices to meet the needs and desires in the era of industrial revolution 4.0 (Tapscott, 2008; Dahal et al., 2020). In the era of industrial revolution 4.0, there are a lot of different kinds of ICT tools used by students when they interact with each other and with teachers (e.g., flipped classrooms, mobile apps, and clickers devices) (Dahal et al., 2022b). GeoGebra is one of the available online and offline tools for visualizing mathematical concepts or the concepts of the circle.

GeoGebra application has become a part of the curriculum in higher secondary education in many countries (Shrestha, 2017a; Shrestha, 2017b; Dahal et al., 2019). The advantages of using GeoGebra, according to Dikovic (2009), are its user-friendly interface, ability to encourage students in project and discovery learning, ability to help students to enjoy ownership of their own creation, ability to encourage collaborative learning, ability to help in visualization of abstract nature of mathematics and ability to engage students in the active construction of knowledge by manipulating variables in GeoGebra (Dahal et al., 2022a). So, GeoGebra has been used to teach mathematics as a pedagogical tool (Putra et al., 2021). A significantly higher achievement was observed among GeoGebra-taught students compared to the control group. Moreover, experimental group students' perceptions of GeoGebra usage were favorable (Joshi & Singh, 2020). Likewise, students can work freely in GeoGebra software without the help of any teacher, which encourages students to discover learning and take ownership of their creations. In this regard, GeoGebra has grown in popularity as an interactive mathematics learning tool. Students may use this application to draw links between symbolic and visual representations. Likewise, GeoGebra is ideal for applying visualization techniques to mathematical concepts. As it can be widely used in teaching and learning mathematics from the primary stage of schooling to higher levels of education in the maximum field of mathematics, it helps students widely analyze mathematics concepts.

Similarly, it can be used in a dynamic way to learn mathematics. GeoGebra is known to be an open-source dynamic mathematics software for innovative learning and teaching purposes for all levels (Kllogjeri & Kllogjeri, 2014). GeoGebra offers a user-friendly environment with an easy-to-use interface, multilingual languages, commands, and a helpline (Hohenwarter et al., 2020).

In addition to the above, GeoGebra is a computer and web-based program that helps teachers and students study mathematics, particularly geometry, algebra, and statistics. Numerous features of the GeoGebra software suggest that it could be a valuable tool for assisting students in quickly, accurately, and efficiently visualizing abstract geometrical concepts (Tamam & Dasari, 2021). This plays a vital role in relating mathematics to the student's everyday learning experiences and lives by creating graphs, images, and animations. In Nepal, mathematics' significance is increasing daily in society because of the use of mathematics in various other fields. No doubt, mathematical concepts are indeed attached to every socio-economic activity. The need for mathematics and its application in our day-to-day life has always been an important part of the educational sector. The mathematics teachers of this twenty-first century are supposed to be innovators, creators, and knowledgeable about today's needs in teaching and learning.

Some studies demonstrated that when GeoGebra was applied in teaching mathematics to illustrate mathematical concepts, it helped students visualize and understand concepts through exploration. Further, GeoGebra has a positive impact on students' understanding of geometry. Dogan (2010) claimed that GeoGebra had positively affected students' learning and achievement so that they are motivated toward learning geometry. Blondal et al. (2013) also revealed that students improved their mathematical understanding after using GeoGebra. Students were able to investigate and generate hypotheses, resulting in higher grades. Some of the studies in Nepal have evaluated the impact of GeoGebra in teaching and learning (Shrestha, 2017a; Dahal et al., 2019; Dahal et al., 2022) and showed that secondary-level students demonstrated a better understanding of the concepts while using GeoGebra.

With the support of the second and third authors, this research attempts to help the students while teaching the concepts of circles meaningfully by challenging the traditional teaching methods. This paper aimed to explore the use of GeoGebra in teaching concepts of the circles. Guided by the research question— what role does GeoGebra play in helping students to understand and visualize the concepts of the circle? Starting with the introduction, this paper covers the theoretical framework, method, discussion of findings, and conclusion and way forwards of the study.

THEORETICAL FRAMEWORK

This research is grounded in the theoretical framework of social constructivism. Our study aimed to explore the concepts of the circle using GeoGebra. We implemented teaching experiment methodology (Steffe & Thompson, 2000). This method was chosen to explore the students' firsthand learning and reasoning experiences (Steffe & Ulrich, 2014). Likewise, learning theory of constructivism asserts that rather than passively absorbing information, students actively engage in the process of creating new knowledge. Students build their representations and incorporate new information into their existing knowledge as they experience the world and reflect on their experiences while conceptualizing the concepts of the circle using GeoGebra. Next, social constructivism is an educational philosophy emphasizing learning as a collaborative process. Knowledge develops due to learner's interactions with one another, their culture, and the larger society (Amineh & Asl, 2015).

The teachers must collaboratively guide the learning process to understand students' learning. We are actively involved in regular discussions with students within social constructivism as a theoretical framework. Similarly, we regularly took feedback from the students, motivated them to participate in classroom activities actively, allowed them to operate GeoGebra, and took feedback from witness-researcher about the students' behavioral change and engagements in experiment classes. Within this ethos, the essence of the social constructivism theory is that learning takes place through social interaction. Social constructivism believes social processes are vital for collaborative learning and cognitive development. Vygotsky (1978) believed that classroom community plays an important role in the meaning-making process of the content of any subject matter. It believes that knowledge is socially constructed. “Language, culture, daily activities, material goods, interpersonal engagement, peer interaction, tools, and symbols are all crucial social components in learning” (Dahal et al., 2019, p. 2). Students in the first author's classes are encouraged to participate in peer interaction and conversation. Vygotsky (1978) suggested that cognitive development results from the learners' social interaction.

Group work allows students to explore ideas, beliefs, perceptions, and misunderstandings with their peers and teachers. According to Vygotsky (1978), a difference exists between what a student can do on his own and what the student can do with help from others. Students can do things they would not be able to complete on their own with the support of adults and a greater understanding of others as experts. With the students' pre-existing knowledge of the circle, we incorporated GeoGebra to gain further an in-depth level of understanding of the concepts of the circle. Pictorial images, animations, and adequate illustrations in GeoGebra helped us accomplish the task of making students gain an adequate understanding of the concepts of the circle.

Moreover, social constructivism guided us to make experimental classes collaborative. Our lessons were designed so that students actively discussed the concepts of the circle. We followed the regular schedule of classes and assigned students group assignments to be done at home remotely. Collaboratively they did the assignment at home as well as in the classroom. New forms of ideas emerged through the interaction among the students. So, GeoGebra played the role of catalyst to secure the goals of a social constructivist classroom. Synchronous and asynchronous tools provide a platform for discussion among learners, leading to the social construction of knowledge (Dahal et al., 2020).

METHOD

We employed a well-known method of research known as the "teaching experiment" (Thompson, 1979; Dahal, 2019; Dahal et al., 2019; Dahal et al., 2022). Likewise, the teaching-research cycle helped us to design, implement, assessment & analysis and refine our plan (Czarnocha et al., 2016). Within teaching experiment and teaching-research cycles, GeoGebra in our study investigated how students conceptualized and visualized the circle's concepts and verifications. The circle concepts taught were chord, diameter, semicircle, segments, sector, central angle, arc, inscribed angle, concentric circles, intersecting circle, con-cyclic points, and cyclic quadrilateral in experimental classes. In addition, a few experimental classes included conceptual and procedural verifications and proofs. Similarly, concepts and verifications such as the inscribed angles of a circle standing on the same arc are equal, the central angle of a circle is double the inscribed angle standing on the same arc, and the inscribed angles standing on the same arc of a circle are equal.

The sum of opposite angles of a cyclic quadrilateral were explored using GeoGebra during the experimental classes. The experimental classes were recorded with the help of the witness researcher. We revisited those recordings while generating the meaning and the themes. Our research was conducted with 18 tenth-grade students where ten were boys and eight were girls. This teaching experiment/cycles encompassed two episodes of fifteen classes in which the concepts of circles were taught intensively. In addition, evaluations were an integral part of the circle concept learning that occurred during experiment classes.

We took both formal and informal field notes throughout experimental classes. Many students engaged in conversations and activities during the intensive instruction. The first author proposed numerous possible explanations on integration of GeoGebra in the concepts of the circle. Likewise, using recorded videos and field notes, potential findings were generated. In addition, the researcher-witness assisted the first author in comprehending the emotions and activities of the students. Being a reflective educator/researcher is also crucial when using the teaching experiment method. We expend more energy as reflective practitioners than teachers (Steffe & Thompson, 2000; Dahal et al., 2022a). These activities, termed "retrospective action", are performed by researchers and educators. The retrospective action allows the researcher to be aware of what occurred in the past or to take a fresh look at what occurred in the past. Reflecting on past teaching experience is one of the most crucial aspects of the teaching experiment method regarding making sense of the collected data. In addition to developing research and research instruments, the other two authors contributed to implementation process, the review of the research process, the completion of the research study, and the writing of the paper.

FINDINGS AND DISCUSSIONS

This section of the research examined the findings and discussed them in-depth. Our study was based on the information we gathered from teaching experiments, likewise, the participants and researcher's reflective journals were also used to generate some findings. The main ideas and findings are put together in different parts of the text using interpretation and analysis. The following six major findings were discussed.

GeoGebra Helps Recall and Understand Terminologies of Circle

As a teacher-researcher, the first author became aware about the diverse group of students' interests, knowledge, and ability to learn mathematics in general and "circle" in particular, which is one of the contents related to mathematics. Even though the circle has been introduced from the primary level and continue till higher grades, we teacher researcher become aware of the students' prior knowledge and experiences, on the concepts of circle and its basic terminologies, and so on, which are required to explore abstract ideas and concept particular in grade ten. It is important to know students' pre-knowledge and experiences not only in teaching circle but also in every topic related to mathematics. Prior knowledge and experiences enable students "understand new concept more quickly, retain their current knowledge more effectively, and transfer their knowledge to new situations more easily" (Klosterman, 2018, p. 12).

Thus, considering the importance of students' prior knowledge and experiences related to circle, the author first recalls the basic terminologies. For this, the first author uses simple question answer method and mathematical quiz incorporating the question from circle from previous grades and the grade ten itself. Such as what is circle? What is chord? How do you differentiate between chord and diameter? What is radius? What is the relation between radius and diameter? If diameter of a circle is 6 cm than what is the radius? How to calculate the area of a circle if the diameter is 10 cm? How do you differentiate between area and circumference of a circle? And so on. In addition, we will use many vocabulary terms when talking about the circle, which is essential to visualize. As shown in figure 1 the first author has demonstrated the basic terminologies of circles through the GeoGebra based on the questions asked them to know their pre-knowledge about the circle. Based on the activities, it has been found that most students can conceptualize and visualize the basic terminologies for the circle that they learned in previous grades. Some students were able to remember the terminologies of the circle because of the activity.

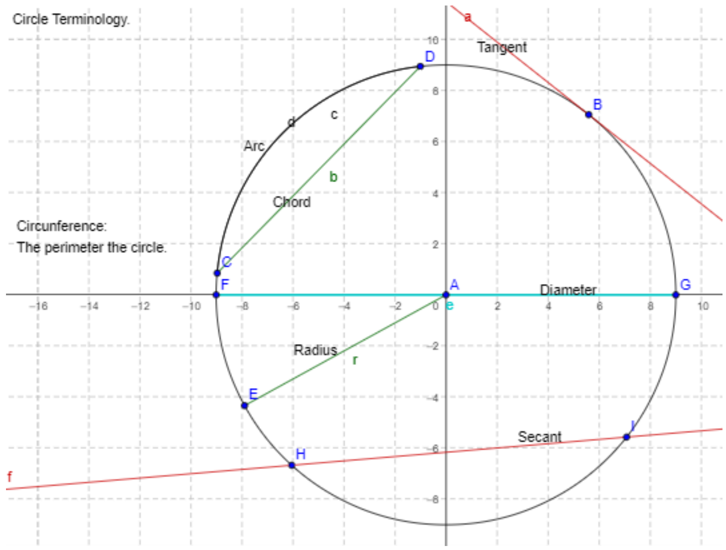


Figure1: Terminologies of the Circles

The activity was helpful for students who had missed a few circle classes in a previous grade. For instance, one of the boy students shared "*sir, this activity has helped me to visualize the concepts of the circle when radius is changed*". As the students did not get an opportunity to learn and understand the basic terminologies of circle through GeoGebra in previous grades, they found the very first day class of circle with recalling the experiences using GeoGebra was very effective. During the time of the reflection one of the average boy students reflected that "*this was my very first day class of circle using GeoGebra. Even though it is the introductory class of circle, I enjoyed throughout the discussion and activity. I got an opportunity to explore basic concepts related to circle and visualized that through GeoGebra. I hope to explore and solve more abstract problems related to circle through GeoGebra in our upcoming class".* Such kind of the reflection of the students encouraged the first author to conduct and to experiment the problems and theorems related to circle using GeoGebra.

GeoGebra Encourages Engaged Learning through Group Works

It could be one of the days of October 2018, the first author started the second day of the experiment class with great interest to demonstrate the concepts—chord, diameter, semicircle, segments, sector, central angle, arc, inscribed, angle, concentric circles, intersecting circle, con-cyclic points, and cyclic quadrilateral of the circles. These concepts were recalled during the first day of circle class in grade ten. The first author has spent more than one and a half years teaching this group of students. In this class, first author attempts to teach the students by integrating innovative pedagogies with the help of GeoGebra application as done in day one. In contrary, the first author was quite nervous about the students' reaction and behaviors when he presented the contents of mathematics of grade ten in a very different way. The first author was preparing for the session with great anxiety and curiosity. For instances, on particular class, the first author has to manage many new teaching materials such as a projector and a laptop.

For that day, the first author's planning was to make students visualize in group activities splitting the whole class in five different groups based on the concepts, namely, central and inscribed angle, cyclic quadrilateral; some experimentally verification that equal arcs of a circle subtend equal angles at the center (to mention).

With the help of witness-researcher, as shown in figure 2, the first author demonstrates the concepts of the circle, central angle, and its corresponding arcs with the following steps as students were already habitual for handling the GeoGebra application:

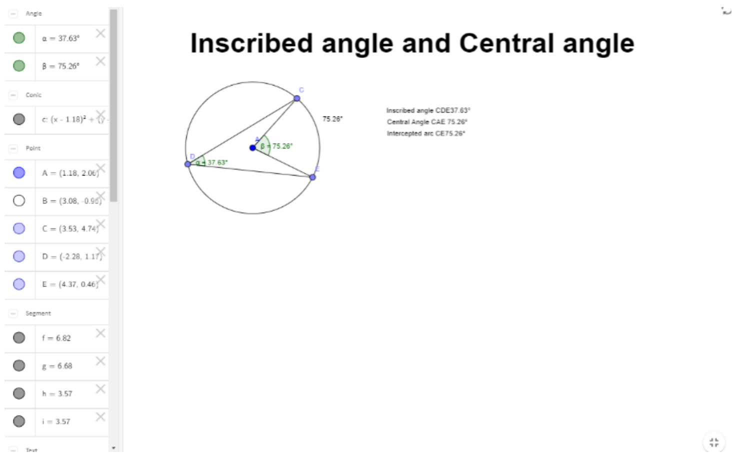


Figure 2: Screenshot of experimental verification

Step 1: Open GeoGebra and hide the axes.

Step 2: Create two circles' radii more than 3 cm with center O.

Step 3: Choose the point C on circumference of both the circles.

Step 4: Create a line between points A & C and B & C.

Step 5: Measure angles ACB and Ref. angle AOB. What do you notice about their measures?

Step 7: Create an arc a, between points A and B in both the circles. What do you notice?

.

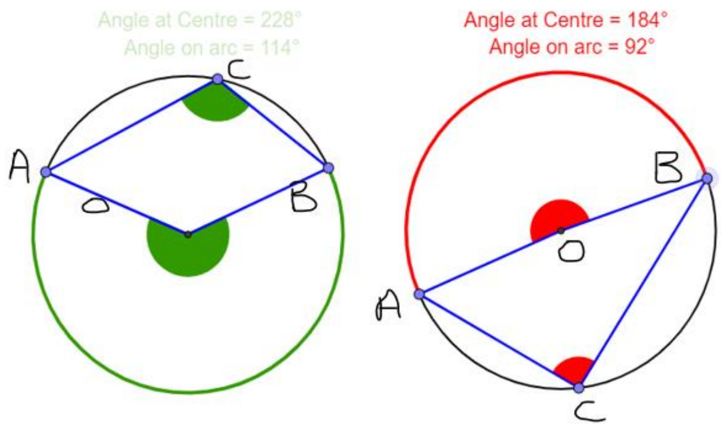


Figure 3: Screenshot of experimental verification

After demonstrating the concepts and responding the concerns raised by the students associated to center angle and inscribed angle standing on same arc. As shown in the figure 3 of two different figures, the first author divided the students into five groups each containing five students and gave them A4 papers. The first author assigned each group the following tasks.

Task I

- Draw a circle of radius 5 cm
- Draw two equal angles at the center
- Measure corresponding arcs of each central angle
- Draw your result.

Task II

- Draw circle of radius 5 cm
- Draw two equal arcs
- Measure corresponding angles at center subtended by each equal arcs
- Draw your result.

After the allocated time was over, the researcher asked each group to share their results. To surprise, the first author found that each team concluded the correct result: in a circle, equal central angles subtend equal arcs and equal arcs of a circle subtend equal central angles.

First author has started teaching decade before obtaining a post graduate diploma (PGD) in education. First author, as a teacher, used to think that curriculum is merely a collection of content material to be delivered to the students within the academic year. Students' interest and motivation are often ignored in his class. While teaching mathematics by integrating GeoGebra and collaborative teaching methods, we found students engaged in doing mathematics and deriving results from different activities rather than listening to talk and staring at the white board. They could share their ideas among their peers and work in a miniature knowledge society. We realized that students learn many life skills form the group works. The following figure 4 demonstrates the students' skills as the form of engaged learning of the concepts of the circle.

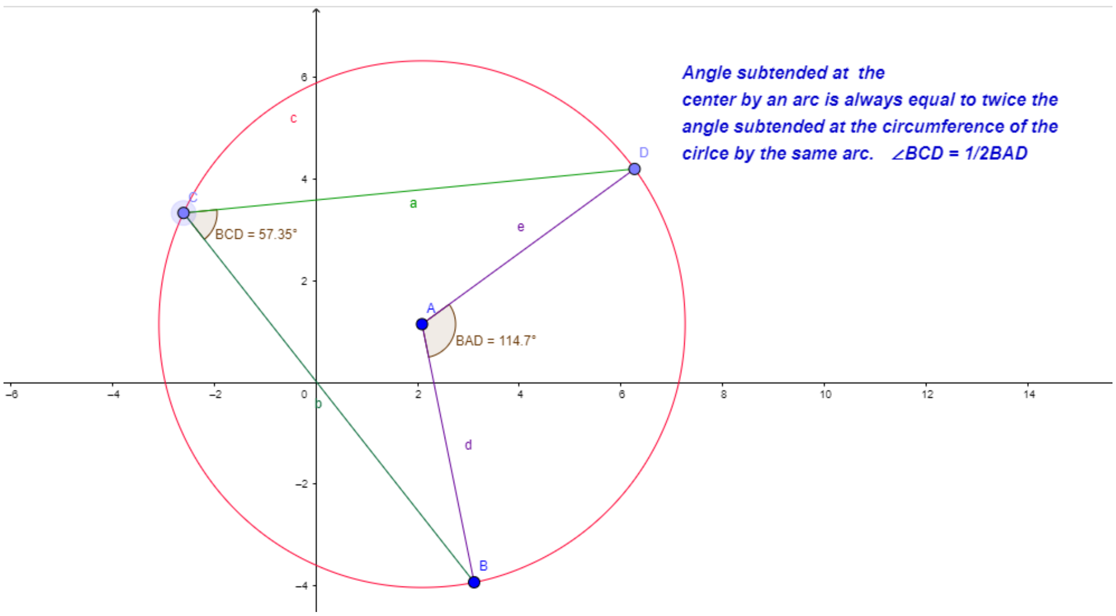


Figure 4: Screenshot of students' creation

The creative task done by students shown in figure 4, as a teacher we may not be able to solve other problems in our profession, but at least we can change our teaching by incorporating cooperative learning. We know that loss of students' interest is one of the major causes of students' failure. Among the strategies, integrating the GeoGebra while teaching general mathematics and circle is particularly likely to motivate the students to learn mathematics in unique ways.

In collaborative ICT class, first author, performed as a mediator. He mediates learning through dialogue and collaboration. We managed the proper environment and physical structure within the classroom. The availability of all the teaching/learning materials is the most important factor for the students to collaborate, and we had to prepare a lot in advance. We managed the structures so that the students may easily interact and perform.

In applying collaborative learning methods with GeoGebra, we encountered a number of obstacles, including the pace at which students followed the first author's instructions, distractions from peers, and a lack of participation on the discussion are among others. Most parents have a traditional learning approach in mathematics, so they expect their children to be educated similarly. They think that their children are deprived of learning mathematics from the teacher and their friends. They believe there should be competition among the students, not cooperation while learning mathematics (Panthi & Belbase, 2017). Students sometimes become arrogant and do not follow the instructions given to them. Some students are passive learners by nature, and some are noisy. Beside all these obstacles, it is worthy to adopt a collaborative method of learning mathematics integrated with technology.

GeoGebra Promotes Meaningful Learning

First author was comfortable compared to the previous day of day four. He did not have to run for materials and arrange rooms for the session. He planned to visualize central angle, inscribed angle, major segment, and minor segment with the help of GeoGebra in the experimental class as in a form revision where some of the concepts were illustrated in the previous class. Major task for the session was to verify experimentally, angles in the same segment of a circle are equal.

As planned by first author, we were waiting for students of grade ten in ICT room with the required teaching aid. Most of the students came in time except a few. Taking the situation easily, we started the sharing the task that was plan for day five with the help of GeoGebra.

After drawing pictures, first author instructs students to do the following activities:

1. Move point D. Measure the central angle, inscribed angle, and the intercepted arc.
2. Move point C. Measure the central angle, inscribed angle, and the intercepted arc.
3. Move point E. Measure the central angle, inscribed angle, and the intercepted arc.
4. What is the relationship between the central angle and the inscribed angle?
5. What is the relationship between the central angle and corresponding arcs?
6. What is the relationship between the inscribed angle and the corresponding arc?

We collected information from all the students, and we found that student's active engagement during the teaching and learning circle using GeoGebra to determine the relationships between the central angle, its opposite arc, and the inscribed angle, its opposite arc, angles in the same segment of a circle are equal produces satisfactory outcomes. For example, one of the female students stated, "*Sir, this is a completely novel and innovative learning experience for me. I can now see the relationships between central angles, their opposite arcs, and inscribed angles and their opposite arcs*". Then we told students to do the other activity in their note copy with the help of a compass, ruler, and pencil. They did the activity enthusiastically and the session was concluded by wrapping up the session with feedback from the students.

GeoGebra Promotes Conceptual Learning

It was first author's regular class and the sixth day of the experimental class in grade ten and his students were eagerly waiting for him. When he entered ICT room, students greeted him with a smile. As a teacher, we have felt that it is very difficult to precede the class if the students show indifference in our presence. His priority is always to maintain harmony and warm relationships with students so that teaching does not remain the only means of livelihood. In the session, as per first author plan was to visualize concyclic points of a circle, cyclic quadrilateral, angles of a cyclic quadrilateral, opposite angles of a cyclic quadrilateral.

Before starting the session, the first author tried to recall the contents discussed in previous classes. First author requested the students to illustrate with the figure of the circles—central angle, inscribed angle, intercepted arc, and their relationship. Most of the students could explain and illustrate the concepts with the figure of the terms mentioned above of a circle. We were very happy at that movement.

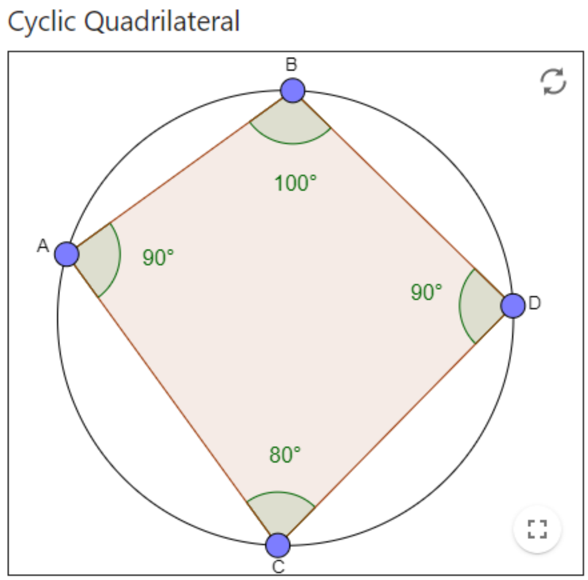


Figure 5: Screenshot of cyclic quadrilateral

First author started the session with a GeoGebra window where a cyclic quadrilateral was drawn as shown in figure 5 alongside. He asked the students to draw two sets of different circles in their notebook. He measured all the angles of the cyclic quadrilateral and told the students to do the same. On the GeoGebra window, we found that the sum of opposite angles of a cyclic quadrilateral is two right angles. First author asked the students to conclude the result. Most of the students got the expected result. Some students could not measure angles with a protractor. We assisted them in measuring the angles. They were happy when they were able to get the desired result. When the students do any activity with satisfaction it gives immense pleasure. Students understood that cyclic quadrilateral, by definition, is any quadrilateral that can be inscribed inside a circle. Also, they learned that the sum of opposite angles of a cyclic quadrilateral is two right angles.

After proving the theorem experimentally in the notebook and on the GeoGebra sheet, we proceed to prove the theorem theoretically. We proved the theorem easily fulfilling the required procedure for the examination. We should always keep in mind that we are teaching students in Nepal, and they are evaluated on three hours written test with specific requirements of a solution. This is an important in the case of teaching secondary level students. We were happy that first author could make the students learn mathematics conceptually.

GeoGebra Promotes Learner-Centered Teaching

We are in the middle of Dashain and Tihar of 2019 AD. Dashain and Tihar are the national festivals of Nepali. All the people seemed to be busy preparing for the festival. They have just enjoyed the festival of joy and are preparing for the festival of light and sweets--Tihar. We were busy collecting data for the research. First author was implementing lesson plans hoping to do the best. Sometimes, we rewarded through students' success and sometimes we have to regret choosing this profession when we see students failing in the subject.

Writing about first author, he has been teaching from a long decade. However, on the experimental class, he taught students to prove theoretically that standing on the same arc, the angle in the center is double that of the angle at the circumference. He proved the theorem that states that the angle in the semicircle is one right angle with the help of GeoGebra as shown in figure 6 below.

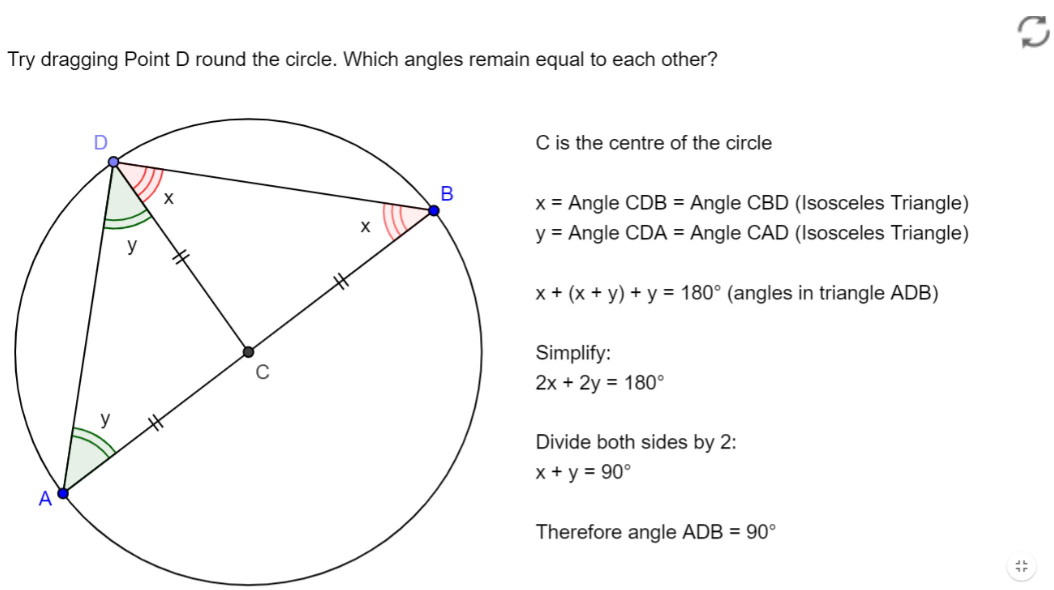


Figure 6: Proof of inscribed angle of the semicircle is one right angle

For instance, examinations in Nepal, most of the students must demonstrate the exact solution that was taught and/or illustrated in the examples. However, we could see the students struggling to maintain the required standard of proving theorems theoretically. This standard includes—proof statements, clear figure, given statements to prove, construction (if any), and a proof table including statements and reasons. We had to help them in proofing as per the standards. Nevertheless, GeoGebra only illustrates the visual form of the concepts.

For the proof of the theorem, first author asked for a volunteer role from one of the students. After all, we believe that learning is an active process. One student was happy to solve the theorem on the whiteboard. She explained the solution to her friends. Everyone understood the theorem. We could see the brightness in students' eyes, which we think is the most important indication of students' understanding of the content. In such a context, the student who solved the problem seemed to be excited and happy. First author asked her, *"How are you feeling now?"* She shared, *"Sir, I generally was not encouraged to involve myself in the classroom activities of GeoGebra. I hope this is true for other friends. Solving problems on the whiteboard and showing my talent were not practiced. This is probably my first time standing in front of my friends and you to solve the problem with the help of GeoGebra. I am feeling honored and responsible, sir"*. Being teachers, we must celebrate the progressive methods of teaching that encourage us to implement learner-centered methodologies in teaching and learning activities. By keeping learners at the center of learning and activities, students should be given opportunities to think, plan, implement, and execute by actively participating in the process. Keeping students engaged in the learning process is challenging, but the tasks with rich learning materials and resources simplify the learning process and help in a learner-centered learning environment. Moreover, in today's digital world, the use of technology and materials developed by using digital technologies seem to engage students effectively. GeoGebra is among those ICT applications for encouraging students' active participation in learning to the circle concepts.

GeoGebra Motivates Students in Learning

GeoGebra helps teachers motivate students in learning mathematics. Those leanings are of the curriculum and/or outside of the curriculum. Role of GeoGebra is very important for students of various backgrounds. The students with less motivation can be attracted towards learning. In one of the experimental classes, we explore how changes in arcs changes the center and inscribed angle. One of the students, motivated by replying that *this is exact learning with visualizing the concepts of arcs*. However, traditional classes only focus on the content of the study. But the ICT integrated classes to some extent motive students for leaning mathematics. Role of GeoGebra is very important for visualizing and animating the concepts of the circles.

Some of the glimpses of the experimental classes are as follows:

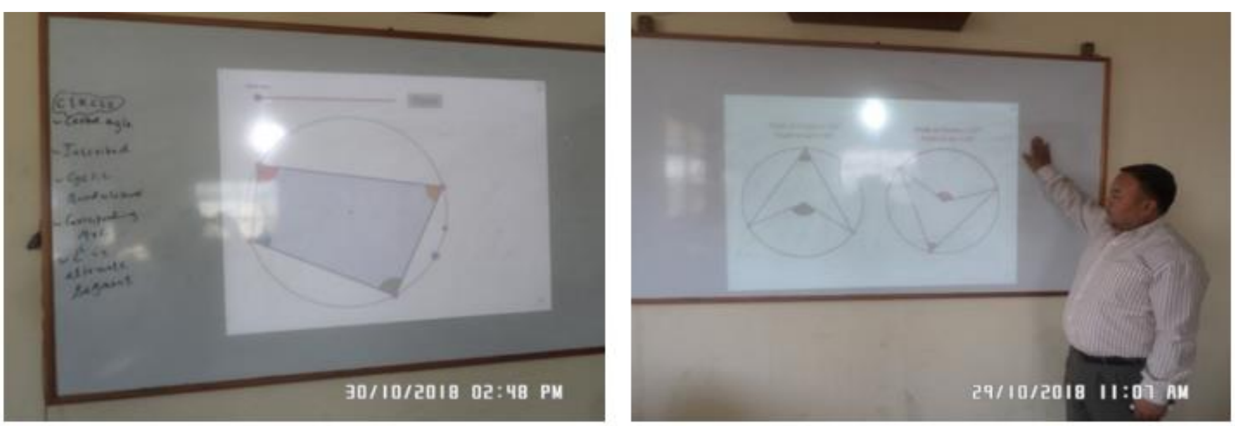


Figure 7: Glimpses of the experimental classes

Above glimpses demonstrate some of the concepts of the circles using GeoGebra that plays an important role for mathematics teaching and learning to visualize the concepts of the circles. The notion of mathematics would not have made sense without integrating GeoGebra in mathematics class, and it is very helpful for a strengthening the concept of mathematics especially the concepts of the circles. GeoGebra excels the learners for deep learning in mathematics, which helps develop logical thoughts and imagination. No doubt, GeoGebra plays an important role in teaching.

It is likely to be claimed that GeoGebra is a powerful visualizing tool that provides students with various experiences demonstrating the essential abilities to select when and how to utilize it in teaching and learning. Near to the final, from experiment classes, we can conclude that GeoGebra plays a vital role in visualizing the concepts of the circle namely, chord, diameter, semicircle, segments, sector, central angle, arc, inscribed, angle, concentric circles, intersecting circle, con-cyclic points, and cyclic quadrilateral as of the motivating ICT tool to name and among other verifications and proofs.

CONCLUSIONS AND WAY FORWARDS

The paper discussed the circle's concepts of chord, diameter, semicircle, segments, sector, central angle, arc, inscribed, concentric circles, intersecting circle, con-cyclic points, and cyclic quadrilateral. In addition, concepts and verifications as inscribed angles of a circle standing on the same arc are equal, the central angle of a circle is double of the inscribed angle standing on the same arc, the inscribed angles standing on the same arc of a circle are equal and the opposite angles of a cyclic quadrilateral are supplementary were explore using the GeoGebra during the experimental classes. The purpose of the experimental classes was to see how GeoGebra could be integrated in mathematics classrooms especially while introducing the concepts of the circle for learning and assessments. In this study, the first author, in collaboration with the second and third authors, utilized a teaching experiment to determine students' mathematical comprehension of circle concepts. The second and third authors also collaborated to determine how students conceptualized changes in the circle concepts using animations features. For the purpose, we employed a teaching experiment approach as the form of teaching research cycle to understand students' engagements during the experimental classes based on the research question, what role does GeoGebra play in helping students to understand and visualize the concepts of the circle?

Similarly, we put our effort into understanding students' conceptual understanding on the concepts of circles of intensive teaching in two episodes of fifteen classes. In Euclidean geometry, a circle is a basic shape. It is the set of all points in a plane that are at a given distance from a given point, the center; alternatively, it is the curve traced out by a point that moves in such a way that its distance from a given point remains constant. On contrary, drawing the various concepts of the circles on the white paper limit the students' conceptual understanding, but GeoGebra helps to visualize the changes made and its exact figure of the circles and associated concepts. These engagements offer the students for meaningful and conceptual learning of the concepts of the circle.

Further, various individuals have concluded that there should be a variety of approaches to mathematics learning and instruction, including the use of teaching tools proven to increase students' interest in mathematics. This is one of the most significant discoveries to date. Math teachers should have access to this software to offer students a broader perspective on mathematics and to help them develop their critical and creative thinking skills. GeoGebra is one of the mathematical software programs on this list. As a result, expanding the use of ICT applications in mathematics classes could assist teachers and students in contextualizing mathematical concepts.

During the research process, we saw students actively participating in experiment classrooms. Similarly, we found that GeoGebra is user-friendly software. The findings of the study namely, GeoGebra promotes meaningful learning, GeoGebra encourages conceptual learning, GeoGebra promotes learner-centered teaching, and GeoGebra motivates students to learn mathematics concluded that using GeoGebra in the classroom can improve students' conceptual and procedural understanding on the concepts of the circle. Similarly, students' engagements in class and their replies after that revealed that GeoGebra can assist students for visualizing the abstract geometrical concepts of the circles quickly, correctly, and effectively.

Likewise, GeoGebra helps improve students' understanding of the circle concept. Integration of GeoGebra allows students to become active learners in the classroom. It helps teachers to make student-centered mathematics classrooms. It also helps to minimize unnecessary distraction in the classroom. All these benefits of using GeoGebra in mathematics classroom makes mathematics classroom effective and meaningful. So, as a teacher-researcher, we suggest that all the mathematics teachers to use GeoGebra in their classrooms. In this paper, we explored the use of GeoGebra in teaching geometry circles. There are many other mathematics contents where GeoGebra can be used effectively. Many algebraic concepts and transformation can be made visible and easy for students. So, we recommend mathematics teachers explore the possible use of GeoGebra in teaching other concepts on the contents of the school mathematics and beyond.

REFERENCES

[1] Agyei, E. D., & Agyei, D. D. (2021). Promoting interactive teaching with ICT: Features of intervention for the realities in the Ghanaian physics senior high school classroom. *International Journal of Interactive Mobile Technologies (iJIM)*, *15*(19), 93–117. https://doi.org/10.3991/ijim.v15i19.22735

[2] Amineh, R. J., & Asl, H. D. (2015). Review of constructivism and social constructivism. *Journal of Social Sciences, Literature and Languages*, *1*(1), 9-16.

[3] Battista, M. T. (1999). The mathematical miseducation of america's youth: Ignoring research and scientific study in education. *Phi Delta Kappan*, *80*(6), 425-433.

[4] Blondal, T., Nielsen, S. J., Baker, A., Andreasen, D., Mouritzen, P., Teilum, M. W., & Dahlsveen, I. K. (2013). Assessing sample and miRNA profile quality in serum and plasma or other biofluids. *Methods*, *59*(1), S1-S6.

[5] Curriculum Development Center [CDC]. (2015). *Secondary level curriculum 2075 (2015)*. Authors.

[6] Czarnocha, B., Baker, W., Dias, O., & Prabhu, V. (Eds.). (2016). *The creative enterprise of mathematics teaching research: Elements of methodology and practice–from teachers to teachers*. Springer.

[7] Dahal, N. (2019). Integration of GeoGebra in teaching mathematics: Insights from teaching experiment. In *Proceeding of Seventh National Conference on Mathematics and Its Applications* (pp. 60-71), Butwal, Nepal.

[8] Dahal, N., Luitel, B. C., Pant, B. P., Shrestha, I. M., & Manandhar, N. K. (2020). Emerging ICT tools, techniques and methodologies for online collaborative teaching and learning mathematics. In *Mathematics Education Forum Chitwan* (Vol. 5, No. 5, pp. 17-21). https://doi.org/10.3126/mefc.v5i5.34753

[9] Dahal, N., Manandhar, N. K., Luitel, L., Luitel, B. C., Pant, B. P., & Shrestha, I. M. (2022b). ICT tools for remote teaching and learning mathematics: A proposal for autonomy and engagements. *Advances in Mobile Learning Educational Research*, *2*(1), 289–296. https://doi.org/10.25082/amler.2022.01.013

[10] Dahal, N., Pant, B. P., Shrestha, I. M., & Manandhar, N. K. (2022a). Use of GeoGebra in teaching and learning geometric transformation in school mathematics. *International Journal of Interactive Mobile Technologies (iJIM)*, *16*(08), pp. 65–78. https://doi.org/10.3991/ijim.v16i08.29575

[11] Dahal, N., Shrestha, D., & Pant, B. P. (2019). Integration of GeoGebra in teaching and learning geometric transformation. *Journal of Mathematics and Statistical Science*, *5*(12), 323-332.

[12] Dikovic, L. (2009). Applications GeoGebra into teaching some topics of mathematics at the college level. *Computer Science and Information Systems*, *6*(2), 191-203.

[13] Ghory, S., & Ghafory, H. (2021). The impact of modern technology in the teaching and learning process. *International Journal of Innovative Research and Scientific Studies*, *4*(3), 168–173. https://doi.org/10.53894/ijirss.v4i3.73

[14] Hohenwarter, M., Preiner, J., & Yi, T. (2020). Incorporating GeoGebra into teaching mathematics at the college level. In *Proceedings of the International Conference for Technology in Collegiate Mathematics*. Markus Hohenwarter.

[15] Joshi, D. R., & Singh, K. B. (2020). Effect of using GeoGebra on eight grade students'

understanding in learning linear equations. *Mathematics Teaching Research Journal*, *12*(3), 76-83.

[16] Kllogjeri, P., & Kllogjeri, Q. (2014). *GeoGebra: A vital bridge linking mathematics with other sciences*. In OCWC Global.

[17] Klosterman, P. (2018). Connecting new knowledge to old: Uncovering hidden premises in mathematical explanations. *Australian Primary Mathematics Classroom*, *23*(2), 23-26.

[18] Maharjan, M., Dahal, N., & Pant, B. P. (2022). ICTs into mathematical instructions for meaningful teaching and learning. *Advances in Mobile Learning Educational Research*, *2*(2), 341-350. https://doi.org/10.25082/AMLER.2022.02.004

[19] Panthi, R. K., & Belbase, S. (2017). Teaching and learning issues in mathematics in the context of Nepal. *European Journal of Educational and Social Sciences*, *2*(1), 1-27.

[20] Putra, Z. H., Hermita, N., Alim, J. A., Dahnilsyah, D., & Hidayat, R. (2021). GeoGebra integration in elementary initial teacher training: The case of 3-D shapes. *International Journal of Interactive Mobile Technologies (iJIM)*, *15*(19), 21–32. https://doi.org/10.3991/ijim.v15i19.23773

[21] Shrestha, D. (2017a). *Use of GeoGebra in learning geometric transformation* [Unpublished Master Dissertation]. Kathmandu University School of Education, Dhulikhel, Nepal.

[22] Shrestha, P. (2017b). *Evaluation of interactive learning materials for universal design: Case of GeoGebra in Norwegian high schools* (Master's thesis). Oslo and Akershus University College of Applied Sciences.

[23] Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. *Handbook of research design in mathematics and science education*, 267-306.

[24] Steffe, L. P., & Ulrich, C. (2014). Constructivist teaching experiment. In *Encyclopedia of mathematics education* (pp. 102-109). Netherlands: Springer.

[25] Tamam, B., & Dasari, D. (2021). The use of GeoGebra software in teaching mathematics. In *Journal of Physics: Conference Series* (Vol. 1882, No. 1, p. 012042). IOP Publishing.

[26] Tapscott, D. (2008). *Grown-up digital*. McGraw-Hill Education.

[27] Thompson, P. (1979). The constructivist teaching experiment in mathematics education research. In *research reporting session, annual meeting of NCTM,* Boston.

[28] Vygotsky, L. (1978). Interaction between learning and development. *Studying on the Development of Children*, *23*(3), 34-41.

Concept of Polygon: Case Study of Elementary Students' Difficulties

Herawati[1,3], Tatang Herman[2], Didi Suryadi[2], Sufyani Prabawanto[2]

[1]Departemen of Primary Education, Schools of Postgraduate, Universitas Pendidikan Indonesia, Jl. Dr. Setiabudi No. 229, Bandung 40154, Indonesia, [2]Departemen Pendidikan Matematika, Universitas Pendidikan Indonesia, Jl. Dr. Setiabudi No. 229, Bandung 40154, Indonesia, [3]FTK, Universitas Islam Negeri Ar-Raniry, Banda Aceh, Indonesia

herawati@upi.edu

Abstract: This study aims to determine students' difficulties in understanding the concept of polygons in elementary schools. This research is qualitative research with a case study method. The subjects of this study were 30 grade 5 elementary school students from two different schools in the city of Bandung. The instruments used are test and non-test. The technique of tests is asking some questions about polygons, while the non-test is in the form of interviews. The data collected were analyzed using the three stages of the Miles and Huberman model, including reduction, presentation, and conclusion. The findings in this study indicate that students have difficulty understanding the concept of polygons, namely difficulties in identifying polygons properties, polygons rules and regulations, and determining polygons' names. One of the things that teachers can do to follow up on the problems from these findings is to design learning based on didactic situations that are appropriate and according to the problem faced by students.

INTRODUCTION

Geometry is an important material widely used in various disciplines (Loc et al., 2017). By studying geometry, students can solve various problems in everyday life from different perspectives, build relationships and use geometric representations to simplify abstract concepts (Biber et al., 2018; Filiz & Gür, 2021; Jones, 2002; Sopany & Rahayu, 2019). One of the goals of learning geometry for students is to have basic 21st-century skills, namely reasoning, problem-solving, and critical thinking skills (Erşen et al., 2021; Herbst et al., 2017). Geometry lessons have been taught from kindergarten to university. Geometry learning is very well taught early because students consistently interpret geometric shapes based on how they move their bodies (Douglas H. Clements et al., 2004). Thus, an understanding of the basic concepts of geometry has been instilled in students from an early age (France, 2004; Hallowell et al., 2015). Learning geometry starts at an early age at the kindergarten level, where students perceive the differences between geometric shapes by observing the objects they see in the environment and trying to find aspects of the similarities between them. However, as they age, they continue to study geometry at a higher level from a view of induction and deduction. Students can experience various errors and misunderstandings in understanding geometric material during this process.

Geometry concepts must be taught hierarchically and sequentially to avoid mistakes and misunderstandings (Filiz & Gür, 2021). Because if students cannot understand the concepts of geometries well at the elementary level, there are critical problems in studying geometry at the next level. Geometry is one of the most challenging materials for students to master (Aksu, 2013; Anggraini et al., 2021; Fajari, 2020; Gal & Linchevski, 2010; Rogers, 1995). That is shown by Indonesian students' achievements in geometry from elementary school until universities are still low (Alawiyah et al., 2018; Hartono, 2020; Puspasari et al., 2015; Sholihah & Afriansyah, 2017). Geometry is considered a complex material to understand because geometry's characteristics require visual abilities or imagination and high analytical skills to understand unreal objects.

In contrast, elementary school students are at the concrete operational stage and must use concrete objects to understand something (Fajari, 2020). One of the geometry materials taught in elementary schools is polygons. Polygon material is one of the basic materials in geometry. One of the topics that must be understood in this material is the context of elements and properties of polygons, so if students still experience limitations in understanding this material, it will be an obstacle to students when understanding and using concepts to solve other geometric problems. Several studies on understanding the concept of polygons reveal the fact that the level of students' geometric perception is not at the expected level as in the quadrilateral material, which is considered one of the most problematic materials for students (Ayvaz et al., 2017; Bernabeu et al., 2021; Biber). et al., 2018; Fernigil L. Colicol et.al., 2017). Among the problems often encountered are students having difficulty describing a shape based on its characteristics (Hidayat, 2019). That also impacts naming polygons, so students do not realize the hierarchical relationship between plane shapes (Fujita, 2012). In addition, students also have problems defining plane shapes (Fujita, 2012; Fujita & Jones, 2007).

Bernabeu et al. (2021) also researched the concept of polygons and the relationship between polygons in third-grade elementary school students. Focus on introducing polygons, the relationship between polygons, and giving reasons to state examples of polygons and non-examples of polygons. As for this study, the researcher analyzed the students' difficulties in understanding the polygon concept, which focused on understanding the concepts of polygons, regular and irregular polygons, and naming polygons in elementary schools. It also addresses the following research questions: What kinds are students' difficulties in mathematics who have learned polygons in geometry material? The hope is that this research can be a reference for teachers and other researchers in designing learning by minimizing the various problems in studying geometry, especially polygon material.

METHOD

This research uses a qualitative approach with a case study method. According to (Gall et al., 2009), the case study method is a method used to explain certain phenomena, whether in the form of processes, individuals, programs, and so on. Thus, the case study method can be used as an appropriate method to explore students' difficulties in understanding the concept of polygons in elementary schools.

The subjects in this study were fifth-grade elementary school students from as many as 30 students (aged 10 – 11 years) from two different schools in the city area. The details are as follows: 16 State Elementary School students in large schools and 14 State Elementary School students in small schools. The selection of student groups was because the polygon material had been given in the previous class. The instruments used include test and non-test instruments. The test instrument is in the form of questions related to the polygon concept, and non-test instrument is in the form of in-depth interviews to strengthen the data obtained.

The researcher traced the students' difficulties from the aspect of understanding the polygon concept by giving tests to 30 fifth-grade students who had studied polygon material. A total of 9 questions have been given to students related to the concept of polygons. These questions are designed to identify difficulties in students' understanding of polygon material related to understanding the concept of polygons. The questions given consisted of one question about recognizing polygonal shapes, one question about regular and irregular polygon shapes, and seven questions about names of plane shapes. The instruments given to students have gone through a qualitative validation process in the realm of material, construction, and language involving two mathematics education experts and two elementary school teachers. After the validation process, the instrument was declared suitable for use according to research needs.

Question Number	Criteria	Category
1	Have more than three sides	Understanding polygons
	Has an angle of more than three	
	Closed curve	
	All sides are line segments	
2	All sides are the same length	Understanding regular and irregular polygons
	All angles are the same	
3	Saying the name of the wake correctly and correctly	Mention the names of plane shapes/polygons
	Name the shapes based on the number of sides, such as triangles, rectangles, etc.	

Table 1: Criteria for polygon questions

Miles and Huberman models are used in analyzing the data of this study. These stages consist of data reduction, data presentation, and conclusion drawing (Miles, M. B. & Huberman, 1994). At the data reduction stage, the researcher recorded student responses in answering questions related to polygons material. Furthermore, the data is presented starting by grouping the types of student responses based on their level of difficulty. In the last step, the researcher drew conclusions after going through the process of analyzing the difficulties experienced by students with polygonal material.

RESULTS

The analysis was carried out based on the component aspects of students' understanding of the characteristics of polygons. Next is to determine the percentage of students who have difficulty understanding polygons. The following shows the percentage of students who have difficulty answering the questions related to the polygon material.

Number of Questions	Percentage of correct answers (%)	Percentage of incorrect answers (%)
1	50	50
2	30	70
3.a.	76.7	23.3
3. b.	33.3	66.7
3. c.	40	60
3. d.	3.3	96.7
3. e.	26.7	73.3
3. f.	30	70
3. g.	13.3	86.7

Table 2: Percentage of student answers

Based on the table above, the details of the findings in this study consist of understanding the concept of polygons, the concepts of regular polygons and irregular polygons, and determining polygon names.

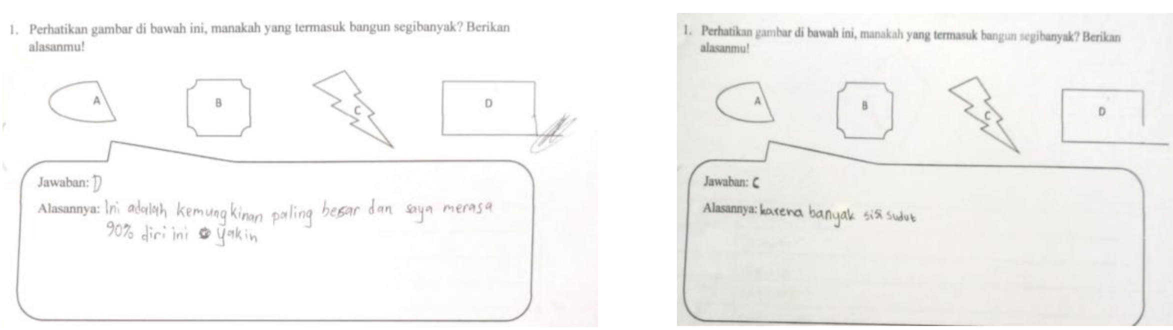
Difficulty in the Concept of Polygons

Students' understanding of the polygon concept has shown in question Number 1. Based on the percentage of students' answers, some students can show or determine the polygon shape from several other shapes that are not polygonal. However, out of 50% of the students, not all could mention the right reasons for determining polygons. The following are some of the reasons given by students for the correct answer choices.

Question Number 1 Correct answer choice C	Reason:
S1	because it has 7 polygons
S2	because it has many angles
S3	has seven sides
S4	because it has many sides
S5	because it has more angles
S6	because it has 11 polygons
S7	because it looks much and looks good.
S8	because it has 11 facets
S9	because it is different from the others
S10	The C-sided shape has more angles than the other side shape.
S11	because the arrangement is more numerous than the others
S12	reasons because they are different from each other.
S13	because the C has more facets than the others.
S14	because it has more squares.
S15	because the highest number is C.

Table 3: The reasons given by students for the correct answer choice for question number 1

Based on the table above, it can be seen that there are still many students who cannot mention all the characteristics of polygons. In this question, students do not understand the nature of polygons, so most students choose the correct answer but give the wrong reason. The reasons given by students only meet one of the characteristics or properties of polygonal shapes such as the nature of having many sides and many angles. As for the nature that the polygon must be a closed curve, none of the students mentioned it. So, for the choice of a shape in the form of an open curve, students take the initiative to draw a line or close the shape. An example of a student's answer can be seen in the following picture:



(a) (b)

Figure 1: (a) and (b) Example of student answers for question number 1

Subject 1 chose the shape "D" as shown in Figure 1(a) as a polygon by connecting the broken lines. The reasons given are:

Researcher: what about question number 1? Can you answer it?

Subject 1: yes Miss.

Researcher: what is asked in this question?

Subject 1: about polygons, Miss, asked to choose among the existing shapes, Miss.

Researcher: what answer did you choose?

Subject 1: "D" Miss.

Researcher: why is the shape said to be polygonal?

Subject 1: because this shape (while showing the picture on the question sheet) is neater and has many sides, Miss

Students assume that polygons are only quadrilaterals that are neat and orderly in shape. For students who choose the correct answer, namely "C" (as shown in Figure 1(b)). The responses given are:

Researcher: do you have difficulty answering question number one?

Subject 2: mmmm…, (while smiling)

Researcher: what answer did you choose?

Subject 2: "C" Miss

Researcher: why did you choose C?

Subject 2: because it has many sides, Miss…

Researcher: do you have any other reason?

Subject 2: no Miss

That shows the students still have limited knowledge about the polygon concept even though there are 50% of students chose the correct answer. However, the reason is not right, so the student assumes that the polygon shape is a shape that only has many sides without paying attention to the properties involved else. Students have difficulty determining the properties of polygons.

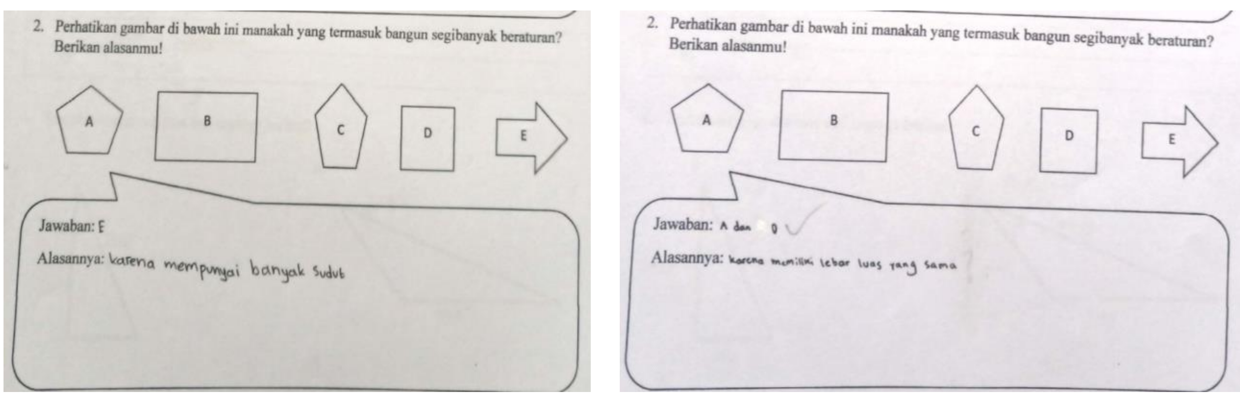
Difficulty in the Concept of Regular Polygons

Regular polygons are part of the polygon material that students can understand after students can determine and distinguish polygon and non-polygonal shapes. To find out students' understanding of the regular polygon can be seen in the student's ability to answer question number 2. The percentage of students who can answer the question by determining the regular polygon from several polygon shapes given is only 30% of 30 students. From this percentage, not all students gave the right and correct reasons for determining the answer. The following are some of the reasons given by students regarding choosing the correct answer:

Number of Questions	Answer Choices and Student Reasons
2	Only choosing A, the reason being because it has 5 sides Only choosing A, the reason being because there are more sides. Only choose D, the reason is because it looks neat. Only choose D, the reason is because it has many polygons and is not too small. Choose A and D, the reason is Because they have the same width. Only choose A, the reason is Because the plane A has many sides but is not regular like E Only chose A, the reason is because it has five sides and is regular Only chose A, the reason is Because A has many regular sides Just chose A, the reason is Because it is neater and has many

Table 4: The reasons were given by students for the correct answer choice for question number 2

Based on the table above, students chose the correct answer but could not give a good reason to state that the shape was a regular polygon. In this question about regular polygons, most students (46.7%) stated the E shape as a polygon for various reasons. One of the reasons expressed by the students was because the shape E has many angles, this shows that students think that a regular polygon is only a shape that has many angles when compared to other shapes. The examples of student answers are presented as follows:



(a) (b)

Figure 2: (a) and (b) Example of student answers for question number 2

Difficulty in Determining the Name of the Polygonal Shape

Students' ability to determine polygons' names is still having difficulties, as found in this study. The plane shapes that are presented to be named are rectangle, square, right triangle, rhombus, parallelogram, kite, and isosceles trapezoid. The results obtained are as follows:

Number of questions	Student Answers
3. a	Rectangle, rectangle.
3. b	Square, regular square, rectangular shape, square, square, rectangle, square, block, rectangle, square.
3. c	Equilateral triangle, oblique triangle, flat shape triangle, isosceles triangle, long triangle, triangle, right triangle, triangle, *bermuda* triangle (names given by students based on their own words), irregular triangle, inverted triangle, square, acute angle triangle, rectangle.
3. d	Square, quadrangle, quadrilateral, square 5, pentagon, quadrangle, cone, rhombus, rhombus, parallelogram, square kite, *lontong* (rhombus-shaped food name), dubus (The name given by the student is based on his own words because it sounds almost the same as a cube in Indonesian), kite quadrilateral, square triangle, square crystal.
3. e	Parallelogram, slanted square, oblique rectangle, oblique rectangle, parallelogram, irregular rectangle, parallelogram, slanted rectangle, rhombus, oblique rectangle, cube, flat shape, square side, square, oblique facet, facet, facet 4.
3. f	Kite shape, flat shape down, square, kite square, long cone, kite drawing, kite, kite, parallelogram, rectangular kites, kites, kites, rectangles, flat triangles, kites, quadrilaterals, crystal terms, kite squares, triangles, rectangles.
3. g	Trapezoid, square box, rectangular shape, flat shape, hollow square, quadrilateral, radial symmetrical/parallel square, quadrangle, inclined square, pentagon, oblique, square, square, steel front cage, square facet, triangle, square, square, square.

Table 5: Student's answer to question number 3

Students still have difficulty in determining the name of the plane shape given to the problem, both in mentioning a special name or the name of the shape based on its many sides. Based on the student's answers above, some students mention the name of the plane shape that relates to the objects around them, such as the rice cake to build a rhombus. This is because of the rice cake (ketupat) that they often encounter in their daily lives. And some students mention kites as “crystal facets” because the kite pictures given are associated with crystal shapes in two dimensions. The following is an example of the answers given by students:

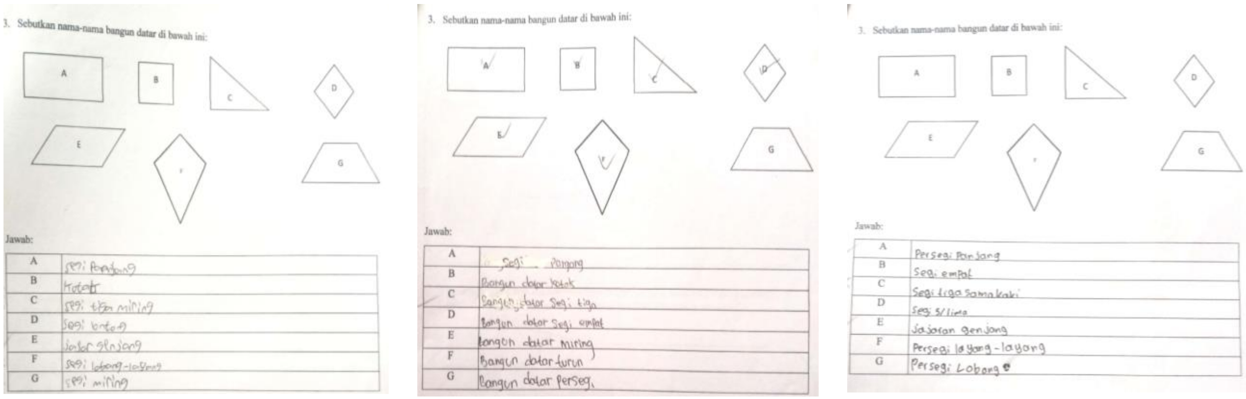


Figure 3: Example of student answers for question number 3

DISCUSSION

This study investigates students' conceptual understanding of polygons. In particular, students understand the concepts of polygons, regular and irregular polygons, as well as the names of plane shapes that have been taught in elementary school. The findings of this study indicate that some students (50%) succeeded in determining the polygon shape but could not give the right reasons for the choice answers that they are chosen. This study shows that students do not understand the characteristics of polygons. Misunderstanding students happen because some students lack a basic conceptual understanding of geometry (Chiphambo & Feza, 2020). In addition, students have several misconceptions and lack background knowledge and reasoning in studying geometry material (Özerem, 2012).

The concept of plane shapes, especially the concept of polygons, is one of the concepts that must be taught to students so that it is easy to understand the following material in learning mathematics, especially geometry material. Because when students do not understand the characteristics in terms of many, students will have difficulty in solving problems related to plane shapes, such as determining the area or circumference of a given polygon (Sholihah & Afriansyah, 2017). The difficulty of students in using concepts is the inability of students to express the meaning of terms that represent the concept of polygons and the inability of students to remember a condition that is sufficient for an object to be expressed in terms that represent the concept of polygons (Fauzi & Arisetyawan, 2020).

In addition to the concept of a polygon related to the properties of the polygon itself, students still have difficulty distinguishing the shape of a regular polygon and an irregular polygon. This situation is in line with the expression (Fitri & Lena, 2021) that there are several obstacles for students in studying polygons, including difficulties in understanding the types of plane shapes, sorting out the properties of regular and irregular polygons, and difficulties in determining names based on their properties. The concept of geometry is complex for students. That is because conceptual development in geometry involves several skills and mental constructions that build on each other in complicated ways (Walcott et al., 2009). If students do not understand the names of the given shapes, students will experience problems in classifying plane shapes. As revealed in research (Hallowell et al., 2015; Žilková et al., 2015), the students have difficulty identifying the properties of plane shapes correctly, and the effects are students having difficulty in classifying geometry shapes. Fujita et al. (2019) suggest that a systematic stage is needed in defining and classifying geometric shapes. That is because students' interpretation of the meaning of mathematics is subjective and temporary but, in the process, becomes more subtle and objective.

The findings in this study also show that students use informal language (own language) in determining the names of geometric shapes. For example, when students state that the rhombus is a *lontong* (food name "ketupat"), this shows that students can integrate real life with geometric shapes. However, the actual concept of students does not understand how the characteristics of the rhombus itself. Likewise, with parallelogram shapes, students use non-mathematical language when mentioning the name of the shape. Students use the term "sloping rectangle or rectangle with beveled edges" to construct a parallelogram (Walcott et al., 2009). Although there are students who understand geometry in formal language, they talk about it informally (Budiarto & Artiono, 2019). In studying geometry, students not only have to understand theorems but also need students' ability to understand terms in geometry so that they do not become obstacles in learning geometric concepts (Chiphambo & Feza, 2020) because in geometry many subjects are interconnected. Therefore, geometry teachers need to investigate the understanding of their students to provide meaningful learning experiences at certain developmental levels. (Feza & Webb, 2005).

Based on Van Hiele's thinking stage, the students in this study had not yet reached the Analysis stage. This problem is because there are still many students who are not able to identify the properties of polygons (Žilková et al., 2015). in addition, several studies revealed that the students' geometric perception level was not at the expected level (Clements & Battista, 1992). Several factors cause students not to be at the right stage in learning geometry, including a lack of understanding of concepts and properties of polygons, a lack of welling understanding of the previous material, and a lack of student skills in using geometric ideas in solving mathematics problems. (Sholihah & Afriansyah, 2017).

Thus, the findings of this study are not in line with the expectations (The National Council of Teachers of Mathematics (NCTM), 2000), which claims that "In Pre-K to grade two all students must recognize, name, construct, draw, compare, and sequence two- and three-dimensional shapes". However, most of the children in this study did not succeed in identifying the names of polygons. Therefore, teaching polygons material is necessary to cultivate basic concepts. Teachers can direct students to gradually understand a concept from polygonal shapes and provide examples and non-examples of polygonal shapes so that students can describe polygons, both regular and irregular polygonal shapes. This problem is in line with what was expressed by (Bernabeu et al., 2021) in teaching the concept of polygons, and it is necessary to define the relationship between perceptual, discursive, operative, and sequential understandings as teaching objectives that explicitly pay attention to the properties of existing shapes. In a sense, to stimulate understanding of the polygon concept, it is expected to recognize examples and non-examples of polygon and be able to turn non-examples into examples of polygonal shapes by explaining and drawing.

CONCLUSIONS

The conclusion that can be summarized from the discussion above is that students have difficulty understanding the concept of polygons. The students' problem with polygons concepts is that students find it complicated to describe the properties of polygons, so they have difficulty defining the shape of polygons. The impact of students' misunderstanding of the properties of polygons is the difficulty in determining regular and irregular polygons. In addition, the problem that occurs to students is that students give inappropriate reasons to state the nature of the shapes. The next problem is that there are still students who cannot mention plane shapes with the correct names, resulting in students having difficulty learning the following geometric material.

The problems of students are the incompleteness of students understanding of a concept, whereas it becomes a part of learning the following concepts. For example, if students do not know the names of the shapes given by the teacher, this will result in students cannot describe the properties of these shapes. In other words, students do not complete the stage of thinking level one from the level of thinking based on Van Hiele's theory and will have problems continuing to the next level of thinking. Giving a diagnostic test before learning is one of the ways to solve these problems. The diagnostic test is necessary to find students' difficulties regarding the prerequisite material for further instruction. Thus, based on the test results, we can design learning activities based on the students' levels of thinking.

Understanding the concept in the study of geometry is necessary so that it does not become a problem in learning other mathematical materials. Thus, teachers should be able to make learning designs with appropriate didactic situations to make students understand the concepts being taught by paying attention to the problems faced by students. This research is limited to polygon material. The recommendation for further researchers is to conduct similar research on other materials. This research can be a discourse of knowledge that may be useful for future researchers to conduct research with similar themes so that, for example, they can generalize the conclusions of richer research results.

REFERENCES

[1] Aksu, A. D. (2013). Predicting the Geometry Knowledge of Pre-Service Elementary Teachers. *Cumhuriyet International Journal of Education-CIJE*, *2*(July), 15–27.

[2] Alawiyah, A., Waluya, S. B., Priyono, A., & Prasetyo, B. (2018). Didactical Situations of Students' Mathematical Reasoning Based on the Learning Obstacle on Quadrilateral

Areas. *Unnes Journal of Mathematics Education Research, 7*(2), 196–203.

[3] Angraini, P., Prahmana, R. C. I., & Shahrill, M. (2021). The Innovative Learning of Square and Rectangle using Macanan Traditional Indonesian Game. *Southeast Asian Mathematics Education Journal*, *11*(2). https://doi.org/10.46517/seamej.v11i2.129

[4] Ayvaz, U., Gundus, N. and, & Bozkus, F. (2017). Understanding of Prospective Mathematics Teachers of the Concept of Diagonal. *Journal on Mathematics Education, 8*(2), 165–184.

[5] Bernabeu, M., Moreno, M., & Llinares, S. (2021). Primary school students' understanding of polygons and the relationships between polygons. *Educational Studies in Mathematics, 106*(2), 251–270. https://doi.org/10.1007/s10649-020-10012-1

[6] Biber, A. Ç., Tuna, A., Korkmaz, S., & Aliustaoğlu, F. (2018). Analysis of Defining and Drawing Skills of Secondary School Students: Parallelogram Example *. *MATDER Journal of Mathematics Education, 3*(1), 32–40.

[7] Budiarto, M. T., & Artiono, R. (2019). Geometri Dan Permasalahan Dalam Pembelajarannya (Suatu Penelitian Meta Analisis). *JUMADIKA: Jurnal Magister Pendidikan Matematika*, *1*(1), 9–18. https://doi.org/10.30598/jumadikavol1iss1year2019page9-18

[8] Chiphambo, S. M., & Feza, N. N. (2020). Exploring geometry teaching model: Polygon pieces and dictionary tools for the model. *Eurasia Journal of Mathematics, Science and Technology Education*, *16*(9). https://doi.org/10.29333/EJMSTE/8358

[9] Clements, D. H., & Battista, M. T. (1992). *Geometry and spatial understanding. Handbook of research mathematics teaching and learning. (Edt: D. A. Grouws)*. McMillan Publishing Company.

[10] Clements, Douglas H., Sarama, J., & DiBiase, A.-M. (2004). *Engaging Young Children in Mathematics: Standards for Early Childhood Mathematics Education*. Lawrence Erlbaum Associates, Inc., Publishers.

[11] Colicol, F. L., JR, R. L. R., & Galarosa, L. R. (2017). Performance of Grade VI Pupils on Angles and Polygons. *International Journal of Educational Science and Research*, *7*(2), 155–164.

[12] Erşen, Z. B., Bülbül, B. Ö., & Güler, M. (2021). Analysis of Solved Examples in Mathematics Textbooks Regarding the Use of Geometric Habits of Mind. *Turkish Journal of Computer and Mathematics Education (TURCOMAT)*, *12*(1), 349–377. https://doi.org/10.16949/turkbilmat.850882

[13] Fajari, U. N. (2020). Analisis Miskonsepsi Siswa pada Materi Bangun Datar dan Bangun Ruang. *Jurnal Kiprah*, *8*(2), 113–122. https://doi.org/10.31629/kiprah.v8i2.2071

[14] Fauzi, I., & Arisetyawan, A. (2020). Analisis Kesulitan Belajar Siswa pada Materi Geometri di Sekolah Dasar. *Kreano, Jurnal Matematika Kreatif-Inovatif, 11*(1), 27–35. https://doi.org/10.15294/kreano.v11i1.20726

[15] https://doi.org/10.24247/ijesrapr201718

[16] Feza, N., & Webb, P. (2005). Assessment standards, Van Hiele levels, and grade seven learners' understandings of geometry. *Pythagoras*, *62*(Desember), 36–47.

[17] Filiz, A., & Gür, H. (2021). The Effect of the Cognitive Learning Model Integrated with ARCS Categories on the Learning and Motivation Levels of Students About Polygons and Triangles. *Necatibey Eğitim Fakültesi Elektronik Fen ve Matematik Eğitimi Dergisi, 15*(1), 186–215. https://doi.org/10.17522/balikesirnef.907736

[18] Fitri, E. D., & Lena, M. S. (2021). Pengaruh Video Pembelajaran Terhadap Hasil Belajar Sifat-Sifat Segi Banyak Beraturan dan Tidak Beraturan di Kelas IV SD Wilayah II Kabupaten Pasaman. *Jurnal Pendidikan Tambusai, 5*(2), 4373–4380.

[19] French, D. (2004). *Teaching and Learning Geometry*. Continuum.

[20] Fujita, T. (2012). Learners' level of understanding of the inclusion relations of quadrilaterals and prototype phenomenon. *Journal of Mathematical Behavior, 31*(1), 60–72. https://doi.org/10.1016/j.jmathb.2011.08.003

[21] Fujita, T., Doney, J., & Wegerif, R. (2019). Students' collaborative decision-making processes in defining and classifying quadrilaterals: a semiotic/dialogic approach. *Educational Studies in Mathematics, 101*(3), 341–356. https://doi.org/10.1007/s10649-019-09892-9

[22] Fujita, T., & Jones, K. (2007). Learners' understanding of the definitions and hierarchical classification of quadrilaterals: Towards a theoretical framing. *Research in Mathematics Education, 9*(1), 3–20. https://doi.org/10.1080/14794800008520167

[23] Gal, H., & Linchevski, L. (2010). To see or not to see: Analyzing difficulties in geometry from the perspective of visual perception. *Educational Studies in Mathematics, 74*(2), 163–183. https://doi.org/10.1007/s10649-010-9232-y

[24] Gall, J. P., Gall, M. D., & Borg, W. R. (2009). *Applying Educational Research*. Prentice Hall PTR. https://books.google.co.id/books?id=m3uJoAEACAAJ

[25] Hallowell, D. A., Okamoto, Y., Romo, L. F., & La Joy, J. R. (2015). First-graders' spatial-mathematical reasoning about plane and solid shapes and their representations. *ZDM Mathematics Education, 47*(3), 363–375. https://doi.org/10.1007/s11858-015-0664-9

[26] Hartono, S. (2020). Effectiveness of Geometer's sketchpad learning in two-dimensional shapes. *Mathematics Teaching-Research Journal, 12*(3), 84–93.

[27] Herbst, P., Fujita, T., Halverscheid, S. and, & Weiss, M. (2017). *The Learning and Teaching of Geometry in Secondary Schools a Modeling Perspective*. Routledge.

[28] Hidayat, C. (2019). Epistemological obstacle on the topic of triangle and quadrilateral. In *Journal of Physics: Conference Series* (Vol. 1157, Issue 4). https://doi.org/10.1088/1742-6596/1157/4/042110

[29] Jones, K. (2002). Aspects of Teaching Secondary Mathematics. In *Issues in the Teaching and Learning of Geometry* (pp. 121–139). https://doi.org/10.4324/9780203165874

[30] Loc, N. P., Tong, D. H., & Hai, N. T. B. (2017). The Investigation of Primary School Students' Ability to Identify Quadrilaterals: A Case of Rectangle and Square. *The International Journal of Engineering and Science, 06*(03), 93–99. https://doi.org/10.9790/1813-0603019399

[31] Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis*. Sage Publication.

[32] National Council of Teachers of Mathematics (NCTM). (2000). *Principles and Standards for School Mathematics, 2000. Math Standards and Expectations: Geometry Strand.*

[33] Özerem, A. (2012). Misconceptions in Geometry and Suggested Solutions For Seventh Grade Students. *Procedia - Social and Behavioral Sciences, 55,* 720–729. https://doi.org/10.1016/j.sbspro.2012.09.557

[34] Puspasari, L., Zulkardi, Z., & Somakim, S. (2015). Desain Pembelajaran Luas Segi Banyak Menggunakan Tangram Berpetak di Kelas IV. *JINoP (Jurnal Inovasi Pembelajaran), 1*(2), 150. https://doi.org/10.22219/jinop.v1i2.2566

[35] Rogers, V. L. C. (1995). Teaching geometry in the elementary classroom. *Theses Digitization Project. 1044.*

[36] Sholihah, S. Z., & Afriansyah, E. A. (2017). Analisis Kesulitan Siswa dalam Proses Pemecahan Masalah Geometri Berdasarkan Tahapan Berpikir Van Hiele. *Mosharafa, 6*(2).

[37] Sopiany, H. N., & Rahayu, W. (2019). Analisis miskonsepsi siswa ditinjau dari teori kontruktivisme pada materi segiempat. *Jurnal Pendidikan Matematika, 13*(2), 185–200.

[38] Walcott, C., Mohr, D., & Kastberg, S. E. (2009). Making sense of shape: An analysis of children's written responses. *Journal of Mathematical Behavior, 28*(1), 30–40. https://doi.org/10.1016/j.jmathb.2009.04.001

[39] Žilková, K., Guncaga, J., & Kopácová, J. (2015). (Mis)Conceptions about Geometric Shapes in Pre-Service Primary Teachers. *Acta Didactica Napocensia, 8*(1), 27–35.

A visualization approach to multiplicative reasoning and geometric measurement for primary-school students: a pilot study

Sonal Jain[1], Ho-Hon Leung[1*], Firuz Kamalov[2],

[1,2]Department of Mathematical Sciences, United Arab Emirates University, Al Ain, United Arab Emirates

[2]Faculty of Engineering, Applied Science and Technology, Canadian University Dubai, Dubai, United Arab Emirates

*Correspondence: hohon.leung@uaeu.ac.ae

sonaljainmaths@gmail.com, firuz@cud.ac.ae

Abstract: Understanding the concept of area requires an understanding of the relationship between geometry and multiplication. The multiplicative reasoning required to find the areas of regular figures is used in many courses in elementary mathematical education. This paper explores various methods in which multiplicative reasoning is incorporated into the measurement of area. The main goal is to provide tasks that encourage the application of multiplicative reasoning when students are asked to measure the areas of geometric figures. Student performance is analyzed in two pilot studies of the relationship between geometric measurement and multiplicative reasoning.

Keywords: Area measurement, geometric measurement, matrix array structure, mathematics education, multiplicative reasoning.

INTRODUCTION

In mathematics education, multiplicative reasoning (MR) is a well-studied topic. According to Jacob and Willis (2003), "MR results in a multiplicative reaction to a circumstance by identifying or constructing a multiplicand, a multiplier and their simultaneous coordination in that context". It involves paying attention to the multiplicative relationship between magnitudes and quantities, as well as the ability to deal with such situations numerically. MR can be used to understand the inverse relationship between multiplication and division, the part-whole relationship, fractions and proportions, among other things. Recent studies (Al Farra et al., 2022; Khairunnisak et al., 2021; Lyublinskaya, 2009; Migon & Krygowska 2007; Putrawangsa et al., 2021) demonstrated that various in-classroom activities, by implicitly incorporating MR, help students' learning on various mathematical concepts in elementary and middle schools. By contrast, the measurement of area is less studied, with even fewer studies discussing directly the link between measuring and MR (Cavanagh, 2007; Chen & Herbst, 2013; Moyer, 2001; Nohda, 2000; Schoenfeld, 2016; Susac, et al., 2014; Yeo 2008).

In geometry, a measurement involves calculating a new quantity---`the number of units'---by deriving it from known quantities, such as the unit's magnitude and the magnitude of the space to be measured. The target magnitude is the product of the unit and the space to be measured. The concept of geometric measurement is usually based on MR rather than direct counting, and several authors have argued that elementary students are being taught how to measure rather than how to develop the mathematical concept behind measuring certain objects (Carpenter et al., 2003; Ellis, 2007; Fernández et al., 2014; Fujii & Stephens, 2001; Fujii & Stephens, 2008; Greens & Rubenstein, 2008; Hunter 2007; Kieran, 2004; Lins & Kaput 2004; Medov et al., 2020; Molina & Ambrose, 2008; Mulligan et al., 2009; Naik et al., 2005; Tan Sisman & Aksu, 2016; Widjaja & Vale, 2021). Smith III et al. (Smith III et al., 2013) stated that only some students understood that the unit of measurement might be broken down further into smaller subunits to improve precision. In the present study, we look at how MR is used in geometric area measurement, with a particular focus on developing tasks that link the use of MR to finding the area of geometric figures.

In the literature on MR, most of the situations and contexts examine proportionality and involve a linear relation between two one-dimensional measures (Abramovich & Pieper 1995; Ayalon et al., 2016; Cheng et al., 2017; Carpenter et al., 2003), such as those between weight and cost, wage and time, distance and speed, etc. Each of these one-dimensional measures is analogous to length, and many of the issues examined in the context of proportionality have their geometric equivalents in the case of length measurement. For example, in proportional reasoning, unitisation (the process of cognitively chunking discrete units into a bigger, more convenient unit or dividing a unit into smaller units) is crucial (Lamon 2007), and flexible unitisation is used in operations that demand the creation of a `unit of units' (Reynolds & Wheatley 1996). In proportional reasoning, the number line---which is a direct representation of length---can be useful. Curry, Mitchelmore, and Outhred (Curry et al., 2006) outlined the following five measuring principles: (i) the need for congruent units, (ii) using an appropriate unit, (iii) using the same unit for comparing objects, (iv) the relation between the unit and the measure, and (v) the structuring of unit iteration. Each of the five principles listed above requires an understanding of the multiplicative relationships between various quantities in the context of geometric measurement.

As we move from linear measurements to other types of geometric measurement, we discover that MR is used in new ways. In the case of area measurement, MR is first required in the same manner as in length measurement: (i) using sub-units and chunked units (unit of units) in estimating area and (ii) the inverse relationship between size of the unit and the measure. On the other hand, MR also appears in ways in which length measurement does not, such as the array structure of units in the case of rectangles or other two-dimensional figures, leading to the area formula as the product of length and width. There is also a multiplicative relationship between the rectangle's area and the unit, as well as among the area, the length and the width. Similarly, the area and the magnitude of the area unit---which depends on the length and width of the unit---have an inverse relationship. In addition, the transition to non-rectangular figures involves triangulation, and the base case for triangulation, i.e. finding the area of a right-angled triangle by taking half of a rectangle, naturally requires a multiplicative relationship. Consequently, we see that MR is used explicitly and implicitly in various ways in various geometric measurements.

Geometric Measurement of Area: Two Pilot Studies

The present study is part of a larger one aimed at enhancing teachers' knowledge of students' thinking by designing tasks that support reflective teaching. The phase of the study reported herein explores students' algebraic reasoning when exposed to early algebraic ideas through contexts such as number sentences, pattern generalization, proof and justification, etc. Based on students' reasoning, we plan to prepare student cases for discussion among mathematics teachers and teacher educators. By analyzing students' strategies for area measurement, the present study aims to examine the importance of MR in understanding the relation between spatial attributes (e.g. area) and the numerical values assigned to them. The study aims to develop tasks that have the potential to stimulate different modes of MR in area measurement.

As an example, in the study by Reynolds and Wheatley (Reynolds & Wheatley 1996), students were asked how many cards of a given size (3 cm x 5 cm) are required to cover a rectangle of dimensions 15 cm x 30 cm. The researchers were initially unaware of the fact that the students had understood that the two dimensions of the small card have to completely divide those of the large rectangle to ensure that the number obtained by dividing the bigger area by the smaller area of a card is a whole number. This gives an instance in which the unit is related to the target measure of area of the space in terms of not only the multiplicative relationship between the magnitude of the unit and that of the target measure but also that between either the length or breadth of the unit and the target measure.

Herein, we report two pilot projects labeled 1 and 2 that were done by means of task-based interviews of students. The aim of the two pilot projects was to create tasks that explore MR in activities related to area measurement, and gather preliminary data in the form of student replies and their performances in the given tasks. We picked two different groups of students for these two pilot studies.

METHODOLOGY

Pilot study 1. Data Collection

The first task in the study involved tiling exercises in which students had to determine the number of cards of given size required to cover the area of a rectangular piece of paper. The interviews of the students were recorded with their permission and used for analysis. The concept behind pilot study 1 was that it involved not only the numerical relationship between the unit and the measure but also the spatial structuring of the units.

The sample for pilot study 1 comprised ten grade-5 children in two classes from the same school, which was BSC Public Secondary School Kheroda in Udaipur, Rajasthan, India. Five children were picked randomly from each class, and of the ten children, five were labeled as above average, three were labeled as average and two were labeled as below average, with the labeling based on evaluation by their class teachers.

Tasks involved in Pilot Study 1

We constructed three tiling tasks (see Table 1) that are comparable to those used by Reynolds and Wheatley (Reynolds & Wheatley 1996). In the first two tasks (T1 and T2), the chosen tile area was a factor of that of the given rectangle to be covered, but a dimension of a tile was not necessarily a factor of the corresponding dimension of the given rectangle; for example, each of the tiles used in the second task (see case S4 of task T2) had dimensions of 3 cm x 2 cm, and the tile width (2 cm) was not a factor of the length (19 cm) of the given rectangle (see Figure 1).

Task	Material provided	Dimension of rectangle	Shape and dimensions of tile(s)
T1	Rectangular paper sheet and three different paper tiles	21 cm x 12 cm	Rectangle 2 cm x 2 cm (S1), 3 cm x 4 cm (S2), 6 cm x 2 cm (S3)
T2	Dimensions of rectangle and tiles	19 cm x 6 cm	Rectangle 3 cm x 2 cm (S4)
T3	Dimensions of rectangle and tiles	15 cm x 8 cm	Right-angled triangle height: 5 cm, base: 2 cm (S5)

Table 1: Description of tasks in pilot study 1

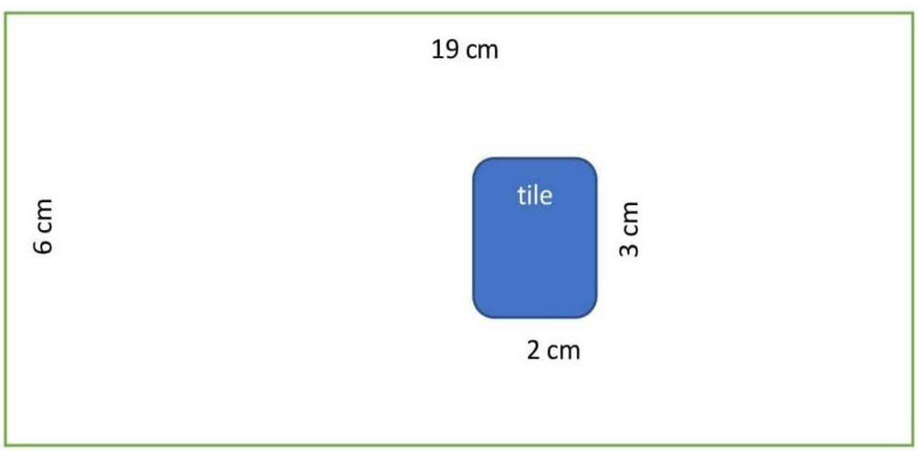


Figure 1: The tile's width (2 cm) is not a factor of the rectangular length (19 cm)

In each of the tiling tasks, students were asked whether the given tiles, glued over and over, might cover the rectangle and about the number of tiles required to do that. In task T1, students were given a rectangular sheet of the given size and three different types of paper tiles (see S1, S2 and S3). In tasks T2 and T3, students were simply informed verbally about the sizes of the rectangle and tiles, with no tangible objects with which to work; in task T3, students were told to use right-angled triangular tiles to cover the rectangle.

In task T1, students were given physical objects with which to work. In case S2 of task T1, both the length and width of the given rectangle are clearly divisible by the corresponding length and width of the given tiles, so they can tile the rectangle. In cases S1 and S3 of task T1, although the area of the given rectangle is divisible by that of the given tiles, they cannot tile it because the given tiles have even-numbered dimensions and can cover only rectangles of even-numbered dimensions. Specifically, in cases S1 and S3 of task T1, the dimensions of the tiles are 2 cm x 2 cm and 6 cm x 2 cm respectively. Hence, they can tile rectangles of even-numbered length and even-numbered width only. But, in the task T1, the given rectangle has an odd-numbered length (21 cm). As a result, the rectangle given in task T1 cannot be covered by tiles in cases S1 and S3.

The most challenging task is T2. Although the length of the given rectangle is divisible by neither the length nor the width of the given tiles, they can still tile it. Doing so requires a special arrangement of the tiles in the rectangle, and Figure 2 shows a solution for task T2.

In case S5 of task T3, it is obvious that the given tiles can tile the given rectangle. Two of the given right-angled triangles can be glued together to form a 5 cm x 2 cm rectangle, and clearly these rectangles can tile the given 15 cm x 8 cm rectangle.

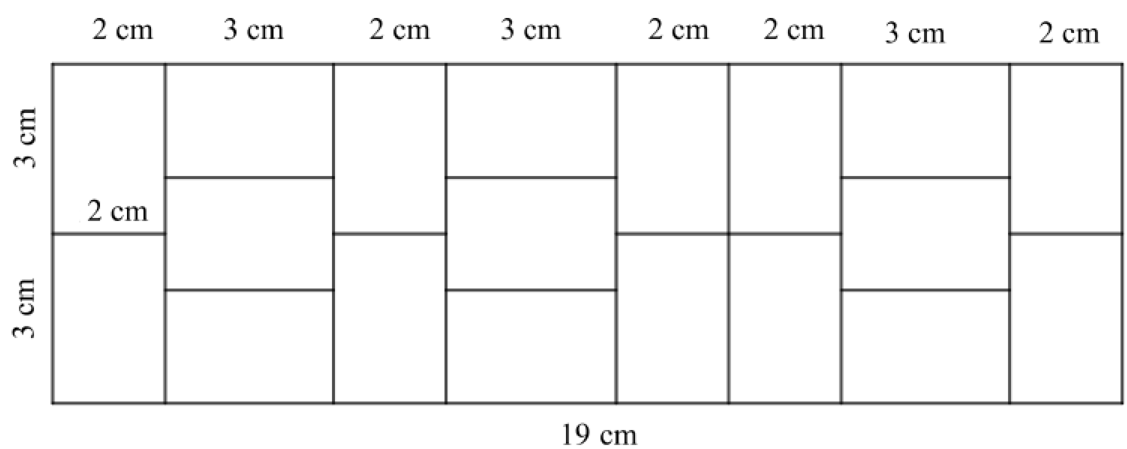


Figure 2: A solution for task T2.

Pilot Study 2. Data Collection

In pilot study 2, the student interviews were analyzed with the agreement of the children and their parents, and the data from the interviews were collected and analyzed. The sample comprised eight grade-5 students from the same school as in pilot study 1, and all the students were given identical tasks on which to work.

Tasks involved in Pilot Study 2

There were four tasks in total, which are described below.

Task U1: The students were asked to compare two sets of rectangular sheets that differed slightly in either length or width but not both. The goal of this activity was to observe the tactics used by the students in the given task, which were either (i) direct comparison of the sizes of the two sheets by physically trying to cover one with the other or (ii) comparison of one-dimensional quantities such as the lengths or widths of the two sheets. For example, in Figure 3, students were given two sets of rectangular sheets. The first one is a 5 cm x 6 cm rectangle; while the second one is a 5 cm x 7 cm rectangle.

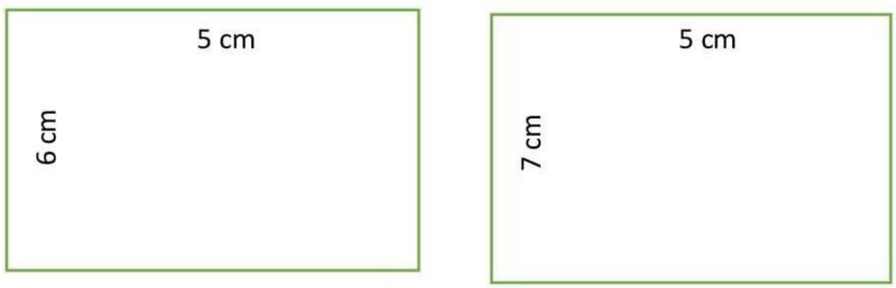


Figure 3: Task U1: Comparison of two rectangle sheets that differ in length but not in width.

Task U2: This task consisted of three closely related but different exercises.

In the first exercise, each student drew randomly from a set of composite numbers and then was asked to collect that number of 1 cm x 1 cm cards. Next, they were asked to construct a rectangle by using all the cards.

The second exercise was essentially the same as the first one, but the students had to complete the task without being given any physical objects: each student drew randomly from a set of composite numbers and then was asked to say whether a rectangle could be constructed by using exactly that number of 1 cm x 1 cm cards.

In the third exercise, each student drew randomly from a set of both prime and composite numbers and then was asked to say whether a rectangle could be constructed from that number of 1 cm x 1 cm cards; if their answer was yes, then they were asked to describe the dimensions of such a rectangle.

The first exercise gave students the opportunity to connect the number of cards with the size of a rectangular array, while the second and third exercises gave students the opportunity to strengthen their mental arithmetic skills. Implicit in this challenge is the multiplicative relationship between the given number and its factors as the length and width of a rectangle. Because the first exercise required physical action by the students, we could determine whether they were implicitly aware of the stated multiplicative relationship, even if they could not articulate it explicitly.

Task U3: Students were asked to compare the sizes of a 7 cm x 7 cm sheet and an 8 cm x 6 cm sheet. They were also given a box full of 1 cm x 1 cm cards to use if necessary (see Figure 4). This task allowed us to investigate the various tactics used by the students to compare the sizes of the sheets (e.g. array structuring, complete covering, etc.). Physically trying to cover one sheet by the other is not helpful in reaching the correct answer, and we investigated whether the students had understood the mathematical concepts from the two preceding tasks (U1 and U2), such as whether they used multiplicative relationship or repetitive addition to measure the two different regions.

Task U4: Students were given an A4 page for this task and were free to use any of the items from the preceding task. They were asked to use either the rectangular sheets or the 1 cm x 1 cm cards from the preceding task to calculate the size of the A4 sheet, and then were asked to use it to calculate the size of a table. This activity was designed to assess whether the students could apply their knowledge of area measurement to larger regions. The use of tangible materials allowed the students to work with the items in whatever way they wished, and the use of repeated multiplicative operations was required in this task to reduce the number of steps needed to reach the final answer.

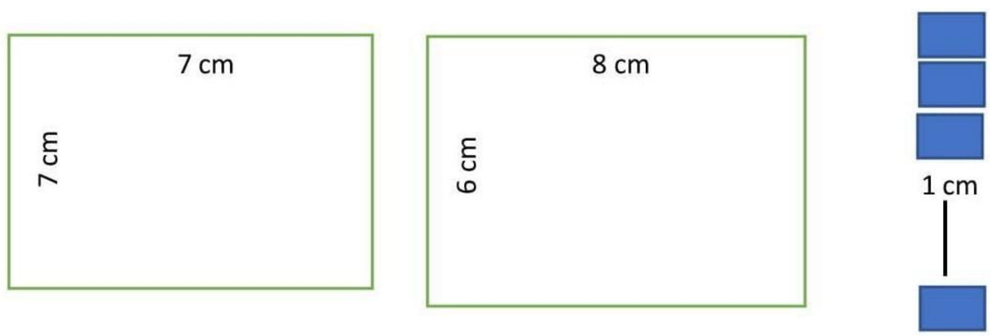


Figure 4: Task U3: Comparison of two rectangular sheets. Students can choose to complete the task by using 1 cm x 1 cm cards.

RESULTS AND DISCUSSION

Results and Analysis of Pilot Study 1

After all students had completed the tiling exercises, we noticed that most had completed tasks T1 and T2 using two different tactics, which are explained as follows.

(1) Dividing the area of the given rectangle (determined by numerically multiplying its length by its width) by that of a tile to find out whether the tiles can be used to cover the rectangle fully.

Four students completed all the tasks by using this method. Therefore, in the cases in which a tile dimension is not a factor of the corresponding rectangle dimension (see cases S1 and S3 of task T1), those students provided incorrect answers to the exercises. The tricky point in these situations is that the given tiles (of even-numbered dimensions) cannot cover the given rectangle of length 21 cm, even though the area of the given titles (4 square centimeters in case S1 and 12 square centimeters in case S3 respectively) divides the area of the given rectangle (252 square centimeters).

In particular, in task T1, they did not take advantage of being given physical objects with which to work (a piece of rectangular paper and the tiles of given size). Their performance showed that they did not fully understand the importance of the dimensions of the objects in such tiling tasks, nor did they understand the close relationship between the spatial structures of the tiles and that of the rectangular paper.

(2) Checking the dimensions of the tiles along the dimensions of the rectangular sheet.

Four students used this strategy in all the tiling tasks, giving all the answers correctly in task T1 but not in task T2. After pilot study 1, several students indicated that they would have solved task T2 correctly had they been given physical objects with which to work. Clearly, those students understood the importance of measuring the dimensions of the objects in such tiling tasks, and they showed a deeper understanding of the geometrical structures of the objects. Also, a few of them may even have understood that the rectangular sheet can be covered fully by arranging the tiles in special ways, as mentioned in the `Remarks' section.

Interestingly, two students started task T1 by using the second strategy but then switched to the first strategy for tasks T2 and T3. This shows that some students may be inclined to look at the dimensions of the objects when they are given physical objects with which to work (task T1), rather than merely the numerical values of the areas of the figures. They solved task T1 correctly, which may have been simply by trial and error with the physical objects in their hands; it is unclear whether they really understood the importance of the dimensions of the objects in these tiling exercises. Further investigation and studies may be required to see whether these students understood implicitly the multiplicative relationship between the dimensions and areas of the figures.

None of the students solved task T2 completely correctly, which is perhaps understandable given that they were given this task only verbally; it may be necessary to give students the physical objects with which to play. Tasks in this format (with no physical objects given to the students) may be too challenging even for students with a good understanding of the dimensions of objects in tiling exercises.

In task T3, every student stated initially that the triangular tiles could not be used to cover the rectangle entirely, but two students changed their minds later after realizing that two such triangles could be combined to form a rectangle. This showed that a few of them were at least implicitly aware of the multiplicative relationship based on the geometric division of a figure into two equal parts.

These examples show that it is not enough to understand the multiplicative relationship between the area of a tile and that of the rectangular paper: it is also necessary to understand the multiplicative relationship between the dimensions of the tiles and those of the rectangular paper. In particular, as shown in cases S1 and S3 in task T1, it is clear that the area of the given rectangle (252 square centimeters) is divisible by the area of the tiles (4 square centimeters in case S1 and 12 square centimeters in case S3 respectively). But the given tiles (of even-numbered dimensions) cannot be used to cover a rectangle which has an odd-numbered length (21 cm).

In addition, in task T2, the students' performance could have been much better had they been given physical objects. Further investigation may be required to understand how physical exercises such as this can improve students' learning in mathematics education in a classroom setting.

Discussion about Pilot Study 1

Before pilot study 1, eight of the ten students had learned the formula for the area of a rectangle (as the product of its length and width), and so they calculated the area by using the product formula during the tasks. Furthermore, they divided the area of the rectangular sheet by that of a tile of given size so that they could relate the tiling problem to the areas of the objects. However, they did not give correct answers to all the tiling tasks (in particular, cases S1 and S3 of task T1). This was mainly because they failed to check whether the tile dimensions divided the rectangle ones. This shows that several students may have learned how to compute the area of a figure simply by memorizing the formula correctly, without a deeper conceptual understanding of the multiplicative relationship between the dimensions and area of a figure.

As a side note, it is worth comparing the students' academic performance in school with their results in these tasks. All the students labeled above average or average (eight in total) had learned the formula for the area of a rectangle, but two used the first strategy to tackle all the tasks. In particular, they did so by ignoring the dimensions of the objects in task T1, for which the physical objects were provided. This shows that several students who achieved good academic results in mathematics may not have fully understood the concepts behind the formulas learned in class. To foster students' conceptual understanding in mathematics education, it may be important to provide physical tasks (such as those in pilot study 1) more frequently to students; such hands-on experience may help students to understand the multiplicative relationship between the dimensions and area of a figure.

It is unclear whether the students who used the formula for the area of the rectangle were aware of the array of unit squares that covered the rectangle. The data from pilot study 1 show that more research is needed into students' understanding of the relationship between MR and area measurement. In particular, it is necessary to create challenges that could elicit such thinking.

In pilot study 2 discussed next, we devised a series of activities focused on the array structure of square units in rectangles of various dimensions.

Results and Analysis of Pilot Study 2

In the following subsections, we report the results pertaining to the students' performances in pilot study 2. We make detailed observations on their behavior throughout the tasks, and we analyze their performances as follows.

Task U1: The difference in length or width (but not both) was not visually clear, and the students could not easily tell the difference between the paper sheets by simply glancing at them. When the rectangular pieces of paper were laid flat on the table next to each other, all but one of the eight students tended to compare them by length or width. Later, they compared the sheets by overlapping them. To compare the sizes of the rectangles, the natural tendency of most students was to compare their sides, which reflects an intuitive awareness of the relationship between the dimensions and area of a rectangle.

Task U2: After a few failed attempts, four of the students gradually understood the relationship between the factors of the given composite number and the rectangle that could be constructed from it. However, the remaining four students were unable to spot this relationship during the exercises. It seems that several of the students used the multiplicative relationship between the number of cards and their arrangements in a two-dimensional array in some of these exercises. On numerous occasions, we found that students used a factor of the given composite number to make the first row of cards, but they were unable to explain why they chose that particular number to start with. The conceptual understanding of the multiplicative relationship between numbers and their factors was not solid for these students; this was clear from the fact that they did not use the same technique consistently to solve the exercises.

In one instance, a student constructed a 4 cm x 3 cm rectangle and a 6 cm x 2 cm rectangle by using 12 square cards for each of them; he also said that a 3 cm x 5 cm rectangle could be created by using 15 cards. However, when asked afterwards about the dimensions of rectangles that could be created with 10 and 13 cards, he responded incorrectly that they would be 3 cm x 7 cm and 3 cm x 10 cm, respectively. Another student, who correctly formed a 7 cm x 2 cm rectangle by using 14 cards, said that an 8 cm x 6 cm rectangle could be formed by using 14 cards. Apparently, he wrongly thought of the relationship between the area and dimensions of the rectangle as being 6+8=14; this mistake arose even though he answered correctly in the preceding exercises. Interestingly, several students attempted to form rectangles with fractions as their widths and lengths. For example, two of the eight students cut a few of the cards in half in an attempt to create rectangles: one student was able to make a 7.5 cm x 2 cm rectangle out of 15 cards, while another student created a 5.5 cm x 4 cm rectangle out of 22 cards. It seems that some students attempted the exercises using this strategy when the given composite number (i.e. 15 and 22 in these cases) had fewer non trivial factors.

After completing the tasks, one of the eight students stated explicitly that he was looking at the factors of the number in order to create the rectangle. We also observed that three students used multiplication tables explicitly and consistently. The remaining four students switched between additive and multiplicative relationships throughout the exercises, without sticking to the same strategy consistently. This shows that the latter four students may not have had a solid understanding of MR. More specifically, we note some interesting facts about the latter four students. Two of them began with a factor of the provided number. One of them was unable to explain why he chose that particular number of cards. Another student claimed that the number occurred to her without her having to think about it. On another occasion, one of the students said that a square can be created using a product of the same number, such as 6 x 6, 1 x 1, 10 x 10 and 3 x3. This shows that a few of them understood the relationship between a number and a square, even though they may not have had an excellent understanding of MR in geometric measurement. It appears that those students (i.e. the first four) who had a better understanding of the concepts of multiplication were able to make the measurement--multiplication connection faster than those who were unaware of it. To certain extent, the remaining four students may have understood multiplication partially, but we cannot confirm that their understanding was solid and complete.

The exercises in task U2 are intriguing because the students had the option to break down a given number by using either additive or multiplicative relationships, and the two ways of thinking---at least in the context of the given exercises---are related somehow to a certain degree. The students may have viewed rectangles in two different ways: either as a two-dimensional array or as the border of an empty rectangle. The latter way of viewing a rectangle was observed in two students, who started to put the cards on the sides of the given rectangle while leaving its interior empty. The impact of these two distinct ways of visualizing a geometric figure while learning about area is intriguing, and it would be interesting to investigate this subject further.

Based on pilot study 2, it is true to say that the students' performances at the tasks improved significantly when they were allowed to work on the exercises physically with the given items. A few of the students may have obtained the correct answers simply by trial and error, without a full understanding of the MR required for the exercises, but the fact that several students took quite a bit of time to get the correct answers when they were given the physical objects to work with suggests that a few of them may have gradually understood the required MR when they worked through the exercises.

Task U3: One student compared the additional space left on both the square and the rectangle after the two sheets were placed one above the other; he discovered that the width of the extra space left on the square was one unit wide, while the length of the extra space left on the rectangle was also one unit long. Consequently, he stated incorrectly that both sheets had the same size. However, he failed to take into account the fact that the extra unit of length left in the square could be filled by using seven 1 cm x 1 cm cards, while the extra unit of width left in the rectangle could be filled by using only six 1 cm x 1 cm cards. The remaining students completed this task correctly but by using different approaches.

The first approach was to cover one sheet by the other and then use the 1 cm x 1 cm cards to measure the size of the extra space left over on the square and the rectangle. Four students took this approach.

The second approach was to find the total number of 1 cm x 1 cm cards required to fill up each sheet. Three students took this approach. To calculate this number, the students first filled the outermost sides of the two sheets with the square cards, then most of them repeated this action until the whole sheets were filled with the cards. This essentially means that these students obtained the area of the sheets by the process of repetitive addition. Only one student calculated the total number of cards required by multiplying the numbers of cards used on the length and width. This is essentially the process of getting the area of a rectangle by the product of its length and width.

Interestingly, most of the students preferred to measure the size of a rectangle by using the given square cards to fill its empty space repeatedly. This is essentially an additive process rather than the more advanced multiplicative process. Even among the students who completed task U2 by using MR, they preferred the additive approach to task U3. The reason for their preference is unclear because they did not mention any specific reason in the interviews after the tasks; it is probably because students of this age group prefer to complete such tasks physically in the most elementary way (if the objects are all given), rather than thinking about the hands-on tasks mathematically.

Task U4: Six students completed this task, while the remaining two students did not. Three of the students who completed the task used the multiplicative relationship, while the remaining three used repetitive addition; the students who did not complete the task simply did not understand the MR required for it, and they were also the students who performed poorly in task U2. For example, the students who used the multiplicative relationship to complete the task computed the total number of cards required to fill the whole table as the product of 10 x 100 once they realized that an A4 sheet could be filled by 100 cards and a table could be filled by ten A4 sheets.

Note that the students were required to use a nonstandard chunked unit (i.e. the A4 paper) rather than a standard square unit to measure the area of the table, and we consider task U4 to be more challenging than task U3. In task U4, the understanding of geometric division of the measure in terms of this new multiplicative unit is required. Put another way, the students had to understand the connection between the numerical and geometrical aspects of the given figures to complete the task. A student was considered to have excellent understanding of MR if they completed this task by using the multiplicative relationship between the chunked unit and the standard square unit.

Discussion about Pilot Study 2

In this pilot study, the students were given opportunities to connect area measurement to multiplication directly. As well as the results stated above, we list the following three key findings.

(1) Several of the students tended to focus on the sides of a rectangle only, rather than on the area that it covered. Consequently, some students tended to compare the sides in task U1, and in task U2, a few of them occasionally failed to fill the inside of the rectangle.

(2) In task U2, when the students were asked to construct rectangles with a given number of cards, a few of them incorrectly used the additive relationship between the numbers rather than the multiplicative one. We consider this error to be a conceptual one, and several of the students clearly lacked the understanding of the MR required in area measurement.

(3) It seems that several of the students understood the multiplicative relationship between numerical quantities and the area of a geometric figure but were unable to explain it explicitly. Consequently, they were unable to articulate the multiplicative concepts involved in the physical tasks of area measurement, even though they got the answer correctly.

From a methodological perspective, the tasks used in pilot study 2 allow students to examine the array structures of geometric figures and their relationships to the multiplication of numbers. Students are also able to explore the concepts of MR based on physical objects in hands, even when they are unable to articulate it.

We recorded and analyzed the students' performances in each of the four tasks separately in this pilot study. However, we did not study how exposure to one task, such as task U2, may have influenced the students' performance in other tasks, and further study may be required in that regard.

CONCLUSIONS

The use of MR has been investigated in various fields in mathematics education. The present pilot studies showed that by connecting geometric measurement to MR, students may improve their understanding of geometric measurement. The two pilot studies allowed students to explore various geometric quantities and structures and their connections to the addition and multiplication of numbers. Providing tasks similar to those in our pilot studies may be useful in fostering students' learning in elementary mathematics education. We note that the sample sizes taken in our pilot studies were small. Inspired by the results shown in these studies, our next goal is to conduct similar studies on a larger sample size of students in classroom settings. On the other hand, it would be equally important for education researchers to conduct similar studies on students chosen from other countries. In the future, we would be interested in collaboration with school teachers in the region. Our main focus is to understand how students, regardless of their nationality and social upbringing, connect MR to different tasks of geometric measurement.

Acknowledgments

The authors are grateful to the financial support given by the Ministry of Education in the United Arab Emirates. The authors are also grateful to the support given by the United Arab Emirates University.

Declarations

Author contribution: Sonal Jain: Conceptualization, Writing - Original Draft & Editing, Formal analysis, and Methodology; Ho-Hon Leung: Writing - Original Draft, Review & Editing, Formal analysis, and Supervision; Firuz Kamalov: Writing - Review & Editing.

Funding Statement: This research was funded by the Collaborative Research Program Grant 2019 (CRPG 2019), which is supported by the Ministry of Education in the United Arab Emirates.

Conflict of Interest: The authors declare no conflict of interest

REFERENCES

[1] Al Farra, N. K., Al Owais, N. S., & Belhase, S. (2022). Computational, Logical, Argumentative, and Representational Thinking in the United Arab Emirates Schools: Fifth Grade Students' Skills in Mathematical Problem Solving. *Mathematics Teaching-Research Journal*, 14(1), 215-252. Retrieved from https://commons.hostos.cuny.edu/mtrj/wp-content/uploads/sites/30/2022/04/v14n1-Computational-Logical-Argumentative.pdf

[2] Abramovich, S., & Pieper, A. (1995). Fostering recursive thinking in combinatorics through the use of manipulatives and computing technology. *The Mathematics Educator*, 6(2). Retrieved from https://openjournals.libs.uga.edu/tme/article/view/1789/1697

[3] Ayalon, M., Watson, A., & Lerman, S. (2016). Progression towards functions: Students' performance on three tasks about variables from grades 7 to 12. *International Journal of Science and Mathematics Education*, 14(6), 1153-1173. https://doi.org/10.1007/s10763-014-9611-4

[4] Battista, M. T. (2007). *The development of geometric and spatial thinking*. In F. K. Lester (Ed.), Second Handbook of Research on Mathematics Teaching and Learning (pp. 843-908). Charlotte, NC: Information Age.

[5] Cheng, S.-C., She, H.-C., & Huang, L.-Y. (2017). The impact of problem-solving instruction on middle school students' physical science learning: Interplays of knowledge, reasoning, and problem solving. *Eurasia Journal of Mathematics, Science*

and Technology Education, 14(3), 731-743. https://doi.org/10.12973/ejmste/80902

[6] Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking Mathematically: Integrating Arithmetic & Algebra in Elementary School.* Portsmouth, NH: Heinemann.

[7] Carraher, D.W. & Schliemann, A. D. (2007). Early Algebra and Algebraic Reasoning. In F. Lester (Ed.) *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 669-705). Charlotte, NC: Information Age.

[8] Cavanagh, M. (2007). Year 7 students' understanding of area measurement. In K. Milton, H. Reeves, & T. Spencer (Eds.), *Proceedings of the 21st biennial conference of the Australian Association of Mathematics Teachers Inc.* (pp. 136-143). Adelaide: Australian Association of Mathematics Teachers.

[9] Chen, C.-L., & Herbst, P. (2013). The interplay among gestures, discourse, and diagrams in students' geometrical reasoning. *Educational Studies in Mathematics*, 83(2), 285–307. http://doi.org/10.1007/s10649-015-9652-9

[10] Curry, M., Mitchelmore, M., & Outhred, L. (2006). Development of children's understanding of length, area and volume measurement principles. In J. Novotna, H. Moraova, M. Kratka, N. Stehlikova (Eds.), *Proceedings of the 30th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 377-384), Prague, Czech Republic. Retrieved from https://research-management.mq.edu.au/ws/portalfiles/portal/17070807/mq-10586

[11] Ellis, A. B. (2007). Connections between generalizing and justifying: Students' reasoning with linear relationships. *Journal for Research in Mathematics education*, 38(3), 194-229. https://www.jstor.org/stable/30034866

[12] Fernández, C. H., De Bock, D., Verschaffel, L., & Van Dooren, W. (2014). Do students confuse dimensionality and "directionality"? *Journal of Mathematical Behavior*, 36, 166-176. http://doi.org/10.1016/j.jmathb.2014.07.001

[13] Fujii, T., & Stephens, M. (2001). Fostering an understanding of algebraic generalisation through numerical expressions: The role of quasi-variables. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the 12 th ICMI Study Conference* (Vol. 1, pp. 258–264). Melbourne, Australia.

[14] Fujii, T., & Stephens, M. (2008). *Using number sentences to introduce the idea of variable.* In *Algebra and algebraic thinking in school mathematics: Seventieth* Yearbook (pp. 127-140). National Council of Teachers of Mathematics.

[15] Greens, C. E., & Rubenstein, R. N. (2008). *Algebra and Algebraic Thinking in School Mathematics; Seventieth Yearbook.* National Council of Teachers of Mathematics.

[16] Hunter, J. (2007). Relational or calculational thinking: Students solving open number

equivalence problems. In J. Watson & K. Beswick (Eds), *Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 421-429), Adelaide: MERGA.

[17] Jacob, L. & Willis, S. (2003) The development of multiplicative thinking in young children. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia* (Vol.1, 460–467), Geelong, Vic: MERGA.

[18] Khairunnisak, C., Johar, R., Yuhasriati, Zubainur, C. M., Suhartati, & Sasalia, P. (2021) Learning Trajectory of Algebraic Expression: Supporting Students' Mathematical Representation Ability. *Mathematics Teaching-Research Journal*, 13(4), 27-41. Retrieved from https://commons.hostos.cuny.edu/mtrj/wp-content/uploads/sites/30/2022/01/v13n4-Learning-Trajectory-of-Algebraic-Expression.pdf

[19] Kieran, C. (2004). Algebraic thinking in the early grades: What is it? *The Mathematics Educator*, 8(1), 139-151. Retrieved from https://citeseerx.ist.psu.edu/viewdoc/download;jsessionid=7ED762CE131CB4011BCE097E747328C6?doi=10.1.1.513.5908&rep=rep1&type=pdf

[20] Lins, R., & Kaput, J. (2004). The early development of algebraic reasoning: The current state of the field. In In K. Stacey, H. Chick, & M. Kendal (Eds.). *The Future of the Teaching and Learning of Algebra: The 12th ICMI Study* (Vol. 8, pp. 45-70), Springer, Dordrecht: Kluwer Academic Publishers. https://doi.org/10.1007/1-4020-8131-6_4

[21] Lyublinskaya, I. (2009). Developing Number Sense with Technology-Based Science Experiments: Reflections on Classroom Practice in Primary Grades and Pre-service Education. *Mathematics Teaching-Research Journal*, 3(2). Retrieved from https://commons.hostos.cuny.edu/mtrj/wp-content/uploads/sites/30/2018/12/v3n2-Developing-Number-Sense-with-Technology.pdf

[22] Medová, J., Bulková, K. O., & Čeretková, S. (2020). Relations between Generalization, Reasoning and Combinatorial Thinking in Solving Mathematical Open-Ended Problems within Mathematical Contest. *Mathematics*, 8(12), 2257. https://doi.org/10.3390/math8122257

[23] Migon, J., & Krygowska, P. (2007). Diagnostic procedure in algebra course of the first grade of the middle school. *Mathematics Teaching-Research Journal*, 1(2). Retrieved from https://commons.hostos.cuny.edu/mtrj/wp-content/uploads/sites/30/2018/12/v1n2-Diagnostic-procedure-in-algebra.pdf

[24] Molina, M., & Ambrose, R. (2008). From an operational to a relational conception of the equal sign. Thirds graders' developing algebraic thinking. *Focus on Learning Problems in Mathematics*, 30(1), 61-80. Retrieved from https://core.ac.uk/download/pdf/12341174.pdf

[25] Moyer, S. P. (2001). Using representations to explore perimeter and area. *Teaching Children Mathematics*, 8(1), 52-59.

[26] Mulligan, J., Vale, C., & Stephens, M. (2009). Understanding and developing structure—Its importance for mathematics learning. *Mathematics Education Research Journal*, 21(2), 1-4. https://doi.org/10.1007/BF03217541

[27] Nohda, N. (2000). Teaching by Open-Approach Method in Japanese Mathematics Classroom. In T. Nakahara, & M. Koyama (Eds.) *Proceedings of the Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 39–53), Hiroshima, Japan. Retrieved from https://files.eric.ed.gov/fulltext/ED466736

[28] Naik, S., Banerjee, R., & Subramaniam, K. (2005). Understanding Student's Reasoning while Comparing Expressions. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.) *Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia* (Vol. 2, pp. 569–576), Sydney: MERGA.

[29] Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (Vol. 1, pp. 629-668). Charlotte, NC: Information Age Publishing.

[30] Putrawangsa, S., Febrian, & Hasanah, U. (2021). Developing Students' Understanding of Percentage: The Role of Spatial Representation. *Mathematics Teaching Research Journal*, 13(4), 4-26. Retrieved from https://commons.hostos.cuny.edu/mtrj/wp-content/uploads/sites/30/2022/01/v13n4-Developing-Students-Understanding.pdf

[31] Reynolds, A., & Wheatley, G. H. (1996). Elementary students' construction a coordination of units in an area setting. *Journal for Research in Mathematics Education*, 27(5), 564-581. https://doi.org/10.2307/749848

[32] Smith III, J. P., Males, L. M., Dietiker, L. C., Lee, K., & Mosier, A. (2013). Curricular treatments of length measurement in the United States: Do they address known learning challenges? *Cognition and Instruction*, 31(4), 388-433. https://www.jstor.org/stable/23525183

[33] Schoenfeld, A. H. (2016). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. *Journal of Education*, 196(2), 1-38. https://doi.org/10.1177/002205741619600202

[34] Susac, A., Bubic, A., Vrbanc, A., & Planinic, M. (2014). Development of abstract mathematical reasoning: the case of algebra. *Frontiers in Human Neuroscience*, 8, 679. https://doi.org/10.3389/fnhum.2014.00679

[35] Tan Sisman, G., & Aksu, M. (2016). A study on sixth grade students' misconceptions and errors in spatial measurement: length, area, and volume. *International Journal of Science*

and Mathematics Education, 14(7), 1293-1319. https://doi.org/10.1007/s10763-015-9642-5

[36] Widjaja, W., & Vale, C. (2021). Counterexamples: Challenges Faced by Elementary Students When Testing A Conjecture about the Relationship between Perimeter and Area. *Journal on Mathematics Education,* 12(3), 487-506. http://doi.org/10.22342/jme.12.3.14526.487-506

[37] Yeo, J. K. (2008). Teaching area and perimeter: Mathematics-Pedagogical-Content-Knowledge-in-Action. In M. Goos, R. Brown, & K. Makar (Eds.) *Proceedings of the 31st Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 621-627). Brisbane: MERGA.

A study on the effectiveness of some cognitive activities in teaching integrals in secondary school

Anass El Guenyari[1], Mohamed Chergui[2], Bouazza El Wahbi[1]
[1] University Ibn Tofail, Morocco, [2] Regional Center of Education Jobs and training (CRMEF-RSK),
Kenitra, Morocco
anass.elguenyari@gmail.com, chergui_m@yahoo.fr, bouazza.elwahbi@uit.ac.ma

Abstract: Several studies have pointed out difficulties of various types encountered by students in both secondary and high schools to acquire the exact meaning of the integral and to master the ability to invest it correctly in different domains.

This contribution attempts to explore the effectiveness of some cognitive activities engaged in teaching integrals on the achievement of the objectives targeted in secondary school curriculum. For this purpose, a cognitivist analysis of the program is performed by focusing on the implementation of the changes of frames and the conversion of registers of semiotic representation during the treatment of this notion. To explore the effectiveness of these cognitive activities on learning the integrals, a sample of secondary school students were tested.

It emerges from this study that the quality of learning integrals of a real valued function and the formation of correct conceptions on this notion are highly correlated to the investment of this concept in various frames and different semiotic registers. In fact, many of the students tested were unable to perform certain algebraic tasks or to interpret correctly some integrals in familiar situations.

INTRODUCTION

The emergence of analysis results from the reflection on problems that preoccupied both mathematicians and physicists. The first ones, interested in geometric problems inherited from the Greek era, found difficult to transpose classical methods in geometry to situations newly generated by the algebraic manipulations invented by Descartes and Fermat. Physicists concerned with solving problems mainly related to mechanics, found no longer valid their approaches based on intuitive representations of certain concepts. These efforts led to the definition of new concepts, such as fluxions and fluents for Newton, differentials and integrals for Leibniz. Thus, a new field was forged known by the name of infinitesimal calculus. This new type of calculation represents the ancestor of modern analysis as it is defined in an abundant bibliography, as for example in Chauvat (1997) or Bloch (2000).

In this epistemological dynamic, the notion of integral has played a very important role, contributing to the development of several fields in mathematics and through its applications in several fields that fall under other disciplines. It has allowed the institutionalization and representation of geometric (length, area, etc.) and physical (speed, acceleration, etc.) quantities.

In secondary school (17-18 years old), the curricula give the mathematical field of analysis a large place. In this field, the notion of integral of a function has not ceased to be present as an essential course. The main objective is to provide the students the basic knowledge, on this notion, essential for generalizations or extensions required for higher education. A pragmatic reason behind this curricular choice is that the concept of integrals is very invested as a tool in the study of certain programmed phenomena in physics. This changing of the status in learning of integrals is also essential for good cognitive functioning (Douady, 1986). Indeed, the meaning of any notion can only be acquired through situations that invest it. In conceptual field theory (Vergnaud, 1990), all of these situations represent a main component in the characterization of a concept. But, to get different interpretations of an integral requires several registers of semiotic representations. For Duval (1993), the ability to perform manipulations in the required registers and conversions from one to the other is necessary for efficient learning.

Thus, it seems natural to deduce that the teaching of the integrals must take into account this inherent plurality in frames and in registers of semiotic representations. Indeed, the implementation of this change of frames favors some cognitive conditions necessary for the conceptualization of this mathematical knowledge by placing it in situations where the dialectic tool/object can arises (Douady, 1986) and to diversify its semiotic representations (Duval, 2005).

In the he present work, we are interested by the importance of the implementation of this plurality in teaching of integrals in secondary school, in terms of the formation of correct conceptions on this notion and the development of different mathematical capacities transferable in different situations. Thus, it will be a question of answering the problem of the effectiveness of the cognitive activities involved in teaching integrals in secondary school. Accordingly, the research questions will be formulated as follows:

1. What place do the cognitive activities of changing frames and converting registers of semiotic representation occupy in official orientations and in school textbooks with respect to the notion of integral?
2. What is the impact of the cognitive activities targeted in teaching of the integral on its learning?

Taking into account, the qualitative exploratory nature of the current study (Corbin & Strauss, 2007), a methodological protocol is implemented in accordance with this type. Thus, we will proceed to an analysis of the teaching program of the integrals starting with the official framework texts and then its implementation through the textbooks accredited by the Ministry of National Education. After, a test is administered to a group of students, in order to diagnose the impact of the cognitive activities engaged by the official program on learning the integrals.

The remainder of this paper is organized as follows. We begin by stating a literature overview on works related to teaching and learning integrals followed by a theoretical framework. Then, the methodology framework and the results obtained as well as the ensuing discussion are provided before presenting some conclusions.

LITERATURE REVIEW

In this section, we present a synthesis of some works that have focused on the teaching and learning of the integral. We would like to point out that the researches carried out in connection with this theme are relatively numerous and they concern secondary and higher schools.

In many empirical studies undertook by several authors, for example (Orton, 1983; Thomas & Hong, 1996; Huang, 2012; Hashemi et al., 2014) , it has been inferred that students at school and at university have fundamental difficulties in understanding the concept of integral. As a consequence of his literature review, González-Martin reported in his thesis work (González-martin, 2005) that although it is relatively easy to teach students techniques for calculating derivatives and integrals, there is a great difficulty to bring students truly into the field of analysis and achieving a satisfactory understanding of the concepts and methods of thought that are central in this mathematical area.

This situation of incapacity is expressed by Serhan, in the context of his research on definite integrals, by the fact that students were limited to the procedural knowledge manifested by the ability to calculate a definite integral, which is not the case when it comes to linking this concept with its different representations (Serhan, 2015, p. 15).

In his study, Jones (Jones, 2013, p. 138) considers that the difficulties encountered by students do not necessarily result from a lack of knowledge, but from the activation of cognitive resources that are less productive compared to others. Difficulties of cognitive type have been very recently evoked by Purnomo et al (Purnomo et al., 2022) in their analysis of problem solving process for integral calculus performed on a group of students of Mathematics Education in a university in Indonesia.

Beyond this cognitivist analysis, Belova (2006) refers to the courses given on integral calculus. Qualifying them classical, she considers that they only develop a very limited conceptual understanding of this notion. In this institutional dimension and in order to contribute to the improvement of the teaching of the notion of integral, Luong (2006) conducted a comparative study on the teaching of integrals in high school in France and in Vietnam. Thus, by referring to the anthropological theory of didactics (Chevallard, 1998), the questions of disparities in the two educational systems were studied between the defined integral scholarly knowledge and the knowledge to be taught, prescribed in the official texts, then between this one and what is produced by the teacher following some didactic choices made for teaching and finally between this latter type of knowledge and what has been taught effectively.

In his thesis project, Haddad (2012) studied the difficulties related to the notion of integral encountered by Tunisian students in secondary school and in the first-year university students. This study was conducted with the aim of offering an alternative teaching in secondary school. The author deduced that it is possible to implement a practice that takes into account the links between area, integral and primitive. In 2013, the same author interested by what students newly in university retain from the notion of integral taught in high school. He explored their ability to identify the integral and the area and the modes of validation of the results they produce (Haddad, 2013). His results are aligned with those of Orton (1983) who noted difficulties in articulating the notions of area, integral and primitive, with a group of students asked to calculate, when it is possible and to give a justification otherwise, the area of a part of the plan illustrated by a figure. Almost no student provided justifications for their answers to this question. It was also observed that the symbols used to write and calculate integrals were a source of difficulty.

This lack of in understanding deeply the integral was also noted by Ely (2017) after observing that students were unable to solve some situations slightly modified from those familiar to them. The same author explained this vulnerability by the fact that students have acquired only a procedural knowledge of integration in terms of techniques, without reaching an adequate conceptual knowledge of the inherent structures (Ely, 2017). This was also stated by Akrouti (2019) in his study on students' conceptions of the definite integral when entering university.

According to this concise review, we can conclude that the teaching of integrals in secondary school poses serious problems manifested by a deficiency in terms of acquiring the meaning of this concept, its modes of operation and the tools for validating learners' productions. Consequently, a natural question arises immediately: how can we overcome the issue of apprehension of the meaning and the transferability of the acquired knowledge by the students in integrals in other domains?

The present work represents an attempt to answer this question/issue. The following section offers a theoretical framework that will help us find relevant answers.

THEORETICAL FRAMEWORK

We begin with the point of view of Douady (1986), according to which *“Mathematical knowledge can be effectively constructed by bringing the tool-object dialectic into play within appropriate frames, by the use of problems responding to certain conditions.”*

By object, Douady (1986) means *“the cultural object having its place in a larger edifice which is scholarly knowledge at a given moment, socially recognized”*. The word object then refers to the formal representation of a concept, to its cultural aspect. For the same author, a concept is qualified as a tool if *“we focus our interest on the use made of it to solve a problem.”* (Douady, 1986).

A frame is *“made of the objects of a branch of mathematics, the relationships between the objects, their various possible formulations, and the mental images associated with these objects and their relationships.”* (Douady, 1986).

The word frame is also used with a wide meaning. It can also refer to a field of knowledge that does not belong to mathematics. Changing frames is a way of obtaining various formulations of the same problem, which are not necessarily completely equivalent since each frame involves specific concepts. This diversity gives the possibility of new access to the difficulties encountered and the implementation of tools and techniques that were not essential in the original formulation.

In the teaching-learning process, this change of frames is intentionally implemented on the initiative of the teacher to achieve the tool-object dialectic in order to confer meaning on the mathematical objects targeted. From an epistemological point of view, this change is imposed by the origins of the problems that mathematics tries to solve. The notion of integral is a good illustration of this point. Indeed, the main idea behind its emergence is of geometric origin and the first attempts of calculation emanated from certain intuitive approaches that were common in the study of some phenomena in physics. Consequently, the meaning of this concept and its usefulness can only arise through its investment in situations from different disciplines. In other words, it is the tool status that contributes crucially to the formation of the meaning of a concept.

On the other hand, R. Duval distinguishes mental representations from semiotic representations (Duval, 1995). By the first, he designates all the mental images or conceptions that one may have about an object or a situation. In order to externalize these mental representations, it is necessary to have tools that perform this function. In this context, the same author (Duval, 1993) defines the registers of semiotic representations as being *"...productions constituted by the use of signs belonging to a system of representations which has its own constraints of significance and functioning, the set of these signs is called the register of semiotic representation"*.

This theory underlines the importance of the role played by semiotic representations in the manipulation of abstract objects since they make them visible and accessible. In mathematics, each object has many registers of semiotic representations, each of which provides partial access to the object it represents and allows certain operations to be performed on it. A semiotic system acquires the status of register of semiotic representations if it allows the following three fundamental cognitive activities. First one is the formation of a representation identifiable as a representing of a given register. This formation is subject to some convenient rules, specific to the system used, not only for reasons of communicability, but also for the feasibility of processing by the tools offered by this system. Second one is the transformation of one representation into another without changing the semiotic system. Final activity is the conversion of a representation into another register while retaining all or only part of the content of the initial representation. Conversion is therefore an activity external to the register of the original representation.

Duval (2005) considers that learning in mathematics cannot be dissociated from the activity of recognizing at least two representations of the same object.

The importance of representations in learning mathematics was revealed in many studies. Janvier (1987) considers that the use of representations in mathematical thinking is fundamental. Mainali (2021) recommended that instructional strategies should be focused on incorporating different modes of representation in order to meet the student's preferences for solution strategies.

In the case of the Riemann integral notion, for establishing the properties of integrals and in the case where formal proofs cannot be presented, geometric interpretations by the notion of area prove to be a didactic alternative which has the contribution of bringing the student to perceive the meaning of these properties and to attribute to them a certain legitimacy although it is intuitive. This interpretation is sometimes practiced by illustrations using graphical representations of functions. It is then a transition from the algebraic frame to the geometric and/or graphic frame.

The application of integrals by employing situations from physics or mathematics is another occasion where the change of frames is manifested. This diversification of frames in teaching or learning of integrals implies naturally the use of different registers of semiotic representations. The representation of the integral by the Newton-Leibniz formula is part of the symbolic register but the interpretations of this notion or its applications require representations in other registers. This is the case, among others, in evaluating a geometric or physical quantity or when in illustration by a curve.

It follows then that the conceptualization of the integral is strongly connected to operating this concept in different frames and registers.

METHODOLOGY FRAMEWORK

In this study, we limit ourselves to the case of the Moroccan curriculum. We focus on the teaching program of integrals in secondary school addressed to scientific and technology classes. Therefore, let us begin with an institutional analysis of the integral.

Institutional place of the integral

According to the framework document for the teaching of mathematics in Morocco (MEN, 2007), the notion of integral (of Riemann) is presented to students in the science and technology classes of secondary school in a chapter entitled integral calculus. The integral of a continuous function is defined by the Newton-Leibniz formula, $\int_a^b f(x)dx = F(b) - F(a)$ where F stands for a primitive of the function f on the interval $[a, b]$. This last concept is a prerequisite for students because it is introduced directly after dealing with the differentiability of a function.

In parallel to these epistemological and didactic choices, the following guidelines are stated:

- to present through examples of some simple functions, the link between the integral of a function over an interval $[a, b]$ and the area of a domain of the plane bounded by the curve of a function f and the lines of equations $x = a$ and $x = b$ and the abscissa axis,
- to restrict evaluating an integral to integration by parts and to the direct use of primitive functions.
- no proof should be presented for the properties of integrals,

- to invest the integrals in situations from physics or mathematics.

The details of the programmed content and the targeted capacities to be built on the learners are presented in Table 1 (MEN, 2007).

Contents of the program	Target capacities
- Integral of a continuous function on a segment. - Properties of the integral: Chasles relation, linearity, integral and order. - The mean value of a function on an interval. - Techniques for calculating integrals. - Calculation of areas and volumes.	- Calculation of the integral of a function by using one of two techniques. - Calculation of the area of a domain of the plane between two curves and two lines parallel to the ordinate axis. - Calculation of the volume of a solid of revolution generated by the rotation of the curve of a function around the abscissa axis.

Table 1: Integral Calculus Teaching Program

Officially, the implementation of this program is manifested by the production of textbooks, published only after an accreditation issued by the Ministry of National Education in accordance with some specific prescriptions drawn up for this purpose.

Methodology

In this subsection, we develop the methodology adopted to provide some answers to our problem and the related questions.

For the first question, which aims to explore the place of the activities of changing frames and converting registers of semiotic representation in official orientations and in school textbooks with respect to the notion of integral, we will analyse the official prescriptions at the light of the theoretical study carried out previously in this paper. It is a question of identifying the frames and the registers of semiotic representation where the activities are supposed to be practiced by the learners as stipulated by the official texts.

We will also undertake an analysis of the activities proposed in the textbooks. The analysis of textbooks is justified by their principal role in the implementation of program prescriptions.

The activities analysed are chosen from mathematics textbooks for science and technology classes, accredited by the Ministry of National Education. There are two textbooks whose data are recorded in the following table 2:

Textbooks	Ministerial accreditation number	ISBN
Fi Rihab Ryadiat	09CB 21207	9954-436-82-0

Al Wadih fi Ryadiat	09CB 21307	9981-30-179-5

Table 2: Identification of textbooks analyzed

In both textbooks, the chapter on integrals is organized as follows: the first part is devoted to preparatory activities for the introduction of new knowledge, the second is dedicated to presenting the main content stipulated by the programs and the last part offers some training activities aiming to consolidate and to invest the new learning.

Our analysis, which focuses on the ten preparatory activities and the 123 training activities proposed in the two textbooks, aims to explore the cognitive activities of frame changes mobilized in the two types of activity. As a result, the preparatory and training activities will be analysed according to the frames and registers of semiotic representations of formulating the activity and those required to produce the expected responses to the instructions.

The analysis will be carried out on the basis of a grid, in Table 3, which gives indicators that allow us to identify in official texts or in the activities of textbooks the frames and registers of semiotic representations involved. The development of this grid was based on the various studies carried out during this research.

	Types of frames or registers	Indicators to identify the frame or the register involved in the activity
Involved frame	Algebraic	The use of algebraic formulas for processing integrals.
	Geometric	The use of geometric concepts to illustrate, interpret or invest integrals.
	Graphic	The use of curves to identify or illustrate an integral.
	Discipline other than mathematics	Physics, biology, …
Involved semiotic register	Graphic register	Formulation of data via curves.
	Algebraic register	The use of function symbols and/or expressions.
	Geometric register	Formulation of data by geometric figures.

Table 3 : Analysis grid of activities according to frames and registers

To answer the second question, a test targeting different abilities related to integral calculus is administered (in French) during the 2021-2022 school year, to a group of students of scientific classes. The test (Appendix) consists of seven questions is analysed as presented in Table 4.

Questions	Abilities assessed
Q 1.	The use of the value of the integral $\int_a^b f(x)\,dx$ to estimate the increase of a primitive of a function f on the closed interval $[a, b]$.
Q 2.	Investment of the independence of the value of the integral of the chosen primitive function.
Q 3.	Determination of the sign of an integral.
Q 4.	Mastering of the status of the independent variable $(x, z, \dots)$ in computing an integral.
Q 5.	Representation of an area by an integral.
Q 6.	Representation of the volume of a solid by an integral.
Q 7.	Investment of the integral for the estimation of a quantity in physics.

Table 4 : Analysis of the administered test

Before the administration of the test and for validation purposes, four mathematics teachers working in secondary school and three researchers in mathematics education were consulted on the test items. After some minor modifications, the test was administered during the first two weeks of May 2022 to students in the aimed classes, who had taken advantage of lessons on the knowledge involved in the test.

Given the constraints imposed by the COVID 19 pandemic, the test was only distributed in 5 secondary schools of the Regional Academy of Education and Training of Rabat-Sale-Kenitra. The total number of participants is 75.

RESULTS

Analysis of the official program

Taking into account the epistemological and didactic choices of the curriculum (MEN, 2007), we deduce the results cited below in connection with the activities of changing frames and registers of semiotic representations.

The naming of the chapter by integral calculus refers, intentionally or not, rather to a field of mathematics where calculations dominate with specific tools and techniques than to a new concept which has its own meaning and which serves in particular to model some geometric or physical objects. Moreover, the change of frames is restricted to calculation of area and volume and in very particular cases.

Regarding the choice to introduce the integral by the Newton-Leibniz formula $\int_a^b f(x)dx = F(b) - F(a)$, which comes under the algebraic frame, we have the following:

- Rather, it attributes legitimacy to the presence of the concept of primitive function in the program without showing the student concretely the need to learn the new concept.

- From a didactic point of view, although this choice makes possible to produce proofs for certain properties accessible by the student in secondary school, it is not invested to avoid excessive recourse to intuition in validating knowledge.

The definition represents the integral of a continuous function f on the interval $[a, b]$ by the symbol $\int_a^b f(x)dx$ bound by an equality relation to the notation $[F(x)]_a^b$, then substituted immediately by the quantity $F(b) - F(a)$, with F is a primitive function of f on $[a, b]$. This generates a real confusion on the exact object referred to in the definition: is it the integral concept, its representation by $\int_a^b f(x)dx$ or the symbol $[F(x)]_a^b$?

We would also like to mention that the official prescriptions (MEN, 2007) do not present any indication on the cognitive importance of the change of frames or registers in the practice of teaching the integral.

Analysis of textbook activities

Using the indicators explained in the analysis grid cited in Table 3, we classified the preparatory and training activities according to the frames or registers of semiotic representations implemented in the formulation of the activity or required to respond to instructions. The results obtained are registered in table 5.

	Frame or register used for formulation of the activity		Frame or register necessary for the formulation of the requested response	
	Preparatory activities	Training activities	Preparatory activities	Training activities
Algebraic	8	118	9	116
Geometric	4	4	0	3
Graphic	3	8	1	5
Discipline other than mathematics	0	6	0	5
Graphic register	4	8	1	5
Algebraic register	18	123	19	116
Geometric register	3	4	0	3

Table 5 : Classification of textbook activities according to frames and registers

Test' results

It is important to state at the beginning that reliability of the test was performed. Using the XLSTAT 2022 to compute Cronbach's α, we found that $\alpha = 0.851$. It is considered that the reliability is acceptable when α is equal or higher than 0.7 (Taber, 2018).

In table 6 below, we present the number of students who succeeded in producing answers and the numbers of correct productions. We point out that for each question, we have considered the answer to be correct when the student manages to produce the object requested in the instruction (value, sign, expression, etc.) and by explaining the approach he has used.

Question	Q1	Q2	Q3	Q4	Q5	Q6	Q7
Number of students who provided an answer	53	63	66	37	35	19	19
Number of correct answers	28	17	12	5	17	2	9

Table 6 : Number of correct answers per question

In addition to these quantitative results, the Table 7 presents some errors committed by the students in each of the questions of the test.

Question	Errors observed in the responses
Q 1.	- Since $\int_1^3 f(x)dx = 7$ then $f(x) = 7$. So, $F(x) = 7x$ and $F(3) - F(1) = 14$
Q 2.	- $G(3) - G(2) = 1/3$. - $G(2) - G(3) = 2x - 3x = -x$. - $G(2) - G(3) = G(-1)$.
Q 3.	- J is negative because $(x - 3)$ is negative and ln (x) is positive on $[1; 2]$. - The sign of J is the one of $(x - 3)$.
Q 4.	- The response is yes without any proof. - No relation between the two integrals. - $\int_a^b f(x)dx = [F(x)]_a^b$ and $\int_b^a f(z)dz = [F(z)]_b^a$ but no relation was provided. - $\int_b^a f(x)dx = -\int_b^a f(z)dz$.
Q 5.	- $\int_0^1 f(x)dy$. - $A = \frac{1}{b-a} \int_0^1 f(x)dx$.
Q 6.	- $\mathrm{V} = \int_4^0 \pi(f(x))^2$. - $\mathrm{V} = \int_1^7 \pi(f(x))^2$. - $\mathrm{V} = \int_{-2}^2 y - f(x)$.
Q 7.	- The average speed is $\int_1^7 \mathrm{v}(t)dt$. - The average speed is $\frac{\mathrm{V}(7)-\mathrm{V}(1)}{7-1}$.

Table 7: A list of some errors made by students

DISCUSSION

The analysis of the official program on the teaching of integrals allows us to identify several results. First, the choice to present the integral of a function by the Newton-Leibniz formula prompts us to wonder if the learner acquires the meaning of the integral or if he appropriates the formula for his calculation. Theoretically, the answer is clear. In fact, only problem situations that highlight both the usefulness and the usability of a mathematical object are effective in acquiring its meaning. On the other hand, presenting the concept of integral of a function by the Newton-Leibniz formula is likely to create in the student an ambiguity in the defined object. In other words, it is unclear whether the expression $F(b) - F(a)$ represents for him a definition of the symbol $\int_a^b f(x)dx$ or of the concept integral of the function f on the interval $[a, b]$. It seems then that the course on integrals represents a continuation on the course on primitives where this last notion is invested.

In the programmed content and concerning the cognitive activity involved, there is a dominance of calculus which also extends to situations of application of integrals. Indeed, the official prescriptions aim at the numerical estimation of the surface and the volume by a mechanical restitution of certain formulas, and not to make use of the concept of integral like a tool of modelling situations resulting from the geometry or the physics or possibly another domain.

In connection with the knowledge targeted in the program, it seems important to point out that the question of integrability is not mentioned at all. Learners are completely exempt from asking themselves this natural and fundamental question in mathematics related to the existence of the object studied. In principle, this lack of awareness to wonder about the existence of the integral is unexplained since it is attached to the concept of primitive function which has continuity as unique sufficient condition presented in the program for its existence. Faced with such a didactic choice, the following important question arises: can the student recognize whether a function is integrable from its graphical representation? This question will be put into perspective for a future work.

Concerning the interpretation of integrals, the interest is particularly focused on area and volume. This questions the fact of restricting, in the mathematics program, only to the geometric frame knowing that physics offers a whole variety of situations likely to improve the perception of the notion of integrals.

The analysis of preparatory activities has clearly shown that they are mainly formulated in the algebraic frame and the use of another frame does not appear to be an objective in itself, to promote the formation of another mental or even semiotic representation of the integral. In addition, the change of frames is only carried out in one direction. Indeed, the unique frame required to produce the answers to the instructions of the activities is the algebraic one. It is also clear when we see that no situation from another disciplinary field is suggested among the preparatory activities. We can therefore conclude that the frames used in these activities do not take into account the issue of diversifying the cognitive activities of the student.

The dominance of the algebraic frame in this type of activity, essentially intended for the formation of mental representations of integrals among students, has another disadvantage for their semiotic practice. The only register involved is the algebraic one, manifested by the use of symbols and expressions since they are necessary for the calculation. This fits well with the algebraic frame used. In fact, the formulation of the activities or the expected answers only requires the exercise of computational tasks. Thus, no diversification in terms of semiotic representations is required in these activities.

The dominance of the algebraic frame extends widely to training activities. Basically, these activities aim to make learners acquire automatism in the use of techniques or procedures and to implement the tool aspect of newly learned mathematical objects. The achievement of these two cognitive objectives can only be the result of students' work in situations emanating from various frames and registers where the activities of change are practiced in all directions. However, the analysis of the items of this last type of activity reveals that apart from the mechanical use of algebraic properties of integrals, no result is the fruit of a specific thought to the concept of integral. In fact, the calculation of integrals, especially when it comes to calculating area or volume, is dominated by the determination of primitive functions and the techniques behind them. Questions on bounding integrals are solvable by the properties of the order in the set of real numbers or numerical functions.

From the analysis of the prescribed program and of the preparation and training activities proposed in the textbooks, it appears that access to the meaning of the integral and its properties is hampered by the abuse of computational tasks which do not allow to vary the cognitive activity required of the student and to develop his capacities in mathematical languages.

The analysis of the students' answers to the test reveals a clear deficiency in terms of restitution of knowledge. The number of correct answers provided for questions Q1 and Q2 related to the relationship between the integral and the notion of primitive functions is too unsatisfactory since the program is quite articulated on this notion. Furthermore, errors revealed in Table 7 concerning Q1 and Q2 show that some students confuse the integral of a function and its primitives.

The inability of the majority of students tested to answer correctly to question Q3 although it only requires the execution of a two-step procedure, determination of the sign of the integrand function and comparison of the bounds of the integral, is completely understandable since the program and the activities that implement it are too centered on the mechanical calculation of the value of an integral by the Newton-Leibniz formula or by integration by parts while the properties of the new notion are little discussed if we dare not say marginalized. Beyond the fact that some students consider $\ln x$ to be positive accordingly to the sign of the variable x, they do not take the order of the bounds into account to determine the sign of the integral.

According to the low number of students who answered the question Q4 and the errors made in this question, we can deduce that the symbolic representation of the integral is not mastered.

For the investment of the integrals in the situations Q5 and Q6 resulting from the geometry, incapacities are well observed. The transition from the geometric frame to the algebraic one then poses difficulties for the students. This last statement is clearly observed from the obtained results about question Q5 where evaluating an area is targeted. The answers to question Q6 presented in Table 7 show also that the students use the integral although it is not necessary.

Finally, from question Q7 where it is proposed to evaluate a mean value in a contextualized situation, it turns out that the students did not master one of the very interesting interpretations of the integral.

Following the results of our analysis and the ensuing discussion, some conclusions emerge. They will be the focus of the following last section.

CONCLUSION

This study starts from the point of view according to which the access of a learner to the meaning of mathematical concepts necessarily passes through their treatment in several frames put into play intentionally. This condition aims at the relevant formation of mental representations on a given concept to facilitate its treatment in different frames. This position motivated us to question the effectiveness of the cognitive activities involved in the teaching of integrals in secondary school on the learning and the investment of this notion.

The choice of the notion of integrals was first motivated by its importance in mathematics and in other disciplines. After their emergence to solve classical problems of geometry or physics, they quickly used other concepts, such as limits and functions, to gain new momentum in the field of mathematics, and then by its place in secondary and higher education mathematics curricula.

To determine what place occupies the cognitive activities of change of frames and conversion of registers of semiotic representation in the official orientations and in the textbooks relatively to the concept of integral, an analysis of the texts of framing in vigour has been achieved. Such an analysis, supported by our bibliographical review, allows us to conclude that there is a total domination of the computational processing of integrals which in turn calls on semiotic representations mainly in the algebraic register. The same conclusion was deduced from the analysis of textbooks. In the preparatory activities proposed therein, it has been revealed that the computational activity takes precedence. And as a result, the algebraic frame is the most favored to bring into play since the manipulations required for the determination of the primitive functions are algebraic. It follows that the choice of preparatory activities is completely subject to the introduction of the integral by the Newton-Leibniz formula. It is a restrictive choice, in the sense that it does not provide a set of frames and registers of varied semiotic representations.

The result is then clear, these activities do not contribute to build an exact conceptions on the integrals. This dysfunction is accentuated by the proposals for training activities that aim in principle, among others, to reinvest knowledge in other contexts so that it acquires a cultural status. From the analysis of this type of activity, it follows that the most dominant ability is the determination of the numerical value of the integral. This statement is also true for the calculation of areas or volumes where the only task assigned to the student is the use of formulas, simple to reproduce, for domains in the plan or in space described in the same way in all activities.

Faced with this assessment and as an answer to the second question, it is difficult to confirm that the proposed training activities contribute to master the sense of the integral or to develop the capacity to invest knowledge on this notion in other disciplines. The results deduced from the test administered confirmed this. Indeed, apparent difficulties in applying integrals in the estimation of certain quantities were revealed.

Overall, after this analysis which focused on the three levels of the curriculum, the design, the implementation and what is supposed to be learned, it turns out that the lack of intentional engagement in the teaching of the integrals of the situations which impose a diversification of frames for processing integrals and of registers to represent it semiotically presents didactic defects manifested by the abusive recourse to computational tasks. This weighs cognitively, access to the meaning of the integral of a function and the acquisition of the ability to use it is to be considered with delicacy.

At the end of this work, we would like to point out that during this work, other questions arose. We have not been interested in them because they go beyond the objectives of this article. But, they have generated great motivation for future work articulated on the following two main questions:

- What is the impact of teaching the definite integral centered on a graphical approach? In particular, can the learner recognize the properties of a definite integral from its representative curve?
- What place occupies the validation and how it is performed in the students' productions about integrals?

REFERENCES

[1] Akrouti, I. (2019). Students’ conceptions of the definite integral in the first year of studying science at university. Proceeding of the 11th Congress of the European Society of Research in Mathematics Education. Utrecht, Netherlands, ERME Publishing. ISBN 978-90-73346-75-8.

[2] Belova, O. (2006). Computer based approach to integral calculus for prospective teachers. Thése, Krasnoyarsk State Pedagogical University.

[3] Bloch, I. (2000). L’enseignement de l’analyse à la charnière lycée/université.Savoirs, connaissances et conditions relatives. Thèse de Doctorat, Université Bordeaux I.

[4] Chauvat, G. (1997). Etude didactique pour la réalisation et l’utilisation d’un logiciel de représentations graphiques cartésiennes des relations binaires entre réells dans l’enseignement des mathématiques des DUT industriels. Thèse, Université d’Orléans.

[5] Chevallard, Y. (1998). Analyse des pratiques enseignantes et didactique des mathématiques : L’approche anthropologique. Actes de l’U.E. de la Rochelle.

[6] Corbin, J., & Strauss, A. (2007). Basics of qualitative research. Sage Publications, pp. 1-312.

[7] Douady, R. (1986). Jeux de cadres et dialectique outil/objet dans l'enseignement des mathématiques. Revue RDM, vol 7(2), pp. 5-32.

[8] Duval, R. (1993). Registre de représentation sémiotique et fonctionnement congnitif de la pensée. annales de didactique et des sciences congnitives(5), pp. 37-65.

[9] Duval, R. (1995). Sémiosis et pensée humaine (registres sémiotiques et apprentissages intellectuels). Berne: Peter Lang.

[10] Duval, R. (2005). Transformations de représentations sémiotiques et démarches de pensée en mathématiques. In J-C. Rauscher (éd.), Actes du XXXIIe Colloque COPIRELEM, Strasbourg : IREM, pp. 67-89.

[11] Ely, R. (2017). Definite integral registers using infinitesimals. The Journal of Mathematical Behavior, 48, 152–167.

[12] Gonzalez-martin, A. S. (2005). La généralisation de l'intégrale définie depuis les perspectives numérique, graphique et symbolique en utilisant des environnements informatiques. thèse.

[13] Haddad, S. (2012). L'enseignement de l'intégrale en classe terminale de l'enseignement tunisien. Université Paris Diderot, Paris 7.

[14] Haddad, S. (2013). Que retiennent les nouveaux bacheliers de la notion d'intégrale enseignée au lycée. Petit x n° 92.

[15] Hashemi, N., Abu, M. S., Kashefi, H., & Rahimi, K. (2014). Undergraduate students' difficulties in conceptual understanding of derivation. Procedia—Social and Behavioral Sciences, 143, pp. 358-366.

[16] Huang, C. (2012). Engineering students' representational flexibility—the case of definite integral. World Transactions on Engineering and Technology Education, 10(3), pp. 162-167.

[17] Janvier, C. (1987a). Translation process in mathematics education. In C. Janvier (Ed.), Problems of representation in the teaching and learning of mathematics (pp. 27-32). Hillsdale, NJ: Lawrence Erlbaum.

[18] Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. Journal of Mathematical Behavior, 32, pp. 122-141.

[19] Luong, C. K. (2006). La notion d'intégrale dans l'enseignement des mathématiques au lycée : une étude comparative entre la France et le Vietnam. Grenoble: Thèse.

[20] Mainali, B. (2021). Representation in Teaching and Learning Mathematics. International Journal of Education in Mathematics, Science and Technology, 9(1), 1-21.

[21] MEN. (2007). Orientations pédagogiques générales et programme des mathématiques pour le cycle secondaire qualifiant. Direction des Curricula, Ministère de l'éducation nationale, Royaume du Maroc.

[22] Orton, A. (1983). Students' understanding of integration. Educational Studies in Mathematics, 14, 1-18.

[23] Purnomo, E. A., Sukestiyarno, Y. L., Junaed, I., & Agoestanto, A. (2022). Analysis of Problem Solving Process on HOTS Test for Integral. MATHEMATICS TEACHING RESEARCH JOURNAL, 14 no 1, pp. 199-214.

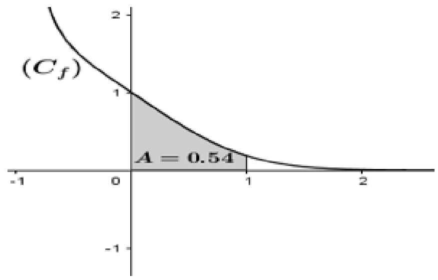
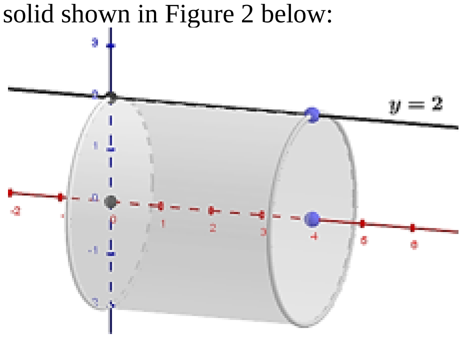
[24] Serhan, D. (2015). Students' understanding of the definite integral concept. International Journal of Research in Education and Science 1(1), pp. 84-88.

[25] Taber, K. (2018). The Use of Cronbach's Alpha When Developing and Reporting Research Instruments in Science Education. Res Sci Educ 48, pp. 1273–1296 .

[26] Thomas, M. O., & Hong, Y. Y. (1996). The Riemann integral in calculus: Students' processes and concepts. Proceedings of the 19th Mathematics Education Research Group (pp. 572–579). Melbourne: In P. C. Clarkson (Ed.).

[27] Vergnaud, G. (1990). La théorie des champs conceptuels (Vol. 10). Recherche en didactique des mathématiques.

Appendix: Test

Q 1.	Let f be a continuous real function on $[1,3]$ such that $\int_1^3 f(x)\,dx = 7$. F denotes a primitive function of f on [1,3]. Evaluate $F(3) - F(1)$.
Q 2.	Let f be a continuous function defined on [2;3] such that $\int_2^3 f(t)\,dt = 1/3$ and G a primitive function of f on $[2;3]$. Give the exact value of $G(2) - G(3)$.
Q 3.	What is the sign of the integral $J = \int_2^1 (x-3)\ln(x)\,dx$? Justify the answer.
Q 4.	f is a continuous function defined on $[a,b]$. Is there any relationship between the following two integrals: $\int_a^b f(x)dx$ et $\int_b^a f(z)dz$? If yes, precise it?
Q 5.	Let f be the function represented by the curve (C_f) in an orthonormal frame $(o;\vec{i};\vec{j})$. A is the area of the domain of the plane colored in gray (Figure 1). Express by an integral the area A and give the value of this integral.  Figure 1
Q 6.	Calculate the volume of the solid shown in Figure 2 below:  Figure 2
Q 7.	Give the value of the average speed between the instants $t_1 = 1$ and $t_2 = 7$ of a moving body whose instantaneous speed is $v(t) = 4t + 3$.

Epistemological Obstacles on Limit and Functions Concepts: A Phenomenological Study in Online Learning

Rini Sulastri[1,2], Didi Suryadi[1], Sufyani Prabawanto[1], Endang Cahya[1]

[1]Department of Mathematics Education, Universitas Pendidikan Indonesia, Indonesia,

[2]Department of Mathematics Education, Universitas Serambi Mekkah, Indonesia

rini.sulastri@upi.edu, ddsuryadi1@gmail.com, sufyani@upi.edu, endangcahya@gmail.com

Abstract: Barriers to learning can affect educators and students in achieving learning objectives. One of them is the epistemological obstacle caused by the limited context used when a concept is first learned. The purpose of this study was to explore the epistemological obstacles of mathematics students in the concept of limits and functions during online learning. Data were obtained from 16 first-year students who took a written test on the concept of limit function. Based on the various answers and the presence of errors, nine subjects were selected to be followed by semi-structured interviews. The qualitative research data with this phenomenology approach were analyzed descriptively. The results of the study indicate that there are several concepts of function limits that are misunderstood by subjects who come from learning experiences in high school and also online learning. Epistemological obstacles that occur include an incorrect understanding of the concepts of real and infinite numbers, the value of a function will always be the same as the limit value for the same function, and the use of the substitution method which is generalized to the limit of the function. To overcome these obstacles, the presentation of the material, especially the basic concepts, needs to be done in-depth so that is easy to understand and to minimize the occurrence of misconceptions.

INTRODUCTION

The global pandemic status since the beginning of March 2020 has affected many countries so various strict policies such as social distancing has been implemented. This has resulted in school closures and even a number of schools being completely closed, Indonesia is no exception (Huang, Liu, Tlili, Yang, & Wang, 2020). The government has limited community mobilization by continuing to call for an agenda to work, study, and worship from home. To deal with these conditions, the government has implemented remote learning and teaching programs such as electronic learning (Mailizar, Almanthari, Maulina, & Bruce, 2020). However, not all schools are able to implement e-learning effectively. Lack of experience with e-learning causes difficulties in using online applications (Zaharah, & Kirilova, 2020).

Research conducted (Mailizar, et al., 2020) on the barriers to implementing e-learning during the

COVID-19 pandemic showed that the highest barriers occurred at the student level related to skills and knowledge, motivation, and e-learning infrastructure. The teacher's background has no impact on the level of the barrier. In contrast, a survey conducted (Mushtaha, et al., 2022) shows that the implementation of online learning has a positive impact on aspects of flexibility related to place and time, and aspects of accessibility and effectiveness in the assessment and communication methods used. This shows that the implementation of online learning has a positive and negative impact on its users. In addition, the impact that appears is also related to the learning process carried out.

Several sources of problems in the learning process include the orientation gap between current learning and philosophical orientation; learning flow problems related to didactical design; the problem of organizing didactic situations that are not in accordance with the nature of mathematics and its learning; the problem of conceptual gaps between educators, students, and scientific conceptions; formation of transpositional knowledge (didactic and pedagogical), and various problems related to didactic design (Suryadi, 2019a). Problems related to didactic design are ontological, didactical, and epistemological (Brousseau, 2002; Suryadi, 2019a). The ontological obstacle is related to the gap between the design demands and the child's capacity. Didactical obstacles can arise due to the sequence and stages of the curriculum and its presentation in class. The limited context used in the didactic design can create an epistemological obstacle. In other words, the epistemological obstacle is the gap between the context of the learning experience that has been passed and the demands of linking learning outcomes with various contexts outside that have been experienced (Suryadi, 2019a; 2019b).

Epistemological obstacle can be found in students' errors in answering or responding to questions given (Brousseau, 2002; Cornu, 1991). These errors can be influenced by prior knowledge (Brousseau, 2002). In a study conducted (Moru, 2009), indicators of epistemological barriers were based on students' errors and difficulties in understanding the concept of limits. In this study, epistemological barriers were also traced from students' mistakes in answering several questions about the concept of limits and functions.

One of the gateways to studying advanced science and mathematics is Calculus (Sebsibe & Feza, 2020; Roble, 2017; Sadler & Sonnert, 2016). The concept of limit acts as a central concept in calculus so it must be understood correctly by every student (Zollman, 2014; Artigue 2000; Bezuidenhout, 2001; Cornu, 1991; Oehrtman, 2002; Szydlik, 2000; Williams, 1991; Ervynck, 1981). In addition, limits will also be used in advanced learning (Beynon & Zollman, 2015). Although considered as the basis for understanding calculus, a complete understanding of the concept of limit is still very limited among students (Davis & Vinner, 1986; Tall & Vinner, 1981; Sierpinska, 1987; Cornu, 1981).

Many kinds of research on the limit of functions have been carried out on students, college students, or teachers. The results of research by Arnal-Palacián & Claros-Mellado (2022) show that pre-service teachers use algorithmic procedures in solving limits, and use an intuitive approach to explaining them to students. The results of the research by Fernández-Plaza &

Simpson (2016) show that most students solve limit problems separately, only a few can relate the basic concepts of limits between the limit of a sequence, the limit of a function at a point, and the limit of a function at infinity. However, they often do it in a way that runs counter to formal mathematics. Reed (2018) investigates students' understanding of personal concepts about limits and their relation to formal definitions of limits after students complete introductory calculus lessons. The result shows that students' ability in defining personal concepts related to limits is in a low category. Thus, a conceptual understanding needs to be developed to be able to master the concept of limits.

According to Vinner (1991), a definition has a special role in the technical use of a construct in contrast to the intuitive use and everyday language. In understanding an idea such as the limit of a function, it is necessary to identify it in different representations such as graphic, numeric, algebraic, and verbal. In addition, the coordination of each of the different representation systems is necessary to gain a comprehensive understanding of the idea (Arnal-Palacián & Claros-Mellado, 2022; Duval, 1998). Kidron (2011) conducted a study on the definition of limit which was constructed using a horizontal asymptote definition approach. In this case, a conflict arises between the concept image and the concept definition of the horizontal asymptote. Students consider an asymptote to be a straight line that a curve approaches so that it becomes closer along the line. In addition, the closer it is understood without anything to do with limits.

Job & Schneider (2014) in their research on the epistemological obstacle in Calculus, used the anthropological approach of Chevallard. The aim is to analyze the history of calculus in building an epistemological model based on pragmatic and define deductive praxeology. More specifically, research on epistemological obstacles to the concept of limit functions was carried out by Moru (2009) with the subjects being first-year mathematics students at the Faculty of Science and Technology. In the results of his research, it was stated that the subject had difficulty in determining or distinguishing between limit values and function values.

Problems that often arise from the concepts of limits and functions relate to whether a function can reach its limits, whether a limit is actually bound, whether a limit is a dynamic process or a static object, and whether a limit is inherently bound to the concept of motion (Williams, 1991). The question "Is a limit reached or not?", according to Moru (2009) as an epistemological obstacle in understanding the concept of limit. According to Tall (1993) some difficulties in understanding the concept of limit include: the use of language and terms such as limits, tends to, approaches, and as small as we please, which has a colloquial meaning contrary to formal concepts; limit processes are not carried out with simple arithmetic or algebra; the process of a variable becoming smaller is often interpreted as a smaller variable quantity; the idea of N getting bigger, implicitly indicates the conception of infinity; confusion about whether a limit is really achievable; and confusion about the part from finite to infinity raises questions about what happens to infinity.

The difficulty in understanding the concept of limit begins with the many informal ideas of the word limit that students already have (Barnes, 1991). Some students understand some aspects of formal mathematical concepts but this can distort their definitions thereby creating a conceptual barrier that makes learning about limits more difficult (Cavanagh, 1996). Students' difficulties regarding the concept of limits are also due to the prerequisite materials that must be mastered such as real numbers, functions, and very small and large numbers (Parameswaran, 2007; Sierpinska, 1987). In addition, the factors that may contribute to students' difficulties regarding limits relate to the predominance of the dynamic and procedural aspects of limits in textbooks and calculus teaching, and students' attitudes towards mathematics (Bezuidenhout, 2001; Parameswaran, 2007; Williams, 1991).

According to Denbel (2014) students have limited conceptions of limit and continuity due to the way the concepts are introduced to students, and the existence of limited concepts related to pre-calculus concepts. Furthermore, the concept of limit is often disputed with a number of questions such as whether a function can reach its limit, whether a limit is actually a limit, whether a limit is a dynamic process or a static object, and whether a limit is inherently related to the concept of motion (Tall, 1993; Williams, 1991).

The results showed that the subject viewed the limit of the function as an unreachable function (Sulastri, et al., 2021). Similarly, research conducted by Denbel (2014) shows that students see limits as unreachable, approximate, limits, dynamic processes, and not as static objects, and are impressed that a function will always have a limit at a point. More deeply students' misunderstandings in understanding limits relate to (a) the relationship between continuous functions and limits, where students think that a function must be defined at a point to have a limit at that point, (b) a function that is not determined at a certain point has no limit value. Students think that when a function has a limit, it must be continuous at that point. (c) The limit is equal to the value of the function at a point, where the limit can be found by the substitution method, and if we get a division of zero by zero, the result is zero. In this case, most students know that any number divided by zero results in undefined.

As for what distinguishes this study from previous research on epistemological obstacles in the concept of limit functions, the focus of the material on the concept of limits in this study is related to the value of the function limit and the value of the function as well as the relationship between these values. In addition, the subjects involved are first-year mathematics students at the Faculty of Mathematics and Teacher Education who have obtained material on function limits in the Differential Calculus course with online learning. This is due to the Covid-19 pandemic, so online learning is applied.

The purpose of this study was to explore the epistemological obstacle experienced by first-year mathematics students in the concept of limit function in online learning. In addition, to reveal the type of error and how to understand the error.

METHOD

This qualitative research uses a phenomenological approach. Epistemologically, the phenomenological approach is based on the paradigm of personal knowledge and subjectivity and emphasizes the importance of personal perspective and interpretation. In other words, phenomenology is concerned with the study of experience from an individual's perspective. The purpose of the phenomenological approach is to clarify specifically and identify and describe certain phenomena through a person's perspective in a situation (Lester, 1999; Creswell, 2012; Freankel, Wallen, & Hyun, 2012). Thus, the purpose of using this approach is to reveal the epistemological obstacles that occur to students during online learning. To reveal this, the researcher analyzed student errors in solving function limit problems and also explored learning experiences on function limits and knowledge relevant to this material.

The subjects in this study were first-year mathematics students from one of the universities in Aceh, Indonesia, totaling sixteen people. They are in a transitional period from secondary education to higher education and have different learning experiences, especially on the concept of limits. At the level of secondary education or equivalent to senior high school, they learn more about the limit of functions procedurally than conceptually because of the demand to pass the final exam. Meanwhile, at the higher education level, which is equivalent to college, the limit of functions is studied in the differential calculus course in the first semester. The learning process in this first year is through online learning due to the COVID-19 pandemic since early 2020.

The data collection instruments in this study were written tests and interview guidelines conducted through the Zoom Meeting and Google Meet applications. This test instrument was designed by the researcher and was adapted from several studies (Moru, 2006; Jordaan, 2005; Denbel, 2014) and refers to the didactic design (Suryadi, 2013) and the didactic situation (Brousseau, 2002). Furthermore, the instrument was validated by mathematicians to get the concept of a limit function that is appropriate and in accordance with scientific concepts. In addition, this test item was tested on high school students (secondary education level with an age range of 16-18 years according to government regulations through the national education system) and also mathematics college students who have studied differential calculus. This is done to determine the readability and accuracy of the questions to be tested on college students. In this study, the test material on the concept of the limit of a function is limited to the relationship between a function and the limit of a function. Two questions or cases in the written test (See Figure 1), the first case is about determining and comparing two functions and their relation to limits, and the second case is about the concept of the relationship between functions and limit functions.

It is known that $f(x) = \frac{x^2 - 9}{x-3}$ and *g(x)* = *x* + *3*. Answer the following questions and their reasons!

Is the function f(x) the same as the function g(x)? Draw the graph!

Find $\lim_{x \to 3} f(x)$ and $\lim_{x \to 3} g(x)$!

What conclusions can be drawn from solution (b)?

Answer the following questions with examples and explanations!

If $\lim_{x \to a} f(x) = L$, is $f(a) = L$?

If $f(a) = L$ and $\lim_{x \to a} f(x)$ exist, is $\lim_{x \to a} f(x) = L$?

Figure 1: Two questions given on a written test to research subjects

Based on various written test answers, nine students were selected (symbolized by M1, M2, …, M9) to take part in an online semi-structured interview. This interview aims to confirm the test results and also to find out their experience in studying limits in the course of differential calculus.

The research data, namely written answers and interview transcripts from the recordings were analyzed descriptively. In addition, the coding of each of these data related to the concept of limits and the relationship between limits and functions is carried out. For the same context, they will be grouped under one code so that several different categories are obtained for the overall written answers and interview transcripts. Furthermore, an analysis of these categories is carried out related to the epistemological obstacles that arise and also based on the learning experience during online learning.

RESULTS

In this section, we will discuss the results of the subjects' answers through written answers and the results of interviews for the two questions or cases of limit functions.

First case: Define and compare two functions and their relation to limit

For the first question of the first point (a), most of the subjects concluded that the two functions f(x) and g(x) are the same functions. From the solution process, it can be seen that the function f(x) is translated by factoring and then dividing regardless of the conditions that result in x+3, which is the same as the function g(x). Thus, the graphs of the two functions are depicted the same (see Figure 2). Most of the subjects did not specifically state what happens at x=3 for the function f(x), only a straight line that passes through x=3 in the graph. From the way of solving the problems and the graphs described, it can be seen that the subject did not understand the concept of functions, especially rational functions correctly, thus ignoring the special rules of these functions.

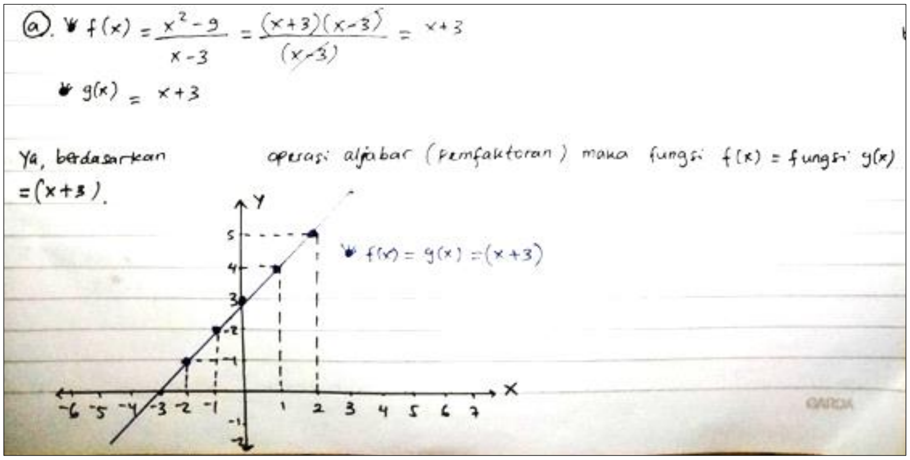


Figure 2: M1 represents f(x) = g(x) so the graph is unified ignoring the special rules of the function f(x)

Only one subject (M9) answered differently that the function f(x) is not the same as g(x). The subject takes any positive and negative x values and x = 0. This also applies to any value of y. Before describing the graph of the function f(x), the subject looks for the turning point with x=1 so that a graph like the one shown in Figure 3. This erroneous understanding of rational functions causes errors in designing the graph of the function. Subjects do not pay attention to the form of rational functions properly, so they assume the function is a quadratic function. The transcript of the interview between the researcher (R) and the subject (M9) is as follows.

R : *Is the function f(x) the same as g(x)?*
M9: *not the same, if the first is a quadratic function, the second is a linear function*
R: *what about the results?*
M9: *for the result, if the linear function directly finds the x value, but for the quadratic function, we have to make a test point first and then we will produce two x values, can it be x twin roots or x is different*
R: *How to find the value of the function f(x)?*
M9: *x squared is simplified one by one, it's still* ($x^2 - 9$) *means that it is first translated to its simplest form to be (x-3)(x+3) equalized (x-3).*
R: *What to do next?*
M9: *(x-3) is the same as (x-3) the denominator can be beheaded.*
R: *Why can be beheaded?*
M9: *Because they are the same. The value is 1. So, the result is (x+3)*
R: *Then, what is the conclusion?*
M9: *It's the same, ma'am, with the first function.*
R: *Why is it the same?*
M9: *Because the numbers are the same as if they can be divided like that, for example like 9 and 3, make them still in one, like multiples are the same, multiples of 3. So, when there is a number that can be divided the result is not 1, so the remainder is (x+3).*

From the interviews conducted, M9's answer changed which stated that the two functions f(x) and g(x) were the same. In the answer sheet, the graphs made are also different, the graph of the function f(x) is in the form of a parabola while the function g(x) is a linear line. Finally, M9 mentions that there is an error in understanding the two functions. Since the two functions are the same, the graph is the same. When asked again whether the function f(x) is equal to g(x), M9 states:

> M9: *It is the same, because after the function f(x) will produce a linear function. But not all. Why can the function f(x) in this number be the same as the function g(x) because the number in the value of the function f(x) is equally divisible by multiples, which is a multiple of 3. So, when we simplify the form it produces a linear function so that it is equal to the function g(x) but when the value of the function f(x) is different, meaning not in multiples, it might not be the same function value as g(x).*

Although M9 understands the concept of quadratic functions in describing graphs, in this case, M9 does not understand it in depth. The function f(x) is at first glance a quadratic function. Since this is a rational function, the denominator must also be considered. In solving the function f(x) will produce a linear function which means the graph is a straight line.

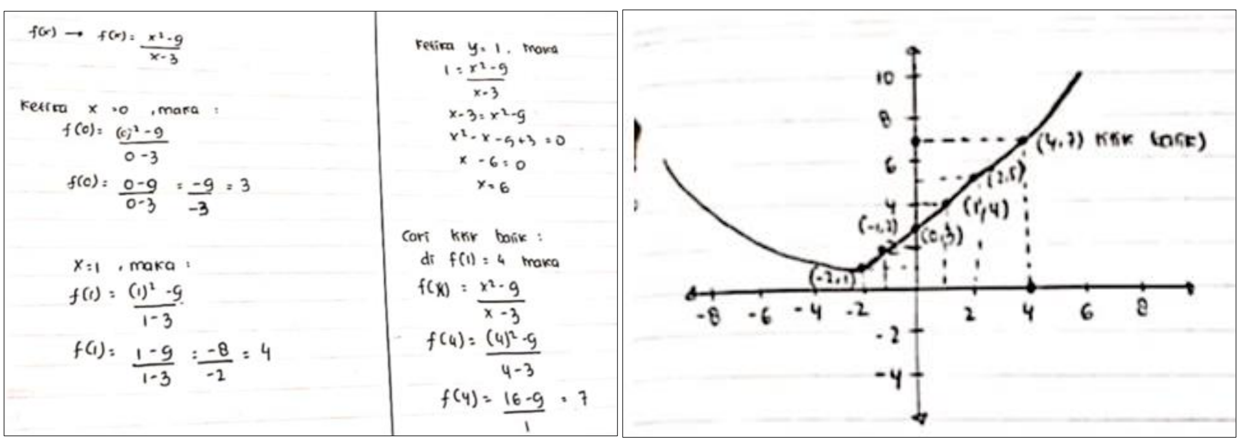


Figure 3: M9's answer to the function f(x)≠g(x) and the graph of f(x)

In determining the limit value of the functions f(x) and g(x), almost all subjects can determine their value in the same way as in the first point (1-a). Where describes $\lim_{x\to 3}\frac{x^2-9}{x-3} = \lim_{x\to 3}(x+3) =$ 6 which is obtained from the substitution x=3. Thus, most conclude that the limit f(x) is the same as the limit g(x) because they are both equal to 6 as x approaches 3, but without paying attention to the special rules of the function f(x). In this case, the concept of limit is not properly understood by the subject. They understand the concept of limit is the same as the concept of a function, directly substituting a value of x into the function. Though this concept is clearly different even though it will get the same value for the function and also the known limit.

There is one subject (M2) who gives reasons for elaborating by factoring in solving the function f(x). The reason is that if the function f(x) is substituted directly for x=3 it will produce $\frac{0}{0}$, so to avoid this, the function in the numerator must be factored into $(x^2 - 9) = (x-3)(x+3)$. This solution is actually not wrong, it's just that the subject does not understand further the rules that must be met for a rational function. When obtaining the result in the form $\frac{0}{0}$, the subject states it as an indeterminate form. That is, the subject understands that dividing zero by zero will result in indeterminacy, but the subject does not relate the results obtained with the concept of limits. Subjects do factor to avoid indeterminate results. Although using other ways to get the limit value, the concept of the limit remains that x approaches 3, not x equals 3. This shows that the limit of the function still has a value but is not continuous when x is equal to three.

In giving conclusions about the limit functions of f(x) and g(x), most of the subjects stated that the two limit functions are the same because they have the same limit value. The variety of subject answers can be seen in Table 1.

Answer category	Answer explanation
The limit value of the function is the same as the value of the function	The limit value of the function f(x) is the same as the limit value of the function g(x)
	f(x) = g(x), the limit solution f(x) is different from g(x) but has the same value
	g(x)=x+3 is a factor or simple form of $\frac{x^2-9}{x-3}$
The limit value of the function is not the same as the value of the function	the function f(x) is not the same as g(x) because the limit values are different, f(x) ≠ 3 and g(x) ≠ -3
	The limit f(x) is the limit of the polynomial, while the limit g(x) is the algebraic limit

Table 1: Various conclusions for the answer to the first question

Based on Table 1, it can be seen that most of the subjects stated that the two limit functions f(x) and g(x) were the same for various reasons. For example, based on the process of solving the function f(x) by factoring which produces the same function as g(x), namely (x+3). Thus, the values of the functions f(x) and g(x) will be the same for all x. In this case, the subject does not understand the concept of function properly, especially rational functions. A rational function will have a function value when it fulfills one of the conditions, namely that the denominator cannot be equal to zero. That is, for x=3 does not satisfy the function $f(x) = \frac{x^2-9}{x-3}$ because it produces a zero divisor. So, it must be a function $f(x) = \frac{x^2-9}{x-3}$, with the condition x≠3.

There were two subjects who answered that the value of the functions f(x) and g(x) was the same but with different explanations. In the function $f(x) = \frac{x^2-9}{x-3}$ where x=3 is not met or f(x) is not defined at x=3, otherwise the function g(x) is defined at x=3. Another subject (M2) stated that *"Although the function f(x) = g(x) but the solution to the limit of the function f(x) is different from the function g(x), it will have the same result"*. In this case, the subject only sees the process of solving the function f(x) by factoring. Where the result of factoring is a function of g(x). In other words, the subject is not careful about the concept of a defined or undefined function at a certain point.

Subject M15 stated that the function f(x) is different from g(x) but the explanation is wrong. Where the subject stated that "*the function f(x) is not the same as g(x) because the limit values are different, f(x)= 3 and g(x)= -3"*. This is not correct because of the concept of a function where all the x domains of real numbers will satisfy the function g(x). In addition, in this first point, what is being asked is the similarity of the two functions, not related to the limit of the function. In other words, the subject does not understand the context of the problem correctly. After being investigated, it turns out that the subject's answers cover all the questions in the second case. In this case, the subject combines the answers to the three questions relating to functions and function limits. The written answers can be seen in Figure 4.

From Figure 4, it can be seen that the subject understands the concept of the limit of a rational function by mentioning the condition for the limit of its function which is not defined at x=3. This is because when x = 3 then the denominator returns zero. On the other hand, the subject made a mistake in substituting the value of x=-3 in the limit of the function g(x) which resulted in the wrong graph of the function being described. In other words, the subject's understanding of how to make graphics still needs attention. Thus, it can be stated that this subject is less thorough in understanding the context of the problem and also the concepts of limits and functions. In addition, there is a subject (M9) who does not directly state the conclusion to the two known limit functions f(x) and g(x) but only provides an explanation. In this case, the subject only concludes with the definition of the type of limit function for the functions f(x) and g(x) along with how to solve it. The subject can distinguish between the two limit functions, but this answer is not what is expected from the conclusion of the process of solving the limit functions of f(x) and g(x).

"Limit f(x) is a polynomial limit, where the solution is to simplify the form of the function by dividing the largest power, after being simple, substitute the value of x into the equation of the function f(x). While the limit of the function g(x) is an algebraic limit where the only way to solve it is to substitute/replace the value of x into the equation of the function. If the substitution method doesn't work, then use the factoring method or multiply by common roots".

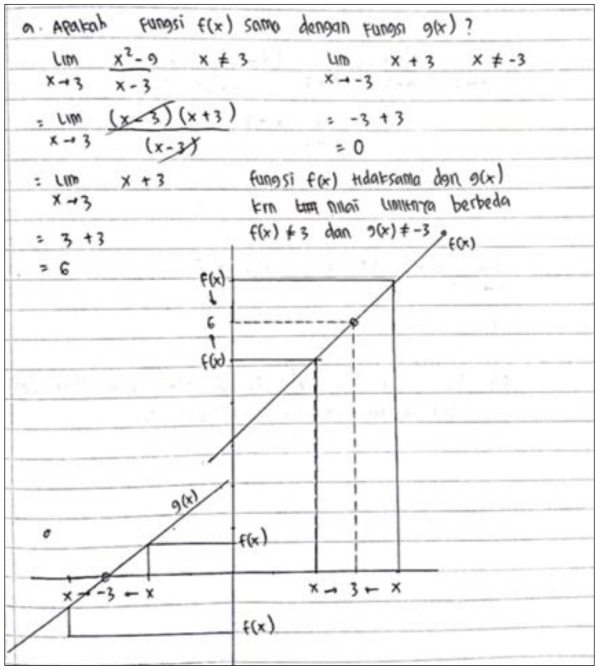


Figure 4: The answer of the subject who does not understand the context of the question and the explanation is wrong

In addition, another subject mentioned a way of solving a function that is to "*calculate the limit value x→3 with the denominator ≠a. If* $\lim_{x\to a} X$ *then the result is a. Using the factoring method if* $\lim_{x\to a}\frac{x}{x-a}, x-a\neq 0$". Based on the subject's answer, information was obtained that the subject had not correctly understood the concept of function for all domain points. This resulted in an understanding of the concept of limit is wrong.

In solving the function and the limit of this function, it is necessary to pay attention to the known form of the function. For the function f(x) there is a condition that must be met, namely x≠3 because for x=3 it will produce $\frac{0}{0}$, which means the function f(x) is not defined at x=3. In contrast to the function g(x) which satisfies all x real numbers. That is, the two functions are only the same for all domains f(x) i.e. real numbers except when x=3. In terms of limits, for the limit functions f(x) and g(x) have the same value when x→3 is close to 6.

Second case: Concept of the relationship between function and limit of a function

Based on all of the subject's answers to these questions, three categories of answers can be grouped, namely statements that are true, statements that are not necessarily true, and statements that are different. In the first question, there are 50% of the subjects answered correctly that if the limit value of a function is L then the function value is also L, and six people answered not necessarily correct because it depends on the known function, while the other subjects answered differently for the known limit and function values. For the second question, most of the subjects (81%) answered correctly for the known statement that if the function value is L and the limit value exists, then the limit value is the same as the function value for the same function.

In answering the first case related to the relationship between the function's limit value and the function value, *"If* $\lim_{x\to a} f(x) = L$*, then f(a) = L"*, most of the subjects stated it was true if the function's limit value was L then the value of the function will also be L. This value is obtained from the process of substituting the value of an into the function f(x) (see Figure 5). In this case, the subject did not understand the concept of function and function limit and their relation correctly. In other words, the subject's understanding of the types of functions is limited to simple functions. Where each domain that is substituted into the function easily produces a function value. The concept of the limit of a function is considered the same as the concept of a function. To get the limit value, they do it by substituting a value that is approximated without meaning to be approximated. As with functions, taking any number of domain members is then substituted into the known function. Whereas the concept of limit is very different from the concept of function. The following is a transcript of the interview between the researcher and M1 related to understanding the concept of function and function limit.

M1: *In my opinion, it depends on the conditions. Sometimes there is a limit value but the limit value is not the same as the function value. So, it could be that the limit value of a function has a limit value and there is a limit value that is not the same as the function value. Now for my explanation, I give an example where the limit value exists and has the same function value as the limit value. So that when the limit value of a function exists, it is the same as the function value, which is equal to L. … And it will have the same value when the a value is substituted into the function.*

R: *Means for the first case, is the statement true?*

M1: *Yes, because I gave an example where the limit value and function are the same. The limit value of a function, if it has a limit value, the limit value is the same as the function value, then it will be the same as the function value.*

R: *So, is this a conclusion for all or just a special case?*

M1: *Especially for this question, ma'am.*

R: *So, what is the conclusion for the statement if the limit is L, is the function value also the same as L?*

M1: *In my opinion yes, ma'am, when the limit value is L then the function value will also be L.*

Based on the answers during the interview, M1 added that the written answer was a true statement with the examples given. In this case, M1 has not been consistent in providing explanations regarding the concept of limits and functions. Even though at the beginning of the conversation, M1 has shown the correct concept that the statement can be true or false depending on the function given. However, at the end when asked again, M1 states that the limit value will always be the same as the function value.

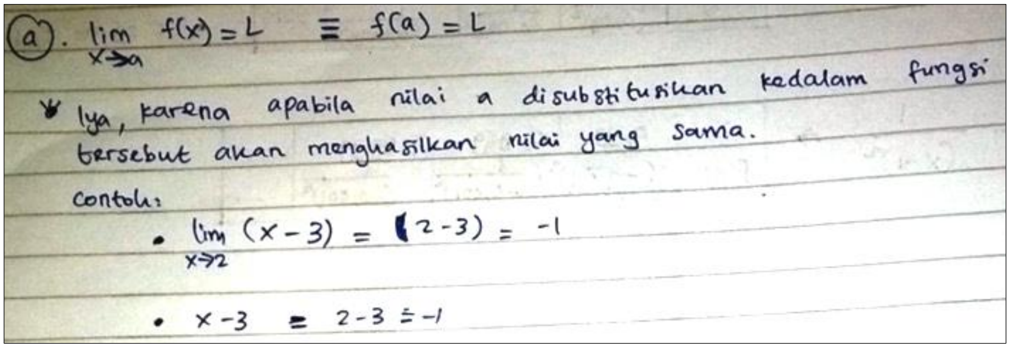


Figure 5: M1 answer for function value equal to limit

Another subject also stated the same for the statement that the value of the limit is the same as the value of the function with the reason "*when the limit is L, the left and right limits are defined in L, then the function will also be worth L for the same point (x=a)*". One of the subjects gave an explanation related to continuity that "*lim f(x) = L, f(a) = L then lim f(x) = f(a) =L means that the function f is defined in a and the limit value is the same as the function value then f(x) is continuous at a. Must meet continuous conditions, (1) defined in a or f(a) exists, (2) lim left = lim right or limit value exists, (3) f(a) = limit value*". From this explanation, it can be seen that the subject understands the concept of continuous limit but is not yet firm for the answer in this first case.

On the other hand, there is a subject (M2) that states that the value of the function will be equal to the value of the limit of the function depending on the known form of the function, but this does not apply to all functions. In the answer, it is not necessarily written that f(a) = L because in $\lim_{x \to a}$ f(x) may be worth $\frac{0}{0}$ so it needs to be changed. The subject mentions an example, namely $f(x) = \frac{x^2-9}{x-3}$. In this case, the subject understands the concept of the limit of a function but is less precise in the concept of a function for a known domain, for example when x=3. In solving the limit case $f(x) = \frac{x^2-9}{x-3}$ by factoring $(x^2 - 9) = (x - 3)(x + 3)$ to get $\lim_{x \to 3} \frac{x^2-9}{x-3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x \to 3}(x + 3) = 6$. On the other hand, for the value of the function f(x) when x=3, the subject only substitutes the value of x=3 into the function without factoring and providing an explanation for the value of the function obtained is $\frac{0}{0}$. This is not correct because the function f(x) can also be factored in to get the value of the function. During the interview, M2 re-explained the example given where the example function can also be changed or factored in to avoid the $\frac{0}{0}$ form. M2 states that the function is x≠3 because when x=3 the function has no value. In this case, f(x) is defined for all values of x except when x=3.

Several other subjects also stated that if the value of a limit function is L, then the function value is not necessarily L as well. M4 only includes one simple example, namely $\lim_{x \to 2} f(x) = 5$ and $f(2) = 5$ without knowing the form of the function with the result being a number. From this explanation, it can be seen that the subject understands the concept of the limit of the function where the value of the limit of the function will not be right at that point but approaches it from both sides, namely the left side and the right side. In the written answer, the subject does not explain the concept of function. Similarly, M7 mentions that there are two possibilities for the known statement. The statement will apply to function values equal to the limit, and also applies to function values different from the limit. The subject explained that "*there is a graph of a continuous function where the limit value and the function value are the same. There is a graph of a discontinuous function where the limit value and the function value are not the same*". In this case, M7 adds examples for both statements accompanied by graphs (see Figure 6). This shows that M7 understands the concept of limit especially in distinguishing between continuous and discontinuous limits. Although the example given is very simple without any explanation of the form of the function given.

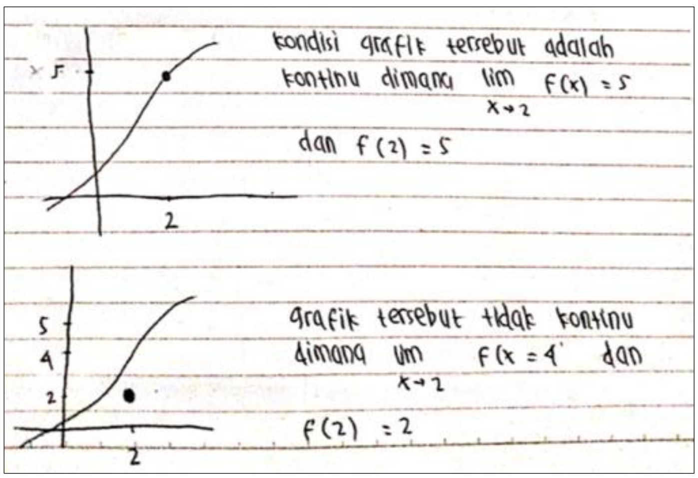


Figure 6: M7 gives an example of a function value equal to limit and vice versa

There are subjects who answer that the limit value is different from the function value. According to M9, the limit of a function is not the same as a function because "$\lim_{x \to a} f(x) = L$ *is a limit function where when a function x or f(x) where x⟶a is defined in L, while f(a) = L is a function (a) which has a value of L*". The subject provides an example of a simple function by way of substitution. This shows that the subject does not fully understand the concept of limits and functions and is limited to simple functions.

For the statement, if $\lim_{x \to a} f(x) = L$, then it is not necessarily the value of *f(a)=L*. It is based on the known form of the function. The limit value of the function will be the same as the function value if the known function does not require certain conditions as in the rational function. Rational functions must be considered in the denominator, if the denominator returns zero then the function is not defined at the specified point.

For the second statement, it is known that the value of the function *f(a)=L* and $\lim_{x \to a} f(x)$ exists, then $\lim_{x \to a} f(x) = L$. Most of the subjects stated that the statement was true based on the examples included with the substitution method. When searching through interviews, M1 stated that the answer was different from the written answer. According to M1 the limit value and function value will be the same or different depending on the given function. The transcript of the interview with M1 is as follows.

M1: *For this second condition, it depends on the function. Sometimes there is a function value when we substitute it, it is not the same as the existing limit value. That is, there is a function that does not have a limit value but has a function value*. (Next, the researcher asked M1 to write down an example in a notebook, as shown in Figure 7).

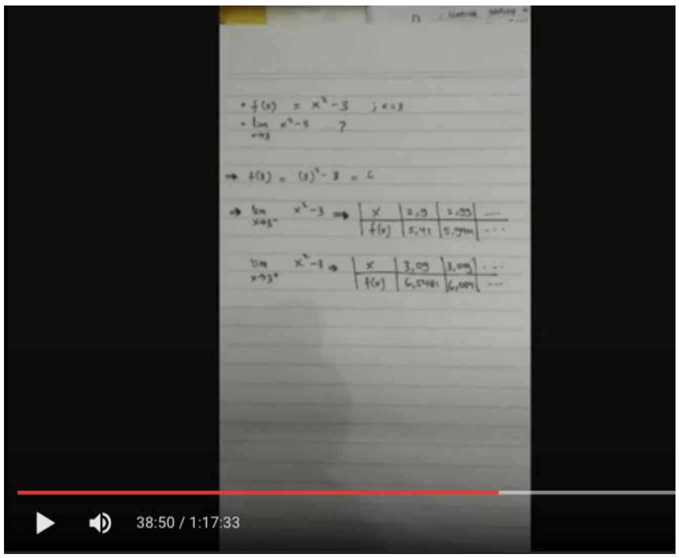


Figure 7: M1 gives an example of a function value not equal to the limit value

One of the subjects gave a reason with the answer "*yes, it means a continuous function where the function value and the limit value are the same*". Several other subjects answered the statement correctly by giving examples of rational functions so that the solution was by factoring. On the other hand, there is a subject (M9) who states that the statement is not the same as the reason "*the difference between f(a) – L with* $\lim_{x \to a} f(x)$ *or* $f(a) = L \neq \lim_{x \to a} f(x)$". From the results of the interview, M9 did not understand the context of the question because it had never been studied. M9 stated that there was no relationship between limits and functions. According to him, the function was correct. maps only one pair of domains to codomains while limits cannot be explained. In everyday life, limits are seen as limits such as debit card limits, while in the context of mathematics, limits are something that is close.

Based on the answers and explanations of the subjects regarding the relationship between functions and function limits, most of the subjects did not understand the concept properly. Especially when faced with cases like this that require analysis of the concepts of limits and functions. Even in this case, the subject is still limited in providing examples for each statement. The included examples are still in the form of simple algebraic functions. Even though there are still many forms of functions and function limits that can justify the statement or cancel it.

From the two statements, the same conclusion is obtained that if it is known that $\lim_{x\to a} f(x) = L$, then the value of f(a) is not necessarily the same as L. On the other hand, if we know the value of the function of $f(a) = L$ and $\lim_{x\to a} f(x)$ exists, then $\lim_{x\to a} f(x)$ is also not necessarily equal to L. Based on this, it can be said that the value of the function and the limit value of the function can be the same or different values. This is because the value of the function and also the value of the limit of the function depend on the known function.

DISCUSSION AND CONCLUSIONS

In building the concept of limits, there is at least three knowledge that must be possessed, namely actions, processes, and objects (Cottrill, et al., 1996). The development stages include actions internalized into processes and processes encapsulated into objects. This is investigated in the learning process using computers in studying the concept of limits. On the other hand, Moru (2009) investigated the development of the limit concept without using a computer. The results show that the subject understands whether or not there is a limit to the indeterminate, $\frac{0}{0}$ for functions represented algebraically. On the other hand, in this study, the subject erred in the concept of a limit which resulted in the division of a number n (where n≠0) by zero. Subject assumes that the division results in undefined also applies to limits. This concept only applies to division and differs in the limit case (Sulastri, et al., 2022). For geometric functions, the subject denies that there is a limit to undefined functions. According to the subject the limit value can only be determined for algebraic functions. In addition, the subject is confused about the concept of the limit value and function value (Moru, 2009). Likewise, with the results of this study where the subject also has not been able to understand the concept of limits and functions correctly.

A limit is indicated by the symbol →. Most of the subjects understand that in determining the limit value, for example when x→0 it is replaced by x=0 in a known expression. This is one of the sources of confusion and errors experienced by the subject in determining the limit value. The use of this generalized substitution method applies to continuous functions. In addition to this context, it is referred to as an epistemological barrier. According to Tall (1991), generalization is one of the epistemological barriers. Similarly, Bachelard states that several types of epistemological barriers include the tendency to generalize, the barrier caused by natural language, and the tendency to rely on mistaken intuitive experience (Herscovics, 1989). The epistemological aspects that are problematic in the concept of limits occur in the relationship between graphical and arithmetic representations of mathematical content, processes and objects, static and dynamic interpretations, and intuitive ideas and mathematical specifications (Hofe, 2003).

In relation to functions, there are limitations in manipulating algebraic functions (Sebsibe, Dorra, & Beressa, 2019), for example being faced with a division by zero situation which results in no function value. This interpretation is assumed to be the same as the concept of limit (Fischbein, 1999; Moru, 2009). This understanding causes problems in understanding the concepts of limits and functions. A limit is seen as a point that is approached without reaching it, or a point that is approached and reached (Taback, 1975; Cornu, 1991). In this study, the subject did not explicitly state whether the limit was being approached without being reached or when it was being approached.

Based on the process of solving the function limit problem carried out by the subject, most of them understand and do it operationally, namely explaining in the act of calculating. This shows that the limit is associated with calculations, which come from previous learning experiences when high school (Moru, 2009). Learning at school, where the limit of function material is in the second grade even semester. Learning is not optimal because students are focused on being able to solve mathematical problems operationally to face the national exam. In addition, the subject matter of school limits was obtained through online learning due to the COVID-19 pandemic in the first year of the lockdown.

The application of online learning has positive impacts such as flexibility in place and time, accessibility, and effectiveness of assessment and communication methods (Musthaha, et al., 2022). Findings from research by Qutishat, Obeidallah, & Qawasmeh (2022) show the success of implementing online learning with material understood by students, but student interaction during learning is low. Difficulties in online learning can occur if schools do not have experience with electronic learning such as teachers not understanding how to use online applications (Zaharah & Kirilova, 2020). The same difficulty also occurred to the subjects in this study which was their first experience. In addition, the lack of understanding of basic concepts such as the concept of real numbers, inequalities, functions, and others also greatly affects the study of function limits. In the teaching process, the lecturer no longer explains in detail the concept because it is considered to have been studied in basic mathematics courses such as Elementary Algebra. The results of research by Mailizar, et al. (2020) show that the barriers that have the highest impact on electronic learning are at the student level. A teacher needs to ensure an accurate representation of mathematical tools in selecting technology to use in the classroom. Thus, in the online and offline teaching and learning process, special attention must be paid to overcoming the difficulties and misconceptions identified in learning a material (Sebsibe, et al., 2019).

Subjects easily understand a function or limit which is represented in a simple algebraic form such as the examples given for known cases. In determining the value of a function, the subject performs by substitution of some arbitrary real number or value specified. The same way is also done to find the limit value of a function. In this case, the subject generalizes the direct substitution method to find the limit value. This mistake made the concept of function and limit and their relation not understood properly. In other cases, there are subjects who understand that the limit cannot be reached or is close to the specified point (not exactly at that point), compared to the concept of a function that can be substituted right at that point. This is similar to William's (1991) statement that there are several confusing concepts related to limits, including whether a function can reach its limit; whether limit is limit; whether limit is a dynamic process or a static object; and whether limits are inherently tied to the concept of motion. These problems lead to incomplete conceptions of limits (Denbel, 2014). According to Williams (1991), this conception relates to the process of limiting by the mathematical community before Cauchy's formal definition of the definition of limit using epsilon-delta.

In the nineteenth century, Weierstrass wrote a formal definition of limit that requires a high level of logical and syntactic knowledge to understand, namely $\lim_{x \to a} f(x) = L$ where $\forall \epsilon, \exists \delta; 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$. Easily understanding this definition can be done with an informal interpretation that is, a limit is a number whose y-value of a function can be as close as desired, by choosing x in the interval x as small as necessary (Sarvetani, 2011).

Several studies that discuss the formal definition of limits (Cornu, 1991; Cottrill et al., 1996; Ervynck, 1981; Fernández, 2004; Tall & Vinner, 1981; Vinner, 1991; Williams, 1991) show that there are various reasons students have difficulty communicating a coherent understanding of the formal definition of limits. One of them is the struggle of students in understanding algebraic notation in the definition of limit ε–δ (Cornu, 1991; Cottrill et al., 1996; Ervynck, 1981; Fernández, 2004). Students' difficulties also occur in understanding what ε and δ represent; the relationship between the variables (and parameters) in the definition; and why |x – c| must be positive, while |f(x) – L| is not (Fernández, 2004). Another difficulty in the formal definition of limits is due to the struggle students have with quantification (Cottrill et al., 1996; Dubinsky, Elterman, & Gong, 1988; Tall & Vinner, 1981).

The findings of Beynon & Zollman (2015) show that many students do not use a formal definition of limit to solve limit problems. In this case, their understanding of the definition of the concept of limit is inconsistent with the definition of the formal concept. Only students who have high abilities have openness to using formal mathematical concepts. The same thing is also obtained from other studies (Tall, 1990) that students have strong procedural knowledge compared to conceptual understanding of a mathematical concept. This causes students not to be able to solve problems by applying stage of problem-solving. Other causes are the lack of mathematical literacy skills and imperfect mathematization processes (Purnomo, et al., 2022).

In the case of the value of a function with a limit value, most of the subjects stated that the value of the function will always be equal to the limit value for a function that is known to be the same. Other errors also occur in the prerequisite material for limit functions such as the concept of real numbers and the concept of infinity. The subject has a limited conception of the basic concept. To overcome these errors, the presentation of the material, especially the basic concepts, needs to be done in-depth so that is easy to understand and to minimize the occurrence of misconceptions.

REFERENCES

[1] Arnal-Palacián, M., & Claros-Mellado, J. (2022). Specialized content knowledge of pre-service teachers on the infinite limit of a sequence. *Mathematics Teaching Research Journal*, *14*(1), 169-189.

[2] Artigue, M. (2000). Teaching and learning calculus: What can be learnt from education research and curricular changes in France? *CBMS Issues in Mathematics Education 8*, 1-15, A.M.S.

[3] Barnes, M. (1991). *Investigating change*. Melbourne: Curriculum Cooporation.

[4] Beynon, K. A., & Zollman, A. (2015). Lacking a formal concept of limit: advanced non-mathematics students' personal concept definitions. *Investigations in Mathematics Learning*, *8*(1), 47-62.

[5] Bezuidenhout, J. (2001). Limits and continuity: some conceptions of first-year students. *International Journal of Mathematical Education in Science and Technology*, *32*(4), 487–500. https://doi.org/10.1080/00207390010022590

[6] Brousseau, G. (2002). *Theory of didactical situations in mathematics. In theory of didactical situations in mathematics*. Kluwer Academic Publishers. https://doi.org/10.1007/0-306- 47211-2

[7] Cavanagh, M. (1996). *Student understandings in differential calculus*. Macquarie University. Retrieved from https://trove.nla.gov.au/version/27235403

[8] Cornu, B. (1981). Learning the concept of limit: Models and spontaneous modes. *Proceeding of fifth Conference Psychology of Mathematics Education*. Grenoble, France: Psychology of Mathematics Education. 322-326

[9] Cornu, B. (1991). 'Limits', in Tall, D (ed). *Advanced mathematical thinking*, Dordrecht, the Netherlands, Kluwer academic publishers, pp 153-166.

[10] Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., and Vidakovic, D. (1996). Understanding the limit concept: beginning with a coordinated process schema. *Journal of Mathematical Behavior*, *15*, 167-192

[11] Creswell, J.W. (2012). *Educational research: Planning, conducting, and evaluating quantitative and qualitative research*. Educ Res (2012), 10.1017/CBO9781107415324.004

[12] Davis, R., & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *Journal of Mathematics Behavior*, 5, 281-303.

[13] Denbel, D. G. (2014). Students' misconceptions of the limit concept in a first calculus course. *Journal of Education and Practice*, *5*(34), 24–41.

[14] Dubinsky, E., Elterman, F., & Gong, C. (1988). The student's construction of quantification. *For the Learning of Mathematics, 8*(2), 44–51.

[15] Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. Educational Studies in Mathematics, 61, 103–131.

[16] Ervynck G. (1981). Conceptual difficulties for first year students in the acquisition of the notion of limit of a function. *Actes du Cinquième Colloque du Groupe Internationale* PME, Grenoble, 330-333.

[17] Fernández, E. (2004). The students' take on the epsilon-delta definition of a limit. *Primus*, *14*(1), 43–54. doi:10.1080/10511970408984076

[18] Fernández-Plaza, J.A., Simpson, A. (2016). Three concepts or one? Students' understanding of basic limit concepts. *Educational Studies in Mathematics, 93*, 315–332. https://doi.org/10.1007/s10649-016-9707-6

[19] Fischbein, E. (1999). Intuitions and schemata in mathematical reasoning. *Educational Studies in Mathematics, 38*, 11–50.

[20] Fraenkel, J. R., Wallen, N. E., & Hyun, H. H. (2012). *How to design and evaluate research in education* (8th ed.). New York: Mc Graw Hill

[21] Herscovics, N. (1989). Cognitive obstacles encountered in the learning of algebra, in Wagner, S. and Kieran, C (eds), *Research issues in the learning and teaching of algebra*. Reston, VA, Lawrence Erlbaum for NCTM, pp. 60-86

[22] Hofe, R. vom. (2003). Epistemological problems with the limit concept a case study on communication and argumentation within a computer-based learning environment. *Thematic Working Group 4, CERME 3: Third Conference of the European Society for Research in Mathematics Education*, 28 February - 3 March 2003 in Bellaria, Italy. 1-11. http://www.mathematik.tu-dortmund.de/~erme/CERME3/Groups/TG4/TG4_list.php

[23] Huang, R., Liu, D., Tlili, A., Yang, J., & Wang, H. (2020). *Handbook on facilitating flexible learning during educational disruption: the Chinese experience in maintaining undisrupted learning in covid-19 outbreak*. Beijing: Smart Learning Institute of Beijing Normal University.

[24] Job, P., & Schneider, M. (2014). Empirical positivism, an epistemological obstacle in the learning of calculus. *ZDM: the international journal on mathematics education 46*(4):635-646. https://doi.org/10.1007/s11858-014-0604-0

[25] Jordaan, T. (2005). *Misconceptions of the limit concept in a mathematics course for engineering students*. University of South Africa.

[26] Kidron, I. (2011). Constructing knowledge about the notion of limit in the definition of the horizontal asymptote. *International Journal of Science and Mathematics Education. 9*(6), 1261–1279. https://doi.org/10.1007/s10763-010-9258-8

[27] Lester, S. (1999). *An introduction to phenomenological research*. Taunton UK. Retrieved from www.devmts.org.uk/resmethy.pdf

[28] Mailizar, Almanthari, A., Maulina, S., Bruce, S. (2020). Secondary school mathematics teachers' views on e-learning implementation barriers during the COVID-19 pandemic: The case of Indonesia. *EURASIA Journal of Mathematics, Science and Technology Education*, *16*(7), em1860. https://doi.org/10.29333/ejmste/8240

[29] Moru, E.K. (2006). *Epistemological obstacles in coming to understand the limit concept at undergraduate level: a case of the National University of Lesotho*. Unpublished doctoral thesis. Cape Town: University of the Western Cape.

[30] Moru, E. K. (2009). Epistemological obstacles in coming to understand the limit of a function at undergraduate level: A case from The National University of Lesotho. *International Journal of Science and Mathematics Education*, *7*(3), 431–454. https://doi.org/10.1007/s10763-008-9143-x

[31] Mushtaha, E., Abu Dabous, S., Alsyouf, I., Ahmed, A., Abdraboh, N.R., (2022). The challenges and opportunities of online learning and teaching at engineering and theoretical colleges during the pandemic. *Ain Shams Engineering Journal*, *13*(6), https://doi.org/10.1016/j.asej.2022.101770

[32] Oehrtman, M. (2002). *Collapsing dimensions, physical limitation, and other student metaphors for limit concepts: An instrumentalist investigation into calculus students' spontaneous reasoning* (Dissertation the University of Texas)

[33] Parameswaran, R. (2007). On understanding the notion of limits and infinitesimal quantities. *International Journal of Science and Mathematics Education*, *5*, 193–216.

[34] Purnomo, E. A., Sukestiyarno, Y. L., Junaedi, I., Agoestanto, A. (2022). Analysis of problem solving process on hots test for integral calculus. *Mathematics Teaching Research Journal (MTRJ), 14*(1), 199-214

[35] Qutishat, D., Obeidallah, R., & Qawasmeh, Y. (2022). An Overview of Attendance and Participation in Online Class During the COVID Pandemic: A Case Study. *International Journal of Interactive Mobile Technologies (iJIM), 16*(04), pp. 103–115. https://doi.org/10.3991/ijim.v16i04.27103

[36] Reed, S. D. (2018). *Student personal concept definition of limits and its impact on further learning of mathematics*. Bowling Green State University, Ohio, USA

[37] Roble, D. B. (2017). Communicating and valuing students' productive struggle and creativity in calculus. *Turkish Online Journal of Design Art and Communication*, *7*(2), 255-263. https://doi.org/10.7456/10702100/009

[38] Sadler, P. M., & Sonnert, G. (2017). *Factors influencing success in introductory college calculus. The role of calculus in the transition from high school to college mathematics*. Retrieved from https://www.maa.org

[39] Sarvestani, A. K. (2011). *Contemplating problems taken from the history of limits as a way to improve students' understanding of the limit concept.* Thesis. Universiteit van Amsterdam

[40] Sebsibe, A.S., Dorra, B.T., & Beressa, B.W. (2019). Students' difficulties and misconceptions of the function concept. *International Journal of Research - Granthaalayah*, *7*(8), 181-196. https://doi.org/10.5281/zenodo.3381160.

[41] Sebsibe, A. S., & Feza, N. N. (2020). Assessment of students' conceptual knowledge in limit of functions. *International Electronic Journal of Mathematics Education*, *15*(2). https://doi.org/10.29333/iejme/6294

[42] Sierpińska, A. (1987). Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics, 18*(4), 371-397.

[43] Sulastri, R., Suryadi, D., Prabawanto, Cahya, E., Siagian, M.D., and Tamur, M. (2021). Prospective mathematics teachers' concept image on the limit of a function. SEA-STEM 2020. *Journal of Physics: Conference Series*. 1882 (2021) 012068 https://doi.org/10.1088/1742- 6596/1882/1/012068

[44] Sulastri, R., Suryadi, D., Prabawanto, S., Cahya, E., Fitriani. (2022). Zone of concept image differences in infinite limits at undergraduate level. *Jurnal Didaktik Matematika 9*(1), https://doi.org/10.24815/jdm.v9i1.24709

[45] Suryadi, D. (2019a). *Landasan filosofis penelitian desain didaktis* (DDR). Bandung: Pengembangan DDR Indonesia.

[46] Suryadi, D. (2019b). *Pengetahuan transposisi sebagai konektor pendidikan akademik dan pendidikan profesi guru* (PPG) matematika. Universitas Pendidikan Indonesia

[47] Suryadi, D. (2013). Didactical Design Research (DDR) to improve the teaching of mathematics. *Far East Journal of Mathematics Education 10*(1), 91-107 .

[48] Szydlik, J. E. (2000). Mathematical beliefs and conceptual understanding of the limit of function. *Journal for Research in Mathematics Education*, *31*, 256–276.

[49] Taback, S. (1975). The child's concept of limit, in Rosskopf, M. (ed), C*hildren's mathematical concepts,* New York. NY, Teachers College Press, pp. 111-144

[50] Tall, D. (1990). Inconsistencies in the learning of calculus and analysis. *Focus on Learning Problems in Mathematics*, 12, 49–63. 1, 2.1

[51] Tall, D. (1991). The psychology of advanced mathematical thinking, in Tall, D. (ed), *Advanced mathematical thinking*, Dordrecht, The Netherlands, Kluwer Academic Publishers, pp. 3-21

[52] Tall, D. (1993). Students' difficulties in calculus. *In Proceedings of Working Group 3* (pp. 13–28). Québec, Canada.

[53] Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151–169.

[54] Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In A. J. Bishop (Managing Ed.) & D. Tall (Vol. Ed.), *Mathematics Education Library*: Vol. 11. Advanced mathematical thinking (pp. 65–81). Boston, MA: Kluwer. doi:10.1007/0-306-47203-1_5

[55] Williams, S. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education 22*(3), 219-236

[56] Zaharah, Z., & Kirilova, G. I. (2020). Impact of corona virus outbreak towards teaching and learning activities in indonesia. SALAM: *Jurnal Sosial dan Budaya Syar-i*, *7*(3). https://doi.org/10.15408/ sjsbs.v7i3.15104

[57] Zollman, A. (2014). University students' limited knowledge of limits from calculus through differential equations. *The mathematics education for the future project: Proceedings of the 12th International Conference*, (pp. 693-698).

Ethnomathematical Exploration on Traditional Game *Bahasinan* in Gunung Makmur Village the Regency of Tanah Laut

Abdul Jabar[1], Rahmita Yuliana Gazali[1*],Ajeng Ayu Ningrum[1], Muh. Fajaruddin Atsnan[2], Rully Charitas Indra Prahmana[3]

[1] Mathematics Education, STKIP PGRI Banjarmasin, Indonesia [2]Mathematics Education, Universitas Islam Negeri Antasari Banjarmasin, Indonesia, [3]Mathematics Education, Universitas Ahmad Dahlan, Indonesia

abduljabar@stkipbjm.ac.id, rahmitayg@stkipbjm.ac.id, ajengayuningrum09@gmail.com, fajaratsnan@uin-antasari.ac.id, rully.indra@mpmat.uad.ac.id

Abstract: Bahasinan is a traditional game, a maritime cultural heritage, which still exists today in the area of South Kalimantan, Indonesia, precisely in the village of Gunung Makmur. Bahasinan itself is a traditional game that should be played in groups. In Bahasinan, people can find many aspects that have strong correlation with the learning of mathematics. For example, in Bahasinan children should make up their arena by drawing six rectangles that should be combined with one another. This drawing of rectangles alone has contained mathematical concepts. Thus, the object of the recent study is to identify which part of the traditional game Bahasinan that contain mathematical concepts and to identify what mathematical concepts that have been contained in the traditional game. In conducting the study, the method that has been selected is qualitative study using ethnography. The data are gathered from observation, interview, and documentation with the local public figures and two children as the subject for the study. After the data have been collected, the data are analyzed by using data reduction, data presentation, and conclusion drawing or verification. In order to assure the data validity, triangulation method has been implemented so that the data that will be resulted are consistent. The results of the study show that the parts of the traditional game Bahasinan that contains mathematical concepts are game arena, game procedures, player position and player gesture whereas the mathematical concepts that can be found in the traditional game Bahasinan consist of two-dimensional figure, translation, reflection, inter-line connection, congruency, and mathematical activities.

INTRODUCTION

Indonesia is a country with abundant cultures embedding mathematical ideas, ways, and techniques using mathematical modeling. This creates opportunities in mathematics education to use local contexts and to boost students' critical reasoning and interest by reinventing mathematics rooted in students' culture existing in their surroundings to get the benefit from it (Prahmana, Rosa, & Orey, 2021). Comprising of thousand islands has become one of the factors that lead to the increasingly different cultures throughout the country. It is this cultural diversity that has shaped the pluralistic society, meaning that Indonesian people have different behaviors, customs, histories, and religions. Since culture refers to the general or universal phenomenon, this term can serve to accommodate the universal elements that can be found in every corner of the global society. Despite being general and universal, cultures have different shapes and forms. It is these different shapes and forms that make each culture unique (Maran, 2019).

Mathematics is one of the subjects that have been studied from elementary school until university for a long time and has been regarded as a fundamental subject (Charles et.al., 2019). Mathematics holds an important role in the daily life. For example, our activities such as waking up and going to bed cannot be set apart from concepts in mathematics. Unfortunately, sometimes Mathematics becomes a peculiar difficulty for some students and one of the reasons for this difficulty is that it is difficult for the students to comprehend the abstract concept within mathematics. Therefore, during the learning process in the classroom, the teachers should strive to make mathematics more interesting for the students. In addition, the students should be able to trigger the learning interest of the students especially the ones with less proficiency on certain materials. The teachers should be able to use the appropriate method within the learning process. Then, one of the ways to motivate the students to learn mathematics is encouraging the students to use the knowledge that they have and provide an opportunity for the students to reinvent and rebuild the mathematical concepts on their own. This end can be achieved by benefitting games. In fact, many researchers have opinion that to develop educational games for learning mathematics have shown that their games could facilitate mathematics performance, enjoyment, and growth self-efficacy (Ku et.al., 2014; MacLaren et al., 2017). Through this initiative, the students can both play and learn.

Mathematics that has association with culture is known as Ethnomathematics. Even many activities are identified as ethnomathematics, not all of them can be a good context in learning mathematics (Syahidah et.al., 2021). Ethnomathematics can be defined as part of mathematics that has been practiced by certain cultural groups such as urban people and rural people / villagers, children from certain age groups, customary communities, and alike (Rachmawati, 2012). An Ethnomathematics review contains several characteristics of mathematical activities in which there is a process of abstracting concrete experience into mathematics and vice versa and consists of grouping, calculating, measuring, devising tools or constructing buildings, making patterns, counting, finding location, playing and explaining.

The product of Ethnomathematics are the results of mathematical activities that have been possessed or developed in the middle of Indonesian society by inserting the mathematical concept into the cultural heritage such as temples and inscriptions, potteries and traditional tools, and also batik and embroidery patterns. The traditional games alone have been in existence throughout Indonesia since the ancient time. Almost all regions in Indonesia have peculiar traditional games respectively. Traditional games thus can be considered as an activity that develops out of the habits from certain society (Apriyono & Kholil, 2018).

Traditional games are very interesting to play because these games do not only contain fun but also cultural values and mathematical concepts. The reason is because traditional games, not only include physical skills, but also brain and strategy designing skills (Vardani & Astutik, 2020). Mathematical concepts as abstract ideas for children can help them in grouping or identifying objects as the learning examples. One of the traditional games that have been often played by the children in Gunung Makmur Village, South Borneo, is *Bahasinan*. The word *Bahasinan* comes from the terms *baha* and *masinan*. The word baha, which is then written as *baha* means to play freely, while *masin* means spear, as a result *Bahasinan* means to play freely to avoid spears or middle players called *hasin*. *Bahasinan* itself is a traditional game that should be played in groups. The area of the game is rectangular and with lines as the border. In order to play this traditional game, there should be two teams. One team serves as the defensive one and the other serves as the offensive one. Unfortunately, with the rapid development of technology, the traditional games such as *Bahasinan* have been rarely played since the children prefer the modern games such as video game, PlayStation, and online games more than the traditional ones. Modern games can result in numerous negative impacts for both health and psychology of the children. In addition, modern games can also cause addiction, resulting in significant loss of time. Those situations are different from the ones that can be found among the children who play the traditional games. Usually, playing the traditional games will leave unforgettable impression among the children. In addition, the existence of traditional games really helps children to grow a good social spirit towards others and the surrounding environment (Kurniawan, 2018).

Several previous research results say that a number of traditional games in Indonesia have a contribution as a starting point in learning mathematics. The results of Fadila's research (2021), said that in the *Lompat Tali* game there are mathematical concepts that are usually learned at the elementary school level. Starting from counting natural numbers, measuring lengths, measuring distances, forming angles and lines. Another study, from Kuswidi et.al (2021), said that traditional *Layangan* games has many mathematical concepts, including the concepts of line, angle, circumference, area, and congruence.

Traditional games such as *Bahasinan* have been gradually eroded by the turn of the century; therefore, the traditional games should be developed and preserved so that these games can deliver multiple benefits for the children. One of these benefits, especially by playing *Bahasinan*, is that the children can learn about two-dimensional figure. In order to play *Bahasinan*, the children should make six rectangles next to each other. This activity holds the mathematical concept that the children should master so that they can draw the six rectangles with the same shape and proportion. This activity alone has shown the strong correlation between *Bahasinan* and mathematics. If people explore more, there will be more associations that can be found. Departing from these associations, the researchers are encouraged to study and explore the parts of *Bahasinan* that contains mathematical concepts and the mathematical concepts that can be found in *Bahasinan*. So, very interesting to explore many concepts mathematics in *Bahasinan*, because there has been no previous research that examines *Bahasinan*.

METHOD AND DESIGN

The method used in this research is an ethnographic method, which is a method that describes the culture of a community (Spradley & McCurdy, 1989). Ethnography was chosen as the method in this study because it is linier with the aims of Ethnomathematics which study ideas, methods, and techniques in a particular culture from the original view of members of that culture (Prahmana, Rosa, & Orey, 2021).

The subjects for the study are two school-age children (6 – 12 years old) and one local figure namely the expert of *Bahasinan* traditional game. The study has taken place in the District of Takisung, the Regency of Tanah Laut, the Province of South Borneo. The data for the study are gathered through observation, interview, and documentation. The observation is used in order to have direct view on the phenomenon under the study by directly visiting the location without getting involved into the activity. In this study, what will be observed is the *Bahasinan* game arena, how to play it, the position of the players when playing the game, and the player's hand movements when playing. The observation instrument used is an observation sheet which contains indicators related to the *Bahasinan* game such as the *Bahasinan* playing arena, how to play the *Bahasinan*, the player's position when playing, and also the player's hand movements while playing.

Then, the interview technique that has been implemented in the study is the unstructured interview. In this occasion, the researcher has interviewed the local figure and the two school-age children in order to gather the necessary data. The interview instrument used was an interview guide sheet which contained indicators related to the *Bahasinan* game such as the *Bahasinan* playing arena, how to play the *Bahasinan*, the position of the players while playing, presented in Table 1.

Aspects (indicators) interviewed about the *Bahasinan* game	Interview Questions
Area to Play *Bahasinan*	. "What is the shape of the traditional *Bahasinan* playing arena?" . "Is there an official measure that has been set?"
How to Play *Bahasinan*	. "How is the procedure for the *Bahasinan* game?" . "Is there a specific officially defined way of playing?"
Player Position When Playing *Bahasinan*	. "What is the position of the players when playing the *Bahasinan* game?" . "Is there a special position when playing that has been set?"
Hand movements when players play *Bahasinan*	. "How are the players' movements when playing the *Bahasinan* game?" . "Are there any preset moves during play?"

Table 1: List of Questions

Last but not the least, documentation is also implemented in the study in order to gather the supporting data. Documentation related to the shape of the game arena, how to play and the position of the players when playing the game and hand movements when playing the game is discussed, of course, in-depth documents are needed to reveal it by collecting data. Documents in the form of photos are very important because from here the researchers relate the traditional *Bahasinan* games which will be analyzed for the concepts and mathematical elements contained in them.

After the data have been gathered, the data are analyzed by using qualitative data analysis by using three stages namely data reduction, data presentation, and conclusion-drawing or verification. A qualitative study is said to be scientific if there are levels of data trustworthiness. Consequently, the researchers use the data validity technique in order to take responsibility over the study. Furthermore, in order to test the data validity, the researcher has implemented the method triangulation. Sugiyono (2017) states that method triangulation refers to the data testing and comparison from the both the data source and the data observation. Method triangulation can elicit the data so that the data can be held trustworthy and the method triangulation is conducted by checking the data that have been retrieved from the study through several sources and observations. The data sources in the study consist of participatory observation, in-depth interview, and documentation.

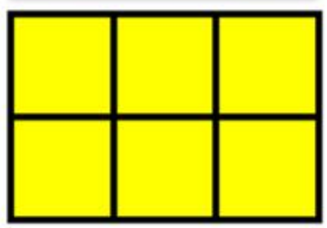
RESULTS

Based on the results of the interview (table 2) and the observation (table 3) with one of the local figures and two respondents of 6-12 years old, the game *Bahasinan* is practically associated with Mathematics. The elements of Mathematics that have been contained in the traditional game is reflected through the game area, the game procedures, the player position in each movement, and the hand movement in playing *Bahasinan*. The results of both the interview and the observation show that the traditional game *Bahasinan* can improve the affective and psychomotor capacity.

Questions	Answers
How is the shape of the area for the traditional game *Bahasinan?*	The shape of *Bahasinan* game area is usually rectangle drawn in equally proportional partition. For the partition alone, there are six rectangles or eight rectangles since this number has been the most frequently used for playing *Bahasinan*. Then, for the length of the game area, children usually use 15 x 9 meter in adjustment with the playing area or with the player agreement. *Bahasinan* can be played anywhere and children usually use sticks, stones, or chalks for drawing the lines.
Can you explain the procedures or are there are procedures that have been officially made?	First of all, players should define their team first namely the offensive team and the defensive team. Each team consists of four players and they will decide who will be in the offensive team and in the defensive team by using *hompimpa*. The defensive team should guard the *Bahasinan* game area by stepping onto the lines under his or her spot or by drifting his or her leg. Then, the offensive team should break through the defensive team completely from the front to the back and get back to the front again. If the offensive team successfully does this, then they will be awarded with 1 point. However, if the offensive team is touched by the defensive team, then the game will be suspended or stopped. As the offensive team begins the game, they should say "*Masuk Kadadas*" and when they can return to the front line they should say "*Masin*."
How is the position of the players when they play *Bahasinan*? Is there any specific position	Usually, the position of the players in defensive team is limited. They can only shift

Questions	Answers
that has been set?	to the right, to the left, or to the front. On the contrary, the position of the players in offensive team is better since they can move to the front, to the back, to the left, or to the right within the same rectangle. In *Bahasinan* game, the position of the players is certainly changing.
How is the hand movement of the players when they are playing *Bahasinan*? Is there any specific hand movement that has been defined?	The players from the defensive team usually stretches their arm and they slightly raise their left or right arm with the hands stretching forward.

Table 2: Interview Results

No	Aspects (Indicators under observation for *Bahasinan*)	Observation Picture (Documentation	Description on Observation Results
1	*Bahasinan* Game Area	  	In *Bahasinan* game area, the elements of Mathematics that can be found are two-dimensional figure, division and multiplication, reflection, lines, and congruency. Then, the activities of measuring and numbering in making the game area are also related to Mathematics. Both activities should always be performed whenever children want to play *Bahasinan*.
2	*Bahasinan* Game Play		The number of players and their assignment into the team in *Bahasinan* game is related to the operation of division or subtraction and there are also the activities of numbering or calculating the number of the players. In addition,

		the rectangle in *Bahasinan* also contains the element of calculation as having been shown by the sequence of the rectangle and the number of defensive players that should be passed by the offensive players. Every offensive player who has successfully passed the whole line from the front to the back to the front again will be awarded with 1 point. Based on this statement, the element of Mathematics that can be found is the operation of summation namely when the overall scores from all players in the offensive team are summed into the team score.
3 Player Position during *Bahasinan*		Judging from the position of the player during *Bahasinan*, the position of the defensive players is limited. They can only move to the left, to the right, and to the front. On the contrary, the position of the offensive players is more flexible. They can move to the front, to the back, to the left, or to the right. Learning from the movement above, the elements of Mathematics that have been contained in *Bahasinan* are translation / movement and reflection.

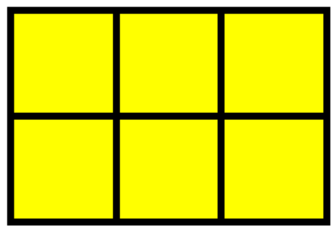
4	Hand Movement during the *Bahasinan*	    	Judging from the hand movement during the game, the movement made by the defensive players should be given attention. In order to defense their base, the defensive players usually stretch their arms by slightly raising their left or right arm and reaching their hand to the front. By doing so, the right angle, the acute angle, the obtuse angle, and the straight angle can be made.

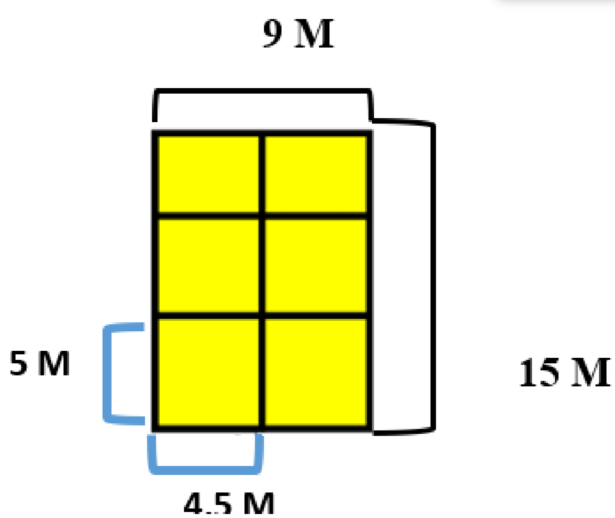
Table 3. Observation Results

Based on the results of interview, observation, and documentation, the researchers have found several findings that can be analyzed as follows.

a. *Bahasinan* Game Area

The game area of *Bahasinan* consists of several rectangles with equal size. The total size of *Bahasinan* game area is 15 m x 9 m that consists of 6 rectangles with similar size namely 5.00 m x 4.50 m.





In the game area of *Bahasinan*, the mathematical concepts that can be seen are as follows:

1) Two-Dimensional Figure

Two-dimensional figure is one of the geometrical concepts that have been recorded in the game area of *Bahasinan* (figure 1). The game area of *Bahasinan* is made of 15 m x 9 me rectangle. Every square in the rectangle has been divided into six smaller rectangles with 5.00 m x 4.50 m in size.

Figure 1: The Mathematical Concept Rectangle Over the Game Area of *Bahasinan*

2) Division and Multiplication

As having been explained, the game area of *Bahasinan* is made of 15 m x 9 m rectangle. Then, the rectangle is divided into six equal rectangles with the length divided by 3 and width divided by 2. As a result, the length of each square is 15 m ÷ 3 = 5 m and the width of each square is 9 m ÷ 2 = 4.5 m. On the contrary, the calculation can involve multiplication in which multiplication is the inverse mathematical operation of division.

3) Reflection

The square in *Bahasinan* also contains the element of reflection. This can be shown by the symmetrical shape of the game area. As it can be seen, there are symmetrical axes that cut the game area of *Bahasinan* into two equal parts (figure 2).

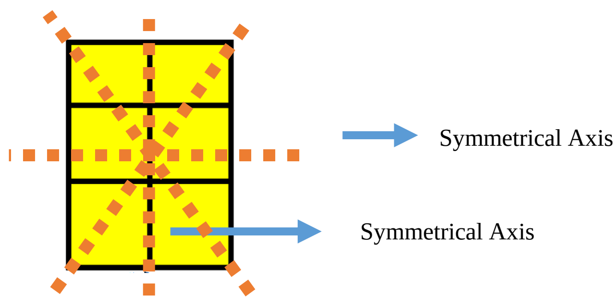


Figure 2. The Illustration of Reflection on the Square of *Bahasinan*

4) Lines

Lines refer to the indefinite clusters of nodes. The game area of *Bahasinan* has also show the lines connection, which consists of parallel, intersecting, and perpendicular lines (figure 3).

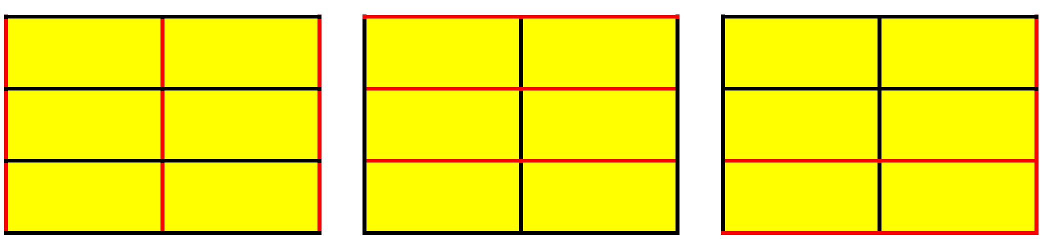


Figure 3: Illustration (1) and (2) refers to the parallel line while Illustration (3) refers to intersecting line

and perpendicular line in *Bahasinan*

5) Congruency

The squares in *Bahasinan* holds the elements of congruency. The statement is confirmed by the six squares of *Bahasinan* game area that has the same size and shape (figure 4).

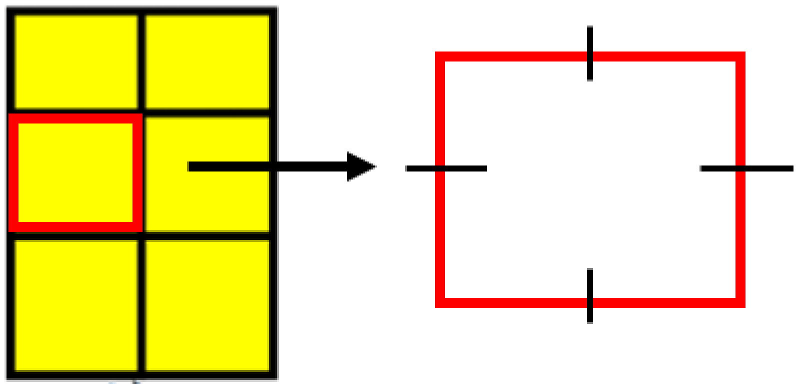


Figure 4: The Illustration of Congruency over the Square of *Bahasinan*

6) The activities of measuring and numbering are also contained in the game because the game area of *Bahasinan* should be measured and numbered first.

b. How to Play *Bahasinan*

In order to play *Bahasinan*, at least there should be six players and the six players should be divided into two groups namely the offensive group and the defensive group. The group is divided by performing *hompimpa*. The number of players in *Bahasinan* and the task division of these players are related to the activities of dividing, subtracting, numbering, and calculating the players. In addition, the square in *Bahasinan* also contains the elements of calculation, which has been shown by the sequence of the squate and the number of the defensive players that the offensive players should pass. In this regard, a player can count each square and each opponent that he or she has or has not passed. On the contrary, pertaining to the score of *Bahasinan*, the mathematical element that has been uncovered is calculation. Each offensive player who has passed the whole lines from the front to the back to the front again will be awarded with the score 1. Based on the statement, the mathematical element that has been uncovered is summation, namely when the score of each player is accumulated into the score of the team. Thus, the team with the highest score shall be the winer of the game.

c. Player Position during *Bahasinan*

Based on the position of the players during the game *Bahasinan*, the position of the defensive players is limited since they can only move to the right, to the left, or to the front. In the meantime, the position of the offensive players is more flexible since they can move to the front, to the back, to the left, or to the right. Such movement contains the concept of transformation geometry namely translation and reflection. The transformation geometry is illustrated further through Figure 5 and Figure 6.

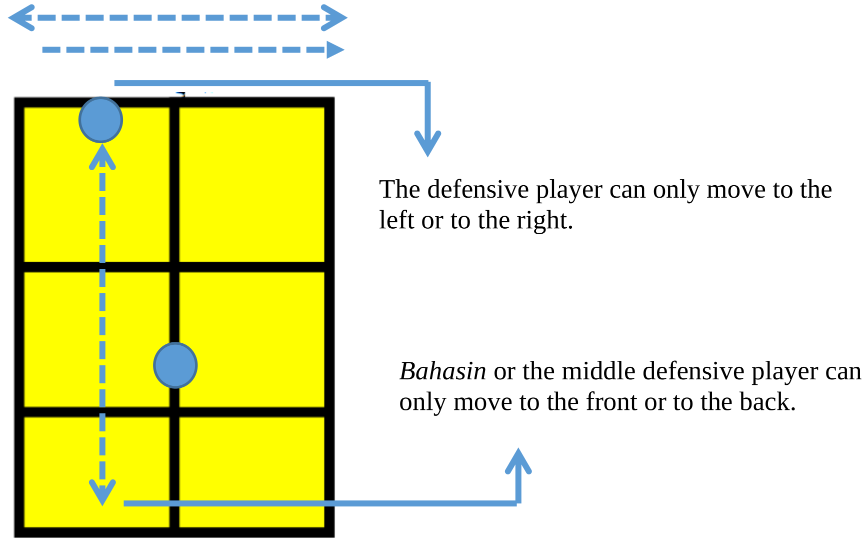


Figure 5: The Illustration of Translation on *Bahasinan* Players

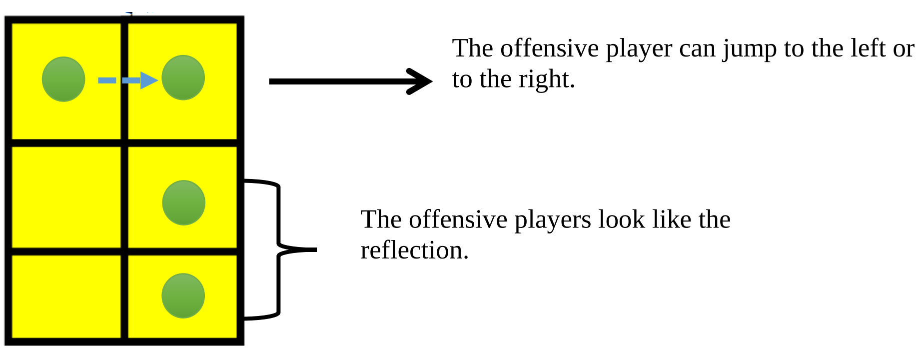


Figure 6: The Illustration of Reflection on *Bahasinan* Players

d. Hand Movement for the *Bahasinan* Players

From the results of the observation, in order to guard their opponents, the defensive players usually stretch their arms with the left arm or the right arm slightly raised as they stretch both arms to the front. Such hand movements will shape the right angle, the acute angle, the obtuse angle, and the straight angle. These angles are better illustrated in Figure 7 below.

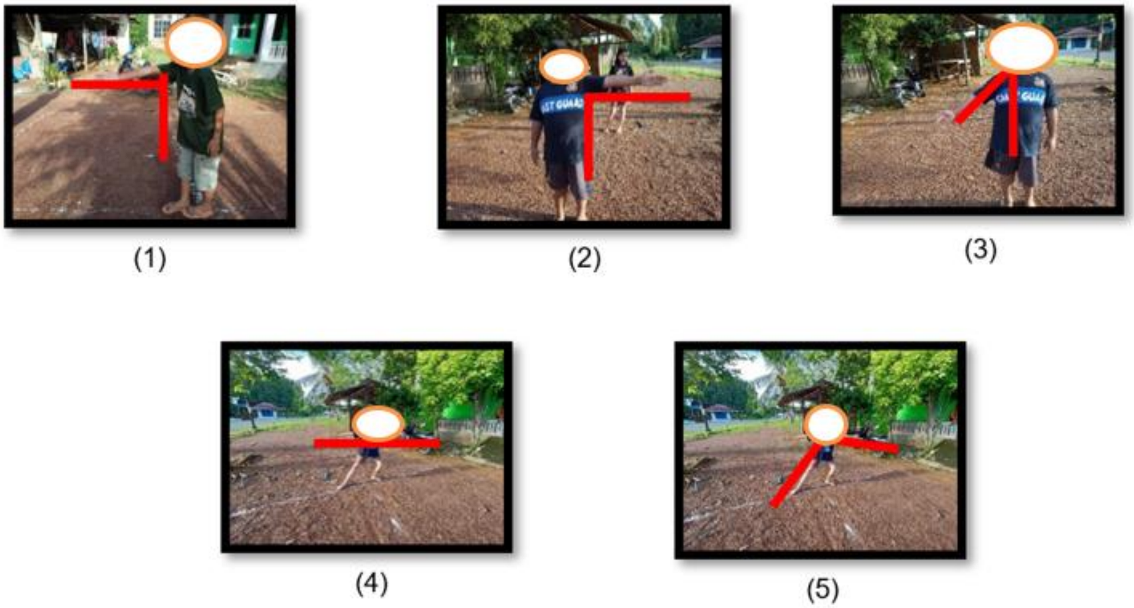


Figure 7: The Illustration of Right Angle ((1) and (2)), Acute Angle (3), Straight Angle (4), and Obtuse Angle (5)

Departing from the results of interview, observation, and documentation that have been elaborated, and also based on the results of the data analysis, it is found that *Bahasinan* holds numerous benefits. In addition to preserving the Indonesian cultural heritage, playing *Bahasinan* is definitely useful for the players. According to Achroni, the benefits of playing *Bahasinan* are namely delivering enjoyment, eliciting strategy-devising capacities, building responsibility and sportsmanship, exercising the morale, and training leadership (Een et.al, 2020). In addition, the psychomotor skills that can be improved through playing the game are strength, concentration, speed, flexibility, and endurance. Eventually, the ethnomathematical findings that have been uncovered in the traditional game *Bahasinan* can be further elaborated in Table 4.

Game Area of *Bahasinan*	In the game area of *Bahasinan*, the mathematical elements that can be seen in terms of shape is two-dimensional figures. The square of *Bahasinan* contains the mathematical operations such as division and multiplication, reflection, lines, and congruency. In addition, the game area also encourages the activities of measuring and numbering since the game area should be made by means of measurement. Furthermore, prior to the game, the players should define the location of the game area first and this activity also involves measurement.
Game Procedures of *Bahasinan*	The number of players and the division of their task in *Bahasinan* are associated with the mathematical operations such as division and subtraction; in addition,

	there are also the activities of numbering or calculating the players and the game. Furthermore, each square in *Bahasinan* contains the elements of calculation as having been shown by the size and the sequence of the square as well as the number of the defensive players that each offensive player should pass. Each offensive player who has made their way to from the front to the back to the front again will be awarded with the score 1. Based on the statement, the mathematical element that has been found in the mathematical operation name summation. Summation takes place when the score of each player is accumulated into the score of the team.
Player Position During *Bahasinan*	The player position during the traditional game *Bahasinan* is different between the defensive players and the offensive players. The defensive players can only move to the left, to the right, or to the front. In the meantime, the offensive players can more to the front, to the back, to the left or to the right. Learning from such movement, it can be concluded that the traditional game *Bahasinan* contains the mathematical concept named transformation geometry namely translation and reflection.
Hand Movement of the Players During the Traditional Game *Bahasinan*	The hand movement during the traditional game *Bahasinan* is also different between the defensive players and the offensive players. The defensive players usually stretch their arms with their left or right arm slightly raised and their hands moving forward. Such movement will shape the right angle, the acute angle, the obtuse angle, and the straight angle.
Character Education	The traditional *game Bahasinan* can be one of the tools for internalizing communality, social relationship, and teamwork.

Table 4. The Ethnomathematical Elements in the *Bahasinan* Game

DISCUSSION

We find a lot of mathematical concepts in the *Bahasinan* game. Especially in Table 2 has been shown numerous mathematical concepts in the elements of the traditional game *Bahasinan*. For example, the findings on the concept of two-dimensional figure are in line with the theory proposed by Unaenah et al., which asserts that two-dimensional figure refers to the two-dimensional objects over the planes that have been bordered by the straight line or the curve line with the concept of angle and the concept of liens in use (Een et.al, 2020). Then, the ethnomathematical findings over the concept of geometry transformation are in line with theory proposed by Ulum, Budiarto & Ekawati, which asserts that geometry is a branch of Mathematics that studies the relationship among lines, nodes, angles, planes, two-dimensional figures and three-dimensional figures (Ulum & Ekawati, 2017). Last but not the least, the activities of numbering, defining location, and playing are also in line with the theory proposed by Sirate which asserts that ethnomathematical elements can be found in the context of daily life (Sirate, 2012). The cultural products in a community that have been forged through the traditional game such as *Bahasinan* contains mathematical concepts known as cultural-based mathematics or also known as Ethnomathematics. After we find many mathematical concepts related to the language, the next task is how to integrate them in learning to make it more meaningful. The application of ethnomathematics can be a means of motivating, stimulating students, and overcoming boredom and learning difficulties, so that learning becomes more meaningful (Gazali, 2016).

CONCLUSIONS

Bahasinan is one of the traditional games that can be played in groups. This traditional game is integrated into the elements of Mathematics such as the concept of geometry and numbers. The integration of Mathematics into culture itself is known as Ethnomathematics. The elements of ethnomathematics for the concept of geometry and numbering in the traditional game *Bahasinan* can be reviewed in terms of game, player numbers, game procedures, and scoring system. The concepts of geometry that has been found in the traditional game are namely two-dimensional figure, translation, reflective, lines, and congruency. On the contrary, the concepts of numbers that have been found in the traditional game are namely summation, subtraction, multiplication, and division. with the presence of the current study, the ethnomathematical approach can be developed within the mathematical learning in the schools. These findings can be used as an introduction to learning materials by teachers, so that students appreciate the usefulness of mathematics in everyday life. Departing from the statement, the future researchers are suggested to explore the association between Mathematics and other traditional games, especially in the context of Indonesian culture, on a wider basis.

REFERENCES

[1] Abi, A.M. (2017). Integrasi Etnomatematika dalam Kurikulum Matematika Sekolah. *JPMI (Jurnal Pendidikan Matematika Indonesia)*, 1(1), 1-6. https://dx.doi.org/10.26737/jpmi.v1i1.75

[2] Apriyono, Fikri, Mohammad Kholil. (2018). Identifikasi Konsep Matematika Dalam Permainan Tradisional di Kampung Belajar Tanoker Ledokombo Jember. *Indonesian Journal of Islamic Teaching*, 1(1), 62-75. https://jurnalpasca.iain-jember.ac.id/ejournal/index.php/IJIT/article/view/3

[3] Yeh, C.Y.C., Cheng, H.N.H., Chen, ZH., Liao, C. C. Y, & Chan, T.W. (2019). Enhancing

achievement and interest in mathematics learning through Math-Island. Research and Practice in Technologu Enhanced Learning, 14(5), 1-19. https://doi.org/10.1186/s41039-019-0100-9

[4] Een Unaenah et. al. (2020). Teori Brunner Pada Konsep Bangun Datar Sekolah Dasar. *Jurnal Nusantara*, 2(2), 327-349. https://doi.org/10.36088/nusantara.v2i2.840

[5] Ekayati, I. A. S. (2015). Pengaruh Permainan Tradisional '*Bahasinan*' terhadap Kecerdasan Intrapersonal dan Interpersonal pada Anak Usia Dini. http://digilib.unmuhjember.ac.id/gdl.php?mod=browse&op=read&id=umj-1x-ifaaristia-3394

[6] Fadila, Rifina Wildati. (2021). Eksplorasi Etnomatematika pada Permainan Tradisional Lompat Tali. *Jurnal Penelitian Guru Sekolah Dasar*, 9(4),1-21. https://ejournal.unesa.ac.id/index.php/jurnal-penelitian-pgsd/article/view/39922

[7] Ku, O., Chen, S. Y., Wu, D. H., Lao, A. C., & Chan, T. W. (2014). The effects of game-based learning on mathematical confidence and performance: high ability vs. low ability. *Journal of Educational Technology & Society*, 17(3), 65–78. https://www.semanticscholar.org/paper/The-Effects-of-Game-Based-Learning-on-Mathematical-Ku-Chen/c09d8892c00aacfc1ac135154a6d4712154262c5

[8] Kurniawan, M. R. (2018). Permainan Tradisional Yogyakarta Sebagai Sumber Belajar Alternatif Berbasis Kearifan Lokal Bagi Pembelajaran Di Sekolah Dasar. *Premiere Educandum: Jurnal Pendidikan Dasar dan Pembelajaran*,8(2), 98 – 111. http://doi.org/10.25273/pe.v8i2.2697

[9] Kuswidi, I., Devinta Fajar Lestari, Nurul Arfina, & Raekha Azka. (2018). Eksplorasi etnomatematika pada permainan tradisional layangan (pemahaman materi bangun datar layang-layang dan pengembangan karakter). *Jurnal Pengembangan Pembelajaran Matematika (JPPM)* , 3(2), 129-137. https://doi.org/10.14421/jppm.2021.32.129-137

[10] Maran, J. P. (2019). Etnomatematika dalam Ritus Agama Khatolik Larantuka. Prosiding Sendika Universitas Muhammadiyah Purwakarta. http://eproceedings.umpwr.ac.id/index.php/sendika/article/download/768/653

[11] McLaren, B. M., Adams, D. M., Mayer, R. E., & Forlizzi, J. (2017). A computer-based game that promotes mathematics learning more than a conventional approach. *International Journal of Game-Based Learning*, 7(1), 36–56. https://doi:10.4018/IJGBL.2017010103

[12] Nursyahidah, Farida., & Albab, I.U. (2021). Learning Design on Surface Area and Volume of Cylinder Using Indonesian Ethno-mathematics of Traditional Cookie maker Assisted by GeoGebra. *Mathematics Teaching Research Journal*, 13(4), 79-98. https://files.eric.ed.gov/fulltext/EJ1332350.pdf

[13] Prahmana, R.C.I., Yunianto, W., Rosa, M., & Orey, D.C. (2021). Ethnomathematics: Pranatamangsa System and the Birth-Death Ceremonial in Yogyakarta. Journal on Mathematics Education, 12(1), 93-112. http://doi.org/10.22342/jme.12.1.11745.93-112.

[14] Rachmawati, I. (2012). Penerapan Konsep Etnomatematika masyarakat Sidoarjo. *Penelitian Mathdunesa*,1(1), 1-7. https://doi.org/10.30653/003.201512.23

[15] Gazali, Rahmita Yuliana. (2016). Pembelajaran Matematika yang Bermakna. Math Didactic: Jurnal Pendidikan Matematika, 2(3), 181-190. https://doi.org/10.33654/math.v2i3.47

[16] Riyanto, A. (2017). Pengembangan Garis Lapangan Portabel *Bahasinan* pada Pembelajaran Permainan Tradisional di SD Negeri Sendangadi 1 Mlati Kabupaten Sleman," Program Studi Pendidikan Guru Sekolah Dasar Penjas Jurusan Pendidikan Olahraga Fakultas Ilmu Keolahragaan Universitas Negeri Yogyakarta, Program Studi Pendidikan Guru Sekolah Dasar Penjas Jurusan Pendidikan Olahraga Fakultas Ilmu Keolahragaan Universitas Negeri Yogyakarta.

[17] Ulum, B., Budiarto, M. T., & Ekawati, R. (2017). Etnomatematika Pasuruan: Eksplorasi Geometri untuk Sekolah Dasar pada Motif Batik Pasedahan Suropati." Prosiding Seminar Nasional Intregasi Matematika dan Nilai Islami, 1(1),70-78. http://conferences.uin-malang.ac.id/index.php/SIMANIS/article/view/40

[18] Sirate, F. (2012). Implementasi Etnomatematika dalam Pembelajaran Matematika pada Jenjang Sekolah Dasar. *Lentera Pendidikan : Jurnal Ilmu Tarbiyah dan Keguruan*, 15(1), 41-54. https://doi.org/10.24252/lp.2012v15n1a4

[19] Spradley, J. P., & McCurdy, D. W. (1989). Anthropology: The cultural perspective. Reissued Long Grove, IL: Waveland Press.

[20] Sugiyono. (2017). Metode Penelitian Kualitatif (3rd ed.). Alfabeta.

[21] Sujarno, S. Galba, Th. A. Larasati, dan Isyanti. (2011). Pemanfaatan Permainan Tradisional dalam Pembentukan Karakter Anak. Yogyakarta: Balai Pelestarian Nilai Budaya (BPNP)

[22] "Permainan Tradisional: *Bahasinan* (Galasin/Cak Bur/Margala) – Rumah Anak Mandiri." [Daring]. Tersedia pada: http://www.anakmandiri.org/2016/11/29/permainan-tradisional-gobak-sodor-galasincak-burmargala/ [Diakses: 05-Jan-2020].

Appendix/attachment

About *Bahasinan*

One of the traditional games that can be found in South Borneo, specifically in Gunung Makmur Village, the Regency of Tanah Laut, is *Bahasinan*. *Bahasinan* itself is a traditional game that should be played in teams. There are two versions about the origin of the game. First, the name *Bahasinan* is derived from English phrase *Go Back to the Door*, which means that the offensive player should return to their based through the door. Second, the name *Bahasinan* is derived from two words namely *Baha* and *sin*. *Baha* means free whereas *sin* means spear. Being put together, the name *Bahasinan* means running free like a spear; this also explains why the middle player is known as *masin* (Sujarno et.al., 2011). This traditional game is known under many names in several Indonesian regions. The name *Bahasinan* is known in Java and South Borneo but in Jakarta this traditional game is known as *Galah Asin* or *Galasin*. In Natuna, this traditional game is known as *Galah* while in Riau it is known as *Galah Panjang*. Furthermore, in South Sulawesi this traditional game is known as *Masallo*. In Toba, this traditional game is known as *Margala*. Eventually, in Aceh and Bengkulu this traditional game is known as *Hadang* (www.anakmandiri.org).

Just like the other traditional games, *Bahasinan* consists of game arena, regulations, and players. In this traditional game, many aspects related to mathematical concepts and materials can be found. For example, the arena of *Bahasinan* displays the mathematical concepts and materials. The arena itself looks like rectangle and therefore it can be concluded that *Bahasinan* has the concept of plane and geometry. Based on this explanation, the researchers would like to discuss the game area (fiqure 8), the game regulations, the player position, and the hand gesture in *Bahasinan* from the perspective of mathematical concepts.

Figure 8: *Bahasinan* Game

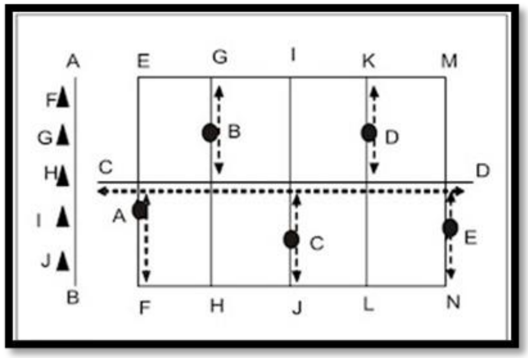


Figure 9: *Bahasinan* Game Scheme

Note from figure 9:

- Line ab : Baseline for the team of players (Playing Team / *Team Laku*)
- Line cd : *Hasinan* Line
- Line ef, gh, ij, kl, and mn: Garis ef, gh, ij, kl, mn : Tranverse
- ←---→ : Guarding line for the defensive team
- ▲ : Players of the offensive team
- ● : Players of the defensive team

Bahasinan is a traditional game of Banjarmasin people especially in the Gunung Makmur Village, the Regency of Tanah Laut. Prior to the era of gadget, the children were very fond of playing the traditional game. The traditional game puts the strategy, the agility, and the physical strength of the playing children on a test. *Bahasinan* is played over a field with 6 rectangles (figure 10) as the playing ground.

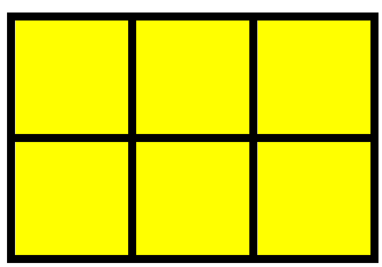


Figure 10: Playing Game of *Bahasinan*

Every line is guarded by one player from the opponent team (Defensive Team). The number of the members for each of the defensive and the offensive team is five people or more.

The defensive team guards the horizontal line and the vertical line. One player may only guard the line under his or her authority. Then, there is a long horizontal line in the middle of the playing field. This line is usually guarded by the best player of the defensive team. The offensive team may move freely inside the rectangle as long as he or she is not touched by the player of the defensive team. The offensive team starts playing from the start line and they have to break through the defense line of the defensive team and return to the start line (figure 11). Every successful movement is awarded 1 point.

Figure 11: Situations During the *Bahasinan* Game

Then, the offensive team may not return to the previous rectangle as the move forward. If the member of the offensive team violates this requirement, then the given team member will be suspended. The member of the offensive team who have been touched by the member of the defensive team for three times will lose the game and, thus, the offensive team should be the defensive team and vice versa. Furthermore, two members of the offensive team may not be in the same rectangle. This situation is termed "*Katuyung Gambir Dua*" and the consequence is that both members are suspended. On the national scale, the traditional game *Bahasinan* is known as "Hadang" and the duration for the traditional game is 2 x 15 minutes.

Despite its wonderful characteristics, the ex of the traditional game has been forgotten. The existence of the traditional game has been slowly replaced by the digital games, which development has been proliferating as the society becomes more individualistic, whereas in the practice the traditional game delivers numerous benefits. In her research, Ekayati has found that the *Bahasinan* game itself has influence over the intrapersonal and interpersonal intelligence of the children (Ekayati, 2015). In addition, playing *Bahasinan* can be one of the ways for preserving the Indonesian traditional games. This culture can be preserved by certain manners, one of which is integrating the traditional game into the learning practice. In learning, the culture of playing *Bahasinan* is closely associated with Mathematics. Mathematics is a science which object is abstract but its practice is practically integrated into the daily life and this can be found in the culture within society (Riyanto, 2017).

Views on Mathematical Giftedness and Characteristics of Mathematically Gifted Students: The Case of Prospective Primary Mathematics Teachers

Gönül Yazgan-Sağ

Gazi University, Ankara, Turkey

gonulyazgan@gazi.edu.tr

Abstract: Mathematically gifted students are mostly getting their education in mixed classrooms. Teachers should be able to recognize mathematically gifted students to be responsive to those students' needs along with other students in their classrooms. The current study aims to reveal prospective primary mathematics teachers' views about mathematical giftedness and the characteristics of mathematically gifted students. The participants of this qualitative study are 11 prospective mathematics teachers who take their education in a four-year primary mathematics teacher education program at a state university in Turkey. Two focus group interviews are conducted with the participants. The row data were analyzed by using descriptive analysis. The findings reveal that prospective teachers associated mathematical giftedness with various concepts such as social environment and effort. The participants' views on the characteristics of mathematically gifted students have varied in line with the literature. It also concluded that prospective primary mathematics teachers emphasized creative acts like re-formulating problems and finding unique solutions to the problems as indicators of being mathematically gifted.

INTRODUCTION

In the field of giftedness since the 1900s, there have been discussions on how this concept can be defined and measured (Dai, 2010; Subotnik, Olszewski-Kubilius & Worrell, 2011; Ziegler & Heller, 2000). Similarly, there is also no consensus on a clear and universally accepted definition of mathematical giftedness (Karp, 2009; Mann, 2006). Mathematical giftedness is traditionally related to scoring above the 95th percentile on various standardized tests (Sheffield, 2003). However, mathematically gifted students demonstrate their talents in different forms (Gavin, Firmender & Casa, 2013). Recent literature also shows other factors as such motivation and persistence can be efficient to describe mathematically gifted students. According to recent studies, mathematical giftedness consists of two components that are mathematical ability and mathematical creativity (Kontoyianni, Kattou, Pitta-Pantazi & Christou, 2013). Mathematical giftedness refers to not just a high ability to do mathematical computations and get high marks in examinations, it also refers to a remarkably high ability to reason and understand in a mathematical context (Miller, 1990). Sheffield (1994) also indicated that there are many abilities and characteristics describing mathematical giftedness. Mathematical ability commonly is manifested in accomplishing a task in the classroom context. However, it can be related to the capacity to learn and master other mathematical ideas and skills namely potential skills. Thus, this ability is not fully observable, only deduced from the performances of the student (Koshy, Ernest & Casey, 2009).

There are various studies examining the characteristics that may indicate giftedness in mathematics (Leikin, 2009; Karp, 2009; Singer, Sheffield & Leikin, 2017; Sriraman, 2004; Sriraman, Haavold & Lee, 2013, Yazgan-Sağ, 2020). Krutetskii (1976) carried out a pioneering work in this field by examining the structure of mathematical abilities for 12 years. He defined giftedness in mathematics as a unique combination of mathematical abilities that emerge during the successful execution of a task. In addition, Krutetskii (1976) used the term "mathematical cast of mind" to describe the tendency of mathematically gifted students to view the world around them through a mathematical lens. "They have a tendency to discover number bonds and mathematical relationships everywhere, to follow their own personal pathways to find a solution, and to produce novel ideas of some value" (Dimitriadis, 2016, p.223). In this sense, gifted students in mathematics have flexibility in mental processes and the ability to generalize mathematical relations and operations quickly and thoroughly (Singer, Sheffield, Freiman & Brandl, 2016). There is also another approach that refers to mathematical promise while considering mathematical giftedness. A mathematical promise is described as a function of ability, motivation, belief, and experience or opportunity (Sheffield et al., 1999). Mathematically promising students are mentioned as potential leaders and problem solvers in the future" (Sheffield, 1999). Discussing whether mathematically promising students are mathematically gifted, "mathematical giftedness is regarded as an emerging promise or high ability with mathematics relative to one's peers" (Reed, 2004, p. 91). Although there are different perspectives on describing mathematically gifted students, the literature mostly agrees that they can do mathematics that older students can do, or engage in mathematical thinking in a different way than their peers. It can be said that the mathematical promise possibly includes mathematical giftedness. Freiman (2003) described the characteristics of gifted students in mathematics according to the studies he examined. Mathematically gifted students

- love math (spending time on mathematics, seeing beauty in mathematics, enjoying doing math);
- want to learn more about mathematics (being highly motivated, being curious, being persistent, having exploratory orientation; being an entrepreneur; having a broad interest);
- think about the situations they encounter mathematically (gathering and organizing information; formulating situations; analyzing facts, patterns, and relationships; generalizing; reasoning abstractly; counting and calculating; interpreting data; proving and explaining logically);
- exhibit behaviors that increase their chances of succeeding in a mathematical task (hard

work; having long attention; having a good memory; being flexible; quickness in thinking; reflecting; critical thinking; focusing; finishing a job; being able to convey the results in verbal / writing; attention to detail; seeing the whole structure; intuitive thinking; being in a race).

However, it should be noted that these indicators should not be used as a set of criteria showing that students are gifted in mathematics. Gifted students in mathematics may not display all of the characteristics. In addition, the indicators of those characteristics may appear at different times depending on the development of students (Singer et al., 2016). Considering the literature, gifted students in mathematics can be described as individuals who have the ability to distinguish, abstract, generalize, and reason mathematical structures; the ability to think flexibly and reverse mathematical operations. They also have the ability to think analytically and intuitively, and to pose alternative problems related to the problems they solve. Lastly, they can perceive and work with highly complex structures (Krutetskii, 1976; Miller, 1990; Sriraman, 2003; Van Harpen & Sriraman, 2013; Yazgan-Sağ, 2019). It has been observed that gifted students especially prefer challenging problems in problem-solving environments (Shore & Kanevsky, 1993). Again, these students are the students who have an intuitive awareness of discovering mathematical proofs and principles (Sriraman, 2004).

The needs of mathematically gifted students differ from other students in a learning context. Mathematics teachers mostly tend to favor high achievers and ignore other indicators that point out giftedness in mathematics. Thus, revealing the characteristics of mathematically gifted students is essential for both the identification and development of the students (Sheffield, 1999). Mathematics teachers should also be able to notice mathematical giftedness indicators to be responsive to the needs of those students and to teach gifted students at an appropriate level (Reed, 2004; Sheffield, 2003). Although the literature review reveals several studies related to the characteristics of mathematically gifted students (Karp, 2009; Singer et al., 2017; Krutetskii, 1976; Sriraman, 2004; Sriraman, Haavold & Lee, 2013), there are limited studies on mathematics teachers' views of those students (Leikin & Stanger, 2011). Mathematically gifted students commonly take their education in heterogeneous classrooms (Reed, 2004). In this sense, thinking and reflecting on mathematical giftedness and mathematically gifted students can trigger prospective primary mathematics teachers' to notice those students in their future classrooms. This study aims to reveal the readiness of prospective primary mathematics teachers who will also teach mathematically gifted students in their classrooms. Therefore, the purpose of this study is to investigate prospective primary mathematics teachers' views related to mathematical giftedness and the characteristics of mathematically gifted students.

METHOD

The current study aims to explore prospective primary mathematics teachers' views on mathematical giftedness and the characteristics of mathematically gifted students. The participants of the qualitative study were prospective primary mathematics teachers studying a four-year teacher education program at the Mathematics and Science Education department of a state university in Turkey. They were attending the "Teaching Methods" course in the 6th semester of the program, and the instructor of that course was the researcher of this study. The researcher asked questions related to giftedness and mathematical giftedness in one of the lessons of this teaching methods course. These questions provided an environment for a discussion about the issues in mathematical giftedness literature. These open-ended questions were "What do you understand from the term 'giftedness'?"; "What do you understand from the term 'mathematical giftedness'?"; "What do you know about the mathematically gifted students? Have you ever had any experience with such students?"; "What could be the characteristics of the mathematically gifted students for you?"; "What could be the characteristics of the teachers of the mathematically gifted students for you?". After this lesson, the instructor asked the prospective teachers whether they want to participate in a study that aims to reveal their views about these questions. Eleven of 23 prospective mathematics teachers voluntarily agreed to participate in this study. While presenting the data, prospective primary mathematics teachers were named Bilge, Ceren, Mira, Kader, Ferhat, Havva, Sibel, Ezgi, Melis, Zeynep, and Filiz (pseudonyms). Two focus group interviews were conducted; one with 5 participants and the other with 6 participants. Those interviews were video-typed and lasted approximately 100 minutes. These focus group interviews allowed all prospective mathematics teachers to reflect on other prospective teachers' thoughts (Patton, 2002). The researchers employed related prompts during the interviews, in this way prospective mathematics teachers stayed focused on these open-ended questions. Descriptive analysis was used while coding the row data. Within the scope of this study, the data regarding mathematical giftedness and characteristics of mathematically gifted students will be presented in the findings.

FINDINGS

The findings will be presented in two sections. The first section aims to provide information about the prospective mathematics teachers' views on mathematical giftedness. Besides, the participants' views of the characteristics of mathematically gifted students are included in the second section of the findings.

Prospective Primary Mathematics Teachers' Views on Mathematical Giftedness

This section of findings describes briefly the prospective primary mathematics teachers' views of mathematical giftedness. The participants associated mathematical giftedness with several concepts. One of the issues that emerged in the focus group interviews was innate ability. Most of them hve argued that being mathematically gifted is innate. Here are a few opinions of the participants:

> Bilge: I think it starts from birth.
>
> Ceren: It comes from birth. […] I mean it is not something that will occur with training afterward. For example, a person has an interest in wrestling; he comes to the highest level in wrestling and becomes gifted in that field. But I don't think it's the same for other disciplines. Am I wrong?

As seen above, Bilge and Ceren both thought that people are born with mathematical talents. However, Ceren added that this may not be true in other disciplines which require muscle strength. The participants also discussed whether mathematical giftedness could be improved. Mira came up with the idea that talent cannot be improved: "I think talent is something innate, I don't think it's something that is developed later. Whatever it is, it doesn't change. It stands still there" As seen from the excerpt, Mira thought that talent is something fixed. On the other side, Filiz objected to this idea by saying "even average people come to a certain level through hard-working, so I believe talent can be improved by hard-working". Similar to Filiz, most participants stressed that effort is an important factor in improving talent.

The dichotomy among the prospective teacher also continued with the loss of mathematical talent. Most of the participants agreed with this statement: if people don't work on their mathematical abilities, then they can lose their talent. However, they highlighted the argument that social environment also has influence on the improvement of mathematical giftedness. Zeynep's explanation is as follows:

> Zeynep: I think the social environment in which mathematically gifted students live is very important. I mean, in the class if the teacher says "don't ask questions like that" or their peers say "what a stupid question", this may affect these students. Let's look at home, what if their families don't care enough or can't provide suitable conditions?

As seen from the above excerpt, the prospective teachers referred to both classroom and house environments in mathematical talent development. Some participants emphasized that describing mathematical giftedness can change according to the historical perspective. For example, Bilge thought that the definition of mathematical giftedness might differ depending on the time and said that "[…] In the past, people may have preferred memorization over writing." In the same manner, Sibel highlighted the historical context by saying "being the first person to memorize certain numbers may have been associated with mathematical giftedness in the past." A number of participants also related mathematical giftedness with IQ scores. Bilge and Ceren only shared their narrow knowledge of the content of the exam in which the IQ scores were determined and the meaning of the scores obtained in the test.

Prospective Primary Mathematics Teachers' Views of Characteristics on Mathematically Gifted Students

This section provides insights into the participants' views of the characteristics of mathematically gifted students. Most of them stated that these students have quick-thinking abilities. For instance, Ferhat expressed this ability by saying: "[…] because they reason and think quickly they can do mental computations and answer quickly". However, Ezgi didn't agree with her friends at this point and explained her thoughts:

> Ezgi: "Being quick is not a sign for me. Let's say these students answer the questions quickly, then I can be skeptical that they may be gifted. But if they can't answer quickly, I don't think that they are definitely not like that".

Ezgi also added that mathematically gifted students enjoy playing with numbers and like very complex operations. Then other participants agreed with the following statement: quick thinking is not the only condition for a being mathematically gifted student. In the same manner, Kader conveyed the statement:

> Kader: I don't think that quick thinking has much to do with giftedness. It may be related, but it may not be. They can catch the subject later than others, but they can develop a method more firmly. […] At first, they may not be able to understand the teacher's method, but later they may develop a method themselves.

As can be understood from Kader's statement, developing a new method and developing that method on their own is more notable than thinking quickly about being mathematically gifted. Then Ceren gave her reasons as follows: "If they don't accept everything and ask questions like 'where does that integral sign come from?', 'where did this formula come from?' then they may be gifted in mathematics." Melis also highlighted abstract thinking with the concepts that may be the characteristics of mathematically gifted students.

The prospective teachers have argued about getting high marks on the math exams. While some of the participants favored that being a hard-working student can be a sign of mathematical giftedness, others prioritized being consistently successful in activities. Here is a vignette that portrayed the thoughts of prospective mathematics teachers concerning students' acts and marks in the classroom context:

> Filiz: In general, they are the most hardworking in the class, I think of them as hardworking. Those are always first in class and school. They always feel like such a leader.

> Ceren: For example, if the students' success is continuous, if they show differences in the class many times, then they may be gifted.

> Sibel: I don't think that a student who consistently achieves the same level of success or even a very high level of success in the lessons and receives 100 continuously does not indicate that the student is gifted in mathematics. It just shows that they are hardworking students; I

don't think they are gifted.

Ferhat: For example, if they are constantly getting 100 without working, should we say this again?

Melis: They can perform above the expected class average. They just listen to the teacher and take high marks, what happens then?

Sibel: Maybe then we should have a look. I think it is unnecessary to even suspect otherwise.

As the dialog shows that participants have different views on the relationship between hard-working students and mathematically gifted students. Sibel insisted on the idea that hard-working and getting high marks in the exams may not be the characteristics of being gifted in mathematics. However, when Ferhat and Melis suggested new conditions such as getting high marks without working and performing above average, then Sibel agreed on such situations can be taken as a signal for the mathematical talent in students.

The prospective teachers denoted that being curious is one of the characteristics of mathematically gifted students. Bilge explained being curious as follows: "I think being more open to learning, and very curious about mathematical knowledge can be some criteria for being gifted". Zeynep remarked that mathematically gifted students are able to modify the conditions of a given problem:

Zeynep: In a given problem, they may reflect on the problem, such as "what would I do if that wasn't here if the problem had asked that". If they can turn a given problem into a different problem by reasoning like "I wonder what I would have done if it hadn't been there or if the problem had asked me something like this", that is, if they are playing with the problem, they may have mathematical talent.

Zeynep focused on a problem-posing act: re-formulating the mathematical problem that is known as one of the creative activities in the literature (Prabhu & Czarnocha, 2013; Silver, 1994). Similarly, other participants related certain creative characteristics to mathematically gifted students. A number of prospective teachers stated that thinking differently from their peers might be one of the characteristics of the mathematically gifted students:

Zeynep: They can get bored very quickly in class. […] They can have a different way of thinking or a different perspective.

Researcher: Different from who or what?

Havva: Different from their age group. For example, when solving a logic problem, they find their way that is something unique, or they think something different. They say different things compared to their peers or people in that society, they produce different solutions and, you know, they break out of the routine.

Zeynep: They don't solve a problem in the way we expect in their age group. [...] And they have found their way, they think about the problem differently than we think, that is, differently from what we expect from that age.

Melis: In fact, to be able to grasp immediately, to be able to bring different perspectives.

Sibel: That is, finding the right solution that is unique to them, originality.

Here, the participants indicated that these students may think more about the solutions to the problems compared to their classmates. Zeynep also asserted that the lessons can be boring for mathematically gifted students. Sibel and Havva revealed finding a unique solution is one of the characteristics of these students. The prospective mathematics teachers noted that flexible movements between subjects can be seen as a mathematical talented behavior. Here is Ezgi's explanation: "It is very important to be able to switch from subject to subject. I think they are able to apply a mathematical subject or formula they learned in chemistry and physics". Flexibility is also relevant to creative acts in doing and learning mathematics (Sriraman, 2005). Only Havva stated her thoughts on creativity explicitly:

Havva: I think they are creative people, I mean, every gifted person is also a creative person.

Researcher: What do you mean by saying creativity?

Havva: Actually, creativity covers what the friends say here. That is, they think differently, and produce different solutions. Also, they do this all the time.

DISCUSSION AND CONCLUSION

This qualitative study examined fourteen prospective primary mathematics teachers' views related to mathematical giftedness and characteristics of mathematically gifted students through focus group interviews. Findings revealed that the prospective teachers have different views on the mentioned concepts. Since the participants did not take any specific course related to (mathematically) giftedness, similar to Leikin's (2011) statement, most of the prospective teachers' views were directly connected with their personal experiences. Prospective primary mathematics teachers thought that mathematical giftedness came from birth. This perspective is relevant to the belief people are born with fixed abilities which is one of the myths (Sheffield, 2017). Nonetheless, the prospective teachers also highlighted that effort will improve the mathematical talent of the students and conveyed the idea that "To develop mathematical talent a person has to work hard" (Leikin, 2020, p. 323). The effort is included while describing mathematical ability (Koshy et al., 2009). Participants' views revealed that one of the concepts that may influence mathematical giftedness was the social environment (Renzulli, 2000). The prospective primary mathematics teachers stressed the significant role of the environment in nurturing mathematical ability in gifted students. They emphasized the historical perspective on describing mathematical giftedness. The participants stated their narrow knowledge of IQ scores which are traditionally accepted as evidence of giftedness (Yazgan-Sağ & Argün, 2020).

The prospective teachers expressed that quick thinking is one of the prominent characteristics of mathematically gifted students. This finding is in line with Paz-Baruch, and her colleagues (2014) demonstrated that speed is inherent to mathematical giftedness. However, some participants stated that the absence of speed does not necessarily mean the absence of mathematical giftedness. Those participants acknowledged developing new methods while solving problems as a sign of being mathematically gifted. Also, participants asserted that mathematically gifted students have characteristics like questioning the reasons, thinking abstractly, and solving complex problems. These cognitive characteristics are also stated in the literature that may be the strong predictors of mathematical giftedness (Davis & Rimm, 2004; Singer et al., 2017; Krutetskii, 1976; Paz-Baruch, Leikin & Leikin, 2022). Another discussion among the prospective teachers was about getting high grades in the mathematics examinations. Some of those argued that getting high marks even without working hard and demonstrating high mathematical performance continuously can be an indicator of being a mathematically gifted student. This statement is also supported by previous research (Krutetskii, 1976). Leikin (2020) suggested, "A student is mathematically gifted if s/he exhibits a high level of mathematical performance within the reference group and is able to create mathematical ideas which are new with respect to his/her educational history," (p. 318). As Leikin stated mathematical giftedness is connected to mathematical creativity in the literature (Mann, 2006; Prabhu & Czarnocha, 2013/2014; Sriraman, 2005). Similarly, the prospective primary mathematics teachers related various creative acts such as posing problems and trying to find a unique solution with mathematically gifted students.

This study revealed that the participants' views of mathematical giftedness and mathematically gifted students are mostly binary. It may stem from the concepts concerning mathematical giftedness that differ from one society to another. For educators, it is critical to know the literature about mathematically gifted students. With proper training, teachers can become successful in meeting the needs of mathematically gifted students (Karsenty, 2014). Both theoretical and empirical studies may help prospective teachers to expand their knowledge, but this knowledge may not be enough to recognize mathematically gifted students. Therefore, "in order to develop the creativity of future teachers, it is crucial to familiarize them with a variety of pedagogical situations that can genuinely expand their conception of what is possible in the classroom." (Karp, 2010, p. 279). Prospective primary mathematics teachers should be in an environment where they can acquire both theoretical knowledge and experience with mathematically gifted students.

ACKNOWLEDGEMENT

The author would like to thank Gazi University Academic Writing Application and Research Center for proofreading the article.

REFERENCES

[1] Dai, D. Y. (2010). *The nature and nurture of giftedness: A new framework for understanding gifted education*. New York, NY: Teachers College Press.

[2] Davis, G. A., & Rimm, S. B. (2004). *Education of the gifted and talented*. Boston, MA: Pearson Education Press.

[3] Dimitriadis, C. (2016). Gifted programs cannot be successful without gifted research and theory: Evidence from practice with gifted students of mathematics. *Journal for the Education of the Gifted*, *39*(3), 221-236.

[4] Freiman, V. (2003). *Identification and fostering of mathematically gifted children at the elementary school.* Master's Thesis. Concordia University, Canada.

[5] Gavin, M. K., Firmender, J. M., & Casa, T. M. (2013). Recognizing and nurturing math talent in children. *Parenting for High Potential, 3*(2), 22-26.

[6] Karp, A. (2009). Teaching the mathematically gifted: An attempt at a historical analysis. In R. Leikin, A. Berman, and B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 11–29). Rotterdam, the Netherlands: Sense Publishers.

[7] Karp, A. (2010). Teachers of the mathematically gifted tell about themselves and their profession. *Roeper Review*, *32*(4), 272–280.

[8] Karsenty, R. (2014). Who can teach the mathematically gifted? Characterizing and preparing mathematics teachers of highly able students at the secondary level. *Gifted and Talented International*, *29*(1-2), 161–174.

[9] Kontoyianni, K., Kattou, M., Pitta-Pantazi, D., & Christou, C. (2013). Integrating mathematical abilities and creativity in the assessment of mathematical giftedness. *Psychological Test and Assessment Modeling*, *55*(3), 289-315.

[10] Koshy, V., Ernest, P., & Casey, R. (2009). Mathematically gifted and talented learners: Theory and practice. *International Journal of Mathematical Education in Science and Technology*, *40*, 213–228.

[11] Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren*. Chicago: University of Chicago Press.

[12] Leikin, R. (2009). Bridging research and theory in mathematics education with research and theory in creativity and giftedness. In R. Leikin, A. Berman & B. Koichu (Eds), *Creativity in mathematics and the education of gifted students* (pp. 383–409). Rotterdam: Sense Publishers.

[13] Leikin, R. (2011). The education of mathematically gifted students: Some complexities and questions. *The Mathematics Enthusiast*, *8*(1), 167-188.

[14] Leikin, R. (2020). Giftedness and high ability in mathematics. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (p. 315–325). Dordrecht, The Netherlands: Springer.

[15] Leikin, R., & Stanger, O. (2011). Teachers' images of gifted students and the role assigned to them in heterogeneous mathematics classes. In B. Sriraman & K. W. Lee (Eds.), *The elements of creativity an giftedness in mathematics* (pp. 103–118). Rotterdam: Sense Publishers.

[16] Mann, E. L. (2006). Creativity: The essence of mathematics. *Journal for the Education of the Gifted*, *30*(2), 236–262.

[17] Miller, R. C. (1990). *Discovering mathematical talent*. Reston, VA: Council for Exceptional Children, ERIC Clearinghouse on Disabilities and Gifted Education. ERIC Document Reproduction Service No: ED 321 487.

[18] Patton, M. Q. (2002). *Qualitative research and evaluation methods.* Newbury Park: Sage Publication.

[19] Paz-Baruch, N., Leikin, M., Aharon-Peretz, J., & Leikin, R. (2014). Speed of information processing in generally gifted and excelling in mathematics adolescents. *High Abilities Studies*, *25*(2), 143–167.

[20] Paz-Baruch, N., Leikin, M., & Leikin, R. (2022). Not any gifted is an expert in mathematics and not any expert in mathematics is gifted. Gifted and Talented International. https://doi.org/10.1080/15332276.2021.2010244

[21] Prabhu, V. & Czarnocha, B. (2013) .Problem posing, problem solving dynamics in the context of teaching- research and discovery method. *Mathematics Teaching-Research Journal Online*, *6*(1&2), 100-122.

[22] Prabhu, V. & Czarnocha, B. (2013/2014). Democratizing mathematical creativity through Koestler's bisociation theory. *Mathematics Teaching-Research Journal Online*, *6*(4), 33-46.

[23] Reed, C. F. (2004). Mathematically gifted in the heterogeneously grouped mathematics classroom: What is a teacher to do? *The Journal of Secondary Gifted Education*,*3*, 89– 95.

[24] Renzulli, J. S. (2000). The identification and development of giftedness as a paradigm for school reform. *Journal of Science Education and Technology, 9*(2), 95–114.

[25] Sheffield, L. J. (1994). *The development of gifted and talented mathematics students and the national council of teachers of mathematics standards.* Storrs, CT: The National Research Center on the Gifted and Talented, The University of Connecticut.

[26] Sheffield, L. J. (Ed.). (1999). *Developing mathematically promising students*. Reston, VA: National Council of Teachers of Mathematics.

[27] Sheffield, L. J. (2003). *Extending the challenge in mathematics: Developing mathematical promise in K-8 students.* Thousand Oaks, CA: Corwin Press.

[28] Sheffield, L. J. (2017). Dangerous myths about "gifted" mathematics students. *ZDM Mathematics Education, 49*(1), 13–23.

[29] Sheffield, L. J., Bennett, J., Berriozabal, M., DeArmond, M., & Wertheimer, R. (1999). Report of the NCTM task force on the mathematically promising. In L. J. Sheffield (Ed.), *Developing mathematically promising students* (pp. 309–316). Reston: NCTM.

[30] Silver, E. A. (1994). On mathematical problem posing. *For the Learning of Mathematics, 14*(1), 19-28.

[31] Singer, F. M., Sheffield, L. J., Freiman, V., & Brandl, M. (2016). *Research on and activities for mathematically gifted students*. New York: Springer Nature.

[32] Singer, F.M., Sheffield, L.J. & Leikin, R. (2017). Advancements in research on creativity and giftedness in mathematics education: Introduction to the special issue. *ZDM Mathematics Education, 4*9(1): 5–12.

[33] Shore, B. M., & Kanevsky, L. (1993). Thinking processes: Being and becoming gifted. In K. A. Heller, F. J. Monks, & A. H. Passow (Eds.), *International Handbook for Research and Development on Giftedness and Talent* (pp. 133–148). London: Pergamon.

[34] Subotnik, R., Olszewski-Kubilius, P., & Warrell, F. (2011). Rethinking giftedness and gifted education: A proposed direction forward based on psychological science. *Psychological Science in the Public Interest, 12*(1), 3–54.

[35] Sriraman, B. (2003). Mathematical giftedness, problem solving, and the ability to formulate generalizations. *The Journal of Secondary Gifted Education, 14*, 151–165.

[36] Sriraman, B. (2004). Gifted ninth graders' notions of proof: Investigating parallels in approaches of mathematically gifted students and professional mathematicians. *Journal for the Education of the Gifted, 27*, 267–292.

[37] Sriraman, B. (2005). Are giftedness and creativity synonyms in mathematics? An analysis of constructs within the professional and school realms. *The Journal of Secondary Gifted Education, 17*, 20–36.

[38] Sriraman, B., Haavold, P., & Lee, K. (2013). Mathematical creativity and giftedness: a commentary on and review of theory, new operational views, and ways forward. *ZDM Mathematics Education, 45*(2), 215–225.

[39] Van Harpen, X. Y., & Sriraman, B. (2013). Creativity and mathematical problem posing: an analysis of high school students' mathematical problem posing in China and the USA. *Educational Studies in Mathematics, 82*(2), 201–222.

[40] Yazgan-Sağ, G. (2019). A theoretical view to mathematical giftedness. *Milli Eğitim Dergisi*, *48*(221), 159–174.

[41] Yazgan-Sağ, G. (2020) Possible interactions with mathematically gifted students: Views of prospective teachers. *Research in Pedagogy*, *10*(2), 121–132.

[42] Yazgan-Sağ, G. & Argün, Z. (2020). Self-reflections of gifted students in the context of mathematical problem solving. *Malaysian Online Journal of Educational Sciences, 8*(2), 14-27.

[43] Ziegler, A., & Heller, K. A. (2000). Conceptions of giftedness from a meta-theoretical perspective. In K. A. Heller, F. J. Mönks, R. J. Sternberg, & R. F. Subotnik, (Eds.), *International Handbook of Giftedness and Talent* (2nd ed., pp. 3–21). New York: Elsevier.

Children's Errors in Written Mathematics

Chin Ying Liew*, Siow Hoo Leong, Nor Hazizah Julaihi, Tze Wee Lai, Su Ung Ting, Chee Khium Chen, Anniza Hamdan

Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA, Cawangan Sarawak, Kampus Samarahan 2, 94300 Kota Samarahan, Sarawak, Malaysia

*Corresponding author: cyliew@uitm.edu.my

Abstract: *Studies of errors in mathematics are essential for mathematics educators to design and contextualize a whole new instruction accordingly. Nonetheless, less attention has been given to the mathematical writing errors when compared to the mathematical conceptual and procedural errors. It is mainly because the former mistakes usually do not affect the final answer or are seemingly irrelevant to students' mathematics knowledge. Most of the studies surveying the mathematical writing errors focus on the undergraduates, followed by middle school students. Little has been done towards the primary school pupils. This paper aims to identify and classify the types of mathematical writing errors committed by primary school pupils in three domains: number and operation, measurement, and statistics. A qualitative approach is employed to identify and classify the mathematical writing errors committed by 29 above-average students from three schools. The results show that six categories of mathematical writing errors are observed. Lack of term or phrase and misuse of mathematical symbols top the overall categories of errors. The close-up views of the solutions with written errors provide some insights to tackle the problem by the mathematics teachers starting from the tender age of the students.*

Keywords: Elementary education; Lack of term or phrase; Mathematics grammar; Misuse of mathematical symbols; Writing error

INTRODUCTION

Studies of errors in mathematics are pertinent in the field of mathematics education. Mathematics educators believe that a whole new instruction could be designed and contextualized accordingly based on the errors identified (Bray, 2013; Rushton, 2018). By realizing the errors committed, students tend to experience a more rewarding learning experience, promoting enjoyment and confidence (Huang & Lin, 2015) and leading to an increased level of positive attitude towards mathematics. These students are more likely to engage in active and constructive mathematical learning processes and hence perform well in mathematics. Students who have performed well in mathematics are reported to have a higher possibility to join the Science, Technology, Engineering and Mathematics (STEM)-related field career (Blotnicky et al., 2018).

Mathematical writing errors, conceptual errors, and procedural errors are the general types of errors committed by the students (Sumartini & Priatna, 2018). Many studies have been done in researching issues related to students' mathematical errors with many of them focusing on the conceptual and procedural aspects (Arhin & Hokor, 2021; Ekawati et al., 2022; Mutambara & Bansilal, 2021; Payadnya et al., 2021; Rushton, 2018). However, Usiskin (2015) has stipulated that familiarity with mathematics language, both spoken and written, is a prerequisite to the understanding of mathematics concepts and procedures. It is commonly associated with the use of pronunciations, symbols, notations, and operations (Khoshaim, 2018; Matthews et al., 2012). The same researcher commented that the negligence of it would cause severe distortion of the intended mathematical meaning and solution, and perpetual bad mathematical writing habits of which one's mathematical professionalism could be greatly impaired.

Seo (2015) considers mathematical writing as an intertwining of three aspects: words, symbols, and images. It is referred to as "a thematic condensation of symbols, terms, and images to convey mathematical knowledge and meaning" (p.135). It implies mathematical writing as an endeavor to communicate mathematics between the students and their audience where the writing should be understandable by the audience. Surprisingly, relatively less attention has been given to the mathematical writing error specifically, when compared to the mathematical conceptual and procedural errors. This is especially true when such mistakes do not affect the final answer or are seemingly irrelevant to the students' mathematics knowledge (Freeman et al., 2016; Guce, 2017; Khoshaim, 2018). Researchers commented that the students who are offered a place in the university generally do not know how to write mathematics (Gunns et al., 2020). Although the students enrolled in STEM-related programs at the varsities show satisfactory performance in the technical aspects of mathematics, their skills in showing and explaining the technical mathematical results draw concerns among the researchers (Arévalo et al., 2021). These students used to self-perceived that they are "good at math" but "poor at writing" mathematics (Li et al., 2019). This may be due to the fact that written mathematics is still yet a common practice in the schools where most of the mathematics classes relied on mathematical skills building and conceptual understanding activities (Urquhart, 2009). Researchers generally regard mathematical writing as an underutilized method of instruction, be it in college or school (Heavner & Devers, 2020b; Urquhart, 2009).

In a study analyzing the mathematical writing errors of students enrolled in a calculus course, Khoshaim (2018) reported that missing a symbol like the equal sign or a notation were the most common inattentive errors observed, followed by the misuse of the equal sign and other symbols. In another study conducted by Guce (2017) that involved students from an advanced calculus course, it revealed that the most committed mathematical writing errors were incorrect grammar and misuse of mathematical symbols. A study on the types of mathematics error among the students attending a ring theory course found that the students had problems in using standard mathematical symbols to express mathematical definitions correctly, and no proper reasons were given for the final answers they obtained (Fatmiyati et al., 2020). Heavner and Devers (2020a, 2020b) claimed that undergraduates should be furnished with mathematical writing skills in order to communicate the mathematical concepts competently, which will eventually contribute to increased understanding of college mathematics. Studies conducted on middle and elementary school students reveal that the majority of students had difficulties communicating mathematical ideas and using the correct mathematical terms or notations (Adnan et al., 2021; Ng, 2020). The primary school pupils' weaknesses in translating the word problems into mathematical terms or notations were noticed (Rosli et al., 2020). In a worrying situation, pre-service teachers were reported to show impediments in understanding the mathematical notations and terminologies (Makonye & Ramatlapana, 2017).

Khoshaim (2018) categorized the mathematical writing errors into three: missing symbol or notation, misuse of symbol or notation, and miss-order of operations, which are errors that occurred due to a lack of attention or inattentive errors. These categories can be refined into a more detailed classification of mathematical writing errors by Guce (2017) with nine categories: misuse of mathematical terms; misuse of mathematical symbols; incorrect notation; incorrect grammar; incorrect capitalization; no or incorrect punctuation; vague term; incorrect term or phrase; and lack of term or phrase. The classification has been adopted in the analysis of written error in the, for example, trigonometry topic (Setiawan, 2021) and online mathematics courses (Heavner & Devers, 2020b). Most of the studies scrutinizing the errors in written mathematics are on the college or university students, followed by middle school students. Studies about types of mathematical writing errors among elementary or primary school pupils are scarce. Findings from studies by Hornburg et al. (2022), Fyfe et al. (2020), and Vincent et al. (2015) imply that formal understanding of mathematical knowledge at the elementary level advances the positive mathematical development of the children at the higher level. This includes inculcating proper mathematics language, focusing on the use of words and symbols in relating the mathematical concepts learned and mathematical procedures performed among the children. Consequently, this study aims to identify the mathematical writing errors committed by primary school pupils when executing their procedural explanations and classify the types of these errors based on the classification of mathematical writing errors by Guce (2017), with quantification of the error's occurrences. The mathematical writing studied here focuses mainly on symbolic representation and the use of words. Knowing the importance of possessing mathematical writing skills, it is thus imperative to nurture primary school pupils at the beginning level of their mathematics education.

The organization of this paper is as follows: Section One gives the introduction and background of this study; Section Two provides the method and material used; Section Three presents the results obtained from this study and its corresponding discussions, and the last section concludes the study together with some recommendations for future research.

METHOD AND MATERIAL

A total of 29 sixth graders participated in this study. They were from three primary schools located among the communities of similar family socioeconomic backgrounds. These schools were referred to as Schools X, Y and Z in this study, with a total of 24, 43 and 41 sixth graders respectively. The students were purposively selected based on their score of at least 65% in an instrument adapted from a set of sample questions released by the national examinations board. The format and scope of this instrument are equivalent to the primary school national public examination of Malaysia. The instrument consists of sixteen questions covering three domains: number and operation (Question 1 and 3 are under the topic of number and operation, while Question 2, 13, 15 and 16 are under the topic of fraction and percentage), measurement (Question 4 and 5 are under the topic of weight, whereas Question 6 and 7 are under the topic of volume of water), and statistics (Question 8, 9 and 10 are under the topic of mean, while Question 11, 12 and 14 are under the topic of pictograph), and the total score is 100%. The students who scored at least 65% were expected to possess adequate mathematical conceptual and procedural skills (Guce, 2017). Therefore, they were selected for the analysis of mathematical writing errors.

A qualitative approach was employed in this study to identify and classify the mathematical writing errors. The mathematical writing error classifications by Guce (2017) was adopted given its conclusive and adequate consideration of all main distinctive errors in written mathematics across all education levels although the findings presented are on undergraduate students. By adopting the same mathematical writing error classifications, the results obtained from this study on primary school pupils add value to the research community where comparisons among different levels of education could be made. In this study, the solutions with the correct final answers from all the students were analyzed for the mathematical writing grammar used. Only the solutions with correct final answers were scrutinized for the written errors because these solutions were assumed to have possessed a satisfactory level of mathematical procedural and conceptual knowledge.

After all occurrences of mathematical writing errors committed by the students were identified and classified accordingly, quantification of the errors was carried out to provide richer information. The number of occurrences of the errors was summarized in total and percentage. Each occurrence of an error category was recorded using its corresponding coding and was counted as one error. The process of identifying and classifying the errors was repeated at least twice for confirmation. It was carried out by two researchers separately and their findings were compared. If the categories of an error classified by the two researchers were different, a discussion was carried out to reach an agreement. Should a dispute arise, and no agreement could be reached, a third researcher was called to finalize the error category. In this paper, the categories of all errors were agreed upon by two researchers. Table 1 provides the codes and brief descriptions of the nine categories of mathematical writing errors used.

Category	Code	Brief description
Misuse of mathematical terms	MMT	Incorrect use of mathematical terms such as mathematical equations and expression.
Misuse of mathematical symbols	MMS	Incorrect use of mathematical symbols such as the equal sign "=" and upper-case and lower-case symbols.
Incorrect notations	IN	Incorrect use of mathematical notations, the symbolic expressions with their established meaning such as (Adnan et al.) and a_i, and $\frac{dy}{dx}$ and $\frac{\partial y}{\partial x}$.
Incorrect grammar	IG	Incorrect use of grammar and spelling when writing mathematics in sentences where standard grammatical rules of writing are violated.
Incorrect capitalization	IC	Incorrect use of capitalization where words either are capitalised when they should not be or are not capitalised when they should be.
No or incorrect punctuation	NIP	Incorrect use or absence of punctuation when writing mathematics in complete sentences.
Vague term or phrase	VT	A vague term refers to a term that is unclear of its meaning in which the distinctness of the mathematical statement becomes ambiguous or vague.
Incorrect term or phrase	IT	An incorrect term or phrase refers to the inappropriate use of a term or phrase in a mathematical statement which causes the inaccurate idea of the statement.
Lack of term or phrase	LTP	Lack of term or phrase refers to the presentation of a mathematical solution for a mathematics problem without any or adequate explanation such as defining a variable used in the solution, causing possible misinformation to the readers.

Table 1: Codes and brief description of mathematical writing errors used

RESULTS AND DISCUSSION

This section presents the results of mathematical writing errors according to each domain for the three participating schools. The frequencies of writing errors from the three schools are first summarized as a whole, and then the frequencies breakdown from each school are tabulated and compared. Each school is analyzed separately to investigate if there is a different trend or pattern of errors across the schools, which could be due to the school environment such as the experience or training of their teachers. Besides, it also considers equal importance for each school as the numbers of students selected from the three schools for analysis of written errors are unbalance. The types of errors committed will thus not be dominated by the school with more students. Then, close-up views of the solutions are shown to explicate how a particular written error was made by the students.

Number and Operation

Table 2 shows the number of various written errors observed in the solutions with correct final answers for the domain of number and operation for the three schools. Five categories of errors were observed for this domain: MMS, IG, VT, IT, and LTP. However, out of the total of 68 errors observed, only one observation was noticed for each of the IG, VT, and IT errors. The majority of the errors recorded was LTP (73.5%), followed by MMS (22.1%). School Y recorded a very high percentage of LTP error (81.1%) when compared with School X (69.2%) and School Z (61.1%). Interestingly, more categories of errors (MMS, VT, IT and LTP) were observed in the solutions of students from School X, followed by three categories of errors (MMS, IG and LTP) were observed for School Y while only two categories of errors (MMS and LTP) were observed for School Z.

		School X (n_65% = 4)						School Y (n_65% = 19)						School Z (n_65% = 6)						Total (N_65% = 29)					
Domain	Q	✓	MMS	IG	VT	IT	LTP	✓	MMS	IG	VT	IT	LTP	✓	MMS	IG	VT	IT	LTP	✓	MMS	IG	VT	IT	LTP
Number and operation	1	4	1	0	1	0	0	17	0	0	0	0	0	6	0	0	0	0	0	27	1	0	1	0	0
	2	4	0	0	0	0	0	18	0	0	0	0	0	6	0	0	0	0	0	28	0	0	0	0	0
	3	4	1	0	0	1	3	14	2	0	0	0	12	6	4	0	0	0	4	24	7	0	0	1	19
	13	4	0	0	0	0	2	8	4	0	0	0	2	3	1	0	0	0	1	15	5	0	0	0	5
	15	4	0	0	0	0	4	19	0	1	0	0	16	6	2	0	0	0	5	29	2	1	0	0	25
	16	0	0	0	0	0	0	3	0	0	0	0	0	1	0	0	0	0	1	4	0	0	0	0	1
Total			2	0	1	1	9		6	1	0	0	30		7	0	0	0	11		15	1	1	1	50
Total Errors					13						37						18						68		
Percentage			15.3	0	7.7	7.7	69.2		16.2	2.7	0	0	81.1		38.9	0	0	0	61.1		22.1	1.5	1.5	1.5	73.5

Note: The domain is listed in the first column of the tables. The second column 'Q' refers to the question number in the instrument. The number of students who had answered each question correctly is given in the column with the column header '✓'. This is followed by columns showing the categories of writing errors classified and the number of errors identified for each category. The last column of the tables, with column header labelled 'Total' presents the overall findings of this study for the domain. The notation 'n_65%' refers to the number of students who scored at least 65% for each school; whereas 'N_65%' refers to the total number of students in this study that had scored at least 65%.

Table 2: Incidences of errors observed for the domain of number and operation

Figure 1(a) shows an example of an MMS error that involves the equal sign ("="), where the subject of the equal sign is missing in the solution of "= 260/2 = 130". This is the most common mathematical writing error observed for MMS in the domain of number and operation. Another common misuse of the equal sign is that the students exploited it as a way to refer to the final answer of their solutions. There was one particular example of MMS that is presented in Figure

1(b), where a student misused the equal sign and wrote “7 = 700 000” to express the final answer of “7” as the digit that carried the place value of hundred thousands in the given number of “2 736 981”. Figure 1(c) presents another serious misuse of mathematical symbols involving the equal sign. Although the vertical calculation shown on the left is correct, the horizontal procedure on the right is incorrect because “13 259 + 461 × 2” was not equal to “27 440”. This error suggested a partial insight error, a type of bit error due to the carelessness or lack of conceptual understanding (Sumartini & Priatna, 2018), on the priority of executing the addition and multiplication operations in this example.

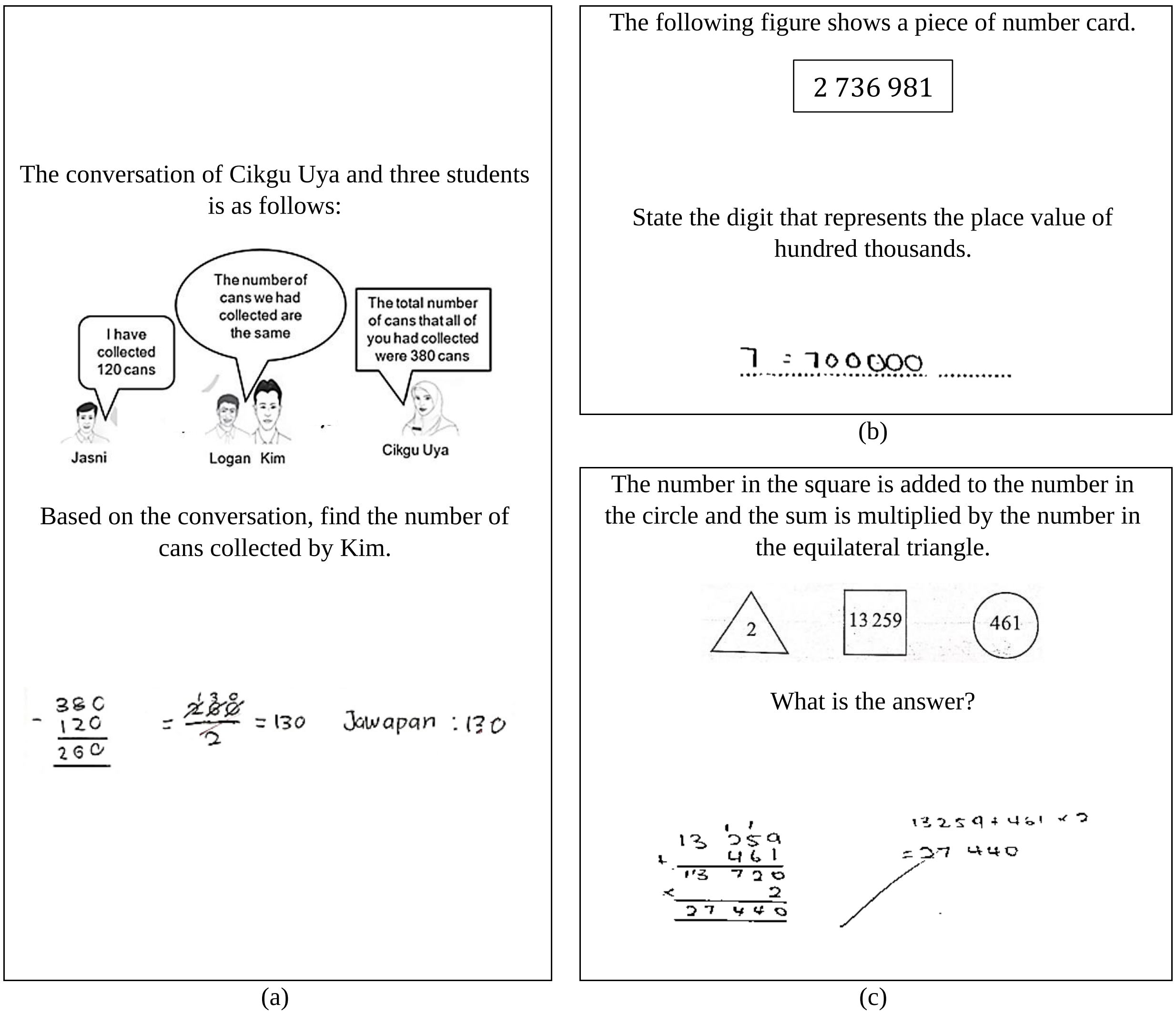
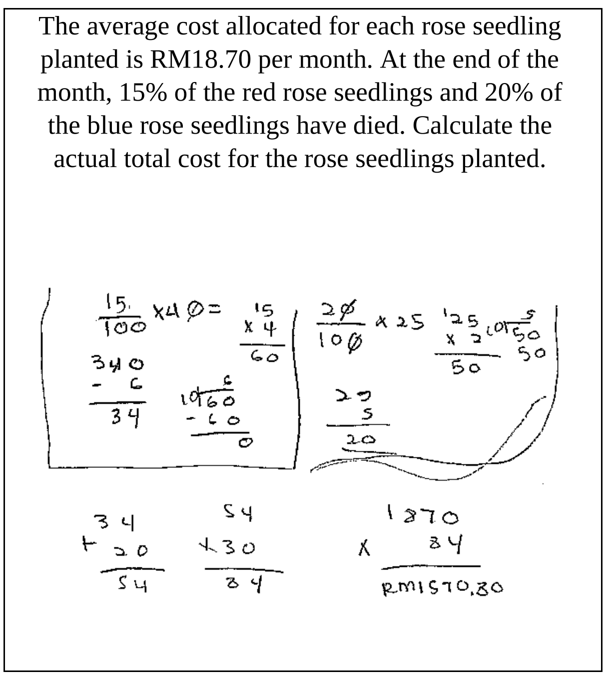
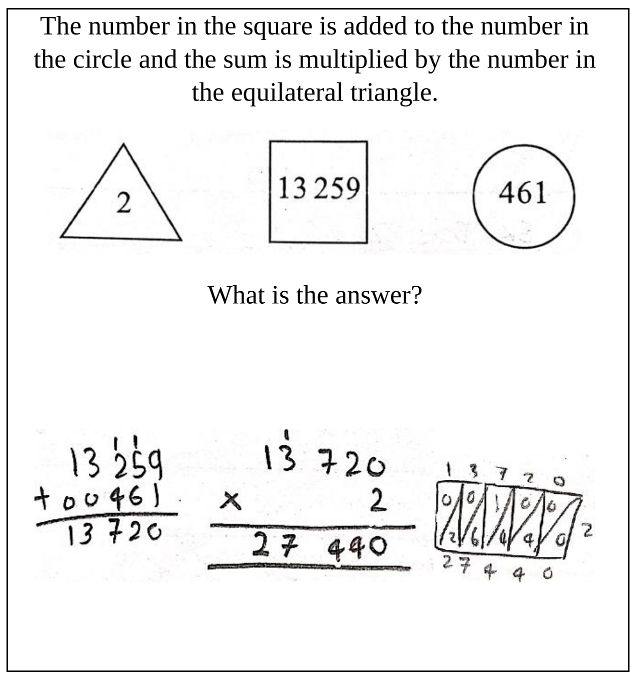


Figure 1: MMS by (a) Z35 (no subject for “=”); (b) X7 (unequal value on both sides of “=”) and (c) Z32 (incorrect horizontal calculation)

On the other hand, the LTP errors recorded in this domain were due to the lack of term or phrase in the solution of the students. There was an absence of terms or phrases to connect and relate the concepts behind the mathematical expressions presented by the students in solving the questions. This is especially obvious when two or more different arithmetic operations are needed to obtain the final answer. In fact, many LTP errors committed by the students in this domain were because the students did not seem to bother to specify and inform the reader which solution was the final answer. Two examples are shown in Figure 2(a) and Figure 2(b). Moreover, the students did not indicate specifically the purpose of each steps written. These are typical examples of mathematical writing errors where "… *presenting only the mathematical solution without any explanation is like assuming that the solution will speak for itself* …" as purported by Guce (2017). An acceptable final answer without LTP error was shown in Figure 1(a), where Z35 had specified and informed that the final answer was 130 by indicating it using "Jawapan: 130" (in Malay), which means "Answer: 130".For the IG error, an example presented in Figure 3(a) showed that the student gave "15 bunga ros" (in Malay), which means "15 roses" as the final answer instead of the quantifier "rose seedlings" used in the question. This had caused incorrect grammar in the answer. For the VT error, the term "or" had been used in "7 or 700 000" for the solution of the question asking for the digit that is in the hundred thousands place value for the number of "2 736 981". This is presented in Figure 3(b), where "atau" (in Malay) means "or". So, by answering "7 or 700 000", the term "or" is vague in this context. As for the IT error, Figure 3(c) shows an example where the student wrote "130 bilangan" (in Malay), which means "130 number" instead of "130 cans" in the solution for a question asking about "number of cans collected". The term "bilangan" (in Malay) or "number" is incorrectly used.



(a)

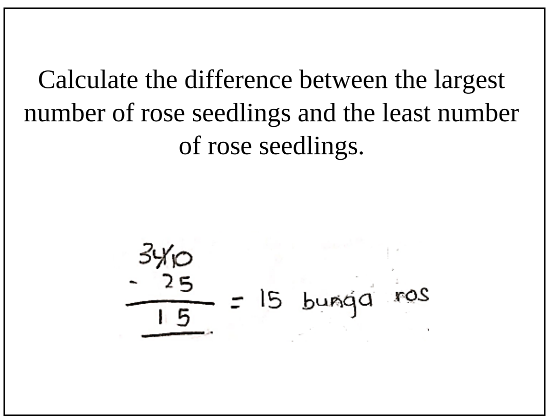


(b)

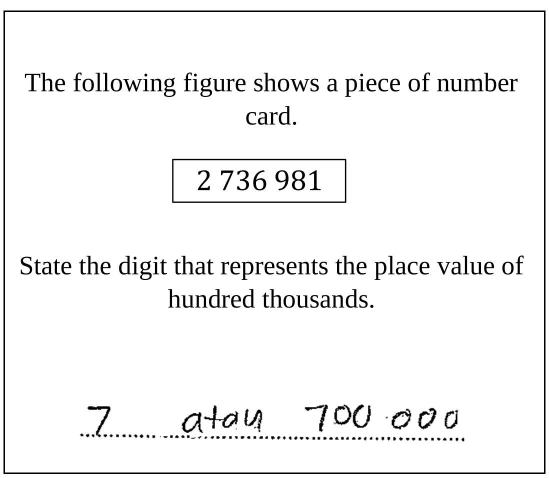
Figure 2: LTP by (a) Y32 and (b) Y36

Measurement

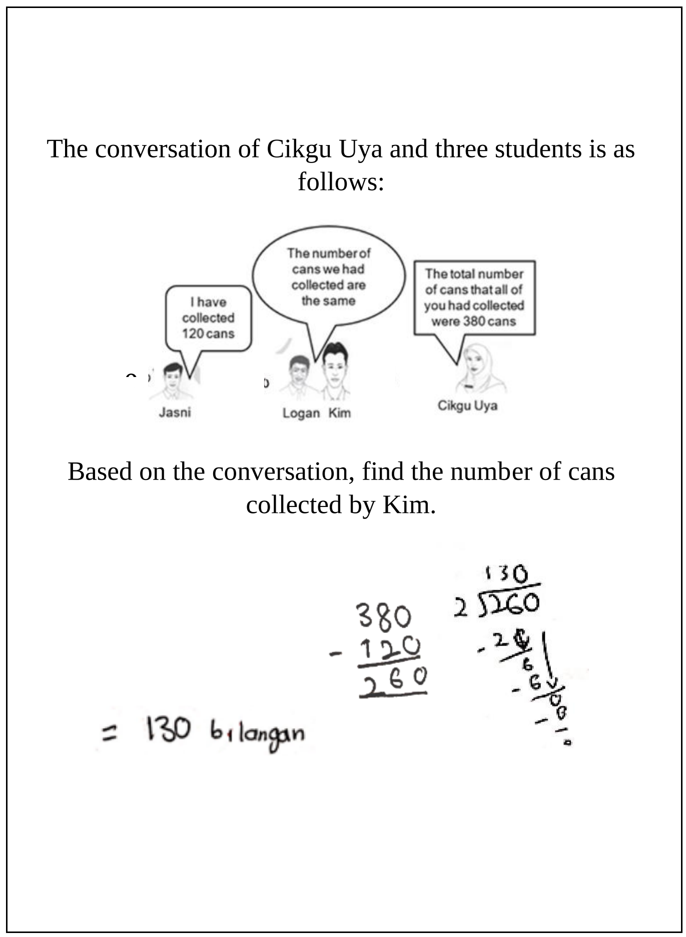
In the analysis of mathematical writing errors in the domain of measurement, the number of written errors observed in the solutions for Questions 4 and 5 from the topic of weight, and Questions 6 and 7 from the topic of the volume of water were grouped respectively in Table 3. The results show that MMS, IN, and LTP were three categories of error observed in the solutions of the questions under this domain. Out of the total of 48 errors observed, there was only one IN error, while MMS error topped the list (77.1%), and followed by LTP (20.8%). MMS was found to be the most common error, which is followed by LTP error across all the three schools in the domain of measurement. It is noticed that only one category of errors (MMS) was recorded from School X, whereas all the three categories of errors (MMS, IN, and LTP) were observed in the solutions from School Y and two categories of errors (MMS and LTP) were observed for School Z. This could be because School Y recorded the greatest number of students with the correct final answer while School X had the least number.



(a)



(b)



(c)

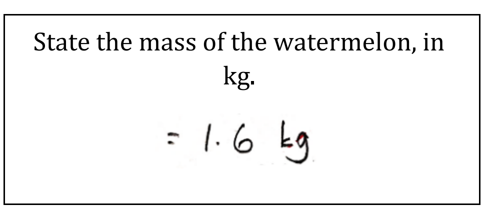
Figure 3: (a) IG by Y32, (b) VT by X15 and (c) IT by X20

		School X (n_65% = 4)				School Y (n_65% = 19)				School Z (n_65% = 6)				Total (N_65% = 29)			
Domain	Q	✓	MMS	IN	LTP	✓	MMS	IN	LTP	✓	MMS	IN	LTP	✓	MMS	IN	LTP
Measurement	4&5	3	1	0	0	18	17	0	1	5	4	0	1	26	22	0	2
	6&7	1	1	0	0	18	10	1	7	4	4	0	1	23	15	1	8
Total			2	0	0		27	1	8		8	0	2		37	1	10
Total Errors				2				36				10				48	
Percentage			100.0	0	0		75.0	2.8	22.2		80.0	0	20.0		77.1	2.1	20.8

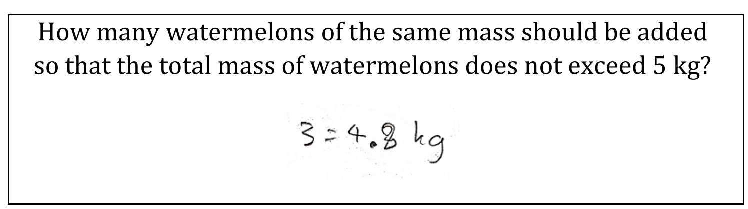
Table 3: Incidences of errors observed for the domain of measurement (See Table 2 for Note)

The common errors observed under MMS were the misuse of the equal sign, especially when the subject was missing in the expressions of the solutions. An example is shown in Figure 4(a). Another instance of MMS error observed was when the students gave "3 = 4.8 kg" (Figure 4(b)) to express that the weight of three (3) watermelons was 4.8 kg. This was a typical MMS error where both sides of the equal signs were not equal.

On the other hand, the common LTP error observed under the measurement domain was similar to the number and operation domain, in which there were lacking terms or phrases to connect the steps of solutions. Two examples are presented in Figure 5(a) and Figure 5(b) where the former shows that the student did not specify which one was the final answer while the latter shows the lack of a phrase for the final answer such as "The mean volume of water is 0.18 liter" or "Answer: 0.18 liter". Besides that, the students always failed to provide a unit of measurement when it was needed in their solutions or missed out the correct arithmetic operation symbol as exemplified in Figure 5(b). As for the IN error, an example is depicted in Figure 5(c). The notations of a milliliter (mL) and liter (L) were erroneously used in the solution where milliliter was used in the question, but liter was given by the student, instead.

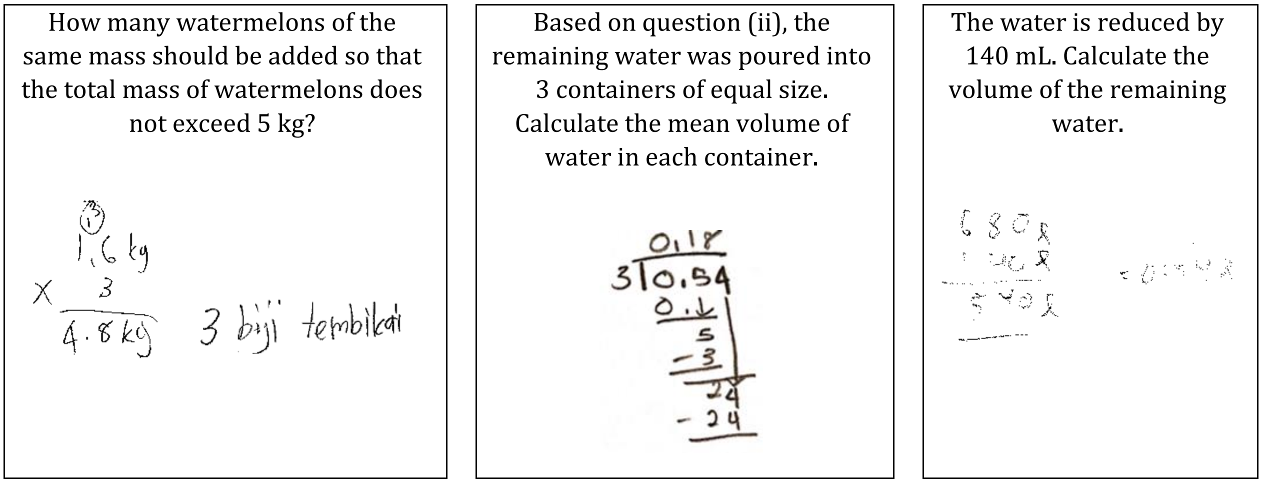


(a)



(b)

Figure 4: MMS by (a) Z37 (no Subject for "=") and (b)Y42 (unequal value on both sides of "=")



(a) (b) (c)

Figure 5: (a) LTP by X15, (b) LTP by Y35 and (c) IN by Y24

Statistics

The number of written errors observed in the solutions for Questions 9 and 10 from the topic mean, and Questions 11 and 12 from the topic pictograph were grouped together respectively in Table 4 for analysis. There were two categories of errors reported for the solutions of questions under the domain of statistics: MMS and LTP, which accounted for 23.9% and 76.1% of the errors respectively. The results reported for School X and School Y were similar, with more than 80% LTP error whereas the scenario was the opposite for School Z with almost 64% MMS error. No other category of errors was observed for this domain. This could be due to the nature of the questions that entail short solutions.

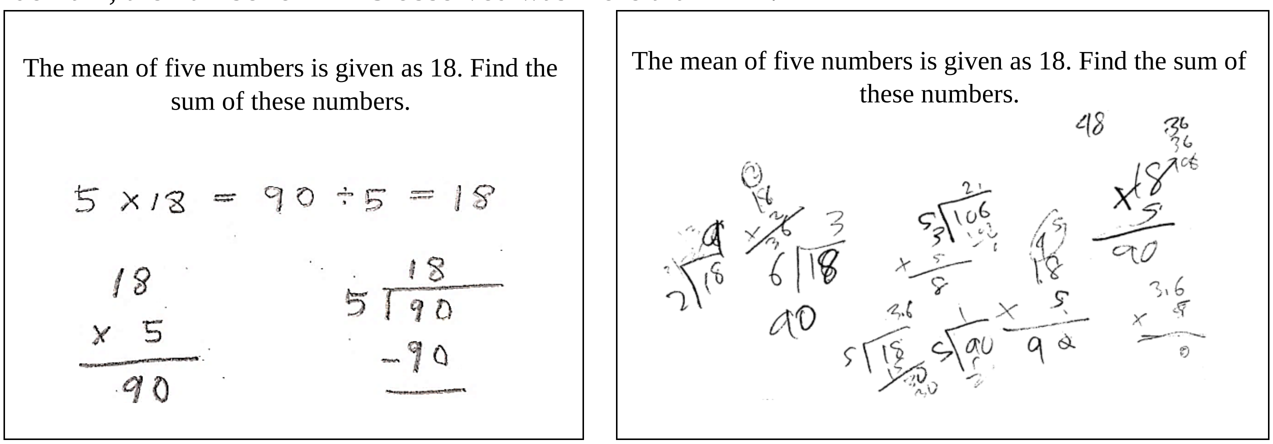
		School X (n_65% = 4)			School Y (n_65% = 19)			School Z (n_65% = 6)			Total (N_65% = 29)		
Domain	Q	✓	MMS	LTP	✓	MMS	LTP	✓	MMS	LTP	✓	MMS	LTP
Statistics	8	1	0	0	14	4	5	5	3	1	20	7	6
	9&10	2	1	1	14	2	21	1	3	0	17	6	22
	11&12	2	0	1	8	2	7	2	0	3	12	2	11
	14	3	0	3	15	0	9	5	1	0	23	1	12
Total			1	5		8	42		7	4		16	51
Total Errors			6			50			11			67	
Percentage			16.7	83.3		16.0	84.0		63.6	36.4		23.9	76.1

Table 4. Incidences of errors observed for the domain of statistics (see Table 2 for Note)

Two main common MMS mathematical writing errors committed in the domain of statistics were the use of equal signs throughout the steps of solution and the missing of a subject for the equal signs written in an expression of the solutions. An example is presented in Figure 6(a) where 5×18 was neither equal to $90 \div 5$ nor 18 although $90 \div 5$ did equal to 18. The solution shown in Figure 6(a) is a typical incidence as commented by Guce (2017) where "… *students tend to use equal signs to connect several lines of solution which are not actually equal*". An example of LTP error is given in Figure 6(b) where the student was "… *presenting only the mathematical solution without any explanation* …" which "… *is like assuming that the solution will speak for itself* …" (Guce, 2017).

Across Domain

Figure 7 presents the summary of overall errors identified and classified in this study across the three domains. Altogether, there were six categories of mathematical writing errors observed. LTP appeared to top the overall errors observed, followed by MMS. Most of the LTP was observed in the number and operation domain as compared to statistics and measurement domains. MMS was observed mostly in the measurement domain as compared to number and operation, and statistics domains. The number of LTP observed was more than MMS in the domains of number and operation, and statistics. However, in the case of the measurement domain, the number of MMS observed was more than LTP.



(a) (b)

Figure 6: (a) MMS by Y14 (using "=" throughout) and (b) LTP by X15 (working all over the solution)

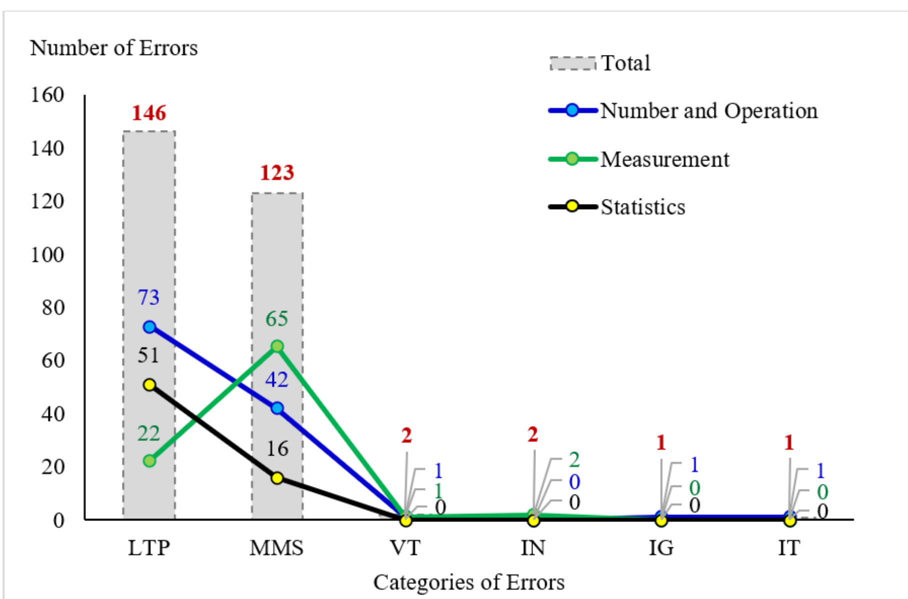


Figure 7: Cognitive process dimension

Observations of the other four categories of errors (IN, IG, VT, and IT) were very minimal and there were no incidents recorded under the category of MMT, IP, and NIP errors. This may be due to the nature of the instrument used in this study where most of the questions do not require lengthy solutions. As explained in Section 2, the instrument of this study consists of typical questions the sixth graders are expected to answer in the national primary school public examination. It is different from the expository-writing-nature instrument used by Guce (2017) where all categories of errors were observed. The findings of this study that involve primary school pupils are in accord with that of Khoshaim (2018) and Guce (2017), whose respondents were students in tertiary education institutes. Misusing mathematical symbols, especially the equal sign appears to be a writing error common to students across the levels of education.

The findings obtained in this study give rise to two implications worth highlighting: a) the correct use of equal signs; and b) the use of appropriate terms or phrases in presenting the mathematics solutions by the primary school pupils. It is vital for young children to possess a sound concept and interpretation of equal sign as this is closely related to mastering the algebraic skills and knowledge (Fyfe et al., 2020; Hornburg et al., 2022). Researchers have commented that algebra is a crucial step to not only mathematics studies at higher level but also higher education (Madej, 2022; Matthews et al., 2012). Henceforth, mathematics teachers are obliged to cultivate the good practice of using equal signs among their students and set a good example; use it only when the left-hand-side and the right-hand-side expressions of an equal sign do mean equality (Powell, 2012; Vincent et al., 2015). Both the relational and operational interpretation of equal sign among the students are important to holistic understanding of the meaning of equal sign and should be equally emphasized (Fyfe et al., 2020; Madej, 2022; Matthews et al., 2012).

At the same time, mathematics teachers ought to show and encourage the use of appropriate terms or phrases in the whole process of solving a mathematical problem, be it routine, partial-routine, or non-routine, particularly when it takes more than one step of solutions to come to the final answer of the problem. The mathematics teachers need to inspire the students to write and explain their solution and final answer. Using correct mathematical grammar is an important mode of thinking and learning for the students (Colonnese et al., 2018). The benefits are twofold: for the teacher to know both the depth and width of the conceptual and procedural understanding of their students.

Subsequently, the mathematics teachers must give equal emphasis to the grammar of the written mathematics alongside the intense attention given to the conceptual and procedural aspects of mathematics (Guce, 2017; Khoshaim, 2018), starting from the tender age of the students. As emphasized by the researchers, no toleration should be given to all possible writing errors. The power of correct mathematical writing should never be neglected in any mathematics instruction (Freeman et al., 2016). Although it may appear at a time that these mathematical writing errors are trivial to the students' conceptual understanding and procedural execution in the classroom, the consequences of continual carelessness and ignorant in mathematical writing are severe (Khoshaim, 2018).

CONCLUSIONS

This study looks into the mathematical writing errors of the sixth graders. It reveals the students' incompetency in written mathematics among the above-average performers. The findings should be taken seriously by educators as it is necessary to emphasize written mathematics besides stressing the importance of mathematical conceptual and procedural, starting from the tender age of the students (Ouyang et al., 2021; Valcan et al., 2020). As the students grow in their mathematics and English (or other language of instruction) language vocabulary at the elementary level, this could be made possible with teachers guiding the students in the class to practice mathematical writing alongside understanding the mathematics concepts and procedure in solving mathematics problems. In promoting good mathematics writing habits, teachers should pay attention to the mathematics writing errors committed in the students' homework and their formative and summative assessments. The students need to be informed clearly and explicitly of the errors and consciously be reminded that these errors are not acceptable. It is the responsibility of the teachers to be the role model and show appropriate mathematical writing in every mathematics class, when showing solutions in front of the class or to the students personally. Although highly identical patterns on the types of errors have been shown by the majority of the students from different schools of the communities with similar family socioeconomic backgrounds and located within the same geographical location, further studies in surveying any possible relationship between the demographic characteristics of students and the mathematical writing errors committed are recommended.

ACKNOWLEDGEMENTS

The authors would like to acknowledge the participating schools for taking part in this study. The authors wish to extend thanks to the faculty members who had rendered excellent support and assistance in the fieldwork.

REFERENCES

[1] Adnan, M., Abidin, Z. Z., Mustam, A. A., Maat, S. M., Sutama, S., Prihaswati, M., & Zulnaidi, H. (2021). The ability and analysis of students' errors in the topic of algebraic expression. *Journal of Physics: Conference Series*, *1988*, 012049. https://doi.org/10.1088/1742-6596/1988/1/012049

[2] Arévalo, M.-J., Cantera, M. A., García-Marina, V., & Alves-Castro, M. (2021). Analysis of university STEM students' mathematical, linguistic, rhetorical–organizational assignment errors. *Education Sciences*, *11*(4), 174. https://doi.org/10.3390/educsci11040173

[3] Arhin, J., & Hokor, E. K. (2021). Analysis of high school students' errors in solving trigonometry problems. *Journal of Mathematics and Science Teacher*, *1*(1), em003. https://doi.org/10.29333/mathsciteacher/11076

[4] Blotnicky, K. A., Franz-Odendaal, T., French, F., & Joy, P. (2018). A study of the correlation between STEM career knowledge, mathematics self-efficacy, career interests, and career activities on the likelihood of pursuing a STEM career among middle school students. *International Journal of STEM Education*, *5*(22). https://doi.org/10.1186/s40594-018-0118-3

[5] Bray, W. S. (2013). How to leverage the potential of mathematical errors. *Teaching Children Mathematics*, *19*(7), 424-431. https://doi.org/10.5951/teacchilmath.19.7.0424

[6] Colonnese, M. W., Amspaugh, C. M., Le May, S., Evans, K., & Field, K. (2018). Writing in the disciplines: How math fits into the equation. *Reading Teacher*, *72*(3), 379-387. https://doi.org/10.1002/trtr.1733

[7] Ekawati, R., Imah, E. M., Amin, S. M., Kohar, A. W., Nisa, K., & Prahmana, R. C. I. (2022). Error analysis of dyslexic student's solution on fraction operation tasks. *Mathematics Teaching Research Journal* *14*(1), 67-79. https://commons.hostos.cuny.edu/mtrj/wp-content/uploads/sites/30/2022/04/v14n1-Error-Analysis-of-Dyslexic.pdf

[8] Fatmiyati, N., Triyanto, & Fitriana, L. (2020). Error analysis of undergraduate students in solving problems on ring theory. *Journal of Physics: Conference Series*, *1465*, Article 012050. https://doi.org/10.1088/1742-6596/1465/1/012050

[9] Freeman, B., Higgins, K. N., & Horney, M. (2016). How Students communicate mathematical ideas: An examination of multimodal writing using digital technologies. *Contemporary Educational Technology*, *7*(4). https://doi.org/10.30935/cedtech/6178

[10] Fyfe, E. R., Matthews, P. G., & Amsel, E. (2020). College developmental math students' knowledge of the equal sign. *Educational Studies in Mathematics*, *104*(1), 65-85. https://doi.org/10.1007/s10649-020-09947-2

[11] Guce, I. K. (2017). Mathematical writing errors in expository writings of college mathematics students. *International Journal of Evaluation and Research in Education (IJERE)*, *6*(3), 233-242. https://doi.org/10.11591/ijere.v6i3.pp233-242

[12] Gunns, J. M. M., Carey, R., Butler, L., & Donald, A. (2020). Teaching students to write and read mathematics. In R. Marks (Ed.), *Proceedings of the British Society for Research into Learning Mathematics* (Vol. 40). https://bsrlm.org.uk/wp-content/uploads/2020/10/BSRLM-CP-40-2-05.pdf

[13] Heavner, S., & Devers, C. (2020a). Building conceptual understanding: Improving mathematics education in online courses. In T. Bastiaens & G. Marks (Eds.), *Proceedings*

of Innovate Learning Summit 2020 (pp. 558-571). Association for the Advancement of Computing in Education (AACE). https://www.learntechlib.org/primary/p/218851/

[14] Heavner, S., & Devers, C. (2020b). Digital tools for mathematics composition: Building an online mathematics course. *International Journal on E-Learning: Corporate, Government, Healthcare, and Higher Education*, *19*(4), 373-381. https://www.learntechlib.org/primary/p/216944/

[15] Hornburg, C. B., Devlin, B. L., & McNeil, N. M. (2022). Earlier understanding of mathematical equivalence in elementary school predicts greater algebra readiness in middle school. *Journal of Educational Psychology*, *114*(3), 540-559. https://doi.org/10.1037/edu0000683

[16] Huang, Y. C., & Lin, S. H. (2015). Development and validation of an inventory for measuring student attitudes toward calculus. *Measurement and Evaluation in Counseling and Development*, *48*, 109-123. https://doi.org/10.1177/0748175614563314

[17] Khoshaim, H. B. (2018). Inattentive students' errors in mathematics: Alarming repetition. *International Journal of Learning, Teaching and Educational Research*, *17*(11), 135-153. https://doi.org/10.26803/ijlter.17.11.9

[18] Li, C. Q., Randi, J., & Sheffield, J. P. (2019, 15 – 19 June). *An exploratory study of engineering students' misconceptions about technical communication* ASEE Annual Conference and Exposition: Charged Up for the Next 125 Years, Tampa. https://peer.asee.org/32061

[19] Madej, L. (2022). Primary School Students' Knowledge of the Equal Sign—the Swedish Case. *International Journal of Science and Mathematics Education*, *20*(2), 321-343. https://doi.org/10.1007/s10763-020-10144-z

[20] Makonye, J. P., & Ramatlapana, K. (2017). Mathematics education prospective teachers' errors patterns on Grade 12 mathematics. *International Journal of Educational Sciences*, *9*(2), 151-161. https://doi.org/10.1080/09751122.2015.11890305

[21] Matthews, P., Rittle-Johnson, B., McEldoon, K., & Taylor, R. (2012). Measure for measure: What combining diverse measures reveals about children's understanding of the equal sign as an indicator of mathematical equality. *Journal for Research in Mathematics Education (JRME)*, *43*(3), 316-350. https://doi.org/10.5951/jresematheduc.43.3.0316

[22] Mutambara, L. H. N., & Bansilal, S. (2021). A case study of in-service teachers' errors and misconceptions in linear combinations. *International Journal of Mathematical Education in Science and Technology*, 1-19. https://doi.org/10.1080/0020739X.2021.1913656

[23] Ng, B. S. (2020). Newman error analysis for errors in mathematical word questions among year three students in Sekolah Kebangsaan Taman Kluang Barat. *International Journal of*

Novel Research in Education and Learning, *7*(2), 58-63. https://www.noveltyjournals.com/upload/paper/Newman%20Error%20Analysis-2300.pdf

[24] Ouyang, X., Zhang, X., Zhang, Q., & Zou, X. (2021). Antecedents and consequences of young children's interest in mathematics. *Early Childhood Research Quarterly*, *57*, 51-60. https://doi.org/10.1016/j.ecresq.2021.05.005

[25] Payadnya, I. P. A. A., Suwija, I. K., & Wibawa, K. A. (2021). Analysis of Students' Abilities in Solving Realistic Mathematics Problems Using "What-If"-Ethnomathematics Instruments. *Mathematics Teaching Research Journal*, *13*(4), 131-149. https://commons.hostos.cuny.edu/mtrj/wp-content/uploads/sites/30/2022/01/v13n4-Analysis-of-Students-Abilities.pdf

[26] Powell, S. R. (2012). Equations and the equal sign in elementary mathematics textbooks. *Elementary School Journal*, *112*(4), 627-648. https://doi.org/10.1086/665009

[27] Rosli, S., Shahrill, M., & Yusof, J. (2020). Applying the hybrid strategy in solving mathematical word problems at the elementary school level. *Journal of Technology and Science Education*, *10*(2), 216. https://doi.org/10.3926/jotse.965

[28] Rushton, S. J. (2018). Teaching and learning mathematics through error analysis. *Fields Mathematics Education Journal*, *3*(1), 4. https://doi.org/10.1186/s40928-018-0009-y

[29] Seo, B.-I. (2015). Mathematical writing: What is it and how do we teach it? *Journal of Humanistic Mathematics*, *5*(2). https://doi.org/10.5642/jhummath.201502.12

[30] Setiawan, Y. E. (2021). Kesalahan mahasiswa semester pertama dalam menyelesaikan masalah fungsi trigonometri sudut tidak lancip [First semester student errors in solving trigonometric function of non-acute angle problems]. *Jurnal Program Studi Pendidikan Matematika*, *10*(2), 599-614. https://doi.org/10.24127/ajpm.v10i2.3458

[31] Sumartini, T. S., & Priatna, N. (2018). Identify student mathematical understanding ability through direct learning model. *Journal of Physics: Conference Series, 1132*. https://doi.org/10.1088/1742-6596/1132/1/012043

[32] Urquhart, V. (2009). *Using writing in mathematics to deepen student learning*. Mid-continent Research for Education and Learning (McREL).

[33] Usiskin, Z. (2015). What does it mean to understand some mathematics? In S. J. Cho (Ed.), *12th International Congress on Mathematical Education*. Springer. https://doi.org/10.1007/978-3-319-17187-6_46

[34] Valcan, D. S., Davis, H. L., Pino-Pasternak, D., & Malpique, A. A. (2020). Executive functioning as a predictor of children's mathematics, reading and writing. *Journal of Applied Developmental Psychology*, *70*, 101196. https://doi.org/10.1016/j.appdev.2020.101196

[35] Vincent, J., Bardini, C., Pierce, R., & Pearn, C. (2015). Misuse of the equals sign: An entrenched practice from early primary years to tertiary mathematics. *Australian Senior Mathematics Journal*, *29*(2), 31-39.

Division problems with remainder: A study on strategies and interpretations with fourth grade Mexican students

Ever Pacheco-Muñoz[1], Stefany Nava-Lobato[1], José Antonio Juárez-López[1], Manuel Ponce de León-Palacios[2]

[1] Meritorious Autonomous University of Puebla, México, [2] Puebla State Popular Autonomous University, México

ever.pacheco@alumno.buap.mx, stefany.naval@alumno.buap.mx, jajul@fcfm.buap.mx, manuel.poncedeleon@upaep.mx

Abstract: The present research focuses on analyzing how fourth-grade elementary school students (ages 9 to 10) solve and interpret the result of non-routine problems, precisely division measurement and division-partition with remainder. The methodology is qualitative, with a descriptive and interpretative approach. The information was collected using a questionnaire consisting of three problems (two of quotitive division and one of partitive division) and a clinical interview. The results showed the importance of using the division, multiplication, and addition algorithms to give a realistic answer to the problems. In the same way, it was possible to demonstrate the graphic strategy combined with counting to give a realistic answer to the problem. However, students were found to use division correctly but without an interpretation of the remainder or quotient. Likewise, they struggled to choose the correct procedure to solve the problem. These data suggest including realistic problems in mathematics classrooms to make sense of mathematical concepts in real life or the student's context. Likewise, this study provides implications on the mathematical problems that the teacher proposes in the classroom, where not only the division algorithm should be taught mechanically, nor focus on posing routine problems that lead the student to use a single heuristic resolution strategy. Essentially, the teacher is required to include real-world problems, where the student can awaken creativity to represent in different ways the understanding of a problem and, therefore, different strategies to solve it. In addition, that the student has the ability to check the result of the problem, with the conditions, situations or circumstances imposed by reality or everyday life.

Keywords: Division problems with remainder, division-quotitive, division-partitive, resolution strategies, primary education.

INTRODUCTION

To talk about problem-solving is to consider the importance of schools as a suitable setting for learning. It is here where the student manages to develop mathematical strategies to face challenges where the solutions to these do not require an obvious answer. However, different researchers interested in the subject show that the problems brought to the mathematics classroom lack authenticity. That is, they leave aside the students' context and reality. These types of problems do not help students learn to apply mathematical procedures in situations in their daily lives (Chamoso et al., 2014; Jiménez & Ramos, 2011).

Frequently, problem-solving is only a processing of the numerical data that appear in the statement without understanding what is being sought (Eslava et al., 2021). For Vicente et al. (2008), the most significant difficulties in students occur when working with real-world problems. In this sense, one of the most studied non-routine realistic problems, and where difficulties arise, are division problems with remainder (DWR). That is, problems in which the dividend and the divisor are integers, and the division gives rise to a non-integer result. In this type of problem, the result must be interpreted by the real-world constraints that give meaning to the problem (Verschaffel et al., 2009). According to Jiménez (2008), "realistic problems are verbal problems that describe real-life situations and that the application of an arithmetic operation does not simply lead to the solution of the problem" (p. 38). For Dewolf et al. (2014), verbal problems are an important way to bring the real world into the mathematics classroom and to teach mathematical modeling and applied problem-solving. A specific feature of realistic verbal problems is that they often do not contain all the information required to obtain a correct solution (Krawitz et al., 2018). For Payadnya (2021), realistic problems play an important role when you want to learn mathematics. That is, on the one hand students are required not only to understand the concept but also to apply the concept to solve everyday problems. Therefore, in verbal problems, the student must use real-world knowledge to give meaning and coherence to his or her answers. In that sense, the real world is the starting point where the development of mathematical concepts takes on meaning and relevance (Agustina et al., 2021).

An example of a realistic problem found in the literature was: "An army bus holds 36 soldiers. If 1128 soldiers are being transported by bus to their training site. How many buses are needed?" (Carpenter et al.,1983, as cited in Verschaffel et al., 2009). Two situations may arise in this problem. The first is related to not having difficulty in identifying division as the correct solving operation, and the second is linked to the tendency to give an incorrect answer 31.3 buses because they do not emphasize the non-integer quotient, taking into account the situation of the problem (Lago et al., 2008).

The structure of the division problems with remainder allows us to see which element of the division (remainder or the non-integer quotient) the analysis is focused on. According to Fischbein et al. (1985). They propose two intuitive problem models of division: the quotitive and the partitive. The first could also be called division by measure, which seeks to determine how many times a given quantity is contained in a larger quantity. For this type of division it is required that the dividend must be greater than the divisor. That is, it is established how many times a given quantity is contained in a larger one (Lago et al., 2008). An example of this model is shown by Zorrilla et al. (2021) "9 fans want to travel to the stadium in another city. Each cab can carry four fans. How many cabs do they need?" (p. 1322).

In the case of partitive division model we rely on what is established by Fischbein et al. (1985): "an object or collection of objects is divided into a number of equal fragments or subcollections. The dividend must be larger than the divisor; the divisor (operator) must be a whole number; the quotient must be smaller than the dividend (p. 7). Likewise, this refers to the approximation of the partitioning. That is, a given number of equivalent groups is formed to define the number of each group (Lago et al., 2008). For Buforn (2017), split-partite problems refer to problems where the number of objects per group is unknown. An example of this model is also shown by Zorrilla et al. (2021) "A dance academy has distributed in a class eight tickets for a musical. The dancers in the class were three, and all received the same number of tickets. How many tickets did each dancer receive?" (p. 1322).

It should be emphasized that these two types of division models with remainder are the ones we intend to address in the present research. The analysis we want to develop also focuses on how students interpret both the remainder and the non-integer quotient.

In that sense, interpretation plays a vital role in these division problems with remainder. That is, not only is it required that the student uses the mathematical algorithm correctly, but also that the answer makes sense with the real situation of the problem. This implies two situations: in the first one, it must be kept in mind that the existence of the remainder forces the student to recognize the value of the quotient plus one unit as a result. The remainder is not contemplated in the second, but the non-decimal quotient is based on the partition's context (Buforn, 2017; Lago et al., 2008; Parra & Rojas, 2010; Verschaffel et al., 2009).

On the other hand, further research has focused on studying the use of strategies in division problems with remainder in elementary school students (Downton, 2009; Ivars & Fernández, 2016; Sanjuán, 2021; Zorrilla et al., 2021). In that sense, Downton (2009), in his study with third-grade students (ages 8 to 9 years), evidenced the use of modeling, multiplicative thinking, repeated addition/subtraction, and counting that were employed as strategies in both division models.

For their part, Ivars and Fernández (2016) in their research with students from 6 to 12 years old, evidenced modeling and counting strategies in both division models and number fact strategies and multiplication of equal addends in division-measurement problems. In Sanjuán's study (2021) with students aged 6 to 12, strategies such as direct modeling, repeated addition, grouping, combination, and the multiplicative strategy were found in division-measure problems.

As for division-partition problems, non-anticipative, additive, and multiplicative coordination strategies were used. Finally, Zorrilla et al. (2021), in their study, coincide with the strategies observed by Ivars and Fernandez (2016), highlighting a strategy modeling with counting (graphical strategy), additive and subtractive strategies (use of successive addition-subtraction), and known number facts and those derived from addition and multiplication.

In this research, we analyzed how fourth-grade elementary school students solve division problems with quotient and remainder considering the division-quotitive and division-partitive models. With this objective, we seek to answer the following questions:

How do students interpret the quotient and the remainder in non-integer division problems? What strategies do students employ in solving division problems with remainder?

METHOD

The present research is qualitative, descriptive, and interpretative, according to Hernández (2014), since it attempts to make sense of the phenomena in terms of the meanings people give them. A questionnaire validated by a group of researchers was first applied. The participants were 50 fourth-grade elementary school students from a private school in Puebla, Mexico, whose ages ranged between 9 and 10 years old, selected by convenience. This questionnaire is made up of three non-routine problems of multiplicative structure, two division-measurement problems, and one division-partite problem (see Table 1). Verschaffel et al. (2009) propose three situations for these types of problems. The first requires the quotient to be rounded up, the second consists of rounding down, and the last suggests keeping the result of a division with the remainder as a decimal.

In this sense, Zorrilla et al. (2021) classified these situations into three types: the first implies that the presence of the remainder forces to recognize as a solution the value of the quotient plus one unit (type 1). The remainder is not considered in the second but the non-decimal quotient (type 2), and the third is the quotient plus the fractional part of the remainder (type 3). It should be noted that we will focus on type 1 and type 2 situations (see Table 1).

Questionnaire problems	Problem Characteristics
A candle vendor in the Emiliano Zapata market in Puebla, Mexico, packs 30 candles in a box. How many boxes will the vendor need if he has to pack 122 candles?	Quotitive Division. The remainder forces to recognize, as a result, the value of the quotient plus one unit Zorrilla et al. (2021) (Type 1).
A museum has given away 75 tickets to an art exhibition to 14 schools. The schools have received the same number of tickets to be distributed to their best students. How many tickets does each school receive?	Partitive Division. The remainder is not considered. The solution is the non-decimal quotient Zorrilla et al. (2021) (Type 2).
Twenty-two players of the Puebla soccer team want to travel by cab to the training sports venue. Each vehicle can carry four players. How many cabs do they need?	Quotitive Division. The remainder forces to recognize, as a result, the value of the quotient plus one-unit Zorrilla et al. (2021) (Type 1).

Table 1: Measurement and partitive division problems

Secondly, an interview was conducted with the seven selected students to understand how children construct their worlds, think, and work cognitive processes (Ginsburg, 2009). That is, to know how they solve problems and interpret the quotient and the remainder in the division-measure and division-partite problems. It should be noted that the interview was applied to seven students, which were audio-recorded and immediately transcribed for subsequent analysis and triangulation of the information. In addition, the students were assigned codes S1, S2, S3, S4, S5, S6 and S7 for the interviews' excerpts. The letter R stands for the researcher.

RESULTS AND ANALYSIS

The results presented in this section are organized into two sections. The first section presents the analysis of the responses to the division problems with residue, distinguishing their realistic and unrealistic character. The responses were classified into realistic responses with the application of division, realistic responses without the application of division, unrealistic responses applying division, and other responses. The second section presents the strategies used by the students when solving problems 1 and 3 of type 1 of the quotitive division model. The strategies evidenced are realistic responses.

Realistic responses with the application of division.

In these answers, the individual uses division appropriately and considers the realistic aspects of the problem. The solver interprets the remainder or the quotient to give a realistic answer. This response is evidenced in problem 3 of type 1 of the quotitive division model, as shown in Figure 1.

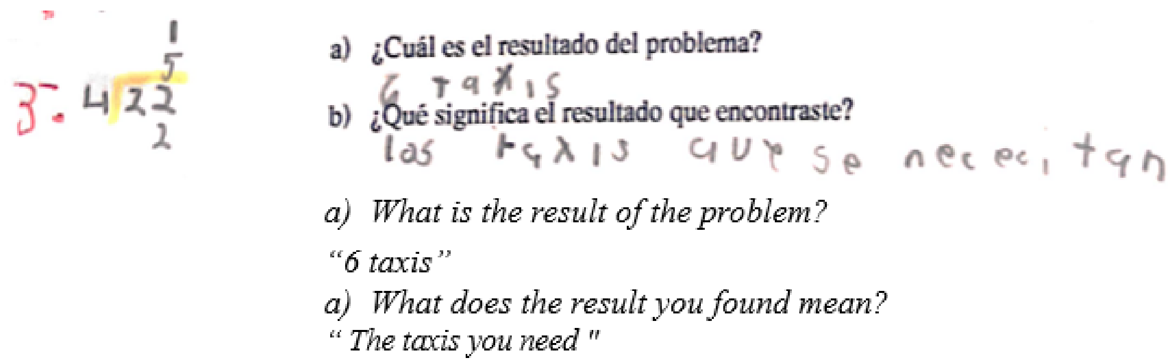


Figure 1. Realistic response to the division operation.
Source: student's response (4th grade)

In the interview, the student was asked the following questions:

R: What is the exercise asking you?

S1: How many cabs do the soccer team players need to go to the sports venue?

R: What procedure or operation did you perform there?

S1: A division.

R: You tell me that they are...

S1: 6

R: Why do you arrive at six cabs? Could you explain it to me?

S1: Because I divided: 22 by 4, it gives me 5, 2 are left, and the other two players leave in another cab. That would be six cabs. Five full cabs and another one with two players (4th-grade student, Interview excerpt, May 13, 2022).

Here the student uses division as an adequate procedure, expressing that to reach the problem result, he had to divide 22 by 4. In addition, he considers the real facts of the problem when he emphasizes that six cabs are needed to transport the 22 players to the sports venue. This is because the student understood the problem and interpreted the remainder of the division (2 players), assigning to this remainder an extra cab. He emphasizes that the division gives five as the quotient, and two are left over, expressing that there are five cabs full of players and another with two players for a total of six cabs.

Another realistic response result applying the division algorithm was evidenced in problem 2 of type 2 of the partitive division model, as shown in Figure 2.

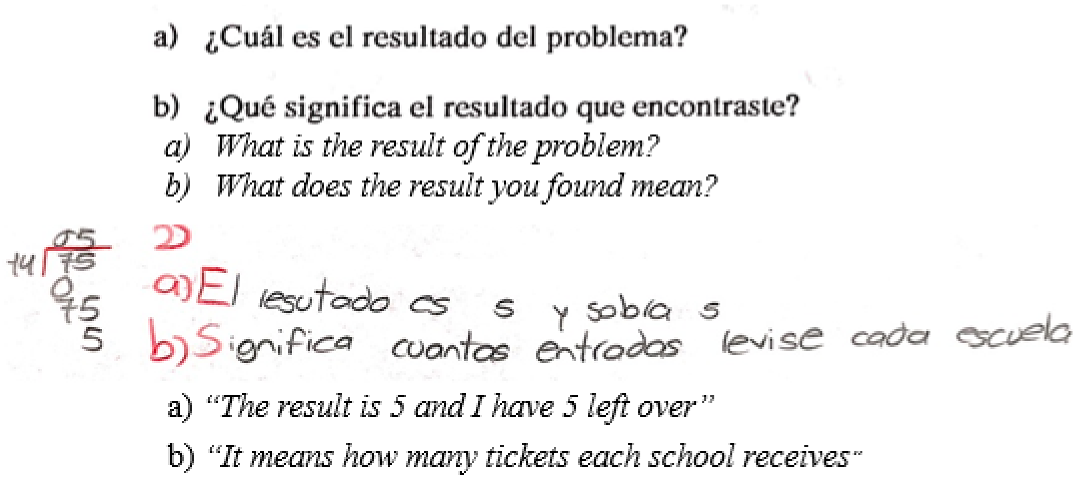


Figure 2: Realistic response to the division operation.
Source: student's response (4th grade)

In the interview, the student was asked the following questions:

R: What is the problem asking of us?

S2: A museum has given away 75 tickets to 14 schools. Each one is going to send its best students. So I divided to find out how many tickets each school gets.

R: Do you think you can do another procedure here?

S2: A sum could be

R: Can you explain how you would use addition in this problem?

S2: Adding until I get a result, but not more than the number of tickets we were given, and putting the remainder as a remainder.

R: Ok, so how many tickets does each school get?

S2: Five tickets

R: Ok, what does this residue mean?

S2: The leftover tickets

R: Would it be convenient to distribute those five tickets among the fourteen schools?

S2: No because it would be unfair

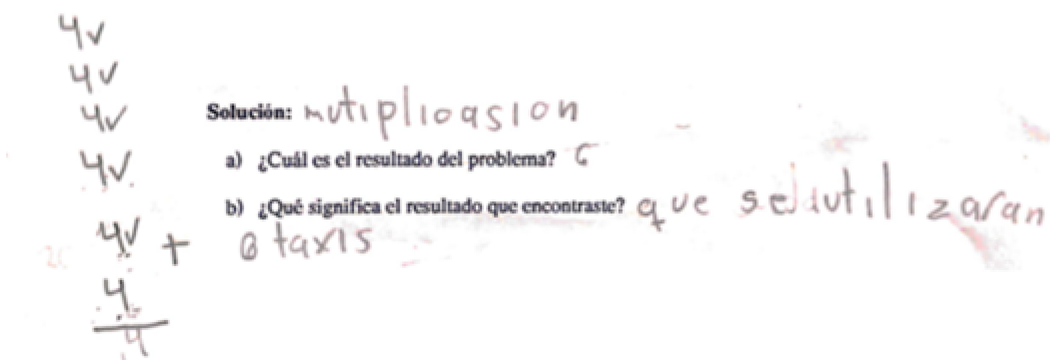
R: What would be the most convenient thing to do with those five tickets?

S2: Give them as a gift to someone else. (4th-grade student, Interview excerpt, May 13, 2022).

Here the student applies the division algorithm as a suitable procedure to answer the problem. In addition, he expresses that the repetitive addition would be another procedure that could be applied as long as it did not exceed the total number of entries and that the leftover entries would be the remainder. She also states that she interpreted the remainder as the five entries that were left over and the quotient as the number of entries that should be distributed to the fourteen schools equally (five entries for each school). In other words, the student interpreted the numerical answers correctly (context of the distribution) and successfully solved the problem.

Realistic responses without the application of division.

These are responses where the subject uses arithmetic operations other than division and manages to interpret the real situation of the problem. That is, the solver uses addition and multiplication as an adequate procedure. Likewise, they interpret the reality or situation described in the problem to solve it. This situation was evidenced in problem 3 of type 1 of the quotitive division model, as shown in Figure 3.



Solution: "Multiplication"

a) What is the result of the problem? "6"

b) What does the result you found mean? "That 6 taxis will be used"

Figure 3: Realistic responses without the application of division.
Source: student's response (4th grade)

In the interview, the student was asked the following questions:

R: What is this problem asking you?

S3: I am being asked if 22 players of the soccer team want to travel in cabs to the training sports venue. Each vehicle can only carry four players. How many cabs are needed? I did multiplication.

R: Using multiplication, how did you arrive at the answer of 6 cabs?

S3: If each cab can carry 4, because 4 times... no I had to divide 22 by 4 and I got six.

R: Explain to me why you got the answer six cabs?

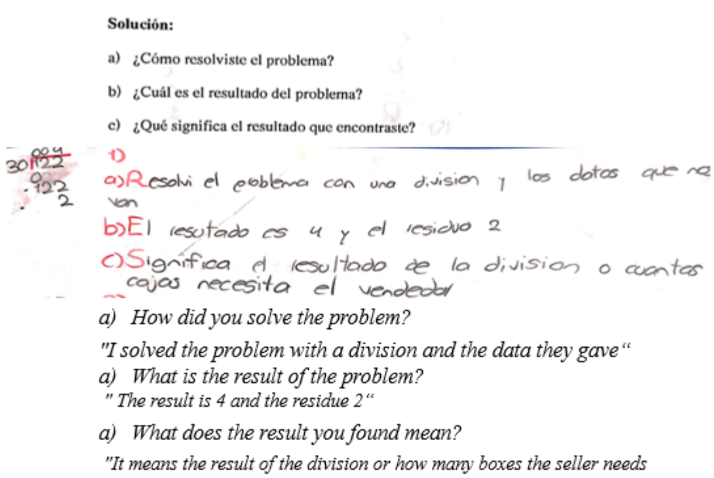
S3: The four players that fit in the cab I added them until I got 22, I also multiplied 4 x 4=16, 4 x 5=20, 4 x 6=24, now 4 x 5=20, the players are complete in the cab, and then I have two players left over.

R: What happens to those two players left over?

S3: They can go in a sixth cab, then they are 4 x 6= 24, but it happens, I know I have only 22 players, but the two missing players can go in a cab that fits 4, and nothing happens (4th-grade student, Interview excerpt, May 13, 2022).

Here the student used addition and repetitive multiplication as the appropriate procedure to answer the problem and gave it the correct interpretation. However, at one point in the questioning, he manages to reflect by stating that he has divided. However, when he begins to explain what he has done, he seems to have used addition and multiplication. That is, the student expresses that to arrive at the answer of 6 cabs, he added four by four until he got 22 players. In explaining how he performed the multiplication, he emphasizes that the product of 4×5=20 represents the players that can travel in the five cabs with four people. He also considered that the two surplus players could travel in a sixth cab, considering the multiplication of 4×6=24, although it exceeds the total of 22 players that must be transported.

Unrealistic responses applying division. Although using division as an appropriate arithmetic operation, the answers given by the subject did not take into account the real part of the problem. The solver does not interpret the division's elements (the quotient and the remainder). An example of this unrealistic response to problem 1 of type 1 is shown in Figure 4.


Solución:

a) ¿Cómo resolviste el problema?

b) ¿Cuál es el resultado del problema?

c) ¿Qué significa el resultado que encontraste?

1)
a) Resolvi el problema con una division y los datos que me dan
b) El resultado es 4 y el residuo 2
c) Significa el resultado de la division o cuantas cajas necesita el vendedor

a) How did you solve the problem?

"I solved the problem with a division and the data they gave"

a) What is the result of the problem?

" The result is 4 and the residue 2"

a) What does the result you found mean?

"It means the result of the division or how many boxes the seller needs

Figure 4: Unrealistic responses applying division
Source: student's response (4th grade)

Here the student divides 122 candles among the four boxes, obtaining; as a result, four boxes in the quotient and two candles as the remainder. Taking into account the above in the interview, he was asked the following questions:

R: In the first problem: What was I asking you to do?

S4: Divide how many boxes the salesman will need to put the candles in, and I already had to divide and get the result for the questions you were asking me.

R: So for this problem, you divided, right?

S4: Yes

R: What's the answer to this question? How many boxes will the salesperson need?

S4: Five boxes, right?

R: Five boxes? Why?

S4: Ah no. Four boxes.

R: And those two that are loose, what happens to them?

S4: They can't go because the boxes are already full. Yes, they could fit, but they would not be well arranged and out of the box.

R: What other option would you give?

S4: Buy another box

R: Apart from the division, do you think you could apply another procedure?

S4: No.

R: Then how many boxes would you need to ship the candles in total?

S4: 5 (4th-grade student, Interview excerpt, May 13, 2022).

In this interview excerpt, the student expresses that she used division to solve the problem. Furthermore, although she correctly applied the division procedure, she could not interpret the remainder (the two leftover candles). That is, the student did not consider the problem's real part. When answering, the student forgets the problem's text and takes the one found with the division algorithm as the correct answer, thus generating an incorrect result. It is worth noting that at one point in the interview, the student reflected that in order to pack the two leftover candles, an additional box was needed. That is, concluding that five boxes were needed.

Other responses.

The solver's responses in this situation are influenced by an inadequate procedure or calculation of the arithmetic operation. In addition, there is no interpretation of the result. An example of these responses was evidenced in problem 1 of type 1 in the quotitive division model, as shown in Figure 5.

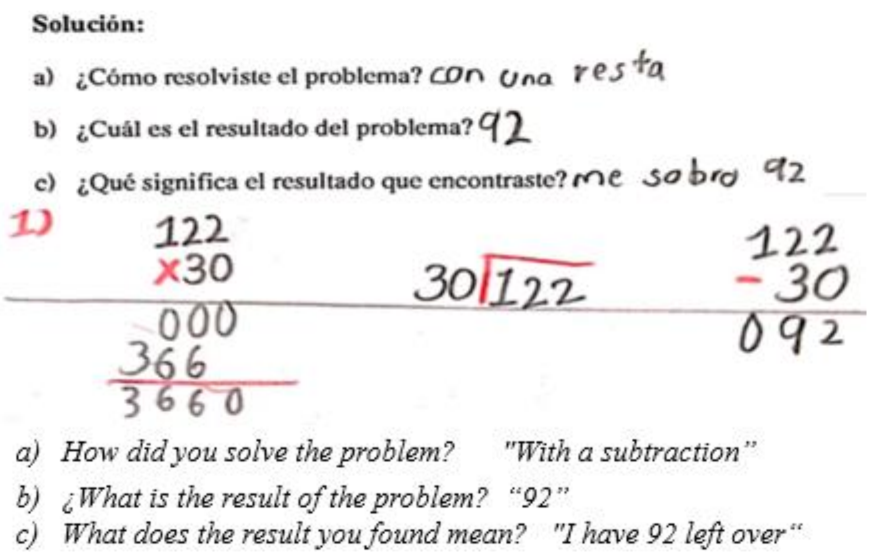


Figure 5: Other responses
Source: student's response (4th grade)

Here the student applies different arithmetic operations without arriving at a reasonable answer to the problem in question. The student performed arithmetic operations: from the application of subtraction and multiplication to division. However, there was no success in solving the problem.

In the interview, the student was asked the following questions:

R: In the first problem, what is he telling you, or what data is he giving you?

S5: In the Emiliano Zapata market in Puebla, Mexico, a candle vendor packs 30 candles in a box. How many boxes will the vendor need if he has to pack 122 candles?

R: What is the problem asking you?

S5: How many boxes will the seller need?

R: What procedure do you think should be done for this problem?

S5: A...

R: Here you said a subtraction. What should be subtracted?

S5: One hundred twenty-two minus thirty

R: Minus 30, which is what you did here.

S5: Yes.

R: But here I see you were going to divide. Why didn't you do that?

S5: I couldn't, so I tried subtraction, giving me a result I thought it would be.

R: Why did you do multiplication?

S5: Because I was starting from multiplication to subtraction, then division, subtraction, multiplication, and addition [A confusing sentence, and the investigator did not go deeper].

R: And why did you stick with the subtraction result?

S5: Because I feel that they are the boxes that are going to accommodate 122 candles.

R: Ok, well, do you think that doing multiplication, division, or addition can also give me the correct result?

S5: No, because multiplication will be a more significant number, the division will be a bigger number, and the addition will be a bigger number.

R: Ok, then it's the subtraction. (4th-grade student, Interview excerpt, May 13, 2022).

In this interview excerpt, the student is convinced that the correct procedure is subtraction. In addition, he justifies that with this answer, the 122 candles can be accommodated and that multiplication, division, or addition would be a more significant number. In this problem, there is the belief that multiplication, addition, and division imply a more significant number and subtraction a minor number without considering what the problem is posing. In this case, the student shows an incorrect understanding of the problem and a lack of clarity when choosing the correct operation. The above could be why students tend to focus on superficial aspects of the problem statement and select inappropriate solution procedures

Strategies used in realistic responses

Graphic strategy. According to Zorrilla et al. (2021), this type of strategy occurs when the subjects offer a solution in which they use a drawing or diagram to solve the problem. For Matalliotaki (2012), drawings are one of how children express a complex phenomenon by facilitating the expression of the spatial relationships of objects. In that sense, the representation created by the students is helpful in interpreting the result, keeping in mind the real part of the problem. An example of this strategy is shown in Figure 6. The student drew the cabs needed to take the players, evidencing that the student is aware of the meaning of the rest.

Figure 6: Example of graphical strategy-counting in realistic responses.
Source: student's response (4th grade)

In the interview, the student was asked the following questions:

R: What does the picture you drew in this problem mean?

S6: The cab that takes the players

R: Well, explain to me what does this drawing consist of? Why do you place six cabs?

S6: Because there are 22 players

R: Ok, how many players are in each cab?

S6: Four players

R: Why do you put two players in the last cab?

S6: Because they are 1,2,3,4,5,6,7,7,8,910,11,12,13,14,15,16,17,18,18,19,20,21 and 22.

R: ok

S6: And here, fit two because they are all complete of 4.

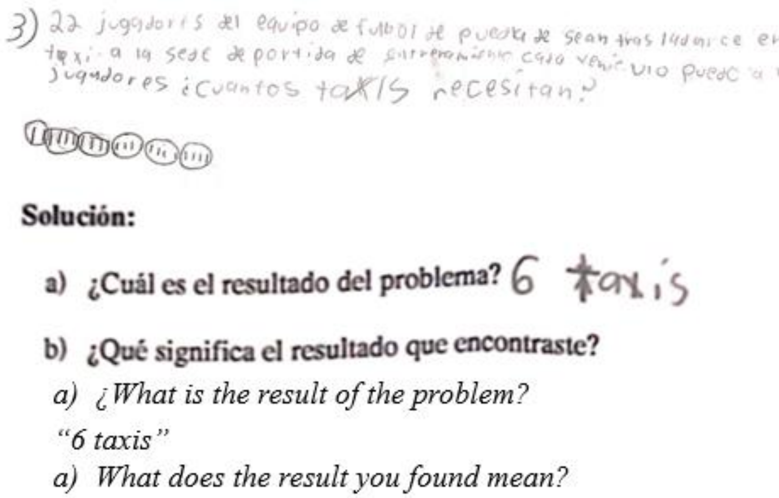
R: Ok, they are complete of what?

S6: Of four

R: Does it matter that the last cab is of two players?
S6: It doesn't matter (4th-grade student, Interview excerpt, May 13, 2022).

Here the student employs the graphing and counting strategy to represent the solution to the problem. In the first instance, the graph made by the student illustrates the number of players with the total number of cabs that can be used to transport the 22 players. For this strategy, the student keeps in mind that the total number of players who can go in a cab is 4. The counting strategy is evidenced when he started counting from 1 to 22, concluding that a sixth cab was used. In addition, he considered that they completed five cabs with 20 players, and the remaining two could travel in a different cab.

Another realistic response where the graphing and counting strategy was used, as evidenced in problem 3 of type 1 of the quotitive division model, as shown in Figure 7.


3) 22 jugadores del equipo de futbol de puebla se van tras ladarse en taxi a la sede deportida de entrenamiento cada vehiculo puede a 4 jugadores ¿cuantos taxis necesitan?

Solución:

a) ¿Cuál es el resultado del problema? 6 taxis

b) ¿Qué significa el resultado que encontraste?

a) ¿What is the result of the problem?

"6 taxis"

a) What does the result you found mean?

Figure 7: Example of graphical strategy-counting in realistic responses.
Source: student's response (4th grade)

In the interview, the student was asked the following questions:

R: In the third problem, what is the problem asking us, what data does it give us?

S7: 22 players of the Puebla soccer team want to travel by cab to the training sports venue. Each vehicle can carry 4 players. How many cabs do they need?

R: What is the problem asking you?

S7: How many cabs do you need to go to the stadium?

R: How many cabs do you need?

S7: Six cabs

R: How did you arrive at that answer?

S7: By drawing

R: What is the meaning of this drawing you made here?

S7: I put twenty-two lines and enclosed in fours.

R: Ok, you enclosed four by four, and in this last one how many were left?

S7: There are two

R: How many players are in this last cab?

S7: Two players

R: So how many cabs are needed?

S7: Six cabs (4th-grade student, Interview excerpt, May 13, 2022).

In this problem the E7 answers the question How did you solve the problem? giving as answer "6 taxis" and the second question What does the result you found mean? The student did not submit any response. First, the student used a graphic strategy or drawing to interpret and solve the problem. In addition, during the questioning, it was possible to identify that in addition to using the drawing, the student made a count by enclosing the lines representing the 22 players, four by four. In this problem, the student formed six groups representing the six cabs where the players should be transported, specifying that there are five groups with four players and one group with two players for a total of 22.

In summary, the results presented are a sample of how students interpret the remainder and the quotient in realistic problems. In that sense, in problems 1 and 2 (See Figures 1 and 2), students show a realistic response to having the division algorithm as a procedure. In these division models, students correctly interpret the result based on the reality of the problem. In the first instance, E1 interprets the remainder as adding one unit to the quotient. On the other hand, S2 interprets the context of division as a non-decimal result without emphasizing the remainder. These results disagree with those obtained by Zorrilla et al. (2021), who state that the algorithm's application does not result in a realistic response.

On the other hand, in some cases, when the student faces a realistic problem, it seems that he/she uses the arithmetic operations he/she handles correctly and uses it to solve the situation. In our study, the written production of S3 (See Figure 3), the procedure did not focus on the division algorithm but on multiplication and addition, thus generating a realistic response.

On the other hand, using division as an adequate procedure does not guarantee the student a correct interpretation of the result (See Figure 4). Rodríguez et al. (2009) pointed out that the choice of division as a solution procedure did not rule out inadequate numerical results. The division algorithm used in problem one by S4 was performed correctly; however, the difficulty was evidenced when associating the answer with the context of the problem. Galvão & Labres (2006) mention that children do not consider the remainder as a component of the division related to the other components. Another difficulty evidenced was considering the result of subtraction as appropriate without first interpreting problem 1 (quotient plus the unit) simply by considering that the result of multiplication, division, and addition is associated with a more significant number (See Figure 4). Jiménez et al. (2011) state that it is striking that, in order to solve problems with an apparently additive structure, many of the errors were produced by the application of subtraction, multiplication, division, or the combination of two arithmetic operations. That is to say, when solving a realistic problem; children tend to have difficulty selecting the arithmetic operation with which they intend to operate and making sense of the answer.

On the other hand, this study is a sample of the different resolution strategies that students use to give meaning and interpretation to the result of the problem. An example is a graphic and counting strategy, where S6 and S7 made a drawing to represent the situation of the problem. In problem three, the lines and dolls represent the players, and the circles and cars are the vehicles that will be used to move. In this representation, the student performs a count in rounds or consecutively. These strategies were also evidenced in research (e.g., Downton, 2009; Ivars & Fernández, 2016; Sanjuán, 2021; Zorrilla et al., 2021).

DISCUSSION

In the answers given by the students in problem 3 of type 1 as shown in Figures 1, 3, 6 and 7, the use of the algorithm of division, multiplication, and the fact of including drawings as a graphic strategy and repetitive counting are evident. These heuristic problem-solving strategies allowed students S1, S2, S6, and S7 to arrive at the correct solution and interpret the remainder or residue in terms of the actual situation or circumstance of the problem. In the case of using the division algorithm properly, Lago et al. (2008), pose that students' answers when applying correct resolution procedures are usually accompanied by correct interpretations. This is evident if we consider both the division model and the types of subtraction.

In the case where the student uses multiplication as a calculation to solve the problem, Downton (2008) asserts that young children can solve complex division problems when provided with a problem-solving learning environment that encourages them to draw on their intuitive thinking strategies and knowledge of multiplication. On the other hand, the fact of including graphic strategies such as drawing and performing successive counts, allowed the students to solve the problems. In this sense, Ivars y Fernández (2016), state that the student graphically represents the sharing process and counts the elements a group has. Likewise, using graphical strategies to represent the problem facilitates students to identify the structure underlying the problem (Santos, 1997, cited in Zorrilla et al., 2021).

In problem 2 of type 2 (See Figure 2) S2 interpreted the numerical answers correctly (context of the distribution) successfully achieving the solution of the problem. This result agrees with that obtained in Lago et al. (2008), where in this type of problem (non-decimal quotient), students always interpret the numerical answer with a high percentage of correct interpretations. However, it disagrees with the findings of Zorrilla et al. (2021), wherein in non-decimal quotient problems, realistic responses decrease from 81.4% to 62.5% between fourth and fifth grade. This decrease coincides with the increase of unrealistic responses in fifth grade, in which students solve the problems by giving the decimal quotient as the solution without considering the distribution context.

In the case of problem 1 type 1 as shown in Figures 4 and 5, on the one hand S4 applies procedure or algorithm of division properly, however it does not interpret the rest in terms of the actual situation of the problem. That is, the student forgets the text of the problem and gives as a correct answer the one found with the division algorithm, thus generating an incorrect result. For Verschaffel et al. (2009), students' weak performance in DWR problem solving is because they provide many mathematically correct but situationally inappropriate answers. Likewise, students present difficulties with problem situations about division. In addition, they require the activation of realistic considerations and sense-making to give an adequate interpretation of a non-integer quotient. Furthermore, Galvão & Labres (2006) posit that, children do not realize the meaning of the remainder in solution processes when it comes to divisions. In some cases, the remainder is seen as something superficial and not part of the problem's interpretation.

İncikabıa et al. (2020) showed that most students using the division algorithm successfully applied the operation steps but had difficulty interpreting the remainder. This result is interpreted from a clause of the didactic contract called formal delegation proposed by D'Amore and Martini (1997). According to the authors, solving a school problem coincides with finding the most appropriate operations, i.e., interpreting the text arithmetically and moving from natural language to arithmetic expression. The result of this operation is interpreted as the answer to the problem. At the end of this phase, the solver forgets the text and focuses on solving the operation.

Finally, in the S5 case at the time of solving problem 1 fails to understand the problem or identify the appropriate heuristic resolution strategy to solve it. That is, the fact of not understanding the problem leads the student to perform different arithmetic operations using trial and error but without reaching the correct solution. For Inoue (2005), students execute arithmetic operations without thinking, without evaluating their actions about common sense understanding of real-life practices. In that sense, Cooper & Harries (2005), when faced with a contextualized division with remainder, students fail to identify and use the appropriate operations. Likewise, S5 considers that multiplication, addition and division imply a larger number and for subtraction a smaller number without taking into account the understanding of the problem. For Jiménez and Ramos (2011), the incorrect beliefs generated by the didactic contract in the classroom seem to be responsible for the greater or lesser difficulty in solving the problems. In that sense, Parra and Rojas (2010) found that students perform any operation, following the rules learned in the solution of school arithmetic problems, in which the important thing seems to be to identify numerical data and use an arithmetic operation to give a numerical answer. In addition, As Rodríguez et al. (2009) state, students usually misinterpret problems from the beginning, and this initial error guides the entire solution process, including the interpretation stage. Also, students' difficulties in DWR problems did not seem to stem from the lack of interpretation of the correct numerical answer but the initial misunderstanding of the problems.

CONCLUSIONS

In elementary school, division problems with remainder (DWR) have been a source of conflicts or difficulties for students in finding the appropriate algorithm and making real sense of the problem. Likewise, the lack of contextualized problems in the classroom means that students are only prepared to solve routine problems. Lagos et al. (2008) argue that mathematical concepts must be perceived as useful in real life and that mathematics classes should favor understanding and reflection. For Inoue (2005), real-life knowledge plays an important role in mathematical thinking. In this sense, it is necessary for teachers' lesson plans to include this type of division problem with remainder and encourage students to use different resolution strategies.

In the problems of division with rest raised in the present research, it would be expected that the students of 4th grade of basic primary would solve these problems taking into account the following aspects: Read the problem carefully in order to understand it. That is, here the student must keep in mind the data or information provided by the problem and therefore the question that arises and the situation or circumstances that the problem presents. In addition, the student must create or design a path for resolution of the same, including heuristic resolution strategies. For example, mental calculations, trial and error, making a representation, outline or diagram, making a table and illustrations or drawings. Once this plan is executed and an answer to this problem is available, the student must relate the situation described in the problem with his reality. That is, the fact of relating it to your daily life will allow you to understand it better.

In that sense, teachers in their classrooms could present their students with different heuristic resolution strategies, making explicit the role played by the quotient and the rest in problems of this type. It is also suggested that students discuss among themselves and with the teacher the different meanings and interpretations that arise when solving this type of problems, specifically those of division with rest. The above could be useful to make it clear to the student that the realization of the algorithm in a strict way without making use of the interpretation of the result and reality is not always correct.

However, the analyzed results show that even though the student uses the division algorithm and correctly interprets the problem, it is still evident that some students use this algorithm without interpreting reality. It should also be noted that using arithmetic operations is not the only way the student uses to solve the problem. This is due to the use of strategies, such as the graphic strategy combined with skip counting or successive counting, where the student shows an answer coherent with the real part of the problem.

A possible limitation of our investigation would be the fact that we have not considered division problems in which, both in the dividend and in the divisor the figure of the units is zero. In the present work, problems related to the division with rest (DWR) were addressed. An example of this could be the following problem: "In a stationery store the employee needs to store 130 pencils in boxes with capacity for 20. How many boxes will you need?" One of the strategies that a student could have used would be to omit the zeros that appear in the quantities and therefore consider that the quotient and the rest would be respectively 6 and 1, showing with it the attachment to the algorithm (formal delegate) (D'Amore, 2006). However, the interpretation of the rest in realistic terms would be that the remaining 10 pencils would merit the use of an additional box, and therefore, the answer to the problem would not be 6 but 7 boxes.

Finally, it is suggested to present students with these types of problems to develop different strategies according to the context and the meaning given to the problem's solution. We consider that this would help them to give meaning to the procedures with arithmetic operations. For Ivars and Fernández (2016), giving students opportunities to solve problems using their strategies allows them to propitiate the confluence of their abilities with more formal approaches.

REFERENCES

[1] Agustina, E. N. S., Widadah, S., & Nisa, P. A. (2021). Developing Realistic Mathematics Problems Based on Sidoarjo Local Wisdom. *Mathematics Teaching Research Journal*, *13*(4), 181-201. https://commons.hostos.cuny.edu/mtrj/archives/volume-13-n-4/

[2] Buforn, Á., Zorrilla, C., & Fernández-Verdú, C. (2017). Un experimento de enseñanza: Mirar profesionalmente el pensamiento de los estudiantes cuando resuelven problemas realistas. En Roig-Vila (Eds). Investigación en docencia universitaria. *Diseñando el futuro a partir de la innovación educativa*. (pp. 88-96). Octaedro. http://rua.ua.es/dspace/handle/10045/71093

[3] Cooper, B., & Harries, T. (2005) Making sense of realistic word problems: portraying working class 'failure' on a division with remainder problem, *International Journal of Research & Method in Education*, *28*(2), 147-169, https://doi.org/10.1080/01406720500256228

[4] Chamoso, M. J., Vicente, S., E. Manchado, E., D. & Múñez, D. (2014). Los Problemas de Matemáticas Escolares de Primaria, ¿son solo Problemas para el aula? *Cuadernos de Investigación y Formación en Educación Matemática*, *12*, 261-279. https://revistas.ucr.ac.cr/index.php/cifem/issue/archive

[5] D´Amore, B. (2006). *Didáctica de la Matemática*. Magisterio Editorial.

[6] D´Amore, B., D'Amore, Bruno; Martini, Berta (1997). Contrato didáctico, modelos mentales y modelos intuitivos en la resolución de problemas escolares típicos. *Números, Revista de Didáctica de las Matemáticas*, *32*(3), pp. 26-42. http://funes.uniandes.edu.co/3172/

[7] Dewolf, T., Van Dooren, W., Cimen, E., & Verschaffel, L. (2014) The Impact of Illustrations and Warnings on Solving Mathematical Word Problems Realistically, *The Journal of Experimental Education*, *82*(1), 103-120, https://doi.org/10.1080/00220973.2012.745468

[8] Downton, A. (2008). Links Between Children's Understanding of Multiplication and Solution Strategies For Division. In M. Goos, R. Brown, & K. Makar (Eds.), *Proceedings of the 31st Annual Conference of the Mathematics Education Research Group of Australasia*, (pp. 171-178). MERGA Inc. https://www.merga.net.au/Public/Publications/Annual_Conference_Proceedings/2008_MERGA_CP.aspx

[9] Downton, A. (2009). It seems to matters not whether it is partitive or quotitive division when solving one step division problems. In R. Hunter, B. Bicknell & T. Burgess (Eds.), Crossing divides: *Proceedings of the 32nd annual conference of the Mathematics Education: Research Group of Australasia*, (pp. 161-168). MERGA Inc.https://www.merga.net.au/Public/Public/Publications/Annual_Conference_Proceedings/2009_MERGA_CP.aspx

[10] Eslava, M., Baltazar, E., & Escobedo, Y. (2021). *Resolución de problemas (Polya) Segundo Grado de Primaria, investigación en posgrado* (ENSEM). [Conferencia] 4º Congreso Nacional de Investigación sobre Educación Normal, Hermosillo, México. https://conisen.mx/Memorias-4to-conisen/index.html

[11] Fischbein, E., Deri, M., Nello, M. S., & Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education*, *16*(1), 3-17. https://doi.org/10.2307/748969

[12] Galvão & Labres, S. (2006). Exploring the role played by the remainder in the solution of division problems. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education* (pp. 153-160). Charles University, Faculty of Education. https://eric.ed.gov/?id=ED496932

[13] Ginsburg, H.P. (2009). *Entering the child's mind: the clinical interview in psychological research and practice.* Cambridge University. https://doi.org/10.1017/CBO9780511527777

[14] Hernández, R. (2014). *Metodología de la investigación*. Industria Editorial Mexicana.

[15] İncikabı, L., Ayanoğlub, P., & Uysal, R. (2020). Sixth-Grade Students' Procedural and Conceptual Understandings of Division Operation in a Real-Life Context. *International Electronic Journal of Elementary Education*, *13*(1), 35-45 https://eric.ed.gov/?id=EJ1272662

[16] Ivars, P. & Fernández, C. (2016). Problemas de estructura multiplicativa: evolución de niveles de éxito y estrategias en estudiantes de 6 a 12 años. *Educación Matemática*, *28*(1), 9-38. https://www.revista-educacion-matematica.org.mx/revista/2016/05/12/vol28-1-1/

[17] Jiménez, L. (2008). *La activación del conocimiento real en la resolución de problemas: un estudio evolutivo sobre los problemas no-rutinarios de adición* [Tesis Doctoral, Universidad Complutense, Madrid].https://dialnet.unirioja.es/servlet/tesis?codigo=93611

[18] Jiménez, L., & Ramos, F. J. (2011). El impacto negativo del contrato didáctico en la resolución realista de problemas. Un estudio con alumnos de 2º y 3º de educación primaria, *Electronic Journal of Research in Educational Psychology*, *9*(25), 1155-1182. https://doi.org/10.25115/ejrep.v9i25.1499

[19] Krawitz, J., Schukajlow, S & Van Dooren, W. (2018) Unrealistic responses to realistic problems with missing information: what are important barriers?, *Educational Psychology*, *38*(10), 1221-1238, https://doi.org/10.1080/01443410.2018.1502413

[20] Matalliotaki, E. (2012) Resolution of division problems by young children: what are children capable of and under which conditions?, *European Early Childhood Education Research Journal*, *20*(2), 283-299, https://doi.org/10.1080/1350293X.2012.681132

[21] Márquez, M., Arredondo, E. H. & García-García, J. I. (2019). Estrategias en la resolución de problemas de división-medida por estudiantes de séptimo básico en Chile. *Espacios*, *40*(33), 10-24. https://www.revistaespacios.com/a19v40n33/19403310.html

[22] Lago, M., Rodríguez, P., Enesco, I., Jiménez, L., & Dopico, C. (2008). Me sobran cuatro y no sé qué hacer con ellos. Un estudio sobre los problemas de división con resto en alumnos de 1º de ESO. *Anales de psicología*, *24*(2), 201-212. https://revistas.um.es/analesps/article/view/42721

[23] Organización de las Naciones Unidas para la Educación, la Ciencia y la Cultura y la Oficina Regional de Educación para América Latina y el Caribe (UNESCO, 2020). *Análisis curricular Estudio Regional Comparativo y Explicativo (ERCE 2019) México. Documento nacional de resultados.*https://es.unesco.org/news/resultados-analisis-curricular

[24] Payadnya, I. P. A. A., Suwija, I. K., & Wibawa, K. A. (2021). Analysis of Students' Abilities in Solving Realistic Mathematics Problems Using "What-If"-Ethnomathematics

Instruments. *Mathematics Teaching Research Journal*, *13*(4),131-149. https://commons.hostos.cuny.edu/mtrj/archives/volume-13-n-4/

[25] Parra, D. & Rojas, C. (2010). Las Interpretaciones de las Soluciones en Problemas Reales. Cuando los Estudiantes solo Calculan [Conferencia]. *Encuentro Colombiano de Matemática Educativa*, Bogotá, Colombia. http://funes.uniandes.edu.co/1097/

[26] Rodríguez, P., Lago, M. O., Hernández, M. L., Jiménez, L., Guerrero, S. & Caballero, S. (2009). How do secondary students approach different types of division with remainder situations? Some evidence from Spain. *European Journal of Psychology of Education*, *24*(4), 529-543. https://www.jstor.org/stable/23421669

[27] Sanjuán, E. (2021). Estrategias de resolución de problemas de división-medida y división reparto con fracciones por estudiantes de Educación Primaria. *Edma 0-6: Educación Matemática en la Infancia*, *10*(1), 77-9. https://www.edma0-6.es/index.php/edma0-6/article/view/166

[28] Vicente, S., Van Dooren, W., & Verschaffel, L. (2008). Utilizar las matemáticas para resolver problemas reales. *Cultura y Educación*, *20*(4), 391-406. https://doi.org/10.1174/113564008786542235

[29] Verschaffel, L., Van Dooren, W., Chen, L., & Stessens, K. (2009). The Relationship between posing and solving division-with-remainder problems among Flemish upper elementary school children. In L. Verschaffel, B. Greer, W. Van Dooren y S. Mukhopadhyay (Eds.), *Words and Worlds*. (pp. 143-160.) https://doi.org/10.1163/9789087909383_010

[30] Zorrilla, C., Ivars, P., & Fernández, C. (2021). Problemas realistas de división con resto Un estudio sobre las estrategias en educación primaria. *Revista Mexicana de Investigación Educativa*, *26*(91), 1313-1339. https://rua.ua.es/dspace/handle/10045/118463

The Problem Corner

Ivan Retamoso, PhD, *The Problem Corner* Editor

Borough of Manhattan Community College

iretamoso@bmcc.cuny.edu

The Purpose of **The Problem Corner** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Hello Problem Solvers, I got solutions to **Problem 8** and to **Problem 9** and I am happy to inform that they were correct, interesting, and ingenuous. By posting what I considered to be the best solutions, I hope to enrich and enhance the mathematical knowledge of our international community.

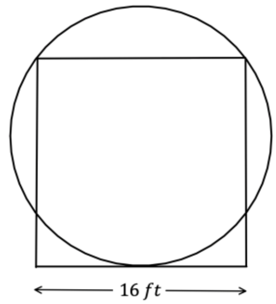
Solutions to **Problems** from the Previous Issue

Interesting Geometry problem.

Proposed by Ivan Retamoso from Borough of Manhattan Community College, City university of New York, USA

Problem 8

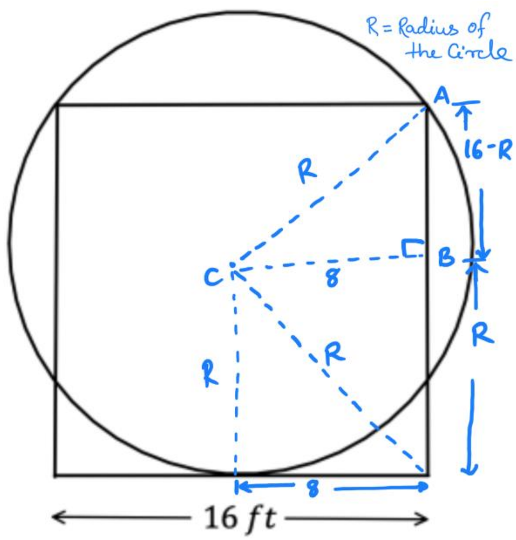
In the figure below find the ratio between the perimeter of the circle (circumference) and the perimeter of the square in exact form.



Solution to problem 8

By Aradhana Kumari, Borough of Manhattan Community College, City university of New York, USA.

This clearly explained solution utilizes the theorem of Pythagoras as a tool to find the radius of the circle, which in turn serves to compute the requested ratio.



Consider the diagram below.

Let C be the center of the circle.

Step1. We want to first find the radius of the circle.

Applying Pythagorean theorem to right triangle CBA, we have

(Length of CB)^2 + (length of BA)^2 = (length of CA)^2

$(16-R)^2 + 8^2 = R^2$

$256 + R^2 - 32\,R + 64 = R^2$

Hence

$320 = 32\,R$

$R = \frac{320}{32} = 10$

Hence the circumference of the given circle $= 2\,\pi \times R = 2\pi \times 10 = 20\,\pi$

The perimeter of the given square is $4 \times$ length of the square $= 4 \times 16 = 64$

Therefore

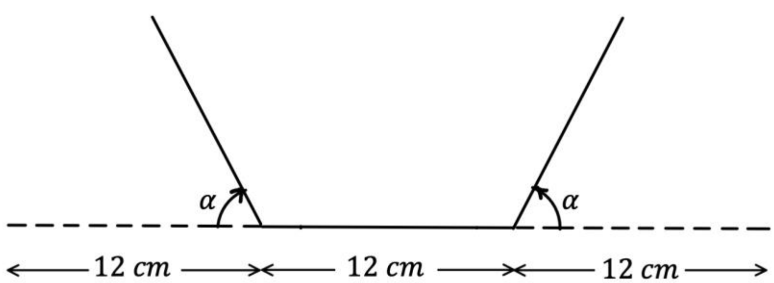
$$\frac{\text{circumference of the given circle}}{\text{perimeter of the given square}} = \frac{20\,\pi}{64} = \frac{5\pi}{16}$$

Interesting Applied Optimization problem.

Problem 9

Proposed by Ivan Retamoso, BMCC, USA.

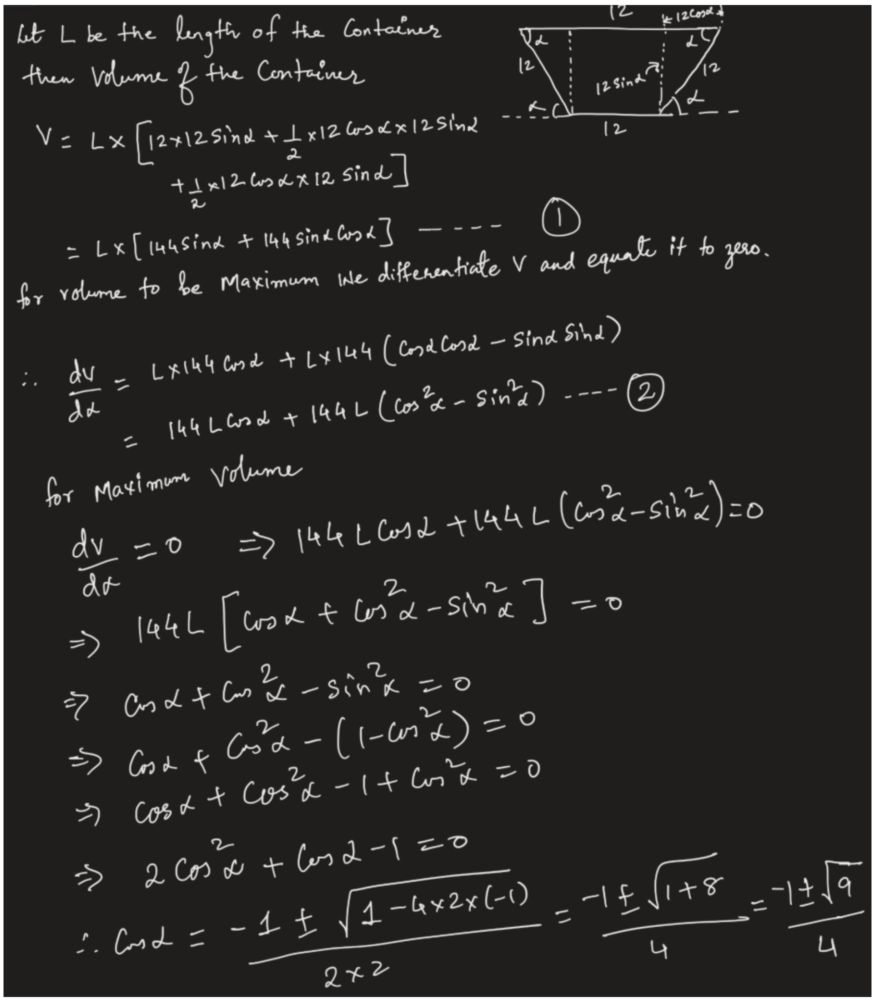
It is needed to construct a rain gutter from a metal sheet of width $36\ cm$ by bending up one-third of the sheet on each side through an angle α. Find the value of α such that the gutter will carry the maximum amount of water.



Solution to problem 9

By Aradhana Kumari, Borough of Manhattan Community College, City university of New York, USA.

Sometimes the solution to a problem is independent of some measurements, when we attempt to solve the problem we feel that information is missing, other times we discover that some given *quantities are not relevant for the final solution of a problem, often we read, as part of the solution "without loss of generality, let a measure be this or that..", problems like this, in my opinion, are important because the solution can be generalized to an arbitrary context. The solution presented below is a great example of how to solve this type of problems, additionally the solver provided a picture that help to visualize the construction of the gutter with the "perfect" angle for the rain.*



Let L be the length of the Container
then Volume of the Container

$$V = L \times \left[12\times 12\sin\alpha + \frac{1}{2}\times 12\cos\alpha \times 12\sin\alpha + \frac{1}{2}\times 12\cos\alpha \times 12\sin\alpha\right]$$

$$= L \times [144\sin\alpha + 144\sin\alpha\cos\alpha] \quad \text{---- (1)}$$

for volume to be Maximum we differentiate V and equate it to zero.

$$\therefore \frac{dV}{d\alpha} = L\times 144\cos\alpha + L\times 144(\cos\alpha\cos\alpha - \sin\alpha\sin\alpha)$$

$$= 144L\cos\alpha + 144L(\cos^2\alpha - \sin^2\alpha) \quad \text{---- (2)}$$

for Maximum volume

$$\frac{dV}{d\alpha} = 0 \Rightarrow 144L\cos\alpha + 144L(\cos^2\alpha - \sin^2\alpha) = 0$$

$$\Rightarrow 144L[\cos\alpha + \cos^2\alpha - \sin^2\alpha] = 0$$

$$\Rightarrow \cos\alpha + \cos^2\alpha - \sin^2\alpha = 0$$

$$\Rightarrow \cos\alpha + \cos^2\alpha - (1 - \cos^2\alpha) = 0$$

$$\Rightarrow \cos\alpha + \cos^2\alpha - 1 + \cos^2\alpha = 0$$

$$\Rightarrow 2\cos^2\alpha + \cos\alpha - 1 = 0$$

$$\therefore \cos\alpha = \frac{-1 \pm \sqrt{1 - 4\times 2\times(-1)}}{2\times 2} = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1\pm\sqrt{9}}{4}$$

$$\Rightarrow 2\cos^2\alpha + \cos\alpha - 1 = 0$$

$$\therefore \cos\alpha = \frac{-1 \pm \sqrt{1-4\times 2\times(-1)}}{2\times 2} = \frac{-1\pm\sqrt{1+8}}{4} = \frac{-1\pm\sqrt{9}}{4}$$

$$= \frac{-1\pm 3}{4}$$

$$\therefore \cos\alpha = \frac{-1+3}{4} \text{ or } \cos\alpha = \frac{-1-3}{4} = \frac{-4}{4} = -1$$

$$\cos\alpha = \frac{2}{4} = \frac{1}{2} \text{ or } \cos\alpha = -1$$

$$\alpha = 60^\circ \text{ or } \alpha = 180^\circ$$

Which α will maximize the volume?

for this we substitute $\alpha = 60^\circ$ in eq^n ①

$$\therefore V = L\left[144\sin 60^\circ + 144\sin 60^\circ \cos 60^\circ\right]$$

$$= L\left[144\frac{\sqrt{3}}{2} + 144\frac{\sqrt{3}}{2}\times\frac{1}{2}\right] > 0$$

If we substitute $\alpha = 180^\circ$ in eq^n ① we get

$$V = L\left[144\sin 180^\circ + 144\sin 180^\circ \cos 180^\circ\right]$$

$$= L\left[(144\times 0) + (144\times 0\times -1)\right] = 0$$

$\therefore$ for maximum volume, α is 60°.

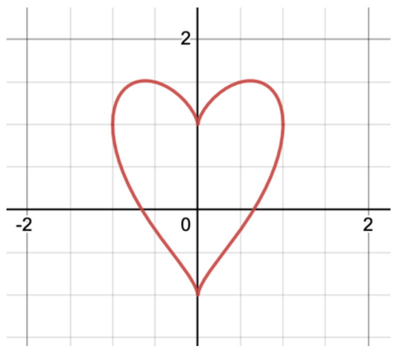
Dear Problem Solvers,

I really hope that you enjoyed and more importantly learned something new by solving problem 8 and problem 9, time to move forward so below are the next two problems.

Problem 10

Proposed by Ivan Retamoso, BMCC, USA.

The Graph of the equation $\left(y - x^{\frac{2}{3}}\right)^2 = 1 - x^2$ is a "heart" and is shown below:



Find the slope of the tangent line of the "heart" at the point $\left(\frac{1}{8}, \frac{2+3\sqrt{7}}{8}\right)$ in exact form.

Problem 11

Proposed by Ivan Retamoso, BMCC, USA.

Find the coordinates of the point (x, y) that belongs to the graph of $x^2 + 18xy + 81y^2 = 144$ that is closest to the origin $(0,0)$ and lies on the third quadrant.