Editorial from Rully Charitas Indra Prahmana, Southeast Asia Editor of MTRJ

Mathematics is a scientific field that can be taught with various approaches, methods, models, and strategies. Mathematics can also be taught by integrating the latest Information and Communications Technology (ICT). Although mathematics is taught using ICT integration (both in learning activities and assessment), the meaningfulness of mathematics can still be obtained by students through the provision of various contextual problems that are close to students' daily lives. In addition to the application of ICT, strengthening student activities in mathematics learning is also the most important part of optimizing the effectiveness of mathematics learning. Student activities in mathematics learning can also be optimized using the Ethnomathematics approach, which uses cultural contexts in mathematics. Therefore, in Vol. 14 No. 4, Fall 2022, MTRJ publishes 12 articles that present different research results on learning mathematics using various approaches, methods, models, and strategies by optimizing student learning activities. In this issue, we also publish one article published in **Problem Corner** section created by the Problem Corner Editor, namely **Ivan Retamoso**.

This issue opens with a paper written by Indonesian researchers Abdul Taram and Fariz Setyawan from Ahmad Dahlan University, Yogyakarta, **Indonesia**, with the title **Stress Tolerance in Probabilistic Thinking: A Case Study.** This research explores students works which scored highest on the stress tolerance dimension could solve problems with simple steps. These findings show that the level of probabilistic thinking depends on students' stress tolerance.

The second paper continues with a study that applied play in learning numerical sense for primary school students. Selepe Mmakgabo Angelinah and Mphahlele Ramashego Shila from the University of Free State and the University of South Africa, **South Africa**, present an article entitled **The Viability of Play in Teaching Number Sense to Grade 3 Learners**. This research offers the use of play in teaching number sense to Grade 3 learners through simple yet quality materials. The research also recommends using indigenous games that are easy to find as a medium for learning numbers.

In this issue, MTRJ publishes two papers that were analyzed using Rasch Measurement analysis. In the first paper, Rahmi Ramadhani, Nuraini Sri Bina, and Edi Syahputra from Universitas Negeri Medan and Universitas Potensi Utama, Medan, **Indonesia** writes the first Rasch analysis paper with the title **Flipped Classroom Assisted Autograph in Calculus Learning for Engineering Students: A Rasch Measurement Study**. This research analyzes calculus learning for engineering students using an Autograph-based Flipped Classroom. Integrating Autograph as ICT media helps engineering students understand Calculus material. The Rasch Measurement-Stacking Analysis results show a change in the logit value of each student's test results. This result shows that the mathematics learning achievement of engineering students increased, and students gave positive feedback after being taught using the Flipped Classroom model assisted by Autograph.

Furthermore, in the second Rasch analysis paper, Suherman and Tibor Vidákovich from the University of Szeged, **Hungary** presents a paper that used Rasch Measurement analysis in analyzing students' creative thinking ability through learning tapis pattern mathematics in the context of Ethnomathematics. The paper entitled **Tapis Patterns in the Context of Ethnomathematics to Assess Students' Creative Thinking in Mathematics: A Rasch Measurement** discuss the Tapis Lampung design used includes geometry concepts that can be used in measuring students' creative thinking skills. Each Tapis pattern also includes local values (i.e., sacred value, social stratification, history and understanding, creativity, inclusivity, and economic value).

The use of technology in mathematics learning is also used in the next article by Ana Katalenić and Zdenka Kolar-Begović from the University of Osijek, **Croatia**, with the title **Prospective Primary School Teachers' Work in Continuous Online Assessments in the Course of Didactics of Mathematics.** This research applies online assessment to the effectiveness of blended learning conducted by prospective primary school teachers. This research is an evaluation conducted during emergency distance learning due to the Covid-19 Pandemic. The findings show that students' learning approaches were strategic and relied heavily on peer support. The results of this research may influence the design of future continuous assessments in blended learning for prospective primary school teachers.

Investigation of students' activities is the focus of further research offered in the next article by Muhammad Daut Siagian, Didi Suryadi, Elah Nurlaelah, Sufyani Prabawanto from Universitas Pendidikan Indonesia and Universitas Islam Sumatera Utara, **Indonesia**, entitled **Investigation of Secondary Students' Epistemological Obstacles in the Inequality Concept.** This research explores the epistemological obstacles students face in the inequality concept by analyzing the errors in solving inequality problems. The results show that there are obstacles experienced by students, which are shown by students' limitations in understanding and interpreting inequality signs in solving inequality problems.

The next paper was written by Hashituky Telesphore Habiyaremye, Celestin Ntivuguruzwa, and Philothere Ntawiha from the University of Rwanda College of Education (URCE), **Rwanda**, with the title **Assessment of Teaching Methods in Mathematical Simplicity and Complexity in Rwandan Schools via Pedagogical Content Knowledge**. This research focuses on assessing the practice of Rwandan mathematics teachers through strengthening pedagogical content. The results find a lack of mastery of content and specialized knowledge at the university level was the cause of low teacher performance. The research recommends providing training in content knowledge to strengthen the teaching practices of teachers, especially those from teacher training colleges.

The next two papers focus on the competencies of prospective and mathematics teachers. The paper entitled **The Investigation of Concept Image towards Derivative Representation: A Case Study of Prospective Mathematics Teacher** was written by Aditya Prihandhika, Didi Suryadi, Sufyani Prabawanto from Universitas Pendidikan Bandung and Universitas Islam Al-Ihya Kuningan, **Indonesia**. This research presents a qualitative research design that investigates the conceptual description of prospective teachers in derivative representation. The results show the conceptual description of all participants on the concept of derivatives was still limited to the representation of functions.

Furthermore, María Burgos Navarro and María José Castillo Céspedes from the University of Granada, Spain, and the University of Costa Rica, **Costa Rica**, present their research entitled **Developing reflective competence in pre-service teachers by analyzing textbook lessons: the case of proportionality**. This study describes the implementation and outcomes of training actions with 45 pre-service teachers to develop reflective competence by analyzing the didactical appropriateness of a lesson on direct proportionality. The implementation results show the evolution of reflective competence in most pre-service teachers, who could make detailed judgments by correctly applying the appropriateness criteria, especially in the cognitive-affective and instructional dimensions.

The last three papers focus on the development of teaching materials and exploring alternative solution for some calculus problem. The first paper entitled **The Development of Inquiry-Based Teaching Materials for Basic Algebra Courses: Integration with Guided Note-Taking Learning Models** written by Merina Pratiwi, Dewi Yuliana Fitri, and Anna Cesaria from the University of PGRI West Sumatra, **Indonesia**. This research produces inquiry-based learning tools with guided notes for Basic Algebra courses that are valid, effective, and practical. Furthermore, Tria Gustiningsi and colleagues from Sriwijaya University, **Indonesia**, produce a jumping task in the form of a valid and practical student worksheet through research entitled **Designing Student Worksheet on Relation and Function Material for Mathematics Learning: Jumping Task.** The results show that the designed student worksheet could help students understand the instructions or questions in the student worksheet and could be used by students. Lastly, the paper entitled **Heuristic Method for Minimizing Distance without using Calculus and Its Significance** written by Ivan Retamoso from Borough of Manhattan Community College of the City University of New York, USA. In this paper, he provides alternatives for solving some Applied Optimization Problems related to minimizing a distance, without the use of the Derivative from Calculus, and instead, using a "Reflection Principle" based on symmetry, Geometric properties, and heuristic methods.

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Stress Tolerance in Probabilistic Thinking: A Case Study

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Abstract: Probabilistic thinking is a structure of thinking characterized by scenarios that allow one to explore reality. Therefore, the characteristic of probabilistic thinking is problem-oriented that will occur in a future full of uncertainty. Nevertheless, few studies examine the students' probabilistic thinking level based on the Stress Tolerance dimensions. Thus, in this study, researchers aim to describe the students' probabilistic thinking level based on the Stress Tolerance dimension in solving probability problems. It is shown that the smallest Stress Tolerance (ST)-Students consider confirming that the first solution is accurate. In contrast, the students with the highest score in ST-dimensions tend to make a simple step in solving the problem. The students' answers to probability problems characterize authentic risk-based decision-making. When we deal with probabilistic situations in everyday life, we all use a series of decision-making in our everyday estimation of probabilities, which sometimes leads to biases. However, the level of probabilistic thinking depends on the stress tolerance of the students. The students with the smallest stress tolerance score tend to get level 4 in probabilistic thinking. In contrast, the students with the highest stress tolerance score tends to reach level 1 in probabilistic thinking.

INTRODUCTION

Each student has their motivation to study. Students' motivation has an impact not only on their learning outcomes but also on their mental processes (Kasdhan, 2018). A motivation to study occurs if students are curious about the information or experiences as a motivation to learn something. In a world where people are deluged with information and can attain novel experiences with only a few keyboard clicks, curiosity becomes a potent psychological strength (Kashdan, et al., 2018). Kashdan said that curiosity is about seeking information and experiences for their own sake through self-directed behaviour. Curiosity is a desire to acquire new knowledge and sensory experience that motivates exploratory behaviour (Litman, 2005; Berlyne, 1954, 1960; Loewenstein, 1994). A recent set of studies suggests that being curious about other people's feelings, thoughts, and behaviour is distinct from observing other people surreptitiously to acquire new information (e.g., Litman & Pezzo, 2007; Renner, 2006). Students are required to be able to develop their higher-order thinking skills.

Some researchers defined probabilistic thinking as formal thinking characterized by abstraction, hypothetical, deductive, inductive, and logical thinking (Pfannkuch, et al., 2016; Savard, 2014; Borovcnik & Kapadia, 2014). Pfannkuch characterized probabilistic thinking as how one views and thinks about probability, whether from a classical, frequentist, or Bayesian perspective has been presented as crucial to how one engages in probabilistic thinking. Borovnick & Kapadia (2014) described probabilistic thinking as a structure of thinking characterized by scenarios that allow one to explore reality. Therefore, the characteristic of probabilistic thinking is problem-oriented that will occur in a future full of uncertainty.

Jones et al. (1997, 1999) suggested four levels of probabilistic thinking, namely subjective, transitional, informal quantitative, and numerical. Furthermore, Polaki (2002) developed the Jones' probabilistic thinking level in more detail for several subjects or materials in probability theory. In determining the level of probabilistic thinking ability indicators, using the probabilistic thinking rubric as shown in Table 1.

No.	Level	Indicators
1	Level 1 (*Subjective*)	1. Students are always bound to a subjective reason
2	Level 2 (*Transitional*)	2. Students think naively and often change.
3	Level 3 (*Informal Quantitative*)	3. Students can harmonize and quantify their thoughts about the possibilities that will occur.
4	Level 4 (*Numerical*)	4. Students can make precise relationships between the sample space and its probabilities and use numerical measurements appropriately to describe the probability of an event.

Table 1: Level of Probabilistic Thinking (Adopted from Taram, et al., 2019; Polaki, 2002)

However, there are few studies which explore the students' probabilistic thinking level based on the Five-Dimensional Curiosity Scale-Revised (5DCR) curiosity dimensions. The 5DCR distinguishes between experiences of curiosity that differ in emotional valence. The degree to which someone is curious depends on two cognitive judgments. Initially, a person must recognize that an event is exciting and warrants attention. Mysterious, novel, complex, uncertain, and/or ambiguous events tend to elicit interest (e.g., Berylne, 1954, 1960; Silvia, 2008a). Curiosity is initiated if a person notices that an event has novelty potential. A person will only be curious if they also believe they can sufficiently cope with the distress that arises from exploring the novelty potential of a situation (Silvia, 2005, 2008a). If a person believes that a case has novelty and coping potential, a person is said to be curious now (i.e., state curiosity). People who endorse novelty and coping potential with high frequency, intensity, and/or longevity are said to be highly curious (i.e., trait curiosity) (e.g., Silvia, 2008b). From this work on the appraisal components of curiosity, the 5DCR instrument measures a dimension of curiosity referred to as Stress Tolerance—the dispositional tendency to handle the anxiety that arises when confronting the new. Thus, in this study, researchers aim to describe the students' probabilistic thinking level based on the Stress Tolerance dimension in solving probability problems.

METHOD

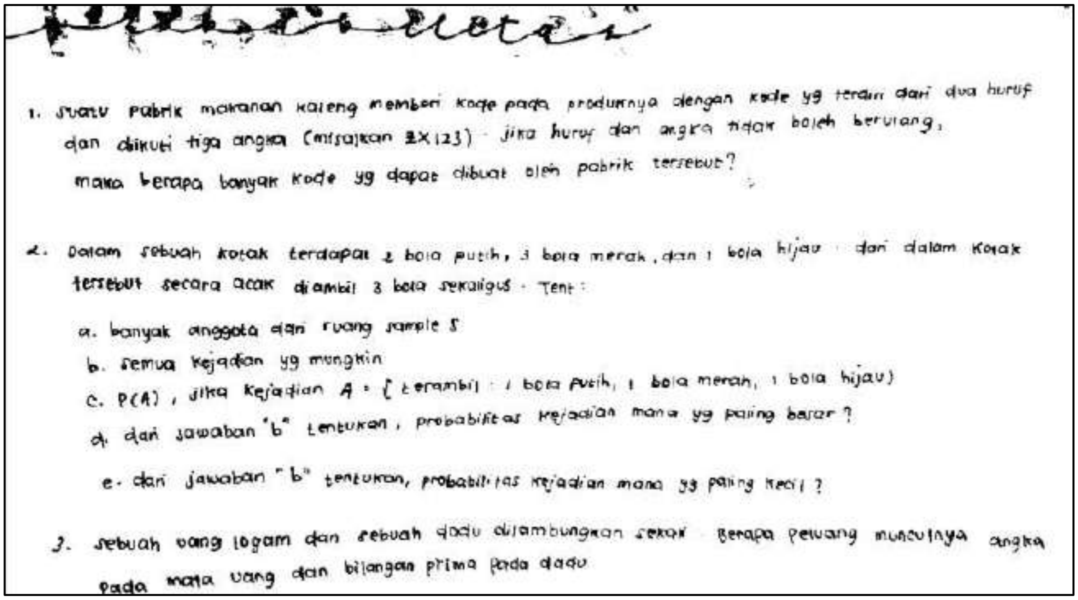
This research was conducted at a university in Yogyakarta, Indonesia. Furthermore, the study was held in May 2022. The number of subjects involved in this study is ten students of 2nd-semester students in the mathematics education department. The subject is chosen by purposive sampling based on the dimensions of curiosity. This study uses two instruments: 1) 5DCR Questionnaire of Curiosity and 2) the Probabilistic Thinking Test. The data was analyzed qualitatively. The researchers give the descriptive explanation about the profile of the students' probabilistic thinking based on their Stress Tolerance dimension.

Stress Tolerance (ST) questionnaire

The Stress Tolerance questionnaire is a part of a five-dimensional curiosity-revised (5DCR) instrument. It is adopted from the Kasdhan, et al. (2018). It consists of 24 questions. There are 4 statements for Stress Tolerance (ST) dimension. The students choose their preferences in the questionnaire from scale 1 (one) to 7 (seven). Scale seven indicates that the statement does not describe him/her at all. In contrast, scale ones mean the statement completely describe his/herself. The scale indicates the degree to which statements accurately describe his/herself.

Probabilistic problem test

In this case, the researcher uses an instrument test to determine the students' probabilistic thinking level. The test is an essay. It is containing three (3) questions that have been validated by 2 validators. The students write the answer directly in the answer sheet that is given and did the test for 60 minutes. First question of the test is asking the number of possibilities of a code can be made. Second is asking about the possibility of taking 3 color balls simultaneously from a pocket. Third question of the test asks about the possibility of a head-tail event from tossing a coin and prime number when tossing a die. The probabilistic problems test is shown in Figure 1.



[illegible]

1. suatu pabrik makanan kaleng memberi kode pada produknya dengan kode yg terdiri dari dua huruf dan dikuti tiga angka (misalkan ZX123). jika huruf dan angka tidak boleh berulang, maka berapa banyak kode yg dapat dibuat oleh pabrik tersebut?

2. Dalam sebuah kotak terdapat 2 bola putih, 3 bola merah, dan 1 bola hijau. dari dalam kotak tersebut secara acak diambil 3 bola sekaligus. Tent :
   a. banyak anggota dari ruang sample S
   b. semua kejadian yg mungkin
   c. P(A), jika kejadian A = {terambil 1 bola putih, 1 bola merah, 1 bola hijau)
   d. dari jawaban "b" tentukan, probabilitas kejadian mana yg paling besar ?
   e. dari jawaban "b" tentukan, probabilitas kejadian mana yg paling kecil ?

3. sebuah uang logam dan sebuah dadu dilambungkan sekali. Berapa peluang munculnya angka pada mata uang dan bilangan prima pada dadu

Translate in English:

1) A food factory gives a code for their products, consisting of 2 letters and 3 numbers (for example, ZX123). If the letters and numbers used were not allowed to be repeated, then how many codes can be made?
2) A bag contains 2 white, 3 red, and 1 green ball(s). Three balls are drawn one after the other without replacement. Determine:
    a) the number of the elements of sample space
    b) all possible event occurs,
    c) the probability that the balls are drawn is white, red, and green,
    d) what is the probability of the most possible event?
    e) what is the probability of the less possible event?
3) A coin and a die are tossed once. What is the probability of getting the head of the coin and the prime number of dice?

Figure 1: Probabilistic Thinking Test

Interview

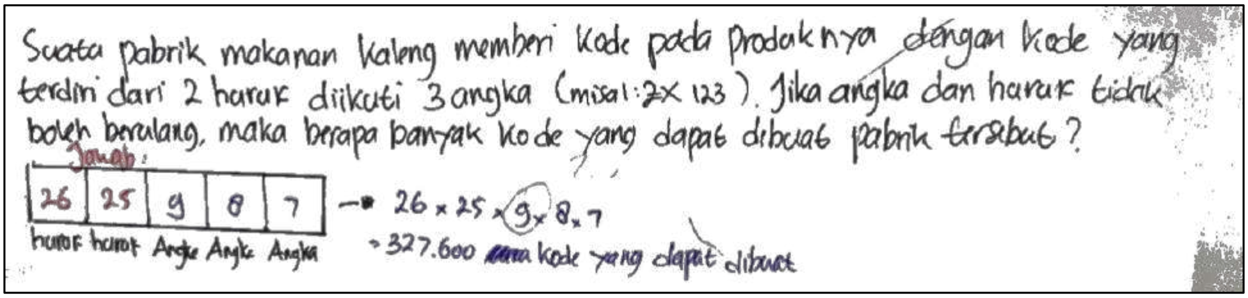
Researchers interviewed the students to confirm their answers. The researchers ask the students what, why and how the step of the probabilistic thinking level. This interview is conducted after the researchers give the test of probabilistic thinking.

RESULTS AND DISCUSSION

In each dimension of curiosity, the researchers choose one student who represents their work in solving probabilistic problems based on the Stress Tolerance dimensions.

The smallest vs the highest ST score of student probabilistic thinking

The student with a minor in ST dimension score writes down the number of possibilities of a code that can be made by combining the alphabetic and numerical code. It is shown in Figure 2 that students write the solution to the first problem in detail.



Suatu pabrik makanan kaleng memberi kode pada produknya dengan kode yang terdiri dari 2 huruf diikuti 3 angka (misal: ZX 123). Jika angka dan huruf tidak boleh berulang, maka berapa banyak kode yang dapat dibuat pabrik tersebut?

Jawab:

26	25	9	8	7
huruf	huruf	Angka	Angka	Angka

→ 26 × 25 × 9 × 8 × 7

= 327.600 kode yang dapat dibuat

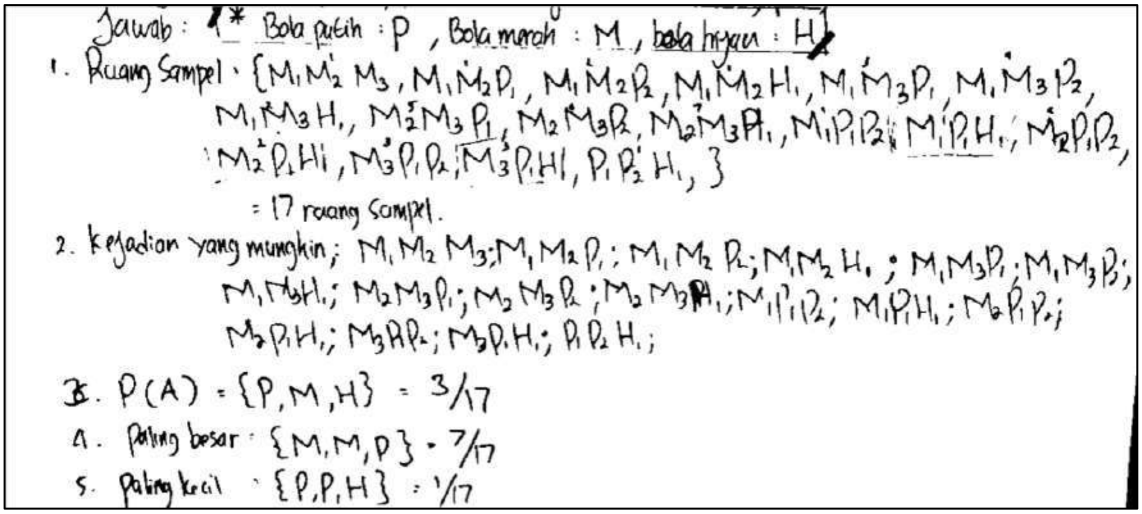
Translate in english:
A food factory gives code for their products that consist of 2 letters and 3 numbers (for example ZX123). If the letters and numbers used were not allowed to be repeated, then how many code that can be made?
Answer:

26	25	9	8	7

Letters Numbers
26 x 25 x 9 x 8 x 7 = 327.600 codes

Figure 2: The smallest ST-Student's solution in number 1

When the researchers interviewed the student about the solution, the student could explain his idea about how he got the answer. He can distinguish the combination of the letters and the numbers for the product's code. Like the solution in number 2, the subject made precise relationships about the sample space and its probabilities and appropriately used numerical measurements to describe the probability of an event (see in Figure 3).

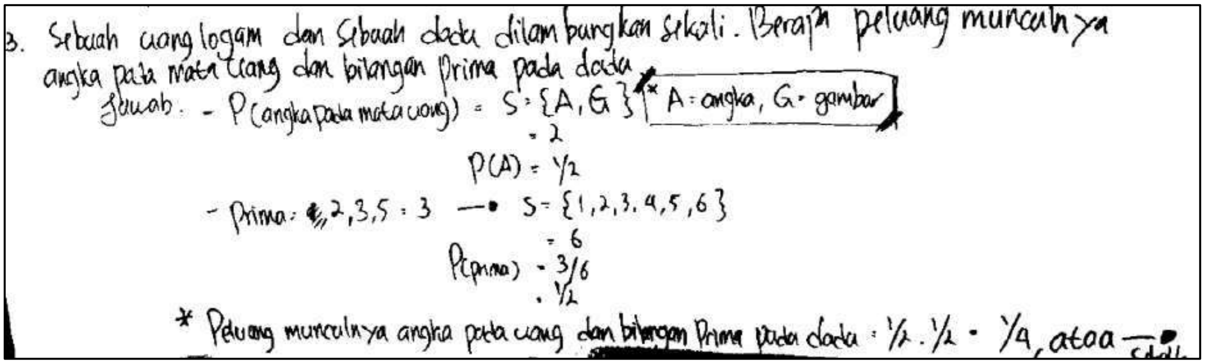


Jawab : {* Bola putih : P , Bola merah : M , bola hijau : H}
1. Ruang Sampel : $\{M_1M_2M_3, M_1M_2P_1, M_1M_2P_2, M_1M_2H_1, M_1M_3P_1, M_1M_3P_2, M_1M_3H_1, M_2M_3P_1, M_2M_3P_2, M_2M_3H_1, M_1P_1P_2, M_1P_1H_1, M_2P_1P_2, M_2P_1H_1, M_3P_1P_2, M_3P_1H_1, P_1P_2H_1, \}$
= 17 ruang sampel.
2. kejadian yang mungkin ; $M_1M_2M_3; M_1M_2P_1; M_1M_2P_2; M_1M_2H_1; M_1M_3P_1; M_1M_3P_2; M_1M_3H_1; M_2M_3P_1; M_2M_3P_2; M_2M_3H_1; M_1P_1P_2; M_1P_1H_1; M_2P_1P_2; M_2P_1H_1; M_3P_1P_2; M_3P_1H_1; P_1P_2H_1;$
3. $P(A) = \{P,M,H\} = 3/17$
4. Paling besar : $\{M,M,P\} = 7/17$
5. Paling kecil : $\{P,P,H\} = 1/17$

Translate in English
Answer: White ball (P), Red ball (M), Green ball (H)
1. Sample Space = $\{M_1M_2M_3, M_1M_2P_1, M_1M_2P_2, M_1M_2H_1, M_1M_3P_1, M_1M_3P_2, M_1M_3H_1, M_2M_3P_1, M_2M_3P_2, M_2M_3H_1, M_1P_1P_2, M_1P_1H_1, M_2P_1P_2, M_2P_1H_1, M_3P_1P_2, M_3P_1H_1, P_1P_2H_1\}$ = 17
2. Possible event = $M_1M_2M_3, M_1M_2P_1, M_1M_2P_2, M_1M_2H_1, M_1M_3P_1, M_1M_3P_2, M_1M_3H_1, M_2M_3P_1, M_2M_3P_2, M_2M_3H_1, M_1P_1P_2, M_1P_1H_1, M_2P_1P_2, M_2P_1H_1, M_3P_1P_2, M_3P_1H_1, P_1P_2H_1$
3. $P(A) = \{P, M, H\} = \frac{3}{17}$
4. The biggest = $\{M, M, P\} = \frac{7}{17}$
5. The smallest = $\{P, P, H\} = \frac{1}{17}$

Figure 3: The smallest ST-Student's solution in number 2

The smallest score of ST-student made solution in number 3 of the probabilistic test showed that there was not only one solution derived to determine the probability of the event in tossing a coin and a die. The student gives two alternatives to find the possibility of event head-tail and prime number in tossing a coin and a dice consecutively. The first solution of the smallest Stress Tolerance students is illustrated in Figure 4.



3. Sebuah uang logam dan Sebuah dadu dilambungkan sekali. Berapa peluang munculnya angka pada mata uang dan bilangan prima pada dadu

Jawab. - P(angka pada mata uang) = S: {A, G} [* A: angka, G: gambar]

= 2

P(A) = 1/2

- Prima: 2,3,5 : 3 → S = {1,2,3,4,5,6}

= 6

P(prima) = 3/6

= 1/2

* Peluang munculnya angka pada uang dan bilangan Prima pada dadu : 1/2 . 1/2 = 1/4, atoa

Translate in English:

3. A coin and a die are tossed once. what is the probability of getting a head of the coin and prime number of a die?

Answer:

- $P\ (tail\ of\ \ a\ coin)$: $S = \{A, G\}$ |A: Head, G: Tail|

$= 2$

$$P(A) = \frac{1}{2}$$

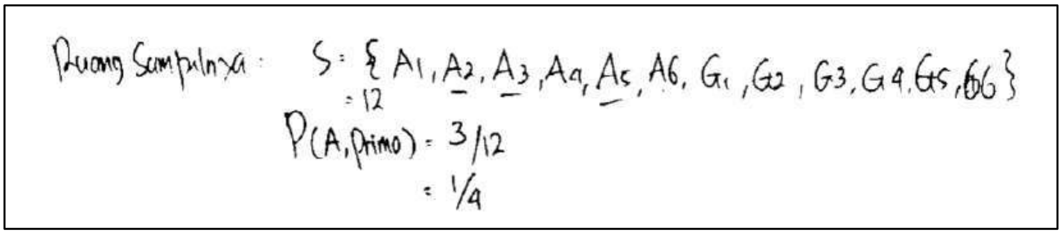
- Prime number = 2,3,5 $= 3 \rightarrow S = \{1,2,3,4,5,6\}$

$= \ 6$

$$P(Prime) = \frac{3}{6} = \frac{1}{2}$$

Figure 4: The smallest ST-Student's solution for Number 3 (First Solution)

Furthermore, the ST student gives two alternative solutions to solving problem number 3. The second answer shows that he considers confirming that the previous solution is accurate. The student wrote the sample space of the event by mentioning the elements of it, one by one, as a set of the probable events (See in Figure 5). In the last of the second solution, the student counts the number of elements of the sample space and then determines the probability of the event. As a result, the level of probabilistic thinking of the subject is categorized as level 4.



Ruang Sampelnya: S: {A1, A2, A3, A4, A5, A6, G1, G2, G3, G4, G5, G6}

= 12

P(A, Primo) = 3/12

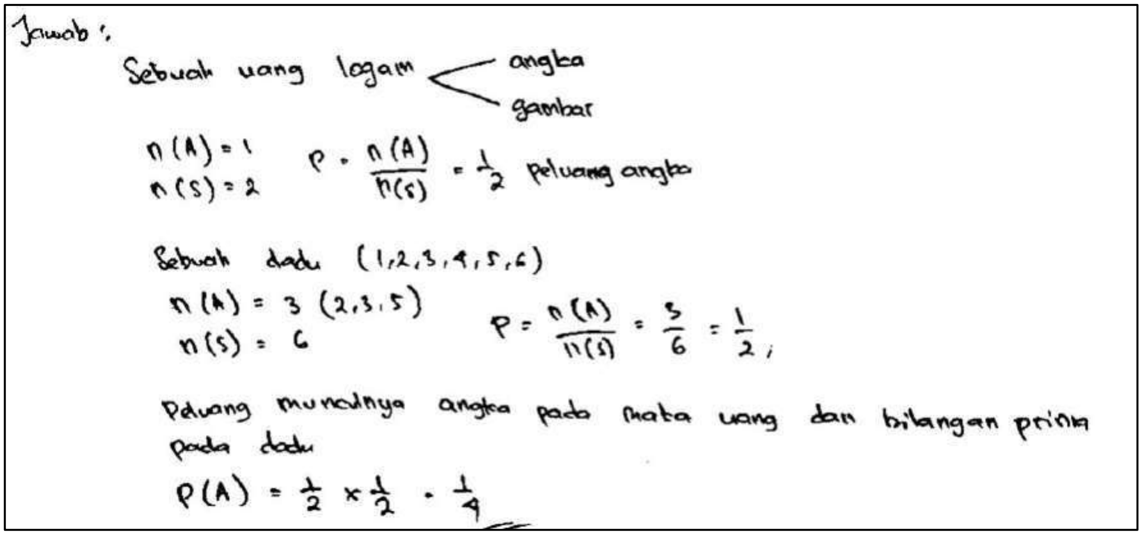
= 1/4

Sample space: $S = \{A_1, A_2, A_3, A_4, A_5, A_6, G_1, G_2, G_3, G_4, G_5, G_6\}$
$= 12$

$$P(A, prime) = \frac{3}{12} = \frac{1}{4}$$

Figure 5: The smallest ST-Student's solution in number 3 (Second Solution)

Besides, another student with the next-to-the smallest score of ST-dimension was writing the solution in detail. She wrote the diagram to ensure the coin has two possibilities (head and tail) events (see in Figure 6). Both are bound to a subjective reason, think naively and often change, harmonize, and quantify their thoughts about the possibilities that will occur, make precise relationships about the sample space and its probabilities, and use numerical measurements appropriately to describe the probability of an event.



Jawab :

Sebuah uang logam < angka / gambar

$n(A) = 1$ $\quad P = \frac{n(A)}{n(S)} = \frac{1}{2}$ peluang angka
$n(S) = 2$

Sebuah dadu (1,2,3,4,5,6)
$n(A) = 3$ (2,3,5) $\quad P = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$;
$n(S) = 6$

Peluang munculnya angka pada mata uang dan bilangan prima pada dadu

$P(A) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Answer:
A coin → Head
A coin → Tail

$n(A) = 1$
$n(S) = 2$ $\quad P = \frac{n(A)}{n(S)} = \frac{1}{2}$ *tail possibilities*

A die (1,2,3,4,5,6)
$n(A) = 3$ (2,3,5)
$n(S) = 6$ $\quad P = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$

Probability of head of a coin and prime number of a die:

$$P(A) = \frac{1}{2} x \frac{1}{2} = \frac{1}{4}$$

Figure 6: Another smallest score of Stress Tolerance-Student's solution in number 3

In contrast, the students with the highest score in ST dimensions tend to make a simple steps in solving the problem. She wrote the simplest way to find the solution to the problem. For example, the key to the number 3 was not noted on the answer sheet. She claimed that the problem was complex for her. She does not know how to start and how to resolve it. It can be seen in the solution of number 2 that she crosses out the first answer. The first solution is shown in Figure 7.

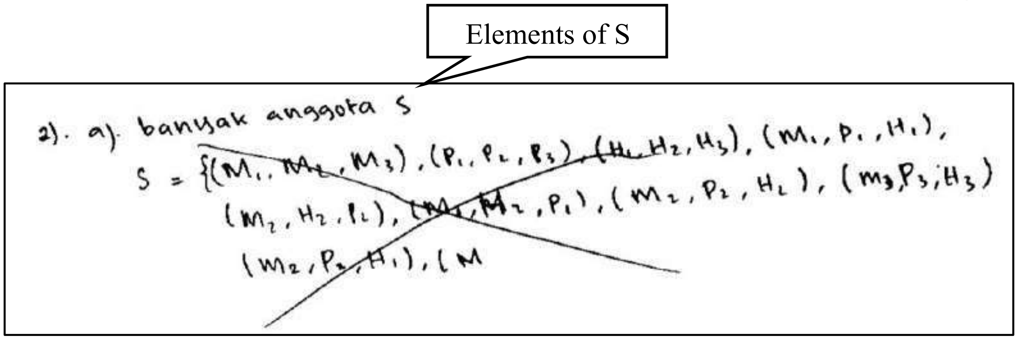


Figure 7: The highest ST-Student's solution in number 2 (first solution) - First attempt

However, the revised answer only mentioned the sample space without determining the possibility of taking three balls simultaneously. The highest ST student wrote the number of the sample space but not the elements of it (see in Figure 8). After the interview, the subject only mentioned that she did not worry about the result. In other words, she does not know the number of possibilities of the event. As a result, the level of probabilistic thinking of the subject is categorized as level 1.

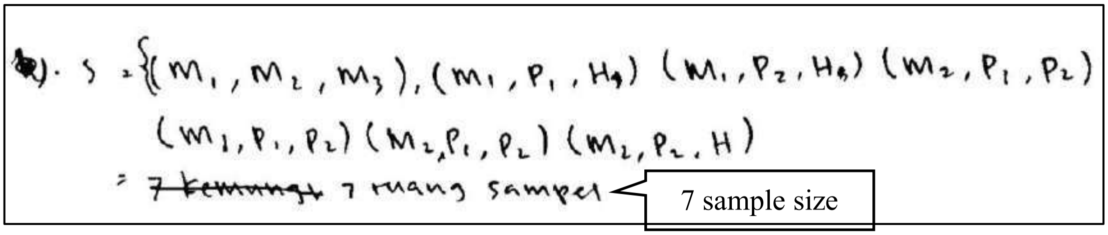


Figure 8. The highest ST-Student's solution in number 2 (revised solution) – Second attempt

Stress Tolerance in Probabilistic Thinking

Stress Tolerance was found to have the most vital links to dispositional mindfulness, work engagement (vigor, dedication, and absorption), and low levels of work burnout, as well as being moderately linked to work-related curiosity and a willingness to defy social norms and express opposing viewpoints. In addition, stress tolerance had the most vital inverse relationship with negative emotionality and positive relationships with extraversion and conscientiousness, respect and trust, psychological needs satisfaction, and the humility to separate intellect and ego (Kasdhan, et al., 2018). However, the relation between stress tolerance and probabilistic thinking was on the structure of thought characterized by scenarios that allow one to explore reality. Mysterious, novel, complex, uncertain, and/or ambiguous events tend to elicit interest (e.g., Berylne, 1954, 1960; Silvia, 2008a). If a person notices that an event has novelty potential, Stress Tolerance occurs. The students will only be stressed if they also believe they can sufficiently cope with the distress that arises from exploring the novelty potential of a situation (Silvia, 2005, 2008a).

All probabilistic analysis is based on the idea that (suitably trained and intelligent) people can at least recognize good probabilistic arguments presented by someone else or discovered or thought of by themselves, but not necessarily generate good assessments. The fact that there was correspondence about the gambles – and occasionally some disputes about them – indicated that people do not automatically assess probabilities in the same way or accurately (e.g., corresponding to relative frequencies or making good gambling choices). Their understanding of mathematical ideas must be developed via experience and exposure to the challenges of solving various issues (Aguilar & Telese, 2018).

Unlike the typical research finding, which sets out to identify underlying the level of probabilistic thinking, the researchers note that this research is founded upon 'clinical' methods where the problem to which subjects answer in probability problems characterize authentic risk-based in decision making. The basic concept is that we all use a series of decision-making in our everyday estimation (explicit or implicit) of probabilities, and this decision-making sometimes leads to biases. Just as associational thinking serves us well in many contexts, so does decision-making. When we deal with probabilistic situations in everyday life, we can often 'muddle through,' but occasionally, not appreciating the comparative nature of valid probabilistic thinking can lead to judgmental disasters. These systematic deviations may be linked to thinking in terms of associations, whereas excellent probabilistic judgment always necessitates comparative thinking.

CONCLUSION

The smallest Stress Tolerance-Student considers confirming that the first solution is accurate. In contrast, the students with the highest score in ST dimensions tend to make a simple step in solving the problem. The student's answers to probability problems characterize authentic risk based on decision-making. The students with the smallest stress tolerance score tends to get level 4 in probabilistic thinking. In contrast, the students with the highest stress tolerance score tends to reach level 1 in probabilistic thinking.

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The Viability of Play in Teaching Number Sense to Grade 3 Learners

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Abstract: Play is one of the most widely used teaching strategies in Foundation Phase. Grade 3 is the exit grade off the Foundation Phase in the South African context. This paper is an output of a Masters' dissertation that explored the use of play when teaching number sense to Grade 3 learners. The dissertation findings encouraged the researchers to explore the viability of play when teaching number sense. The theoretical underpinning of this study was based on Vygotsky's theory of social development and Gardner's theory of multiple intelligence because they both emphasize the importance of play in enhancing social interaction between learners and educators. We used semi-structured interviews, document analysis and non-participant observation to collect data from six Grade 3 teachers from three primary schools in the Capricorn South District's Lebopo Circuit of Limpopo Province in South Africa. The results of this study show that educators lack clear guidelines on how to integrate play in number sense education, from planning to presentation to assessment. Educators need guidelines on how to use play to teach number sense and to be effective. They require curriculum workshops to show them how to use low-cost materials but high quality in lesson preparation. They can play Indigenous games since they require resources that are found in nature.

INTRODUCTION

Number sense is a key notion in early mathematics learning in Foundation Phase (Jordan, Kaplan, Ramineni & Locuniak, 2009) The (Department of Basic Education, 2011) states that Grade 3 learners should leave FP with a secure number sense. Since number sense develops over time through opportunities to explore and play with numbers, educators can use a variety of pedagogies to teach number awareness in FP Learning through play is one of the pedagogies. When teaching learners mathematics in the FP, the Curriculum Assessment and Policy Statement (CAPS) recommends a play-based pedagogy to educators. The use of play-based pedagogy in the classroom has the potential to improve and strengthen children's learning (Zosh, Hopking, Jensen, Liu, Neale, Hirsh-Paseek, Solis & Whitebread, 2017).

In FP, play-based learning strengthens young learners' foundational knowledge, allowing them to achieve math and language proficiency in Grade 12 (UNICEF, 2018). To teach number sense to Grade 3 learners, educators should employ a play-based strategy. Play is an important aspect of FP because it helps young children develop cognitive, physical, social, and emotional skills.

Most learning theories, such as theory of social development (Vygotsky, 1978) and theory of multiple intelligence (Gardner, 2000), promote the use of play in educating young learners.

These two philosophies both emphasize the importance of the educator in the use of play. The role of the educator as a more knowledgeable other (MKO) when using play in the teaching of number sense, is central to Vygotsky's theory. In addition, the relationship between play and social interaction is also vital. Gardner's theory of multiple intelligence advocates the use of play to help young learners improve their logical-mathematical skills in real-world situations (Rosli & Lin, 2018).

The purpose of this article was to present data from a study on the use of play in teaching number sense to Grade 3 learners. The following objectives were established:

- To understand the role of play in teaching number sense to Grade 3 learners
- To establish how play can be used to teach number sense to Grade 3 learners
- To identify types of play that can help to teach number sense to Grade 3 learners
- To recommend guidelines on the use of play when teaching number sense to Grade 3 learners

In 2015, the South African curriculum included the use of play in educating learners in FP. CAPS determined that play should be used as a teaching strategy by FP educators (Department of Basic Education, 2011). The new mathematical curriculum in Grade 3 emphasizes the importance of covering the following five core areas: (i) number, operations, and relationships; (ii) patterns, functions, and algebra; (iii) space and shape; (iv) measurement; and (v) data processing.

The core content area in Grade 3 is number sense, which is generated from numbers, operations, and relationships. In Grade 3, this curriculum area accounts for 58% of the content. If educators can effectively teach number sense in a way that learners comprehend, most learners will be able to pass mathematics. According to(Jordan et al., 2010), number sense contributes to other mathematical curriculum areas in Grade 3.

Theoretical Framework

The framework for this study was built on two theories: Vygotsky's social development theory and Gardner's multiple intelligence. These two theories underpinned the major goal of this study – understanding the use of play in teaching number sense to Grade 3 learners. The theoretical framework and literature evaluation are critical since they provide the foundation for data analysis and recommendations.

The employment of Vygotsky's social development theory in this study was also inspired by (Walshaw, 2017) postulated that mathematical development, in Vygotskian understanding, is conceptualized as a process involving participation, communication, inclusiveness, instructiveness, collaboration and situatedness. (Rosli & Lin, 2018)emphasized the relationship between play and social interaction in the teaching of number sense, noting that parent's participation can promote socialization during play. Regarding Gardner's multiple intelligences, Rosita Dewi Nur, Herman & Mariyana (2019) highlighted that logical-mathematical intelligence equips FP learners with the ability to handle numbers and calculations, patterns, and logical and scientific thinking.

The promotion of social interaction during play is a crucial aspect linking Vygotsky's social development and Gardner's multiple intelligences theories. Both theories emphasize the importance of social interaction through the MKO's role in developing logical-mathematical intelligence. In this study, the link is strengthened using play by the educator to promote social interaction and expand logical-mathematical intelligence. When an educator expands logical-mathematical intelligence while teaching number sense, learner engagement is fostered. Social interaction flourishes when learners are engaged in the activity and participate actively by asking and answering questions.

LITERATURE REVIEW

To gain an understanding of the existing research and debates relevant to the use of play in teaching number sense to Grade 3 learners and to identify knowledge gaps, the objectives mentioned in the introductory section were used to review the related literature.

The role of play in teaching number sense

The role of play in teaching number sense discussed in the literature is the promotion of socialization. Björklund, Magnusson & Palmér (2018) conducted a study in Sweden that concluded that educators should not focus only on pre-prepared activities but spontaneously improvise and invent new activities that employ play to promote socialization. Similar results were found by Dele-Ajayi, Strachan, Pickard, and Sanderson (2019) who explored how the Speedy Rocket game contributed to socialization in rural Ado-Ekiti, Nigeria. In addition, Omidirile, Ayob, Mampane, and Sefotho (2018) found that playing with hula hoops, bean bags, cones, different colours and sizes of shapes and blocks, and flashcards with numbers strengthens learners' group work, performance, class participation and social skills. The second role was to increase learners' logical-mathematical intelligence as per Vogt, Hauser, Stebler, Rechsteiner, and Urech (2018)'s study which stressed the use of play in the form of card and board games.

The use of play when teaching number sense

The related literature reviewed for this study revealed that the use of play in mathematics teaching aligns with different contexts such as culture, communication, and socialization. Tsindoli, Ongeti & Chang (2018) taught number sense through an Indigenous game called Kora which improved learners' social contact. Kora is Kenya's indigenous game which played with pieces of broken pottery or stones. In this game, stones are collected in the palm then thrown into the air. The main purpose is that more than one of the stones thrown into the air must come to rest on the back of the hand.

Scholars like (Sharma, 2017) and (Dicker & Naude, 2019) demonstrated that the main role of play in teaching number sense to Grade 3 learners is socialization. In this study, the use of play as guided Gardner's idea of multiple intelligences promotes logical-mathematical intelligence, The researchers found that there is a significant relationship between the role of play in socialization and the use of play in teaching mathematics.

Types of play that can help to teach number sense

A discussion of several types of play that can be used to teach number sense concluded the literature review of this study. Dicker and Naude (2019) identified four types of play: Physical play, construction, exploratory play, creative play, and wordplay

- Physical play.

  According to (Wonderly, 2017) physical play involves all kinds of physical movements including locomotor and non-locomotor movements. Physical play can be played indoors or outdoors.

- Construction play

  (Reikerås, 2020)describe construction play as the type of play which learners usually use building blocks such as wood and bricks.

- Exploratory play

  Exploratory play is described by (Dicker & Naude, 2019) as the kind of play that allows learners to explore the possibilities of unfamiliar things by experiment with tools and toys.

- Wordplay

  When learners engage in wordplay, they label items according to their appropriate number-words. In this play learners should recognize various quantities labelled with the same number word and other quantities with other number words, they develop the skill of counting (Dicker & Naude, 2019).

Dicker and Naude (2019) explained that these types of play could increase social interaction in the teaching of number sense to Grade 3 learners. All four types of plays can be played both indoors and outdoors. Bose and Seetso (2016) investigated physical play and found that there is a link between physical play and socialization. Physical play includes Indigenous activities that does not necessitate the use of expensive mathematics equipment, which is often lacking in rural schools (Bose & Seetso, 2016). They can also aid in the child development (Omidire et al., 2018).

There is a connection between the MKO's role and construction play. According to Mntunjani et al., (2018), the MKO's role is to guide construction play, which enhances socialization. When learners build with blocks, they interact and communicate with one another. There is a connection between the MKO and exploratory play. According to Sharma (2017), in exploratory play, the MKO's role is to provide a safe, secure, and challenging mathematics environment that will pique learners' interest.

The MKO's purpose in creative play is to aid in the development of logical-mathematical intelligence by providing learners with a variety of engaging materials (Wilmot et al., 2015) and by creating an enticing atmosphere (Sharma, 2017). Even if a teacher does not have a wide range of resources, they can design a creative environment in any situation. Finally, wordplay in the classroom promotes socialization (Frye et al., 2013). These tools, which are used in this form of wordplay, can be made by the educator using flashcards and worksheets. After the learner learns the distinction between more and less, they can play creative games.

METHOD

Researchers employed an interpretive case study design to explore Grade 3 educators' perspectives, perceptions, and observations during the use of play in teaching number sense to Grade 3 learners. Elbardan et al., (2017) and Ponelis (2015) explained that interpretive case study researchers collect data on their own and interact with participants to investigate and comprehend a phenomenon. According to Elbardan et al., (2017), when interpretive case study researchers collect data on their own, it allows them to interact more intimately with participants and dig deeper into difficulties. This entails close conversations with participants to learn more about a phenomenon rather than making assumptions. In this study, the researchers focused on educators' perspectives and perceptions when they use play in the classroom, according to an interpretive case study.

The purpose of this study, which was to understand the use of play in teaching number sense to Grade 3 learners from the perspective of Grade 3 educators, supports the use of an interpretive case study. The research was guided by the following questions.

Research questions

To aid in the selection of acceptable research methodologies, the following study questions were established:

The central question in this study asks:

- What role does play perform in teaching number sense to Grade 3 learners?

Sub-questions

- How can play be used to teach number sense to Grade 3 learners?
- What types of play can help to teach number sense to Grade 3 learners?
- What guidelines do educators follow when integrating play in the teaching of number sense?

Study site and sample

Three public primary schools were selected from the Capricorn South District's Lebopo Circuit, with Polokwane City as their metropolitan center. For anonymity, the schools were named Schools

1, 2 and 3 according to the sequence in which they were visited. Classrooms were labelled Classroom 1 to 6. Two educators were purposively sampled from each of the three schools based on their level of expertise in teaching mathematics in Grade 3, work experience of two or more years, and qualifications in teaching mathematics in the FP.

Data collection methods

Educators' perspectives and experiences of using play in teaching number sense to Grade 3 students were gathered through semi-structured interviews. The researchers also used document analysis to corroborate educators' responses from the semi-structured interviews and, to triangulate the data, the researchers used non-participant observation to attempt to provide “a confluence of evidence that breeds credibility” (Eisner, 2017). The data collection tools (interview schedule, document analysis tool and observation schedule are included in this paper as Appendices A, B and C respectively). For anonymity, as recommended by Chimentão and reis (2019) participants were given pseudonyms according to their school and teaching experience. An educator with the least years of experience (3 years) was labelled Teacher 1 and was from School 3. The pseudonym was S3E1. The second teacher with four years’ experience from School 2 was named S2E2. The third one was from School 1 with five years’ experience and was named S1E3. The fourth educator from School 1 with 10 years’ experience was named S1E4. The fifth educator with 16 years’ experience was from School 2 and was named S2E5. The last educator with 36 years’ experience was from School 3 and was named S3E6.

Data analysis

Data from interviews was accessible in audio format and was transcribed before analysis by a professional transcriber. When all datasets were in text format, they were organized according to the schools. The researchers began by manually coding data, then converting it to pdf format and entering it into [1]Atlas.ti, where it was re-coded electronically. The codes were produced using keywords from the interview questions that corresponded to some of the research questions' keywords. For reliability, the data was co-coded by the two researchers who compared the codes for similarities and differences. Atlas.ti was used to generate categories based on the theoretical framework that underpins this research. The researcher created data analysis outputs such as tables, networks, word clouds, and word lists using the codes, categories, and derived themes. These outputs were used to portray the data following the themes that were created in response to the research questions.

Ethical considerations

Ethical clearance was obtained from UNISA College of Education Research Ethics Committee (Ethical reference number: 2020/10/14/64019209/07/AM). During data collection, the applicable UNISA Covid-19 rules as well as other ethical considerations were upheld. Permission was also requested from the Limpopo Department of Education. Parents were also provided consent letters

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[1] Atlas.ti is a computer-aided qualitative data analysis software that analyses textual, graphical, and multimedia formats including rich text format documents, Word documents, most picture formats, most sound formats and most digital video formats, as well as PDF documents (Archer, Janse Van Vuuren & van Der Walt, (2017).

to allow researchers to conduct observations in the classrooms in the presence of their children. Potential participants were given ample opportunity to read and comprehend the information sheet before signing consent to participate in the study.

RESULTS

Results for this study are presented under four categories, namely: the use of play when teaching number sense, lesson preparations, the role of play when teaching number sense and types of play suitable for teaching number sense.

The use of play when teaching number sense

The first interview question aimed at getting educators' views on the use of play in the teaching of number sense. Three participants said that play was an important method for teaching number sense. Play, according to S1E4, was a tool that is used to enhance learner-centredness. S1E3 clarified that play was used to increase interaction between peers as well as between the educator and the learners.

The second question asked how educators used play when teaching number sense. Educators explained that different types of games were used to educate number sense. Indoor and outdoor games were covered. S3E1 explained that the Covid-19 pandemic was one of the problems in using play to teach number sense to Grade 3 learners. S3E1 went on to say that she preferred outdoor games since she could follow the Covid-19 guidelines. She stated:

> *"I use games, such as snakes and ladders, but it is now difficult because we have to adhere to Covid-19 regulations, and it needs a die and board with numbers, and it will help them in counting."* (S3E1)

Aside from Covid-19, another issue identified in the use of play in teaching Grade 3 learners number sense was a lack of resources. Due to a lack of teaching and learning tools, educators explained that engaging in various types of play in teaching number sense was difficult. S3E6 highlighted that to create hand-made resources, instructors must be creative. She expressed her feelings by saying:

> *"The schools in rural areas are disadvantaged because they do not have resources to participate in other play. Nevertheless, as creative educators, we create our games from boards. I always use outdoor games, for example, the game called "back-to-nought", which helps learners in counting."* (S3E6)

Observations from Classroom 1 verified the scarcity of mathematical materials. To teach number awareness, S1E4 used a card game. S1E4 echoed what S3E6 suggested during conversations about innovation. S1E4 made cards out of cardboard that the learners decorated with watercolours. During mental math, the game was used in teaching the lesson. The learners were divided into six players per team. Each team was given a card and was asked to round the number on the card to the nearest tens. For example, if the card said 3, the learners had to round 3 to the nearest tens.

Since 3 is less than 5 then it becomes 0. (3=0). The team's monitor ensured that all players were participating.

With the observation from Classroom 6, the researcher confirmed the ideas expressed by S3E6 during the interview sessions. To teach learners counting, S3E6 used an outdoor game called 'back-to-nought'. The game was contested by two teams, each with four players. Because S3E6 used a ball manufactured from recycled plastic bags, the issue of insufficient mathematical resources was recognised in Classroom 6. Team 1 threw a ball at each other while Team 2 players ran to the spot in each small square while counting from 1 and avoiding being beaten by the opposite team's ball. If a player miscounted, they would start counting from zero. The winner would be the first learner to count to 30 without miscounting and being beaten by the ball. If the defenders tagged an attacker with the ball, that player was out. The educator used group assessment to allocate the teams' grades according to the points won from the play. The problem of Covid-19 was realised during the game because it was tough for learners to follow the rules of social distancing. The learners enjoyed the lesson and the play, and they did not focus on the fact that they are being assessed through the play

Lesson preparations

During the interviews, when educators responded to the fourth question which asked them if they included play in their lesson preparation, they highlighted that they did not know how to incorporate it. An educator from School 1 explained that the lesson template did not allow for the incorporation of play. Other participants agreed that play was occasionally employed in the classroom. S3E1 stated:

> *"When planning, no. I just involve it when I am busy teaching, maybe if I have a difficult word, learners will not understand, I switch to the play."*

The issue of a lack of mathematical resources was brought up again by educators, who stated that it had a detrimental impact on planning of play activities. S3E6 said that one of the reasons she did not include play in her lesson plan was a lack of resources. S1E4 attributed the effectiveness of play to the fact that she always included it in her preparation.

> *"Since the lesson plan template does has provision to include play, I write it in the lesson presentation section of the lesson plan" (S1E4)*

In responding to the fifth question, the educators emphasized that they did not include play in lesson preparation, which affected the lesson presentations, and assessments. Three educators (S1E4, S1E3, and S3E6) indicated they did not always follow their lesson plans, whereas the other three educators (S2E2, S2E5, and S3E1) said they always did. When the researchers checked their responses to the first question, educators clarified that they used play as an intervention strategy rather than a teaching strategy. S1E3 verified this by saying:

> *"Remember if you are teaching, then you are active with your learners and then you see that your learner is not participating, they don't understand something, you sometimes go out and include some play or something to help learners."*

S3E1 described how incorporating play into the lesson preparations might help with time management and the development of mathematics skills for learners because of the learners' participation and concentration. She gave an example of playing diketo [2]where she explained that she groups the learners into groups of four and allocates 2 minutes for each player to collect as many stones from the play as possible. But if the player makes the mistake of dropping the throwing stone (gho/mokinto) before 2 minutes is over, he/she forfeits the remaining time. In this play, learners learn how to manage time while adding the number of stones to win points.

The classroom observations corroborated what the educators shared during the interviews because the researchers saw that three educators did not adhere to their lesson preparations and used play without preparation. The first two classrooms observed demonstrated the difficulty in time management because of not including play in the lesson preparation but instead using it as an intervention strategy. One example was when learners had difficulty understanding number patterns. In Schools 1 and 2, S1E4 and S2E2 were used to play as an intervention strategy to elaborate patterns. During the follow-up conversation after the classroom observations, all educators agreed that play should be included in lesson preparation to save time.

The document analysis revealed that three educators (S2E5, S3E1, and S3E6) included play in their lesson plan template without being aware that they had included it. That might have been caused by copying and pasting previous lesson plans done by other educators. S2E5's lesson plan, for example, detailed how the game was played as well as its goals and objectives.

The role of play when teaching number sense

The second and eighth interview questions aimed to gather the views of educators on the role and purpose of play in the teaching number sense. The role of play in teaching number sense, according to S2E2 and S3E6, is to increase learners' participation. S3E5 added that play can also be used as a differentiated teaching technique since it allows students with mathematical learning difficulties to participate.

> *"It increases participation in the lesson and motivates learners, especially weak learners or underperforming learners.*" (S3E6).
>
> *"Play makes the learners so much active and interested in the lesson, every one of them will want to participate, even those that are slow."* (S2E2).

The researchers designed a word-cloud (Figure 1) to present the words used to describe the role of play when teaching number sense.

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[2] Diketo is a South African indigenous game played by two or more players using stones, pebbles or marbles. The player throws a stone called "mokinto" into the air and then tries to take out as many stones as possible from the circle before they catch it again with the same hand. Then they put the stones back into the hole one stone at a time, until all ten stones are back in the hole. The player can only move a stone while the "gho"/"mokinto" is in the air and before catching it again with the same hand.

Figure1: The role of play

The word participation was the repeatedly used by the three educators when responding to the question about the role of play. Even though this word-cloud cannot directly prove that play increases participation in the learning of number sense, it was the is the most used combination in the teachers' responses. In addition, S1E4 and S2E2 both stated that using play to teach number sense helps learners understand numbers.

> "*It makes learners understand numbers more.*" (S2E2)

During the observations, researchers witnessed learners who were engaged in play and there was a lot of social interaction. The importance of play was demonstrated in Classroom 5, where a game of skipping rope was used to teach place value. Learners who had difficulty with math were paired with those who had high logical-mathematical intelligence.

Types of play suitable for teaching number sense

Educators concentrated on games that did not necessitate the use of physical-mathematical resources due to shortage of mathematical materials. S3E1 resorted to outdoor games such as "do a number dance" and "hopscotch".

> *"I always use games that are outdoor… that helps learners in counting… I develop gross motor skills.*" (S3E1)

S3E6 also stated that she was discouraged from using other types of play to educate number sense due to a lack of resources. She used cardboard to make a "snakes and ladders game". She commented:

> "*Nevertheless, as creative educators, we create our games from boards.*" (S3E6)

When conducting classroom observations, the researchers witnessed the following types of physical play:

- Tsheretshere – is a South African Indigenous game in which children play by throwing a flat stone which they push with one foot from one box to another while balancing on one leg. Sometimes instead of rectangular boxes, they use circles, which helps learners develop a sense of shapes. The counting system is developed every time a player plays since they must remember where they ended and where to start the next time. The concepts of mass, weight and balancing are also perfected in the process.
- Kgati – is also a South African Indigenous game played by three or more players. Two hold the skipping rope at the ends and they swing it in circular motions while one or more players skip in the middle counting. This game teaches learners how to break numbers into hundreds, tens, and units.

The document analysis revealed that some of the lesson plans showed only physical and outdoor play.

The following section discusses these results thematically in relation to the theoretical framework.

DISCUSSION

This study explored the viability of play in teaching number sense to Grade 3 learners. The findings were discussed thematically using four themes that were formed from the theoretical framework and the categories.

Theme 1: Teaching number sense through play

Teaching number sense through play improves social interaction according to (Rosli & Lin, 2018). They added that play could effectively strengthen the interaction between parents and educators. Based on the findings presented in the previous section, teaching number sense through play helps learners with varying levels of logical-mathematical intelligence, (such as identifying, reasoning, and thinking logically) as well as those who have difficulty learning mathematics. According to S2E2, play helps learners understand number sense. Considering this, the researchers conclude that teaching number sense through play benefits learners with various levels of logical-mathematical competence.

Theme 2: Enhancing social interaction through play

The review of the related literature conducted on the usage of play in several regions, including Europe, Africa, and South Africa revealed a substantial link between the usage of play and social interaction. According to Dele-Ajayi et al., (2019), using games to teach mathematical concepts enhances socialization. As a result, this research asked: how can play be used to teach number sense to Grade 3 learners? It was emphasized that the role of play in the teaching of number sense is to promote social interaction (Wium & Louw, 2015). Tsindoli et al., (2018) argue that culture

(Indigenous games) can be used to teach number sense. They also recommended that educators (MKO) provide supervision in play to improve social interaction.

This study found that educators recognised their roles in using play to improve social interaction in the teaching of number sense. For example, during the classroom observations, S2E5 meticulously introduced a skipping rope game with clear rules, goals, and objectives. Play, according to educators, is learner-centred because learners can engage and share different roles while learning.

Although some of the educators did not integrate play into their lesson plans, according to data from document analysis, they used play as an intervention strategy and to enhance opportunities for social engagement, which aids in the teaching and learning of number sense.

Theme 3: Guidelines for integrating play in teaching number sense

Vygotsky's concepts of the Zone of Proximal Development (ZPD) and the MKO, according to Abtahi (2017), "have been interpreted and re-interpreted in the field of mathematics education at various degrees of depth". In this study, the MKO in this study was viewed as the educator who would integrate play into the teaching of number sense to Grade 3 learners. The findings presented in the previous sections uncovered the non-inclusion of play in lesson planning and the use of play as an intervention strategy than integrating it in the lesson activities. During the interviews, the researchers gathered that there were no guidelines for educators to integrate play into their mathematics teaching and learning activities. Guidelines are imperative for the creation of a practical and ethical framework for decision-making because they instill a sense of responsibility and accountability.

Theme 4: Types of play enhancing Logical-Mathematical Intelligence

Dicker and Naude (2019), Gordon and Browne (2011), Rosli and Lin (2018), and Woolfolk (2014) found that diverse types of play improve logical-mathematical intelligence, which aids in number sense learning. Educators in the selected schools employed mainly musical, wordplay and physical play to enhance learners' logical-mathematical intelligence. The most used type of play was physical. Physical play in mathematics, according to S3E6 and Omidire et al., (2018), can also be used to build gross motor skills.

Indigenous games such as singing songs, 'Tsheretshere', skipping rope, and 'back-to-nought' were recognised as types of play that might enhance logical-mathematical intelligence. Tsindoli et al., (2018) explained that Indigenous games could potentially be used to spread logical-mathematical intelligence. These types of games can help learners gain number sense by assisting them with comparing numbers, counting forward and backwards, and identifying numerical patterns.

This study found that varied types of play are important in the development of Grade 3 learners to enhance perceptual skills (integrated with Life Skills) while wordplay increases communication skills (when combined with the home language). These findings advocate the use of various types of play not only to enhance logical-mathematical intelligence but also to promote social interaction.

CONCLUSIONS

The South African curriculum (CAPS) recommends that educators use play to teach mathematics in the FP. Since number sense is most important for learners to understand numbers and number relationships and to solve mathematical problems, it is imperative for educators to make it fun and easy by using play. This study's findings revealed that Grade 3 educators have no clear guidelines for using play in teaching number sense. The researchers recommend the guidelines that include the following:

- Linking play with the learning outcomes.
- Integrating play into learning.
- Incorporate play in the assessment activities.

The researchers recommend that curriculum workshops should be conducted which will empower Grade 3 educators on how to include play-based approaches in lesson planning. This paper further recommends that if educators do not have access to resources provided by the DBE, they can find play resources in their immediate environment or create their own using recycled material.

This study was conducted in one circuit in the Limpopo province's Capricorn South district. Only six Grade 3 female educators with three to thirty-six years' experience ranging took part in the study, which limited the scope and applicability of the findings to the rest of the province. Nonetheless, the recommendations may be applicable to the rest of the province and other similar areas. For future research, the researchers suggest a longitudinal study that will explore the role of play in the creation of independence of learning and improvement of learner performance. Independence of learning is described by Prabhu (2006) as learning that occurs in practice where learners learn independently.

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APPENDIX A: INTERVIEW SCHEDULE

i. What is your highest qualification?

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ii. How many years have you been a Foundation Phase teacher?

ii. Have you taught any other phase?

---

iv. Till what Grade did you do Mathematics in school?

v. Did you experience difficulty in Mathematics at school?

vii. Do you experience difficulty teaching Mathematics? Elaborate.

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1. What are your views about the use of play when teaching number sense?

2. What role does play perform in the teaching of number sense?

3. How do you use play when teaching number sense?

4. Do you use play when planning your lessons?

5. Do you always follow your lesson plan? If no, explain.

6. Do your lessons always go as planned? Explain.

7. What is the purpose of using play when teaching number sense?

8. What is your opinion about teaching number sense to Grade 3 learners? Is it more important than the other concepts? Elaborate.

APPENDIX B: DOCUMENT ANALYSIS TOOL

Document analysis tool

- **Did the educator use play in the planning of the lesson?**
- **How did the educator use play when planning the lesson**
- **What types of play did the educator use in the lesson plan?**

- **Did the educator give the learner's task that equivalent to what she/he was teaching about?**
- **Did learners score all marks to show that play helped them to understand a concept better?**
- **Does the task given in the classwork book similar to workbooks?**
- **How did the role of play perform in the teaching of number sense?**

APPENDIX C: OBSERVATION SCHEDULE

Observation question	Yes/No	Comment
Did the educator use the planned lesson?		
Did the educator follow the lesson plan?		
Did the educator use play to teach learners?		
Did the educator use the relevant type of play for the topic?		
Where the role of play in teaching and learning number sense evidently?		
Was social interaction among learners and educators evident?		
Did learners fully participate in play?		

Flipped Classroom Assisted Autograph in Calculus Learning for Engineering Students: A Rasch Measurement Study

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Abstract: This project aims to investigate the Flipped Classroom Model Assisted Autograph can help engineering students improve their math skills. Many researchers have discussed how the Flipped Classroom Model has been proven successful for online learning usage, particularly during remote learning because of the Covid-19 pandemic's effects. However, the flipped classroom concept has yet to be implemented in conjunction with other supporting learning media. As a result, this project will combine the flipped classroom concept with Autograph media in calculus instruction. This study is an experimental study using a one-group pretest-posttest design. The data was gathered using essay test instruments and motivation questionnaires that were both valid and reliable. Rasch Model Measurement (Stacking Analysis) was employed as part of the data analysis, aided by the Winstep tool. The change in the logit value in each engineering student test results shows that engineering students' mathematics learning achievement improved. The result indicated that students gave positive feedback on the use of Autograph for mathematics after experiencing Flipped Classroom Model. Based on these findings, a Flipped Classroom Model based on Autograph can improve engineering students' mathematics learning achievement.

INTRODUCTION

The development of Information, Communication, and Technology (ICT) in education has brought many changes in the implementation of educational programs - covering educational curriculum, learning methods, to students' evaluation – which have been the results of the Covid-19 pandemic situation. The transformation of the learning process requires the integration of technology so that learning objectives and achievements can still be obtained optimally. Changes in the implementation of the learning process are supported by the direction of higher education curriculum policies that have flexibility in learning programs. The flexibility provided refers to the application of technology in the learning process. Junaidi et al. (2020) explained that changes in the implementation of learning process at higher education level can take advantage of a learning model that supports distance learning that is integrated with technology (e-learning), namely the flipped classroom model. Online learning is not merely caused by the Covid-19 pandemic, but also the demands of the digital era as well as the Industrial Revolution 4.0 towards Society 5.0. The

Industrial Revolution 4.0 demands technological developments integrated into the process of implementing education. The higher education based on Industrial Revolution 4.0 ensures that the learning process that is transformed from face-to-face to online-based learning (e-learning) does not affect the quality of the educational program. Sumantri et al. (2020) explained that changes in the implementation of the learning process can be utilized as part of improving the competence and quality of educational programs that integrate implementation procedures through technology. The existence of real classes is slowly being replaced with virtual classes assisted by various learning platforms available, as well as those developed by each higher education institution based on the Learning Management System. In addition, the implementation of online learning is also supported by the application of the flipped classroom model.

The definition of the *flipped classroom model* varies in the literature. Several studies explained that the flipped classroom model is a learning model that has two conditions, where students prepare learning materials before the class starts (e.g. watching lecture videos) and follow the learning process in real classes (such as applying problem-solving skills to assignments and structured discussions) (Bachiller & Badía, 2020; Lo & Hew, 2017). Another study explained that the application of the flipped classroom model is identical to combining two learning models, namely face-to-face learning models and online-based learning models by integrating technology (Lo et al., 2017; Ramadhani, 2020; Ramadhani & Fitri, 2020). Several previous studies implemented the flipped classroom model in the learning process before the Covid-19 pandemic and evaluated the integration of the model. The results show that the integration of the flipped classroom model is an active learning strategy that can improve students' learning experiences (Crosier et al., 2000; Herreid & Schiller, 2013; Meyers & Jones, 1993; Siegle et al., 2013) and is more effective than the traditional learning model which are based on lectures and discussions (Baytiyeh, 2017; MoK, 2014). The implementation of the flipped classroom model in learning mathematics in previous studies also showed that students' knowledge, attitudes, and learning achievement in mathematics increased. The flipped classroom model makes the learning process more active and has a positive influence on the student learning environment (Fernández-Martín et al., 2020; Krouss & Lesseig, 2020; Wei et al., 2020). The scheme of flipped classroom model assisted of Autograph in calculus learning cn be seen in Figure 1.

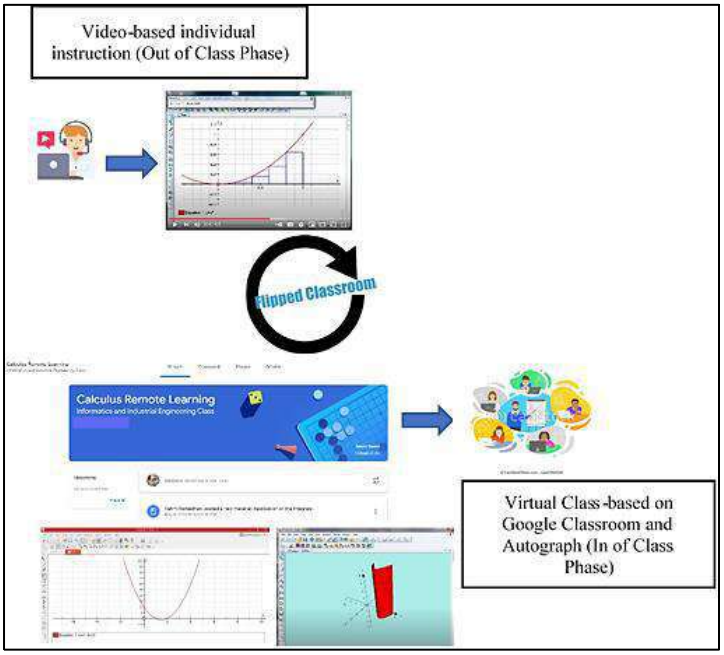


Figure 1: Scheme of Flipped Classroom Model Assisted of Autograph in Calculus Learning

The effectiveness of the flipped classroom model applied during the Covid-19 pandemic situation also contributes positive results in terms of character building, collaboration, communication, critical thinking, and creativity (Latorre-Cosculluela et al., 2021). Learning based on the flipped classroom model attracts students' attention that it can help them improve their performance in during exams. Contrary to Blair et al. (2016) found that no significant increase in terms of student performance and achievement through the flipped classroom model. These findings seem contradictory, but to Tang et al. (2020), effective communication was one of the most important factors in the application of the flipped classroom-based learning. The role of interactive communication built by educators also provides increased motivation and independence of students in participating in a learning process that is far different from previous usual learning experiences. The impression given to students on the implementation of learning based on the flipped classroom model is the first factor that educators need to pay attention to so that the achievement of learning outcomes can be reached successfully. In Attard & Homes (2020), interactive communication between students and teachers supported successful application of the flipped classroom model. Interactive communication also provides opportunities for students to understand their learning styles and gain access to equal learning opportunities that are more in line with student learning needs.

Regardless, the application of the flipped classroom model has not yet been collaborated with other learning support media. Supporting media of technology-based learning provides further convenience for students in exploring mathematics material provided. The application of the flipped classroom model only relies on the Learning Management System (LMS) which contains learning videos that were previously prepared by educators. LMS which contains learning videos has an impact on student satisfaction in acquiring new knowledge as well as student motivation and enthusiasm for learning. Exploration, interest, and creativity of students in developing material content are also limited, which causes the learning process to be centered on educators. Weinhandl et al.(2020a, 2020b) explained that the flipped classroom model can only integrate video or technology in the learning process, but the technology integration discussed is a technology that helps students solve problems, not through direct or direct activities. In addition, Orlando & Attard (2016) added that in the pre-class phase in the flipped classroom model material content transmission is carried out as a source of knowledge as well as training programs that require students to be active and creative in obtaining first knowledge on a material topic before being in the actual class (face-to-face phase). Thus, the flipped classroom model that collaborates with technology-based learning supporting media can facilitate the development of a learning environment and have a positive impact on students related to the findings obtained (Samuelsson, 2006).

One of the supporting media for technology-based mathematics learning that can be collaborated in the application of the flipped classroom model is Autograph. Autograph is one of the dynamic software applications that helps students learn calculus, algebra, and coordinate geometry. Douglas Butler created it to assist students and instructors in visualizing mathematics at the secondary and university levels through dynamically connected objects. In both 2D and 3D, Autograph emphasizes the use of dependent, selectable mathematical objects to assist students in grasping the principles of probability and statistics and coordinate geometry (Isiksal & Askar, 2007). Autograph allows drawing curves (both implicitly and explicitly specified), solving simultaneous equations, and plot derivatives, among other things. It has three modes of operation: 1D for statistics and probability, 2D for graphing, coordinates, transformations, and bivariate data, and 3D for three-dimensional graphing, coordinates, and transformations (Aman et al., 2018; Karnasih & Sinaga, 2015).

In this study, Autograph applied calculus for engineering students. Engineering students study calculus material in the first year of lectures, and several sub-materials require additional visualization as a tool for students to understand the material. The existence of Autograph will provide a lot of convenience in studying calculus consisting of graphs, area problems, to three-dimensional geometry for engineering students. Several previous studies have also shown that Autograph software is useful in terms of improving the effectiveness and quality of teaching and student learning communication (Triana et al., 2019). The use of Autograph software provides convenience in learning mathematics, especially in teaching content related to integral applications in visualization and graphics. Autograph also give the opportunities for students to find the concept of areas with their own, and ultimately improve students' conceptual understanding, mathematical communication, problem-solving, and critical thinking skills (Bina et al., 2021; Ramadhani, 2017). Learning mathematics that requires a lot of interaction, reasoning, and observation requires interactive software, like Autograph (Tarmizi et al., 2009).

Based on the discussion above, the use of Autograph in technology-based learning has not been collaborated in the application of the flipped classroom model, especially in learning mathematics. Therefore, this study aims to implement the flipped classroom model using Autograph in calculus learning for engineering students. The purpose of this study is to assist students in understanding calculus concepts to optimize engineering student mathematics learning achievement. Research questions that will be resolved in this study include:

1. Can the application of the flipped classroom model assisted by Autograph improve the mathematics learning achievement of engineering students-based test scores?
2. How is the motivation to study mathematics for engineering students after being taught using the flipped classroom model assisted by Autograph in learning calculus during the Covid-19 pandemic?

RESEARCH METHODS

This research an experimental study with the type of one-group pretest-posttest design. The sample of this research comprises a section of 24 first-year students of Informatics Engineering and Industrial Engineering majors. Demographic details of the research sample are depicted in Table 1.

Demographic Background	Research Sample	
	n	%
Gender		
Male	16	66.67%
Female	8	33.33%
Age		
17-19 years old	18	75%
20-22 years old	5	20.83%
> 22 years old	1	4.17%
Study of Program		
Informatics Engineering	13	54.17%
Industrial Engineering	11	45.83%

Table 1: Demographic Background of Research Sample (Engineering Students)

Table 1 shows that the demographic indicators of the research sample are used to refer to gender and the type of educational program taken by engineering students. The description of gender and type of educational program from research sample is needed to see a significant increase in student mathematics learning achievement per individual student level. The research data that measures engineering students of mathematics learning achievement used are the results of the calculus test on the sub-material of the application of definite integrals which are arranged in the form of a description test totaling 5 questions. The description test instrument is arranged based on the expected learning outcomes in these sub-materials. The assessment on the test instrument is arranged based on a rating scale that is adjusted to the stages of problem-solving per each test. The highest score for each question is 5 and the lowest score for each question is 1. If the student does not answer the test, then no score is given (no assessment is done). Meanwhile, the research data that measures students' learning motivation after using the flipped classroom model assisted by Autograph was compiled using Motivated Strategies for Learning Questionnaire (MSLQ) (Jackson, 2018; Pintrich et al., 1991) which has been modified according to the learning situation during the Covid-19 pandemic.

The two test instruments that had been developed are then tested for validation and reliability using Rasch Model Measurement analysis assisted by the Winstep application. The analysis was carried out using Winsteps using the Joint Maximum Likelihood Estimation (JMLE) equations that transform the raw data into interval data (logit) (Chang et al., 2020). The logit scale can indicate a person's skill and item's difficulty from positive infinity to negative infinity. The validation results show that the five description questions (the learning achievement instruments) and the learning motivation questionnaire developed are valid (according to the criteria for valid items regarding item response theory), namely the OUTFIT MNSQ value is in the range .5 to 1.5, the OUTFIT Z-STANDARD (ZSTD) value is in the range -2 to +2 and the Point Measurement Correlation (Pt. Measure Corr) value is in the range .4 to .85. The item's mean measure (logit) is .00, and the standard deviation (SD) is 34, indicating that item measurement variance in item difficulty was broad throughout the logit scale. For students, the mean measure was .66 logit, indicating that everyone was very enthusiastic about the study. Despite this, the person SD was .95, which is close to 1, demonstrating that person variation is suitable for data analysis. The five description test instruments were also declared reliable by referring to the results of the summary statistics test (Rasch Model Measurement) at the Reliability value (alpha = .68) in category enough for item and Reliability value (alpha = .76) in category good for a person (Sumintono & Widhiarso, 2015). Likewise, the learning motivation questionnaire was declared reliable by referring to the results of the summary statistics test (Rasch Model Measurement) at the Reliability value (alpha = .75) in category good. The summary statistics of item test and person can be seen in Table 2.

Statistic Test	Test Group	
	Person	Item Test
N	48	5
Measure	.66	0
Mean	17.2	165.2
SD	.95	.34
SE	.17	.20
Mean Outfit MNSQ	.99	.99
Mean Outfit ZSTD	.01	-.03
Separation	1.47	1.77
Reliability	.68	.76
Cronbach's Alpha	.69	

Table 2: The Summary of the Statistics Based on Pearson and Items

Questionnaire of student learning motivation in learning calculus using the flipped classroom model assisted by Autograph consists of 31 statements spread over three major domains, namely the value component, the expectation component, and the affective component. Each of these components is subdivided into six smaller domains which can be seen in Table 3.

Domain	Questionnaires Items
Value	
Intrinsic Goal Orientation	1, 16, 22, 24
Extrinsic Goal Orientation	7, 11, 13, 30
Task Value	4, 10, 17, 23, 26, 27
Expectancy	
Control of Learning Beliefs	2, 9, 18, 25
Self-Efficacy for Learning and Performance	5, 6, 12, 15, 20, 21, 29, 31
Affective	
Test Anxiety	3, 8, 14, 19, 28

Table 3: The Domain Rubric of Mathematics Learning Motivation Questionnaires

Questionnaires are given after the complete calculus learning using a flipped classroom model assisted by Autograph. The time needed to complete the questionnaire is less than 45 minutes. Students of Informatics Engineering and Industrial Engineering must read and answer the statements as prepared by the researchers. Next, they have to answer a questionnaire and choose a response option based on their learning experience.

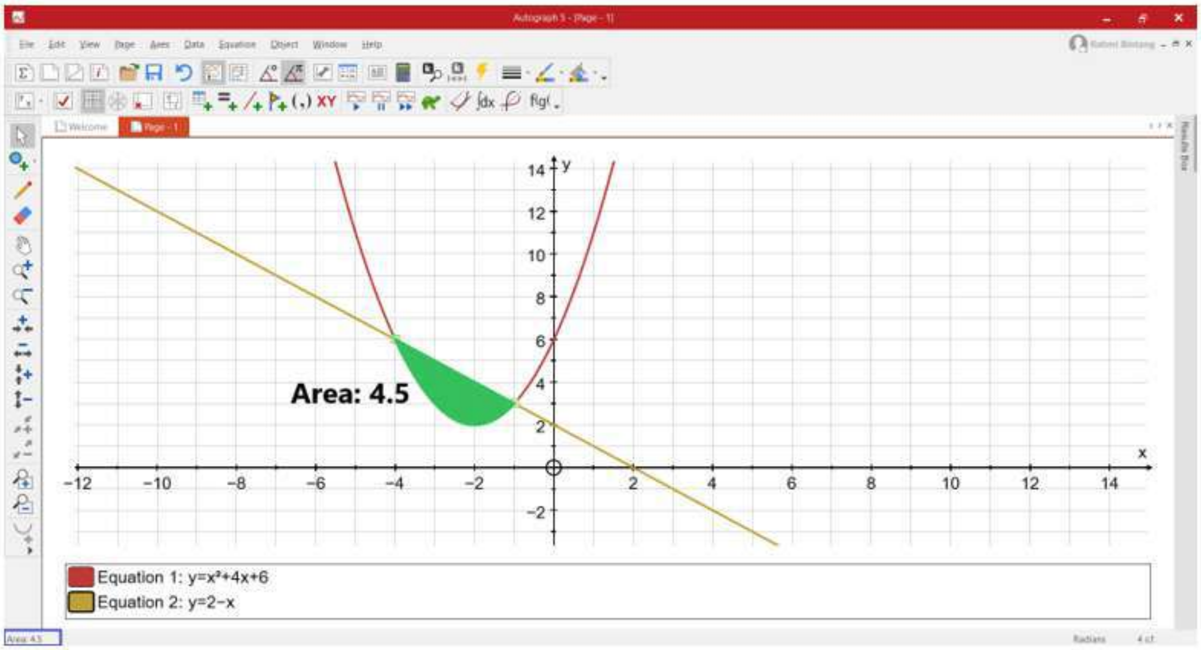
Next, the test instrument that has been valid and reliable is used as a test tool in the application of the flipped classroom model using Autograph in calculus learning. The data obtained are data before (N=24) and data after being given flipped classroom model learning using Autograph (N=24). The data used is not the final score obtained by each student, but the raw score on each description question that has been obtained by the student (both raw scores on the pretest and the posttest). The raw data is then analyzed using stacking analysis techniques. Analysis with data stacking techniques is used for intervention research designs that are pretest-posttest, where one group of respondents or subjects are tested at two different times (Arnold et al., 2018). The next data analysis will be seen based on two indicators, namely the mathematics learning achievement of each engineering student and the student's learning motivation after being taught using the flipped classroom model assisted by Autograph.

RESULTS AND DISCUSSION

The implementation of mathematics learning in many universities around the world during the Covid-19 pandemic was carried out using the flipped classroom model (Collado-Valero et al., 2021; Khan & Abdou, 2021; Tang et al., 2020). The flipped classroom model applied in this study uses lecture videos uploaded on a sharing site, to name YouTube, for the in-class phase and uses the LMS, to name Google Classroom, as a phase outside the classroom. Both phases of learning continue to use the Autograph application as a learning tool that aims to make it easier for engineering students to understand calculus. The students can make visualization on the abstract concepts of definite integral calculus in 2D and 3D design. They may calculate the area between the curves and the volume by typing the function in the Autograph's function box, then selecting the analysis option to calculate it. The interactive features of Autograph allow students to become engaged in explorations where the students themselves find the answers. One of the tests asked about finding the area between curves using Autograph can be seen in Figure 2.

Find the area between curves $y = x^2 + 4x + 6$ and $y = 2 - x$!

We can illustrate the graph from the function $y = x^2 + 4x + 6$ and $y = 2 - x$ in cartesian filed using Autograph.



Autograph can help us to find the area between curves (Area: 4.5)

Figure 2: The Test of Calculus Solving Using Autograph

Students can improve their mathematical abilities by using visualization in Autograph to calculate the area between curves, as shown in Figure 2. In addition, the students compare their Autograph solutions to manual computations. They can figure out how to calculate the area between curves without using a calculation. Before the learning is carried out, engineering students will be given a pretest consisting of calculus intending to see how far the engineering students understand the given calculus, especially definite integral. The same test will be given again after engineering students have finished learning using the flipped classroom model assisted by Autograph. The test results of both pretest and posttest were analyzed using the Stacking Analysis method on the Rasch Model Measurement with the help of the Winstep application. The results of the analysis will show the logit value obtained by each student based on the assessment per test item that has been given before and after learning. The logit value is the value of the logarithm function (logarithm odd unit). The use of logarithmic functions in measuring the improvement of students' mathematics learning achievement abilities is carried out on Rasch modeling which will be used as an analytical technique in this study. Rasch modeling not only measures the number of correct answers obtained by the students but also calculates the probability odds ratio for each item of the test instrument (Sumintono & Widhiarso, 2015). The use of stacking analysis in this study will provide information on how impactful the difference in the logit value is in the pretest and posttest conditions. The differences in mathematics learning achievement examined through the person logit data of each student in the pretest and posttest conditions can be seen in Figure 3.

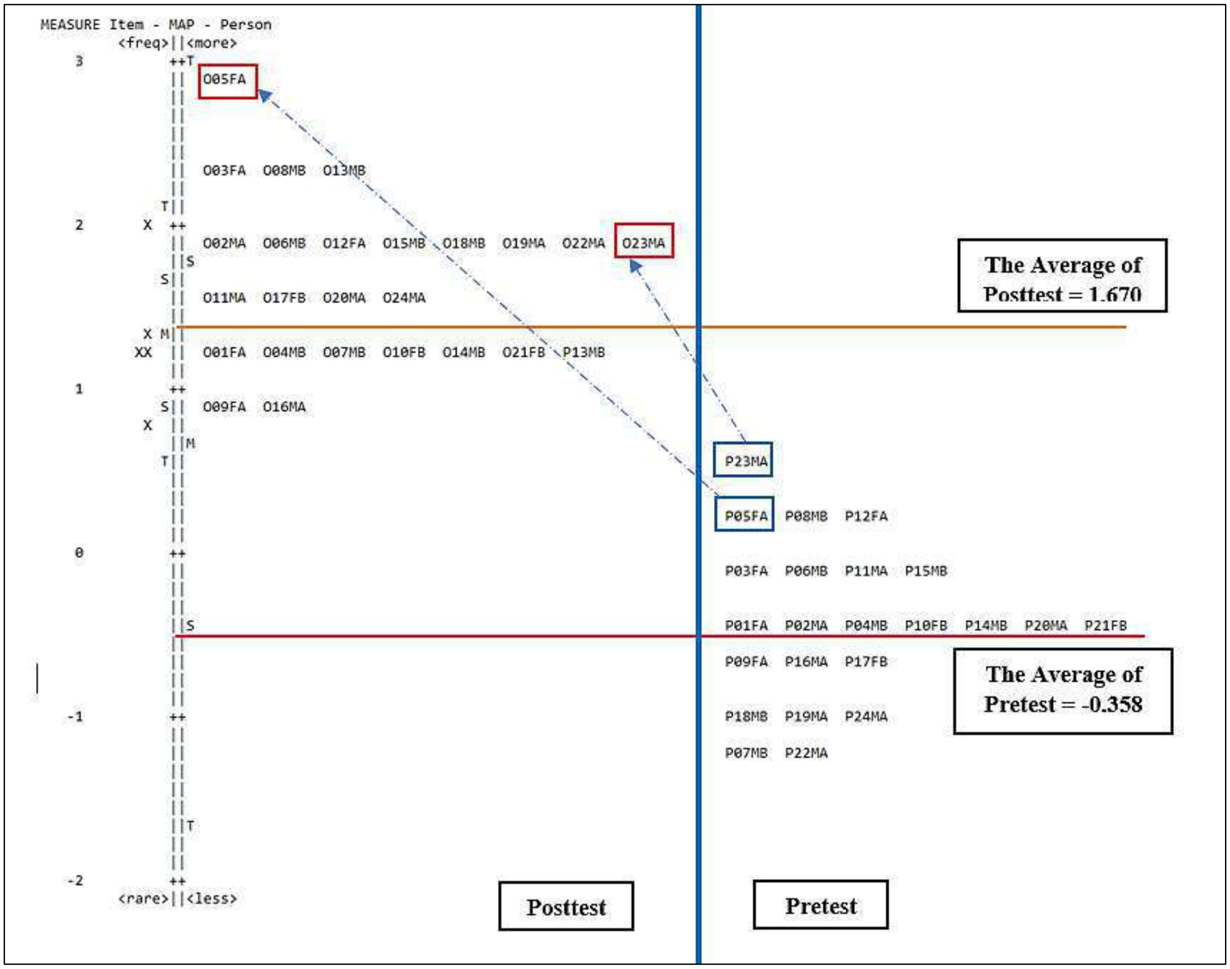
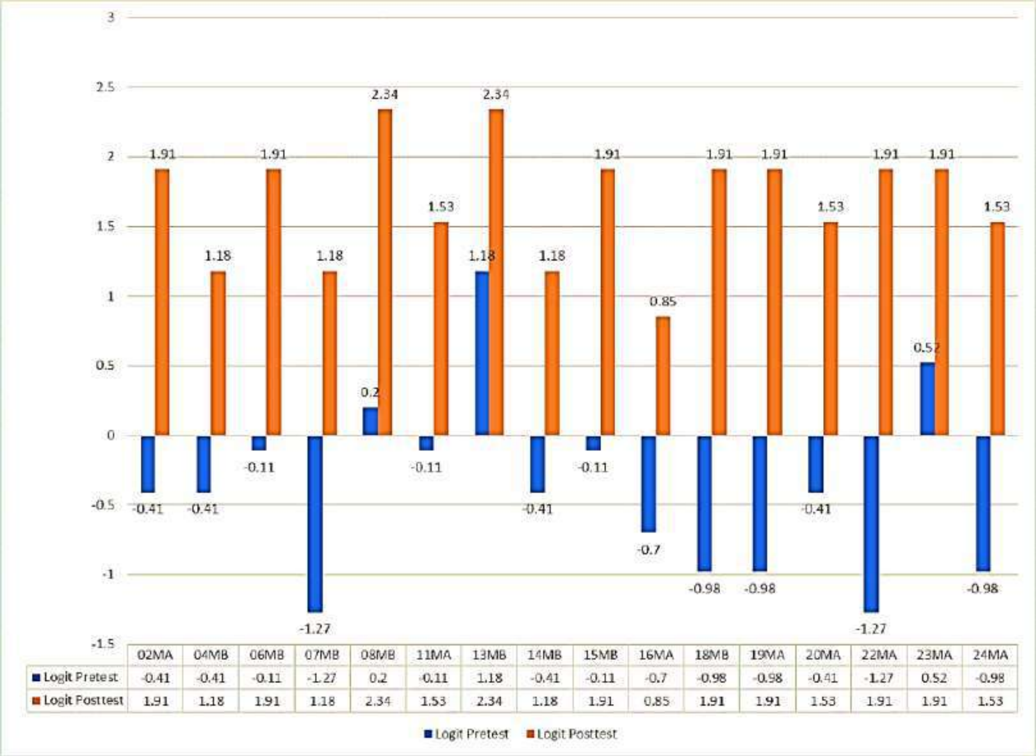


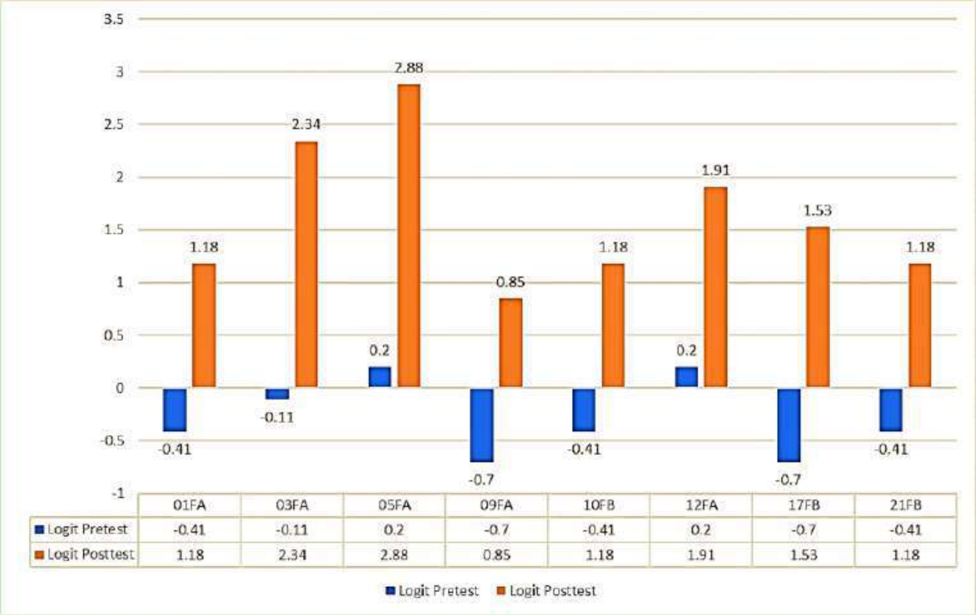
Figure 3: Pretest and Posttest Differences in Calculus Using Map Wright

Based on Figure 3, it can be seen that there are differences in the level of difficulty of each question given to engineering students. The question code is marked with the symbol X, and it can be seen that the level of difficulty of the questions given varies, there is 1 question in the category easy (Logit Value < +1), there are 3 questions in the category medium (+1 < Logit Value < +2), and 1 question in the category difficult (Logit Value > +2). It can also be seen in Figure 3 that there is an increase in the average logit value on the test results of engineering students before and after taking calculus using the flipped classroom model assisted by Autograph. The average logit value on the mathematics learning achievement of engineering students in the pretest obtained a value of -0.358 and experienced a significant increase in the posttest with an average logit value of 1.670. These results show that the level of understanding of engineering students in calculus is better after receiving instruction with a new model even though they are in the Covid-19 pandemic condition. Students with code P23MA had the highest logit value (+ .52) in the pretest, while students with code O05FA had the highest logit value (+2.88) in the posttest. In Figure 3, it can also be seen that the increase in the calculus achievement of one of the engineering students with the code 05FA experienced an increase in the logit value, from a logit value of less than +1 to a logit value greater than +2. The same changes were also experienced by almost all engineering students. This change in logit value also provides evidence that changes in the new learning environment and the use of appropriate supporting learning media can maximize student learning achievement (Cimermanová, 2018; Coman et al., 2020). More detail on how the graph of the increase per individual student based on the logit value obtained before and after taking calculus learning using the flipped classroom model assisted by Autograph can be seen in Figure 4, Figure 5, and Figure 6.



	02MA	04MB	06MB	07MB	08MB	11MA	13MB	14MB	15MB	16MA	18MB	19MA	20MA	22MA	23MA	24MA
Logit Pretest	-0.41	-0.41	-0.11	-1.27	0.2	-0.11	1.18	-0.41	-0.11	-0.7	-0.98	-0.98	-0.41	-1.27	0.52	-0.98
Logit Posttest	1.91	1.18	1.91	1.18	2.34	1.53	2.34	1.18	1.91	0.85	1.91	1.91	1.53	1.91	1.91	1.53

Figure 4: Graph of Logit Values-Based Changes in Mathematics Learning Achievement Before and After Treatment for Male Students



	01FA	03FA	05FA	09FA	10FB	12FA	17FB	21FB
Logit Pretest	-0.41	-0.11	0.2	-0.7	-0.41	0.2	-0.7	-0.41
Logit Posttest	1.18	2.34	2.88	0.85	1.18	1.91	1.53	1.18

Figure 5: Graph of Logit Values-Based Changes in Mathematics Learning Achievement Before and After Treatment for Female Students

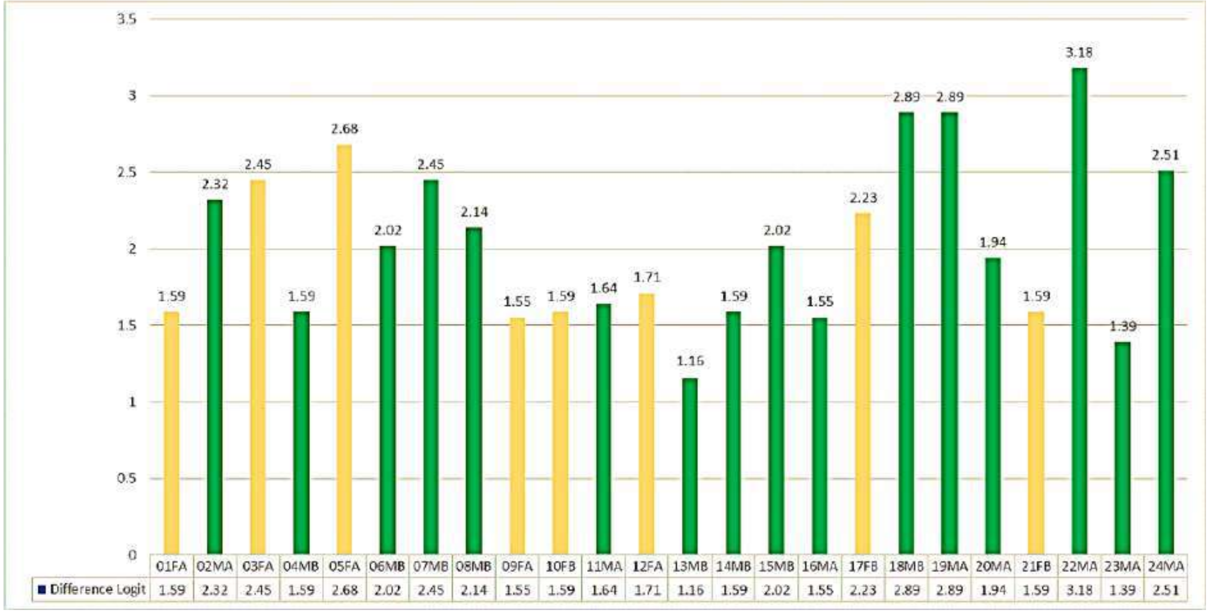


Figure 6: Graph of Difference Logit Values in Mathematics Learning Achievement

Based on Figure 6, it can be seen that the highest change in logit value is at +3.18 (code 22MA) and the lowest change in logit value is at +1.16 (code 13MB). In Figure 4 and 5, we can see that engineering students in the Informatics major (Code A) have a higher average change in calculus achievement than engineering students in the Industrial study program (Code B). The average change in the logit value for informatics engineering students shows a logit value of +2.108 while the industrial engineering student shows a logit value of +1.18. Based on the gender demographics, male students experienced a change in logit value (increased calculus learning achievement) both in informatics engineering students and industrial engineering students. The results of this analysis clearly show that both informatics engineering students and industrial engineering students simultaneously experience positive logit value changes which increase calculus learning achievement. The same thing applies to male students and female students as they experienced a significant increase in learning achievement. Although the change in the highest logit value was obtained by male students, female students obtained the highest logit value in both pretest and posttest. Indicators of differences in majors and genders on engineering students do not have a significant impact on changes in calculus learning achievement using the flipped classroom model assisted by Autograph. Changes in calculus learning achievement occurred due to the effects obtained from new learning treatments in the environment of engineering students during the Covid-19 pandemic. The flexibility, independence, and creativity created by the application of the flipped classroom model (Fernández-Martín et al., 2020; Quinn & Aarão, 2020) as well as the application of Autograph (Moksin et al., 2018; Zubainur et al., 2018) proved to have a real effect in improving students' mathematics learning outcomes.

The converging results are also obtained from the analysis of students' learning motivation after participating in learning using the flipped classroom model assisted by Autograph. Student responses regarding the motivation to learn calculus after participating in learning the flipped model with the help of Autograph can be seen in the Figure 7.

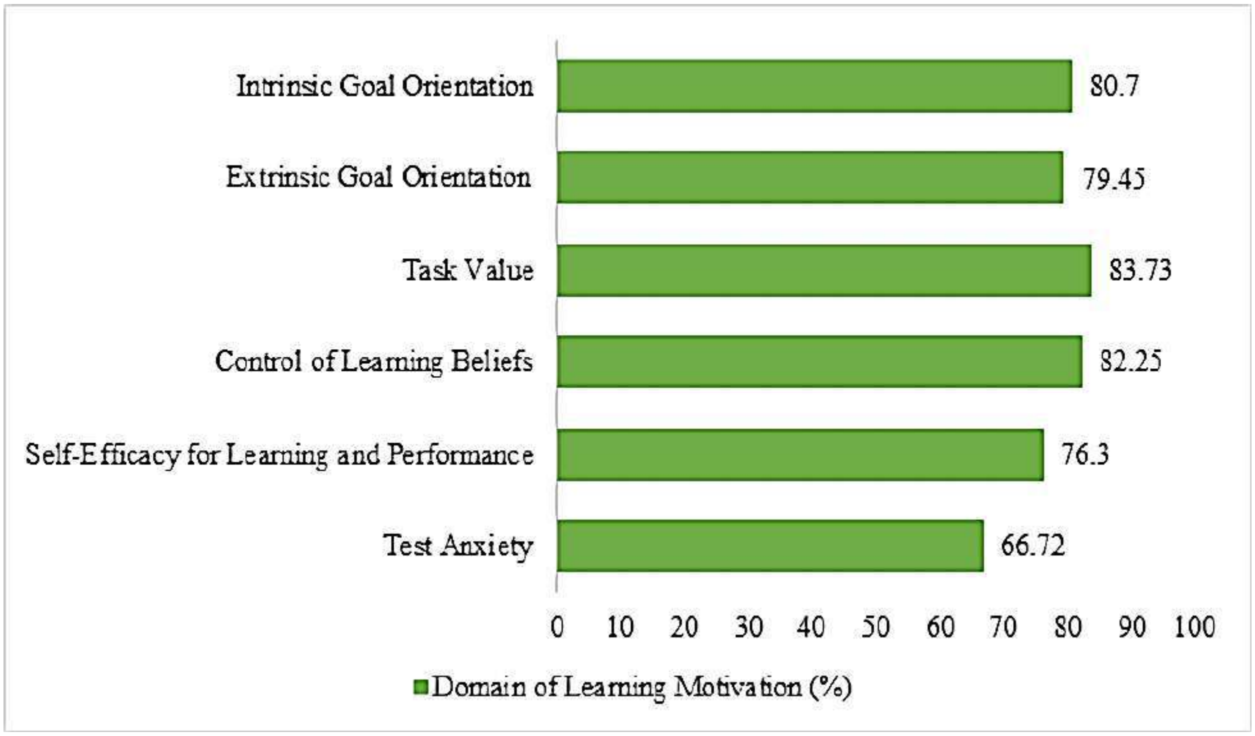


Figure 7: Graph of Student Learning Motivation Achievement

Figure 7 shows that the Task Value domain has a higher percentage value compared to other learning motivation domains, which is 83.73%. These results indicate that learning using the flipped classroom model assisted by Autograph gives special interest to engineering students and is very helpful for engineering students in completing the assignments given. The high task value also shows that students have progressed to develop their learning abilities (Colomo-Magaña et al., 2020). The results of the learning motivation questionnaire also show that the learning obtained by students provides a flexible learning atmosphere, increases collaboration between students and educators (McCarthy, 2016), can adapt to learning styles (Molnar, 2017; Mousa & Molnár, 2020), and the level of learning independence possessed by each student (Kim et al., 2017). The anxiety factor in dealing with the questions given as well as other external factors, such as personal, family, and self-control over the implementation of technology-based distance learning did not affect student learning achievement (Ramadhani et al., 2020; Tsai et al., 2017).

Based on the results of the questionnaire analysis of student learning motivation, students were very interested in participating in calculus learning using the flipped classroom model with the help of Autograph. Students are greatly helped in understanding calculus material, exceedingly definite integral calculus, and its application in learning conditions during the Covid-19 pandemic. Students also remain motivated to follow a series of learning stages that have been designed in this study. The results obtained when evaluated from the lecturer's perspective are not dissimilar to the analysis results on student learning motivation. During the Covid-19 epidemic, lecturers were excited about using the flipped classroom style to teach students. For lecturers who want to explore learning in the Covid-19 pandemic situation, the flipped classroom style is beneficial. Lecturers deliver information in the form of videos and maintain a collaborative and productive learning atmosphere which provides an organized discussion area with the help of the LMS. For lecturers, using Autograph as a learning medium in integral calculus gives a unique teaching experience. Lecturers can assist students in improving their mathematical reasoning skills by allowing them to find their answers using graphic visualization-assisted Autograph. Cevikbas & Kaiser (2020) strongly agreed that the use of LMS and mathematics-specific tools (mathematics video, dynamic software) are crucial if affordance enabling flipped classroom models are to be realized.

Flexibility, online courses, study management, technology, classroom learning, and online interaction were determined to be six markers of successful learning utilizing the flipped classroom approach by the mathematics instructors. Flipped classroom model accommodated those indicators, no wonder it makes learning successful. Flipped classroom model is one of the approaches suggested by various educationalists and research scholars throughout the globe, which will provide the learners with a powerful learning experience (Saboowala & Mishra, 2021). The researchers recommended that the flipped classroom model-assisted Autograph be implemented in other mathematics learning according to the research result.

CONCLUSIONS

Learning calculus using the flipped classroom model assisted by Autograph provides a new learning experience for engineering students. Graphic visualization presentations were designed by engineering students assisted with Autograph and collaborations through the flipped classroom model, proved to help students understand the concept of calculus integral. By that means, the teaching materials are designed in this study to be flexibly implemented in online learning, especially during the Covid-19 pandemic. Communication that exists between the engineering students and the educators can also be carried out well with the support of LMS-Google Classroom which is used in the classroom phase as a virtual class instead of a face-to-face class. The flipped classroom model's effectiveness is further aided by solid student-teacher interaction, autonomous learning, and the use of technology-integrated learning media Autograph, which keeps the virtual-based learning environment appealing and inspires students to participate in the learning process.

Other results are also shown by responses related to learning motivation questionnaires given to engineering students after the learning is done. Positive responses were given, and on the task value dimension, the students answered that the use of the flipped classroom model assisted by Autograph helped them develop their learning skills, thus gaining new knowledge. The results of this study are expected to be supporting data for educators in developing a flipped classroom model that is supported by other technology-integrated learning media. This research is also expected to be a benchmark for other researchers in conducting further experimental studies that focus on student learning activities after applying the flipped classroom model in learning mathematics.

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Tapis Patterns in the Context of Ethnomathematics to Assess Students' Creative Thinking in Mathematics: A Rasch Measurement

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Abstract: Mathematics is employed in cultural activities in traditional and nontraditional societies. Ethnomathematics refers to mathematical ideas integrated into a culture. The culture can be used as a transformation effort to explore mathematical concepts in order to bring the mathematics closer to the reality and understanding of its people. Moreover, culture can be used as a groundwork for school mathematics. This study investigated ethnomathematics as geometry context illustrations of the patterns of Tapis Lampung in Indonesia. With an ethnographic approach and Rasch measurement that is to measure of persons and items on the same scale, this research is a quantitative study. Data were collected through test and documentation with tapis pattern results. It was discovered that the designs of the Tapis Lampung include geometric concepts that can be expressed as translations, rotations, reflections, and dilations. Moreover, students have different results of the creative thinking in mathematics. Each Tapis pattern also includes local values (i.e. sacred values, social stratification, history and understanding, creativity, inclusiveness, and economic value). Tapis Lampung can be used to disseminate and inform the world about Indonesian local wisdom and potentially as a source of contextual mathematics in rural schools and urban areas.

INTRODUCTION

Indonesia is a country with diversity in cultures and religions. Indonesia's population consists of indigenous people, descendants of Chinese, Egypt, India, and the Indo or Eurasian groups engaged in Indonesia and Europe. Indonesia has more than 500 ethnic groups and more than 600 languages (Roslidah et al., 2017). These must be maintained and managed by promoting the values of diversity so that no ethnicity stands as a closed and independent entity but rather interacts and interdepends on and mutually influence one another (Ewoh, 2013).

Lampung is a province in Indonesia, and it is strategically located. It lies at the southern end of the island of Sumatra, making it a gateway to the island of Sumatra. This makes Lampung the busiest due to migrants from various tribes. Therefore, Lampung people are not limited to Lampung in the Lampung province and those in Sumatra Island. Moreover, indigenous peoples of Lampung are divided into two Lampung customs and dialects, namely, Pepadun with O dialect and Paminggir (Saibatin) with A dialect. Papadun areas include Abung, Way Kanan, Sungkai, Tulangbawang, and Pubian, and the tribes under Paminggir include Paminggir Belalau/Ranau, Paminggir Krui, Pesisir Semangka, Pesisir Teluk, Pesisir Rajabasa, and Pesisir Melinting-Meringgai. The sixth customary entity inhabit the coastal West, South, and East Lampung. Thus, Lampung is diverse in culture.

Considering cultural diversity, transforming culture must be known to preserve national culture and cultural education (Nugraha, 2019). Regarding ethnomathematics, diverse cultures can be explored in education, especially in mathematics (Hartinah et al., 2019; Kieran et al., 2013). Despite its expansive scope, ethnomathematics is frequently confounded with ethnic or indigenous mathematics. In this article, I argue that ethnomathematics research should not be limited to the mathematical knowledge of culturally distinct people or people engaged in daily activities. The focus could be on academic mathematics, with an emphasis on the social, historical, political, and economic factors that have shaped mathematics into what it is today. With this background, ethnomathematics research has provided new and refreshing insights into the field of mathematics education, not only regarding ethnic or indigenous mathematical knowledge, but also regarding ethnomathematics approaches to mathematics and its education (Pais, 2011).

Mathematics is well-known in both traditional and nontraditional cultural activities of the societies (Kelly, 2018). Thus, this activities refers to cultural mathematics ideas and acknowledges that each culture and person develops unique ways and complex reasons to understand and modify their own realities (Presmeg et al., 2016; Rosa & Orey, 2017; Rubel, 2017). Furthermore, the ethnomathematics perspective is connecting mathematical concepts and local character value. These perspectives are contained in the 2013 Indonesian curriculum integrated the concept of education based on a character through culture, ethnicity, and values to promote by the Government of Indonesia (Suryadi et al., 2019).

Culture has many aspects that can be beneficially integrated into education. Cultural-based mathematics help students develop a greater interest in mathematics, enabling them to understand that mathematics extends beyond the classroom (Brown et al., 2019). Furthermore, because it investigates how mathematical ideas and practices are processed and used in daily activities, ethnomathematics shows how various cultural groups organize their realities (Brown et al., 2019; Rosa & Gavarrete, 2017; Rosa & Orey, 2016). Ethnomathematics is a dynamic, holistic, transdisciplinary, and transcultural field of study. Its evolution would benefit academic mathematics because it advances in a way that is much closer to reality and the agents immersed in reality (D'Ambrosio, 2020).

Furthermore, ethnomathematics can be seen as the process by which people from a particular culture use mathematical ideas and concepts to deal with quantitative, relational, and spatial aspects of their lives (Bender & Beller, 2018; Supiyati & Hanum, 2019; Widyastuti et al., 2021, p. 2). To describe the mathematical practices of identifiable cultural groups, d'Ambrosio (1985) coined the term ethnomathematics. It is defined as the study of mathematical ideas found in a culture. This perspective of mathematics validates and affirms all people's mathematics experiences by demonstrating that mathematical thinking is inherent in their lives due to the relationship between mathematics and culture (Balamurugan, 2015; Hannula, 2012; Pathuddin et al., 2021). In this context, mathematical development in different cultures is based on common problems encountered within a cultural context, according to an ethnomathematical perspective (Borko et al., 2014; Yuliani & Saragih, 2015).

Ethnomathematics have been widely researched in many countries worldwide. "Exploration of Ethnomathematics at the Margin of Europe – A Pagan Calendar" is one of the reported studies on ethnomathematics (Bjarnadóttir, 2010). In the study, in 930 in Iceland, researchers discovered a system for recording time, or a calendar influenced by the environment, specifically by observing celestial bodies, including the sun and moon. This is an example of empirical adjustment of mathematical models in which the length of the calendar year is adjusted to natural observations. d'Ambrósio, (2006) defined ethnomathematics as follows: "In the same culture, individuals provide the same explanations and use the same material and intellectual instruments in their daily activities." Nyoni (2014) reported that because a mathematics game known as "mutoga" in the local language of South Africa is played every day at home and in school, ethnomathematical epistemology can be implanted. Responding to curriculum and practice assessments, according to researchers, must be mediated by cultural pedagogy.

In Indonesia, few researchers still explore the mathematical concept in unique and rare patterns in traditional woven. However, some researchers have explored fabric patterns worldwide. Regarding the batik patterns, in Yogyakarta batik, the concept of geometry transformation is employed to make Yogyakarta's unique batik motif (Prahmana & D'Ambrosio, 2020). This research is an ethnography study, which shows moral, historical, and philosophical values. The author stated that the mathematics concept can be implemented for students who live in rural and urban areas. Additionally, previous studies have shown that mathematics basic concepts can be explored as Sundanese ethnomathematics (Muhtadi & Charitas Indra Prahmana, 2017). This study was focused on the activities of indigenous people and explored Sundanese culture. Unfortunately, the standards for the application of mathematical concepts in measuring the activity of mathematical rules have not been met. Furthermore, mathematics concepts can be described with ethnomathematics on Dayak Tabun traditional tools (Hartono & Saputro, 2019). This study focused on the aspect of motif as not only as geometry but also algebra and trigonometry concepts. Some researcher try to connect between local culture and students' official cognitive. As Pais (2011) argued that this resource, concerned in establishing a "bridge" between local and school knowledge, is prevalent in ethnomathematics research. This "bridging" of local and school mathematics knowledge is viewed as a way of valorizing students' cultures while also allowing students to gain a better understanding of formal mathematics through their own not yet formalized knowledge. Based on previous studies, this study investigates the local culture “Tapis Lampung” as an exploration ethnomathematics approach into the classroom to assess students’ mathematical creative thinking. The Tapis Lampung will be explored in more detail in the literature review. Tapis Lampung was investigated as a geometry concept for the indigenes of Lampung, which is an ethnomathematics concept. Therefore, it is decided to develop an open-ended test supported by the Rasch measurement model in order to identify and assess the development of students' creative thinking in relation to grade level and gender (Soeharto, 2021). Rasch Analysis (RA) is a one-of-a-kind mathematical modeling technique based on a latent trait that achieves stochastic (probabilistic) conjoint additivity (conjoint means measurement of persons and items on the same scale and additivity is the equal-interval property of the scale) (Granger, 2008).

THEORETICAL BACKGROUND

Ethnomathematics approach

A fundamental change in mathematical instruction is required to account for the continuous change in the demographics of students enrolled in mathematics classes (Rosa & Orey, 2011). Numerous scholars have developed culturally relevant pedagogical theories that take a critical look at the teaching and learning process by incorporating cultural elements and values into mathematics (Fouze & Amit, 2017). It is required for the integration of a culturally relevant mathematics curriculum into the existing mathematics curriculum. From this point of view, it is critical for culturally relevant education because it proposes that teachers contextualize mathematics learning by connecting mathematics content to the student's culture and real-world experiences (Matthews, 2018).

According to the Rosa & Orey (2016) approach, culturally relevant mathematics should be centered on the sociocultural context, incorporating ethnomathematical concepts and ideas, and solving contextual problems from an ethnomathematics perspective. Additionally, ethnomathematics studies are increasingly being conducted, in which culture is linked to mathematical concepts and examples of the cultural context in mathematics are described (Barton, 1996, 2007). Following the recent Indonesian curriculum's emphasis on integrating culture into the curriculum, ethnomathematics may be a promising approach for assisting students in exploring their culture in order to generate mathematical concept ideas while also appreciating the cultures of others in a multicultural country (Peni & Baba, 2019). Additionally, schools must be established to teach the official knowledge while leaving the community's indigenous knowledge on its own (Pais, 2011). Therefore, including cultural aspects in the mathematics curriculum will benefit students in the long term; cultural aspects help students recognize mathematics as a part of everyday life, increasing their ability to make meaningful connections, and deepening their understanding of mathematics (Adam, 2004). This perspective was based on the numerous facets that the culture can be incorporated into the delivery of education to benefit students. The learners gain an understanding of how mathematics is used in everyday life, which improves their ability to make meaningful mathematical connections and broadens their understanding of all types of mathematics (Begg, 2001).

Tapis Lampung as Indonesian Traditional Pattern

One of the well-known and most respected cultural items is tapis. Tapis is an item of Lampung women’s clothing; a shaped sarong made from cotton yarn woven with a motif or decoration material and silver or gold thread with embroidery. However, due to the increasingly modern times, Tapis Lampung can also be used as clothing for men (Figure 1). Tapis Lampung is rich in mathematical concepts, making it suitable as an alternative learning resource in teaching mathematics, especially material related to geometry (Figure 2). Geometric decorations found on tapis fabrics, in general, have firm contours with several line elements, such as straight lines, curves, zigzags, and spirals, and various shapes, such as triangles, rectangles, circles, kites, regular polygons, and geometric transformations.

Figure 1: Tapis Lampung in Wedding Ceremony

The nobility used Tapis Lampung in the past, but now, it is also used by ordinary people in Lampung. In the Lampung community, tapis is a source of income and a staple. It is a local commodity and source of revenue that needs to be preserved. The beauty of tapis is appreciated in the artistic forms spread on cloth sheets (Suherman et al., 2021). Current tapestry shapes show rhythmic regularity or pattern when closely observed. To create some forms of order on the rug, geometric transformations are employed. Euclidean geometry, in contrast, is used to identify forms made by human beings, including rectangles, circles, spheres, and triangles (Suherman et al., 2018). Thus, the structure of Tapis Lampung, which serves as a geometric representation of mathematical concepts, needs to be explored and made known globally.

We illustrate how cultural practices have been incorporated into mathematics education, with designs that adhere to the cultural geometry model, either explicitly or implicitly. These examples demonstrate the potential for Indigenous students to use their culture when designing and implementing school activities through the lens of the cultural geometry model.

Cultural Geometry: Integrating Ethnomathematics into Indigenous Students in the School

Mathematics constructed and evolved historically because of cultural norms or generally accepted and agreed upon practices. Consider how geometry developed during the Babylonian and ancient Egyptian civilizations between 5000 BC or 4000 BC to 500 BC (Muhtadi, 2017). Ancient civilizations made extensive use of visible geometry in their constructions, such as irrigation, flood control, swamp drainage, and large structures. In ancient Egypt, the geometry was used to define land boundaries along the Nile's banks because of flooding. Floods continue to strike the Nile's banks, erasing the boundaries of land owned by the indigenous community. Egyptians sought to redefine land boundaries while maintaining ownership of previously owned land. Later on, the Egyptians discovered a lengthy and extensive measurement system for community-agreed land boundary demarcation and for resolving the problem of tilled flooded land.

Additionally, the Babylonian and ancient Egyptian civilizations are regarded as the forerunners of the birth of the mathematical branches of knowledge, specifically geometry. The knowledge that appears first is cultural, such as experimentation, observation, assumption/estimation, or intuitive activities, which then evolved into standard and universal knowledge. Geometry then reaches a golden age during Euclid's (300 BC) era, when knowledge of geometry is constructed using an axiomatic system. Basic geometric shapes have been widely used as primitive concepts in previous community cultures (knowledge base, a concept which is not defined). The connections between these concepts resulted in the development of definitions, postulates/axioms, and theorems that comprise a deductive system. The deductive system is then accepted as mathematical knowledge, with geometry being classified as a subfield of mathematics.

Fundamentally, the development of human civilization is inextricably linked to the development of culture and mathematics. Nonetheless, because the method of obtaining it is unique, many people appear skeptical that culture cannot be separated from mathematical activity, but also cannot be considered separately or as a source of illumination for the development of mathematics today. In this context, culture encompasses a broad and distinct perspective, as well as being bound to the people's customs, such as gardening, playing, creating, and solving problems, as well as how to dress.

Integrating ethnomathematics into school mathematics for Indigenous students is viewed as significant because it demonstrates the existence of alternative forms of mathematics (Gerdes, 1985). However, the approach favored by these authors is:

> *An integration of the mathematical concepts and practices originating in the learners' culture with those of conventional, formal academic mathematics. The mathematical experiences from the learner's culture are used to understand how mathematical ideas re formulated and applied. This general mathematical knowledge is then used to introduce conventional mathematics in such a way that it is better understood, its power, beauty and utility are better appreciated, and its relationship to familiar practices and concepts made explicit. In other words, a curriculum of this type allows learners to become aware of how people mathematise and use this awareness to learn about a more encompassing mathematics (Adam et al., 2003).*

Diverse perspectives on cultural traditions and practices enable a more nuanced understanding of how Tapis Lampung patterns become valued in general. As a result, while ethnomathematics has been hailed as a means of enriching students' understandings of mathematics through the use of contexts familiar to Indigenous students and enabling them to see themselves and their communities as mathematicians, concerns have been raised about how this integration may have unintended consequences. Even if Indigenous students gain mathematical insights through interaction with familiar cultural practices, the intrinsic value of the culture may be diminished if it is used merely to transmit mathematical ideas.

For many years in Indonesian educational discourse and on the school curriculum, indigenous culture was limited to the recognition of visual elements, such as signs, images, and iconography, that are immediately identifiable as representing indigenous culture and books of Indonesian myths. Cultural traditions and practices, on the other hand, should be valued in and of themselves.

Implementing the cultural geometry model in mathematics classrooms is challenging because all of the issues raised in each step must be considered concurrently. While mathematical concepts can contribute to cultural comprehension, if they are merely presented as representations of "Western mathematics," the possibilities for discussing Indigenous cultural artifacts and processes are likely to result in cultural imperialism (Bishop 1990). Rather than that, striking a balance between Indigenous cultural knowledge, including language, and mathematical cultural knowledge entails reflecting on the cultural geometry model's highlighted aspects.

Translation on tapis Lampung motif

According to Martin (2012), the mapping α is expressed as

$$\begin{cases} x' = ax + by + c, \\ y' = dx + ey + f, \end{cases}$$

This means that $(x', y') = \alpha((x, y))$ for each point (x, y) in the Cartesian plane, where $a, b, c, d, e, and\ f$ are numbers. A translation is a mapping having equations of the form

$$\begin{cases} x' = x + a, \\ y' = y + b, \end{cases}$$

Theorem: Given points P and Q, there is a unique translation taking P to Q, namely, $\tau_{P,Q}$.

Thus, if $\tau_{P,Q}(R) = S$, then $\tau_{P,Q} = \tau_{R,S}$ for points $P, Q, R, and\ S$. Note that the identity is a special case of a translation as $l = \tau_{P,P}$ for each point P. Also, if $\tau_{P,Q}(R) = R$ for point R, then $P = Q$ as $\tau_{P,Q} = \tau_{R,R} = l$.

An image that depicts the translation of geometric transformations, for instance, is the slope motif. Slope motifs are the most common type of motivation in Lampung society. For this reason, fabric tapestry is always created with slope patterns as a job by craft/art teachers for students in schools (Lampung Province). Movements on the slopes are often the key reason for filter manufacturing. This motif is red, black, and yellow at the back, and the carpet is golden to make it more beautiful and harmonious. The pitch motions are regarded as simple motifs.

The form shown in Fig. 2 also results in the combination of the basic forms in the previous figure with vertical lines. The motif of the path is a variation of the type of fundamental motif that repeats or changes. The repetition of the motif moves over the desired range of the filter fabric. The length should be between 2 and 3 meters when a shawl fabric is considered. The following form of the next motif is generated by translation vectors $T_1 = \begin{pmatrix} 0 \\ -b \end{pmatrix}$ when a pitch is a motif positioned on the Cartesian axis. The fundamental form of the pathway movement is shown in Fig. 4 as a form of translation. If the form is shifted the $T_n = \begin{pmatrix} 0 \\ -nb \end{pmatrix}$ vector formulas can be used to convert the form geometrically to show the slopes' motifs.

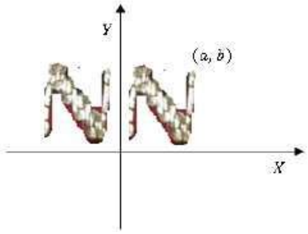


Figure 2: Cartesian coordinate of slop motif

Rotation on Tapis Lampung motif

Rotation is a transition in which a figure is rotated in a specific direction around a fixed point through an angle θ. In other words, a turning point around C by a directed angle θ is a turning point that sends every other C point to P so that P and P' have the same distance from the fixed C-point. A rotation with center C through an angle θ is usually denoted by $\rho_{C,\theta}$. It means that the image of any point P under $\rho_{C,\theta}$ is given as: $\rho_{C,\theta}(P) = \begin{cases} C, if\ P = C \\ P', \ if\ P \neq C, s.t. \overline{CP} = \overline{CP} \end{cases}$

Theorem. A rotation is an isometry.

For distinct points C and P, circle C_P is defined as the circle with center C and radius CP. Thus, $\overline{CP}$ is a radius of the circle C_P, and point P is on the circumference of the circle. Then, $\rho_{C,180} = \sigma_C$ follows that each transformation fixes point C and, otherwise, sends any point P to a point P' such that C is the midpoint of P and P' (Fig. 3).

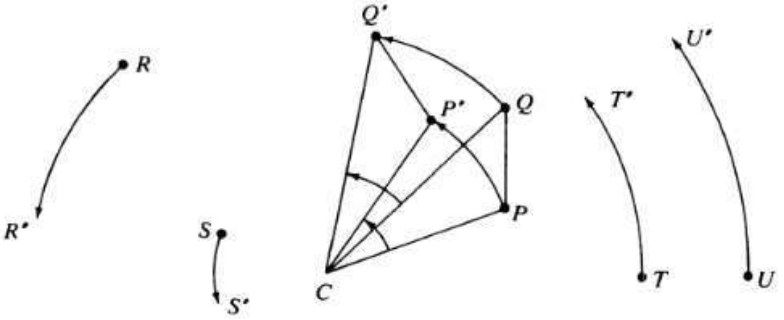


Figure 3: Rotation of illustration

To illustrate a rotation, the following is a motif of square on Tapis Gajah Meghem. If the image is rotated at angles of $90^0, 180^0, 270^0$, and 360^0 produces the original image. The rotation form of the basic shape in Fig. 10 are shown below.

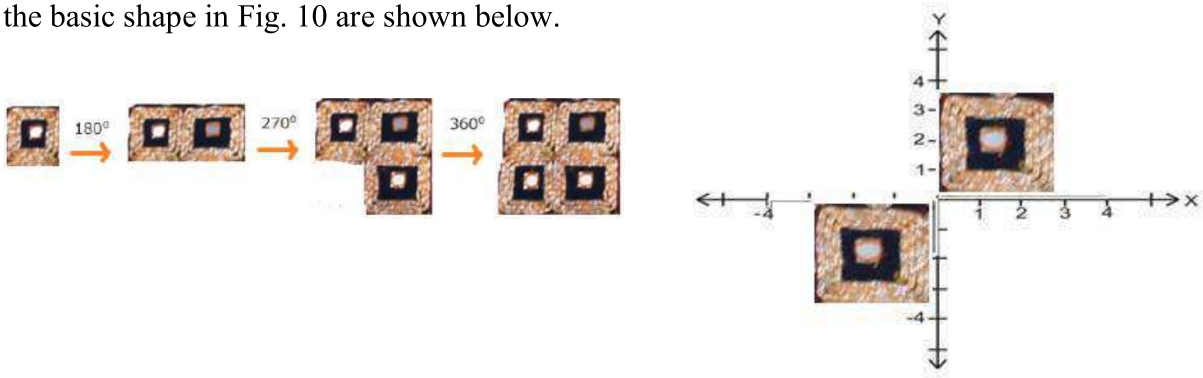


Figure 4: Rotation of Gajah Meghem motifs. The left figure was rotation on angle and the right figure was rotation on Cartesian coordinate

Reflection on Tapis Lampung motif

Given a line l and a point P, then, P' is a reflection image of P on the l if and only if $\overline{PP'}$ is perpendicular to l and $\overline{PM} = \overline{P'M}$, where M is the point of intersection of $\overline{PP'}$ and the l. In other words, P and P' are located on different sides of l but at equal distances from l. In this case, P' is said to be the mirror image of P and the l is said to be a line of reflection or an axis of symmetry. Reflection on l is usually denoted by S_l.

$$S_l(P) = \begin{cases} P, if\ P \in l \\ P', \ if\ P \notin \ \text{and}\ l\ is\ the\ perpendicular\ bisector\ of\ \overline{PP'} \end{cases}$$

Theorem: An isometry is a collineation that preserves betweenness, midpoints, segments, rays, triangles, angles, angle measure, and perpendicularity.

Consider the symmetries of the rectangle in Fig. 5. The axes of the plane are lines of symmetry for the rectangle, and the origin is a point of symmetry for the rectangle. Denoting the reflection in the $x -$ and y-axis by σ_h and σ_v, respectively, we have that σ_h, σ_v, σ_o, and l are symmetries for the rectangle. Note that l is a line of symmetry for any set of points. Since the image of the rectangle is known once which of A, B, C, D is an image of A is determined, the four lines are the only possible symmetries for the rectangle.

This is a geometric illustration of an elephant's transformation on a ship. A nongeometric motif is a motif applied to the elephant tapis. Elephant motif, human motivation, human motif for boat riding, and link motifs are all elements of the form. Application composition is taken from a plant and combined to make it attractive to animal, handler, and human motifs on board boats and chains. The main motif is the elephant animal motif, which stands directly between the motifs of the operator and the person. The above motif is a vessel filter motif with elements, such as bamboo shoots, single boats, handlers, and elephants. The motif of the ship is a ship with freight and elephants in terms of its characteristics in the woven tissue. Reflections show the shape of the elephant motif on the ship. The form of reflection can be guided.

Figure 5: Reflection on the y-axis in the left side, Siger motif and reflection in the right side

There are also filters due to reflection beside the above motif. The remaining elements are siger. The picture above shows two Siger Lampung motifs that are the result of the y-axis reflection, the results are similar in images, reflected on both the x- and y-axis. The result of the reflection. Filters with Tajuk Berayun motifs are also available, as shown in On Tapis Pucuk Rebung, the Tajuk Berayun motif is usually used. It is placed on the motif edge of the swinging canopy ornament. The swinging headers are placed side-by-side. It is obtained from young bamboo plants. The application of this form element has the significance of fertility because fertile natural effects exist. The bamboo-shooting motif is closely connected to the social (value) and religious systems. This motif also depicts the relationship between humans and God, and people and the environment. Tapis Tajuk Berayun motifs (Fig. 6) are used for wedding, graduation, circumcision, and many others ceremonies. Geometrically, the Tajuk Berayun motif illustrates reflections. This is another motif that can illustrate a geometrical reflection.

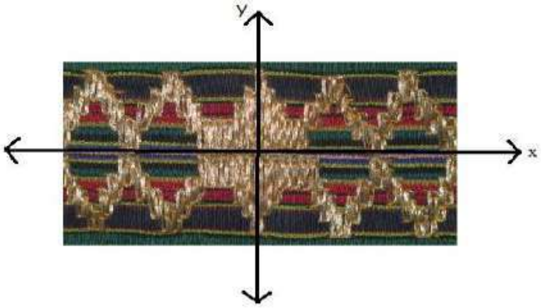


Figure 6: Tajuk berayun motif and reflection

Dilatation on Tapis Lampung

A transformation f is said to be a collineation if and only if the image of any line l under f is a line. In other words, for any point $P \in l$, the image $f(P) \in f(l)$. Furthermore, f is a dilatation if only if the image of any line l under f is a line parallel to l. That is, $f(l) \parallel l$ whenever f is colleneation, then f is said to be a dilatation.

Theorem: A dilatation is a translation or a dilatation

To show that there is a similarity in taking one triangle onto any similar triangle, suppose $\Delta ABC \approx \Delta A'B'C'$, as shown in Fig. 16. Let δ be the stretch about A such that $\delta(B) = E$ with $AE = A'B'$. With $F = \delta(C)$, then $\Delta AEF \cong \Delta A'B'C'$ by ASA. Since there is isometry β such that $\beta(A) = A'$, $\beta(E) = B$, and $\beta(F) = C'$, then $\beta\delta$ is a similarity taking $A, B, and\ C$ to $A', B'and\ C'$, respectively. If α a is similarity taking $A, B, and\ C$ to $A', B'and\ C'$, respectively, then $\alpha^{-1}(\beta\delta)$ fixes the noncollinear points and must be the identity. Therefore, $\alpha = \beta\delta$.

Dilation (multiplication) is a transformation that moves a geometry point, which depends on the dilation center and factor (scale). Thus, shades in a dilated geometry vary in size (small or big). The motif for Jung Sarat, for instance, is the motif for Mato Kibaw. The following motif can consider an extension. To achieve an attractive shape, the motif is enlarged. For the form shown in Fig. 7, the motifs are presented in part. The motif below can partially (separately) be considered as a group of Mato Kibaw motifs, originating from a square building with a white dot of fine zinc sheets at its center. The motifs below are of different sizes. If extended, this form produces a dilation or multiplication with a constant k to a partial shape, as shown in Fig. 18, which is considered a result of the positive real number k.

Mathematically, if k is a dilated factor, it applies to the following relationship. As a result, the shape of the Mato Kibaw motif is a dilated form of the center point of $O(0,0)$ by mapping.

$$[O, k]: P(x, y) \rightarrow P'(kx, ky)$$

The matrix of the equations is given as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Based on the analysis, another motif, a form of dilation or multiplication, can be displayed.

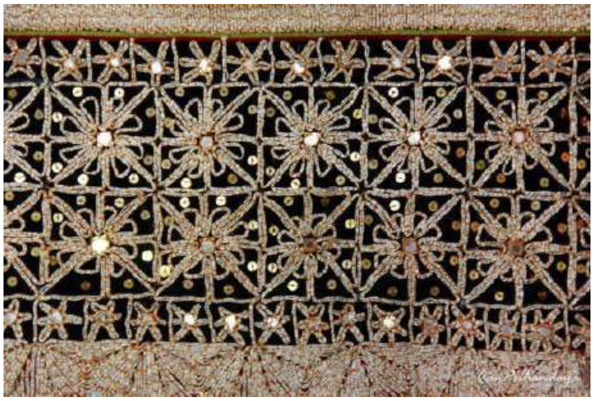


Figure 7: Flaura motif (Mato Kibaw)

METHOD

Participants

Participants were 157 secondary school students' (56% female), ages 12 to 14 years (*M* = 13.9; *SD* = .87). All students came from private and public school.

Procedure

An ethnographic approach was employed in this study (Gobo & Marciniak, 2011), aiming to provide an in-depth description and analysis of culture through intensive and prospective fieldwork research on culture (Huff et al., 2020; Person et al., 2013). This research focused on exploring culture while incorporating elements of Tapis Lampung as a symbol of users in a given culture. It gives insight into users' thoughts and actions, as well as the sights and sounds that they encounter during their activities. It clarifies the culture and symbolization of ethnomathematics. The framework stages are listed in Table 1.

Generic Question	Initial Answer	Critical Construct	Mastery Activity
Where should it look?	In the activities of making Tapis Lampung where there are mathematical practices	Culture	Interviewing indigenes who have the knowledge of Tapis Lampung or those who create Tapis motifs in Lampung.

	in it.		
How is it to look?	Investigating and exploring Tapis motif of the Lampung people concerning mathematics concept.	Alternative thinking and prior-knowledge	Determine what ideas are included in the making of Tapis Lampung in relation to mathematical concepts.
What is it?	Evidence (The outcomes of alternative thinking in the previous procedure)	Philosophical mathematics	Identifying the characteristics in the process of Tapis Lampung
What does it mean?	Significant outcomes of mathematics and culture.	Anthropology	Describe the relationship between the two mathematical knowledge and cultural systems. Describe the mathematical concepts in the activity of making Tapis Lampung for the Lampung people.

Table 1: Design of the Ethnomathematics research

Instruments

The instruments was about figural on the tapis patterns in the context of ethnomathematics. The ethnomathematical test items for testing creative thinking in mathematics. The figural test is about picture construction. This means that the participant starts with a fundamental shape and builds on it to make a picture (J. C. Kaufman et al., 2008). Scores are assigned based on classified responses that include elaborations score. Each response is further considered for its elaborateness and given either two or one points. Below is an example of figural test an ethnomathematics content in the table 2. Furthermore, the results of the items were presented in the table 3.

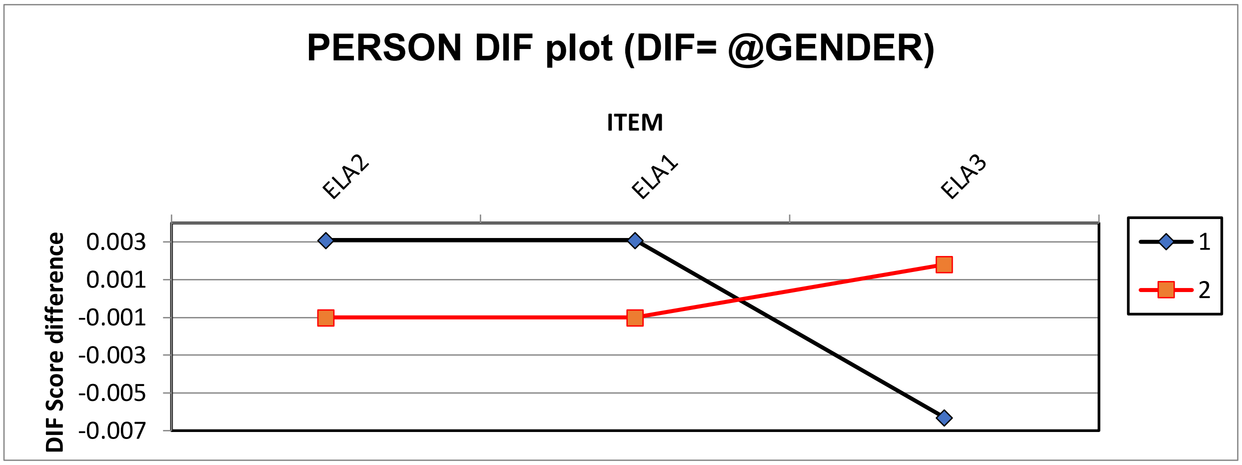
Questions	Picture
The pictures are part of Tapis Lampung with geometry motifs. a. Make a list of any flat shapes that you find in the Tapis Lampung motif! b. Draw any pictures from your findings using at least one flat shape that you found in number a. You can combine 2 or 3 or more flat shapes to create a unique image. Then name the image you have made.	

Table 2: An Example of Ethnomathematics-based Test

Item Measure	Test Group	
	Persons	Item
N	157	3
Measure	.53	.00
Mean	4.8	252.7
SD	.45	.68
SE	1.1	10.4
Mean Outfif MNSQ	.98	1.00
Mean Outfif SZTD	.05	-.23
Separation	.25	2.37
Reliability	.86	.85
Cronbach's Alpha	.86	

Table 3: The Summary of the Statistics Based on Pearson and Items

Based on Table 3, the reliability parameter in Rasch measurement for person and item are was .86 and .85, respectively. The statistics representing good reliability (more than 0.67) (Fisher, 2007). Furthermore, the Cronbach Alpha was 0.86. Rasch's measurements correspond to Outfit MNSQ in person ranging from 0.84 to 1.30 and Outfit SZTD ranging from -1.57 to 2.44. The item based on DIF is calculated for male (1) and female (2). There is no bias for item DIF has shown in the Figure 8.



Note: 1 = male; 2 = female; ELA1 = Item Elaboration no.1; ELA2 = Item Elaboration no.2; ELA3 = Item Elaboration no.3

Figure 8: The DIF item-based gender

Data Analysis

Data were collected through task and documentation. The objects observed include the steps in making Tapis Lampung, from the selection of tools to weaving the Tapis Lampung. As part of the documentation in this study, photographs of the task results by students of Tapis Lampung weaves were taken. To investigate the relationship between Tapis Lampung motifs and mathematical concepts, data were analyzed using Winstep software for Rasch measurement. This research is limited to the rules for determining the couple's matchmaking.

RESULTS

First, the 2-score items were examined using the Winstep format, one that compares the crossover, equal probability points “thresholds point’’ using parameters of the partial credit model (Figure 9). The category probability curves of two items are demonstrated in Figure 9 (item 1 and item 3). The category probability curves indicate that items were like item 3 that equal to thresholds point. Therefore, the category probability has measure relative to item difficulty like item 1.

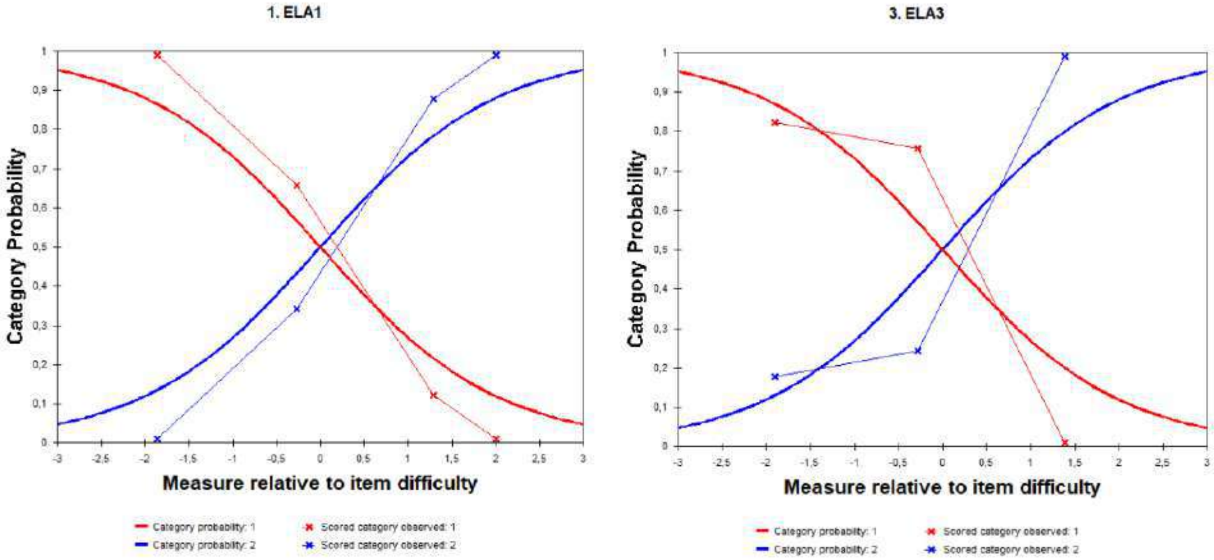


Figure 9: Category probability curves of the items 1 and 3.

Analysis of student answer patterns on creative thinking in mathematics in the context of Ethnomathematics has already presented. Further analysis was conducted to see how the pattern of answers of students with high statistical mathematical creative thinking (MCT) abilities, namely students with code 23MSMP and 116MSMP. The pattern of student answers can be seen in Table 4.

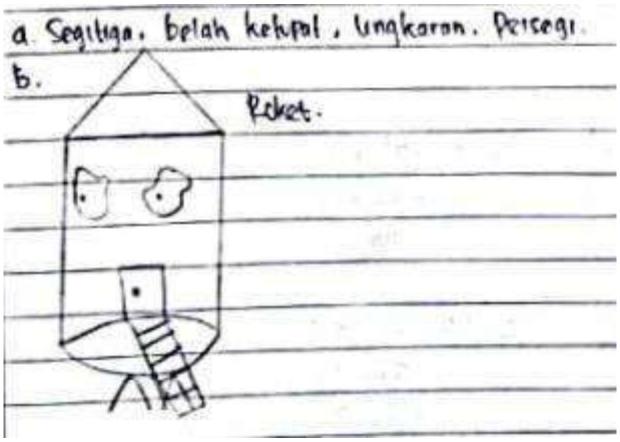
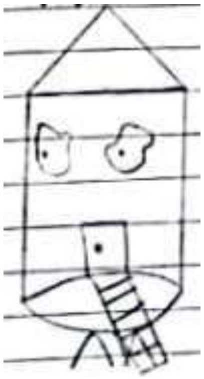
 a. Segitiga. belah ketupat. Lingkaran. Persegi. b. Roket.	**a. Triangle, rhombus, circle, square** **b. Rocket** 
 a) Segitiga, Belah ketupat, dan eksagonal b) Angry Bird	**a. Triangle, rhombus, and heksagonal** **b.** 

Table 4: Students' Answer of Tapis Pattern

Based on students' answers to the code 23MSMP on test number 3, it could be seen that the students' extracted information about the questions. The students were listed of the Tapis pattern in four shapes: triangle, rhombus, circle, and square. Additionally, students can draw pictures using shapes that were seen on the pattern. The picture name is rocket. In contrast, students' answer with code 116MSMP was only 3 kinds of the shapes, triangle, rhombus, and hexagonal, respectively. Then, can draw the angry bird picture.

Multiple Analysis

The regression test is satisfied if the covariate and dependent variable have a linear relationship. The results of multiple regression for the analysis as below in Table 5. Based on Table 5, we can see that the table describes the variance percentage explained by the included independent variables. The statistics results have explained that the independent variables can explain 2.9% of variance in the dependent variable. The total variance explained is 2.9% if we consider only the independent variables that significantly contribute to the regression model, $R^2 = .29$, $p = .21$. That can expect of item test number 3 not seems to exert the strongest elaboration, on the other hand may have an impact on the developmental level of learning creativity. It is also influenced by ethnic students' which task characteristics may explain. Regarding the coefficients analysis, the ethnic score (t = 1.12, p < .001), is the significant explanatory variables. Whereas, the schooltype (t =1.9, *p* = .06), and living place (t = -56, p = .58) does not have a significant contribution to the regression model.

Independent variables	***r***	***ß***	***r·ß*·100**	***p***
Ethnic	.07	.09	.69	<.001
School type	.14	.16	2.31	.06
Living Place	.02	-.05	-.001	.58
Total variance explained			2.89	

Table 5: Results of multiple regression analysis for score no.3 as a dependent variable

Differential Item Functioning (DIS) based Gender

The DIF analysis confirmed that students with cross ethnic had fixed pattern of answer. This can be seen in the table 10 about DIF Measures. The students' have score different items in the own ethnic. In other words, the results of the DIF analysis in figure 10 conclude that although students have supportive demographic factors, such as gender, ethnic (i.e., Lampung, Java, Sundanese, Manado, Batak, Bugis, Munang, and others) in accordance with the given ethnomathematics-based test, they do not provide benefits for students in improving learning outcomes, especially those closely related to improving students' creative thinking in mathematics ability. However, it cannot be omitted that students' initial mathematical abilities also have their own role to support the development of other mathematical abilities.

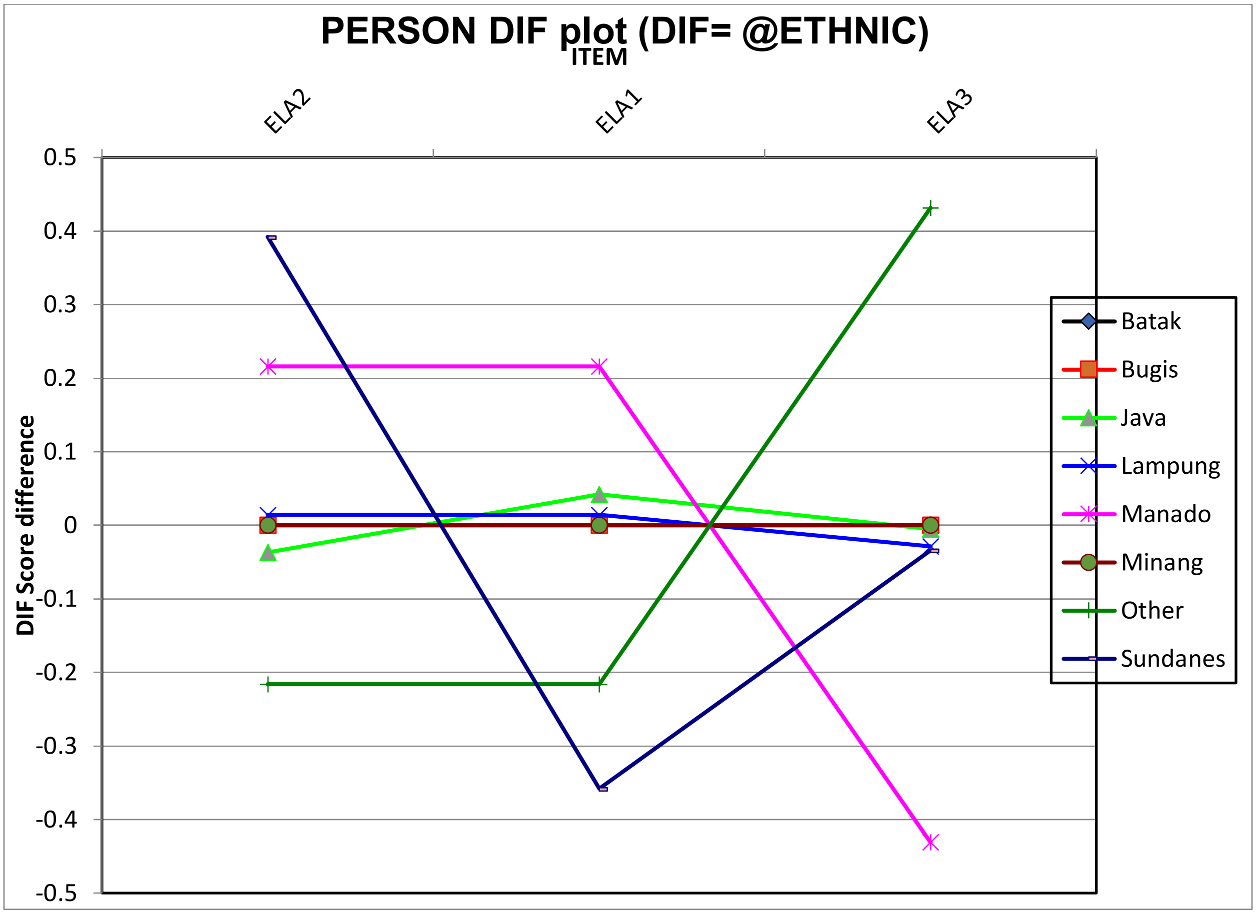


Figure 10: Person IF Students Answer on the Tapis Pattern Based Ethnic

DISCUSSION

Regarding the students answer, they can imagine to drawing the picture related to their own experienced. The picture based on the geometry pattern in the ethnomathematical context. The context has similar with the literature review about transformation geometry. Some examples of transformation geometry applications in the Tapis Lampung pattern are provided. The patterns on filtering motifs can have sacred values, social stratification, history and understanding, creativity, inclusiveness, and economic value (Matthews, 2018). For sacred values, traditional weaving cloth often indicates pure Lampung people in traditional ceremonies. The sacred value sources are motifs containing symbolic philosophical implications, such as constructions. Tapis woven fabric is regarded as a cloth with high symbolic value by the indigenes of Lampung. One of them symbolizes purity, which can protect the wearer from all external dirt. It is typically used in traditional and religious ceremonies to represent sacred values and functions. Ship decoration, for example, is a prominent feature of Lampung traditional tapis woven fabric. Ship shapes and colors also have different meanings. Red ship motifs represent sacredness and relationship with the upper world, whereas blue ship motifs indicate a relationship with the profane underworld. The third world is the middle, which includes humans and their natural environment, fauna, and flora (Nurdin & Damayanti, 2019).

For social stratification, traditional fabrics, usually owned by the local community, are maintained by indigenous Lampung families. Each cloth has a function, significance, and social status, such that some clothes are only allowed to be used by certain groups according to the social status of the ethnic groups. Tapis woven cloth is also an indicator of the social status of an individual (Isbandiyah & Supriyanto, 2019). Thus, a person's social status is known by looking at the woven tissue of the people by tapis. Members of the Lampung community wear tapis in traditional ceremonies, and each cultural group has different patterns and motifs of tapis. The patterns and motifs depend on the ceremony's purpose, and the tapis' patterns describe the users' position in the Lampung society's social hierarchy.

In the results, we found that the answer of students' creative thinking has different for each other. While they can answer the easy and difficulty of the item. We investigate in more detail using regression test to see whether independent variables (i.e., ethnic, school type, and living place) has effect on the creative thinking. Statistically, we found that only 29% of the creative thinking in mathematics contribute to the independent variables. Moreover, 73% was explained by other variables. Additionally, the item test was about figural on the tapis patterns in the context of ethnomathematics which can be seen that the figural covering of elaboration in more detail are fluency, flexibility, and the strategic retrieval and manipulation of knowledge may be among the more fundamental cognitive processes underlying g and divergent thinking (Beaty & Silvia, 2012; S. B. Kaufman et al., 2016). These skills appear to be more essential for mathematical creativity, which requires the application of reasoning and proofing ideation to an existing rational system and problem-solving (Huda et al., 2020). Practically, creative thinking in mathematics is needed to develop students' abilities. It can be understood by focusing on the responses of problem-solving students with out-of-the-ordinary thought processes and examining divergent production by determining the criteria of results (Haylock, 1997; Suherman & Vidákovich, 2022).

CONCLUSION

The Rasch Model analysis is important in checking for possible biases in student response patterns based on demographic factors. Rasch's analysis made it possible to further explore biases on demographic factors other than students' creative thinking in mathematics, gender, ethnicity, and student background. Other factors such as the level of affective factors and socio-cultural factors such as giving different results, can be explored further using the Rasch Model analysis by providing optimal analysis and students' results.

This study giving problems with ethnomathematics contexts was proven to help students' understanding to the problem presented. Tapis Lampung may be expressed by translations, rotations, reflections, and dilations as a geometrical example of transformation. The results of this study will help teachers prepare the most appropriate strategy for improving the mathematical concepts and students' skills, especially in local cultures. This will aid in effective teaching, learning, and assessment of mathematics. However, there is a need for further studies on the empirical use of geometry learning in mathematics.

This research has limited findings, where the ethnomathematics context presented uses the cultural context and tapis pattern in Lampung, Indonesia, and the number of the research subject is also small. Therefore, further research will continue by paying attention to the demographic-focused factors used and the existence of socio-cultural factors so that the findings obtained can provide significant results.

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Prospective Primary School Teachers' Work in Continuous Online Assessments in the Course of Didactics of Mathematics

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Abstract: This study aimed to examine students' work in online assessments to gain more understanding for designing continuous assessments in a blended learning environment for prospective primary school teachers. The study took place during emergency remote teaching due to the COVID-19 pandemic. Course work for prospective primary school teachers in the didactics of mathematics course included continuous, obligatory, non-graded, online assessments. We performed a qualitative content analysis of their answers. Students' work was examined regarding the content knowledge and requirements in the questions, based on the categories of Subject Matter Knowledge and Mathematical Assessment Task Hierarchy taxonomy. The results showed that students' study approach was strategic, relying heavily on peer support. Their work differed concerning the content and requirements of the questions. Students were more engaged in questions that required creating examples, discussing definitions and properties, and solving contextual problems. Questions related to horizon content knowledge were most challenging for students. We discussed how the results of our study could affect the design of continuous assessment in a blended learning environment for prospective primary school teachers.

INTRODUCTION

Prospective primary school teachers in Croatia are educated in different subject areas, physical sciences, mathematics, computer science, arts and humanities, psychology, and other social sciences. They have university-level service courses and didactics courses related to each primary level subject, including mathematics. Students enter university with different secondary education profiles and interests and struggle with the composite nature of their studies, with mathematics courses. There might be different reasons for their difficulties and failure in mathematics, math anxiety, overladen syllabus, weak prior knowledge, and inappropriate study approach. Procrastination and postponing work just before the examination, a common practice among the student population, seemed a particular issue.

Due to the COVID-19 global pandemic, education on all levels was disrupted and abruptly shifted to emergency remote teaching (ERT). There were many challenges for educational institutions and their stakeholders, for example, the dependence on technology and issues with socio-economic equality, compatibility of educational content and achievements with channels for remote teaching and learning, supervision and adaptation of the assessments in a digital environment, increased teachers and students' workload, teachers' professional and digital competencies, students' digital competencies and their engagement (Aldon et al., 2021; Tanujaya et al., 2021; Tay et al., 2021). When digital resources are integrated in a meaningful way, they can enhance the environment by providing alternative, multiple and interactive representations, accessibility of resources, communications in all channels, and increase students' self-efficacy through differentiation, self-paced opportunities, and personalized feedback (Attard & Holmes, 2020; Borba et al., 2016; Jamil et al., 2022; Tanujaya et al., 2021; Wang, 2021). Our intention to use the knowledge gained in these unprecedented circumstances to improve regular and online education motivated this study.

The concern about students' activity in mathematics courses increased during the COVID-19 lockdown. Our lecture-based course in didactics of mathematics went completely remote to the digital environment. To compensate, as a part of the course work, we engaged students in continuous, obligatory, non-graded, online assessments with individual feedback on their work. Research indicates that frequent formative assessment could engage students in continuous work and move them from learning for examination to active learning. Korhonen et al. (2015) reported that constant workload eased the burden before the final exam, contributed to students' understanding, and prompted small group collaboration. Cusi and Telloni (2019) found that university students valued the effectiveness of a designed individualized online path with feedback to support their learning. Continuous work in the form of non-graded assignments and writing in mathematics can contribute to learning and teaching mathematics (Flesher, 2003; Kuzle, 2013). Building on this novel experience of organizing course work, we questioned if such assignments synchronized with course content could complement the lectures into a blended learning environment, as a combination of online and face-to-face activities (Borba et al., 2016; Dio, 2022; Tanujaya et al., 2021).

The purpose of this study was to analyse students' work in online assessments implemented during COVID-19 remote teaching. The results would provide understanding for the future design of continuous assessments in a blended learning environment for university mathematics education of prospective teachers.

The paper has five sections. The literature review contains research about continuous assessment and theoretical constructs used in the analysis of students' work, and it ends with stating research questions. Regarding the methodology section, we describe the context of the study, the study instrument supported with the theoretical constructs and the data analysis process. The section with results is organized according to the stated research questions. In the discussion section, we connect and lead the results toward the study goal, that is, the issue of designing online, continuous assessments for prospective teachers. We also discuss limitations, implications and ideas for further research. The final section is the conclusion.

LITERATURE REVIEW

Continuous assessment and active learning

Assessment is an inevitable part of education. Including different aspects of the notion addressed in educational research, Joughin wrote that assessment is “to make judgements about students’ work, inferring from this what they can do in the assessed domain, and thus what they know, value, or are capable of doing” (2009, p. 16). Goos (2014) added the intention of using assessment results to plan further educational actions. There are multiple assessment purposes, but the authors emphasised evaluating students’ knowledge and supporting their learning (Goos, 2014; Hernández, 2012; Hughes, 2008; Trotter, 2006). The effect of assessment on student learning was extensively researched yet unclear (Joughin, 2009; Rust, 2002). Students’ approach to learning depends on the mode of assessment, but also the teaching pedagogy, learning environment, attractiveness and relevance of the course content, personal attitudes and goals, and others (Darlington, 2019; Joughin, 2009).

In mathematics education, approaches to learning are described with students’ engagement and achievement goals (Dahl, 2017; Darlington, 2019; Jukić Matić et al., 2013). A surface approach to learning is a low-demanding approach focused on avoiding failure. Students memorise the whole material to perform a particular task without understanding. Students who approach the content intending to understand, actively engage in the study and make connections between material have a deep learning approach. A strategic approach to learning is using the least demanding study organization to achieve the best possible examination result. It is important to motivate students to attempt to understand and connect mathematical content, rather than rely on reproducing or memorising facts and even solutions as a part of a surface or surface-strategic approach (Darlington, 2019).

The continuous, formative, learner-oriented and criterion-referenced assessment had a positive impact on students learning (Hernández, 2012; Nair & Pillay, 2004; Patterson et al., 2020; Rust, 2002; Shorter & Young, 2011; Trotter, 2006). Some arguments for effective continuous assessment follow:

- Frequent assignments optimise workload and encourage regular work.
- Smaller-scope and relevant (real-life) tasks raise interest and engagement.
- Prompt, constructive and criterion-related feedbacks are useful.

In the context of our study, a continuous assessment was a part of course work to engage students to regularly reflect on course material and advance based on the teachers’ feedback about their productions (Shorter & Young, 2011).

Prospective teachers work on assessment items

The term work, following Joughin’s definition, referred to students’ productions in assessment items, from which we made inferences about their knowledge and skills. We evaluated students’ work from two perspectives: the category of the knowledge at stake and the category of the requirements in the task.

There is a general agreement that teachers' knowledge should be multidimensional; the theoretical and empirical research differentiate, among others, the content and pedagogical aspects of teachers' knowledge (Schwarz & Kaiser, 2019). In this study, we used the mathematics knowledge for teaching framework proposed by Ball et al. (2008). They described the knowledge required for teaching with several categories distributed among subject matter knowledge and pedagogical content knowledge. Pedagogical content knowledge (PCK) includes knowledge about content and students and knowledge about content and teaching, which refer to peculiarities of learning and instruction for particular mathematical content in a particular educational setting, and knowledge of content and curriculum. Subject matter knowledge (SMK) includes common content knowledge (CCK) as knowledge and activities used in any non-educational context, including formal mathematics, specialized content knowledge (SCK) as knowledge and activities used in teaching mathematics, and horizon content knowledge (HCK) as awareness and ability to vertically correlate mathematics knowledge and activities or observe school content from an advanced point of view. Categories of SMK focus on the work, choices and actions grounded in mathematics whereas PCK relates to pedagogically oriented ideas and choices in teaching.

Teachers' activities related to SCK are presenting and discussing mathematical ideas, examining, selecting and connecting representations, constructing and modifying examples and problems, and interpreting and justifying solutions (Ball et al., 2008; Hill et al., 2004). Mathematical courses for prospective teachers should incorporate assessment items which promote mathematical work relevant for teaching, related not only to CCK but also to HCK, and in particular to SCK (Patterson et al., 2020; Selling et al., 2016).

Questions about the same content can be formulated with different requirements for students' skills. We used a modification of Bloom's taxonomy for structuring assessment tasks to categorise questions based on their requirements (Smith et al., 1996). The Mathematical Assessment Task Hierarchy (MATH) taxonomy consists of eight categories organised into three groups (Table 1). Each category has descriptors of skills and activities required for solving tasks. The categories are not hierarchical and do not relate to the complexity of the mathematical content or the subjective difficulty a student might face with a given task, but the focus is on the mathematical demand of the task (Darlington, 2014). The MATH taxonomy was used to compare examinations (Darlington, 2014; Kinnear et al., 2020), and analyse course material (Bennie, 2005), and researchers suggest it could be used when developing assessments and curricula.

Research questions

It is problematic to assume that continuous assessment promotes a deep approach to learning, but the cost of implementing continuous assessment in a blended learning environment is worth the potential benefits for students learning (Attard & Holmes, 2020; Trotter, 2006). Evaluating and redesigning assessments regarding students' work can contribute to a more balanced, effective and meaningful assessment of and for learning (Hughes, 2008). This study expands the literature about prospective teachers' content knowledge, in particular, regarding the combination with MATH taxonomy. The results of the study contribute to understanding the opportunities of continuous assessment in university mathematics education.

	MATH category	*Descriptors of required abilities*
Group A	1. Factual knowledge	Recall previously learned information
	2. Comprehension	Decide the adequacy of a simple definition, interpret and substitute into a formula, recognise examples and counterexamples
	3. Routine procedures	Use procedures in a familiar context beyond factual recall
Group B	1. Information transfer	Decide adequacy of a conceptual definition, apply a formula in a different context, summarize in non-technical terms, explain relationships between objects, etc.
	2. Application in a new situation	Model real-life settings, extrapolate known information to new situations, etc.
Group C	1. Justifying and interpreting	Recognise the limitations in a model and unstated assumptions, discuss the significance of examples and counterexamples, etc.
	2. Implications, conjectures and comparison	Make inductive or heuristic argumentation, prove by rigorous methods, deduce the implications of a given result, construct examples and counterexamples, etc.
	3. Evaluation	Judge the material for a given purpose based on definite criteria

Table 1: Categories in the MATH taxonomy with corresponding descriptors from Smith et al. (1996)

We aimed to evaluate students' work in online tests to gain more understanding for composing supportive and effective continuous assessments in a blended learning environment for mathematics education of prospective primary school teachers. For that purpose, we state the following research questions:

RQ 1: How can students' work in the continuous online assessment be described? What were their achievements in the assessments compared to formal assessments?

RQ 2: How did their work differ concerning the mathematical knowledge at stake and requirements in the questions from the online assessment?

METHOD

Context of the study

This didactic of mathematics course is obligatory for third-year students of teacher studies at our institution. It is allotted two hours of weekly lectures with a cohort of approximately 60 students. The study took place during eight weeks in the second semester during the ERT due to the COVID-19 pandemic. We utilized Moodle for archiving and disseminating lecture notes, literature and other digital materials, synchronous communication through integrated video conferencing tool and live chat, asynchronous communication through forums and direct messages, and online assessments with HTML-based tests. During this time the course content covered scientific methods in mathematics education, that is, induction and deduction, analyses and synthesis, analogy, generalization and specialization, abstraction, and concretization (Kurnik, 2008). The lectures included a definition and description of each method, an explanation of its advantages and limitations and examples of its use in mathematics and mathematics education. This reflected both content and pedagogical aspects of teachers' knowledge.

The obligatory online assessments were in the form of Moodle tests with questions given in advance. Students had a week to complete them. We examined and evaluated their answers and provided feedback. Students' work in the tests was not a part of the formal assessment in the course. Both formal assessments were pen and paper exams, one before the ERT at the end of the first semester and the second under given epidemiological measures at the end of the second semester of the course.

Participants in this study were 60 students in their third year of university studies for primary school teachers during didactics of mathematics.

Study instrument

Moodle tests contained questions in four different mathematical topics from the course content – inductive reasoning in algebraic and geometric context (Ir1-8), mathematical analogy in algebraic and geometric context (An9-12), method of analysis and synthesis in algebraic and geometric context (As13-18), and the area-perimeter problems in a mathematical and real-life context (Ip19-22 in Appendix). Questions were open-ended and related to different aspects of content knowledge and with different requirements. While Moodle tests allow using a variety of question types, closed questions evaluate the correctness of answers, while open-ended questions allow evaluating the whole solution process (Jamil et al., 2022). Students were required to elaborate their solutions and reasoning in writing which promotes mathematical understanding (Kuzle, 2013).

The SMK and MATH category of the questions from Moodle tests are given in Table 2. The placements in categories are based on theoretical consideration regarding literature review and elaborated in Appendix.

SMK categories	*MATH taxonomy categories*					
	A1	*A2*	*B1*	*B2*	*C1*	*C2*
CCK	Ir1, Ir7	Ir2, Ir8	As13	An11, As14		
SCK			An9, Ip19	Ip20, Ip21		Ir3, Ir5
HCK			An10		Ir4, Ir6, As15, As17, Ip22	An12, As16, As18

Table 2: Questions from the Moodle tests regarding SMK and MATH taxonomy categories

Data analysis

The data in this study consisted of students' answers collected from the Moodle tests as obligatory online assessments and their achievement scores in two formal assessments. We performed a qualitative content analysis of students' answers. It is a step-by-step coding procedure where well-defined categories are assigned to each unit of analysis in several cycles (Kuckartz, 2019; Mayring, 2015). We present our analysis according to steps suggested by Kuckartz (2019) with examples of resulting categories.

Step 1: Preparing the data

We exported students' answers from the Moodle tests into xls tables with the textual format. The files students additionally uploaded were downloaded and labelled with the student name and question number. The text from the files was typed and figures were briefly described in the corresponding cell of the xls tables. In some cases, we were unable to access students' answers, because they did not upload a file or they copied a broken link to a file.

The data prepared for content analysis was a matrix with columns corresponding to each question from Moodle tests and rows corresponding to each student's answers.

Step 2: Forming main categories and units of analysis

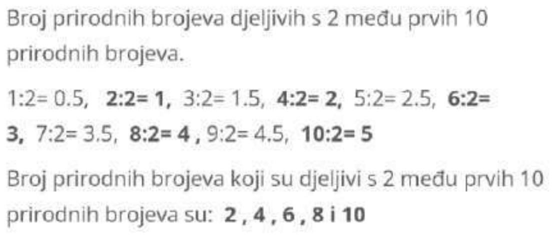
The unit of analysis was each student's answer to each particular question. Students' answers to a mathematical task can be judged by the appropriateness and correctness of the solution. We decided on the structuring procedure by assessing the units with predetermined ordinal categories (Mayring, 2015): 2 assigned for a correct answer, 1 assigned for a partially correct answer, with an appropriate idea but errors in the solution, and 0 assigned for an inappropriate idea, hence also an incorrect solution.

Since the questions from the tests were open, we expected students' answers to vary in presentation and approach to the solution. We decided on the reductive, summarizing procedure by assigning descriptors of the unit that justify the assigned ordinal category, appropriateness and correctness of the answer, and characterise the solution.

Step 3: Coding data with the main categories

We chose to analyse the data by questions. One researcher, the coder, worked through the units coding with predetermined ordinal categories and assigning descriptors (Figure 1).

Student's answer to Ir2



Broj prirodnih brojeva djeljivih s 2 među prvih 10 prirodnih brojeva.

1:2= 0.5, **2:2= 1,** 3:2= 1.5, **4:2= 2,** 5:2= 2.5, **6:2= 3,** 7:2= 3.5, **8:2= 4**, 9:2= 4.5, **10:2= 5**

Broj prirodnih brojeva koji su djeljivi s 2 među prvih 10 prirodnih brojeva su: **2 , 4 , 6 , 8 i 10**

Ordinal category
1 - partially correct
Descriptors
Vague mathematical statement
Calculations $a:d$, $a \leq n$.
Listing numbers divisible by d among first n numbers

Note. We translated the text: •*Number of integers divisible by 2 among the first 10 integers. •1:2=0.5 … 10:2=5 •The number of integers divisible by 2 among the first 10 integers are: 2, 4, 6, 8 and 10.*

Figure 1: Example of student's answer in question Ir2 and coding procedure in Step 3

Step 4: Forming and revising categories and final working through material

The reliability of a qualitative content analysis depends on the categories described for the coding procedure (Kuckartz, 2019). After the first working through units, the two researchers discussed the ordinal categories and descriptors retrieved in the coding process. Researchers made a settling agreement about ordinal categories in each question. They used the descriptors to create data-driven categories, that is inductive category formation, in several cycles until saturation occurred (Kuckartz, 2019; Mayring, 2015). As a final coding procedure, researchers organised, labelled and described ordinal and qualitative categories that cover all instances of students' answers (Table 3). In this way, researchers consensually constructed detailed and precise coding categories and one coder made a final working through units.

Label[1]	*Ordinal category*	*Descriptors of the qualitative category*
P1	2	The volume of an upright three-sided prism with a right triangle base Proof of the true statement
P4	1	The volume of an upright three-sided prism with a right triangle base "The statement is true because it is an analogue"
P5	0	No spatial analogue, reference to the area of a right triangle
L1	1	Spatial analogues of the right triangle copied from literature
I1	1	The volume of a prism imprecisely named as "right three-sided prism"

Note. [1] Qualitative category are characterised and labelled by the origin of students' work in three ways as literature (L), peer (P) or individually (I) oriented work.

Table 3: Examples of several categories in question An12 in the final coding procedure in Step 4

Step 5: Category-based analyses and presenting results

The methodology of this study was grounded on qualitative content analysis. Categorising students' answers allowed for quantitative analysis regarding students' achievement and frequencies of answers. We used descriptive statistics to analyse the data obtained from content analysis. The nature of ordinal categories was fitting to analyse students' achievements. The values 0, 1 and 2 from the ordinal categories in questions from Moodle tests were used as the cumulative scores in online assessment. The qualitative categories and their frequency were used to analyse students' work concerning MATH categories and the content of questions from the Moodle tests.

RESULTS

Students' work and achievement in assessments

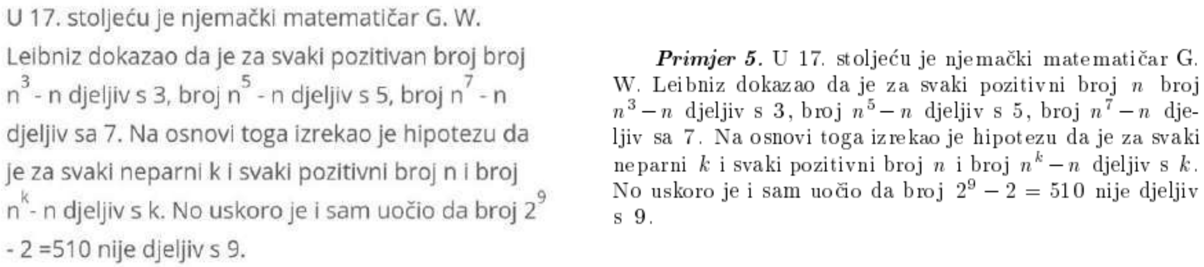
The test interface in the Moodle platform allows typing text, including mathematical typesetting, and inserting figures into the answer field. Students often (in particular in questions As14-18, Ip20-21) skipped these options and separately uploaded photographs of their pen and paper work (Figure 3). This meant additional work for lecturers, with downloading and viewing files in a desktop programme, compared to viewing, evaluating, and providing feedback within the Moodle test environment.

The first working through units in Step 2 of the analysis suggested that students' answers could be characterised by the origin of their work in three ways:

- A student copied the excerpts from the lecture notes or literature in their answer to a question – this is *literature-oriented work (L)*. Such categories were easily recognised since students retyped the text word to word or inserted the screenshot of the original material (see Figure 2).
- Two or more students wrote the same answer to a question – this is *peer-oriented work (P)*. The peer-oriented categories were inductively formed as described in Step 4 of the content analysis procedure. We opted that students' answers were assigned in the same peer-oriented category if the answers were identical (see Figure 3 and Figure 4).

– A student had an original answer to a question – this is *individually oriented work (I)*.

A student typed the solution | Excerpt from a web document



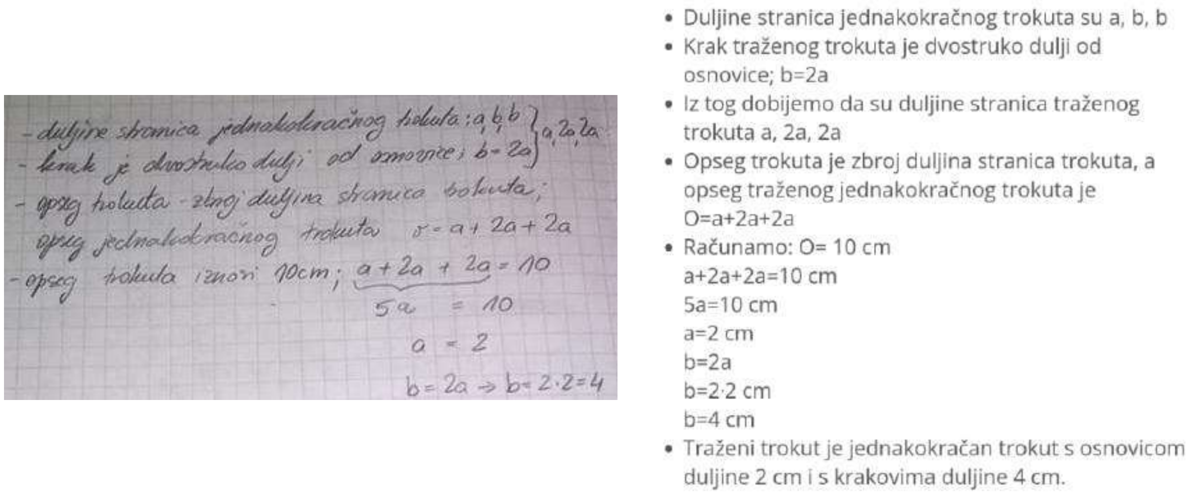
U 17. stoljeću je njemački matematičar G. W. Leibniz dokazao da je za svaki pozitivan broj broj n^3 - n djeljiv s 3, broj n^5 - n djeljiv s 5, broj n^7 - n djeljiv sa 7. Na osnovi toga izrekao je hipotezu da je za svaki neparni k i svaki pozitivni broj n i broj n^k - n djeljiv s k. No uskoro je i sam uočio da broj 2^9 - 2 =510 nije djeljiv s 9.

Primjer 5. U 17. stoljeću je njemački matematičar G. W. Leibniz dokazao da je za svaki pozitivni broj n broj $n^3 - n$ djeljiv s 3, broj $n^5 - n$ djeljiv s 5, broj $n^7 - n$ djeljiv sa 7. Na osnovi toga izrekao je hipotezu da je za svaki neparni k i svaki pozitivni broj n i broj $n^k - n$ djeljiv s k. No uskoro je i sam uočio da broj $2^9 - 2 = 510$ nije djeljiv s 9.

Note. We translated identical texts: *In the 17th century German mathematician G. W. Leibniz proved that for any positive number n, the number* $n^3 - n$ *is divisible by 3, the number* $n^5 - n$ *is divisible by 5, the number* $n^7 - n$ *is divisible by 7. Based on those he stated a hypothesis that for any odd n and every positive number n, the number* $n^k - n$ *is divisible by k. Soon he noticed that the number* $2^9 - 2 = 510$ *is not divisible by 9.* The text on the right-hand side is an excerpt from the web source *Princip potpune indukcije [The principle of complete induction]* (n.d.). Element. https://element.hr/wp-content/uploads/2020/06/unutra-15008.pdf

Figure 2: Student's input coded with the literature-oriented qualitative category in question Ir8

A student uploaded a photograph of the solution | A student typed the solution



- duljine stranica jednakokračnog trokuta : a, b, b } a, 2a, 2a
- krak je dvostruko dulji od osnovice ; b = 2a
- opseg trokuta - zbroj duljina stranica trokuta;
opseg jednakokračnog trokuta o = a + 2a + 2a
- opseg trokuta iznosi 10cm; $\underbrace{a + 2a + 2a}$ = 10
5a = 10
a = 2
b = 2a → b = 2·2 = 4

- Duljine stranica jednakokračnog trokuta su a, b, b
- Krak traženog trokuta je dvostruko dulji od osnovice; b=2a
- Iz tog dobijemo da su duljine stranica traženog trokuta a, 2a, 2a
- Opseg trokuta je zbroj duljina stranica trokuta, a opseg traženog jednakokračnog trokuta je O=a+2a+2a
- Računamo: O= 10 cm
a+2a+2a=10 cm
5a=10 cm
a=2 cm
b=2a
b=2·2 cm
b=4 cm
- Traženi trokut je jednakokračan trokut s osnovicom duljine 2 cm i s krakovima duljine 4 cm.

Note. We translated identical texts: •*Lengths of the sides of an isosceles triangle are a, b, b* •*The leg of the triangle is twice the length of its base; b=2a* •*Therefore, the lengths of the sides of the triangle are a, 2a, 2a* •*The perimeter of a triangle is the sum of the lengths of its sides, and perimeter of an isosceles triangle is o=a+2a+2a* •*We calculate o=10 cm … b=4 cm* •*The triangle is an isosceles triangle with base length 2 cm and legs length 4 cm.*

Figure 3: Students' inputs coded with the same ordinal and qualitative category in question As15

A student typed the solution coded with P1 in qualitative categories

Ispod su ispisani brojevi koji odgovaraju podacima a=1 cm, b=23 cm, P=23cm^2.
a=2 cm, b=22 cm, P=44cm^2.
a=3 cm, b=21 cm, P=63cm^2.
a=4 cm, b=20 cm, P=80cm^2
a=5cm, b=19 cm, P=95cm^2
a=6 cm, b=18 cm, P=108cm^2
a=7 cm, b=17 cm, P=119cm^2
a=8 cm, b=16 cm, P=128cm^2
a=9 cm, b=15 cm, P=135cm^2
a=10 cm, b=14 cm, P=140cm^2
a=11 cm, b=13 cm, P=143cm^2
a=12 cm, b=12 cm, P=144cm^2
a=13 cm, b=11 cm, P=143cm^2
a=14 cm, b=10 cm, P=140cm^2

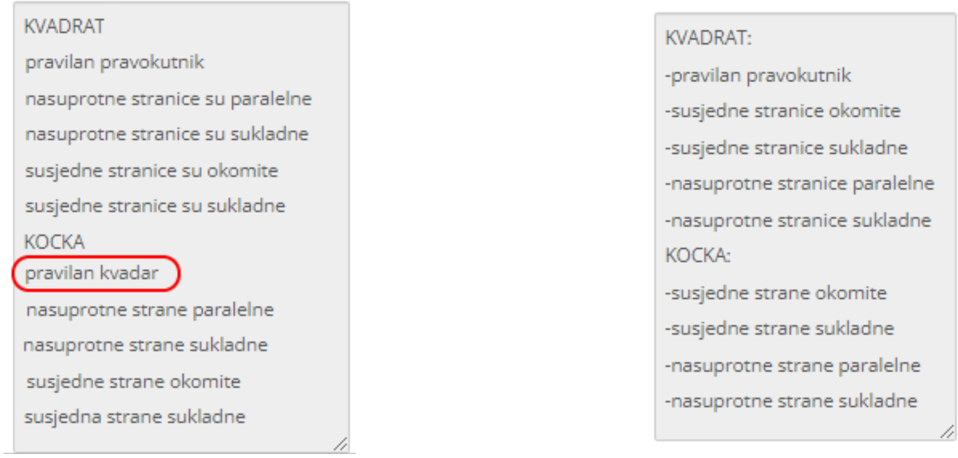
"Below are written numbers that fit the data"

A student typed the solution coded with P1 in qualitative categories

a= 1 cm, b= 23 cm, P= 23 cm^2
a= 2 cm, b= 22 cm, P= 44 cm^2
a= 3 cm, b= 21 cm, P= 68 cm^2
a= 4 cm, b= 20 cm, P= 80 cm^2
a= 5 cm, b= 19 cm, P= 95 cm^2
a= 6 cm, b= 18 cm, P= 108 cm^2
a= 7 cm, b= 17 cm, P= 119 cm^2
a= 8 cm, b= 16 cm, P= 128 cm^2
a= 9 cm, b= 15 cm, P= 135 cm^2
a= 10 cm, b= 14 cm, P= 140 cm^2
a= 11 cm, b= 13 cm, P= 143 cm^2
a= 12 cm, b= 12 cm, P= 144 cm^2
a= 13 cm, b= 11 cm, P= 143 cm^2
a= 14 cm, b= 10 cm, P= 140 cm^2

Figure 4: Students' inputs coded with the same peer-oriented qualitative category in question Ip19

Each qualitative category was enumerated according to the type of work with a reference to the ordinal category. For example, in Table 3, P4 stands for the fourth among peer-oriented qualitative categories and the answer is partially correct. If a student made additional errors in their peer-oriented work, we assigned them to the inherent qualitative category and corresponding ordinal category (Figure 5).



A student typed the solution coded with P1 in qualitative categories and 2 in ordinal categories

KVADRAT
pravilan pravokutnik
nasuprotne stranice su paralelne
nasuprotne stranice su sukladne
susjedne stranice su okomite
susjedne stranice su sukladne
KOCKA
pravilan kvadar
nasuprotne strane paralelne
nasuprotne strane sukladne
susjedne strane okomite
susjedna strane sukladne

A student typed the solution coded with P1 in qualitative categories and 1 in ordinal categories

KVADRAT:
-pravilan pravokutnik
-susjedne stranice okomite
-susjedne stranice sukladne
-nasuprotne stranice paralelne
-nasuprotne stranice sukladne
KOCKA:
-susjedne strane okomite
-susjedne strane sukladne
-nasuprotne strane paralelne
-nasuprotne strane sukladne

Note. We translated identical texts (except for one circled sentence): •*SQUARE* •*regular rectangle* •*opposite sides are parallel* •*opposite sides are congruent* •*adjacent sides are perpendicular* •*adjacent sides are congruent* •*CUBE* •*regular cuboid* •*opposite sides are parallel* •*opposite sides are congruent* •*adjacent sides are perpendicular* •*adjacent sides are congruent*

Figure 5: Students' inputs coded with the same qualitative and different ordinal categories in An10

The median scores in the formal assessment before ERT, online assessment and formal assessment after ERT were 60%, 67% and 46%, respectively. We compared students' achievement in the formal assessments and the online assessment from their distribution in the quartiles by their scores in each assessment (Table 4). Though students scored lower in the second formal assessment, they mainly scored in the same quartile of two formal assessments (18 students did not participate, 18 students scored in the same, 13 students scored in the lower, 11 students scored in the higher quartile of second assessment). Students scored higher in the online assessment than in formal assessments, in particular those students who did not participate and those who scored lower in the formal assessments. Students' scores in the formal assessment after ERT were slightly more aligned with their scores in the online assessment (14 students were in the same quartile) compared to their scores in the formal assessment before ERT (10 students were in the same quartile).

Quartiles in online assessment	*Quartiles in the formal assessment before ERT and after ERT*[1]														NP[2]	Total
	1st	2nd	1st	2nd	1st	3rd	2nd	1st	4th	3rd	2nd	4th	3rd	4th		
	1st	1st	2nd	2nd	3rd	2nd	3rd	4th	2nd	3rd	4th	3rd	4th	4th		
1st	2	3		1	1						1				10	18
2nd	2					1			1	2		1		1	7	15
3rd	2			2		1	2		1	1		1	2	4	1	17
4th	1	1	1		1	1		1	1		1	1	1			10
Total	7	4	1	3	2	3	2	1	3	3	2	3	3	5	18	60

Notes. [1] The ordinals in the upper row correspond to the quartile in the formal assessment before ERT and ordinals in the lower row correspond to the quartile in the formal assessment after ERT. [2] Label NP refers to students who did not participate in either of two formal assessments.

Table 4: Distribution of students regarding achievement in the formal and online assessments

Majority of students dominantly used peer-oriented work (Table 5). Almost all students who scored lower in the formal than the online assessment dominantly used peer-oriented work in online assessment. Students who dominantly or evenly used individually oriented work in the online assessment were mainly in the higher quartiles in the formal assessments. Inspecting for qualitative categories showed that students from lower quartiles in both formal assessments who dominantly used peer-oriented work mainly provided the same answer as a student from higher quartile in either of the two formal assessments.

Dominant origin of work	*Achievements in the formal assessments compared to online assessment*[1]									*Total*
	Both lower			*Same or one lower*				*One higher*		
	2nd	*3rd*	*4th*	*1st*	*2nd*	*3rd*	*4th*	*1st*	*2nd*	*3rd*	
Peer	7	5	5	8	1	3	4	7	3	4	47
Individual				1			1		3	2	6
Mixed	1			1		1		1		2	7
Total	8	5	5	10	1	4	5	8	6	8	60

Notes. [1] Label in the upper row suggests students scored in lower or higher quartile in the formal assessments than the quartile in online assessment marked with ordinal in the lower row.

Table 5: Distribution of students regarding compared achievement in assessments and dominant origin of their work

Students' work concerning content and requirements in questions

Students' answers concerning the type of work (literature-L, peer-P or individual-I oriented) and values (correct-2, partially correct-1 or incorrect-0) varied across questions (Figure 6). The number of qualitative categories with peer-oriented work also varied across questions (Figure 7).

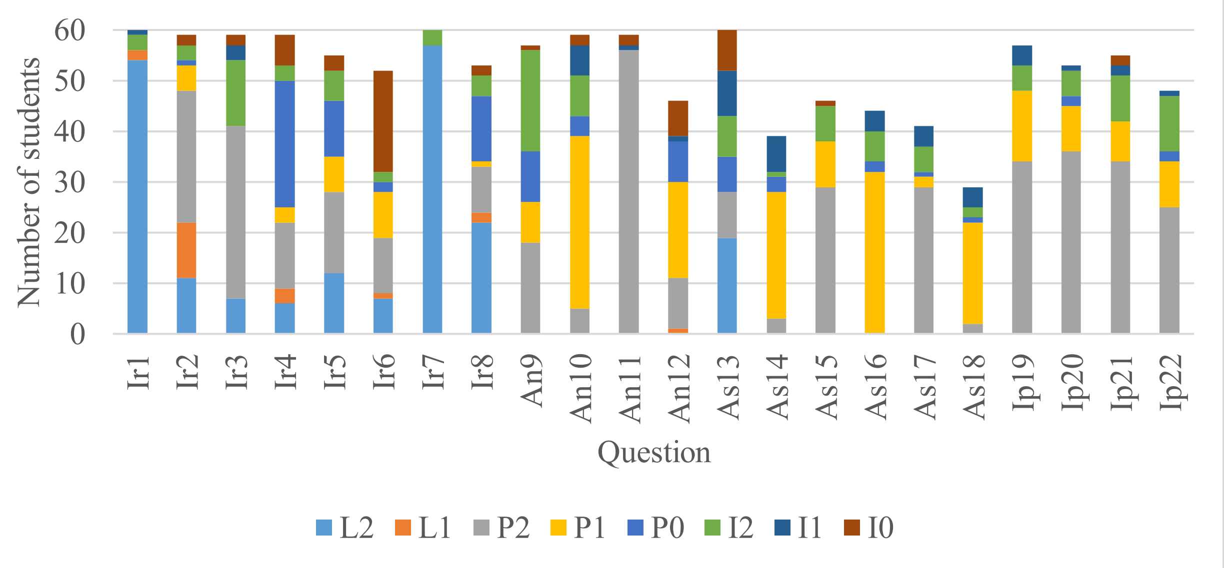


Figure 6: Distribution of students concerning the type of work and value of their answers across questions

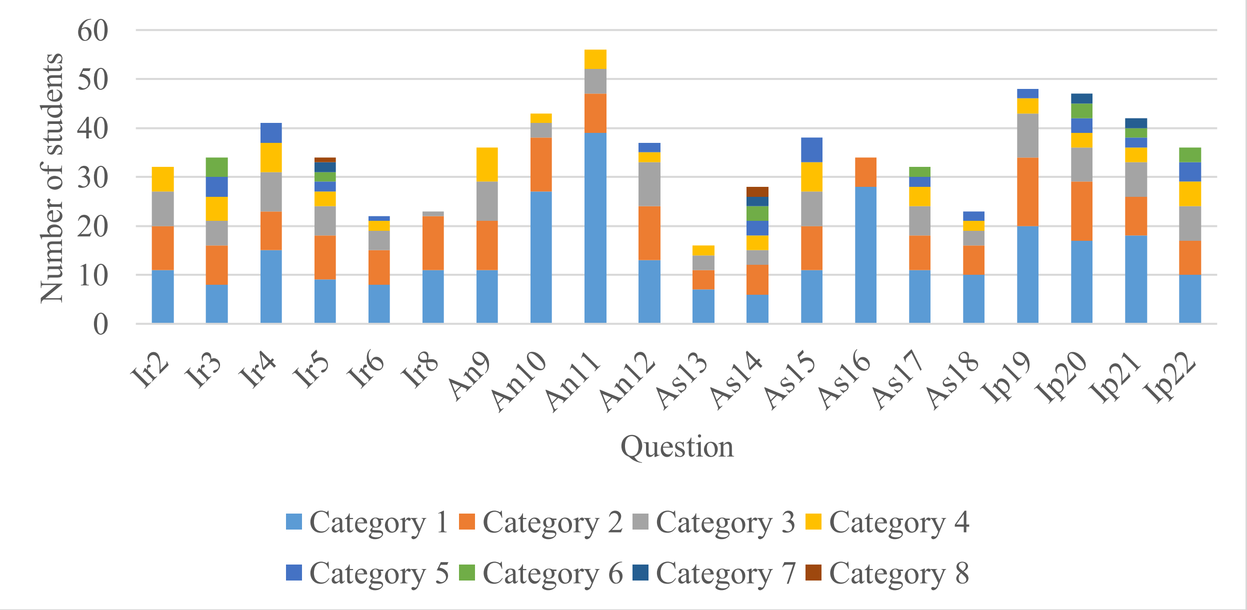


Figure 7: Distribution of students concerning different peer-oriented categories across questions

Questions about inductive reasoning in CCK and A1, Ir1 and Ir7, had almost all literature-oriented correct answers, with students correctly retrieving the definitions from the literature. Questions Ir2 and Ir8 in CCK and A2 had a significant number of literature and peer-oriented answers. In question Ir2, students provided, in a full or partial account, correct examples of complete (finite) induction from literature, and in each of the four peer-oriented categories, a correct example analogous to some examples given in lectures. In question Ir8, students provided correct counterexamples of incomplete (infinite) induction from literature and correct counterexamples in one of the three peer-oriented categories.

Other questions about inductive reasoning (Ir3-6) had a share of literature, and peer and individual-oriented answers. In questions Ir3 and Ir5 categorised in SCK and C2, some students retrieved examples of patterns from literature. Others constructed a variety of analogous and original examples in individual and peer-oriented work, which were mainly appropriate in the numerical infinite context in the former, but not the geometric infinite context in the latter question (Figure 8). In questions Ir4 and Ir6 categorised in HCK and C1, the number of correct answers was smaller than in questions Ir3 and Ir5. Students wrote an incorrect algebraic expression for the general term or vague explanation of the pattern, especially in the case of the non-analogue examples they gave in the preceding questions.

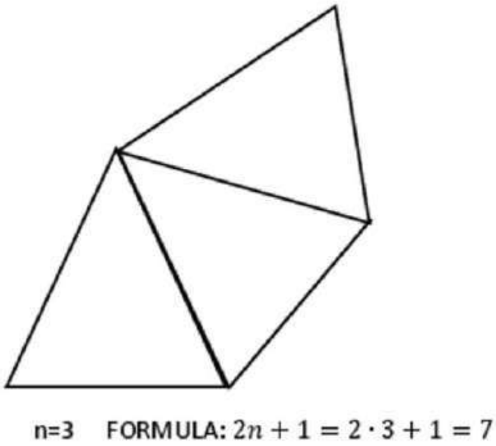


Figure 8: Student's incorrect example for growing geometric pattern in question Ir5

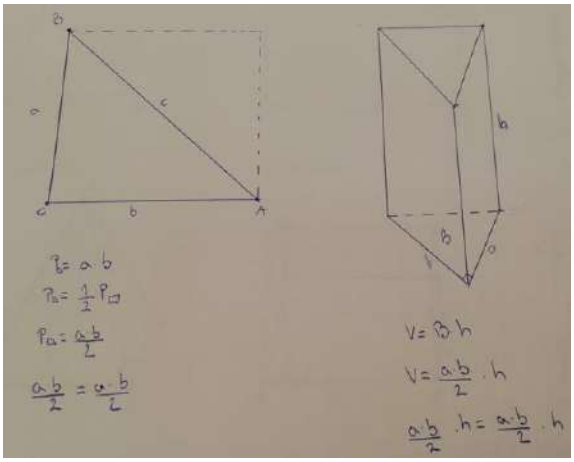


Figure 9: Student's proof of analogy (P1 qualitative category) in question An10

Questions about analogous geometric objects, An9 in SCK and B1 and An10 in HCK and B1, had dominantly peer-oriented answers with few different categories and a significant number of individually oriented answers. Students' answers in An9 differed in minor features, the shapes (particularly the pair triangle-tetrahedron), sketching and labelling geometric figures and solids. Students' answers in An10 differed significantly with a variety of well-observed analogous properties of square and cube. The dominant peer-oriented partially correct answer was focused on one analogues property, that square and cube both have congruent sides. In both questions, students mainly erred in terminology related to the properties of geometric objects.

Question An11 about analogous expression in CCK and B2 had the largest proportion of peer-oriented correct answers. Almost all students made the same correct mathematical analogy of given algebraic equality, and qualitative categories had the same mathematical content with different representations. Question An12 about an analogous statement in geometry in HCK and C2 had two peer-oriented categories with a correct analogy (P1 in Table 3), but only some students provided proof (Figure 9) which counted for a complete, correct answer. Other peer and individual-oriented answers were partially correct due to imprecise statements or missing justification (P4 and I1 in Table 3).

Question As13 about the method of analysis and synthesis in CCK and B1 had differently oriented answers and the largest share of individually oriented answers. Some students retrieved parts of the definition of the method of analysis and synthesis from the literature. Others described the method in their own words, revealing different conceptions. For example, a student's individually oriented partially correct answer was focused on the algorithmic procedure as applicable in question As14, and the dominant peer-oriented answer, "what we do by analysis, we can check by synthesis", did not correctly convey the idea. In question As14 about analysis and synthesis in CCK and B2, we were unable to access all students' answers. Students provided the calculation part of the analytic-synthetic procedure of proving an algebraic inequality. Their answers differed in the order of algebraic manipulation, and they were partially correct due to omitting some elements in the overall procedure.

In the geometric construction problems, we were unable to access all students' answers. The answers were dominantly peer-oriented, mainly correct in questions As15 and As17 in HCK and C1, and partially correct in questions As16 and As18 in HCK and C2 category. In the former, students followed the correct idea in the analysis of the geometric problem and erred in terminology and calculation. In the latter, when justifying the construction, students mainly focused on the dominant property of the geometric figure, that is, the type of the triangle or the perimeter of the triangles. The larger number of categories in questions As17 and As18 than As15 and As16 did not come from students' work with properties of the triangles but their use of different measuring units.

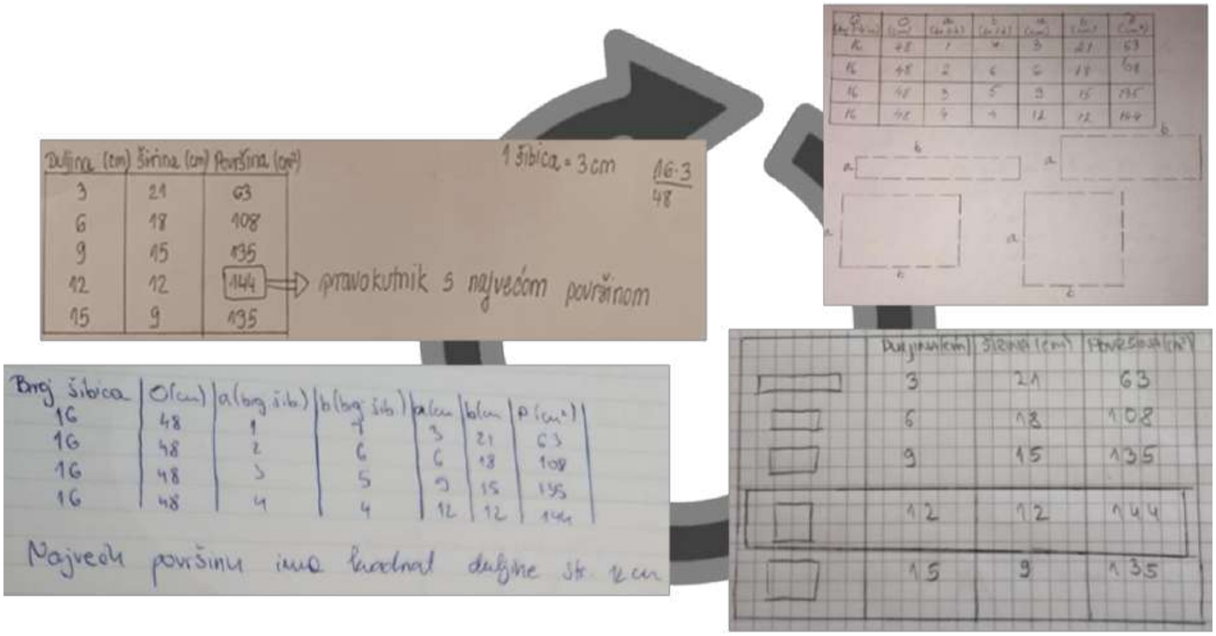
Questions about area-perimeter problems (Ip19-22) mainly had peer-oriented correct answers. The difference between questions was in the variety of answers. Question Ip19 in SCK and B1 had fewer different individual and peer-oriented answers, with students appropriately using the formula and systematically organizing the data similar to the example given in lectures. In questions Ip20 and Ip21, both in SCK and B2, students approached solving the problems in a real-life context in different ways, by solving equations, drawing, tabulating outcomes or calculating values (Figure 10). In question Ip22 in HCK and C1, there was a variety of peer and individual-oriented answers. Students had different focuses when discussing the limitation of the task in question Ip19 – providing examples and non-examples, calculating the values, discussing the properties of figures, and applying mathematical statements or reasoning intuitively. Following are examples of students' answers from different categories.

P1: „If it is an even number not divisible by 4, the lengths of the sides of the rectangle with the maximum area **cannot be calculated** directly. If it is an odd number, we cannot use **integers**."

P2: „ If it is an even number not divisible by 4, the solution is not an **integer**. Eg., *o*=50 cm, *P*=12.5·12.5=156.25 cm^2. The student would investigate sides with **integer** lengths and conclude the maximum area is 156. If it is an odd number, the side lengths are not **integers**. If *a* and *b* are integers, then *o=2(a+b)* is divisible by 2! This task is **not appropriate** for students in primary education."

P3: „ If it is an even number not divisible by 4, the lengths of the sides of the rectangle with the maximum area **cannot be calculated** directly (SQUARE). If it is an odd number, for a rectangle with the maximum area we cannot use **integers** but **real numbers**."

P4: „ If it is an even number not divisible by 4, the solution is **not a square but a rectangle** with approximate side lengths. If it is an odd number when investigating for a rectangle with the maximum area the resulting side lengths are real numbers."



Duljina (cm)	Širina (cm)	Površina (cm²)
3	21	63
6	18	108
9	15	135
12	12	144
15	9	135

1 šibica = 3 cm

16·3
48

144 ⇒ pravokutnik s najvećom površinom

Broj šibica	O(cm)	a(broj šib.)	b(broj šib.)	a(cm)	b(cm)	P(cm²)
16	48	1	7	3	21	63
16	48	2	6	6	18	108
16	48	3	5	9	15	135
16	48	4	4	12	12	144

Najveću površinu ima kvadrat duljine str. 12 cm

	DULJINA (cm)	ŠIRINA (cm)	POVRŠINA (cm²)
	3	21	63
	6	18	108
	9	15	135
	12	12	144
	15	9	135

Figure 10: Students' different representations within different qualitative categories in question Ip21

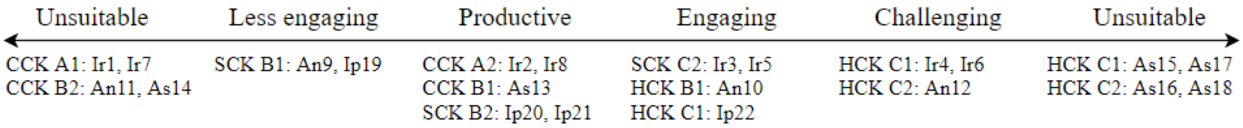
DISCUSSION

During the ERT, students of primary teacher studies in the course of didactics of mathematics worked in continuous, obligatory, non-graded, online assessments in the form of Moodle tests. The purpose of the assessments was to engage students in continuous independent work and provide them with feedback; they would reflect on the course content and develop from feedback. Though the goals of the course work were aligned with the idea of effective continuous assessment there was no evidence that the assessments affected students' achievement in the course. Overall lower achievement in the second formal assessment could be due to different course content, the unprecedented circumstances of the ERT or the lack of preparation. We found that students approached the online assessments in different ways; they relied on literature, and peer or individual-oriented work. Students who scored higher in either of the two formal assessments were more engaged in individual work in the online assessments. The peer-oriented work could have been produced in two ways – as a collaborative student's work or as a copied work from one engaged student. The majority of students worked peer-oriented and their answers were mainly equal to a high achieving student's answers. Thus, high achieving students appeared as managers of peer work, and other students appeared to have invested minimal effort in the online assessments. In the context of study approaches, most students had a surface-strategic learning approach, which was to complete the assignment by submitting the best possible answers and relying on peer-oriented work. Students with a deep learning approach submitted original answers, not necessarily correct since they invested time and effort in individual work. Their approach was aligned with the purpose of the exercises. Reflecting on the results in general, though students completed the frequent assessments which balanced their workload, they did not engage in regular, independent, active work that would be a prerequisite of a deep approach. Focusing on students' work in particular questions, we gained information on choosing appropriate and interesting tasks with supportive feedback to increase students' interest, motivation, and engagement, directing them towards a deep approach to learning.

Questions from the Moodle tests could be judged for their fitness for online assessment and continuous assessment. Evaluating open questions in the Moodle environment, especially files uploaded rather than embedded, was time-consuming, compared to the automated evaluation of closed questions in Moodle tests or evaluating pen and paper assessments. Some of the open questions could be rearranged into closed questions without loss in the requirements and with corresponding, pre-defined feedback (Jamil et al., 2022). For example, questions Ir1 and Ir7, that was recalling the definition, could be `select missing words´ question type, question An9, which was deciding about analogue objects, could be `matching´ question type, and questions An11 and As14, that was extrapolating known information to different situation, could be `multiple choice´ question type to select correct expression and `drag and drop´ to arrange steps of the procedure correctly. Open, `essay type´ questions that were easy to evaluate in Moodle environment were questions Ir2 and Ir8, questions Ir3-6, that were about retrieving or constructing examples and counterexamples, questions An10 and As13, that were explaining the relationship between objects, and summarizing mathematical discourse in non-mathematical terms, and question Ip22, that was discussing limitations of a mathematical task. Feedback in these questions could be criterion-referenced feedback valued against a set of pre-defined requirements and additionally, by the provision of content analysis, general reflective feedback could be constructed for dissemination and discussion. Other questions, solving geometric construction problems, making and proving conjectures and modelling real-life problems seemed less appropriate for the online environment and independent work.

Questions in formal and informal assessments should appropriately reflect the course content and be of different types with varying degrees of difficulty (Hughes, 2008; Korhonen et al., 2015). The subjective difficulty of questions could be assessed from the frequencies of ordinal categories and the diversity of qualitative categories in each question. Questions with almost all correct answers, dominant literature-oriented answers or one dominant qualitative category are unsuitable for differentiation, constructive feedback or promoting a deep approach. Questions with almost all partially correct or incorrect answers are also unsuitable either for assessing prospective primary school teachers' knowledge or for this type of assessment. Questions which engaged students in work were those with varied ordinal categories which were also challenging for students and those with varied qualitative categories which were productive for students.

There were differences in students' engagement in questions in different SMK and MATH taxonomy categories (Figure 11). Students seemed least engaged when reproducing formal definitions of a mathematical notion (Ir1,7 in CCK and A1). But they seemed more engaged when required to propose examples for the same notions (Ir2,8 also in CCK but in A2), or to reflect on the definition of a mathematical notion (As13 also in CCK but in B1). When transferring mathematical knowledge, tasks set in a real-life context appropriate for primary education (Ip20,21 in SCK and B2) appeared more engaging than tasks set in a purely mathematical context (An11, As14 in CCK and also in B2). In the former, students used various representations in their solution, and in the latter, they used presupposed form of the solution. Working with primary level notions and applying primary level formulas (An9 and Ip19 in SCK and B1) was not as productive as constructing examples of numeric and geometric patterns appropriate for primary education (Ir3,5 also in SCK but C2) nor as challenging as describing general, mathematical properties of a primary level notion (An10 in HCK and also in B1). Similarly, discussing the conditions and limitations of a primary level task seemed challenging and productive for students (Ip22 in HCK and C1). Solving geometric construction tasks appropriate for primary mathematics education using the method of analysis and synthesis did not appear productive for students (As15-18 in HCK and C1 or C2). In particular, students recognized the solution in the context of primary education but were not able to communicate the solution in the context of formal mathematics. For students, the most challenging seemed to be generalizing, deducing, and justifying formally about primary level content (Ir4,6 in HCK and C1, An12 in HCK and C2). Corresponding questions had a larger proportion of partially correct or incorrect answers compared to the questions with similar content and different requirements. In particular, students had more difficulties with questions in geometric and infinite contexts than questions with algebraic and finite contexts.



Unsuitable	Less engaging	Productive	Engaging	Challenging	Unsuitable
CCK A1: Ir1, Ir7 CCK B2: An11, As14	SCK B1: An9, Ip19	CCK A2: Ir2, Ir8 CCK B1: As13 SCK B2: Ip20, Ip21	SCK C2: Ir3, Ir5 HCK B1: An10 HCK C1: Ip22	HCK C1: Ir4, Ir6 HCK C2: An12	HCK C1: As15, As17 HCK C2: As16, As18

Figure 11: Distribution of questions regarding suitability and engagement

Questions with a larger proportion of individually oriented answers and multiple categories of peer-oriented work are preferable. Students' answers to such questions provide a variety of discourses, examples, or strategies that can be shared and discussed among peers thus enriching their communication, conceptions and strategies related to mathematical notions. Such are the following types of questions:

Type 1: Retrieving (Ir2 and Ir8) and creating examples (Ir3 and Ir5) for mathematical and primary level notions,
Type 2: Describing, discussing, and making judgments, in particular, summarizing in non-mathematical terms a definition of a mathematical notion (As13), comparing mathematical properties of primary level objects (An10), making and proving mathematical conjectures about primary level objects (An12), and discussing mathematically primary level problems (Ip22),
Type 3: Solving contextual problems related to a particular primary level problem (Ip20-21).

Type 1 and 2 questions gave insight into students' (mis)conceptions about mathematical notions, that is, their concept image as a whole collection of ideas, representations, examples, and relations, formed mentally about the notion (Tall & Vinner, 1981). Such questions are essential to learning and understanding a mathematical notion by developing a comprehensive and suitable concept image (see Horzum & Ertekin (2018), Ulusoy (2021), Vinner (1991)). Type 2 and 3 questions revealed different strategies and focus in students' mathematical work, their mathematical thinking style as a preferred way of understanding, presenting and thinking about mathematical notions (Borromeo Ferri, 2010) or using and connecting different representations as an indicator of specialized content knowledge (Steele, 2013). The visual or analytic style was reflected in discussing geometric or measurable properties of figures, and the problem-solving approach used in the contextual problem (Figure 10). For example, students drew figures - visual thinking style or calculated different outcomes - analytical thinking style, to answer the question.

Limitations, implications, and further research

The participants of the study and the content of the questions were limited by our particular context. Though the results of our study are not generalized, they contribute to practice and research by considering the categories of assessment questions. Students' work agreed with the requirements in the assigned MATH category. Questions with different content in our study were assigned to a few categories in the MATH taxonomy. Any assessment should strive to contain questions across all mentioned categories. More information on students' work in questions from different categories would additionally support the discussion about appropriate types of questions for continuous online assessment. In the context of our study, that option was dismissed since it would have increased students' workload significantly.

The results of our study were inconclusive about how continuous online assessment affected students' achievements in the course. For one, many students relied heavily on peer-oriented work hence the issues of supervision and plagiarism arose, and their cumulative achievement calculated from the ordinal category assigned to their answer might not be their achievement. Second, the underlying idea of the online assessments was that students would prosper from the feedback regardless of the correctness of their answers hence students' lower achievement in online assessments need not imply lower achievement in the formal assessment. Questionnaires and interviews with students about their knowledge, experience and attitude could provide additional insight into their work in online continuous assessment.

The course content limited the nature of questions regarding mathematical knowledge for teaching. The questions were inclined toward the formal mathematical knowledge of prospective teachers.

The results of this study implicate that designing assessments for prospective teachers might include two-dimensional choices, reflecting on content knowledge and task requirements. The MATH taxonomy seemed compatible with SMK categories but no conclusions about its compatibility with PCK categories can be made. Steele (2013) also suggested designing tasks to access interactions between different categories of mathematical knowledge for teaching. This is the direction for our further practice and research.

CONCLUSION

Online assessments in the form of Moodle test implemented in our study provided information about prospective teachers' engagement in continuous work, designing online tests in mathematics education, choosing questions for continuous assessment, and some issues in students' mathematics knowledge. The methodology used in this study can be adapted to different contexts. Content analysis of students' answers proved an invaluable tool, in particular, by forming the qualitative categories which described the origins, correctness and characteristics of students' answers. Categorizing questions regarding mathematics knowledge for teaching and MATH taxonomy seemed fitting and useful for assessing prospective mathematics teachers, both for achieving the goal of the study and for implementing it in our teaching practice.

The results of our study are aligned with the study by Lebeničnik et al. (2015) who found that future teachers are more inclined to passively receive information than actively engage and collaborate on educational tasks and Tanujaya et al. (2021) who reported inactive collaboration and copying other students' solution as issues with online learning. However, continuous, non-graded, online assessments with engaging tasks which complement the regular lectures in a blended learning environment might be such activities that promote deep learning. Students participate without pressure to obtain a particular grade and they learn from the teacher's feedback, while the teacher reorganizes the teaching and discusses different answers and approaches. This kind of work is very demanding and time-consuming hence it is important to thoughtfully formulate questions and choose the format of the assignment.

The results of our study suggest that the questions that require creating examples, discussing definitions and properties, and solving contextual problems prompted students' active engagement and provided insight into students' conceptions and thinking styles founding for rich and constructive feedback. It was the case for both mathematical and primary level content. These types of tasks can be included in frequent, individual assignments with formative, individual feedback and reflective, comprehensive feedback on examples and non-examples of solutions students provided. The tasks that required argumentation or generalization about primary level content from ab advanced point of view (horizon knowledge) were challenging for students. Such questions can be implemented as occasional, intensive, group work with peer evaluation.

Students approached asynchronous online tests strategically and worked with peers to complete them. In that light, more effective continuous assignments can be designed in two ways, as easy-to-evaluate independent, engaging assessments or as collaborative activities. Each of these elements could be incorporated as a blended learning environment to engage students in continuous work.

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APPENDIX

Questions from Online Assessments

	Question from the online assessment	*Requirements from MATH taxonomy category*	*SMK domain*
Ir1	Explain briefly complete induction.	A1 Recall the definition of complete induction	CCK includes knowledge of complete induction as mathematical content
Ir2	Provide an example for complete induction.	A2 Recognize an example for complete induction (finite context)	CCK includes knowledge of complete induction as mathematical content
Ir3	Create a rule for a sequence of integers. Write several first terms in the sequence.	C2 Construct an example of an infinite integer sequence	SCK includes constructing examples for integer sequence as school content
Ir4	Describe the rule for the sequence with words and the n-th term with symbols.	C1 Discuss the properties of an infinite integer sequence	HCK includes generalization about integer sequence as mathematical work with school content
Ir5	Create a rule for a growing geometric pattern. Draw several first figures by the pattern.	C2 Construct an example of a growing geometric pattern	SCK includes constructing examples of geometric pattern as school content
Ir6	Describe the rule for the growing geometric pattern with words and the value for the n-th figure with symbols.	C1 Discuss the properties of a growing geometric pattern	HCK includes generalization about geometric pattern as mathematical work with school content
Ir7	Explain incomplete induction.	A1 Recall the definition of incomplete induction	CCK includes knowledge of incomplete induction as mathematical content
Ir8	Find an example of a false claim obtained by incomplete induction.	A2 Recognize counterexample for incomplete induction (infinite context)	CCK includes knowledge of incomplete induction as mathematical content
An9	Write out all the geometric plane figures mentioned in primary mathematics education. For each figure write the name of its spatial analogue. Draw the pairs of analogues objects. Name the vertices and sides of the figures and solids.	B1 Decide about the analogy between corresponding plane figures and solid shapes	SCK includes presenting ideas and selecting representations of geometric objects as school content
An10	Explain why a square and cube are analogues objects.	B1 Explain analogous properties of square and cube	HCK includes argumentation about geometric objects as mathematical work with school content
An11	Write the analogue of the equality $\|ab\| = \|a\|\|b\|$.	B2 Extrapolate known algebraic relation to a different setting by analogy	CCK includes stating analogous algebraic equality as mathematical work

	Question from the online assessment	*Requirements from MATH taxonomy category*	*SMK domain*
An12	State the analogue of the claim “The area of a right triangle equals the half of the product of its catheti lengths”. Is the analogous claim true? Explain.	C2 Make a conjecture by stating a spatial analogue of a known result in the planar geometry and formally prove or disprove the conjecture	HCK includes stating analogy for the area of triangle as mathematical work with school content
As13	Explain (in your own words) why analysis and synthesis make a unique method.	B1 Summarize in non-mathematical terms the relationship between analysis and synthesis in the scientific method	CCK includes analysis and synthesis method as mathematical work
As14	Let a and b be positive real numbers. Prove the inequality $\frac{a}{b}+\frac{b}{a}\geq 2$.	B2 Extrapolate known procedure of proving an algebraic inequality by the method of analysis and synthesis to a different setting	CCK includes proving algebraic equality as mathematical work
As15	Construct an isosceles triangle with 10 cm perimeter, and with its legs length equal twice the base length. Analyse the problem.	C1 Recognize and interpret assumptions by analysing geometric construction problem	HCK includes analysing simple construction problem as observing school content from advanced point
As16	Construct an isosceles triangle with 10 cm perimeter, and with its legs length equal twice the base length. Synthesise the solution to the problem.	C2 Deduce solution validity by synthesising the geometric construction	HCK includes synthesizing simple construction problem as observing school content from advanced point
As17	Construct an equilateral triangle with a perimeter equal to the perimeter of an isosceles triangle with 1 dm legs length, and the 85 mm base length. Analyse the problem.	C1 Recognize and interpret assumptions by analysing geometric construction problem	HCK includes analysing simple construction problem as observing school content from advanced point
As18	Construct an equilateral triangle with a perimeter equal to the perimeter of an isosceles triangle with 1 dm legs length, and the 85 mm base length. Synthesise the solution of the problem.	C2 Deduce solution validity by synthesising the geometric construction	HCK includes synthesizing simple construction problem as observing school content from advanced point
Ip19	Explore the rectangles with perimeter *o*=48 cm, and integer length of its sides. Write out all the rectangles. Include sides lengths and area.	B1 Apply formulas for the perimeter and area of a rectangle in a particular context	SCK includes presenting ideas and connecting representations about area/perimeter relation as school content

	Question from the online assessment	*Requirements from MATH taxonomy category*	*SMK domain*
Ip20	George has 144 concrete panels shaped like a square with a 1-meter sides length. He will use it to pave a part of his yard shaped like a rectangle. He will surround that part with an expensive fence. What should be the length and width of the concrete part of the yard so that it requires the least fencing?	B2 Model real-life setting with perimeter and area of a rectangle	SCK includes discussing ideas, selecting representations, interpreting solutions about area/perimeter relation as school content
Ip21	Vita is making rectangles using matches with 3 cm length. She has 16 matches. Which of the rectangles has the largest area?	B2 Model real-life setting with perimeter and area of a rectangle	SCK includes discussing ideas, selecting representations, interpreting solutions about area/perimeter relation as school content
Ip22	Note that the value of the perimeter given in Ip19 is a multiple of 4. What if the value of a given perimeter is an even number that is not a multiple of 4? What if the value of a perimeter is an odd number?	C1 Recognizing the limitations occurring in the solution of a mathematical task related to the area and perimeter of a rectangle by changing the initial values in the task	HCK includes argumentation about conditions in area/perimeter task as mathematical work with school content

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Investigation of Secondary Students' Epistemological Obstacles in the Inequality Concept

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Abstract: An inequality concept has an important role; even in advanced mathematics, inequality assists in analysis and proof. However, students' understanding of inequality is not satisfying. The fact shows that students experience difficulties and errors in solving inequality problems. These difficulties and errors are not intentional or do not result from students' carelessness in carrying out solutions or students' ignorance (misconceptions). Instead, these difficulties and errors are caused by epistemological obstacles. Therefore, this study explores the epistemological obstacles students face in the inequality concept by analyzing errors found in solving inequality problems. The qualitative research design with a phenomenological approach was chosen to achieve the research objectives by involving 29 eleventh-grade secondary students. The researchers employed a test on the inequality concept to explore students' epistemological obstacles, which consisted of three problems. A one-to-one unstructured interview was also conducted to investigate students' ways of thinking and understanding based on their answers. Furthermore, the data were analyzed using an inductive approach, combining systematic data management methods through reduction, organization, and connection. The data obtained are then presented in the form of narratives and figures. The results showed that the inequality rules, the absence of semantic and symbolic meanings of inequalities, interpreting solutions, and generalizations in the inequality rules become sources of students' errors in solving inequality problems. Thus, we found epistemological obstacles in the inequality concept based on these four types of errors. The obstacles are indicated by students' limitations in understanding and interpreting inequality signs as they solve inequality problems.

INTRODUCTION

Inequality is an expression showing that two quantities are not equal (Frempong, 2012; Gellert, Kustner, & Hellwich, 1975; Gustafson & Frisk, 2008; Postelnicu & Coatu, 1980). In algebra, expressions of two unequal quantities are connected by symbols (Davies & Peck, 1855). Inequality is a scientific discipline with a highly crucial role, contributing to mathematical discoveries from Classical Greek Geometry to Modern Calculus (Fink, 2000). Inequality is also considered a subject that is difficult to define precisely, in which some parts belong to 'algebra' and the others to 'analysis' (Hardy, Littlewood, & Pólya, 1934). It implies that inequality has an important position at a higher mathematics level. Inequality assists in analysis and proof.

However, several previous studies have shown that students and undergraduates still encounter difficulty understanding or applying inequalities. Taqiyuddin et al. (2017) study revealed that students have difficulty solving problems in linear inequalities, e.g., when students are given a linear inequality $9x + 1 > 9x - 2$. While solving the linear inequality, students tend to incorrectly use algebraic operations, such as in $9x + 1 > 9x - 2$, $9x - 9x > -2 - 1$, and $x > -3$. A similar result is also found by Botty et al. (2015), reporting that students have difficulty solving linear inequality problems, e.g., when students draw a graph of linear inequality and identify the area represented by the given inequality. The results obtained an average of 22%, indicating that the test is challenging for students.

In addition, another fact demonstrates that difficulties in solving inequalities are experienced by not only secondary students, but also undergraduate students (Rowntree, 2007), specifically in the following four areas: a) inequalities as equations (Blanco & Garrote, 2007; Vaiyavutjamai & Clements, 2006), b) limited understanding of the terms “greater than” and “less than” and the appropriate relational symbols (Warren, 2006), c) difficulty in connecting and using different problem-solving techniques (Blanco & Garrote, 2007; Tsamir & Almog, 2001), and d) interpreting solutions (Tsamir & Bazzini, 2004). Besides, other studies (Bicer, Capraro, & Capraro, 2014; Blanco & Garrote, 2007; Booth, McGinn, Barbieri, & Young, 2017; Ellerton & Clements, 2011; El-Shara' & Al-Abed, 2010) found that students made basic arithmetic errors due to insufficient knowledge about inequality rules which tend to alter the inequality sign, such as when dividing the inequality by a negative sign.

Several studies have been conducted related to the difficulties faced by students in solving the concept of inequality. However, the studies mentioned above only focused on the difficulties and errors faced by students in solving the concept of inequalities. And how is the impact of applying a model or learning method in learning the concept of inequalities. In this study, we carried out an update by looking at and exploring students' experiences in understanding and interpreting the concept of inequalities by investigating the epistemological obstacles experienced by students in the concept of inequalities.

Concerning this phenomenon, we used the approach of epistemological obstacle analysis and investigation to student difficulties inherent in structuralist thinking. The concept of epistemological obstacles was first introduced by Bachelard (1938), which appeared in his philosophy of science work. Bachelard (1938) explained that “The problem of scientific knowledge must be posed in terms of obstacles [...] we will illustrate sources of standstill and even regression in the very act of knowing, and this is where we will discern causes of inertia that we will call epistemological obstacles.” Gutting (1989) points out that the center of Bachelard’s philosophy of science is his model of scientific change, which is built around four epistemological categories: ruptures, obstacles, profiles, and acts. Bachelard employs the concept of an epistemological break in contexts. He indicates that, in the first term, scientific knowledge separates from common sense and even the contradiction, and in the second term, ruptures also occur between scientific conceptual elaborations. The end of ruptures, in turn, suggests that there is an obstacle that must be destroyed. Bachelard thus introduced the notion of an epistemological obstacle, understood it as any concept of the method that prevents an epistemological rupture. The idea of an epistemological profile consists of an analysis that reveals the degree to which the individual's understanding of a concept involves elements from various stages of its historical development. Finally, the concept of an epistemological act counterbalances the obstacle and refers to the leaps that the scientific genius introduces in scientific development.

Brousseau (2002) offers the concept of epistemological obstacles as the meaning of knowledge (instead of lack of knowledge) that has been considered effective previously, even in certain contexts, which at some point begins to produce answers that are judged to be wrong or inadequate and raise contradictions. Moreover, epistemological obstacles are resistant and appear sporadically even after being overcome; dealing with them requires deeper knowledge which generalizes the known context and requires students to be aware of the obstacles explicitly (Brousseau, 2002). According to Brousseau (2002), such thinking can be applied to analyze the historical origin of knowledge or teaching or spontaneous cognitive development of students' understanding. The search for epistemological obstacles is conducted using two approaches: first, according to Bachelard, historical research by adopting an epistemological perspective, and second, tracing repeated errors in learning mathematical concepts. The two approaches are interrelated: historical-epistemological developments can assist in identifying possibilities of hidden models and suggest the construction of appropriate learning situations to overcome the obstacles found. Besides, students' difficulties and repeated errors indicate an epistemological obstacle. Relevant to this, Brousseau (2002) has proposed methods to find out these epistemological obstacles, including (1) finding repeated errors and asserting the errors are part of knowledge, not ignorance; (2) investigating obstacles in the history of mathematics; and (3) comparing obstacles with history and determining their epistemological characters. Finding epistemological obstacles to a mathematical content can be carried out through historical analysis and analysis of students' ways of understanding as the epistemological obstacles are not related to the way or method the teacher uses in learning; they are results of the nature and characteristics of the mathematical concept instead (Cornu, 2002).

Additionally, epistemological obstacles can be identified by noticing the tendency to generalize certain understandings to all situations. Sierpinska (1987) explained that the duality of epistemological obstacles provides another clue; if the presence of epistemological obstacles in students is associated with beliefs, overcoming these obstacles does not mean replacing their existing beliefs with the opposite ones. It will double the obstacles. Instead, students have to rise from what they believe to analyze, from the outside, ways they use to solve problems, formulate the hypotheses they have understood, and become aware of possible rival hypotheses.

The epistemological obstacle analysis in this study is conducted to explore the obstacles faced by secondary students in understanding and interpreting the inequality concept by exploring their experiences. This analysis is essential since it is a starting point for teachers to develop teaching strategies (Hausberger, 2017). This epistemological obstacle analysis is supported by a "didactic situation theory" (Brousseau, 2002). According to this theory, knowledge construction refers to the implication of interaction between students and problem situations (broader: milieu); that is, dialectical interaction in which students use prior knowledge to revise, modify, complement, or reject constructs of new knowledge. Students' knowledge is attained by adapting their ways of thinking to an environment. In other words, students' errors emerge due to the adaptation process of ways of thinking and understanding in the learning environment (Brousseau, 2002). Therefore, this current study investigates the epistemological obstacles students encounter in the inequality concept through error analysis on the given problems. In addition to that, the researchers analyzed ways of thinking and understanding of each students' errors. An epistemological obstacle is a well-constructed piece of knowledge that is so practical and valuable that it forces the problem-solver to employ the approach it proposes. We took advantage of the chance to explore the concept of epistemic obstacles more thoroughly in order to find a productive approach for the teachers by offering a variety of obstacles and taking into account their relative difficulties. In order for educational practitioners to use this study's findings as a guide in predicting learning obstacles for students in the idea of the inequalities concept. Additionally, it can be utilized as a guide or source of information when proposing the best design in light of current scientific knowledge of the inequalities concept.

RESEARCH METHODOLOGY

This study aims to investigate the epistemological obstacles in the concept of inequalities of secondary students, by exploring their experiences after receiving/through learning in the concept of inequality. Thus, a qualitative research design with a phenomenological approach was chosen. This approach is defined as a qualitative method focusing on understanding and interpreting human life experience as a topic according to its framework, relating to meaning, and how it is obtained from experience (Grbich, 2007; Langdridge, 2007; Suryadi, 2019). The phenomenon being studied was the epistemological obstacle of secondary students in the inequality concept. Further, the phenomenological facts were linked with normal interpretative and relevant theory (pragmatic interpretation). The research techniques and study layout that make up a phenomenological research study's methodology are described below.

1. *Participant selection*

Selecting research participants who have substantial and meaningful experiences with the issue being examined is a requirement for researching the essence of lived experience (Polkinghorne, 1989). Composition and sample size of the study are additional factors to be taken into account when choosing participants. Because the goal of phenomenological research is to collect descriptions of experience rather than generalizable conclusions, participants' representativeness of the general population is not a priority (Cilesiz, 2011; Seidman, 2006). The subjects might be chosen using the purposive sampling technique. This sort of sample is nonprobability (Alhazmi & Kaufmann, 2022). Due to the in-depth nature of the study, sample sizes for phenomenological research are typically not huge; while recommendations for sample size vary, a sample of 3–10 people are typically seen to be suitable (Creswell, 1998; Polkinghorne, 1989).

2. *Data collection through phenomenological*

In general, phenomenological research, data consist of descriptions of life experiences, which can be collected through interviews, observations, or written self-descriptions (van Manen, 1997). In this study, data were collected through giving tests related to the concept of inequalities and followed by interviews to explore experiences formed from the process of learning the concept of inequalities and how students interpret it.

3. *Phenomenal data analysis*

The goal of phenomenological research is to identify and dissect the structures, logic, and connections that exist within the experience being studied. The central phase of phenomenological research is data analysis. Data analysis was carried out by adopting a phenomenological approach developed by Hycner (1985) and modified by (Groenewald, 2004), which acknowledges the researcher's interpretive involvement with the data.

4. *Validity considerations for phenomenological research*

In qualitative research, the term "validity" typically refers to a study's rigor to ensure that the findings are the product of the proper application of methodologies and that the research delivers relevant information based on its epistemology (Guba & Lincoln, 1982; Lincoln, 1995; Merriam, 1995).

5. *Ethical considerations in phenomenological research*

Participants' privacy and confidentiality must be protected in these situations since failing to do so could harm their reputation or have other negative effects. By using pseudonyms in place of identifiable information such as references to names and localities and by allowing participants to read the final report and point out any weaknesses or objectionable portrayals, it is possible to protect participants' privacy. Another important ethical factor in phenomenological research is reciprocity. According to calls for reciprocity, both the researcher and the subject of the research should gain from the act of doing it. Researchers are requested to share some of the perks associated with their privileged and intellectual positions in exchange for the opportunity to share sensitive details about their lives, even though there is no financial compensation for doing so (Lincoln, 1995).

The subjects involved in this study were 29 eleventh-grade secondary students in Medan City. The test was administered to all students, but from the test results, the subject was then reduced to nine students, and in the final stage, it was reduced again to three students. According to Miles and Huberman (1994), data reduction refers to the process of selecting, focusing, simplifying, abstracting, and changing data that appears in field notes or written transcriptions. Data not only needs to be summarized for easy management but also needs to be modified so that it can be understood concerning the problem being addressed. In other words, at this data reduction stage, there will also be a process of coding, summarizing, and also partitioning or creating parts. In addition, data reduction can also be interpreted as a form of analysis that sharpens, classifies, and directs research objectives. The data for this article was derived from students' learning experience in the inequality concept.

Data were collected through tests and interviews designed to explore students' experiences and how these experiences influence students' perceptions of the inequality concept. The test given to the students was related to the inequality concept. Also, a one-to-one interview was conducted after the researchers analyzed the test results to ascertain the experiences and obstacles encountered by the students. In detail, the test is presented in Table 1.

Problem Type	Problems
1	Please solve the linear inequalities below and present the solutions using interval notations: a. $3(2x-9)<9$ b. $-4(3x+2)\leq 16$.
2	Please use the inequality notation to describe the expression "there exist all values of x within an interval $(-3,5]$".
3	Please explain all values of x with the distance of 4 from number 5. Sketch this expression in a number line, state it using an inequality, and find the solutions.

Table 1: Solving Inequality Problems

All data in this study were transcribed, and a pseudonym was given to each participant. The data were analyzed using an inductive approach, which combines systematic data management methods through reduction, organization, and connection (Dey, 1993; LeCompte, 2000), and the data obtained are then presented in the form of narratives and images. Overall, data analysis was carried out by adopting the phenomenological approach developed by Hycner (1985) and modified by (Groenewald, 2004), which acknowledges the researcher's interpretive involvement with the data. In this study, data analysis was carried out through five steps that have been simplified by Groenewald, including 1) Bracketing and phenomenological reduction; 2) Delineating units of meaning; 3) Clustering of units of meaning to form themes; 4) Summarising each interview, validating it and where necessary modifying it; and 5) Extracting general and unique themes from all the interviews and making a composite summary. As previously stated in the introduction section, epistemological obstacles can be identified by analyzing students' errors in solving inequality problems. In addition to that, this study analyzes other phenomena that appear in students' answers. A theme was made based on patterns and types of errors found to classify student errors. The researchers also reviewed the results of previous studies as our assumptions about ways of thinking and understanding aspects of each error type. After the classification process and descriptive analysis of students' errors, a one-to-one task-based interview was conducted. This interview aims to strengthen the descriptive aspects of ways of thinking and understanding behind the errors obtained.

RESULTS

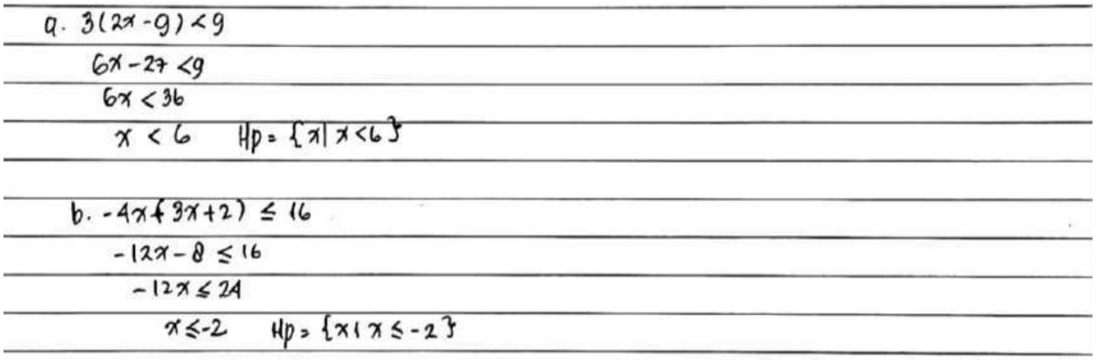
The results are presented in three stages. First, it presents categories of student errors based on themes formed by the patterns and types of errors. Second, besides the categories of errors, patterns, and types of errors, students' way of thinking and understanding behind each underlying error is also analyzed. Third, the epistemological obstacles to the inequality concept are based on students' ways of thinking and understanding. Based on the analysis of student errors in solving the concept of inequality, the discussions with lecturers in Mathematics Education, and previous research, several themes of the source of student errors have emerged, namely: (a) inequalities rules (Bicer et al., 2014), (b) the absence of semantic and symbolic meanings of inequalities (Blanco & Garrote, 2007), (c) interpreting solutions, and (d) generalizations in inequality rules. Descriptions of student errors in the task of inequality concept are summarized in Table 2.

Types of errors	Descriptions of errors	Ways of Thinking	Ways of Understanding
Inequalities rules	Students do not understand the rules/reasons for changing the direction of inequalities when multiplying or dividing inequalities by negative numbers.	Multiplication and division by negative numbers do not change the sign of inequality.	The result of multiplying or dividing an inequality by a negative number does not affect the inequality sign.
The absence of semantic and symbolic meanings of inequalities	Students do not understand or misunderstand notation in inequality	• Students understand the form of inequality notation "≤", "≥" is equal to equality sign "<", ">"	There is no difference in the semantic and symbolic meaning of the inequality notation.

		• There is no difference between interval and closed notation in expressing in the form of inequalities.	
Interpreting solution	Students cannot write their solutions in interval notation through the solution of inequalities and think that only one value makes the inequality correct.	The final result of the operation of solving an inequality is the inequality solution, for example $3(2x-9) < 9$, $6x - 27 < 9$, $6x < 9 + 27$, $6x < 36$, $x < 6$ For the students, $x < 6$ is the solution for $3(2x-9) < 9$	Student assume that the solution set cannot be written in an interval or a finite set; students also, write closed intervals like $[-1, \infty)$
Generalization in the rule of inequalities	Students use the absolute value inequality rule in solving linear inequalities.	Students generalize the inequality process used in absolute value inequalities in solving linear inequalities.	Students generalize the understanding of absolute value inequality to linear inequality.

Table 2: Descriptions of student errors

The main reason why the students in this study committed basic arithmetic errors was that they did not have sufficient knowledge about the rules of inequality that change the direction of inequalities when multiplying or dividing the inequalities by negative numbers. It is seen in the results of students' work of task 1 in Figure 1.



a. $3(2x-9)<9$
$6x-27<9$
$6x<36$
$x<6$ $Hp=\{x \mid x<6\}$

b. $-4x(3x+2) \le 16$
$-12x-8 \le 16$
$-12x \le 24$
$x \le -2$ $Hp=\{x \mid x \le -2\}$

Figure 1: Example of Student Errors in Inequality Rules

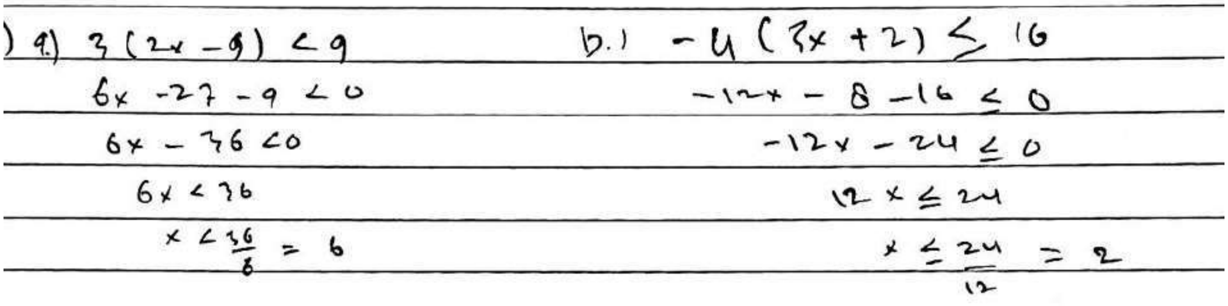
Figure 1 shows that operationally students have solved the inequality of $-4(3x+2) \leq 16$ correctly. Yet, due to the lack of understanding of the inequality rules, students did not change the direction of the inequality when multiplying or dividing by negative numbers. Thus, this student error influenced the solution set of the inequality. The teacher needs to explain and discuss the reasons why we change the direction of the inequality when multiplying or dividing the inequality by negative to overcome this difficulty. Understanding the logic of inequality instead of only memorizing the rules should allow students to understand the inequality concept.

The student errors in understanding the inequality concept were also caused by the students' way of thinking that inequality is equal to equality. It is found based on interviews between researchers and respondents, as summarized below.

Researcher : What do you think is inequality?
Respondent: Statements compare two or more variables
Researcher : What does that meaning of comparing?
Respondent: Greater than, less than
Researcher : Ok, let's pay attention to Task 1; try to read the two forms of inequality.
Respondent: a) three times $2x$, minus nine is less than nine; b) negative four times $3x$ plus two is less than or equal to sixteen.
Researcher : Ok, let's see your work of part b!
Respondent: Ok
Researcher : Do you think that your solution is correct?
Respondent: Yes, the form of $-4(3x+2) \leq 16$, is simplified to $-12x-8 \leq 16$. Then, 8 is moved to the opposite side because it equals 16. Then, we get $-12x \leq 24$, and the result is $x \leq -2$, so the solution set is $\{x|x \leq -2\}$
Researcher : Try to recheck whether the solution is correct!
Respondent: Hmm, it seems correct because it is the same as when we solve linear equations
Researcher : Ok, I see! What do you think happens when an inequality is multiplied or divided by a negative number?

Respondent: The result is negative
Researcher : Then, what about the inequality sign?
Respondent: It remains the same

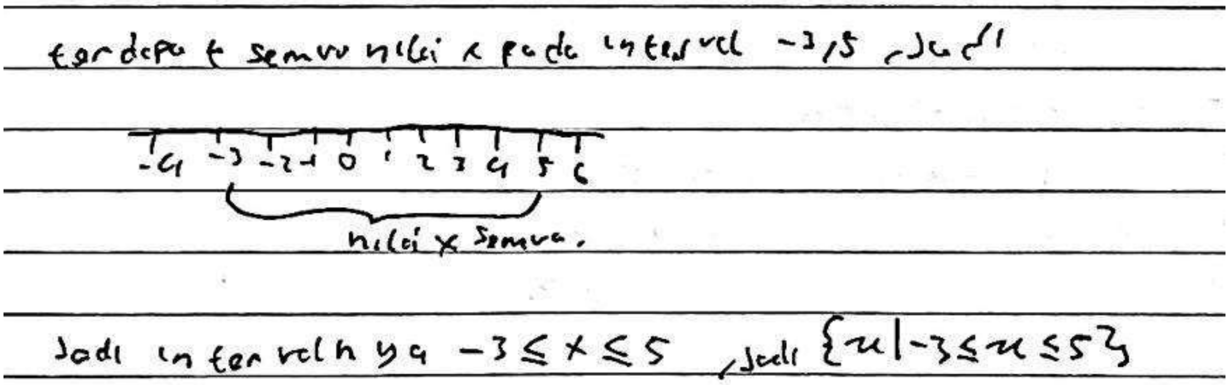
The interview indicates that students believe that multiplying or dividing an inequality by negative numbers does not affect the sign of the inequality. Students' errors in understanding the rules of inequality are also related to the absence of the semantic and symbolic meaning of the inequality, as indicated by the student's assumption that inequality is equal to equality. Tamba and Saragih (2020) stated that students see signs as having no semantic meaning other than connecting two members of the inequality. They solve inequalities by replacing the sign “=” with the inequality sign “<”, “>”, “≤”, or “≥”. The absence of semantic and symbolic meaning of inequality is also indicated by student errors in understanding the notations in the concept of inequality, as shown in Figure 2.



) a) $3(2x-9) < 9$
$6x - 27 - 9 < 0$
$6x - 36 < 0$
$6x < 36$
$x < \frac{36}{6} = 6$

b.) $-4(3x+2) \leq 16$
$-12x - 8 - 16 \leq 0$
$-12x - 24 \leq 0$
$12x \leq 24$
$x \leq \frac{24}{12} = 2$

(a)



terdapat semua nilai x pada interval −3,5 jadi

Jadi intervalnya $-3 \leq x \leq 5$, jadi $\{x \mid -3 \leq x \leq 5\}$

(b)

Figure 2: (a) and (b) Examples of the Absence of Semantic and Symbolic Meaning Errors

Figure 2(a) shows that students tend to replace the inequality notation “<”, “≤” with “=”, likewise in understanding the interval notation in writing the set of solutions or vice versa in changing the form of interval notation to the form of inequalities. On the other hand, Figure 2(b) indicates that students interpret the interval notation representing the inequality notation "≤", "≥" as equal to "<", ">" sign. It is also confirmed by the results of interviews as follows.

Researcher : Look at Task 1; what should you do?

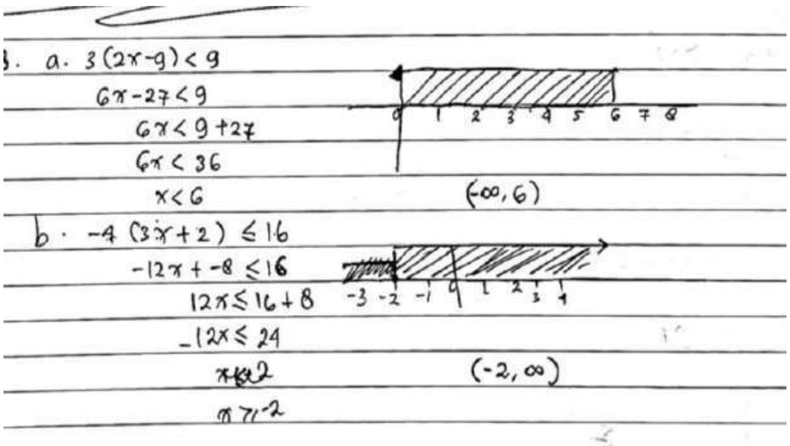
Respondent: Determining the set of solutions in interval notation of the two forms of inequality

Researcher : Ok, what is your solution?

Respondent: For part (a) it is obtained $x < 6$, part (b) is $x \geq -2$

Researcher : Then, what about the set of solutions in interval notation?

Respondent: a is $(-\infty, 6)$, and b is $(-2, \infty)$



a. $3(2x-9) < 9$

$6x-27 < 9$

$6x < 9+27$

$6x < 36$

$x < 6$

$(-\infty, 6)$

b. $-4(3x+2) \leq 16$

$-12x + -8 \leq 16$

$12x \leq 16+8$

$-12x \leq 24$

[illegible]

$x \geq -2$

$(-2, \infty)$

Figure 3: Student's Work in Interpreting Solutions

Researcher : Please recheck; is it correct?

Respondent: Yes

Researcher : In your opinion, what is the set of solutions in interval notation?

Respondent: Writing the set of solutions in brackets

Researcher : I see, then what do a $(-\infty, 6)$, and b $(-2, \infty)$ mean?

Respondent: It means that the value of x is between what I wrote

Researcher : Then, there is no difference between the sign of "<" and "≥", isn't it?

Respondent: Hmm, yes.

The interview excerpt describes the absence of semantic and symbolic meaning of the inequality so that it is one of the sources of student errors in interpreting the solution. Most students view that the set of solutions cannot be written in intervals or finite sets; also, writing closed intervals like $[-1, \infty)$. The absence of semantic and symbolic meanings of inequalities also causes students to make errors and even be unable to change the context of the problem into a mathematical model or form of inequality because they do not understand well the expressions that can represent the inequality notation "<", ">", "≤", and "≥". As seen in Task 3, students were asked to explain all the values of x with the distance of 4 from number 5. Students must at least be able to write the form of the inequality to get the correct solution. However, most students could not do it, as seen in Figure 4.

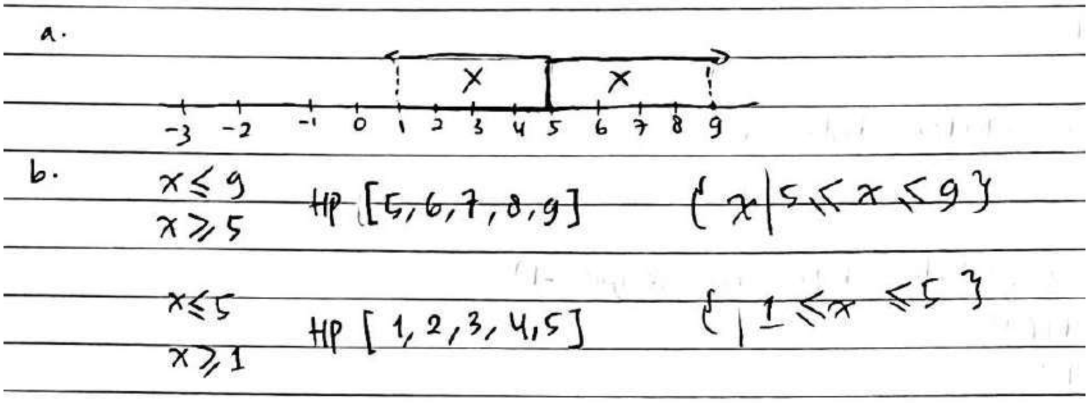


Figure 4: Students’ Limitations in Changing the Context of Problems to Inequality

Based on students' understanding, “$x \leq 9$”, “$x \geq 5$” and “$x \leq 5$”, “$x \geq 1$” are correct forms of inequality to explain all values of x with the distance of 4 from number 5. Another type of student error in completing Task 1 is the generalization of inequality rules. Students tend to think that the rules of the inequality concepts apply to all forms of inequality, as illustrated in Figure 5.

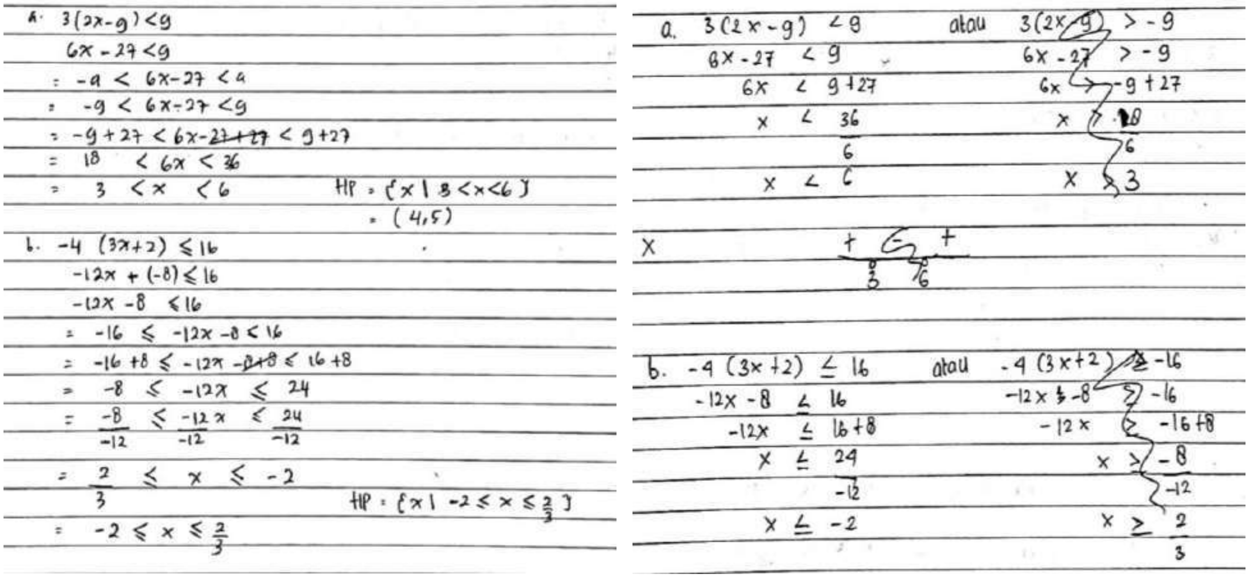


Figure 5: Errors in Generalizing the Inequality Rules

DISCUSSION AND CONCLUSIONS

The students' results showed their difficulty interpreting mathematical concepts and processes related to inequalities. The difficulties and errors found in students did not arise by chance but rather from a stable conceptual framework of students based on their previous knowledge. Radatz (1980) explained that students' errors in mathematics education are not only the result of ignorance, situational accident, insecurity, carelessness, or unique condition, as was assumed in behavioristic educational theory, but also are determined causally, systematically, persistently. It will last for some time unless there is pedagogically action from adults (teachers).

There are some pivotal aspects highlighted in this paper. Many students do not understand the concept of inequality. Most of them have not found a difference between the concept of inequality and equality. Based on students' understanding, the difference is only between the symbols. For example, the symbol "=" is for equality, and one of the symbols "<", ">", "≤", or "≥" is for inequality. These symbols have no semantic meaning for the students because they are used only as a liaison between two inequality members. In understanding the rules of inequality, students tend to view inequality as the same as equality, and for them, this is not an error. This fact is in line with Symonds (1922) that students do not see it as a mistake because they do not understand the symbol's meaning and do not understand its significance.

As an illustration, in solving the inequality of $-4(3x+2) \leq 16$, the students did not make mistake arithmetically since the results given by most of the students were $x \leq -2$ and some even wrote $x = -2$. This mistake is due to the limitations of students in understanding the concept of changing the inequality symbols when an inequality is multiplied or divided by a negative number. As explained earlier that understanding the logic of inequality should allow students to gain a deeper comprehension of the concept of inequality rather than just memorizing the rules. Related to the previous inequality, an understanding of the change in inequality symbol could be given in the following way $-4(3x+2) \leq 16$; $-12x - 8 \leq 16$ (add both side by (8)); $-12x - 8 + 8 \leq 16 + 8$; $-12x \leq 24$ (add both side by $(12x)$); $-12x + 12x \leq 24 + 12x$; $0 \leq 24 + 12x$ (add both side by (-24)); $0 + (-24) \leq 24 + (-24) + 12x$; $-24 \leq 12x$ (multiply both sides by $(\frac{1}{12})$); $-24\left(\frac{1}{12}\right) \leq 12x\left(\frac{1}{12}\right)$; $-2 \leq x$ is equal to $x \geq -2$.

Another method based on Nebesniak (2012) is that besides focusing on procedures and computations, a teacher must include conceptual comprehension related to prior knowledge while encouraging students' understanding and ability to think mathematically. In this case, to consistently solve inequalities correctly, students need to understand the reason behind the rule. For example, Nebesniak mentioned that the teacher could start the lesson with a correct statement $4 < 6$, draw the two numbers on a number line, and discuss what will happen to the statements and graphs if positive numbers are added to both sides of the inequality. The discussion continues by adding the negative numbers on both sides and subtracting the positive and negative numbers.

The final stage is multiplying and dividing by positive and negative numbers into both sides and discussing the inequality symbol.

In addition, another reason for students' errors is because of the absence of inequality semantic and symbolic meaning. Students often see the symbol only as a link between two inequalities in interpreting the statement. Students mistakenly add or exclude values in the solutions without the inequality semantic and symbolic meaning. For example, in task 1, some students wrote the solution $x \leq -2$. They add -2 to their solution by writing open brackets like $(-2, \infty)$. This fact shows that students do not have an efficient semantic meaning of inequality such as “less than” or “less than and equals to”.

To overcome this problem, Rubenstein and Thompson (2001) suggested that some math words need to be emphasized by the teacher to understand the semantic and symbolic meanings. As Usiskin (1996) stated, mathematical symbols are how we write mathematics and communicate mathematical meaning. Tent (2000) also explained that one way that can be done to increase students' semantic and symbolic meaning about inequalities is by reading an inequality in more than one way. For instance, $x < -2$ means x is smaller than -2, x is not greater than -2 or equal to -2, x is neither greater than -2 nor equal to -2.

Students' errors due to the absence of inequality semantic and symbolic meaning are also related to their difficulties in interpreting solutions. They have limitations in interpreting the solution whether to use (1) set notation, (2) number line, or (3) interval notation. As an illustration, in completing task 3, students were required to change the context of the problem into a mathematical model (inequality form) to interpret the solution correctly. To interpret the solution correctly, students must understand that the inequality symbol “less than” has a different meaning from “less than or equal to”. Furthermore, dealing with numbers and algebra gives students a semiotic challenge because symbols act as processes and concepts (Tall, 2008).

Mathematical notations or symbols create the basis of mathematical communication; therefore, students must understand them and relate them to meaning. This statement is due to the diversity of symbols and their meanings in different contexts (Mutodi & Mosimege, 2021). Symbol load, unfamiliarity, and greater density, according to the study, may confront students with difficulties when learning mathematics. Extensive research on secondary students' understanding of mathematical symbols revealed that symbols' clarity and abstraction could be a learning obstacle. This study adds to that debate by highlighting the time required to master certain symbols and the lack of appropriate instructional strategies to promote competence with mathematical symbols. Mitigating the obstacles of mathematical symbolism is still a complex topic for teachers to incorporate into their lessons (Bardini & Pierce, 2015).

Another student's error in solving inequalities is the generalization of the inequality rules. Students assume that every rule in the concept of inequality can be used in all forms of inequality. As shown in Figure 5, to solve the inequality “$3(2x-9) < 9$” " students used the inequality rule “if $x \in R$,

$a \in R$, and $a > 0$, then $x < a$, if and only if $-a < x < a$", the inequality "$3(2x-9) < 9$" in task 1 asked students to determine all possible sets of solutions in interval notation so that the statement "$3(2x-9) < 9$" is true. Meanwhile, applying the inequality rule "$-a < x < a$" to the absolute value of inequalities will produce the wrong solution. In the absolute value concept, a number x can be considered its distance from zero on the number line, regardless of its direction.

Overall, at least two research findings are based on the investigation results of students' experiences in solving inequality problems. First, students make mistakes in completing the task of inequalities such as inequalities rules, the absence of inequalities semantic and symbolic meanings, interpreting solutions, and generalizing the inequality rules. Some research results also explained that there are difficulties for students in solving problems related to the concept of inequality (such as Abu Mokh, Othman, & Shahbari (2019); Almog & Ilany (2012); Konnova, Lipagina, Postovalova, Rylov, & Stepanyan (2019); Lo & Hew (2020); Makonye & Shingirayi (2014); Switzer (2014)).

Second, based on the analysis results, it was found that there were epistemological obstacles in the concept of inequality, which are reflected in the limitations of students in understanding and interpreting inequality symbols. The limited meaning of this inequality symbol is a recurring error in solving the inequality concept in this study. For example, students only interpreted the notation "$\leq$" as something smaller than or equal. Thus, students cannot use these inequality notations correctly when different situations arise.

As seen in task 3, there was given a statement, "Explain all values of x with the distance of 4 from number 5". In this context, students faced difficulty in solving it. This difficulty is due to the limitation of students in understanding and interpreting the inequality symbol. The word "within 4" is a keyword that can be represented by the inequality symbol "$\leq$". Since x is with the distance of 4 from number 5, it means that x is less than or equal to 4, so with the distance from x to 5 can be represented as $|x-5|$ then$|x-5| \leq 4$. Furthermore, epistemological obstacles in the concept of inequality are also found in another research, (such as Bicer et al., 2014; Blanco & Garrote, 2007b; Makonye & Shingirayi, 2014; Nyikahadzoyi, Mapuwei, & Chinyoka, 2013; Tamba & Saragih, 2020)

This study reveals the value of novelty related to the concept of inequality, namely the existence of epistemological obstacles that cause students' errors. Students' errors in solving inequality occur because of their limitations in understanding and interpreting inequality symbols. According to the perspective of "epistemological obstacles", one of the most important goals of historical studies is to find problems and systems of constraints (situation fundamentals) that must be analyzed to understand existing knowledge where the findings are related to the solution of these problems. In some languages, the word inequality can assume two different versions. For example, in French, these words are *inégalité* (in Italian: *disuguaglianza*) and *inéquation* (*disequazione*). Concerning these words, the differences will be summarized as follows: an *inéquation* is a mathematical statement of an *inégalité*. Both from a logical point of view and an educational point of view, there is a big difference between inequalities like "$x + 2 < 3$" and inequalities like "$1 + 2 < 5$", the epistemological status is different (Bagni, 2005).

Besides the limitations of students in understanding and interpreting inequality notation, we see that students' knowledge of prerequisites before arriving at the concept of inequality also acts as an epistemological obstacle. As an illustration, students' knowledge of the concept of numbers greatly contributes to understanding the concept of inequality comprehensively. Based on the facts of research findings, students often make mistakes in operating the results of inequalities that are multiplied or divided by negative numbers. This error can also be identified because students forget to change the inequality sign, but not a few of these errors are also because students do not understand the concept of a comprehensive number. The concept of numbers has an important role in helping students to understand more advanced mathematical ideas (Schröder et al., 2022), therefore the concept of numbers and the concepts associated with them are central to learning mathematics starting from elementary school to intermediate (ages 6 to 17 years) (Elias et al., 2020). Rips et al. (2008) explained that the concept of numbers plays an important role in many mathematical activities, for example, counting and arithmetic. In addition to its practical role, the concept of numbers also has a central place in mathematical theory.

Therefore, analyzing epistemological obstacles is not only focused on how the mistakes made by students, it is also necessary to explore how the historical-epistemological development of a concept is. Obstacles are something that cannot be separated from the learning process for students. So, reflection becomes important to overcome these obstacles in changing the learning model as important content in the learning process (Maknun et al., 2022). So that way we can help identify possible hidden models and suggest the construction of appropriate learning situations to overcome the obstacles found. In terms of the overall concept of inequality, and particularly as it pertains to solving algebraic inequalities in which a student must multiply or divide both sides of the inequality by a negative number, we can add a numerical explanation that uses a concrete example to help solidify the concept. For example, we know that $1 < 2$ (1 is to the left of 2 on the number line). If we multiply both sides by "-3," we would get $-3 < -6$ if we forget to "flip" the inequality sign. The issue is that -3 is to the right of -6 on the number line, which means we should now have $-3 > 6$. I've found this example tends to help students conceptualize this idea in a conceptual manner when using only numbers and the idea with the number line (numbers to the right are greater than numbers to the left).

An epistemological profile consists of an analysis that reveals the extent to which an individual's understanding of a concept involves elements from various stages of its historical development (Bachelard, 1938). So based on this it can be understood that in anticipating epistemological obstacles, it is necessary to conduct a thorough search of students' understanding related to the concepts they are learning with previous concepts related to the concept. So that when students are faced with different situations, they no longer have obstacles to handle them. Trouche (2016) explains that what is definitely important for learning mathematics, is the conceptual component of the schema, i.e., operational invariance: concept-in-action and theorem-in-action, i.e., implicit properties, which are not necessarily true but appear as relevant in certain situations domain. For example, when learning to multiply two integers, students usually develop a strong theorem in action as 'the product of two numbers is a number that is greater than the initial two numbers; and the powerful concept-in-action as 'multiplication is the engine for increasing numbers. Such operational invariants are relevant in a particular domain (which is the reason for their rejection) and turn into a bottleneck as soon as the mathematical context exceeds this domain. For example, when a positive integer is multiplied by a negative number, students tend to be confused about whether the product is greater or less than the initial number.

While doing a math task, a student may point out an error. Error is not only the effect of ignorance, uncertainty, and chance, but also the effect of previous knowledge which is interesting and successful, but is now exposed as wrong or irrelevant. Errors of this type are erratic and unpredictable. These errors can be identified by reviewing the results of student work. So, when making mistakes in understanding inequality notation in solving inequalities. Several questions arise (do students not understand the meaning of inequality notation? Do students consider inequality the same as equality? Do students not understand the nature of inequality? Does the error originate from students' prior knowledge?). Of course, students have reasons or ideas that support these answers. Students may not realize that what they are doing is wrong, because it makes sense to them. Errors are not always the effect of ignorance, uncertainty, or chance; they can result from an interesting and successful application of a piece of prior knowledge, but in other contexts exposed as errors or simply not adapted (Brousseau, 2002). In other words, students use concepts in certain contexts and apply them to other contexts (Brousseau, 2002). This is in line with the view of Modestou and Gagatsis (2007) that the obstacles of epistemological origin manifest in mistakes that are not made by chance and can be reproduced and persisted. Epistemological obstacles are characterized by their appearance both in the history of mathematics and in everyday mathematical activity.

The results showed four types of students' errors in solving inequalities: inequalities rules, the absence of inequalities semantic and symbolic meanings, interpreting solutions, and generalizations of the inequality rules. The concept of equality is still a reference for students in solving inequality problems. Then, the limitation of understanding and interpreting inequality symbol is the leading cause of failure to comprehensively understand the concept of inequality. Meaninglessness is also one of the main problems in dealing with inequality. For that reason, it is necessary to consciously pay attention to how the concept of inequalities is introduced to avoid learning inequalities being reduced to mere mechanical tasks. Each solution should enable students to understand the meaning of the process they follow to reach the correct solution of an inequality. Otherwise, the procedures that they learn will be a source of error.

In addition, this study has research limitations. Where this research only focuses on epistemological obstacles, which are one of two types of learning obstacels, namely ontogenic obstacles and didactic obstacles. So that students' obstacles to the concept of inequalities are only seen from the student's point of view and mathematical characteristics (the concept of inequalities) through the exploration of their experiences in learning the concept of inequalities starting from junior high school to high school. So that some of the information needed relating to student knowledge of the concept of inequalities cannot be explored optimally and thoroughly about past involvement with aspects of inequality. With this limitation, it is hoped that it can be improved in further research.

These findings contribute in several ways to our understanding of the epistemological obstacles faced by students in the concept of inequalities and provide a basis for knowing what material most lead to student misunderstandings. Some practical recommendations for educators and further researchers to follow up on these findings are (1) epistemological obstacles are interpreted as knowledge, so that when there are obstacles to students in understanding and interpreting a concept. The teacher should not ignore it, instead the teacher should use it to provide insight for the student to analyze from the outside the way he used to solve the problem to formulate the hypothesis that he has understood so far, and become aware of possible rival hypotheses; (2) providing a comprehensive understanding of the meaning of inequality notation is very important, so that when students are faced with different contexts and situations, they do not have difficulty applying it; (3) students' mastery of prerequisite knowledge really needs to be emphasized to make a better understanding in understanding the concept of inequalities.

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Assessment of Teaching Methods in Mathematical Simplicity and Complexity in Rwandan Schools via Pedagogical Content Knowledge

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Abstract: The teaching practices of mathematics are still little-known in Rwandan schools though the Competence-Based Curriculum (CBC) and many pedagogical documents recommend how to assess, what to assess, and when to assess the effective teaching of mathematics. This study aimed to assess Rwandan mathematics teachers' practices through pedagogical content knowledge (PCK). We sampled 14 mathematics teachers having similar educational backgrounds. Seven of them were sampled from Teacher Training Colleges (TTC), and the other seven were non-TTC teachers (selected from general secondary schools). We adopted a pedagogic approach to analyze the assessment practices and the tasks proposed by teachers to students. We also analyzed 35 items related to pedagogical content knowledge for teaching (PCK) among these two groups of teachers. Although the results about differences between two groups of seven teachers were not considered robust differences, it was still revealed that in all item categories related to PCK, TTC teachers have less performance than non-TTC teachers. The lack of mastery of content and specialized knowledge at the university level was found to cause this. We also found challenges related to teachers' assessment skills, especially mathematical complexity, as indicated by the interview results. We, therefore, recommend that all teachers, especially TTC teachers, be offered training in content knowledge so that they strengthen their teaching practices.

INTRODUCTION

Recently, in Rwanda, the poor assessment of students' learning has become an issue that concerns the various actors in education and the educational institution [1], [2]. This specific interest has been expressed in several learning areas and different grade levels of education. Most importantly, the policy of implementation of the Competence-Based Curriculum (CBC) in Rwandan schools introduced prescriptions on the assessment practices and emphasized the learning goals by insisting on how to assess the knowledge, skills, and progress of students' understanding in their learning areas [3].

The report produced in 2020 by the Project for Supporting Institutionalizing and Improving the Quality of School-Based In-service Teacher Training (SBI) activity (SIIQS) also revealed that the main learning activity in Rwanda schools is based on group work activities in almost all lessons observed [4]. Regardless of subjects and grade levels, teachers provide assessment questions that require recalling definitions when assessing. Teachers also assess students by using closed questions to confirm whether the answers are correct or wrong, which is against the prescriptions identified in education policy based on CBC for teaching. However, this type of assessment cannot reveal whether students master the content or not because it is superficial. Therefore, there is a need to develop open questions to dig into students thinking [5]. This kind of higher level of questions to assess students' understanding is critical and crucial in mathematics education in Rwanda and worldwide [6]–[8].

The findings based on observation by Moh'd et al. (2021) also revealed that pedagogical content knowledge (PCK) in classroom practices is low. Johar et al. (2021) argued that teachers' lack of content knowledge influences teaching strategies and leads to students' misconceptions instead of developing students' conceptual understanding (Putrawangsa & Hasanah, 2021). The assessment practices in mathematics teaching are essential, especially when an assessment is done by considering the aspect of mathematics teaching. The assessment practice is essential and helpful because it identifies teaching and learning situations for the different grade levels [10]. Otherwise, the lack of knowledge in assessment practices may lead teachers to use the same practice to assess the same content in different grades, while the curriculum frameworks indicate the additional knowledge to be focused on the flow of each grade level.

However, based on these issues of pedagogical content knowledge for teaching revealed in literature, the current study envisions analyzing teachers' PCK and mathematical tasks in the assessment of students learning. We analyzed them using a specific tool that integrates different pedagogical work in mathematics and characterizes the level of performance in mathematical simplicity and mathematical complexity. We tried to study and understand how teachers in Rwandan schools conceive their assessments and creations of knowledge for their students but did not report all the results in this article since this is a portion of a Ph.D. project. We have therefore chosen to assess teaching methods in mathematical simplicity and complexity in Rwandan schools via pedagogical content knowledge.

Context of the study and description of the analysis tool

Analysis of assessment tasks with a pedagogic approach implies considering the relationships between teaching, learning, and content, between assessment and the construction of subject content [10], [11]. These relations give us the fundamental impression to study the assessment practices of Rwandan teachers in mathematics from a pedagogic point of view. For the present study, we have chosen to analyze and present the mathematical tasks proposed by teachers in schools and compare teachers' results of pedagogical content knowledge. We present below some theoretical elements relating to the analysis of mathematical tasks proposed in the assessment. We then describe how the tool developed and utilized in previous studies is applied to produce results for the current study in Rwandan schools. Hill et al. (2004) utilized a tool to analyze the proposed assessment tasks according to the pedagogic approach. Specifically, this tool focused on strategies, methods, and various teaching techniques associated with instructions. The tool has been designed to analyze teachers' pedagogical content knowledge for teaching and assessment practices in mathematics in the U.S. However, we believe it remains relevant to use it in Rwanda because it is standard for mathematics education. We would therefore like to assess its use in this context. It integrates these two dimensions (teachers' pedagogical content knowledge for teaching and assessment practices in mathematics) to analyze mathematical tasks. These tasks include knowledge of mathematical simplicity and mathematical complexity.

Mathematical simplicity: understanding of the task

The language level of the statement, nature and amount of information processed by the student, and an example are considered to determine the level of mathematical simplicity of the task. The example below was taken from a proposed assessment practice by teachers for senior four (S4) students. Instructions for this example were not clear, and as a result, it led to a less explicit understanding of the task for the student. An example was "*find the solution of* $2x^2 = 6$ ". Even if this algebraic equation problem was not clear in instruction for being answered, the task of finding the solutions to this example is explicit. Without any further indication, the student must understand that any quadratic equation produces none, one, or two real solutions. To this end, in mathematical simplicity, students provide shorter proofs or more straightforward calculations. Thus, some students may find one solution instead of two or obtain the solutions without showing work. When the teacher uses this kind of mathematical simplicity in mathematics teaching, Hill et al. (2004) acknowledged that a teacher teaches common content knowledge.

Mathematical complexity: understanding of the task

In this paper, mathematical complexity refers to the various works conducted in mathematics pedagogics in the different domains concerned by the assessment practices to determine the mathematical complexity tasks of students' mathematical learning [13]. For example, the task of *comparing fraction and decimal numbers* can be more or less complex by playing on different pedagogic variables (size, presentation of numbers, and presence of zeros). Here is an illustration of the different levels of complexity of the task taken from different assessments collected during our study. We referred to the mathematical content in the book of mathematics in senior one (S1) (see REB, 2020, p. 54 and Ndyabasa et al., 2016).

Level 1: With $\frac{1}{10}$ and 0.1, the fraction number making up this decimal number, is the same. The student simply applies the comparison rule studied in class.

Level 2: With $\frac{1}{100}$ and 0.01, the zeros presented in both fraction and decimal numbers increase the complexity of the task, even though a student has certainly already performed this type of comparison.

Level 3: With $\frac{1}{1000}$ and $0.01 - 0.009$; these fraction and decimal numbers are not presented similarly. The student must re-compose the second decimal numbers before the comparison can be performed. The comparison task is therefore made to be more complex here.

The mathematical complexity is not intended to discuss the relevance of students' knowledge or the difficulty of carrying it out but to determine complexity levels that allow this knowledge to be considered in the task analysis. Our definition of complexity is inspired by Kontorovich et al. (2012), and we consider the cognitive of mathematical complexity defined by some other authors [16]. This cognitive complexity level determines a demonstration and the availability of the knowledge that students mobilize when they carry out a mathematical task.

We illustrate another example through the mathematical tasks of dashing an area that corresponds to the fraction to indicate this complexity. The example is "*Shade the area that corresponds to the fraction of* $\frac{1}{4}$ *in Figure A and Figure B, and then do the same to shade the area corresponds to a fraction of* $\frac{5}{10}$ *in Figure C"* (see Figure 1).

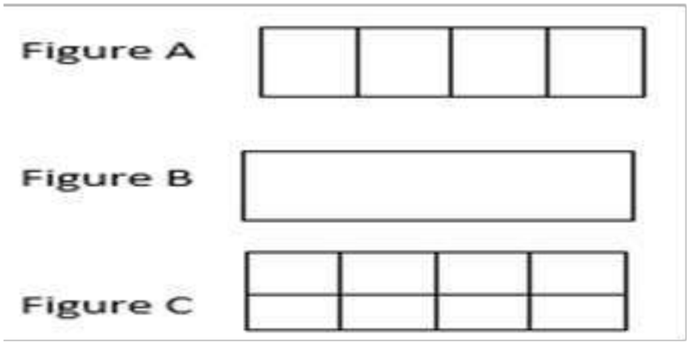

Figure 1: An assigned task for the S1 student

This task in figure A is a low level of mathematical complexity since the students simply apply their knowledge to perform a highly usual task. Figure B allows this task to be done easily because the student has already performed this type of task before in figure A. This time, the student must divide the square into four equal parts. Regarding figure C, students are faced with a highly complex task since they must first realize that the fraction of $\frac{5}{10}$ is equal to the fraction of $\frac{1}{2}$ that he was able to shade in the four rectangles corresponding to half of the square. The students are not guided to perform this task and have never likely performed the task before.

These examples are illustrative of the different levels of complexity as we define it. At the same time, they illustrate the different levels of complexity in different learning areas. Mathematical simplicity and complexity differ in assessment practices context. A task designed for mathematical simplicity requires solving the algebraic equation, and it does not include the context. The task does not include any situation or assessment context except to represent the answers, which are commonly found in the Rwandan curriculum or textbooks, especially in lower-level grades. The task in mathematical simplicity is arguable because the absence of the context in assessment practice limits the opportunity to understand the concept in depth. Contrariwise, the mathematical complexity-designed task incorporates a written teaching scenario. The scenario guides teachers in considering whether students can solve mathematical problems. Clement (1982) advised that teachers must already be aware of misconceptions and confusions students may hold. Based on the above context of teaching and assessment practices, either mathematical simplicity or complexity, the current study is aimed to answer two questions.

Research questions

i. *How do mathematics teachers in Rwandan teacher training colleges and non-TTC teachers who teach in secondary schools perform the items related to pedagogical content knowledge for teaching?*

ii. *To what extent do these teachers understand the assessment practices towards students' understanding of mathematical simplicity and complexity in Rwandan schools?*

This study is the first to assess Rwandan teachers in the African context. It informs researchers on how teachers are skilled in a range of PCK-related fields such as common content knowledge (CCK), specialized content knowledge (SCK), knowledge of content and students (KCS), knowledge of content and teaching (KCT), and content knowledge (KC). It gives room to policymakers on which area needs special input and planning.

METHODOLOGY

A mixed-method research design was used for this study [17]. Quantitative data were collected via a Google form, while Qualitative data were collected face to face in the field. Before collecting data, we applied and got ethical clearance from the research and innovation unit at the University of Rwanda College of Education (URCE). This clearance helped us to seek permission to do research in schools.

Participants

We, at the end of April 2020, invited many teachers through phone calls, and we asked them to participate in the study. We set the schedule together with those who agreed. We selected 14 mathematics teachers in total. The seven mathematics teachers were sampled from Teachers Training Colleges (TTC). The other seven teachers were sampled from the regular or general secondary schools (SS)—here named non-TTC—in Rwanda. All these teachers are experienced in teaching mathematics at the secondary school level. Since Rwanda teachers are using the new curriculum, CBC, they also received many different pieces of training in pedagogical content knowledge for teaching.

Data collection and Procedures

In order to analyze the PCK-related items and the assessment practices of mathematics, teachers who agreed to participate were sent a Google form questionnaire called Measures of Teachers' Mathematical Knowledge for Teaching, MTKT, to answer the survey questions. This process lasted three months (from May to July 2020). This questionnaire comprises 35 items related to PCK (see Hill et al., 2004), and some of them were modified—such as Rwandan names to contextualize the situation—and used in some tasks designed to explore more teachers' knowledge of PCK and the personal dimension of the assessment practices.

The authors developed MTKT to shed light on various teacher's knowledge assessments that were debated around the end of the 20th century in the U.S. Although MTKT was designed for elementary school teachers in the United States, our portion target sample also is characterized by teachers who train primary school teachers in Rwanda. Thus, comparing this part with another part of teachers who teach in general secondary schools can depict how delicate it is to train primary school teachers. The tasks covered mathematical topic areas such as geometry (GEO), rational numbers (RAT), number concept and operations (NCOP), pattern function and algebra (PFA), and proportional reasoning (PR). Box 1 explains the item categories, for example, what kind of knowledge can be categorized as specialized content knowledge, etc.

Box 1: Example of the item category

Question-1 depicted from NCOP is an example of **CCK** is about "0 is even", "0 not a number", and "8 is 008." It says: Ms. Diane was working with a new textbook and she noticed that it gave more attention to the number 0 than her old book. She came across a page that asked students to determine if a few statements about 0 were true or false. Interested, she showed them to a colleague who is also a teacher and asked her what she thought. Which statement(s) should the teachers select as being true? (Mark YES, NO, or I'M NOT SURE for each item below.)

	Yes	No	I'm not sure
(a) 0 is an even number	1	2	3
(b) 0 is not really a number. It is a placeholder in writing big numbers	1	2	3
(c) The number 8 can be written as 008	1	2	3

SCK-related question (in FPA) was about explaining reversing inequalities as **question-34** asks: Ms. Alicia was teaching a lesson on solving problems with inequality in them. She assigned the following problem. [$-x < 9$] Marcie solved this problem by reversing the inequality sign when dividing by -1, so that $x > -9$. Another student asked why one reverses the inequality when dividing by a negative number; Ms. Alicia asked the other students to explain. Which student gave the best explanation of why this method works? (Mark ONE answer)

a) Because the opposite of x is less than 9.
b) Because to solve this, you add a positive x to both sides of the inequality.
c) Because $-x < 9$ cannot be graphed on a number line, we divide by the negative sign and reverse the inequality.
d) Because this method is a shortcut for moving both the x and 9 across the inequality. This gives the same answer as Marcie's, but in a different form: $-9 < x$.

Question-18 is related to **GEO** (Geometry), and is an example of **KCS**. It asks: At the close of a lesson on reflection symmetry in polygons, Ms. Mukesha gave her students several problems to do. She collected their answers and read through them after class. For the problem below, several of her students answered that the figure has two lines of symmetry, and several answered that it has four. How many lines of symmetry does this figure have?

Which of the following is the most likely reason for these incorrect answers? (Circle ONE answer.)
a) Students were not taught the definition of reflection symmetry.
b) Students were not taught the definition of a parallelogram.
c) Students confused lines of symmetry with edges of the polygon.
d) Students confused lines of symmetry with rotating half the figure onto the other half.

KCT-related question was depicted from **RAT (**rational numbers) and is presented in **question-28**. Mr. Shadad is using his textbook to plan a lesson on converting fractions to decimals by finding an equivalent fraction. The textbook provides the following two examples: Convert 2/5 to a decimal: 2/5 = 4/10 = 0.4 Convert 23/50 to a decimal 23/50 = 46/100 =0.46.

Mr. Shadad wants to have some other examples ready in case his students need additional practice in using this method. Which of the following lists of examples would be best to use for this purpose? (Circle ONE answer.)
a) 1/4 8/16 8/20 4/5 ½
b) 1/20 7/8 12/15 3/40 5/16
c) 3/4 2/3 7/20 2/7 11/30
d) All of the lists would work equally well.

Question-18 is related to **PR (**proportional reasoning)**,** and it is an example of **KC**. It asks: Mr. Mutabazi's students were comparing different rectangles and decided to find the ratio of height to width. They wondered, though, if it would matter whether they measured the rectangles using inches or measured the rectangles using centimeters. As the class discussed the issue, Mr. Mutabazi decided to give them other examples to consider. For each situation below, decide whether it is an example for which different ways of measuring produce the same ratio or a different ratio. (Circle PRODUCES SAME RATIO, PRODUCES DIFFERENT RATIO, or I'M NOT SURE for each.)

	Produces the same ratio	Produces different ratio	I'm not sure
a) The ratio of two people's heights, measured in (1) feet or (2) meters.	1	2	3
b) The noontime temperatures yesterday and today, measured in (1) Fahrenheit or (2) Centigrade.			
c) The speeds of two airplanes, measured in (1) feet per second or (2) miles per hour.			
d) The growths of two bank accounts, measured in (1) annual percentage increase or (2) end-of-year balance minus beginning-of-year balance.			

Apart from the MTKT, researchers provided a set of two tasks (one for solving a quadratic equation and another for solving a word problem) for qualitative data collection. We used both MTKT to generate quantitative data and follow-up them with two tasks to generate qualitative data. This helped us to triangulate our results. After submitting their responses, we selected four teachers (two from TTC and two from non-TTC) for an interview discussion in June 2021. They were selected from the most that had misconceptions while solving two tasks provided. These teachers were visited by researchers, and a semi-directive interview was conducted. By interview discussion or semi-directive interview, we mean that researchers sat together with each of the selected participants, presented them with tasks, asked him/her to perform, and discussed how the teacher performed the task.

Items Selection and Data analysis

The items for use were selected based on the related topic areas that are generally found crossing grade levels in the content of Rwandan textbooks and competence-based curricula. These topic areas include geometry (GEO), rational numbers (RAT), number concept and operations (NCOP), pattern function and algebra (PFA), and proportional reasoning (PR). Table 1 represents the item category.

Item category	GEO	RAT	NCOP	PFA	PR
Common content knowledge (CCK)			2		
Specialized content knowledge (SCK)		1	6	4	
Knowledge of content and students (KCS)	1		9		1
Knowledge of content and teaching (KCT)	4	1	4		
Content knowledge (KC)			1		1
Total	5	2	22	4	2

Table 1: Number of items category selected from MKT

Based on this procedure, we could identify questions that could be asked during the interview to help us understand the performance of the 35 items and tasks given to participating teachers. Each interview was transcribed, and the teachers' answers were recorded. We did this because the teacher's answers allowed us to identify elements of practice that we felt were essential. Thus, interpretive and descriptive statistics were used to deliver results.

To have an image of how the item category and assigned tasks are performed, the item difficulties were calculated to clear up whether we measured similar constructs of two groups of teachers. All teachers' responses were entered into SPSS, version 25. Thus, we estimated the point biserial correlation to rate the number of right and wrong answers that teachers gave on the items and the total scores that the teachers received when summing up the scores across the items. The answer of each teacher was recorded for each question. Firstly, the number of teachers in each group who performed well in each item category was computed. Secondly, the average scores for each teacher along each MKT test item were computed. Then, multivariate analysis of MANOVA was estimated to compare multivariate sample means teachers' performance in the item category between the non-TTC and TTC teachers. The separate ANOVA was also estimated to find out whether there is a significant difference between dependent variables (mean scores) on the item categories. Since it is not convincible to report statistical differences between such small groups of 7 each, we discussed the differences on a qualitative basis. We also analyzed the teacher's understanding of the assigned tasks related to the item categories through the interview. Therefore, the conclusion was drawn based on the results obtained from the analysis.

RESULTS

Regarding the first question of the study *[How do mathematics teachers in Rwandan teacher training colleges and non-TTC teachers who teach in secondary schools perform the items related to pedagogical content knowledge for teaching?],* we found that the overall performance of the item category was less performed by TTC teachers (M=2.69, Std=1.737) comparing to the non-TTC (SS) teachers (M=3.41, Std=2.091). With the descriptive statistics, the performance of the item category through the mean (M) scores and standard deviation (Std) is indicated in Table 2. Note that the average reflects on the number of teachers (in each group, there are seven teachers). The most performed skill by both groups of teachers was common content knowledge (CCK), although teachers teaching in general secondary schools outperformed (M=5.25 out of 7 teachers, Std=1.708) those teaching in primary teacher training colleges (M=3.50, Std=1.914). Similarly, the least performed skill was content knowledge (CK) by both groups, and non-TTC teachers outperformed (M=2.50, Std=1.773) TTC teachers (M=1.88, Std=1.727) too.

	TTC		**Non-TTC**	
Item Category	Mean of teachers	Std.	Mean	Std.
CCK	3.50	1.915	5.25	1.708
SCK	2.33	2.140	2.75	2.364
KCS	3.47	0.964	4.05	1.615
KCT	2.33	1.225	3.78	1.922
CK	1.88	1.727	2.50	1.773
Total	2.69	1.735	3.41	2.091

Table 2: Descriptive statistics of mean and Std. of teachers in two groups who performed for each item category

Since our sample was small, we also chose to analyze the individual performance in each item category. The results showed that almost all non-TTC teachers performed better in the CCK item category with the total mean (M = 86%) except for one teacher (non-TTC4) who performed 33% in this item category. On the other hand, except for one teacher (TTC7) who performed CCK 100%, TTC teachers have performed this item category at the average mean of (M = 57%). Generally, non-TTC teachers performed in all item categories at the average mean (M = 57%), while TTC teachers performed at the average mean of (M = 40%). Table 3 presents the individual performance of each teacher across each item category (skills).

CODES	CCK	SCK	KCS	KCT	CK	Mean
TTC1	33	33	53	56	38	42
TTC2	67	33	32	33	38	40
TTC3	67	25	37	0	25	31
TTC4	33	42	42	44	25	37
TTC5	33	38	74	44	38	45
TTC6	67	29	63	44	25	46
TTC7	100	25	47	11	0	37
Mean	57	32	50	33	27	40
Non-TTC1	100	38	16	44	25	45
Non-TTC2	100	50	74	44	38	61
Non-TTC3	100	38	79	67	38	64
Non-TTC4	33	29	63	44	25	39
Non-TTC5	67	50	68	67	50	60
Non-TTC6	100	33	47	56	38	55
Non-TTC7	100	33	58	56	38	57
Mean	86	39	58	54	36	54

Table 3: Descriptive statistics (%) of the individual performance of each teacher in each item category

The teacher who performed well in specialized content knowledge (SCK) was found among SS (non-TTC) teachers (SS-2 and SS-5) and got 50% mean scores. In the knowledge of content and students (KCS), all teachers are known as the average was 58% among SS teachers and 50% among TTC teachers. This was similar to knowledge of content and teaching (KCT); however, content knowledge displayed the lowest scores for all teachers in both groups. Only one teacher (SS-5) could get 50%, while others got below the average of the total score (50%).

We also estimated the MANOVA to determine whether there are significant differences between two groups of teachers (TTC and Non-TTC or SS) on the item categories. Table 4 presents the results through the Tests of Within-Subjects Effects.

Source	Dependent Var	Type III Sum of Sq	df	Mean Square	F	Sig.	Partial Eta Squared
Corrected Model	TTC	23.805a	4	5.951	2.116	0.09	0.125
	SS	39.685b	4	9.921	2.483	0.053	0.144
Intercept	TTC	314.717	1	314.717	111.894	0.000	0.655
	ss	578.908	1	578.908	144.879	0.000	0.711
group	TTC	23.805	4	5.951	2.116	0.090	0.125
	SS	39.685	4	9.921	2.483	0.053	0.144
Error	TTC	165.945	59	2.813			
	SS	235.753	59	3.996			
Total	TTC	652.000	64				
	SS	1018.000	64				
Corrected Total	TTC	189.75	63				
	SS	275.437	63				

a R Squared = .125 (Adjusted R Squared = .066)
b R Squared = .144 (Adjusted R Squared = .086)

Table 4: Analysis of ANOVA of dependent variables on item categories

A separated ANOVA was also conducted for dependent variables, with each ANOVA evaluated at α level of $.05$. There was no significant difference between TTC and SS (non-TTC) on item categories $\mathrm{F}(4,59) = 2.116, \mathrm{p} = .090,$ partial $\eta^2 = .125$, with an estimated marginal mean for TTC (M = 3.500, SD =.839) for CCK; (M = 2.333, SD =.342) for SCK; (M = 3.474, SD =.385) for KCS; (M = 2.333, SD =.559) scoring less than SS (M = 5.250, SD =.999) for CCK; (M = 2.750, SD =.408) for SCK; (M = 4.053, SD =.459) for KCS; (M = 3.778, SD =.666) for KCT; and (M = 2.500, SD =.707) for CK. Therefore, Figure 2 presents the marginal mean of MANOVA between two groups on each item category.

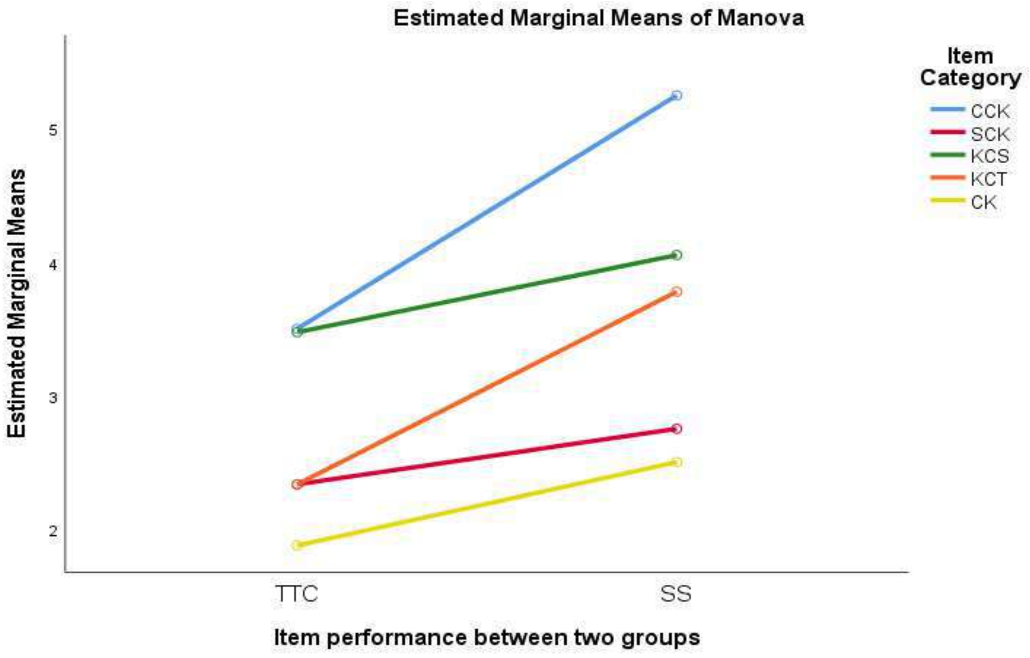

Figure 2: Performance between two groups on each item category

We had five groups in variables: the item categories and dependent variables, in two groups: SS (Non-TTC) and TTC. Within each item category, the SS teacher has performed highly better than the TTC teacher. To find out whether there was a significant difference between dependent variables on the item categories, a multivariate analysis of variance revealed that there was no significant difference between TTC and SS teachers when considered jointly on the variables in item categories, Wilk's $\Lambda = .798, \mathrm{F}(8,116) = 1.73, \mathrm{p} = .099,$ partial $\eta^2 = .107$

Regarding the second question *[To what extent do these teachers understand the assessment practices towards students' understanding of mathematical simplicity and complexity in Rwandan schools?],* we noted that most of the items offered to participating teachers were at a high level of complexity. Though all these items were set to be answered in multiple-choice, teachers who participated in the study revealed weak knowledge of assessment in mathematics teaching, as indicated through the interviews. Though they tried to give explanations, they revealed mistakes in choosing right or wrong answers. This result is reassuring that teachers must understand or know what students have to do in presenting the tasks to assess students' knowledge. Mathematical complexity tasks are often tasks where teachers ask the students to do tasks that are not explicit, and it is up to students to understand and discover ways to perform the tasks. To understand how teachers perceived the designed task from mathematical simplicity and mathematical complexity, we present the following designed task as one of the examples used throughout the interview.

Task 1: Quadratic equation
To solve an equation of $2x^2 = 6$,*? which one of the following will yield the correct answer? (Circle ONE answer.)*

a) Divide both sides by 2, *which gives* $x^2 = 3$, *and divide both sides again by* x *to get* $x = 3$

b) Simplify x^2 *and* x. *Then divide both sides by* 2 *and get the answer of* $x = 3$

c) Take the square root of both sides after diving by 2.

d) Use the method of sum and product of roots to find solutions.

To do this task, teachers did not yield good results from it. We found that only seven out of 14 teachers could give the correct answer. Six out of 14 answered that the correct answer is to divide both sides by 2, which gives $\mathrm{x}^2 = 3$, and divide both sides again by x to get $\mathrm{x} = 3$. There was also one out 14 participants (see SS1 (Non-TTC1) in Figure 3) who said that to answer this question correctly, x^2 and x must be simplified first and later, divide both sides by 2 and get the answer of $\mathrm{x} = 3$. Therefore, based on the results of this task, it seems that teachers are not concerned about offering their students tasks that may lead them to think explicitly. This task was also designed based on mathematical complexity in relation to PCK knowledge. In an interview, teachers revealed less knowledge of assessing mathematical complexity.

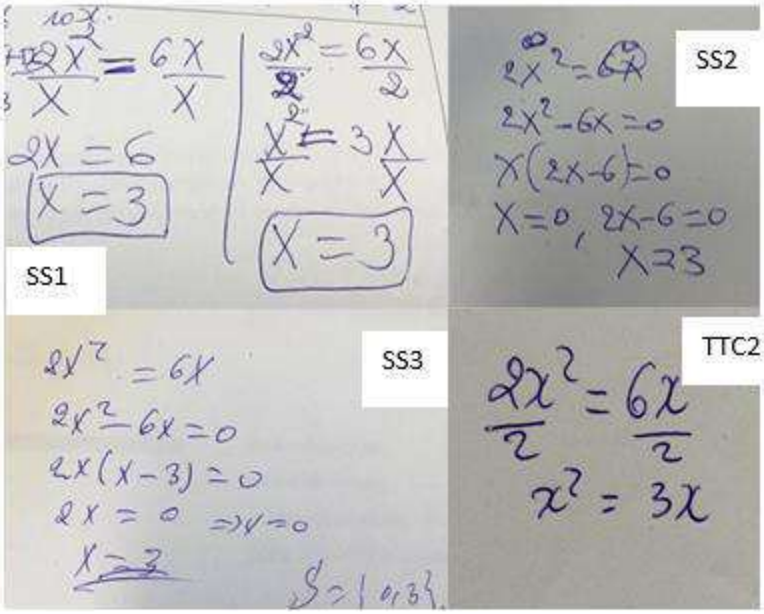

Figure 3: Sample of teachers' performance during quadratic equation solving. Note that SS refers to Non-TTC

Task 2: Word problem

To illustrate the differentiation of the PCK knowledge relative to the same tasks proposed, it seems relevant to mention the sort of tasks proposed in the interview. The following example was also used to analyze the task related to word problem tasks. Teachers were asked to write an equation from the following statement *"there are 4 times as many chalks as pencils"* using the alphabet C as the number of chalks and P as the number of Pencils. Figure 4 illustrates how the participating teachers wrote an equation from the statement.

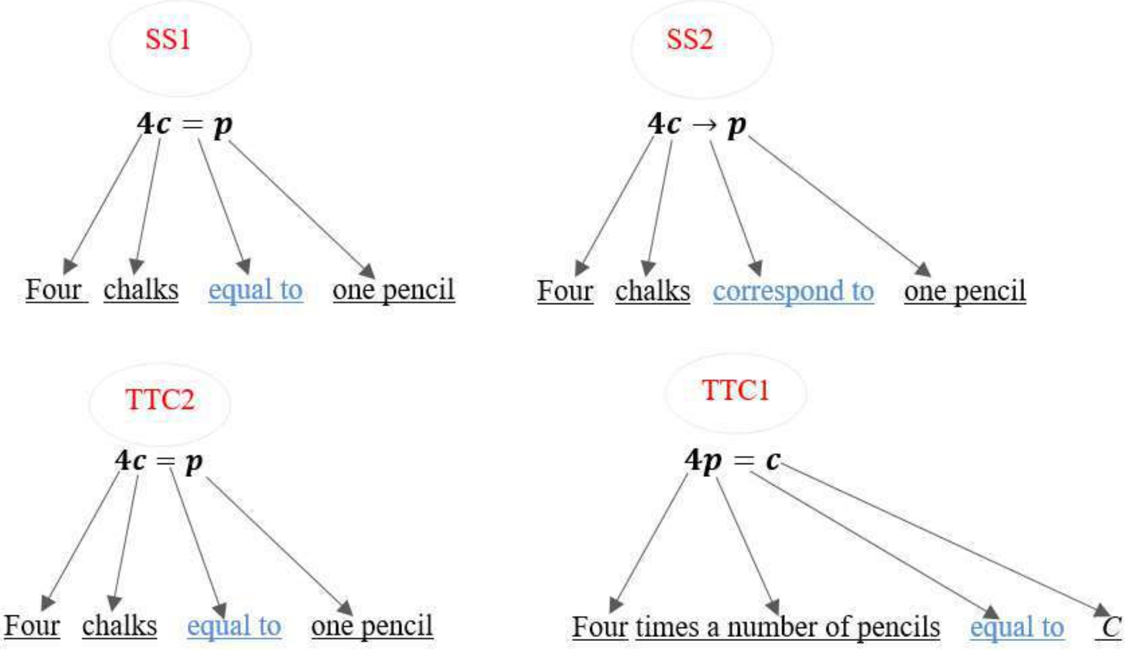

Figure 4: Demonstration of how the interviewed teachers wrote an equation from the statement

Results showed that most participating teachers did not perform this task as it was in a more complex situation that applies to the higher complexity level. We found that teachers were not aware that complexity is not only based on the mathematical knowledge they want to assess but also on methodological skills.

It is true for only equation number 4 (TTC1) that the equation fits the statement. Equation number 2 (SS2) was identified as a function (referring to mapping each element of the domain to exactly one element of the range). This characteristic of equation number 2 allows for one-to-one and many-to-one relationships. However, some participants through interviews have even included one-to-many relations, which are not functions. Through the interview, participating teachers provided examples to disprove the statement by justifying the written answers. They made misconceptions, specifically by using the following logic.

Wrong interpretation. We asked, if there are 100 chalks, how many pencils will be there? In their explanations, many of them said,

"c and p can represent something you can multiply. They wrote an equation as $100c = 25p$ and said, if there are 100 chalks, there will be 25 pencils. Then to prove this, it is just a matter of simplifying the number here. They point a finger on 100 and 25. So if you divide both sides by 25, then you will get $4c = p$ which describes the statement to be correct."

More than (60%) of participants committed this mistake $4c = p$, and a few of them (14%) made the same mistakes by not identifying an equation as the reversal equation but thinking that it is the function (direct variation).

Correct interpretation. On the other hand, the interviewees who gave correct answers also gave relevant explanations for their written work. For example, one teacher explained how he proved the statement as follows.

"The number of chalks is bigger than the number of pencils, right? Then, if there are 100 pencils, the number of chalks will be 100 times four. He wrote an equation, $100c = 400p$*. Therefore,* $C = 4P$*."* We then asked him how he came up to knowing that, and he said, "*to be equal, the number of chalks would be equal to the number of pencils times four.*"

About (84%) of all participating teachers did not provide any relevant explanations about the relationships between the two quantities (C and P). Through the analysis, we found that only 16% of participating teachers could cognitively identify this relationship and explain why the given answer is correct.

DISCUSSION

In general, teachers were found to be good at common content knowledge (CCK), knowledge of content and students (KCS), and knowledge of content and teaching (KCT) because of in-service teachers' training offered in 2016 during the implementation of the competence-based curriculum (CBC) by various partners such as Rwanda Basic Education Board (REB) with various partners such as VVOB, British Council, IEE, JICA, SOMA UMENYE, BLF [18] to mention few. Teachers are well trained in different active learning techniques to cater to learners and the classroom atmosphere. However, specialized content knowledge (SCK) and content knowledge (CK) were found problematic among Rwandan teachers. Both TTC and SS teachers did not get half of the total score (50%) on related items. It appears that math education, in general, does not equip students with the "necessary and enough knowledge" in math. It may also be explained by the fact that students are given many subjects to cover; for instance, some students are currently given to complete three subjects (such as Mathematics-Physics-Education) in only three years at URCE.

We found a significant disparity among participating teachers and the complexity of the proposed tasks. However, this led us to question whether teachers are always aware of the complexity of the tasks they give to students in assessment. Even though on the whole, few complex tasks were proposed, we realized, after interviewing teachers, that those who had proposed them were not necessarily aware of their actual level of complexity and that this could impact the results of the weakest students. To illustrate our point, we discuss the points based on the presented results obtained from the analysis. The task concepts used and presented in the section require prerequisite knowledge of algebraic equations. Though it is still a mathematically questionable statement, from the fundamental theorem of a quadratic equation of the form $ax^2 + bx + c = 0$, as a second-order polynomial, it can be guaranteed that it has two solutions in a single variable. Many common mistakes and misconceptions in this task were to think first about simplifications. The respondents did not realize that if we simplify—using techniques such as factorizing—quadratic equations, we get a linear equation that is guaranteed to have one solution, no solution, or infinitely many solutions but not two solutions. Another thing that is realized from some participants is that, for example, one teacher, through the interview, identified all steps by finding solutions to the quadratic equation, but he selected the wrong answer.

To estimate this level of complexity, we also discussed the concepts of fraction and decimal numbers prepared by teachers. The given fraction and decimal numbers, as the example, were not pre-sentenced in the same way as the others. However, Level 3 makes the comparison task more complex. Definitely, that unusual comparison of the fraction and decimal numbers requires the student to put the proposed decimal numbers in the same "format" before applying any comparison rule. However, without being indicated to the student and being alerted to this fact of higher levels of complexity, the concept of the fraction and decimal presented was in mathematical simplicity.

Therefore, through the interview, we sought to determine whether the interviewed teacher had deliberately chosen to make these tasks more complex by questioning him about level 3. His response was: "*I did not realize this problem. It is just a typo*." This answer is doubly problematic. This teacher was not aware of the complexity generated by the unusual presentation of these decimal numbers of level 3. However, as [12] advised, the choices teachers make in the presentation of the example or exercise generated are decisive for the success or failure of students. Teachers' PCK knowledge is essential to ensure the validity of students' success or failure in his/her learning. If teachers cannot determine the complexity of their students' evaluative tasks, they cannot even design valid assessments. In fact, there should be a link between PCK and knowledge of students and mathematics because a teacher who has a strong knowledge of students' learning should also have a basic knowledge of the mathematics they study (Hill et al., 2004). Thus, PCK is basically not enough for math teachers. They need content knowledge too.

CONCLUSION

Our pedagogical approach allowed us to take a detailed look at the mathematics assessment practices of a sample of 14 mathematics teachers, considering more precisely the nature and complexity of the tasks proposed in assessment practices. For triangulation purposes, we used quantitative and qualitative data collection. Thirty-five tasks (as a sample of these tasks was presented in box 1) adapted from Hill et al. (2004) generated quantitative data. Based on these results, we formulated two tasks (one for solving a quadratic equation and another for solving a word problem) and generated qualitative data. We realized that the responses to assessed tasks proposed by these teachers were of a low level of complexity. Considering the PCK knowledge and the level of assessment practices identified by the participating teachers, we question the overall validity of the assessment practices, even if an individual analysis would be worthwhile to affirm this with more certainty. By this, we mean that the assessments proposed in mathematics by the teachers in our sample do not allow us to determine what their students know. Moreover, passing low complexity tasks does not provide information on the level of resistance or confrontation of students' knowledge and skills to more complex tasks. We believe assessing teachers reflects the students' learning and acquiring knowledge and skills. The results obtained from the study are unique but not conclusive, as our approach was also unique. Based on the results, it seems that from the pedagogical freedom granted to Rwandan teachers, they lack specialized content and content knowledge training. In fact, majoring in a specific and specializing in one subject is needed among higher teacher training institutions.

Therefore, we recommend that URCE and other teacher training colleges cater to the researchers in this field should use a combination of theories and make interventions to investigate why TTC teachers perform less than non-TTC teachers while all teachers have similar attributes. An increase in teachers' samples is needed for researchers to compute inferential statistics and figure out the real difference between these teachers.

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The Investigation of Concept Image towards Derivative Representation: A Case Study of Prospective Mathematics Teachers

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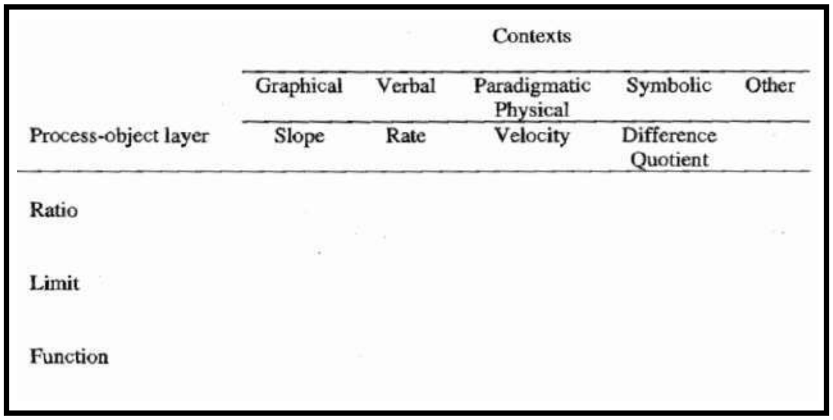
Abstract: Derivative concept is one of the essential studies in calculus, which is studied in teaching mathematics. Prospective mathematics teachers who have completed their studies and later become teachers will teach derivative concepts to their students at school. Therefore, knowledge of derivative concepts is vital in transforming knowledge to students. This study aimed to investigate concept images of prospective mathematics teachers on derivative representations. The research design in this study used a qualitative with case study approach. The participants were prospective mathematics teachers at a university in West Java, Indonesia (N=29). The research data was obtained from the test and clinical interview. The findings of this study show that the concept image of all participants on the derivative concept is still limited in function representation. Concerning the meaning of the derivative concept, most participants only view the derivative concept as a tool to solve procedural problems. It concluded that the representation of participants still did not support conceptual understanding of the derivative concept. It is the impact of the teaching design that given. Based on these findings, educators are expected to be able to improve the quality of teaching derivative concepts in the future by using various contexts or representations so that the concept image formed is more comprehensive to support conceptual understanding in learning of derivative concepts.

INTRODUCTION

Over the last few years, research on calculus has developed a lot, not only examining how students understand and thinking processes, but also examining the educator's point of view and effective teaching methods in the construction of knowledge (Thompson, et al., 2008; Park, 2015; Satianov, 2015; Dagan, M., et al., 2018; Villalobos-Camargo, 2021). Among other concepts in Calculus, derivative is considered a difficult concept because of its definition and requires understanding of other concepts such as functions, quotient differences, and limits (Thompson, 1994; Zandieh, 2000; Park, 2013; Figuero & Campuzano, 2013; Arnal-Palacián & Claros-Mellado, 2022). In fact, the derivative concept is a central concept that is very important to study and understand because it is a fundamental tool in various disciplines involving changes and variations in magnitude

(Vrancken & Engler, 2014; Fuentealba, 2019; Moru, 2020). The problem of understanding derivative concepts is still one of the biggest challenges for teaching mathematics at the university level, and a constant concern for higher education institutions (Ferrini-mundy & Graham, 1991; Bressoud, et al., 2015). The findings in previous studies add that the knowledge taught and knowledge learned about derivative concepts still determines the fundamentals, especially in understanding concepts and meanings of derivative concepts (Orton, 1983; Aspinwall, et al., 1997; Bezuidenhout, 1998; Sierpinska, 1992; Zandieh, 2000; Asiala, et al., 1997; Voskoglou, 2017). This finding is in line with the research conducted by Ferrini-Mundy and Graham (1994) which found the phenomenon that students can easily derive a function but cannot relate and interpret the results of its decline in other contexts. Therefore, Zandieh (2000) presented a framework to explore students' understanding of derivative concepts with multiple representations to obtain complete knowledge so that it can be used to solve various kinds of problems. NCTM (2000) suggests that the term representation refers to processes and products that should be viewed as essential elements to support an individual's understanding of mathematical concepts. The derivative concepts can be represented by graphically as the slope of the tangent line to a curve at a point or as the slope of the line a curve seems to approach under magnification, verbally as the instantaneous rate of change, physically as speed or velocity, and symbolically as the limit of the difference quotient (Borji et al., 2018; Huang, 2011; Tokgöz, 2012). Outline of the framework for exploring multiple representations of derivative concepts as presented by Zandieh (2000) is presented in Figure 1.



	Contexts				
	Graphical	Verbal	Paradigmatic Physical	Symbolic	Other
Process-object layer	Slope	Rate	Velocity	Difference Quotient	
Ratio					
Limit					
Function					

Figure 1: Outline of The Framework of Derivative Concept

That framework with multi-representation can be used as a measuring tool for practitioners to review several conditions including; each individual's understanding of the concepts agreed upon by the mathematics community, comparative understanding of concepts between one individual and another, efficiency of teaching strategies based on the introduction of various aspects of a concept, effectiveness of teaching practices with reference to the curriculum used, and evaluation of essential concepts that must be given through a set of teaching materials that have been planned in the curriculum. In line with Zandieh (2000), Giraldo, et all., (2003) provide a view of the derivative concept that is commonly used, in the context of functions and geometry, derivatives for functions $y = f(x)$ at the point a of the domain is described as the slope of the tangent to the graph of the function at point $(a, f(a))$. In addition, the derivative concept is also described as the rate of change of the function $f(x)$ to x. In a physical context, the derivative concept deals with the instantaneous velocity when an object is moving under the action of a constant force. As for the formal concept definition of derivatives $y = f(x)$ defined by Leibniz notation $f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ or $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$. The limiting process is consolidated to an instantaneous rate of change, which can then be represented by $\frac{dy}{dx}$. The review is an attempt to organize teaching about derivative concepts to be more comprehensive and meaningful. Zandieh's statement is supported by several studies that focus on mental construction, where the results of these studies confirm that participants show different representations in learning derivative concepts (Borji et al., 2018; Huang, 2011; Tokgöz, 2012; Moru, 2020). According to Duval (2006) and Bressoud (2016), research related to the construction of cognitive structures refers to a review the notion of concept image and concept definition as a theoretical framework for analyzing research findings. This is based on Vinner & Dreyfus (1989) which states that understanding in mathematics more often uses concept images and concept definitions than formal concept definitions as mathematical concepts that are agreed upon and used in the community of mathematicians or scientists. Concept image and concept definitions are two mental entities to explore the extent to which individuals understand a mathematical concept (Vinner & Hershkowitz, 1980; Tall & Vinner, 1981). In principle, concept image and concept definition examine the differences between mathematical concepts as formally defined and the total cognitive representation of individuals associated with these concepts. Concept image is the total cognitive structure associated with the concept that contains all mental images, properties, and processes including words, symbols and pictures related to mathematical concepts. While the concept definition is part of the concept image that relates to individual definitions in the form of words to explain or interpret a concept more specifically (Vinner & Hershkowitz, 1980; Vinner, 1983; EMS, 2014).

In Indonesia, several studies that have been conducted on understanding derivative concepts at the school and university level show that students are accustomed to solving routine problems without being able to provide conceptual meaning based on connections between ideas such as gradients, functions, limits, and continuity (Prihandhika, 2018; Mufidah, 2019; Desfitri, 2016; Destiniar, et al., 2021). This condition has the potential to cause difficulties in the learning process where students are unable to determine solutions to problems that are different from the contexts that have been given previously (Prihandhika, 2020; Nurwahyu & Tinungki, 2020). Tall (1996) said that the obstacles experienced by students in learning can be caused by weak concept images and concept definitions. Edwards & Ward (2004) argues that the formation of a concept image in an individual's cognitive structure can occur because of memorizing a formal concept definition without going through a process of meaning to the concept so that the concept image shown by an individual through a concept definition may not be relevant to the existing formal concept definitions. To minimize potential difficulties to solve the conceptual problem and strengthen the relevance between concept image and formal concept definition in teaching derivative concepts, practitioners should pay special attention to mastery of concepts, plans for essential concepts to be taught, and teaching strategies to be provided (Sbaragli, et all., 2011; Park, 2013). Based on the background and literature study that has been presented, this study aims to investigate the representation of the derivative concept based on the participant's concept image concept. Therefore, the research questions that are the focus of this research are as follows, how are participant's representations in the derivative concept based on concept image?

METHOD

In this study, research design used qualitative with case study approach to describing and analyzing the phenomena, attitudes, beliefs, perceptions, and thoughts of participants based on their own life and experiences by holistically and more deeply (Creswell, 2015). The theory of concept image used in this case study to investigate whether participants' understanding of derivative concepts shows the same phenomenon as some previous studies which stated that students' understanding of derivative concepts at the school and university level is still in the context of procedural understanding to solve routine problems without knowing some representations used in understanding derivative concepts (Prihandhika, 2018; Mufidah, 2019; Desfitri, 2016; Destiniar, et al., 2021). The Participants in this study were prospective mathematics teachers attending a bachelor's program (N=29) at a university in West Java, Indonesia. Participants were determined using a purposive technique based on certain criteria (Creswell, 2015). The criteria for participants in this study are students who have obtained a differential calculus course in the first semester of lectures. The research data is generated through the provision of a test in the form of a description of four questions about the concept of derivative with reference to indicators including: 1) Participants can derive a function; 2) Participants can understand the gradient of the tangent line as the first derivative of the function; 3) Participants can understand the derivative as instantaneous speed in the field of physics; 4) Participants can understand the derivative as an gradient of function approximation. The preparation of question indicators refers to the framework for derivative concepts presented by Zandieh (2000). The questions given to participants are as follows:

1. Show how you perform a one-time derivation of a function on $f(x) = x^2 + x - 4$?
2. It is known that the line g intersects the curve f at points A (2,4) and point B (3,9).
   a. Find the slope of the line AB and determine the function of the line g!
   b. Find the equation of the tangent line at point A on the curve f with $f(x) = x^2$
3. If a bullet from a firearm is fired into the air with an angle of elevation θ so that the distance from the origin to the point after t second is $(-t^2 + 12t)$ meter. Then specify:
   a. Average speed over time interval 3.00 ≤ t ≤ 5.00 second
   b. Instantaneous speed at t = 4.00 second
   c. The maximum height that the bullet can reach
4. Based on the table below, show the comparison results of $\frac{f(1)-f(x)}{1-x}$ which is the gradient of the secant line with an approximation of 1 to the function $f(x) = x^2$ and determine the derivative $f(x)$ for $x = 1$

x	0,59	0,99	0,999	1,001	1,01	1,1
$\frac{f(1)-f(x)}{1-x}$						

The test results were then confirmed by clinical interviews to find out the participants' thinking and meaning of derivative concepts with a sequence of easy questions to difficult questions (Hunting, 1997). The research data that has been obtained through tests and interviews are then analyzed with three streams of activities carried out simultaneously, including reducing and presenting data, and drawing conclusions (Mayer, 2015). In the process of drawing conclusions, a theoretical framework concept image is used to interpret the meaning of research findings related to the representation of derivative concepts that exist in the cognitive structure of participants (Tall & Vinner, 1981; Vinner, 1983). The research design used facilitates researchers to investigate participants' thought processes more deeply to find out various kinds of meanings and representations based on the concept image and concept definitions possessed by each participant about the derivative concept. The concept image is personal and will continue to grow depending on the knowledge and experience gained by everyone in teaching. Therefore, the concept image between one individual and another individual can show various outputs. In this study, participants' concept images can be seen and analyzed based on concept definitions indicated by mental pictures, procedures, properties which explained in answering questions given by researchers through assignments and clinical interviews (Nurwahyu, 2020), presented in Table 1.

Component of Concept Image	*Descriptions*
Mental Pictures	*All the information imagined by the participants in the cognitive structure based on the experience that has been obtained for further use in explaining the concept and designing procedures to the solving problems given.*
Procedures	*All the syntax chosen by the participants in translating the mental picture to solve the problem.*
Properties	*All axioms, definitions, lemmas, theorems, formulas, or mathematical rules used in the process of explaining concepts and solving problems.*

Table 1: Component of Concept Image

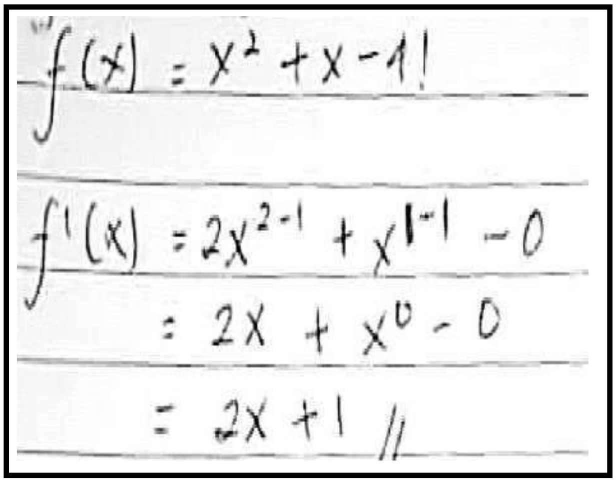
In addition, the research design also supports the interpretation of various phenomena found during the study. However, the various findings that have been obtained cannot be generalized to a wider population scale. This is because prospective mathematics teachers who are participants in the study with prospective mathematics teachers in other places may have different conditions. Therefore, the research carried out is specific and limited and the results only apply to the place that is the subject of the study.

RESULTS

In this section, a description of the participants' answers to each question is presented. A total of 10 participants gave answers related to the derivative concept. The answers are then grouped based on the type of procedures and properties used. Furthermore, the representations used by participants in solving problems categorized using the Zandieh (2000) framework.

Participants' Answer of Questions 1

The first question was asked with the aim of investigating participants' concept image when they were asked to derive a function. The working activity on the first question shows that all participants carry out the process of decreasing function $f(x) = x^2 + x - 4$ use the formula $f'(x) = n.x^{n-1}$ with result $2x + 1$. When confirmed, in the process of decreasing function, all participants imagined two properties in their mental picture, namely the power rule $f'(x) = n.x^{n-1}$ and limit form $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. However, the limit form is not chosen as a procedure to obtain a solution because it has a complicated solution step when compared to the power rule of the derivative concept. Based on the tendency of the procedure used to derive a function, it was found that the participant's concept image about the form $f'(x) = n.x^{n-1}$ is very dominant. The following is the answer of the participants shown in Figure 2.



$f(x) = x^2 + x - 4$

$f'(x) = 2x^{2-1} + x^{1-1} - 0$

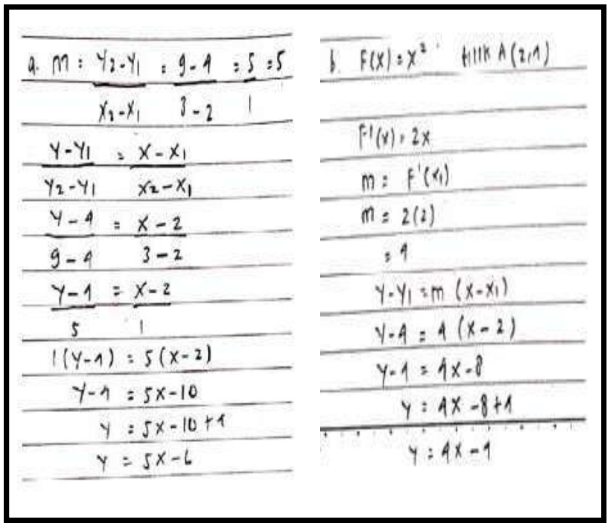
$= 2x + x^0 - 0$

$= 2x + 1$ //

Figure 2: Participants' Answer with $f'(x) = n.x^{n-1}$

Participants' Answer of Questions 2

The second question aims to investigate the participant's concept image about concepts related to derivative concepts, especially those that build geometric representations including the concepts of functions, line equations, gradients, and tangent lines. As many as 43% of participants were able to define procedures to try to solve problems and only 26% of participants were able to carry out procedures to obtain solutions to all the questions given by using properties relevant to the formal concept definition, such as the formula for determining gradients by using $m = \frac{\Delta y}{\Delta x}$ or $m = \frac{y_2 - y_1}{x_2 - x_1}$, the formula to find the equation of a line based on one of the coordinates is to use $y - y_1 = m\ (x - x_1)$ and the equation of a line based on two coordinates by using $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$, and find the equation of the tangent to the curve of $f(x) = x^2$ by using the formula $y - y_1 = m\ (x - x_1)$ for $m = f'(x)$. The following is the answer of the participants shown in Figure 3.



a. $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9-4}{3-2} = \frac{5}{1} = 5$

$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$

$\frac{y-4}{9-4} = \frac{x-2}{3-2}$

$\frac{y-4}{5} = \frac{x-2}{1}$

$1(y-4) = 5(x-2)$

$y-4 = 5x-10$

$y = 5x-10+4$

$y = 5x-6$

b. $F(x) = x^2$ titik A(2,4)

$F'(x) = 2x$

$m = F'(x_1)$

$m = 2(2)$

$= 4$

$y - y_1 = m\ (x - x_1)$

$y - 4 = 4\ (x-2)$

$y - 4 = 4x - 8$

$y = 4x - 8 + 4$

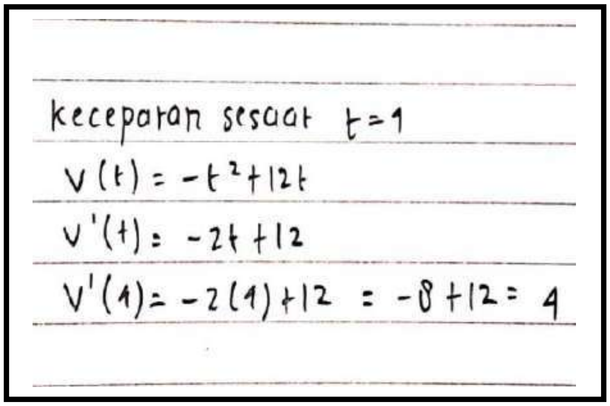
$y = 4x - 4$

Figure 3: Participants' used slope of tangent

Participants' Answer of Questions 3

The third question aims to investigate the participant's concept image about the derivative concept of the paradigmatic physical representation, especially velocity. There are three aspects that want to be seen in the third question, namely the understanding and meaning of participants about the average speed at a certain time interval, the instantaneous speed at t seconds, and the maximum height or stationary point on the function $s(t) = -t^2 + 12t$. Regarding the question of average speed, as many as 8% of participants used procedures to solve problems with properties $v = \frac{s}{t}$ or $v = \frac{\Delta s}{\Delta t}$ the formula for finding velocity by comparing changes in distance and time. Furthermore, regarding the question of instantaneous velocity, as many as 13% of participants used a procedure to answer questions with the derivative concept property, namely velocity on $t = 4$. In the last aspect, as many as 20% of participants used properties $s'(t) = 0$ to determine the maximum height or stationary point of an object. Meanwhile, as many as 54% of participants directly substituted the function as a procedure to solve problems without paying attention to the conditions contained in each question.

Based on these findings, it is generally known that the concept image of participants still has a large gap with the formal concept definition so that the percentage of participants who are able to solve derivative concept problems on the paradigmatic physical representation is still below 50%. The following is the answer from the participants shown in the Figure 4, Figure 5, and Figure 6.



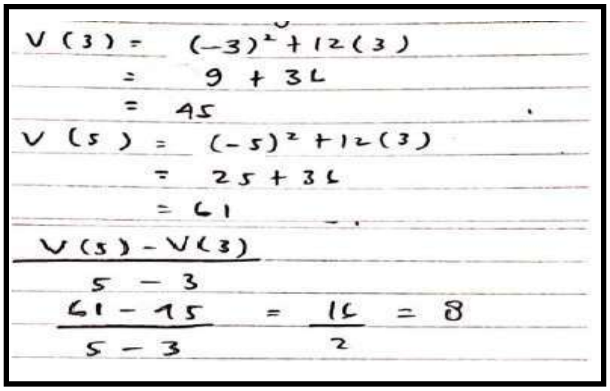
keceparan sesaat $t=1$

$v(t) = -t^2+12t$

$v'(t) = -2t+12$

$v'(1) = -2(1)+12 = -8+12 = 4$

Figure 4: Participants' used average of velocity



$V(3) = (-3)^2 + 12(3)$

$= 9 + 36$

$= 45$

$V(5) = (-5)^2 + 12(3)$

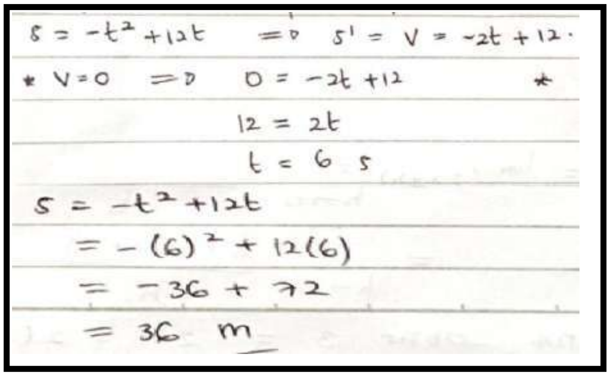
$= 25 + 36$

$= 61$

$\dfrac{V(5) - V(3)}{5-3}$

$\dfrac{61-45}{5-3} = \dfrac{16}{2} = 8$

Figure 5: Participants used instantaneous



$s = -t^2 + 12t \Rightarrow s' = v = -2t + 12.$

$* \; v = 0 \Rightarrow 0 = -2t + 12 \; *$

$12 = 2t$

$t = 6$ s

$s = -t^2 + 12t$

$= -(6)^2 + 12(6)$

$= -36 + 72$

$= 36$ m

Figure 6: Participants used maximum height

Participants' Answer of Questions 4

The fourth question aims to investigate the participant's concept image about the derivative concept based on the $\frac{\Delta y}{\Delta x}$ approximation of a function. These calculations can help construct an understanding of the derivative concept based on the relationship between the limit concept and the ratio concept. However, based on observations, none of the participants can determine the right procedures based on the properties that can be imagined in the mental picture to solve the given problem. These findings indicate that the verbal representation of the rate of change in studying derivative concepts in class has not been recognized by the participants.

Participants' Response of Clinical Interview

The implementation of clinical interviews aims to re-confirm the participant's concept image through the concept definition presented during the interview, explore the thought process, and examine the participants' meaning of the various representations used in the derivative concept. According to Hunting (1997), clinical interviews can be a tool to observe, assess and interpret individual mathematical behavior. Clinical interviews generally consist of open-ended questions and assignments with the aim of knowing how to respond and the underlying thought processes of individuals (Heng & Sudarshan, 2013).

Clinical interviews were conducted with all participants to investigate the concept image of the derived concept representation. However, the presentation of the interview script in this section is only shown from two participants who represent the tendency and uniqueness of the answers related to the derivative concept. This is due to the limitations of the concept image shown by most of the participants so that researchers have difficulty exploring deeper than the results of clinical interviews conducted. In the interview script, R is the researcher, P1 is the first participant and P2 is the second participant. The following are the results of clinical interviews with P1 and P2 which are shown in dialogue below.

R: can you explain, what is a derivative concept?
P1: derivative concept is a concept to reduce the exponent of a function, for example I have a function $f(x) = x^2$*, then the derivative of the function is* $2x$.
R: how do you get $2x$*?*
P1: with using $f'(x) = nx^{n-1}$*, if function of* $f(x) = x^2$ *then the derivative is* $f'(x) = 2x^{2-1} = 2x$.
R: did you use the same method to determine the derivative of $f(x) = x^2 + x - 4$*?*
P1: yes, that's right, I used the same way to answer question 1, the answer is $2x + 1$*. Each term in the function is derived, the derivative of* x^2 *is* $2x$*, the derivative of* x *is 1 because* $x^0 = 1$*. While the derivative of 4 as a constant is 0.*
R: is there another way you can think of?
P1: yes, by using the formula $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ *but the process is long.*
(Clinical interview between R and P1)

R: what is derivative concept in your understanding?
P2: the derivative is concerned with the derivation of the function denoted by $f'(x)$ *or* $\frac{dy}{dx}$.
R: what does this notation mean?
P2: on notation $f(x)$, *the number of accents indicates the number of decreases in the function.* $f'(x)$ *means that the function is derived once,* $f'(x)$ *down twice, and so on. The meaning of notation* $\frac{dy}{dx}$ *is the derivative of y was taken with respect to* x.
R: is there any other notation you know?
P2: no, I only know those two notations.
R: any other ideas you can share about the derivative concept? P2: wait, to determine the derivative of a function using the formula $n.x^{n-1}$ *or* $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.
(Clinical interview between R and P2)

DISCUSSIONS

Based on the results of clinical interviews with P1 and P2, it was found that the concept image based on the concept definition conveyed was very dominant in verbal representation, especially in the process-object layer of function. P1 and P2 explain that the derivative concept is a concept to derive a function with the formula $f'(x) = n.x^{n-1}$. P1 and P2 also have properties about the formula $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ on their mental picture. However, P2 confirms that it cannot use those properties as a procedure to derive a function. After the researcher reconfirmed the mental pictures of P1 and P2 through follow-up questions about the meaning of the derivative concept, P1 conveyed the concept definition of the derivative concept by using a paradigmatic physical representation of the process-object layer of ratio for the context of velocity. P1 explains that the derivative concept can be used to solve instantaneous velocity problems in physics. Meanwhile, P2 presented a concept definition of derivative concept by using a graphical representation of the process-object layer limit for the slope context. P2 explains that the formula $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ is obtained from the concept of the slope of the tangent line and the formula $f'(x) = n.x^{n-1}$ is obtained based on the elaboration of the previous formula. P2 also conveys some of the notations used in the derivative concept. The following is a representation of P1 and P2 which is shown in Table 2.

Process-object layer	*Representations*				
	Graphical Slope	*Verbal Rate*	*Paradigmatic Physical Velocity*	*Symbolic Difference Quotient*	Other
Ratio			○		
Limit	○	○			
Function		●			

Table 2: Derivative representations of P1 and P2

In Table 2, referring to the framework presented by Zandieh (2000), the black close circle indicates the dominance of the representation that is understood and used by the participants in solving derivative concept problems. The close circle is in the verbal representation which shows that the participant has a mental picture of the formula $f'(x) = n.x^{n-1}$ as the main property to determine procedures for solving various problems in the derivative of function. Meanwhile, the white open circle indicates a conceptually incomplete understanding of the representation described through the concept definition. The open circles are in graphical representations, and paradigmatic physical representations. The open circle shows that participants have a mental picture of the formula $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ as the rate and slope of the tangent line and formula $\lim_{h \to 0} \frac{\Delta y}{\Delta x}$ as a procedure to solve problems on velocity.

Based on the results of the answers to the questions given and the results of clinical interviews, it is known that participants experience epistemological barriers when obtaining derivative concept problems that require an understanding of graphical representations, paradigmatic physical representations, and symbolic representations. The following shows the interplay between concept image and concept definition in Figure 7 to analyze the thought processes that cause difficulties experienced by participants in solving the problems in question 4.

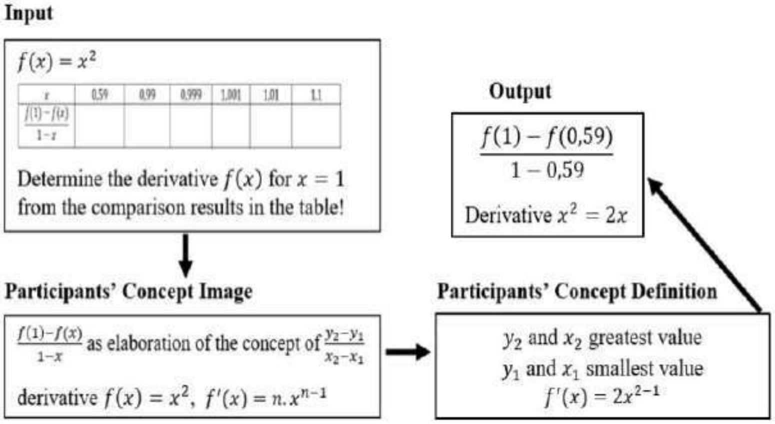


Figure 7: Participants' interplay between concept image and concept definition

Based on the interplay between concept image and concept definition of the participants who tried to answer question 4, it was found that they tried to recall the mental picture of the property difference quotient related to the information contained in the question. However, when asked to determine the derivative of $f(x) = x^2$ for $x = 1$, they again chose the procedure $f'(x) = n.x^{n-1}$ in solving the problem. The procedure is not relevant to the information contained in the question. This procedure was chosen because the participants had difficulty in determining the result $\frac{f(1)-f(x)}{1-x}$ of $f(x) = x^2$ for each x. As a result, the concept definition presented and the output answers given by the participants were not as expected. This may be due to the condition of the participant's concept image regarding verbal representation and symbolic representation that has not been formed in a meaningful way so that the mental picture of $f'(x) = n.x^{n-1}$ appears as a very dominant concept definition in understanding derivative concepts.

CONCLUSIONS

The findings of the study showed that the verbal representation in the context of the rate with the process-object layer of function was very dominant in the participants' understanding of the derivative concept. The properties of the formula $f'(x) = n.x^{n-1}$ are so embedded in the mental picture to determine procedural problem-solving procedures. This condition triggers a difficulty for participants in solving other problems, especially on conceptual problems. Based on the results of the answers to the questions given, only a few of the participants were able to determine the completion procedures relevant to the formal concept definition for questions involving graphical representation in the slope context and paradigmatic physical representation in the velocity context. In addition, the results of clinical interviews show that participants' meaning of the derivative concept is still limited to tools to reduce the number of powers of a function. These findings indicate that the participants' concept image regarding multiple representation in the derivative concept has not been fully formed. The use of multiple representations for teaching derivatives in the calculus domain is highly emphasized as a way to develop individual understanding (Orton, 1983; Aspinwall, et al., 1997; Bezuidenhout, 1998; Sierpinska, 1992; Zandieh, 2000; Asiala, et al., 1997; Voskoglou, 2017). According to Zandieh (2000), the concept of derivative can be represented with graphically as the slope of the tangent line to a curve at a point or as the slope of the line a curve seems to approach under magnification, verbally as the instantaneous rate of change, physically as speed or velocity, and symbolically as the limit of the difference quotient.

The dominance of one of the representations in understanding the derivative concept that triggers the difficulties can be caused by teaching that is less comprehensive in explaining a concept being taught. This statement is reinforced by Figueroa & Campuzano (2012) who said that in a first course of calculus, teachers and students generally did not examine the definition of the derivative as a tangent line obtained from the previous equation so that students had difficulties when solving problems involving graphical representations. In addition, Borji & Alamolhodaei (2016) also conveyed research results showing that calculus teaching in their country tends to emphasize symbolic representations and ignore graphic representations. Apparently, other studies from various countries also show a similar phenomenon (Park, 2015; Moru, 2020). The impact of teaching that emphasizes one representation causes most individuals to be highly skilled in solving routine problems, but have difficulty solving cross-context problems that require conceptual understanding, especially about the correlation between the slope of the tangent to the difference quotient boundary, and the application of derivative concepts in various contextual aspects such as velocity in physics and marginal value in economics (Baker, et al., 2000; Moru, 2020).

The findings of this study using qualitative methods based on case study approach with the aim of examining the phenomenon specifically about the concept image and meaning of the participants, where in this study were prospective students of mathematics teachers at one of the universities in Indonesia, showing the same results. in line with previous research. It seems that problems in teaching and understanding about derivative concepts, as previously stated, are still a common phenomenon in many places. Therefore, a review of the formation of the concept image and participants' meaning of the multiple representations used to understand the derivative concept is a strategic step to obtain alternative solutions to problems that are often found in research. According to Tall and Vinner (1981), there are several possible individual responses in the process of forming a concept image, including replacing the concept image that has been formed with a new information or knowledge, maintaining the old concept image, or separately use both concept images to understand the formal concept definition. Based on the results of this study, the complexity of the concept image owned by the individual strongly supports a comprehensive and meaningful understanding of a concept being studied. Therefore, special attention is needed from educators on teaching practices at the school and university level to involve various contexts or representations in the process of transforming derivative concept knowledge in order to form or improve individual concept images for the better in the future.

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Developing Reflective Competence in Preservice Teachers by Analysing Textbook Lessons: The Case of Proportionality

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Abstract: Teachers must often analyze and select the educational resources they consider relevant for their students, being textbooks one of the curricular materials of preferential use. A textbook lesson shows the instructional process planned by the author to promote the learning of a given content by potential students, so it is essential that the teacher is competent to analyze and assess what happens in this process. Reflection on the relevance of a lesson provides didactic-mathematical knowledge to guide the teacher in making decisions about the management of the text. This paper describes the design, implementation, and results of a training action with 45 preservice teachers, oriented to the development of reflective competence through the analysis of the didactic suitability of a lesson on direct proportionality. In the initial evaluations of the lesson made by the preservice teachers, we found some features of didactic suitability indicators in different components. However, the reflections elaborated by the participants are vague or ambiguous. The results of the implementation show an evolution in reflective competence on the part of most of the preservice teachers, who were able to make a detailed assessment by correctly applying suitability criteria, mainly in the cognitive-affective and instructional dimensions. The participants recognise the importance of the training received to guide their reflection on teaching practice, which they need to complement with content-specific didactic-mathematical knowledge to achieve adequate competence in analysing instructional resources.

INTRODUCTION

Various approaches in teacher training propose reflection on teaching practice as a fundamental competence for professional development and the improvement of teaching (Gellert et al., 2013; Ramos-Rodríguez et al., 2017). Developing reflective competence requires adopting conceptual and methodological frameworks that allow us to address this objective, such as the Lesson Study (Fernández and Yoshida, 2004), Professional noticing (Fortuny and Rodríguez, 2012; Llinares, 2012; Mason, 2016) or the competence of didactic analysis in the Ontosemiotic Approach (OSA) to knowledge and mathematical instruction (Godino et al., 2019; Godino et al., 2017). Within the latter framework, the importance of designing and implementing formative actions to promote, among others, the competence of didactic suitability analysis is highlighted. This competence is aimed at global reflection on teaching practice, its assessment and management for its progressive improvement. The didactic suitability criteria show consensus on what a good mathematics teaching should be like, so they act implicitly as regularities in the discourse of teachers when they have not yet received training on the use of this construct as a guide for their reflection (Breda et al., 2018; Breda et al., 2021; Hummes et al., 2019). However, it is noted that teachers need specific tools and training to direct their attention to the multiple and intertwined factors that affect teaching and learning processes (Seckel and Font, 2020; Sun and van Es, 2015). This has led in recent years to a great deal of research in the field of teacher education using the didactical suitability theory, its components and indicators (Godino, 2013) to organise teachers' systematic reflection on their own or others' practice, develop the competence to evaluate planned or implemented instructional processes, and make informed decisions for improvement (Breda et al., 2018; Burgos et al., 2020; Burgos et al., 2018; Castillo et al., 2021a; Esqué and Breda, 2021; Font et al., 2018; Giacomone et al., 2018; Hummes et al., 2019; Morales-López and Araya-Román, 2020; Pino-Fan et al., 2013).

Teacher guides, textbooks and digital resources are essential tools for teachers that serve as a link between the notions set out in the intended curriculum and the very different and complex world of the classroom (Valverde et al., 2002). Teachers interpret and serve as mediators of the content included in the lessons of the texts they use, so they should have the necessary knowledge and skills to make appropriate use of these resources, considering the needs of their students (Kim, 2007; Lloyd, 2002). Carrying out a critical analysis to support the management of textbooks is a professional teaching task that can be difficult and requires specific training (Beyer and Davis, 2012; Godino et al., 2017; Nicol and Crespo, 2006; Shawer, 2017).

This work is part of a general research focused on the study and development of formative strategies to qualify preservice primary (Castillo and Burgos, 2022) and secondary (Castillo et al., 2021a; 2021b; 2022a) schoolteachers in the critical and constructive analysis of mathematics textbook lessons. The results of these previous interventions showed that participants did not justify their judgements and revealed gaps in didactical-mathematical knowledge that prevented them from adequately interpreting some of the suitability indicators. However, even when the assessment of the didactical suitability indicators was not accurate, the application of the guide for the analysis of textbook lessons helped participants to reflect on the overall suitability of the lesson (Castillo et al., 2021a; Castillo and Burgos, 2022).

In line with these previous experiences, but considering the results obtained for improving the design of the new experimentation cycle, we address the following research question:

> *Does methodology based on the didactical suitability construct help preservice teachers to develop critical and constructive analysis of mathematics textbook lessons?*

To respond to this question, we analyse how reflective competence evolves in preservice primary school teachers, using specifically designed guides based on the facets, components, and criteria of didactical suitability (Godino, 2013), as an instrument to assist their reflection. In this new cycle, a first phase of initial exploration was envisaged, to detect the participants' prior conceptions about the features they consider adequate in a textbook lesson, and to be able to assess more reliably how the training allows them to develop their reflective competence. In addition, the analysis of didactical suitability is requested across the lesson as a whole. To reinforce reflection and justification by the participants, the instrument is improved so that the assessment of the degree of compliance of the suitability indicators is both qualitative and quantitative. Finally, we analyse in this new intervention whether the formative experience influences preservice teachers' beliefs about what aspects are important in a textbook lesson.

As in these previous experiences, we focus on proportionality for several reasons. Firstly, due to the importance of this content in primary and secondary school curricula. Secondly, because proportionality does not usually receive adequate treatment in school mathematics textbooks at this stage (Ahl, 2016; Burgos, Castillo et al., 2020; Shield and Dole, 2013). Indeed, most textbooks emphasise the rote learning of routines such as the rule of three and avoid arguing about the conditions that allow this procedure to be applied when solving a given problem, which prevents the development of an adequate proportional reasoning (Fernández and Llinares, 2011; Lamon, 2007; Riley, 2010). Thirdly, due to the difficulties in teaching proportionality-related concepts shown by both pre-service and in-service teachers (Ben-Chaim et al., 2012; Berk et al., 2009; Buforn et al., 2018; Van Dooren et al., 2008). These deficiencies can be diagnosed and corrected through reflection on the instructional processes envisaged in textbook lessons, thereby generating teacher learning (Nicol and Crespo, 2006; Remillard and Kim, 2017).

FRAMEWORK

In the following, we present the theoretical-methodological notions of the OSA fundamental to our research: the theory of didactical suitability, which connects descriptive-explanatory didactics with effective intervention in the classroom (Godino, 2013), and the model of categories of mathematics teacher knowledge and competences assumed in this framework (Godino et al., 2017).

Didactical suitability theory

The Theory of Didactical Suitability arises within the framework of OSA, from the need for a theoretical-methodological tool to guide teachers in decision-making in the phases of design, implementation, and evaluation of teaching practice (Godino, 2013). This tool can be applied to examine (partial and global) aspects of the teaching and learning process, for example, to the analysis of curricular programmes, of the planning or implementation of a didactic sequence, or of manuals or textbook lessons, among others.

When the teacher plans an instructional process on a mathematical object (e.g., proportionality) for students at a given educational level (e.g., sixth grade of primary education), he/she must first delimit what this object represents for mathematical and didactic institutions. To do so, he/she will refer to what experts consider to be the operational and discursive practices inherent to the object whose instruction is being pursued. With all this, the teacher will determine the system of mathematical practices that we designate as the *institutional meaning of reference* of the object (by extension, of the mathematical content). On the other hand, the *intended institutional meaning* is given by the system of practices that are planned on a mathematical object for a certain instructional process. Thus, considering a lesson in a textbook as a potential or planned instructional process about a certain object, the intended institutional meaning is described by means of the sequence of mathematical and didactic practices proposed by the author for the study of the subject in question. Finally, the system of practices that the student manifests in the resolution of the mathematical tasks in which the object is involved determines the *personal meaning* achieved by the student[1].

The didactical suitability of an instructional process is understood as the degree to which this process (or a part of it) meets certain characteristics that allow it to be qualified as optimal or adequate to ensure the adaptation between the personal meanings achieved by students (learning) and the intended or implemented institutional meanings (teaching), taking into account the circumstances and available resources (environment) (Breda et al., 2017; Godino et al., 2016). It involves the coherent and systemic articulation of the six facets or dimensions (Godino et al., 2007): epistemic, ecological, cognitive, affective, interactional, and mediational.

A mathematics instruction process has a higher degree of *epistemic suitability* if the intended or implemented institutional meanings represent well the reference meaning (Breda et al., 2017; Godino, 2013). This requires the presence and interconnection of diverse partial meanings of the corresponding content (Godino et al., 2017) through the inclusion of a representative and well-articulated sample of problem situations; that multiple representations are involved; that the definitions, procedures and propositions fundamental to the topic are clearly and adequately presented; and that the proposed tasks allow students different ways of approaching them and require them to interpret, generalise and justify the solutions, among other aspects.

The degree to which the contents and their development correspond to the curricular guidelines, and how they are related to other disciplinary contents defines its *ecological suitability*.

Cognitive suitability refers to the degree to which the intended (implemented) meanings are in the learners' potential development zone, as well as the proximity of the personal meanings achieved to the intended (implemented) meanings. An adequate degree of cognitive suitability requires that the contents presented in the instructional process have a manageable difficulty for the educational level at which it is aimed, and that the proposed situations respond to different levels of difficulty.

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[1] OSA assumes that learning involves the appropriation of the intended institutional meanings by students through participation in the community of practice generated in the classroom.

It is also important to promote the use of different resolution strategies and that students are warned of possible difficulties and errors.

Affective suitability is related to factors that depend on both the institution and the learner. A high degree of affective suitability requires the existence of motivational elements (illustrations, humour, etc.), and the selection of situations responding to learners' interests and allowing them to assess the usefulness of the content. In addition, it involves promoting attitudes of perseverance and responsibility towards mathematics, in particular the flexibility to explore mathematical ideas and alternative methods of problem solving.

Mediational suitability involves the availability and adequacy of material resources, and that the sequencing of content and activities is appropriate (in particular, sufficient time is devoted to the content more difficult to understand). The degree to which the modes of interaction allow for the identification and resolution of conflicts of meaning and foster autonomy in learning defines the *interactional suitability* of the teaching and learning process.

For each of these facets, systems of components and general empirical indicators are developed as a guide for analysis, providing criteria for the progressive improvement of teaching and learning processes (Breda et al., 2018; Godino, 2013). These suitability indicators should be enriched and adapted according to the specific mathematical content to be taught (Breda et al., 2017), but also to the type of instructional process. Thus, in Castillo et al. (2022b), Godino's (2013) system of components and indicators of didactic suitability is reviewed and particularised to develop a Mathematics Textbook Lesson Analysis Guide (TLAG-Mathematics for short) as a resource to lead reflection on the instructional processes planned in textbook lessons. Later, in Castillo et al. (2022c), the TLAG-Mathematics is adapted to the topic of proportionality, which leads to the generation of a new guide (TLAG-proportionality for short) in which explicit indicators are introduced for this content that fundamentally affect the epistemic, cognitive, and instructional (interactional-mediational) facets. These indicators are based on an exhaustive theoretical review of research results and expert judgments assumed by the academic community (Breda et al., 2017) in relation to the teaching and learning of proportionality.

Teacher's didactical-mathematical knowledge and competence model

In the Didactic-Mathematical Knowledge and Competences (DMKC) model proposed by the OSA (Godino et al., 2017), it is considered that the two key competences of mathematics teachers are the *mathematical competence* and the *competence of didactic analysis and intervention*. The latter consists, in essence, of "designing, applying and assessing their own and others' learning sequences, using didactic analysis techniques and quality criteria, to establish cycles of planning, implementation, assessment and proposals for improvement" (Breda et al., 2017, p. 1897). This global competence of analysis and didactic intervention of the mathematics teacher is articulated by means of five sub-competences, associated with conceptual and methodological tools of the OSA: *competence of global meaning analysis* (identification of types of problem-situations and practices involved in their resolution); *competence of ontosemiotic analysis of practices* (recognition of the network of objects and processes involved in practices); *competence of management of didactic configurations and trajectories* (identification of the sequence of interaction patterns between teacher, student, content and resources); *competence of normative analysis* (recognition of the network of norms and meta-norms that condition and support the instructional process); *competence of didactical suitability analysis* (assessment of the instructional process and identification of potential improvements). A detailed description of all these sub-competences can be found in Godino et al. (2017).

Previous works describe the results of interventions with preservice primary (Castillo and Burgos, 2022) and secondary school teachers (Castillo et al., 2021a; 2021b; 2022a), aimed at developing the competence of didactical suitability analysing of the planned instructional process in proportionality textbook lessons by applying the TLAG-proportionality. The results of both interventions showed that participants found it difficult to rigorously assess cognitive, affective (especially in relation to beliefs), interactional and mediational indicators.

METHOD

A methodology characteristic of design research (Cobb et al., 2003) is applied, based on the planning, implementation, and retrospective analysis of an intervention, in a real classroom context. In addition, content analysis (Cohen et al., 2011) is used to examine the response protocols of preservice teachers who took part in the formative experience.

Research context and participants

As part of the research, a first research cycle had been implemented as a pilot test in 2020 with a group of third-year Primary Education students (Castillo and Burgos, 2022). The second cycle (described here) was implemented in 2021 with 45 students also in the third year of the Primary Education degree. In both cases, the training experience takes place within the framework of the course Design and Development of the Primary Mathematics Curriculum (sixth semester), at a Spanish university. In previous courses of the degree, preservice teachers receive specific training on epistemic aspects (mathematical contents that are part of primary education), cognitive (mathematical learning, errors and difficulties), instructional (tasks and activities, materials and resources) and curricular aspects, so that by the time the experience takes place, participants are expected to be able to put into practice the knowledge acquired to design and evaluate didactical units in any subject of primary mathematics. In addition, this course specifically contemplates the use and analysis of textbooks as a resource in the mathematics classroom, as well as the evaluation of teaching and learning processes. Two lessons are given twice a week: one theoretical lesson (two hours long) in a large group and another practical class, in reduced groups, working in teams (of four or five members).

In this paper we analyse the information collected in this second cycle from the observer/researcher's notes and the written responses of the preservice primary school teachers (PPTs onwards) to the assessment task proposed at the end of the course.

Design and implementation

The intervention is organised in four phases that include different didactic resources and moments of individual, group, and final evaluation work.

Initial exploration: What do PPTs value in a textbook lesson?

PPTs are asked, as a voluntary activity prior to the formative session on textbook analysis, to carefully read the lesson "Proportionality and Percentages" by González et al. (2015) and answer the following questions through the Moodle platform of the course:

a) What aspects do you consider most important in a mathematics textbook lesson?
b) What did you think of the lesson you have just read? What positive features would you highlight? What negative characteristics do you observe?

The aim of this diagnostic task is to detect the participants' prior beliefs and conceptions about the features they consider appropriate in a textbook lesson and to involve them in a first reflection on the assessment of these features in a specific unit devoted to proportionality. This will allow us to assess how their competence in didactic reflection is evolving.

Introduction to the didactic analysis of mathematics textbook lessons

The training on the didactic analysis of textbook lessons is developed with the PPTs through two class sessions (each lasting two hours)[2]. In the first session, the textbook lesson analysis is presented as a means of identifying potentially conflicting elements, from the point of view of: (a) the mathematical knowledge students are expected to achieve, (b) the prior knowledge they require to understand the lesson, (c) the instructional process it proposes. Next, a methodology for analysing textbook lessons based on the OSA tools is described.

1. General description of the lesson and division into didactic configurations (elementary units of analysis).
2. Onto semiotic analysis. For each of the didactic configurations into which the lesson is divided, it consists of
    a) detailing the mathematical practices proposed,
    b) identifying the mathematical objects involved in them,
    c) describing the main mathematical processes.

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[2] Due to the suspension of face-to-face classes due to the pandemic during the implementation of the experience, these sessions were delivered virtually through the Google Meet platform.

3. Assessment of the didactical suitability of the lesson in the epistemic-ecological, cognitive-affective, and instructional (interactional-mediational) dimensions.

The onto semiotic analysis of a lesson is exemplified using the text "Percentages and proportionality" from Ferrero et al.'s (2015) textbook for the sixth year of primary school. After this first training session, the corresponding practice session of the course (work in teams) is devoted to carrying out the first part of the analysis (general description and onto semiotic analysis of the different configurations) of Gonzalez et al.'s (2015) proportionality lesson for sixth year of primary school.

Didactical suitability as a tool for reflection

In the second formative session, didactical suitability is presented as a global criterion for assessing an intended, planned or implemented instructional process (or part of it), which requires both the analysis of previous practices, objects, and processes and the didactical-mathematical knowledge of that content (in our case, proportionality)[3]. Finally, the TLAG-Mathematics is introduced as an instrument to guide the analysis of textbook lessons, highlighting the need to adapt it to the specific content of the lesson by incorporating didactic-mathematical knowledge on that topic (in our case, proportionality).

Assessment of the didactical suitability analysis competence

After the second training session, the PPTs work individually on the analysis of the didactical suitability of the González et al. (2005) proportionality lesson. It is the same lesson considered in the collaborative work session to carry out the analysis of practices, objects, and processes.

Students are provided with the tables that make up the TLAG-proportionality (adapted from Castillo et al., 2022c) with the components, subcomponents, and indicators of suitability in the following facets: epistemic-ecological (content and adaptation of the lesson to the curricular guidelines), cognitive-affective (learning, attitudes and interests), instructional (material resources, interactions that are promoted in the tasks and their sequencing, etc.). The indicators measure the degree of maximum suitability in each component, so that the lesson will be more suitable with respect to a component to the extent that the corresponding indicators are met in a greater number of configurations.

On this occasion, considering the results of the first research cycle[4], the lesson is considered globally and the suitability in each didactic configuration of the lesson is not analysed.

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[3] Given that during the course the PPTs were required to develop a teaching unit on a specific topic (different for each regular work team) as a main learning objective of the course, the teacher used the content of proportionality to exemplify the type of epistemic-ecological, cognitive and instructional analysis that guides the development of the teaching units.

[4] In previous experiences, participants were asked to decompose the text by units of analysis, applying TLAG-proportionality to each of them. The prospective teachers did not relate some configurations to others and found the breakdown to be excessive (Castillo et al., 2021a;2021b).

Furthermore, to organise the assessment of the lesson, two columns were added to the right of the indicators in the TLAG-proportionality tables. In the first column, PPTs must include a numerical assessment of the degree of compliance with the indicator: 0 (the indicator is not met), 1 (it is partially or sometimes met), 2 (it is fully met); in the second column they give the necessary justifications for the score assigned. Then, considering what has been observed through the assessment of the indicators, the PPTs must make a reasoned judgement on the didactical suitability of the lesson (low, medium, or high) in each of the facets.

RESULTS

To show the progress achieved in reflective competence, in this section, we confront the pre-training (initial descriptions of the characteristics that a good textbook should meet, and prior assessment of the lesson) and post-training (application of the TLAG-proportionality and reasoned judgement of the didactical suitability) analyses of the lesson. We also checked whether they their beliefs about the characteristics that a "good mathematics textbook lesson" should have, changed after the training experience.

Initial meanings of the characteristics that a good textbook lesson should have

When analysing the PPTs' reflections on the aspects they consider most important in a mathematics textbook lesson, it is possible to identify descriptions that can be associated with indicators of the different components of didactical suitability included in the TLAG-Mathematics. Of the 45 PPTs, 30 made some reference to epistemic-ecological aspects, 37 to cognitive-affective and 29 to instructional characteristics. Table 1 shows the TLAG-Mathematics indicators most frequently referred to in each dimension and component. In the category “other”, we include those references to indicators with a low frequency (less than 4), of which the most common are cited.

Components	Indicator	Freq.
	Epistemic-ecological dimension	**40**
Problems	A representative and an articulated sample of problem-situations that allows the contextualization, exercising, amplification, and application of mathematical knowledge	9
Languages	A wide repertoire of representations is used to model mathematical problems, analysing the relevance and potential of each type of representation.	7
Concepts	The fundamental concepts of the subject are presented in a clear and correct way and are adapted to the educational level to which they are addressed.	5
Adaptation to the curriculum	The objectives, contents, their development, and evaluation correspond to the curricular guidelines	7

Intra/Interdisciplinary connections	The contents are related to other intra- and interdisciplinary contents (cross-cutting themes, history of mathematics, …).	5
Others	Procedures are adequately justified, the student is encouraged to justify, content is presented without error	7
	Cognitive-affective dimension	**85**
Individual differences	Expansion and reinforcement activities are included.	10
Increasing difficulty	Situations with different difficulty levels are foreseen.	9
Evaluation	Evaluation and self-evaluation instruments are proposed.	8
Emotions	Tasks and the contents involved are of interest to students. There are motivating elements.	23
Values	The student is encouraged to value the accuracy and usefulness of mathematics in daily and professional life.	14
Others	Necessary previous knowledge is considered, the intended contents are of manageable difficulty, alternative methods of problem solving are encouraged.	21
	Instructional dimension	**39**
Author-student interaction	The author makes an adequate (clear and well-organized) presentation of the topic, emphasizes the key concepts	18
Students interaction	Tasks are proposed that encourage dialogue, communication and debate among students.	5
Sequencing	The content and activity sequencing is adequate, devoting sufficient space to the contents which are more difficult to understand.	6
Others	The use of manipulative materials is promoted, the use of visualisations to introduce concepts, etc. is encouraged.	10

Table 1: Key aspects and frequency (Freq.) highlighted by PPTs in a textbook lesson for each dimension

As it can be seen in Table 1, the cognitive-affective dimension is the one with the highest number of references to indicators.

In the epistemic aspect, the PPTs consider it important that the lesson includes multiple examples and sufficient and varied problems ("sufficient number of activities and focused on the most important", PPT20). They also stress the importance of "clear and simple" language (PPT28) and that the text should be accompanied by " enough and useful" images (PPT20) to "explain the contents" (PPT16), always "related to the exercises" (PPT28). They also mention that the contents should be explained in a clear way ("The content is very important. It is essential that a lesson provides clear explanations so that students can understand it", PPT12). In the ecological aspect, they emphasise the importance of considering the curricular guidelines ("they must be in line with the contents appearing in the primary education curriculum", PPT21) and of considering transversal (and "socio-cultural aspects", PPT29) themes, something with which the PPTs are familiar in the course in which the experience was developed.

In the cognitive-affective aspect, more than half of the appraisals have to do with the interest and usefulness of the contents and tasks proposed to the students, as well as with the inclusion of motivational elements. For example, PPT26 considers it

> relevant that it contains activities related to the daily life and personal environment of the students, so that they can observe the usefulness of mathematics in a real context [...], that the activities and tasks proposed in the textbook are dynamic and attractive for the students, to encourage their motivation.

To a lesser extent, some PPTs appreciate that the lesson includes activities that serve as a review and self-evaluation ("a recap of the lesson as a self-evaluation prior to the exam", PPT3) and that the progression in learning difficulties is considered, advancing "from the simplest to the most complex" (PPT43).

Finally, most of the PPTs' reflections on the instructional aspect mention the attractive design and appropriate presentation of the content, which they tend to interpret as an explanation accompanied by solved examples and illustrative drawings. They also consider the distribution of content to be important ("the order and sequencing of both activities and content, which, for me, should go from the general to the particular", PPT33) and that "activities that encourage cooperative and group work" (PPT28).

Initial analysis of the textbook lesson on proportionality

For each of the dimensions, epistemic-ecological, cognitive-affective, and instructional, we include in Tables 2, 3 and 4, respectively, the types of characteristics and their frequencies when they are valued as positive (PV) or negative (NV) by the PPTs. These are classified by means of the TLAG-proportionality indicators to which they relate, according to the components of each dimension. As before, in the category “other”, we collect those references to indicators with low frequency (less than 4).

Component	Indicator	PV	NV
Problems	A representative and articulated sample of tasks is presented to contextualise and apply proportionality and percentages.	12	8
	Mental arithmetic situations involving proportional reasoning are proposed.	9	-
Languages	The language level is appropriate for the students.	8	-
	Different types of representation (graphical, symbolic, tables of values, manipulative, etc.) are used to model proportionality situations.	7	-
Concepts	The fundamental concepts of proportionality for the corresponding educational level are presented clearly and correctly.	-	6
Others		6	6
Total		42	20

Table 2: Aspects positively or negatively valued by the PPTs in the epistemic-ecological dimension

Of the 45 PPTs, 24 indicate positively, and 17 negatively, some feature in the epistemic-ecological dimension. The PPTs positively value the great variety of exercises in each configuration of the unit, the inclusion of mental arithmetic activities (something they had not reflected on in the first task), and the appropriateness of the language and especially the use of graphs and the proportionality table as a means of representation ("explains the relationship between two magnitudes through a proportionality table, a resource that I find easy to apply with the students", PPT5). In a lesser extent (category "other" in Table 2, PV), they refer to the fact that students are encouraged to create problems ("they are asked to invent and formulate problems themselves", PPT12) or to the alignment of the objectives and contents with the curriculum. Besides, the PPTs highlight as shortcomings in the lesson aspects related to the problem component (also the most positively valued in this facet). Most of them consider that "too many problems" are proposed or that these are not appropriate or are decontextualised. They also find deficiencies in the definition of fundamental concepts. For example, PPT25 states "as a negative aspect, I would highlight that not all the concepts that can help the student to understand the content, for example, proportional magnitudes, are present". A minority (category "other" in Table 2, NV) mention that procedures are not presented correctly or that various types of reasoning are not used in the justification of the statements.

Component	Indicator	PV	NV
Previous knowledge	The necessary previous knowledge (fractions and their equivalence, magnitudes and their measurement) is covered.	-	4
	The intended contents are of manageable difficulty.	7	4
Individual differences	Expansion and reinforcement activities are included.	20	-
Increasing dificulty	Situations with different difficulty levels are foreseen (with integers and non-integers, divisibility relations between quantities, ...).	7	-
	Warns of possible conceptual or procedural errors: illusion of linearity, assuming necessary conditions as sufficient, using erroneous additive strategies.	9	-
Evaluation	Evaluation and self-evaluation instruments are proposed.	9	-
Emotions	Tasks and the contents involved are of interest to students. There are motivating elements.	23	6
Values	The student is encouraged to value the accuracy and usefulness of mathematics in daily and professional life.	13	4
Others		6	9
Total		94	27

Table 3: Aspects positively or negatively valued by the PPTs in the cognitive-affective dimension

In the cognitive-affective dimension, 43 out of the 45 PPTs indicated some positive and 21 some negative characteristics. In the first case, they highlight as potentialities of the lesson those qualities that they considered desirable in the previous task, i.e., the presence of reinforcement (through the final review) and extension activities (in the "test yourself"), and that the proposed situations include elements to capture the students' interest ("use of a character as a common thread to motivate students", PPT19; "workshops and challenges appear to motivate students", PPT20). In addition, they think that including problems related to shopping and sales can show the usefulness of proportionality in everyday situations. They also point out that the "keep in mind" labels included in the lesson can help students with their possible difficulties. Infrequently (category "other" in Table 3, PV) they rate positively that students are supported in learning ("help students in doing the activities", PPT39) or that practical or realistic work is encouraged.

When they express limitations in this facet, they also focus on the degree of interest of the tasks for the students ("the situations are out of the context of the student", PPT26; "problems not very realistic for the context of an 11-year-old child", PPT42) or their usefulness ("not practical exercises, not applicable to the students' daily life", PPT38). Although not frequent, some more specific reflections on the learning of the mathematical content covered in the lesson are observed. For example, in relation to prior knowledge, some PPTs consider that there is a lack of an adequate relationship with fractions or that the conversion of units in magnitudes should be remembered. They also mention that the degree of difficulty of the content on percentages or scales is not adequate ("the content which, from my point of view, can cause the greatest difficulty in learning is the calculation of the percentage of a quantity and scales", PPT23).

Component	Indicator	VP	VN
Author-student interaction	The author makes an adequate (clear and well-organized) presentation of the topic, emphasizes the key concepts	14	7
Students' interaction	Tasks are proposed that encourage dialogue, communication, and debate among students.	4	7
Sequencing	The content and activity sequencing is adequate, devoting sufficient space to the contents which are more difficult to understand.	7	-
Material resources	The use of manipulative (Scali meter, pantograph, proportion compass) and informatics (Geogebra, Excel, ...) materials is encouraged.	4	4
Others		-	6
Total		29	24

Table 4: Aspects positively or negatively valued by the PPTs in the instructional dimension

Of the 45 PPTs, 24 indicated some positive and 18 negative characteristics in relation to the instructional dimension. As when they specified the characteristics, they considered essential in a textbook lesson, the clarity and appropriateness of the presentation of the lesson was again the most prominent positive aspect ("clear and simple explanations", PPT12; "presents an explanation with examples in each section", PPT18). The relevance of the sequencing also stands out ("the contents are structured in an appropriate way, going from the simplest to the most complex", PPT29). Regarding the weaknesses of the lesson, the PPTs also focus their attention on the interactive aspects, where in this case they highlight that the explanation is insufficient or could be improved, and that the fundamental concepts are not included ("the theory and its explanation are insufficient", PPT30; "key concepts are not explained", PPT15). They also consider the degree of interaction between students to be inadequate, mentioning, for example, that an expository method is followed which constrains collaborative work ("It did not seem to me to be a lesson far removed from traditional mathematics teaching", PPT17).

By comparing the results in Tables 1, 2, 3 and 4, we observe a greater number of indications of what is considered appropriate or inappropriate in the planned teaching process when a specific textbook lesson is proposed as a specific means of reflection. It is interesting to note that all those characteristics that they identify as positive or negative in the cognitive-affective or instructional dimensions, had been pointed out as qualities that a mathematics textbook lesson should have in the previous question. However, in the assessment of the proportionality lesson, we recognise some new attributes in the epistemic-ecological aspect. In addition, in general, fewer characteristics are indicated negatively than those indicated positively.

Didactic suitability analysis of the lesson by means of TLAG-proportionality

The a priori analysis of the lesson according to the different facets, components and indicators of didactic suitability by means of TLAG-proportionality was carried out independently by the researchers and an external expert collaborator. The analyses were then compared in order to reach a consensus on an overall assessment. As a result, the didactic suitability was rated as low in each of the dimensions: epistemic-ecological (score 15 out of 62, 17 out of 31 indicators rated 0), cognitive-affective (score 7 out of 24, 6 out of 12 indicators rated 0), and instructional (score 3 out of 18, 6 out of 9 indicators rated 0). However, as can be seen from Table 5, the majority of the PPTs considered the degree of suitability to be medium to high in the epistemic-ecological and cognitive-affective dimensions (73.34% and 51.11% respectively). The assessment was better in the instructional facet, where 53.33% considered the suitability degree as low. To a lesser extent, 28.89% of the PPTs correctly rated cognitive-affective suitability as weak.

Suitability degree	Epistemic-Ecological	Cognitive-affective	Instructional
Low	2	13	24
Low to medium	10	9	4
Medium	26	17	16
Medium to high	4	4	0
High	3	2	1

Table 5. Frequencies in the assessment of the didactic suitability by dimensions (n=45)

The judgements used by PPTs to justify their allocation are of two types.

a) Quantitative. Based on the number of indicators in each facet rated 0, 1 or 2 points, or their average. For example, PPT12 rates instructional appropriateness as low indicating:

> Most of the indicators have a rating of 0, many others have a 1 and there are no indicators rated with 2 points, therefore, we can say that the instructional suitability is low.

b) Descriptive. Description of the positive or negative aspects of the lesson, understood as the degree of compliance with the indicators. For example, PPT34 assigns a low suitability in the instructional aspect, due to:

> A large part of the proposed tasks are very mechanical, and manipulative materials that would allow for more meaningful learning are not used. In some of the didactic

> configurations, I think that more time should be devoted to them, such as the one dedicated to proportional magnitudes. It is appropriate to spend enough time on this before moving on to work on the rule of three. In addition, it should be noted that there are few exercises that encourage collaboration between classmates and group reflection. Nor do we find that the teaching unit works in an interdisciplinary way, it does not relate to any other specific topic.

Although there is no significant difference between the justifications of one type and those of another (47.41% quantitative, 52.59% descriptive), we do observe most of the descriptive type in the evaluations as low (correct) or low to medium (partially correct) in the three dimensions.

Those PPTs that incorrectly assessed the didactic suitability of the lesson in the epistemic-ecological dimension did not identify significant deficiencies in relation to the mathematical content. Thus, a high percentage of PPTs rated positively (2 points) some indicators which in the a priori analysis we identified as having no compliance (0 points) and which represent serious shortcomings in the planned teaching process on proportionality. Among these, we highlight the following:

- Adequate representations are used to distinguish multiplicative relationships within and between quantities. A total of 42.22 % of the PPTs gave this indicator a score of 2 points. These either simply said that "the indicator is fully met" or considered only the presence of pictorial or graphical representations ("drawings and graphs are used in magnitude problems", PPT28), without assessing whether these representations allow distinguishing between both types of relationships, as recommended for an adequate development of proportional reasoning (Fernández and Llinares, 2011; Shield and Dole, 2013).
- Fundamental procedures (arithmetic strategies, reduction to unity, rule of three, calculation of percentages) are presented clearly and correctly. As before, 42.22 % of the PPTs scored 2 points. In this case, when justifying their grading, they mentioned that these procedures have a specific place in the lesson ("a specific section is dedicated to the reduction to unity and the rule of three", PPT35) or that the student is taught the procedure to follow to solve the proposed problems ("the explanations make it clear which procedures to follow in carrying them out", PPT16).
- Proportionality appears to be related to rational numbers. In total 55.56% of the participants gave this indicator a score of 2 points. Most of them did not justify it and when they did, they based their assessment on the inclusion of fractions in the rule of three procedure, percentages and the scale ("percentages, the rule of three and scales are closely linked to rational numbers", PPT44).
- The relationship of proportionality with magnitudes is made explicit (the numerical values involved in proportionality situations correspond to measures of quantities of magnitudes). Up to 62.22% of the PPTs rated this indicator with 2 points, mainly due to the presence of

> different magnitudes in the examples and activities proposed ("the relationship between different magnitudes is presented: euros and tickets, km and time...", PPT5).

Those PPTs who did rate the adequacy in this facet as correct or partially correct (i.e. low or low to medium, respectively), recognised the above indicators as partially or not at all fulfilled, and identified some other shortcomings in the lesson. For example, they noted that the lesson does not state the fundamental properties of the proportionality relation and the procedures are not justified ("we do not find arguments by the author, justifications are omitted", PPT5). They also found that the application situations are not very diverse and representative of the contents (the majority of missing value), do not make explicit the multiplicative relationship in proportional situations, do not allow distinguishing multiplicative comparisons from additive ones, and mainly involve internal ratios ("scarcity of situations involving the use of external ratios", PPT8) i.e. between different values of the same magnitude and not external ones, between values of different magnitudes. Furthermore, they recognise that the representations do not allow them to identify and distinguish the multiplicative relationships that are established within and between proportional quantities. Finally, they also point out that the proportional nature of percentages and scales is not established ("percentages are included, but despite this, there is no section or explanation in the book that relates them to proportionality", PPT29).

From a cognitive-affective point of view, the lesson does not take individual differences into account and does not encourage flexibility and creativity in the resolution on the part of the pupils. Thus, the research team scored 0 points for the corresponding indicators in the TLAG-proportionality. However, 51.11% of the PPTs consider that the lesson takes individual differences into account "through the ‘Unit review’ where there are activities to revise and reinforce the contents" (PPT32) and the presence of two procedures (reduction to the unit, rule of three) to solve different problems, scoring 2 points for both indicators in the TLAG-proportionality. However, the activities proposed in the "Unit review" or "Test yourself" sections are not considered as reinforcement or extension, and no other strategies (progressive construction, additive, multiplicative, etc.) than those described by the PPTs are used.

Those participants who rated the lesson adequately in this dimension identify shortcomings in relation to prior knowledge, attention to individual differences, development of flexible thinking and proposal of appropriate evaluation or self-evaluation instruments. For example, PPT5 indicates:

> At the cognitive-affective domain, the flexibility to explore mathematical ideas and alternative methods of problem solving is not encouraged. In terms of the difficulty of the content presented, it is manageable for the educational level at which it is aimed, but the use of different strategies and situations with different levels of difficulty is not promoted. Nor is prior knowledge considered.

Finally, in the instructional aspect, we observe that the presentation is not entirely adequate. Indeed, concepts (constant of proportionality, direct proportionality relationship) and properties (symmetry of the proportionality relationship, additive and homogeneous character of the linear function) essential to the subject are omitted, various argumentative resources are not used, the student is not given the chance to take the responsibility of the study, and the sequencing does not follow the didactic recommendations on the progression in the teaching of proportionality (Shield and Dole, 2013; Streefland, 1985). More than half of the PPTs recognised most of these shortcomings in the lesson (leading them to rate instructional appropriateness as low). However, 33.33% of the PPTs considered the sequencing of content to be fully satisfactory (2 points) ("the sequencing is adequate, the apparently more difficult content has more exemplification", PPT41), regardless of the relevance of progressing from additive to multiplicative thinking, delaying the introduction of the rule of three until proficiency in other less formal procedures was acquired (Shield and Dole, 2013).

Reanalysis of the characteristics that a good textbook lesson should have. The case of proportionality.

To find out whether the intervention motivates a change in the beliefs, we asked PPTs if they had changed their opinion about the lesson after analysing it by means of the TLAG-proportionality. Of the 45 participants, two did not respond to the question and 16 maintained their initial opinion about the lesson. If this was positive (6 PPTs), they consider it possible to use the material with improvements that they recognise more clearly after the analysis of the didactical suitability. If this was negative (10 PPTs), the participants justify that they initially observed that it was not entirely suitable, although they have been able to carry out a more thorough review. For example, PPT8 states:

> I have now recognised many more shortcomings that I had not noticed before. [...] Not knowing how to do an analysis correctly, I did not identify as many conflicts as I do now, because at that time I only identified the lack of clear explanations that would facilitate the acquisition of the content and the completion of the activities and problems.

A total of 27 PPTs claim to have changed their opinion and consider themselves to have a more critical view of the lesson, which they justify by identifying a greater number of conflicts (17 PPTs) or relevant aspects to consider in a lesson that they had not appreciated before (10 PPTs). In this case, the participants elaborate a detailed report on what are these important features or conflicts that they had overlooked in their initial assessment, referring to the indicators or the evaluation of the didactical suitability in the different facets. For example, PPT21 states:

> My opinion has changed because, in the first analysis I made I thought it was a fairly complete and detailed lesson but, now I think that several elements can be improved (the indicators I have rated with one and include those I have rated with zero).
>
> Similarly, PPT45 comments:
>
> It has changed a lot due to the fact that I have been able to carry out a more complete and thorough analysis which has allowed me to locate the weaknesses and strengths of the book lesson, as well as to establish guidelines to counteract those shortcomings since I can locate the conflicts in a more detailed way and focusing on specific aspects.

In addition, to assess how the training received influenced the PPTs' beliefs about what aspects are important in a textbook lesson, we finally asked them to distinguish the three indicators they considered most important for the assessment of the didactical suitability of a textbook lesson, in each of the dimensions[5]. Table 6 shows the three most frequently selected indicators in each of the facets.

Component	Indicator	Freq
	Epistemic-ecological dimension	
Problems	A diverse and representative sample of tasks (missing value, comparison, tabular, etc.) is used to contextualise and apply the contents of proportionality and percentages.	15
Languages	The language level is appropriate for the students	17
Concepts	The fundamental concepts for the corresponding educational level are presented clearly and correctly: direct proportionality between magnitudes, covariance of quantities, invariance of ratio, constant of proportionality	17
	Cognitive-affective dimension	
Previous knowledge	Previous knowledge of fractions, equivalence of fractions and measurement of magnitudes, necessary according to the educational level, is considered	16
Individual differences	Access, achievement and support for all learners is promoted, e.g. by encouraging the use of a variety of correct strategies (progressive building, additive, multiplicative, etc.).	17
Emotions	Logical reasoning, original ideas or useful, practical or realistic work are encouraged.	15
	Instructional dimension	
Students interaction	Tasks are proposed that encourage dialogue, communication and debate between students in which different solutions are explained, justified and questioned, using mathematical arguments	26
Autonomy	There are opportunities for students to take responsibility for the study: they explore examples to investigate and conjecture; they use a variety of tools to solve problems and communicate them.	28
Sequencing	The content and activity sequencing is adequate, devoting sufficient space to the contents which are more difficult to understand.	26

Table 6: Indicators considered most important in each facet (n=45)

Comparing Table 6 with Table 1, which summarised the key features that the PPTs included in response to the question "What aspects do you consider most important in a mathematics textbook lesson?", we observe that:

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[5] It should be noted that the number of indicators in the epistemic-ecological dimension is much higher than in the cognitive-affective and instructional dimensions.

a) In the epistemic-ecological facet, the importance given to the material including a diverse and representative sample of tasks that allow the content to be contextualised (in this case, that of proportionality and percentages), and that the fundamental concepts appear clearly and correctly (increasing the number of PPTs that indicate this as a fundamental feature of the lesson) is maintained. The third indicator selected as essential to consider the lesson as suitable in this facet relates to the relevance of the language for the educational level. In the previous assessment, more attention was paid to clarity and richness of expression and representation.
b) In the cognitive-affective dimension, the value given by the PPTs to emotions (fostering creativity and practical and useful work on the part of the pupils) is maintained, this being the aspect most highly valued in the initial task. The new indicators selected as most important incorporate prior knowledge and guaranteeing access to all students through flexibility in the use of correct strategies, which had barely been mentioned in the initial reflection.
c) In the instructional facet, which was the least referred to in the initial task (above all they mentioned the importance of clarity in the presentation and organisation of the lesson), the relevance given to interaction between students and sequencing of the contents is maintained, but the PPTs choose as the third indicator the one related to the autonomy component.

The PPTs have selected as fundamental some indicators of the TLAG-proportionality that they had previously assessed incorrectly in the lesson. Therefore, they could judge a textbook lesson as adequate based on these features that they consider to be a priority, when in fact it has shortcomings that they would not consider in the management of the resource. This raises the need to reinforce their didactic-mathematical knowledge in these aspects. In particular, on the cognitive facet, ensuring access for all learners requires knowledge and flexibility in the use of a variety of correct strategies. However, more than half of the PPTs showed in their assessment, a limited vision of the possible ways of dealing with proportionality situations-problems and the need to progress in this type of strategies so that students achieve adequate proportional reasoning. In the instructional aspect, one third of the PPTs considered the sequencing of content to be adequate, focusing their assessment only on the greater exemplification of the most difficult content and without reflection on the relevance of progressing from the intuitive to the formal, delaying the introduction of the rule of three (Streefland, 1985). Also, almost a quarter of the participants felt that the lesson includes moments for students to be autonomous in their learning. In this sense, it is necessary for PPTs to understand that to promote autonomy it is not enough to pose problems for students to solve (as these PPTs considered); it involves incorporating situations that allow students to be spontaneous, dynamic and participatory (Santaolalla, 2014), developing the intellectual capacity to face real problems and new situations (Rey and Penalva, 2002).

CONCLUSIONS

In this paper we have described the design, implementation, and results of a training intervention with preservice primary school teachers, aimed at promoting their reflective competence by analysing the didactical suitability of textbook lessons. Since, without a guide, teachers tend to make descriptive and not very analytical analyses of curricular materials (Nicol and Crespo, 2006), the usefulness of the TLAG-proportionality as a tool to guide and promote reflection on the degree of didactic suitability of a textbook lesson on this subject, is shown.

The results of the experience show the evolution of the participants' competence in didactical suitability analysis. Initially, the PPTs made evaluations of the lesson in which, although they implicitly use didactical suitability criteria, their narratives are superficial and limited to a few aspects that they consider essential. In the final evaluations, critical and detailed judgements can be observed that contemplate a greater number of components of the different facets which appear interconnected in the instructional processes. Thus, in their opinions PPTs explicitly mention didactical suitability indicators, which are correctly assessed by most of the participants in the cognitive-affective and instructional dimensions. Furthermore, although only 12 PPTs rated epistemic-ecological appropriateness adequately, they manage to recognise the importance of argumentation, and are more specific in aspects central to the content of proportionality, for example, by correctly assessing the need to precisely establish the multiplicative relationship between proportional magnitudes or to differentiate between additive and multiplicative comparisons.

In previous experiences (Castillo et al., 2021a), the results reflected inconsistencies between the reasoned judgements about the didactical suitability of the lesson and the ratings of the indicators in the TLAG-proportionality. In this new intervention, all participants have referred to the TLAG-proportionality indicators in their reasoned judgement of suitability (either quantitatively or descriptively) and their judgement is consistent with the indicators' ratings in the instrument. This improvement in the results may be because, on this occasion, the participants applied the guide in a general way to the lesson and were asked to justify the scores given to the TLAG-proportionality indicators, which allowed them to reflect in a more detailed way on their decision.

On the other hand, we observed a significant change in participants' motivation with the TLAG-proportionality utility compared to previous experiences (Castillo et al., 2021a). This improvement may be explained by the fact that in this intervention participants have had the opportunity to find the meaning of the suitability criteria (Schwarz et al., 2008) through the pre-training task of initial exploration. As suggested by Schwarz et al. (2008) the use of analytical tools to help teachers to have a critical stance must have accessible meaning for teachers, otherwise they may acquire a rejecting attitude towards the task or qualify it as too difficult and impractical for their profession.

The results of the intervention show that the analysis of textbook lessons, using didactical suitability as a resource, is a training activity that preservice teachers find useful and interesting. According to the participants, the TLAG-proportionality helps them to select the critical aspects to consider in the analysis of a textbook lesson:

> These [TLAG-proportionality] tables are a guide to know the quality of the textbook in question. Otherwise [without the guides] you would not look at details that are relevant for students' learning or, on the contrary, you would look at details that were not very important (PPT9).
>
> The tables [of the TLAG-proportionality] lead us to go deeper into the most important aspects that must be considered in a textbook lesson, so that it works correctly, as this is the main objective. Not having these tables with the indicators makes it difficult to analyse such a lesson, as there will be aspects that are too important to be left out (PPT38).

However, the results also warn about the importance of prospective teachers consolidating didactical-mathematical knowledge and competences needed to become a reflective education professional. Indeed, the application of the TLAG-proportionality requires the implementation of a series of mathematical and didactic knowledge, which may not be part of the users' previous training. Its systemic structure, by means of descriptors associated with the different components of the various dimensions, allows the teacher trainer to identify difficulties in the interpretation and assessment of specific indicators, and to make decisions on those aspects that need to be reinforced in future training interventions.

The training action described in this article is complemented by action decisions on how the teacher should use the lesson, as well as the proposed resolution of identified conflicts. However, due to space limitations and that the interest of this paper is to show the progress in reflective competence, confronting the pre- and post-training analysis of the lesson, these results will be shown in a forthcoming paper.

We agree with Remillard and Kim (2017) that the type of analysis conducted is especially important for primary school teachers, who typically do not have a broad background in mathematics and are therefore more inclined to rely on curricular resources:

> At the beginning I thought that almost everything would be fine, that there would be no conflict because if there were many conflicts it would not be a book that pupils use at school, etc. [...] One look at the book and it might seem like the best book for your students, but when you analyse the book, it might have more conflicts than you would have imagined. That happened to me (PPT11).

It is common for teachers to consider textbooks as a guarantee of quality and that they can replace the teacher, even "other teachers... have so little confidence in their personal training that they do not dare to make any changes in textbooks or omit anything even if it seems reasonable" (Santaolalla, 2014, p.194). Hence the importance of prospective teachers gaining confidence in their own training, and of having sufficient didactical-mathematical knowledge to be able to detect deficiencies in textbooks and other educational resources. In this respect, we observed that the participants in our experience felt more secure in carrying out these tasks when they had a tool to guide their analysis, acquiring a more critical view of the quality of the text and modifying their initial opinion of the lesson, if necessary.

With a view to future research, it would be desirable to carry out further interventions in which to analyse, over a longer training period, whether reinforcing progressively the didactic-mathematical knowledge supporting the indicators of suitability helps prospective teachers to progress in their reflective competence and how it does so. On the other hand, in this manuscript we have been able to corroborate that prospective teachers are influenced by their convictions when analysing texts (Shawer, 2017). Thus, most of the participants in our experience considered that a good book should have a variety of tasks, but fewer recognised the important role of argumentation or communication of ideas. We believe that analysis guides like the TLAG-proportionality for such content, or the TLAG-Mathematics in general, can help to identify the beliefs that prospective teachers have about teaching and learning mathematics. Given that it is difficult to assess the change in beliefs without analysing how these are manifested in their teaching practice (Lloyd, 2002), it would be interesting, along the lines proposed by Yang and Liu (2019), to infer the beliefs of prospective teachers in the analysis of lessons by contrasting them with a more in-depth study applying a specific questionnaire. It would be also relevant to observe whether with a continuous training, participants manage to develop and materialise rich beliefs about the role of these resources in the teaching of mathematics.

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The Development of Inquiry-Based Teaching Materials for Basic Algebra Courses: Integration with Guided Note-Taking Learning Models

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Abstract: This study induced the students to be more active and able to design discovery activities so that they can communicate the material's concepts well. Students can use their activities to find a solution to problems with inquiry-based learning materials that were supplemented with a guided note-taking model. For this reason, a valid, practical, and effective inquiry-based learning device with a guided note-taking learning model in Basic Algebra courses is required. This study aimed to develop inquiry-based learning tools with a guided note for Basic Algebra courses that are valid, effective, and practical. The development model used in the study was the instructional development institute (IDI). The data sources were three Basic Algebra lecturers, seven small trial students, and 38 students in a large trial class. The validation of the developed learning device involved experts in mathematics, language, educational technology, and peers. The validity values of the developed learning devices were 80.97 and 78.61 with a highly valid category. The practicality value for the module was 90.33 or very practical. The student learning outcomes were improved with a gain score of 0.40 or moderate category. This study produced valid, practical, and effective inquiry-based learning tools with guided note-taking learning models for basic algebra courses.

INTRODUCTION

Mathematics, as it underlies the development of science and technology subjects from kindergarten to university, can be said to be a science that describes the patterns, structure, and integrated relationships, whilst shaping the minds to understand the world (Aufa et al., 2016). As one of the basic sciences, mathematics has an essential role in life (Wardani et al., 2017). Algebra is one of the scientific branches that relate to numbers, letters, and symbols in simplifying and solving mathematical problems. According to David Pimm's discussion, "In a symbol, there is concealment and revelation" (Kereh et al., 2014). For example, x represents a known number and y represents the number to be identified. When Andi has x books and Bob has three more books than Andi, then Bob’s book may correspond as y = x + 3. Using algebra, ones can investigate the patterns of rules and numbers.

Algebra draws conclusions by looking at the objects from above so that a model can be found. Some students have difficulty in understanding arithmetic and algebra (Maharaj & Dlomo, 2018). Algebra is a process skill and requires high-level thinking in mathematics, but this condition tends to make students dislike the challenge in increasing their algebraic skills (Kaput, 1995). Nevertheless, elementary algebra is one of the compulsory subjects in mathematical education courses. A university in West Sumatra has the provisions for student-teaching in the practice of the Court. The abstract nature of mathematics, which is also a character of basic algebra, is one of the factors that cause difficulties in learning it (Kereh et al., 2014) and it is a challenging course to be mastered by students who mainly have low math skills (Suhartati, 2012). This phenomenon is apparent, as students who complete their math studies are still classified as inferior in learning mathematics. The low student’s mathematical skill achievement in learning objectives is due to the insufficiency of learning devices to assist students and facilitate the process of learning (Yuliani & Saragih, 2015). The lecturers are less skilled in creating new learning devices, so the teaching methods tend to be monotone. There are many reasons why students have problems with math, but their difficulties with basic algebra courses are a significant one. From the students' learning results in basic algebra courses, there are students with a below-average rating. It can see from the recap of the basic algebra course 2012–2013 in Table 1.

Mark	**Basic Algebra**	
	Students	**Percentage**
A	20	35.43 %
B	81	
C	107	62.39 %
D	81	
E	21	
Sum of Students	335	100%

Table 1: The result basic algebra course in a mathematical education study program at a university in West Sumatra, 2012–2013

Table 1 showed that most students do not understand basic algebra courses. According to the experience of the researcher, many factors cause low student learning outcomes. One of them is material delivery. When the materials are taught conventionally by lecturers, the students are paying full attention to them. Consequently, the students tend to be reactive rather than active. This lowers the concentration of students from the beginning to the end of the learning process. Then the students are disinterested in acquiring the information and skills required to complete the task. Conventional delivery ultimately does not lead students to analyze, synthesize, evaluate, or solve problems and the students are not involved in the thinking process by discussion, asking questions, or finding solutions to gain new knowledge. Along with the statement by (Carneiro et al., 2016), such classrooms are frequently teacher-centered, thus the students accept the passive role in the class, rarely getting opportunities to lock in the learning materials to create basic thinking and understanding. Our duty as instructors is limited to planning and delivering “good” lectures, and we have no practical assessment of students’ knowledge of the taught materials. In our involvement, this inactivity in learning manifests in the students and they will lose interest in the subject matter gradually.

One way to address the problems is the use of alternative teaching strategies. The adoption of teaching materials that support the learning process and a model of active learning, in which the learners are invited to participate in the learning process mentally and physically; in other words, learning through a method of inquiry. The means of the inquiry learning method is based on the process skills: the students are engaged in learning activities (Indratno & Soepomo, 2016). Some researchers agree to use this method. Lee in Widiastuti et al. (2018) stated that the inquiry-based method provides better opportunities for students to learn, allowing them to improve their learning outcomes. Sanjaya (Rahmi et al., 2018) explained that critical thinking and analytical skills are developed by creating an active learning climate through inquiry-based methods so that students can find answers to problems through a guided investigation by the teacher. The thought process presented in this learning model is similar to the one used by professional scientists to generate new knowledge (Abdi, 2014). Students learn by making connections between scientific facts and existing theories. This method will stimulate students' critical thinking abilities by integrating various information and preparing learning materials that can measure learning performance without the intervention of the lecturer (Neuby, 2010).

Some studies differ from other studies on the model of guided notetaking and inquiry methods integration into learning devices, namely modules and planned lecture activities (SAP or *Satuan Acara Perkuliahan*). According to the substance of the research, guided learning models can improve the students’ concentration and learning outcomes. It is evident from the research conducted by Lazarus (Boyle 2001) that the use of guided note-taking techniques can increase the comprehension and concentration of information provided by the teacher. The same is true with the use of inquiry-based methods. The results of the research conducted by Rodríguez et al. (2019) suggested that the stimulation provided by the inquiry process can improve student learning outcomes. Also, students experience increased satisfaction with their learning experiences.

Guided notetaking is implemented by carrying out a particular action; the teacher prepares a concept map to help students create notes when the teacher instructs on the subject matter. Guided note-taking student gives students opportunities to learn actively, respond, and be involved with the discussed content sot that the students will generate comprehensive and accurate notes. By restudying the notes, the students can acquire high scores (Wardani et al., 2017). Teachers use guided note-taking to improve students’ recall (Kiewra, 1985). Collingwood and Hughes (Russell et al., 1983) stated that the use of guided notes induces students to concentrate more on adopting the subject materials. Guided note-taking can also make it easier for students to understand the material (Tanamatayarat et al., 2017). The advantage of using the guided note-taking learning method is that students can understand the material presented by the teacher easier, which improves student learning outcomes and increases student activity during the learning process (Puspasari, 2017). Education teachers, who focus on using models that successfully meet the demands of their educational assignments, can create powerful lessons that students can collaborate to solve problems (Marbán & Mulenga, 2019).

Based on the presented problems, the researchers will develop learning tools for SAP and inquiry-based modules with guided note-taking learning models. This model is expected to make students be more active and able to design material concept discovery activities by applying the Instructional Development Institute (IDI) model development steps. According to Komalasari (Maisyaroh et al., 2017), "teaching materials are one manifestation of the preparations made by the teacher before making the learning process." In line with that, Trianto (Maisyaroh et al., 2017) stated that "the teaching materials are a device that is needed and used for managing the learning process." Hence, it can be said that planning the teaching materials is essential to achieve a great learning.

METHOD

This study was a research and development study, conducted at PGRI University in West Sumatra, Indonesia. The duration of the study was eight months, from February to October 2016. The subjects were students who took basic algebra courses in the academic year of 2015–2016. This research consists of three stages, discovery/analysis, development, and evaluation stage (Figure 1).

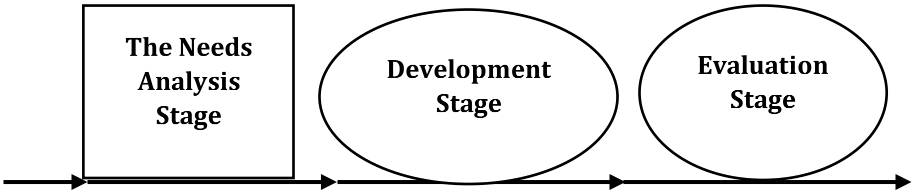

Figure 1: The design of the research

The needs analysis stage was carried out through observation and interview. It was done by analyzing the syllabus and the planned lecture activities to see whether it was teacher-centered or student-centered; analyzing the books on basic algebra to see their contents' conformity with the standards of basic competencies and the competencies that must be achieved by students; reviewing the literature related to the development of learning devices; and interviewing colleagues and students. The interview was about any problems or obstacles encountered in basic algebra courses. The discovery/analysis was used to design and develop products at the development stage.

The development stage was designing a prototype inquiry-based learning device with a Guided Note-Taking learning model based on the discovery/analysis stage results. The designed learning tools were Basic Algebra Modules and its SAP. The creation of this prototype was done through 2 stages, validation test and practicality test. The tested aspects on SAP were identity aspects, formulation of learning goals, selection of learning materials, detailing learning steps and models, and the selected learning resources and assessment. The tested aspects of the module were objective, rational, module content, module characteristics, suitability, module shape, and flexibility. The instruments were validation sheets involving math education experts, educational technology experts, linguists, and lecturers.

The validation results from the validators against all assessed aspects were presented in table form. Next, the average score was calculated using this formula:

$$R = \frac{\sum_{i=1}^{n} V_i}{n} \tag{1}$$

Description:

R = the average assessment from validators,

Vi = the score of the results of the 18th validator assessment,

n = validator number.

Then the obtained average was confirmed with the set criteria. The score ranged from 1 to 5 and the criteria were divided into five levels. The terms were tailored to the aspects in concern. The average range was divided into five interval classes, then the average of all aspects of the module was calculated. To determine the degree of validity of the modules, the following criteria were used:

a. If the average > 3.20 then the module was categorized as very valid.

b. If 2.40 < the average ≤ 3.20 then the module was categorized as valid.

c. If 1.60 < the average ≤ 2.40 then the module was categorized as quite valid.

d. If 0.80 < the average ≤ 1.60 then the module was categorized as less valid.

e. If the average ≤ 0.80, then the module was categorized as invalid.

The practicality tests were done by making observations during the implementation of learning, interviews, and questionnaires. The assessed aspects in SAP and modules were the implementation of learning, ease of use, time, and the equivalence with modules or SAP. The instrument used in this part was a questionnaire.

The practicality of the learning device was described by data frequency analysis techniques with this formula:

$$P = \frac{R}{SM}\, x100\% \qquad (2)$$

Description:
P = Practical value
R = Obtained score
SM = Maximum score

No.	Level of Achievement (%)	Category
1	85 – 100	Very practical
2	75 – 84	Practical
3	60 – 74	Quite practical
4	55 – 59	Less practical
5	0 – 54	Impractical

Table 2: Learning device practicality categories

Based on Table 2, it could be concluded that the device was said to be practical if its practical value was ≥ 75%. This practicality test used a small scale (small group evaluation) involving 7 students. The results of this trial were revised until the product is feasible and practical. The results at this stage were valid and practical modules to use with guided note-taking learning models.

The evaluation stage was carried out to assess whether the developed learning tools are effective in improving the students' learning quality and achievement. The instruments used were pretest and posttest with the indicators of conceptual understanding. The data obtained from the pretest and the posttest were given a score according to the created rubric. The improvement in students' understanding of the mathematical concept was observed through an analysis of normalized gain scores (<g>) compared to the category proposed by Hake (1998) "Normalized gain score i.e., comparison of an actual gain score with the maximum gain score." The actual gain score was the obtained gain score by the student, while the maximum gain score was the highest gain score that the students may get. Thus, the normalized gain score could be expressed by the following formula:

$$\langle g \rangle = \frac{T_1' - T_1}{T_{max} - T_1} \qquad (3)$$

$\langle g \rangle$ was the normalized gain score, T_1' is the *posttest* score, T_1 the *pretest* score, and T_{max} was the ideal score. Good learning is achieved if the normalized gain score was more than 0.4. Table 3 shows the categories that can be seen in the results of normalized gain scores.

Percentage	Classification
$\langle g \rangle < 0.3$	Low
$0.3 \le \langle g \rangle < 0.7$	Medium
$\langle g \rangle \ge 0.7$	High

Table 3: Normalized Gain Criteria

All the calculations were done using Microsoft Excel (Microsoft, USA). Based on effectiveness tests, it was obtained that modules and SAP are effectively used in inquiry-based learning with guided note-taking models. (Elgazzar, 2014) states that products were considered good if they meet the requirements of validity, practicality, and effectiveness.

FINDINGS AND DISCUSSION

The results and discussion contain the development process of inquiry-based learning devices with guided note-taking learning models, the developed products, validity tests, practicality tests, effectiveness tests, and the discussion of the products in improving students' mathematical understanding.

Findings

Need assessment

The analysis of the syllabus and SAP of the basic algebra course showed that the basic competencies have not been able to prepare the students to conduct their experiments, ask questions, search answers on their own, and connect those answers. Basic algebra competencies should provide students with a centralized approach where students are directly involved in asking questions, formulating problems, conducting experiments, collecting, and analyzing data, drawing conclusions, discussing, and communicating the result per the elements of inquiry-based learning. Thus, the students become more active, whilst the role of lecturers is involved in guiding, training, and accustoming students to skilled thinking (minds-on activities).

The stage of inquiry has not been brought up in the reference book. Each presented material has not been able to identify, formulating problems, and describing the discussions between students. Students are not used to making hypotheses, formulate alternative problem-solving, and establish problem-solving when they were provided examples of problems and solutions.

The interviews showed that the lecturers still use the lecturing method, accompanied by the provision of exercises. It is difficult for students to understand and find solutions because of the provided training. The teaching materials were mathematics books of middle and high school level. It was observed that the teaching method did not help the students to understand the material because the questions were not applicable or appropriate. The learning resources used by students were the materials, problem examples, and exercises.

An initial test was provided to measure the student's initial ability in the line and series materials (Figure 2).

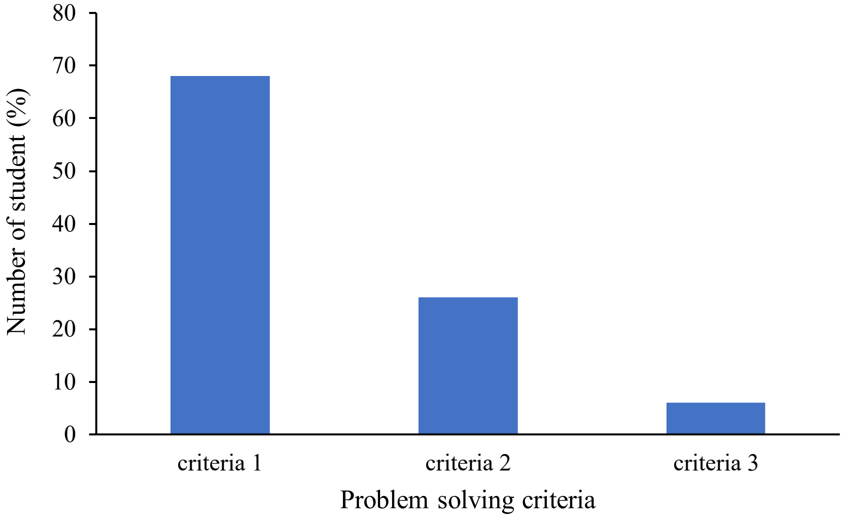

Figure 2: The initial student ability test

Description:

Criterion 1: students can identify problems but have not used any problem-solving approach or strategy
Criterion 2: students can identify the problems correctly, but the used strategy or approach is wrong
Criterion 3: students have done the correct problem solving

Based on the 38 tested students, 68% of students were on criteria 1. Students could identify problems, but they have not used any problem-solving approaches or strategies. As many as 26% of students were on criteria 2. Students could identify problems correctly, but the strategies or approaches were incorrect. Lastly, 6% of students were on criteria 3, who have done the right problem-solving. Most students did not well understand the concept of problem-solving.

Development stage

This stage was drafting the learning device in the form of modules and a guided note-taking learning model for basic algebra courses. Self-evaluation was used to evaluate the modules and SAP. Self-evaluation involved various experts' reviews. Content qualification, presentation, language, and time were evaluated. The results of module validation from expert reviews were in the form of prerequisite materials presented on the module. The results showed that the modules have not directed students to understand the concepts, problem examples, and exercises. Furthermore, the presented materials were not relevant to self-training. SAP validation results from experts were in the form of clarity in lecture scenarios and it was observed that they had not been described in detail and clear. However, all learning devices can be used in large and small trial classes. The results of module validation sheets and SAP can be seen in Table 4.

Components of Assessment	**Validator**		**Sum**	**Validity Value (%)**
	1	**2**		
Material	27	29	56	80
Presentation	28	30	58	82.9
Language	19	21	40	80
Total				242.9
Average				80.97

Table 4: Validation results of the module sheet

The average value of the validation results was 80.97% with a very valid category. This showed that the developed module can be used in trials. Table 5 showed the SAP validation results based on the experts' reviews.

Components of Assessment	**Validator**		**Sum**	**Validity Value (%)**
	1	**2**		
The content presented	15	14	29	72.5
Language	12	13	25	83.3
Time	4	4	8	80
Total				235.83
Average				78.61

Table 5: Validation sheet results of SAPs

The average value was 78.61% for the validation results. This showed that the SAP development has been valid in terms of content, language, and time and can be used in trials.

After the validation of the learning device, a small group evaluation was carried out to improve the quality of modules using questionnaires and interviews so that they have a high level of practicality and effectiveness. The trial involved 7 students from the mathematics education study program. Table 6 displayed the results of the questionnaire analysis.

Components of Assessment	**Student**							**Sum**	**Value**
	1	**2**	**3**	**4**	**5**	**6**	**7**		
Ease of Use	51	49	49	51	47	51	45	343	89.09
Time Efficiency	4	5	5	5	5	4	4	32	91.43
Benefits	14	14	14	12	14	14	13	95	90.48
Total									271
Average									90.33

Table 6: Student ratings of the new modules

The results showed that the new modules were very practical, with a score of 90.33%. This showed that the modules have been practical in the aspects of ease of use, time efficiency, and benefits. The practicality of the modules was supported by the interview result with students to get more accurate information about the student's opinions about the modules. The interviews between lecturers and students showed the practicality of the module.

Lecturer : "How easy was the basic algebra module to use, Indah?"
Indah : "The materials, question examples, questions or instructions, and exercises on the modules have been clearly presented to assist me in finding the concept, Ma'am."
Lecturer : "How is the time efficiency in the module, Ika?"
Ika : "The work of problems in modules is very efficient because of the time allotted."
Lecturer : "Is this module beneficial for Tika?"
Tika : "It is rewarding, Ma'am because I can devote more time to do exercises and study alone."

No	Interviewing aspect	Interview Results
1	Easiness of Use	Materials, question examples, inquiries or instructions, and exercises on the module seem to help students to find concepts; The modules were written in a well-written formal Indonesian language.
2	Time efficiency	The students need about 50 minutes to understand the material in the module, and they can understand the content within the available time.
3	Benefits	The learning modules will lead the students to be more active in listening, speaking, and reading. Thus, the students can study independently and devote more time to working on exercises.

Table 7: Student interview results of the module

According to the interviews with students in Table 7, the modules have made it easier to learn in listening, speaking, writing, and reading. According to (Abu-hardan, 2019) through reading, communication can occur. In this case, learning is the process of interaction between students and the presented material in the module. The presented modules were time-efficient so that the students can understand the materials on the module in a short time frame.

Assessment stage

This stage was done with a large class of 38 students using the revised learning devices. The purpose of the test was to see the effectiveness of inquiry-based learning tools with guided note-taking learning models. The results can be found in Table 8.

Student Learning Outcomes	***Average N-Gain***	***Category***
Experiment	0.40	Medium

Table 8: Student interview results of the module

Table 8 showed that the guided note-taking learning models improved the student's learning outcomes. The learning outcome test score was 0.40, or moderate, indicating that the learning outcomes on algebra courses were improved after an inquiry-based learning device with a guided note-taking learning model was used. Figure 3 showed the distribution of test scores about the achievement of learning outcomes after using the modules. The average score of posttest was 82.1 while the average score of pretest was 67.9. The lowest score on the pretest was 31, while the highest score on the posttest was 93.

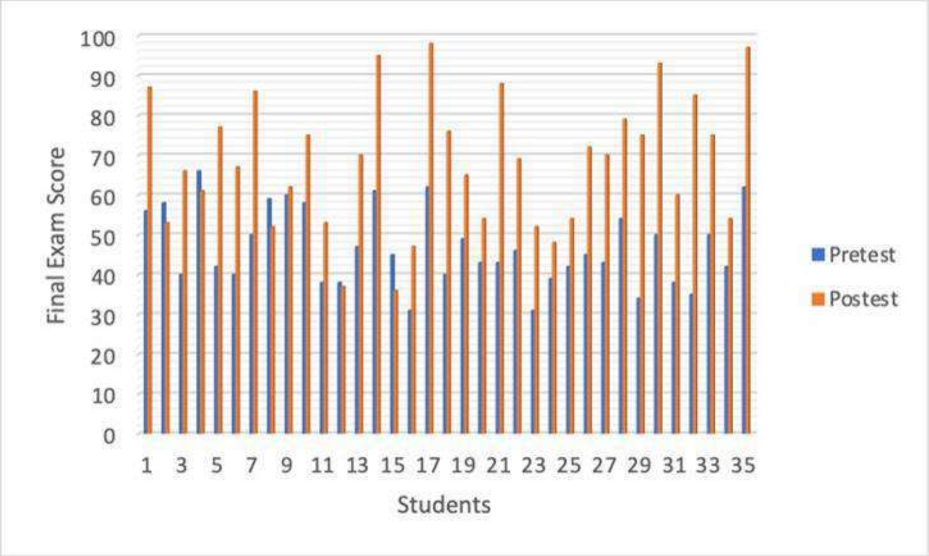

Figure3: Pre-test and post-test score distribution

DISCUSSION

The students have low creative thinking skills and some investigations showed that the learning process has not actively engaged them to solve problems as the teaching materials and devices did not trigger students' interest in learning (Sari et al., 2021). The modules and the learning tools can help students to learn basic algebra more easily (Shofiyah et al., 2021). The modules used in the lecture should engage students to learn on their own. Through this learning device, students can use relevant ways to gather information, use logical analysis (Dharma et al., 2020), and investigate the surrounding phenomena with guidance from the questions, so that they can solve problems using the facts that they find (Risman & Santoso, 2017).

The use of modules can increase students' motivation to learn (Harefa & Silalahi, 2020). Lecturer guides, trains, and educates students through guided notes, taking learning models that are integrated with skilled thinking modules, and mental and physical involvement experience. The learning model can improve students' learning outcomes (Harefa & Silalahi, 2020) and improves verbal and written communication skills through hands-on experience (Schmidt & Kelter, 2017).

The delivery of teaching materials through guided note-taking learning models has received the full attention of students due to the effectiveness of learning devices demonstrated by N-gain scores (Novilia et al., 2016). The effectiveness is shown by the differences in students' critical thinking skills (Irwan et al., 2019). This has an impact on increasing the concentration of students from the beginning to the end of the learning sessions so that they can complete the tasks. This condition causes student responses in analyzing, synthesizing, evaluating, and solving problems. Students are more involved in thinking about new knowledge through the process of discussion, asking, and searching for solutions (Yuhana et al., 2020).

CONCLUSIONS

The inquiry-based learning devices with guided note-taking learning models have been proven to be valid with a score of 80.97 and the SAP had a validity score of 78.61. The inquiry-based learning device was designed with a strong theoretical rationale and its components were internally consistent. The practicality value for the module was 90.33, or very practical, which means that the students, as the module user, considered that the module has met the needs, expectations, and limitations. Students' learning outcomes have been improved with the help of learning devices, indicated by the gain score of 0.40 or categorized as moderate. The learning tools have positive impacts on students.

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Designing Student Worksheet on Relation and Function Material for Mathematics Learning: Jumping Task

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Abstract: This study aims to produce a jumping task in the form of a student worksheet which is valid and practical. Design research in the form of development studies was chosen in this study which consist of preliminary stage and prototyping stage. Data were collected by walk through, test, questionnaire, and interview. Data were analyzed descriptively. Student worksheet is declared valid in terms of content, construct, and language. The results of this study show that the student worksheet is valid and practical. The student worksheet is in accordance with the HOTS level in the taxonomy of Bloom and the PISA framework, in accordance with the curriculum and the material for eight grades, and in accordance with the General Guidelines for Indonesian Spelling (PUEBI) and did not cause multiple interpretations. Then, the students understand the instructions or questions in the student worksheet, and it can be used by students.

INTRODUCTION

Creativity, critical thinking, communication, and collaboration (4C) or often referred to as higher order thinking skills (HOTS) are needed by student in the 21st century (Kemdikbud, 2017; Putri, 2018; Hwang, et al., 2017). Educators, researchers, and various parties state that HOTS is very important for everyone (Bakry & Bakar, 2015; Abosalem 2016; Tambunan & Naibaho, 2019; Elfeky 2019; Lu, 2021; Gustiningsi & Somakim, 2021; Utari & Gustiningsi, 2021; Gustiningsi & Utari, 2021).

There is a relationship between HOTS in Bloom's Taxonomy and PISA framework. In Bloom's taxonomy, HOTS is at a high level, which consist of analyzing, evaluating, and creating (C4, C5, and C6) (Efendi, 2017). In the Program for International Student Assessment (PISA), there are questions at level 4,5,6 which is HOTS level (Setiawan, Dafik, & Lestari, 2014). Based on indicators, there is a match between the Bloom's taxonomy and the PISA framework. Level 4 in PISA describing that students can make model concrete situations, make assumptions, choose, and integrate different interpretations, relate information they have and give arguments or communicate them (OECD, 2018). In Bloom's Taxonomy, level 4 describing that the ability to break down information into pieces of information and detect the relationship of information to one another and to the overall structure and purpose (Anderson and Krathwohl, 2001; McNeil, 2011; Widana, 2017). This description shows that they have the same ability to analysis or relate information to provide arguments and solve problems. Then, level 5 in PISA describing that students can develop models for complex situations, identify constraints and determine assumptions, select, compare, and evaluate appropriate problem-solving strategies, formulate, and communicate their interpretations and reasons (OECD, 2018). In Bloom's taxonomy, level 5 describing that students make judgments or decisions based on criteria and standards (Anderson and Krathwohl, 2001; McNeil, 2011; Widana, 2017). This description shows that they have the same ability to evaluate and identify criteria or several things and make decisions. Then, level 6 in PISA describing that students can conceptualize, generalize, and utilize information based on investigation and modeling of complex problem situations, apply mathematical insights and understanding to develop new approaches and strategies, reflect and formulate as well as communicate appropriately actions and reflections on their findings (OECD, 2018). In Bloom's Taxonomy, level 6 describing that student putting elements together to form a new shape (Anderson and Krathwohl, 2001; McNeil, 2011; Widana, 2017). This description shows that they have the same ability to create something new based on the information possessed by students.

The PISA results show that Indonesian students' mathematical literacy skills are still low (OECD, 2019). In 2018, in the field of mathematics, Indonesia was ranked 72 out of 78 countries (OECD, 2019). Likewise in Trends in International Mathematics and Science Study (TIMSS), in 2016 Indonesia was ranked 44th out of 49 countries (Utomo, 2021). Previous research analyzed students' HOTS abilities, including students' critical thinking skills (Gustiningsi, 2015), students' ability to solve HOTS problems (Abdullah, et al, 2015), and showed that students' HOTS abilities were still low.

HOTS ability can be improved through learning in class. Bakri and Bakar (2015) stated that students' abilities can be developed through activities and mathematics learning. Teachers must pay attention to students so that they can develop students' HOTS abilities (Purnomo, et al., 2021; Pasani & Suryaningsih, 2021).

One of methods that can be done to develop students' HOTS abilities is designing student worksheets to be used in class. Previous research shows that student worksheet is able to improve students' abilities such as students' concept understanding (Nursyahidah, Putri, & Somakim, 2013), students' problem-solving abilities (Fitriati & Novita, 2018) and students' creative thinking skills (Romli, Abdurrahman, & Riyadi, 2018). The student worksheet must meet the HOTS criteria. Sato stated that to improve the quality of learning, the quality of the tasks given is influential, one of the tasks given was in the form of a challenging task or called a jumping task (Saito, 2015).

Sato stated that the HOTS task was related to the jumping task applied in lesson study (Putri, 2018; Putri & Zulkardi, 2019; Hobri, 2020). Jumping tasks are effective in supporting students' HOTS abilities (Putri, 2018), one of which is problem solving skills (Hobri, et al., 2020).

Previous research has designed jumping tasks using the PISA framework (Zulkardi & Putri, 2020; Putri & Zulkardi, 2019), with an RME approach (Sa'id, et al., 2021), and based on an open-ended question (Ummah, et al., 2021). Meanwhile, this study designed the jumping task in the form of a student worksheet using the PISA framework and adjusted the worksheet to the HOTS level in the Bloom's taxonomy. This study aims to produce a valid and practical jumping task-based student worksheet.

RESEARCH METHOD

Design research in the form of development studies was chosen in this study (Bakker, 2019). The research subjects were 20 eight graders with an average age of 14 years old. The stages carried out are the preliminary stage and the prototyping stage (Tessmer, 1993; Zulkardi, 2002; Akker, et al., 2013) as shown in Figure 1.

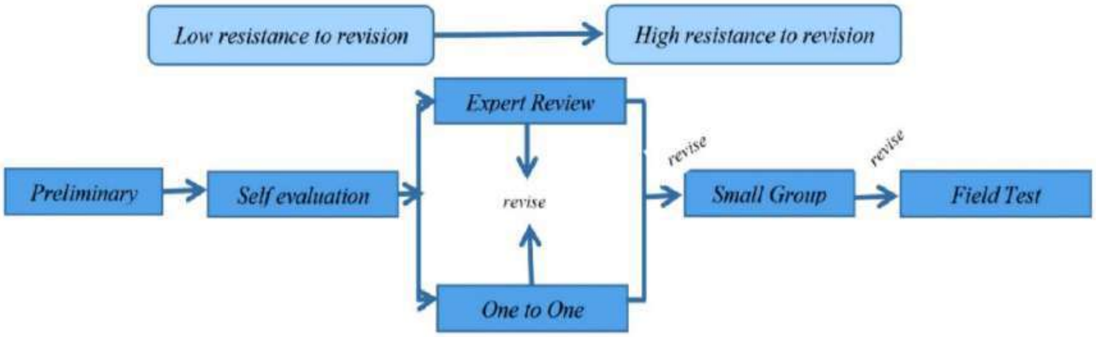


Figure 1: Prototyping Flow (Tessmer, 1993; Zulkardi, 2002; Akker, et al., 2013)

In the preliminary stage, preparations were made to develop a jumping task in the form of a student worksheet, including analyzing eight graders' materials, analyzing research subjects, and analyzing HOTS levels in the taxonomy of Bloom, and analyzing levels in the PISA framework. Furthermore, the prototyping stage starts from the first stage, namely self-evaluation. In the self-evaluation stage, student worksheets were created and self-evaluated (prototype I). The second stage was the stage of expert review and one-to-one. In the expert-review stage, prototype I was validated by experts to see the validity of the student worksheet in terms of content, construct, and language. In the one-to-one stage, prototype I was tested on 2 students who were not research subjects to see the usability of the student worksheet, seen from student answers and student comments in interviews while working on the student worksheet and in the questionnaire after completing on the student worksheet. Then, prototype I was analyzed and revised according to suggestions from experts and according to students' comments and answers. The revised student worksheet is called prototype II. Then, prototype II was tested in the small-group stage on 5 students who were not research subjects. The small-group stage was also aimed to see the practicality of the student worksheet based on the answers and comments of students during and after completing on the student worksheet. Furthermore, the student worksheet that had been tested was analyzed and revised, hereinafter referred to as prototype III. Furthermore, prototype III was tested at the field-test stage to 20 students who were research subjects.

Data were collected through walk-through, questionnaires, interviews, and tests. The walk-through was used at the expert-review stage to ask for advice and comments from the expert, while tests, questionnaires, and interviews were used at the one-to-one, small group, and field test stages. The test was used to see the usability of the developed student worksheet (prototype) and to see the students' way of thinking, while the questionnaire and interviews were used to find out comments, constraints, and difficulties, and explore students' ways of thinking when completing the student worksheet. The walk-through was analyzed descriptively. Comments and suggestions from experts were described and used as materials for revising prototype I. The tests were analyzed based on the scoring rubric, then described. The scoring rubric available in Table 1.

Question	Possible Student Answer	Score
On Monday afternoon at 16.00 West Indonesian Time, Ani wanted to call her friend for about 5 minutes because she had very important business at the time. Ani is confused whether to just call or register for a talkmania package. Finally, Ani choose to sign up for the talkmania package. Is Ani's decision right? Please describe your reason.	Students can relate the information in the table and compare the price of calling with talkmania packages and non-package prices. Then, the students make decisions about which package should be chosen.	1
	Student can't relate the information in the table and compare the price of calling with talkmania packages and non-packages prices.	0
If on Monday afternoon at 16.00 West Indonesian Time, Ani chooses to make a call with a non-package call rate, draw a graph that shows the time between Ani's call and the price she has to pay every minute.	Students can draw a graph that shows the relationship between calling time and costs incurred.	1
	Students can't draw a graph that shows the relationship between calling time and costs incurred.	0
Is the relationship between time and the price of this non-packaged call called a function? Describe your reasons.	Students analyzed answers by paying attention to the definition of a function and relating it to the condition of the relationship between non-packet calling	1

Question	Possible Student Answer	Score
	time and the cost.	
	Students can't analyze answers by paying attention to the definition of a function and relating it to the condition of the relationship between non-packet calling time and the cost.	0

Table 1: The scoring rubric for test

Interviews and questionnaires were analyzed descriptively, then used as supporting information in the development process. The student worksheet is said to be valid in terms of content and construct seen at the expert review stage. The student worksheet is said to be valid in terms of content if it is in accordance with the HOTS level in the Bloom's taxonomy and PISA framework and is valid in terms of constructs if it is in accordance with the curriculum. Student worksheet is said to be practical if it can be understood by students, can be completed, does not cause multiple interpretations, and students are interested in doing it.

RESULTS

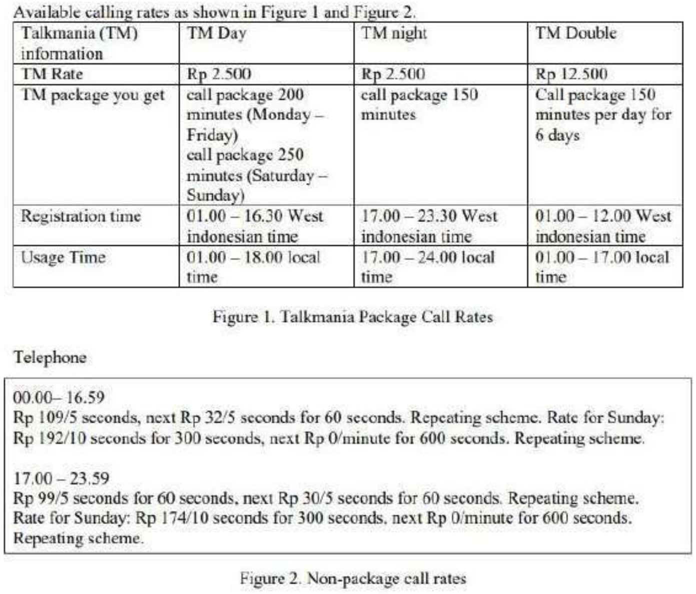
The student worksheet had been developed in two stages, namely preliminary and prototyping. The preliminary stage was carried out by analyzing the curriculum, determining the material, analyzing HOTS criteria in Bloom's taxonomy, and analyzing HOTS criteria in PISA framework. A description of the HOTS levels in Bloom's taxonomy and the PISA framework is provided in Table 2.

Level	Bloom's Taxonomy	PISA Framework
Level 4	Analysis: The ability to break down information into pieces of information and detect the relationship of information to one another and to the overall structure and purpose. (Anderson and Krathwohl, 2001; McNeil, 2011; Widana, 2017)	Students can model concrete situations, make assumptions, choose, and integrate different interpretations, relate information they have and give arguments or communicate them (OECD, 2018).
Level 5	Evaluation: Make judgments or decisions based on criteria and standards (Anderson and Krathwohl, 2001; McNeil, 2011; Widana, 2017)	Students can develop models for complex situations, identify constraints and determine assumptions, select, compare, and evaluate appropriate problem-

Level	Bloom's Taxonomy	PISA Framework
		solving strategies, formulate and communicate their interpretations and reasons (OECD, 2018).
Level 6	Create: Putting elements together to form a new shape (Anderson and Krathwohl, 2001; McNeil, 2011; Widana, 2017)	Students can conceptualize, generalize, and utilize information based on investigation and modeling of complex problem situations, apply mathematical insights and understanding to develop new approaches and strategies, reflect and formulate as well as communicate appropriately actions and reflections on their findings (OECD, 2018).

Table 2: Cognitive level of HOTS category in taxonomy bloom and PISA framework

As Table 2 indicates, there is a match between levels 4,5,6 in the Bloom's taxonomy and the PISA framework. Thus, it can be formulated that the HOTS level consists of the ability to analyze, evaluate, and create. Next, the prototyping stage was carried out in the self-evaluation stage. In the self-evaluation stage, student worksheets were compiled and prototype I was produced. Prototype I is presented in Figure 2.



Available calling rates as shown in Figure 1 and Figure 2.

Talkmania (TM) information	TM Day	TM night	TM Double
TM Rate	Rp 2.500	Rp 2.500	Rp 12.500
TM package you get	call package 200 minutes (Monday – Friday) call package 250 minutes (Saturday – Sunday)	call package 150 minutes	Call package 150 minutes per day for 6 days
Registration time	01.00 – 16.30 West indonesian time	17.00 – 23.30 West indonesian time	01.00 – 12.00 West indonesian time
Usage Time	01.00 – 18.00 local time	17.00 – 24.00 local time	01.00 – 17.00 local time

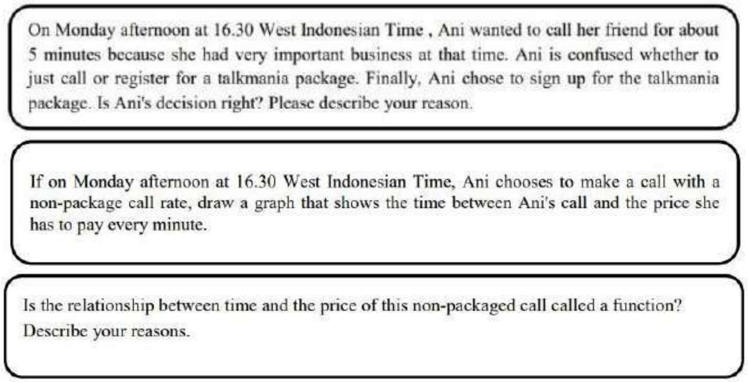
Figure 1. Talkmania Package Call Rates

Telephone

00.00– 16.59
Rp 109/5 seconds, next Rp 32/5 seconds for 60 seconds. Repeating scheme. Rate for Sunday: Rp 192/10 seconds for 300 seconds, next Rp 0/minute for 600 seconds. Repeating scheme.

17.00 – 23.59
Rp 99/5 seconds for 60 seconds, next Rp 30/5 seconds for 60 seconds. Repeating scheme. Rate for Sunday: Rp 174/10 seconds for 300 seconds, next Rp 0/minute for 600 seconds. Repeating scheme.

Figure 2. Non-package call rates



On Monday afternoon at 16.30 West Indonesian Time , Ani wanted to call her friend for about 5 minutes because she had very important business at that time. Ani is confused whether to just call or register for a talkmania package. Finally, Ani chose to sign up for the talkmania package. Is Ani's decision right? Please describe your reason.

If on Monday afternoon at 16.30 West Indonesian Time, Ani chooses to make a call with a non-package call rate, draw a graph that shows the time between Ani's call and the price she has to pay every minute.

Is the relationship between time and the price of this non-packaged call called a function? Describe your reasons.

Figure 2: Prototype I

For the ability to analyze, the student worksheet provides a table of tariff for calling and non-package, which shows the price, registration time and usage time. This table presents information that must be analyzed by students to be able to answer the questions in the student worksheet. Then, for the ability to evaluate, the question "whether Ani's choice is right", is a stimulating question so that the students can judge or choose on the answer based on the information that has been analyzed previously. For creative skills, the question to draw a graph and the question “is the relationship between time and price a function” stimulates students to make a graph and create an argument in the form of a picture or another to answer the question. The relationship between the student worksheet and the HOTS level can be seen in Table 3.

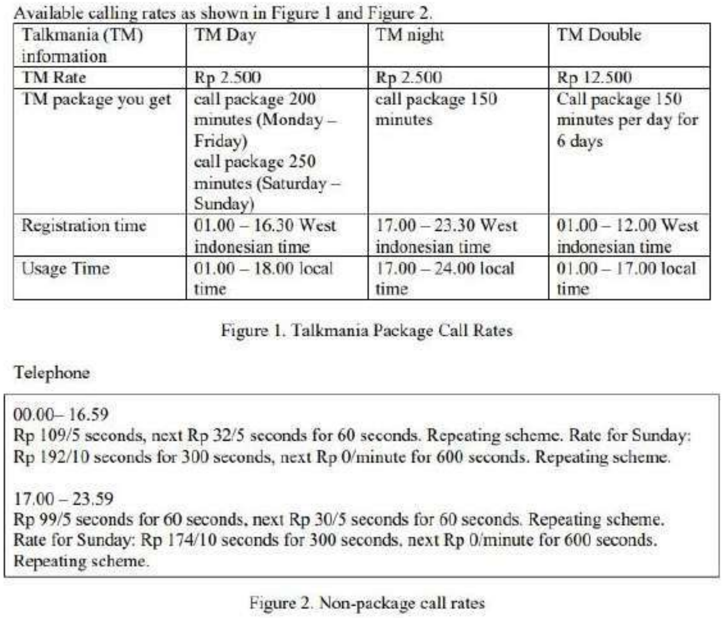
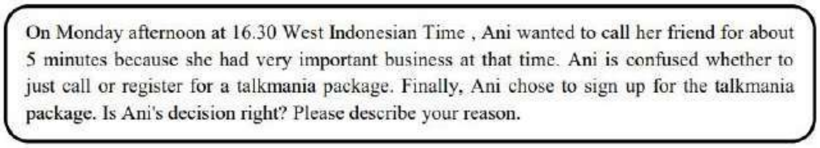
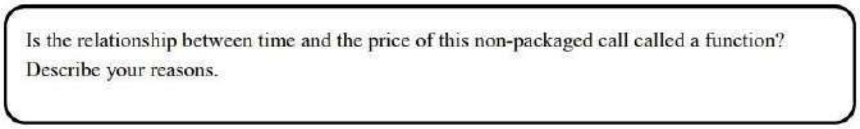
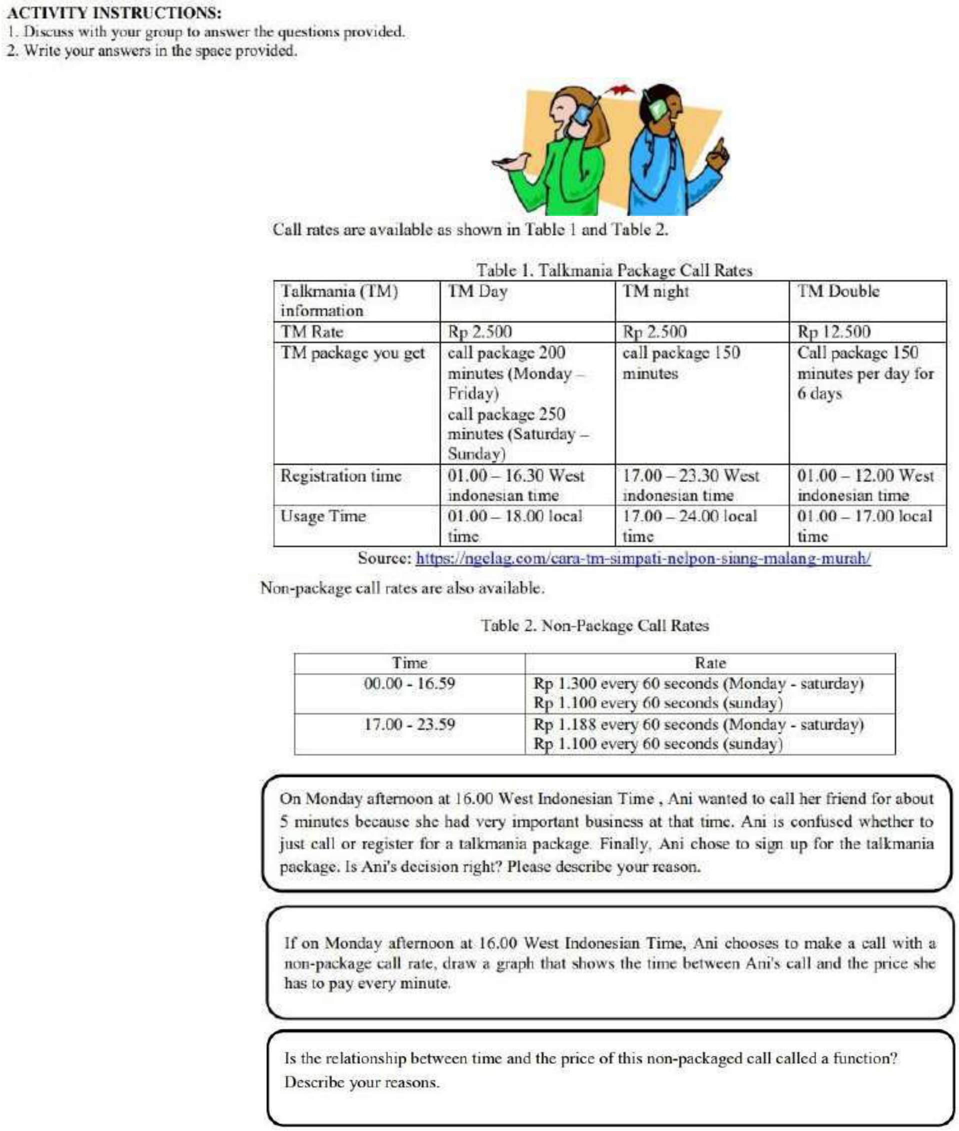
<table>
<tr><th>Student Worksheet</th><th>Level Cognitive</th></tr>
<tr><td>
Available calling rates as shown in Figure 1 and Figure 2.
<table>
<tr><td>Talkmania (TM) information</td><td>TM Day</td><td>TM night</td><td>TM Double</td></tr>
<tr><td>TM Rate</td><td>Rp 2.500</td><td>Rp 2.500</td><td>Rp 12.500</td></tr>
<tr><td>TM package you get</td><td>call package 200 minutes (Monday – Friday)
call package 250 minutes (Saturday – Sunday)</td><td>call package 150 minutes</td><td>Call package 150 minutes per day for 6 days</td></tr>
<tr><td>Registration time</td><td>01.00 – 16.30 West indonesian time</td><td>17.00 – 23.30 West indonesian time</td><td>01.00 – 12.00 West indonesian time</td></tr>
<tr><td>Usage Time</td><td>01.00 – 18.00 local time</td><td>17.00 – 24.00 local time</td><td>01.00 – 17.00 local time</td></tr>
</table>
Figure 1. Talkmania Package Call Rates
Telephone
00.00– 16.59
Rp 109/5 seconds, next Rp 32/5 seconds for 60 seconds. Repeating scheme. Rate for Sunday: Rp 192/10 seconds for 300 seconds, next Rp 0/minute for 600 seconds. Repeating scheme.
17.00 – 23.59
Rp 99/5 seconds for 60 seconds, next Rp 30/5 seconds for 60 seconds. Repeating scheme. Rate for Sunday: Rp 174/10 seconds for 300 seconds, next Rp 0/minute for 600 seconds. Repeating scheme.
Figure 2. Non-package call rates</td><td>Analysis</td></tr>
<tr><td>
On Monday afternoon at 16.30 West Indonesian Time , Ani wanted to call her friend for about 5 minutes because she had very important business at that time. Ani is confused whether to just call or register for a talkmania package. Finally, Ani chose to sign up for the talkmania package. Is Ani's decision right? Please describe your reason.</td><td>Analysis dan Evaluation</td></tr>
<tr><td>
If on Monday afternoon at 16.30 West Indonesian Time, Ani chooses to make a call with a non-package call rate, draw a graph that shows the time between Ani's call and the price she has to pay every minute.</td><td>Analysis, Evaluation, and Create</td></tr>
<tr><td>
Is the relationship between time and the price of this non-packaged call called a function? Describe your reasons.</td><td>Analysis dan Evaluation</td></tr>
</table>

Table 3: Student worksheet relationship with HOTS level

Then, prototype I was validated by the expert at the expert review stage. The expert provided suggestions and comments consisting of 1) Please add pictures that can attract students' attention, 2) Non-package data is simplified to 60 seconds, 3) The call package tariff table should be retyped so that it is clear, 4) It is in accordance with the HOTS level based of taxonomy bloom and PISA framework, 5) It is in accordance with curriculum and material for eight graders.

In parallel, prototype I was tested on 2 students at the one-to-one stage. Based on the students' answers in the one-to-one stage, the students were confused about the non-package price, and they were confused about changing the price per second on the non-package tariff.

Based on the expert- review and one-to-one stage, prototype I was analyzed and revised with the revision decision. The revised decision consists of: 1) Add pictures that can attract students' attention, 2) Non-package data simplified to 60 seconds, 3) Rate table retyped. The revised student worksheet is called prototype II, as shown in Figure 3.



ACTIVITY INSTRUCTIONS:

1. Discuss with your group to answer the questions provided.
2. Write your answers in the space provided.

Call rates are available as shown in Table 1 and Table 2.

Table 1. Talkmania Package Call Rates

Talkmania (TM) information	TM Day	TM night	TM Double
TM Rate	Rp 2.500	Rp 2.500	Rp 12.500
TM package you get	call package 200 minutes (Monday – Friday) call package 250 minutes (Saturday – Sunday)	call package 150 minutes	Call package 150 minutes per day for 6 days
Registration time	01.00 – 16.30 West indonesian time	17.00 – 23.30 West indonesian time	01.00 – 12.00 West indonesian time
Usage Time	01.00 – 18.00 local time	17.00 – 24.00 local time	01.00 – 17.00 local time

Source: https://ngelag.com/cara-tm-simpati-nelpon-siang-malang-murah/

Non-package call rates are also available.

Table 2. Non-Package Call Rates

Time	Rate
00.00 - 16.59	Rp 1.300 every 60 seconds (Monday - saturday) Rp 1.100 every 60 seconds (sunday)
17.00 - 23.59	Rp 1.188 every 60 seconds (Monday - saturday) Rp 1.100 every 60 seconds (sunday)

On Monday afternoon at 16.00 West Indonesian Time , Ani wanted to call her friend for about 5 minutes because she had very important business at that time. Ani is confused whether to just call or register for a talkmania package. Finally, Ani chose to sign up for the talkmania package. Is Ani's decision right? Please describe your reason.

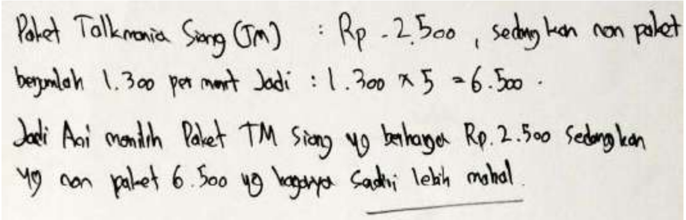
If on Monday afternoon at 16.00 West Indonesian Time, Ani chooses to make a call with a non-package call rate, draw a graph that shows the time between Ani's call and the price she has to pay every minute.

Is the relationship between time and the price of this non-packaged call called a function? Describe your reasons.

Figure 3: Prototype II

Prototype II was tested on students in the small group stage to four students. In the small group stage, it can be seen from the students' answers that none of the students were confused about the questions on the student worksheet and students stated that they needed a high level of thought to determine the answers on the student worksheet. This is a consideration for the researchers not to revise prototype II because it can practically be done by students and requires high thinking. Then, the student worksheet was tested in the field test stage.

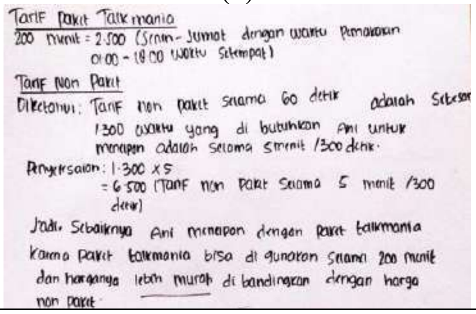
In the field test stage, there were 20 students who were the subjects for the student worksheet test. The students' answers to the first question regarding Ani's decision to choose the talk mania package are as follows.



Paket Talkmania Siang (TM) : Rp -2.500 , sedangkan non paket
berjumlah 1.300 per menit Jadi : 1.300 x 5 = 6.500 .
Jadi Ani memilih Paket TM Siang yg berharga Rp. 2.500 sedangkan
yg non paket 6.500 yg harganya jadi lebih mahal.

Translated to English:
Talkmania Day package: IDR 2.500, while the non-package is IDR 1300 per minute, So 1.300 x 5 = 6.500. So, Ani bought the TM Day package for IDR 2500, while the non-package is IDR 6.500 which more expensive.

(a)



Tarif Paket Talkmania
200 menit = 2.500 (Senin - Jumat dengan waktu pemakaian
01.00 - 18.00 waktu setempat)
Tarif Non Paket
Diketahui: Tarif non paket selama 60 detik adalah sebesar
1300 waktu yang di butuhkan Ani untuk
menelpon adalah selama 5 menit /300 detik.
Penyelesaian: 1.300 x 5
= 6.500 (Tarif non paket selama 5 menit /300
detik)
Jadi, sebaiknya Ani menelpon dengan paket talkmania
karena paket talkmania bisa di gunakan selama 200 menit
dan harganya lebih murah di bandingkan dengan harga
non paket.

Translated to English:
Talkmania package rates
200 minutes = 2.500 (Monday – Friday with usage time are 01.00 – 18.00 local time)

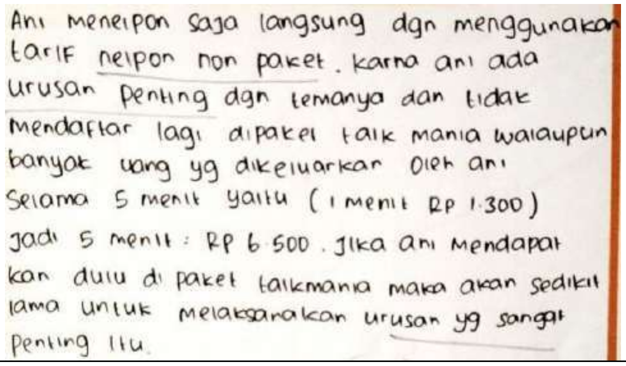
Non-package rates
Known: non-package rates for 60 seconds is 1300, the time which Ani needed to call is 5 minutes or 300 seconds.
Solution:
1.300 x 5 = 6.500 (non-package rates for 5 minutes or 300 seconds). So it's better if Ani call with the talkamnia package because the talkmania package can be used for 200 minutes and the price is cheaper than the non-package price.

(b)

Figure 4: Students' answers to questions about Ani's decision

As Figure 4 shows, the students analyzed package and non-package prices and then compared them. They evaluated it by calculating the non-package price for 5 minutes and it can be seen that the non-package price is more expensive than the price for calling with a talkmania package. Then, the students can decide that Ani's choice is right.

In addition, there were students who answered differently from the answers in Figure 4, as shown in Figure 5.



Ani menelpon saja langsung dgn menggunakan tarif nelpon non paket. karna ani ada urusan penting dgn temanya dan tidak mendaftar lagi dipakel talk mania walaupun banyak uang yg dikeluarkan oleh ani selama 5 menit yaitu (1 menit RP 1.300) jadi 5 menit: RP 6.500. Jika ani mendapatkan dulu di paket talkmania maka akan sedikit lama untuk melaksanakan urusan yg sangat penting itu.

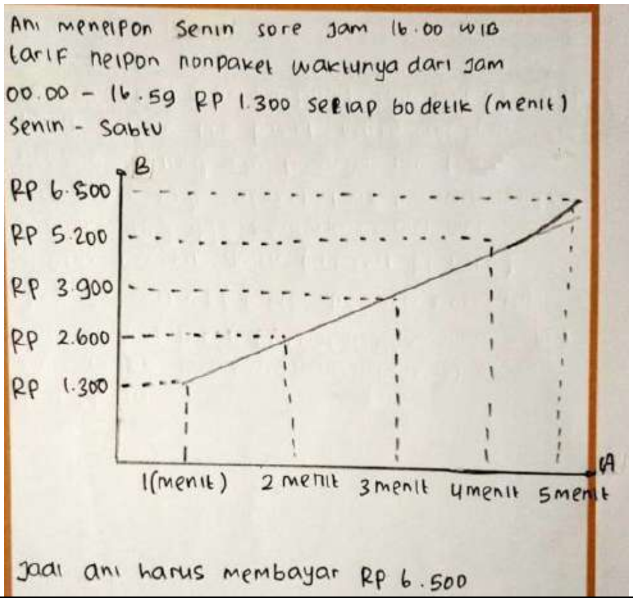
Translated to English:

Ani just called directly using a non-package call rate because Ani had important business with her friend and didn't register for the talkmania package even though Ani spent a lot of money for 5 minutes (1 minute IDR 1.300) so 5 minutes was IDR 6.500. If Ani registers in the talkmania package first, it will take a longer time to carry out that very important business.

Figure 5: Different Students' Answers to the First Question

Based on Figure 5, students answered that Ani should call at a non-package rate because it would save time. In Figure 5, it can be seen the students analyzed the price of package and non-package so that they know the cheaper price for calling is making a call with a talkmania package, it is also seen that the students evaluated Ani's decision by calculating the cost of calling for 5 minutes, but they focused on the sentence "There is an important business right away" which according to the student in Figure 5 that the matter should not be postponed, so students answered the call directly with non-package only with the risk of being more expensive but saving time.

Then, for the second question regarding graphics. Students' answers can be seen in Figure 6.



Ani menelpon Senin sore jam 16.00 WIB
tarif nelpon nonpaket waktunya dari jam
00.00 - 16.59 RP 1.300 setiap 60 detik (menit)
Senin - Sabtu

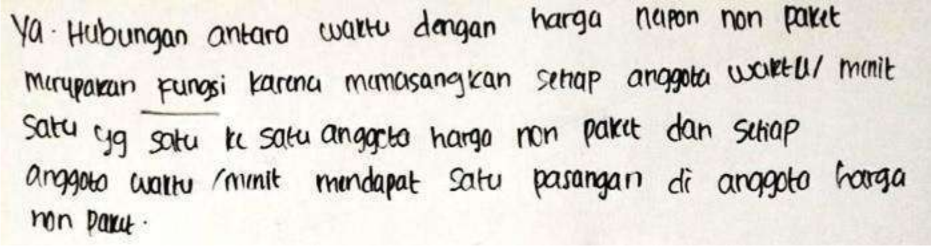
Jadi ani harus membayar RP 6.500

Translated to English:
Ani call at Monday afternoon at 16.00 West Indonesian Time, non-package call rates from 00.00 – 16.59 is IDR 1.300 every 60 seconds.

Figure 6: Student Answers for the Second Question

Based on Figure 6, the students can draw and describe the graph that shows the relationship between the costs incurred when calling with non-package rates and with the length of time calling.

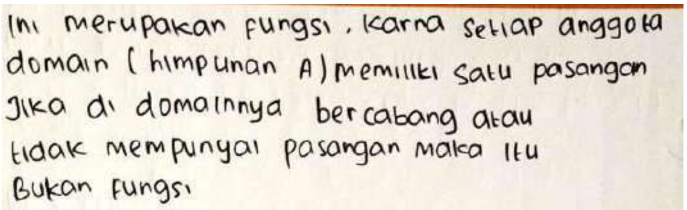
Next is the question of whether the relationship between time and the price of non-packaged calls a function is. The students' answers are shown in Figure 7.



Ya. Hubungan antara waktu dengan harga nelpon non paket
merupakan fungsi karena memasangkan setiap anggota waktu/ menit
satu yg satu ke satu anggota harga non paket dan setiap
anggota waktu /menit mendapat satu pasangan di anggota harga
non paket.

Translate in English:
Yes, the relationship between time and non-package prices is a function because each time member pairs one at a time to a non-packaged price member, the time member gets one pair to a non-packaged price member.

(a)



Ini merupakan fungsi, karna setiap anggota domain (himpunan A) memiliki satu pasangan jika di domainnya bercabang atau tidak mempunyai pasangan maka itu Bukan fungsi

Translate in English:
This is a function because each member of the domain (set A) has one pair. If the domains are forked or have no pairs then it's not a function

(b)

Figure 7: Student Answer for Third Question

Based on Figure 7, the students can determine that the relationship between call time and price is a function. In Figure 7 (b), the students explained the comparison between functions and non-functions. From Figure 7(a) dan 7(b), the students can analyze and evaluate to decide the right answer.

DISCUSSION AND CONCLUSIONS

Akker, et al. (2013) stated that a product is said to be valid if the product developed is based on knowledge or science (content validity) and if the product is consistent with each other or is logical to design (construct validity). Based on the research results, the expert stated that the student worksheet was in accordance with the HOTS level based on the Bloom's taxonomy and the PISA framework. This shows that the student worksheet is valid in terms of content. Then, the expert stated that the student worksheet was in accordance with the 2013 curriculum and that the material chosen was in accordance with the material in class VIII. This shows that the student worksheet is valid in terms of constructs.

Student worksheets are said to be practical if they can be used and are easy to use (Akker, et al., 2013). Based on the development of the prototyping stage from one-to-one to field-tests, students' answers show that they can work on the given student worksheet and are not confused by the questions or information in the student worksheet. This shows that the designed student worksheet is practical.

Students' answers show that the students can relate the information in the table and compare the price of calling with talkmania packages and non-package prices. Then, the students make decisions about which package should be chosen. There are some versions of student answer, there are students who choose the talkmania package as calling with the talkmania package is cheaper than the non-packaged one. However, there is student prefer to call with non-package with the reason that the process is faster, they pay attention to the sentence "there is an important business right away".

The students' answer corresponds to the description of their ability to analyze and evaluate (Anderson and Krathwohl, 2001; McNeil, 2011; Widana, 2017). Judging from the level in the PISA framework, the students' answer show that they can make models from real situations, make assumptions, are able to interpret tables or available information and relate the information provided in tables and questions in the student worksheet to determine answers. This includes capabilities at level 4 in the PISA framework (OECD, 2018). Then, students are also able to determine which calling package decision should be chosen by including their respective arguments. This includes capabilities at level 5 of the PISA framework.

Then, the students can draw a graph that shows the relationship between calling time and costs incurred. To draw a graph, students analyzed the available information about prices and calling times consisting of days and hours. They evaluated the relationship between calling time and the costs incurred for calling with non-package. Then, the students create a new graph. The student's answer about the graph corresponds to the description of the ability to analyze, evaluate, and create (Anderson and Krathwohl, 2001; McNeil, 2011; Widana, 2017). Based on the level in the PISA framework, the students' ability to analyze to draw graphs is included in level 6 because it is seen that the students can create concepts between the x-axis and y-axis, utilize information, and apply insights to develop new strategies (OECD, 2018). It can be seen in students answer that the students present graphs and can read clear information from the displayed graphs.

The next answer show that students analyzed answers by paying attention to the definition of a function and relating it to the condition of the relationship between non-packet calling time and the cost. Then, the students evaluated it by comparing it with the definition of a non-function. The students' answers about the function correspond to the description of the ability to analyze and evaluate (Anderson and Krathwohl, 2001; McNeil, 2011; Widana, 2017). Based on the PISA framework, the students meet ability levels 4 and 5, because it appears that students can model real situations, determine assumptions, interpret the relationship between calling time and cost, relate existing information, and to decide that the relationship between calling time and non-package calling costs is a function with the correct arguments.

Based on students' answers, the designed student worksheet is able to improve students' HOTS abilities. This is in accordance with the jumping task criteria, namely the task given is challenging and requires the HOTS ability to complete it (Saito, 2015; Hobri, 2016).

Based on the student worksheet development, a valid and practical student worksheet has been produced. The student worksheet is declared valid in terms of content because it is in accordance with level HOTS in taxonomy bloom and PISA framework. The student worksheet is valid in terms of constructs because it is in accordance with the material contained in the 2013 curriculum for eight graders. The student worksheet is also practical which shows that it can be used by students, students understand the purpose of activities or problems in the student worksheet, and support student ability namely HOTS.

Suggestion for future learning that this student worksheet can be used for classroom learning to train students' HOTS. For further research, it is possible to develop student worksheets with different materials so that more jumping task-based student worksheet are produced.

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Heuristic Method for Minimizing Distance without using Calculus and Its Significance

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Abstract: A very common Applied Optimization Problem in Calculus deals with minimizing a distance given certain constraints, using Calculus, the general method for solving these problems is to find a function formula for the distance that we need to minimize, take the derivative of the distance function, set it equal to zero, and solve for the input value, that should most of the time, lead to the optimal solution. In this article we provide alternatives for solving some Applied Optimization Problems related to minimizing a distance, without the use of the Derivative from Calculus, and instead, using a "Reflection Principle" based on symmetry, Geometric properties, and heuristic methods.

INTRODUCTION

Let's be clear, it is not our intention to diminish the importance of Calculus as a fundamental tool for solving optimization problems, our purpose is to show that using our alternatives will facilitate uncovering Algebraic-Geometric properties and patterns, that are often overseen by students when using Derivatives to solve applied optimization problems. This properties and patterns when put together can be associated to real-life events, this is pedagogically important since it can help instructors better explain fundamental ideas and principles when teaching Calculus.

Over years teaching Calculus, I realized that students understand the principles of Calculus better when they can visualize the abstract fundamental ideas given in our Lectures as real-life events, this is because, as instructors, building upon something that our students already understand is a pedagogical advantage, based on this, I decided to investigate the relationship between the way light travels as an electromagnetic wave in our universe (see [2]) and the minimization of a distance function given some constraints, which is a very common problem when teaching Applied Optimization Problems in Calculus. Additionally, I investigated the relationship between a billiard ball's path before and after it hits an edge (crease) on an idealized frictionless billiard table (see [1] and [4]), it turns out the two real-life events mentioned before are strongly related to the "shortest path" problem in Calculus and related to each other as well.

METHOD

Since our intention is not to ignore Calculus as a fundamental tool for solving applied optimization problems, let's start by solving a classical basic problem about minimizing a distance using Calculus, then we will solve the same problem using our method, and gradually we will cover more challenging cases.

Problem 1

Paul's house is located at point A, the farm of his grandmother is located at point B, there is a river as shown in the figure below, every morning, Paul needs to go to the river, get water and bring it to his grandmother's farm, what is the length of the shortest path Paul should follow? See figure 1 below.

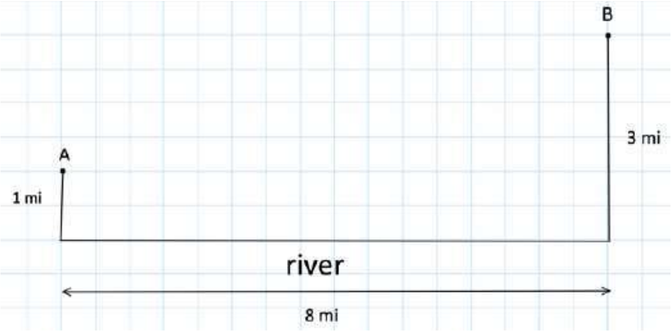

Figure 1: Classical problem in Calculus

A common solution to this classical problem using Calculus goes like this, let C be the point where Paul will reach the river, let x be the distance between O and C, see figure 2 below.

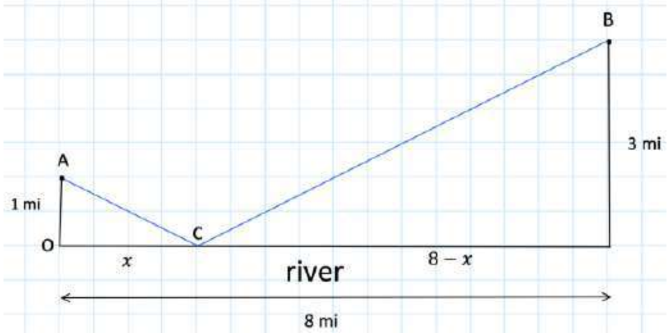

Figure 2: Common solution using Calculus

then we must minimize the distance $AC + CB$, equivalently, we must minimize the function:

$$f(x) = \sqrt{1 + x^2} + \sqrt{(8 - x)^2 + 9} \qquad (1)$$

Taking the Derivative of $f(x)$, setting it equal to zero, and solving for x we obtain:

$$\frac{1}{2\sqrt{1+x^2}}\,2x+\frac{1}{2\sqrt{(8-x)^2+9}}\,2(8-x)(-1)=0 \tag{2}$$

$$\frac{x}{\sqrt{1+x^2}}=\frac{8-x}{\sqrt{(8-x)^2+9}} \tag{3}$$

$$\frac{x^2}{1+x^2}=\frac{(8-x)^2}{(8-x)^2+9} \tag{4}$$

$$x^2(8-x)^2+9x^2=(8-x)^2+x^2(8-x)^2 \tag{5}$$

$$9x^2=(8-x)^2 \tag{6}$$

$$3x=8-x \tag{7}$$

$$x=2 \tag{8}$$

For $0 \le x \le 8$, $f(x)$ is a continuous function.

The extreme values are $x=0$ and $x=8$, evaluating $f(x)$ at the extreme values and the critical point we obtain:

$$f(0)=\sqrt{1+0^2}+\sqrt{(8-0)^2+9}=9.54 miles. \tag{9}$$

$$f(2)=\sqrt{1+2^2}+\sqrt{(8-2)^2+9}=8.94\,miles. \tag{10}$$

$$f(8)=\sqrt{1+8^2}+\sqrt{(8-8)^2+9}=11.06\,miles. \tag{11}$$

The evaluations above show that $x=2$ is a critical point associated to a minimum value of $f(x)$, then we conclude that shortest path is $A \to C \to B$ having as its length $8.94\,miles$.

A graphical verification using OER DESMOS graphing calculator is shown below in Figure 3.

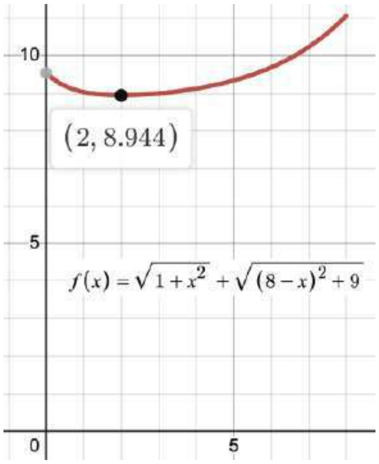

Figure 3: Graphical verification of the shortest path

The Reflection Principle

The minimum distance between two points is the length of a straight path that connects them, if we want to go from a point A to a point B via a sequence of straight paths touching once a line L as shown below, then ideally we may start at point A' the reflection of point A about the line L, this assumption can be made because moving towards line L from A is equivalent, in terms of distance covered, to move towards line L from A' given that the triangles formed above and below L are always congruent, this in turn leads to the equality of the reference angles formed by the paths and the line L, which makes the connection with the principle of reflection in physics, which shows that if a beam of light is aimed at a mirror positioned at line L, it will generate equal angles of incidence an reflection.

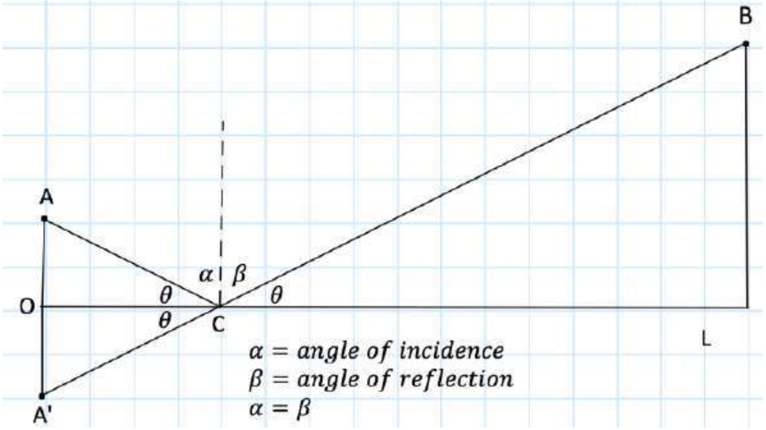

Figure 4: Explanation of The Reflection Principle

Now, we present an alternative solution to problem 1 that does not use Calculus, for this purpose we will use "The Reflection Principle".

Since Paul must reach the river anyway, following "the reflection principle" ideally we may assume that he can start his path at point A' the reflection of point A over the river (over the horizontal line that represents the river), now the shortest path to go from point A' to B is clearly the segment $A'B$, so where $A'B$ intersects the river that must be the point C, also since Triangle AOC is congruent to Triangle $A'OC$ then the path $A \to C \to B$ is the shortest path Paul should follow, see figure 5 below:

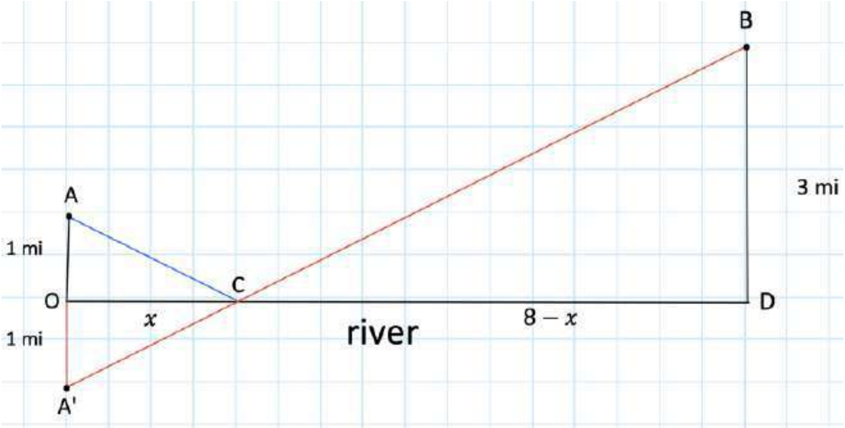

Figure 5: First application of The Reflection Principle.

Now, to determine how far is C from O, since triangle OCA' is similar to triangle BCD then

$$\frac{1}{x} = \frac{3}{8 - x} \tag{12}$$

$$3x = 8 - x \tag{13}$$

$$x = 2 \tag{14}$$

Which leads to the shortest path $A \to C \to B$ having as its length:

$$\sqrt{1 + 2^2} + \sqrt{(8 - 2)^2 + 9} = 8.94\, miles. \tag{15}$$

Our alternative solution using "The Reflection Principle" has the following Educational and Computational advantages:

- It is Geometrically constructive; one can draw the shortest path using only straight edge and compass, this is particularly important because as Calculus instructors we can ask our students to build the "shortest path" as an in-class activity.
- It is algebraically simple, basically, we need to solve a rational equation which reduces itself to a linear equation, this means that this exercise could be given to students in a Precalculus class as a special group project.
- It shows that the reference angles formed by the path components with the horizontal line that represents the river are the same.
- It can be extended via "The Reflection Principle" to the solution of more challenging scenarios as we will show later.

Proposed Educational Activity

1. Give students problem 1 (or a similar problem) for them to solve individually or in groups.
2. Ask students to solve it using Calculus.
3. Explain the idea behind "The Reflection Principle".
4. Ask students to solve the same problem they solved in part 1 using "The Reflection Principle".
5. Ask students to draw "The minimum path" solution using only straight edge and compass.
6. Ask students to verify that the reference angles formed by consecutive paths with the horizontal line of reference are the same.
7. Ask students to measure the total length of the minimum path and compare it to the result they got in part 2.
8. Ask students to compare both methods and share comments about the activity.

Let's consider now a more challenging scenario where in order to go from point A to point B it is needed to touch once two perpendicular lines.

Problem 2

Suppose you have points $A(1,6)$ and $B(5,2)$ on an xy coordinate system, find the length of the shortest path to go from A to B touching first y-axis once and then x-axis once, see figure 6 below.

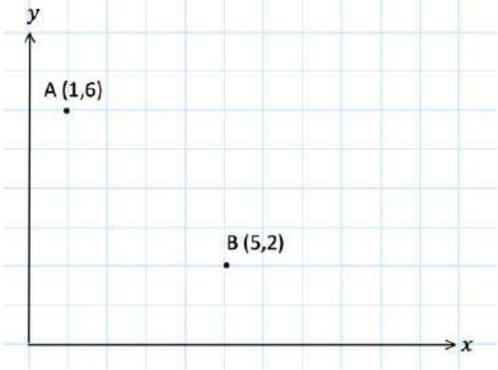

Figure 6: First extension of our basic problem.

Using our "Reflection Principle", ideally we may start at A' and end at B', which are the reflections of points A and B about y-axis and x-axis, respectively, clearly the shortest path between A' and B' is the segment $A'B'$, where $A'B'$ intersects the y-axis let's call that point C and where $A'B'$ intersects x-axis let's call that point D, then $A \rightarrow C \rightarrow D \rightarrow B$ is the shortest path, see figure 7 below.

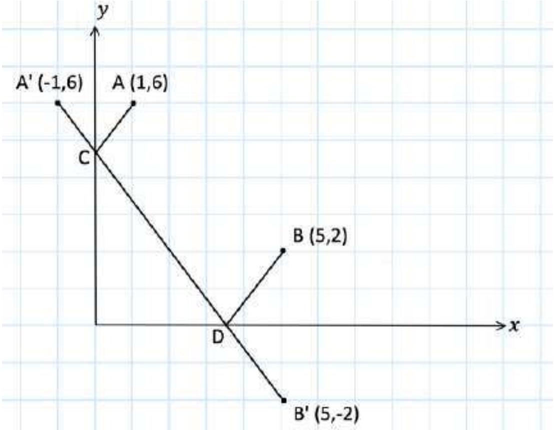

Figure 7: Second application of The Reflection Principle.

The equation of the line passing through the points $A'(-1,6)$ and $B'(5,-2)$ is $y = -\frac{4}{3}x + \frac{14}{3}$ with y-intercept $C = \left(0, \frac{14}{3}\right)$ and x-intercept $D = \left(\frac{7}{2}, 0\right)$ this determines the shortest path, then the length of the shortest path $A \rightarrow C \rightarrow D \rightarrow B$ can be calculated using the distance formula for the points A' and B' as shown below:

$$\sqrt{(5-(-1))^2 + (-2-6)^2} = 10\, units. \qquad (16)$$

Remarks

- Solving Problem 2 using Calculus, may in general, require the minimization of a distance objective function that after being differentiated, could lead to a complicated algebraic equation to solve involving radicals, since we would have to deal with the sum of distance formulas as shown in the first solution of Problem 1.
- Our solution of problem 2 only used knowledge about equations of lines in an xy coordinate system and location of x-intercepts and y-intercepts, which makes problem 2 suitable for a group project once "The Reflection Principle" is explained.

How our solutions can be associated to real-life events

After solving problem 1 and problem 2 using "The Reflection Principle" a pattern emerges and a natural question comes to our minds, are these "shortest paths" related to any events in real life? and the answer to this question is yes, when we consider the path of light as an electromagnetic wave traveling in our universe, it turns out since light travels minimizing time at a constant speed (see [3]) then if we consider the pairwise perpendicular lines in problems 1 and 2 as mirrors, a beam of light from a flash light using the angles we found in the solutions of problems 1 and 2 will follow the same paths as we found for our solutions in problems 1 and 2, this can be confirm experimentally and also using the free OER applet in [2] https://phet.colorado.edu/sims/html/bending-light/latest/bending-light_en.html

Additionally, if we think of the game of billiards (see [1] and [4]) If we hit a billiard ball with the stick, using angles found in our solutions to problems 1 and 2, the paths followed by the billiard ball before and after hitting the edges of the billiard table will be the same as the "shortest paths" that we found in our solutions to problems 1 and 2, this of course, assuming a frictionless billiard table.

There is a way to link how light travels a as wave and the way a billiard ball hits and bounces off a crease:

A ball hitting a crease and bouncing off, acts as a wave of light reflecting off a mirror as seen in figure 8 below:

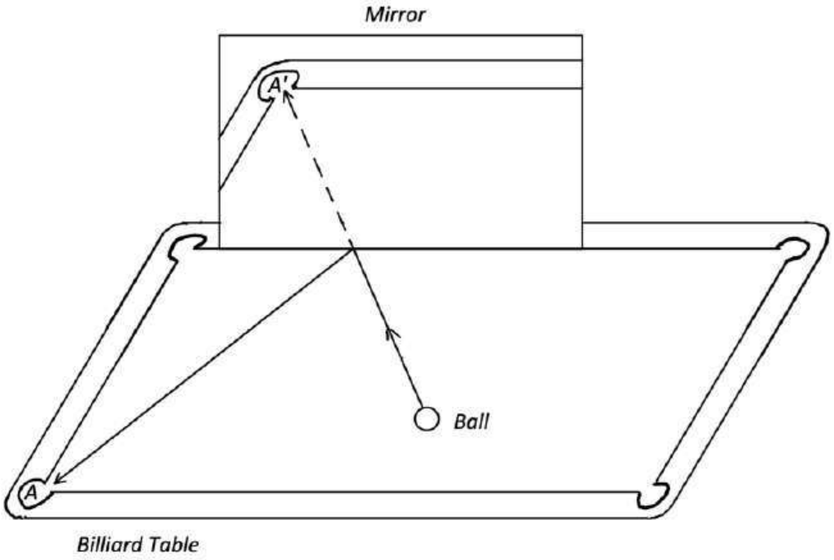

Figure 8: Explanation of the link between the paths of a ray of light and a billiard ball.

Let's consider now a step further challenging scenario where in order to go from point A to point B it is needed to touch once a sequence of pairwise perpendicular lines.

Problem 3

Suppose you have points $A(2,4)$ and $B(6,3)$ on an xy coordinate system, find the length of the shortest path to go from A to B touching first y-axis once, then x-axis once, and finally the vertical line to x-axis y'-axis once, see figure 9 below.

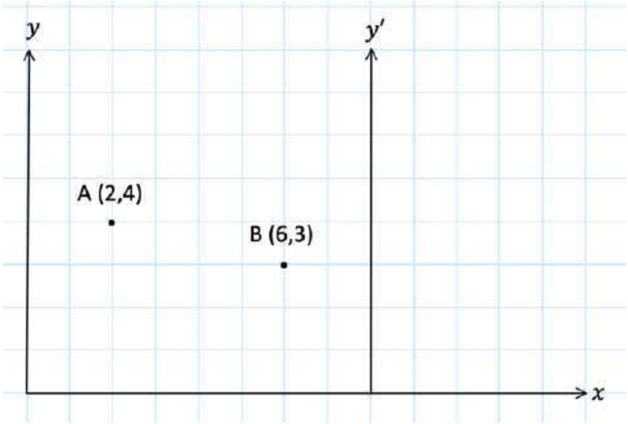

Figure 9: Second extension of our basic problem.

Using our "Reflection Principle", we may ideally start at A' and end at B', which are the reflections of points A and B about y-axis and y'-axis respectively, this in turn is equivalent to ideally starting at A' and ending at B'', where B'' is the reflection of B' about the x-axis. The point where $A'B''$

intersects y-axis let's call it C, and the point where $A'B''$ intersects x-axis let's call it D, the point where DB' intersects y'-axis let's call it E, then the path $A \to C \to D \to E \to B$ is the shortest path, to go from A to B touching once y-axis, x-axis and y'-axis respectively see figure 10 below:

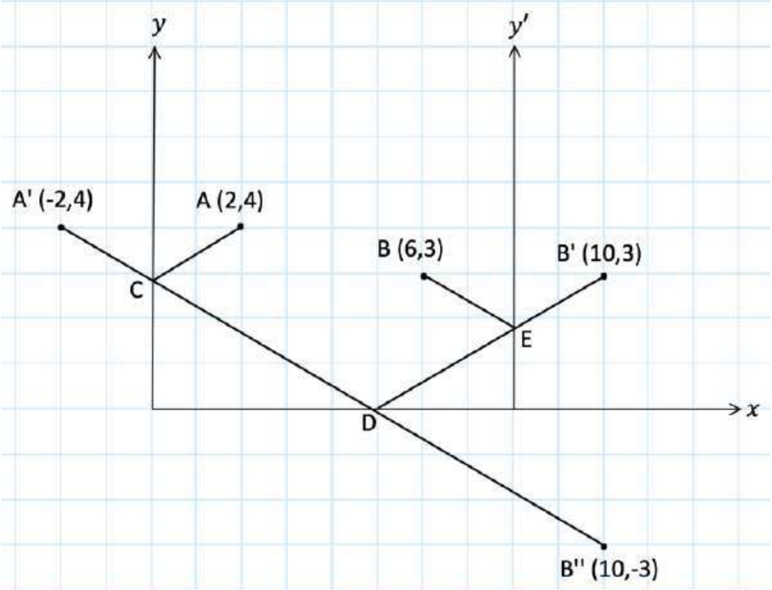

Figure 10: Third application of The Reflection Principle

The equation of the line passing through the points $A'(-2{,}4)$ and $B''(10,-3)$ is $y = -\frac{7}{12}x + \frac{17}{6}$ with y-intercept $C = \left(0,\frac{17}{6}\right)$ and x-intercept $D = \left(\frac{34}{7},0\right)$, the intersection of the line containing the points D and B' $y = \frac{7}{12}x - \frac{17}{6}$ with the vertical line $x = 8$ is the point $E = \left(8,\frac{11}{6}\right)$ this determines the shortest path, the length of the shortest path $A \to C \to D \to E \to B$ can be calculated using the distance formula for the points A' and B" as shown below:

$$\sqrt{(10-(-2))^2 + (-3-4)^2} = 13.89\, units. \tag{17}$$

CONCLUSION

The methods shown to solve the problems presented without the use of Calculus, using "The Reflection Principle" have the following Educational and Computational overall advantages:

All our solutions for problems 1, 2, and 3 using "The Reflection Principle" are constructive, which means that we can geometrically draw the "shortest paths" using only straight edge and compass, this could be the basis for in-class activities, which ultimately, would give our students a "real-world sense" of what the solutions to the "shortest path" problems should be.

The lengths of the "shortest paths" shown in problems 1, 2, and 3 are similar to the paths that a beam of light from a flashlight would follow if we considered the horizontal and vertical lines as mirrors. This would allow students to confirm a principle that comes from physics, which states that light, in our universe, travels following "the shortest path" (see [2] and [3]).

A confirmation (an experimental test) for the "shortest paths" can be performed using a billiard table and a billiard ball, considering the angles found in our constructions of the "shortest paths", to hit a billiard ball towards the edges (creases) of the table and trace how they bounce off, as seen in [1] and [4].

Our method, based on "The Reflection Principle", admits a natural extension to solve more challenging problems related to finding "shortest paths" subject to given constraints, this was shown going gradually through problems 1, 2, and 3.

When finding the "shortest paths", the path components that are not consecutives are parallel, this is a consequence of the angles of "incidence" and "reflection" (see [1] and [4]) being the same when finding the path of the shortest length associated to light traveling as a wave or a billiard ball being hit by a stick on a billiard table towards the edges.

When solving Applied Optimization Problems like problem 1, 2 and 3 our method based on "The Reflection Principle" uncovers properties, patterns, and associations with real-life events, which are overseen by students when they "blindly" follow the algorithm: "take derivative, set it equal to zero, and solve for the variable".

All the above gives Calculus Instructors material to make up activities and projects for students to go over, individually or in groups, so they can see how theoretical principles in Mathematics can be tested in real life which serves as an instructional motivation, especially for undergraduate students who often ask for real-life examples associated to the abstract knowledge given to them in the lectures. For online classes in [2] there is an free OER applet https://phet.colorado.edu/sims/html/bending-light/latest/bending-light_en.html that students can use virtually, to verify the angles and "shortest paths" that we have shown as solutions to problems 1, 2, and 3.

Note: As of now because of the pandemic-related teaching limitations, it is not possible for me to fully implement the activities suggested in this article. Once the pandemic is completely over, I will use my Calculus class for an implementation of the suggested activities. The results could be the basis for a new article.

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[1] Jacobs, H., Mathematics A Human Endeavor. *Third Edition W. H. Freeman (1994)*

[2] Erenso, Daniel., Reflection and refraction of light. *IOP Publishing Ltd (2021)* https://iopscience.iop.org/book/978-0-7503-3715-1/chapter/bk978-0-7503-3715-1ch1

[3] *Moses Fayngold (2008). Special Relativity and How it Works (illustrated ed.). John Wiley & Sons. ISBN 978-3-527-40607-4.*

[4] The Geometric Viewpoint, Periodic Billiard Paths. https://web.colby.edu/thegeometricviewpoint/2014/04/25/periodic-billiard-paths/

The Problem Corner

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The Purpose of **The Problem Corner** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Hello Problem Solvers, I got solutions to **Problem 6** and to **Problem 7** and I am happy to inform that they were correct, interesting, and ingenuous. By posting what I considered to be the best solutions, I hope to enrich and enhance the mathematical knowledge of our international community.

Solutions to **Problems** from the Previous Issue

Interesting algebra problem.

Proposed by Ivan Retamoso from Borough of Manhattan Community College, City university of New York, USA

Problem 6

Given that m and n are real numbers, without solving the equation determine how many real roots the following equation has:

$$(x-m-n)(x-m)=1$$

Solution to problem 6

by Jesse Wolf, Borough of Manhattan Community College, City university of New York, USA.

This solution is concise and utilizes the property of the discriminant of a quadratic equation in relation to the number of real roots of the equation.

A quadratic equation has 2 (distinct) real roots when the discriminant $D = b^2 - 4ac$ is greater than 0.

$(x - m - n)(x - m) - 1 = 0.$

$(1)x^2 + (-2m - n)x + (m^2 + nm - 1) = 0.$

a, b, c are the respective quantities in parentheses.

$D = n^2 + 4 > 0$ for all real m & n.

So, 2 distinct real roots for all m & n.

Solution to problem 6

by Aradhana Kumari, Borough of Manhattan Community College, City university of New York, USA.

For the readers who like complete solutions without skipping steps, here is a solution showing all the details, it is also based on the property of the discriminant of a quadratic equation in relation to the number of real roots of the equation.

First notice that $(x - m - n)(x - m)$ -1

is a second-degree polynomial in the variable x and hence it will have 2 zeros.

(by fundamental theorem of algebra).

Hence either both solutions are real or both solutions are complex.

(Since if z is a complex solution for a polynomial and its conjugate will also be solution of the same polynomial).

Consider the given equation

$(x - m - n)(x - m) = 1$

$(x - m - n)(x - m) - 1 = 0$

$x^2 - xm - xn - xm + m^2 + mn - 1 = 0$

$x^2 - x(2m + n) + m^2 + mn - 1 = 0$

Hence the discrimination is

$(2m + n)^2 - 4(m^2 + mn - 1)$

$= 4m^2 + n^2 + 4mn - 4m^2 - 4mn + 4$

$= n^2 + 4 > 0$

Hence the both roots of the above equation $(x - m - n)(x - m) = 1$

are real.

Interesting applied optimization problem.

Problem 7

Proposed by Ivan Retamoso, BMCC, USA.

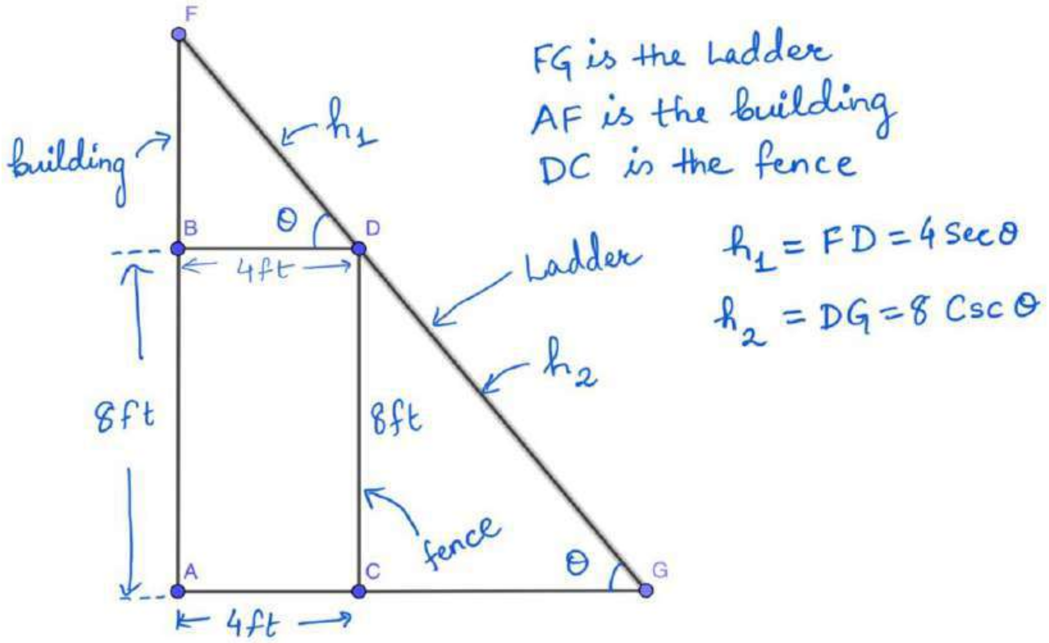
A fence 8ft tall runs parallel to a tall building at a distance of 4ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

Solution to problem 7

by Aradhana Kumari, Borough of Manhattan Community College, City university of New York, USA.

When solving Applied Optimization Problems in Calculus, often we underestimate the power of working with angles, this solution shows that a complicated problem becomes simple by expressing the objective function solely in terms of an angle, the second derivative test is used to identify the critical point as a minimum, the final solution is expressed in exact form.

As per question we have the below diagram



$h_1 = 4 \sec\theta$

$h_2 = 8 \csc\theta$

As per question, length of the ladder $=$ FG

$= h_1 + h_2$

$= 4 \sec\theta + 8 \csc\theta$

Let us assume $f(\theta) = 4 \sec\theta + 8 \csc\theta$

Then

$f'(\theta) = 4 \sec\theta \tan\theta - 8 \csc\theta \cot\theta$

For maxima or minima we equate the derivative of f to zero.

Hence, we have

$4 \sec\theta \tan\theta - 8 \csc\theta \cot\theta = 0$

$4 (\sec\theta \tan\theta - 2 \csc\theta \cot\theta) = 0$

$\sec\theta \tan\theta - 2 \csc\theta \cot\theta = 0$

$$\left(\frac{1}{\cos\theta}\times\frac{\sin\theta}{\cos\theta}\right)-\left(\frac{2}{\sin\theta}\times\frac{\cos\theta}{\sin\theta}\right)=0$$

$$\frac{\sin\theta}{(\cos\theta)^2}-\frac{2\ \cos\theta}{(\sin\theta)^2}\ \ =0$$

$$\frac{(\sin\theta)^3-2\ (\cos\theta)^3}{(\cos\theta)^2\ (\sin\theta)^2}=0$$

Therefore

$$(\sin\theta)^3-\ 2\ (\cos\theta)^3=0$$

$$(\sin\theta)^3\ =\ 2\ (\cos\theta)^3$$

$$(\tan\theta)^3\ =2$$

$$\tan\theta=\sqrt[3]{2}$$

Therefore

$$\theta=\tan^{-1}\left(\sqrt[3]{2}\right)$$

$$\sim 51.56°$$

Now we must check whether $\theta=\tan^{-1}\left(\sqrt[3]{2}\right)\quad\sim 51.56°$ is a maxima or minima.

For this we will find the second derivative of f.

Recall

$$\mathrm{f}\,(\theta)\ =4\sec\theta\ +8\csc\theta$$

$$\mathrm{f}\,'(\theta)=4\sec\theta\ \tan\theta-8\csc\theta\cot\theta=\left(\frac{4}{\cos\theta}\times\frac{\sin\theta}{\cos\theta}\right)-\left(\frac{8}{\sin\theta}\times\frac{\cos\theta}{\sin\theta}\right)$$

$$\mathrm{f}\,''\,(\theta)=\frac{[4\cos\theta\ (\cos\theta)^2)-\{4\ \sin\theta\times 2\times\cos\theta\times(-\sin\theta)\}]}{(\cos\theta)^4}-\frac{8\,[\{\,(-\sin\theta)\times(\sin\theta)^2\}-\{\cos\theta\times 2\sin\theta\times\cos\theta\}]}{(\sin\theta)^4}$$

$$=\frac{4(\cos\theta)^3+8\,(\sin\theta)^2\cos\theta}{(\cos\theta)^4}+\frac{8\,(\sin\theta)^3+16(\cos\theta)^2\ \sin\theta}{(\sin\theta)^4}$$

Since $\theta=\tan^{-1}\left(\sqrt[3]{2}\right)\quad\sim 51.56°$

the angle 51.56° is in the first quadrant. We know that both the $\sin\theta$ and $\cos\theta$ are positive in the first quadrant

Hence

f ’’ $(\theta) = \frac{4(\cos\theta)^4+8(\sin\theta)^2\cos\theta}{(\cos\theta)^4} + \frac{8(\sin\theta)^3+16(\cos\theta)^2\ \sin\theta}{(\sin\theta)^4}$ is positive.

Therefore $\theta = \tan^{-1}(\sqrt[3]{2})\ \ \sim 51.56°$ is a point of minima for the function f.

Hence the length of the shortest ladder is $h_1 + h_2$

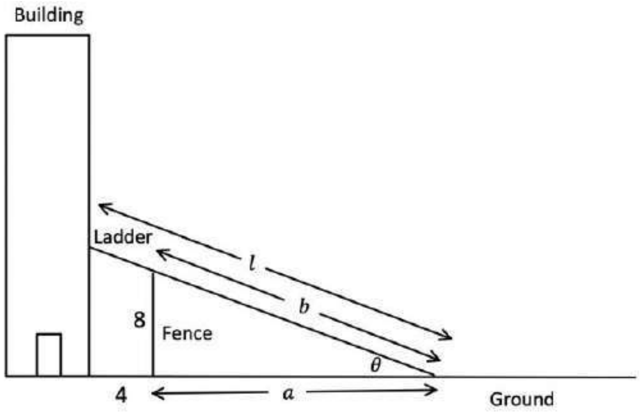
$$= \mathbf{4\sec\theta + 8\csc\theta}$$

$$= \mathbf{4\sec\left(\tan^{-1}(\sqrt[3]{2})\right) + 8\csc\left(\tan^{-1}(\sqrt[3]{2})\right)}$$

Solution to problem 7

by Ivan Retamoso (the proposer), Borough of Manhattan Community College, City university of New York, USA.

This solution confirms the advantage of choosing to work with angles to represent the function that needs to be minimized, for identifying the critical point as a minimum this solution uses the first derivative test, the final solution is computed as a decimal number rounded to two decimal places.



$\frac{a}{8} = cot\theta$ then $a = 8cot\theta$

$\frac{b}{8} = csc\theta$ then $a = 8csc\theta$

By similarity of right triangles

$$\frac{L}{8csc\theta} = \frac{4+8cot\theta}{8cot\theta}$$

Then

$L = 4sec\theta + 8csc\theta$

Setting $\frac{dL}{d\theta} = 0$ and solving for θ then

$4sec\theta tan\theta - 8csc\theta cot\theta = 0$

$4sec\theta tan\theta = 8csc\theta cot\theta$

$tan^3\theta = 2$

$tan\theta = \sqrt[3]{2}$

$\theta = \arctan(\sqrt[3]{2})$

$\theta = 51.56°$

Notice that $0° < \theta < 90°$ then $tan\theta$ is strictly increasing.

For $0° < \theta < 51.56°$ then $\frac{dL}{d\theta} = 4sec\theta tan\theta - 8csc\theta cot\theta < 0$

For $51.56° < \theta < 90°$ then $\frac{dL}{d\theta} = 4sec\theta tan\theta - 8csc\theta cot\theta > 0$

Then L achieves its minimum at $\theta = 51.56°$
Then $L_{min} = 4sec51.56° + 8\csc 51.56°$

Then $L_{min} = 16.65\ ft.$

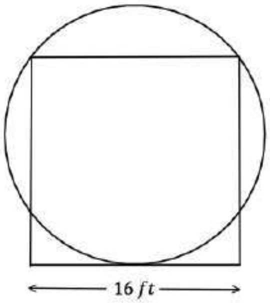
Dear Problem Solvers,

I really hope that you enjoyed and more importantly learned something new by solving problem 6 and problem 7, time to move forward so below are the next two problems.

Problem 8

Proposed by Ivan Retamoso, BMCC, USA.

In the figure below find the ratio between the perimeter of the circle (circumference) and the perimeter of the square in exact form.



Problem 9

Proposed by Ivan Retamoso, BMCC, USA.

It is needed to construct a rain gutter from a metal sheet of width $36\ cm$ by bending up one-third of the sheet on each side through an angle α. Find the value of α such that the gutter will carry the maximum amount of water.