TR52 vol 16 no 2 of the Mathematics Teaching-Research Journal

Editorial from Mónica Arnal-Palacián, Didactics Editor of MTRJ

The Spring issue of the Mathematics Teaching-Research Journal arrives with 11 new articles on teaching and research, alongside the usual The Problem Corner section. Within them, readers will find studies on mathematics education at different levels: university teaching, secondary education, and primary education. Furthermore, and considering various methodological approaches, works are presented in which students, teacher trainees, and active teachers have been sampled.

The Mathematics Teaching-Research Journal aims to continue giving voice to both educators and researchers from different parts of the world. This time, seven countries (Canada, Colombia, Ecuador, India, Indonesia, Morocco and Spain), from four different continents (America, Asia, Europe and Africa) are represented. We hope that each of the studies included in this issue will be of interest to experienced and novice teachers, as well as researchers in mathematics education.

Finally, from the editorial team, we want to express our gratitude for the trust placed in those authors who decided to submit their manuscripts to the journal, making Mathematics Teaching-Research Journal an international reference in mathematics education research.

Empirical Study of Mathematical Investigation Skill on Graph Theory
Karunia Eka Lestari, Mokhammad Ridwan Yudhanegara (Indonesia)
This article analyzes thinking behavior regarding Graph Theory by associating prior knowledge and research in mathematics with second-year Mathematics students in Indonesia.

Investigating Pre-Service Primary School Teachers' Difficulties in Solving Context-Based Mathematics Problems: An Error Analysis
Andi Harpeni Dewantara, Edi Istiyono, Heri Retnawati, Slamet Suyanto (Indonesia)
This manuscript presents students' difficulties in solving context-based problems, with a focus on

pre-service Primary Education teachers. For this purpose, the NEA error classification framework (Newman's Error Analysis) is utilized.

From Informal to Formal Proof in Geometry: a Preliminary Study of Scaffolding-based Interventions for Improving Preservice Teachers' Level of Proof

Sugi Hartono, Tatag Yuli Eko Siswono, Rooselyna Ekawati (Indonesia)

Hartono and colleagues present a design research in which they analyze the level of proof understanding among pre-service mathematics teachers using scaffolding tasks for the proof of the Triangle Theorem.

Exploring Learning Difficulties in Convergence of Numerical Sequences in Morocco: An Error Analysis Study

El Mahdi Lamaizi, Larbi Zraoula,Bouazza El Wahbi (Morocco)

This article addresses the learning difficulties of sequence limits with second-year high school students in Morocco. This occurs after students have been taught about different types of numerical sequences across various educational levels.

Analysis of the Strategies Used by High School Students in Solving Area Problems: A Case Study

Deyner Bolaño, Darwin Peña-González, Roberto Torres (Colombia)

Bolaño and colleagues present in this article activities based on Problem-Based Learning (PBL) methodology with high school students to solve problems related to plane figures.

Direct and Indirect Effect of Self-Efficacy, Anxiety and Interest on Algebraic Problem-Solving Achievement

Imdad Ali, Samiran Das (India)

This article examines the effect of self-efficacy, math anxiety, and interest, all of which are affective constructs, on performance in solving algebraic problems among Secondary Education students.

A case study of proving by students with different levels of mathematical giftedness

María J. Beltrán-Meneu, Rafael Ramírez-Uclés, Juan M. Ribera-Puchades, Angel Gutiérrez, Adela Jaime (Spain)

This research was conducted in Spain with gifted students. Through a didactic proposal involving various types of activities requiring students to provide demonstrations, the relationship between consistency and the level of giftedness is analyzed.

The use of technology and academic performance in the teaching of Mathematics in secondary education

José Fernando Mendoza-Rodríguez1, Víctor Manuel Caranqui-Sánchez (Ecuador)

The authors analyze the use of technology and academic performance in mathematics education, taking a sample of Secondary Education students in Ecuador.

Characterization of Primary School Students' Perceptions in Understanding Negative Integer

M. Qoyum Zuhriawan, Purwanto, Susiswo, Sukoriyanto, Siti Faizah (Indonesia)

In this study, the authors analyze the productions of Primary Education students, particularly focusing on the writing of negative numbers on the number line, resulting in three characteristics of thought.

Development of a Traditional Game-Based Computational Thinking Supplementary textbook for Elementary School Students

If Only Dia, Zetra Hainul Putra, Gustimal Witri, Dahnilsyah, Ayman Aljarrah (Indonesia and Canada)

In this manuscript, the authors develop a complementary textbook to the usual classroom one, aiming to address computational thinking linked to traditional games. The intended audience is Primary Education students in Indonesia.

Teachers' Efforts to Promote Students' Mathematical Thinking Using Ethnomathematics Approach

S. W. Danoebroto, Suyata, Jailani (Indonesia)

The final research study included in this editorial address mathematical thinking skills through local culture. All of this is based on case studies involving three active Secondary Education teachers in Indonesia.

The Problem Corner

Ivan Retamoso (USA)

As in previous issues of MTRJ, Professor Ivan Retamoso proposes two new problems and presents a selection of solutions received by the journal for the problems from previous issues.

Empirical Study of Mathematical Investigation Skill on Graph Theory

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Abstract: *Graph theory allows the student to work on problems that require imagination, intuition, systematic exploration, conjecturing, and reasoning. It implies that mathematical investigation skill is essential to be proficient in Graph Theory. In this study, we conduct empirical research that deals with associational research. There were 97 students selected purposively from sophomore students in the Discrete Mathematics course offered by one of the mathematics education departments in Indonesia. The empirical evidence was analyzed to explain students' thinking behavior on Graph Theory by discovering the association structure between prior knowledge and mathematical investigation skill, then visually depicting its association using the k-means clustering procedure and correspondence analysis. Since in our department, we expect certain prior skills, then this visualization could be used as self-reflection for our department, whether we have gained the results as expected to strengthen certain investigation approaches. Generally, the study concludes some findings that provide novelty and open issues for future research to develop a learning environment supporting mathematical investigation activities.*

Keywords: Correspondence analysis, graph theory, mathematical investigation

INTRODUCTION

Over the last two decades, the mathematics education society has strongly emphasized the essential of investigation-based learning environments (Da Ponte & Pedro, 2007, Leikin, 2014, Yerushalmy, 2009). It suggests that mathematical investigation is an essential skill from an educational point of view—without exception from the Graph Theory point of view as one subject in the Discrete Mathematics course offered by the mathematics education study program in Indonesia. Considering Graph Theory allows the student to work on problems that require imagination, intuition, systematic exploration, conjecturing, and reasoning.

Taylan and Da Ponte (2016) recognize that mathematical investigation as a special form of problem-solving that role in (1) stimulates student engagement for meaningful learning; (2) provides multiple mathematical activities for students at different ability grades; and (3) stimulates holistic thinking that relates a basic condition to many topics for valuable mathematical reasoning.

It implies that a mathematical investigation provides a good context for making the student understand the need to justify their assertions or explain their reasoning.

Barbeau and Taylor (2009) viewed mathematical investigation as a skill required to solve challenging mathematical tasks or problems. It refers to the most inclusive skill that considers different mathematical situations, conjecturing, justifying, explaining the conjectures, proving, inferring, and posing a new question (Leikin, 2007). Particularly, Leikin (2014) argues that conjecturing is the main aspect of mathematical investigation. Mariotti and Pedemonte (2019) state that a conjecture is a statement that is strictly connected to an argument and a set of conceptions where the statement is potentially true because some conceptions allow the construction of an argument that justifies it. However, constructing a conjecture involves a lot of cognition processes, such as organizing and recording data, pattern searching, conjecturing, inferring, justifying, and explaining conjecture (Astawa et al., 2018, Benson et al., 2004, Yeo, 2017). This cognitive process is associated with prior mathematical knowledge.

Some previous studies have been conducted, focusing on developing students' investigation abilities (da Ponte & Pedro, 2007; McCosker & Diezmann, 2009; Quinnell, 2010; Yeo, 2017; Galen & Eerde, 2018). However, no one has studied how mathematical investigation is associated with prior knowledge, even though it is the conceptual foundation for developing mathematical investigation skills. Therefore, this study intends to discover the association structure between prior knowledge and mathematical investigation skills and then visually depict its association using the k-means clustering procedure and correspondence analysis.

In this study, K-means clustering was used to categorize students based on attributes or characteristics of the same prior knowledge level into several groups. Meanwhile, students' mathematical investigation skill is observed and categorized into several aspects based on their measured indicators. This procedure yields two categorical random variables representing the prior knowledge level and mathematical investigation aspects summarized in a two-way contingency table. Furthermore, the association between the two categorical variables was analyzed using correspondence analysis. Correspondence analysis is a powerful statistical tool for the graphical analysis of two categorical random variables that naturally depicts their association structure on a low-dimensional plot, called a correspondence plot (Beh & Lombardo, 2014; Greenacre, 2017; Lestari et al., 2020)

As explained earlier, the main concern in this study is to discover the association structure between prior knowledge level and mathematical investigation aspects. As a limitation, this study led to (1) reveal a significant association between variables using Pearson's chi-squared statistic; (2) visualize their associations through symmetry and asymmetry correspondence plots, and (3) examine the significance of the contribution of each category of variables by constructing an elliptical confidence region. Some findings of this study provide the novelty and open issue for future research to develop a learning environment supporting mathematical investigation activities.

METHODS

This study is empirical research that deals with associational research (also known as correlational research), which investigates the relationships between two variables without any attempt to influence them. In associational research, there is no manipulation of variables, and the existing relationship between variables is described, such that it is also sometimes referred to as descriptive research (Fraenkel et al., 2012; Salkind, 2015). However, the way of describing it is slightly different from other such studies. In this study, the relationship between variables is explained through their dependence. Two variables are said to be unrelated if they are statistically independent. Therefore, the main purpose of this study is to explain students' thinking behavior on Graph Theory by discovering the association structure (dependency) between prior knowledge and mathematical investigation skills.

Participants

The population of this study is sophomore students who are enrolled in a Mathematics Education Study Program at one of the universities in Indonesia. The sample of 97 out of 163 students was selected purposively. The student attends Discrete Mathematics lectures for one semester with three credit points. In other words, students are required to meet a minimum of 136 hours in one semester, which consists of 40 hours for lectures, 48 hours for structured assignments, and 48 hours for private study.

Materials

The study was held during the COVID-19 pandemic, so both lectures and assessments were held online. Online learning is organized using the E-campus platform. This platform provides various learning and teaching resources that allow lecturers and students to interact virtually. Additional features such as the digital library, connection to Google Meet or Zoom, lecture attendance, assignments, quizzes, and exams facilitate teaching and learning activities even in pandemic situations. The assessment and evaluation consist of individual tasks, structured tasks, midterm exams, and final exams.

Since this study was conducted in a specific classroom setting, due to the COVID-19 pandemic, here are practical guidelines for supporting mathematichal investigation skill in other classrooms: (1) review and integrate prior knowledge by providing a conceptual roadmap of the related topics to ensure that students maintain an understanding of concepts and procedures during mathematical investigation activities; (2) incorporate a mix of previously and newly learned problem types during the investigation; (3) propose open problems that students find challenging, which cannot be answered immediately and require them to solve in different ways, arousing their mathematical investigation; (4) expand students' ability to identify relevant information in new contexts by presenting problem information differently; (5) promote discussions that encourage students to offer explanations of their conjecture; and (6) sequence the instructions to allow students' mathematical investigation skills to grow incrementally.

In the Discrete Mathematics course, we discussed discrete objects in mathematics, such as logic, sets, mathematical proofs, and graph theory. Furthermore, we will focus on graph theory as one of the essential materials in this course. The topics include simple graphs, special graphs, isomorphism, invariants, connectivity, coloring, Euler graphs, Hamilton graphs, planar graphs, and trees. The following are definitions of some related concepts in graph theory taken from Ferland (2019).

Definition 1: Simple graph

A graph G consists of a pair of sets: vertex set V and edge set E, denoted by $G = (V, E)$. An edge of G is a function that assigns two vertices, that is $e \in E$ such that $e \mapsto \{u, v\}$ for some $u, v \in V$. The vertices u and v are the endpoints of the edge e. If $e \mapsto \{u\}$ has a single endpoint, then it is called a loop. Two or more edges assigned to the same set of endpoints are called multiple edges. A *simple graph* is a graph $G = (V, E)$ that has no loops and multiple edges. See Figure 1.

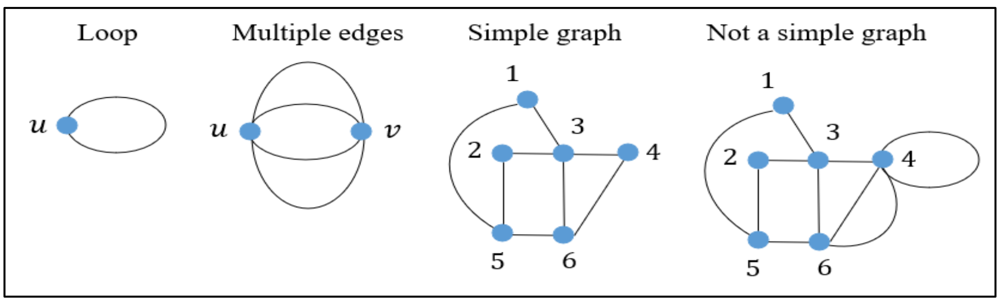


Figure 1. Illustration of the simple graph

Definition 2: Graph isomorphism

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be graphs. A *graph isomorphism* from G to H is a pair of bijections $f_V: V_G \to V_H$ and $f_E: E_G \to E_H$ such that, for $e \in E_G$, the bijection f_V maps endpoints of e to the endpoints of $f_E(e)$. If there exists a graph isomorphism from G to H, then G is isomorphic to H, denoted by $G \cong H$. See Figure 2.

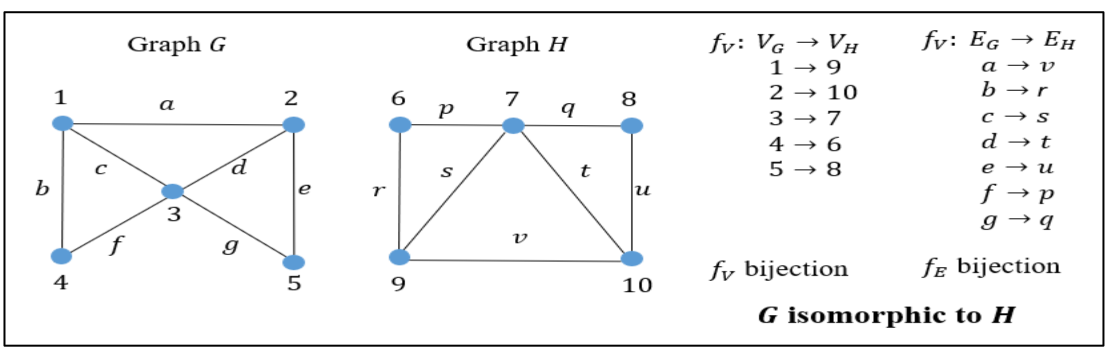


Figure 2. Illustration of a graph isomorphism

Definition 3: Walk, path, circuit, and cycle

A *walk* in graph $G = (V, E)$ is an alternating list of vertices and edges that starts at vertex v_0, end at vertex v_n for $n \geq 0$. A *path* is a walk with no repeated vertices. A *circuit* is a walk of positive length that starts and ends at the same vertex. A *cycle* is a circuit in which the only vertex repetition is $v_n = v_0$.

Definition 4: Hamiltonian graph

Let G be a graph. A *Hamiltonian cycle* in G is a cycle that covers every vertex. A *Hamiltonian path* in G is a path that covers every vertex. Graph G is said to be a *Hamiltonian graph* if it contains a Hamiltonian cycle. See Figure 3.

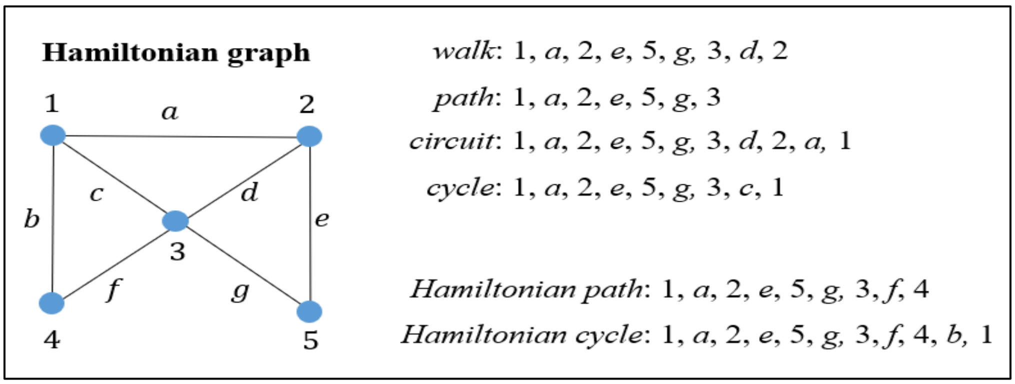


Figure 3. Illustration of definitions 3 and 4.

Definition 5: Subgraph, connected graph and tree

A graph $H = (W, F)$ is a *subgraph* of a graf $G = (V, E)$ if $W \subseteq V$, $F \subseteq E$, and the endpoints of the edges in F all lie in W and the same as in G. A graph G is *connected* if a path exists for any two vertices; otherwise, G is disconnected. A *tree* is a graph that is connected and contains no cycles. See Figure 4.

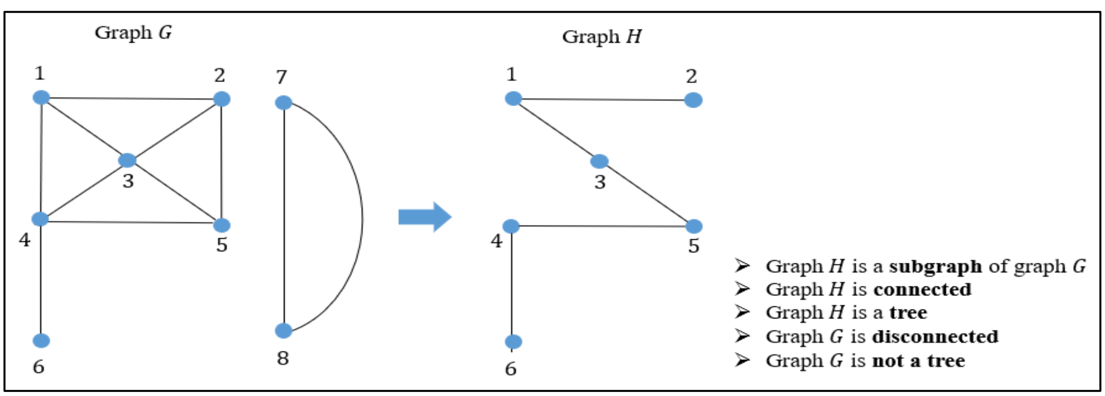


Figure 4. Illustration of a subgraph, a connected graph, and a tree

Definition 6: Weighted graph and minimum spanning tree

A *weighted graph* is a graph $G = (V, E)$ for which the edge has been assigned a positive real number called the weight of the edge. The *weight* of a subgraph is the sum of the weights of the edges in that subgraph. A *minimum spanning tree* for G is a spanning tree with the minimum weight among all spanning trees. See Figure 5.

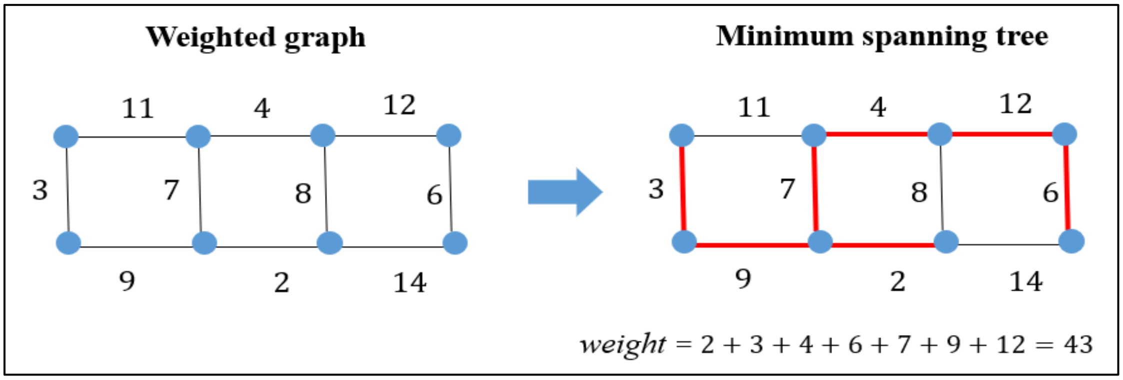


Figure 5. Illustration of a weighted graph and a minimum spanning tree

Measures

Two variables are observed to involve prior knowledge and mathematical investigation skills. Prior knowledge in mathematics is defined as the prerequisite material that students need to know before learning new mathematical concepts. In our study, this knowledge is measured by a preliminary test of basic mathematics such as pre-algebra and number theory. Meanwhile, mathematical investigation skill is measured by final exams. Indicators of mathematical investigation skills that are measured include students' capability in (1) organizing and recording data; (2) pattern searching; (3) conjecturing; (4) inferring, also (5) justifying and explaining conjecture.

Organizing and recording data

Mathematical investigations can begin with organizing and recording data. It involves the ability to integrate several mathematical skills to solve problems. Figure 6 presents the problems given in the final exam that measure students' capability to organize and record data. This problem ordered students to investigate whether the given graph was simple or not by organizing and recording a given set of vertices and a set of edges.

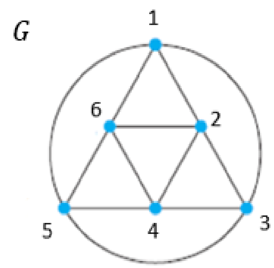
Given graph $G = (V, E)$.
Let $V = \{1, 2, 3, 4, 5, 6, 7\}$
$E = \{\{1,2\}, \{2,3\}, \{1,4\}, \{2,5\}, \{4,5\}, \{4,6\}, \{5,6\}, \{5,7\}\}$
Draw the specified graph and determine whether it is a simple graph.

Figure 6. Mathematical investigation test on organizing and recording data

Pattern searching

Searching for a pattern is often a sensible thing to do at the beginning of an investigation. Finding and describing an observed pattern provides a chance to pursue an investigation beyond the first few minutes (Benson et al., 2004). Figure 7 presents the problems that measure students' capability in pattern searching. In this problem, students must search the pattern for a path or circle that covers every vertex exactly once.

Given the pictured graph $G = (V, E)$.



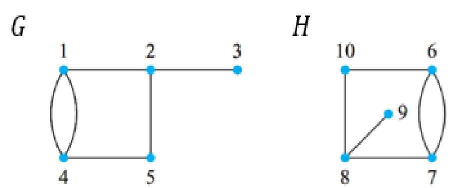
Investigate whether graph G has a Hamiltonian cycle, a Hamiltonian path, or neither. If so, then specify it.

Figure 7. Mathematical investigation test on pattern searching

Conjecturing

In mathematics, one commonly conjectures that a statement follows rule patterns that hold beyond the cases investigated and tries to prove it. It implies that a conjecture bridges someone to investigate the given problem. The following problem measures students' capability for conjecturing. Here, the problem leads students to make conjectures by defining two bijective functions such that graphs G and H are isomorphic. See Figure 8.

Given the two following graphs.



Define a pair of bijection functions $f_v: V_G \to V_H$ and $f_E: E_G \to E_H$ such that G is isomorphic to H.

Figure 8. Mathematical investigation test on conjecturing

Inferring

Mathematical investigations demand students to think through a solution and make inferences (Calleja, 2011). Inference can be viewed as an interpretation or explanation of an investigation through observations involving one's senses. To make an inference, students need to connect what they investigate to prior knowledge and the new information investigated through their senses. The inference can be made from more than one investigation, and it is not just a guess. Therefore, inference can be defined as the process of drawing a conclusion based on the available evidence from an investigation, plus previous knowledge and experience. Figure 9 presents the problems that measure students' capability in inferring.

The graph in the figure below represents an apartment complex for which electric power must be supplied. Each vertex represents an apartment building, each edge represents a potential direct power cable between two buildings, and the weight assigned to an edge reflects the cost of establishing that link.
How to determine the cheapest power grid that connects all of the apartment buildings?

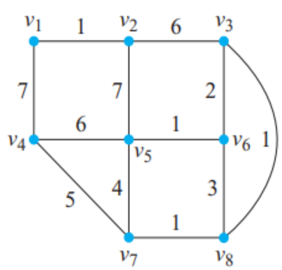
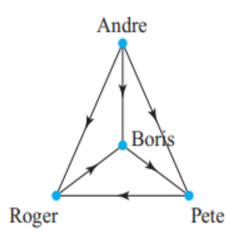


Figure 9. Mathematical investigation test on inferring

Justifying and explaining conjecture

Justification and explanation of a conjecture are a part of the mathematical investigation process that leads the student to give reasons why their conjecture makes sense by investigating a pattern, an algebraic validation, or some other logical methods. The following problem measures students' capability for justifying and explaining conjecture. See Figure 10.

The pictured directed graph is a tournament.



Find a Hamiltonian path through the tournament.
Have the results of this tournament determined a clear ranking of its competitors? Justify your answer.

Figure 10. Mathematical investigation test on justifying and explaining conjecture

Data Analysis

The empirical evidence was analyzed to explain students' thinking behavior on Graph Theory by discovering the association structure between prior knowledge and mathematical investigation skill, then visually depicting its association using the k-means clustering procedure and correspondence analysis. As a first step, the data obtained from the preliminary test that measures students' prior knowledge in mathematics were analyzed using k-means clustering (see Lestari et al., 2022a; Yudhanegara & Lestari, 2019). At this stage, students are classified into five groups (clusters) based on their prior knowledge scores. The level of prior knowledge is defined immediately after the clusters are formed by considering the average in each cluster.

The student's answers in the final test that measured mathematical investigation skills were observed and classified into six categories based on the mathematical investigation aspect, plus one category for “give up”, which represents the students who did not answer the given problem. By doing so, we obtained a two-way contingency table that classified students based on their prior knowledge level and mathematical investigation skill indicator. Both categorical variables were measured on ordinal scales. Finally, a two-way contingency table was analyzed by correspondence analysis to discover the association between prior knowledge and mathematical investigation skills.

Correspondence analysis is a statistical graphical tool to visualize the association between two categorical variables in a two-way contingency table (Lestari et al., 2023). This visualization is displayed in low-dimensional correspondence plots. Each category of two categorical variables is depicted as a coordinate point in the correspondence plot. The association between variables is visually revealed by the relative proximity of the coordinate points of one category to another. The contribution of each category to the association between variables can be determined by constructing its elliptical confidence area. In statistics, an elliptical confidence region is one form of a two-dimensional generalization of a confidence interval. The construction of the elliptical confidence area is determined using Algorithm 1.

Input

Step 1: Read a two-way contingency table

Process

Step 2: Calculate the standardized residual matrix

Step 3: Determine the singular value decomposition of the standardized residual matrix

Step 4: Determine the row and the column principal coordinates

Step 5: Determine the row and the column standard coordinates

Step 6: Determine the major and minor axes of the ellipse confidence area for each row and column principal coordinates

Output

Step 7: Plotting symmetric plot, asymmetric plot, and elliptical confidence regions

RESULTS

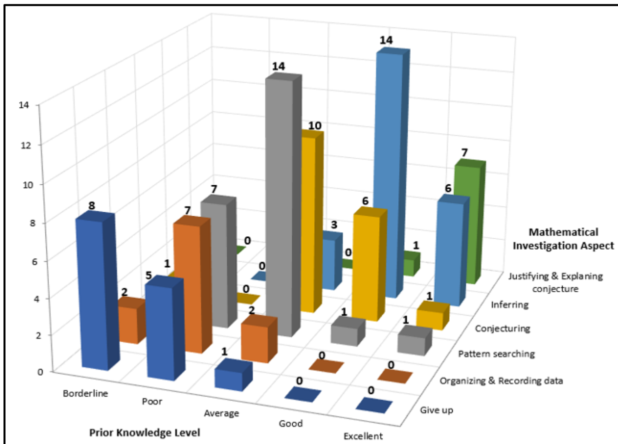
The preliminary test scores reflect students' prior knowledge of mathematics. The data from this test was analyzed using k-means clustering, which yielded five clusters to categorize students' mathematical prior knowledge. By considering the average of each cluster as the centroid, the resulting clusters were used to define five levels of students' prior knowledge, including borderline, poor, average, good, and excellent, with the following descriptions. See Table 1.

Table 1. The prior knowledge level and description based on k-means clustering.

Level	Class centroid	Description
Borderline	34,27	Not quite up to what is standard or expected borderline knowledge in pre-algebra and number theory, such as set theory, basic proof and logic, relations, functions, simplify and solve algebraic equations.
Poor	53,58	Limited knowledge in pre-algebra and number theory and need for application.
Average	61,50	Know and can apply some concepts and procedures in pre-algebra and number theory but need help to develop them.
Good	68,36	Understand the application of some concepts and procedures in pre-algebra and number theory and can develop a simple idea but needs a more detailed explanation.
Excellent	77,93	Strong understanding of concepts and procedures in pre-algebra and number theory, mostly accurate in applying and developing an advanced concept with detailed explanation

Additionally, students' answers in the final test determine the classification of students based on their achievement of the mathematical investigation aspect. Aspects of mathematical investigation are viewed as categories of ordinal-scaled variables ordered from “organizing and recording data” to “justifying and explaining conjecture”, plus one category of “give up” at the lowest order. Each aspect category was assumed to be mutually independent. It means that each student was classified on only one of their highest achievement aspects of mathematical investigation. As an illustration, a student classified in the “pattern searching” aspect means that the student already has the capability of “organizing and recording data”; a student classified in the “conjecturing” aspect means that such a student already has the capability of “organizing and recording data” and “pattern searching”, and so on.

The graphical analysis of association using correspondence analysis needs a contingency table from the cross-classification of two categorical variables. Figure 11 visualize such a table that classifies student by their prior knowledge level and mathematical investigation skill. Each bar in the three-dimensional contingency table reflects the frequency of students who hold characteristics of the joint category represented by the corresponding bar. For example, the height of the bar for the joint category of “excellent-inferring” is 6. It implies that 6 out of 97 students whose excellent prior knowledge could fulfill the inferring aspect of mathematical investigation. Generally, Figure 11 shows that the student who has prior knowledge on average with pattern searching ability and the student who on a good level with inferring ability have the most frequencies.



A two-way contingency table

Prior Knowledge Level	Mathematical Investigation Aspect					
	Give up	Organizing & Recording data	Pattern searching	Conjecturing	Inferring	Justifiying & Explaning conjecture
Borderline	8	2	0	1	0	0
Poor	5	7	7	0	0	0
Average	1	2	14	10	3	0
Good	0	0	1	6	14	1
Excellent	0	0	1	1	6	7

Figure 11. Three-dimensional visualization of the contingency table

Two important aspects of contingency table analysis consider the association within and between variables that are visualized by symmetric and asymmetric correspondence plots. Both plots are obtained by performing correspondence analysis on a two-way contingency table in Figure 11. The correspondence analysis procedure is described in Algorithm 1 and interpreted based on the proximity of a coordinate point from other coordinates' positions and origin. Two categories are strongly associated if the coordinates reflecting those categories are close together. Meanwhile, the category coordinates close to the origin indicate that such a category has a small contribution to the association. Furthermore, the association within the category of variable is displayed on a symmetric plot, as in Figure 12.

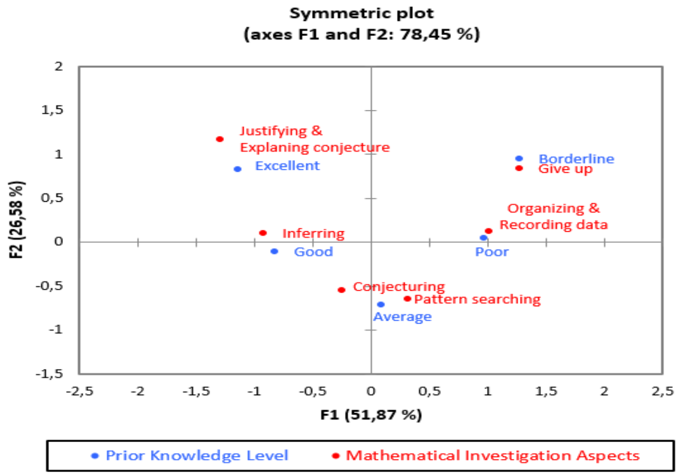


Figure 12. Symmetric correspondence plot

In the symmetric plot, the row-to-row (inter-level of prior knowledge) and column-to-column (inter-aspect of mathematical investigation) distances reflect the approximate chi-squared distance between the respective profiles (Greenacre, 2017). Thus, categories whose frequency is rarely plotted far from the origin, and vice versa (Ginanjar et al., 2016; Lestari et al., 2019a). Moreover, the association between prior knowledge and mathematical investigation skills is displayed on an asymmetric plot, as in Figure 13.

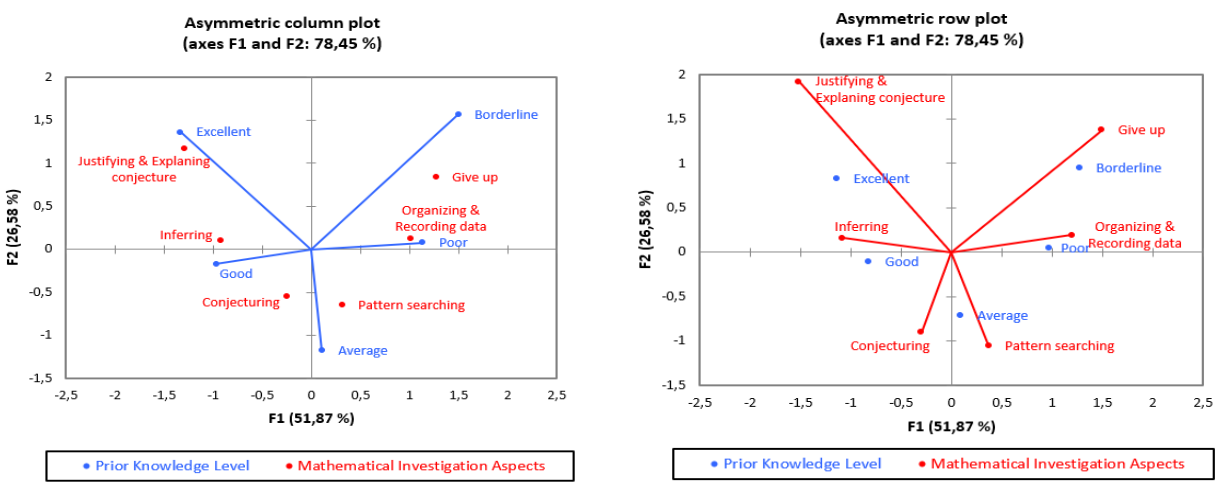


Figure 13. Asymmetric correspondence plot

To seek those categories that make a statistically significant contribution to these association structures can be identified by their proximity of coordinates' positions from the origin. For this reason, we construct an elliptical confidence region for each variable category on the correspondence plot, as shown in Figure 14.

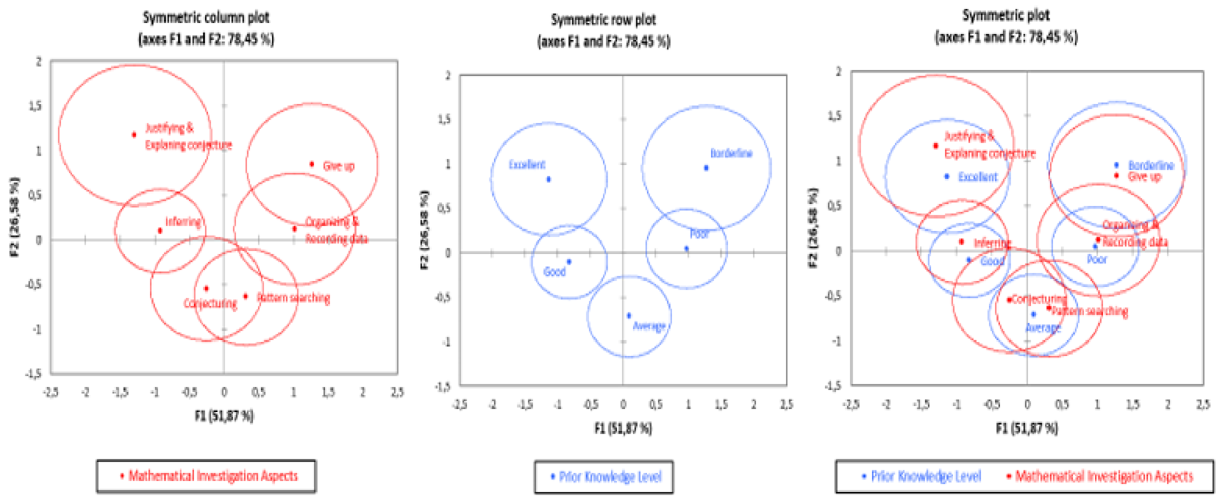


Figure 14. Elliptical confidence regions for each category coordinates

If the origin is included within the ellipse, then the particular category does not contribute to the association structure between the variables (D'Ambra et al., 2020; Lestari et al., 2019b). In other words, an elliptical region that does not include the origin means that, at the specified significance level, the category to which it is related makes a statistically significant contribution to the association structure.

Table 2. Summary statistic of elliptical confidence regions for each category.

Category	Semi-mayor	Semi-minor	χ^2 Statistic	p-value
Borderline	1,2171	0,8712	72,1280	0,0000
Poor	0,9260	0,6629	34,4290	0,0000
Average	0,7370	0,5275	58,4268	0,0000
Good	0,8606	0,6160	29,7193	0,0002
Excellent	1,0422	0,7460	76,1687	0,0000
Give up	1,0788	0,7722	80,4863	0,0000
Organizing & recording data	1,2171	0,8712	22,4563	0,0041
Pattern searching	0,8417	0,6025	39,8883	0,0000
Conjecturing	0,9514	0,6810	22,5812	0,0039

Inferring	0,8417	0,6025	38,6447	0,0000
Justifying & explaning conjecture	1,4271	1,0215	66,8152	0,0000

Taking into the theory underlying the construction of the 100 $(1 - \alpha)$% confidence region of a coordinate point in a correspondence plot, one can estimate the p-value of this point concerning its proximity to the origin. The p-value can be used to assess the statistical significance of each category considered on the association between variables (Beh, 2001; Lestari et al., 2022b). The approximation is determined and derived algebraically based on the elliptical region, as summarized in Table 2.

DISCUSSION

Considering the three-dimensional contingency table in Figure 11, the observed value of Pearson's chi-squared is $\chi^2_{stat} = 135.436$. Its value is greater than the critical value $\chi^2_{\alpha,\nu} = 31.410$ with 20 degrees of freedom. It infers that there exists a statistically significant association between prior knowledge and mathematical investigation skills. The graphical representation of this association is depicted by symmetric and asymmetric correspondence plots (see Figures 12 and 13).

The symmetrical plot in Figure 12 shows that the coordinate position for the “pattern searching” is relatively closer to “conjecture” rather than another aspect. It suggests that “pattern searching” and “conjecture” are strongly associated. As Benson et al. (2004) stated that to make conjectures in mathematical investigations, the student is required to search the patterns first. Conversely, category coordinates of prior knowledge tend to be far from each other, hence they have weak associations. It indicates that each level of students' prior knowledge has different characteristics.

The asymmetric plots in Figure 13 naturally depict the association structure between prior knowledge level and mathematical investigation aspect. The figure shows that the coordinate category for the “borderline” is close to the “give up”. It suggests that students on the borderline level tend to give up when solving mathematical investigation problems. In addition, the “organizing and recording data” aspect is strongly associated with the “poor” level since their coordinates are close together. Similarly, the student on the “average” level tends to be skillful at “pattern searching” and “conjecturing”. The student on the “good” level tends to be capable of “inferring”, and the “excellent” student is qualifying in “justifying and explaining conjecture”.

The relative position of each category of the variable to the origin reflects its contribution to the association (Lestari et al., 2019c). The closer to the origin suggests a more negligible contribution, such that it will not change the association structure if it is omitted. It means that the category coordinate that is close to the origin indicates that the category represented by such coordinate is considered does not contribute to the association structure between variables. In addition, if its elliptical regions contain the origin, thus those categories are not statistically significant contributions to the association between variables. Figure 14 shows that the origin does lie in any elliptical regions. In addition, summary statistics in Table 2 suggest that all categories of prior knowledge levels and mathematical investigation aspects statistically significantly contribute to the association structure between variables since their p-value is less than the level of significance $\alpha = 0.05$.

Furthermore, we discuss a slightly different approach to explaining how prior knowledge and mathematical investigation skills are associated. Here, we reveal the student's tendencies to solve investigation problems based on their level of prior knowledge. The mathematical investigation steps for each given problem are also described in detail.

Consider the problem in Figure 6, the investigation begins with organizing and recording data about the given set of vertices and edge sets by drawing a specified graph in such a way that (1) the only vertex points hit by a curve are the endpoints of the edge it represents; (2) each curve is one-to-one (that is, it does not intersect itself) with the exception that the ends of a loop edge are assigned to a common point; and (3) the images of curves associated with two distinct edges intersect in at most finitely many points (Ferland, 2019). To determine whether the resulting graph is simple, students should involve their prior knowledge; if it has no loops and multiple edges, it is a simple graph. Based on the assessment and evaluation, students with below-average prior knowledge (borderline, poor, and average levels) tend to reach only the first two steps, where the resulting graph contains a crossing, as illustrated in Figure 15.

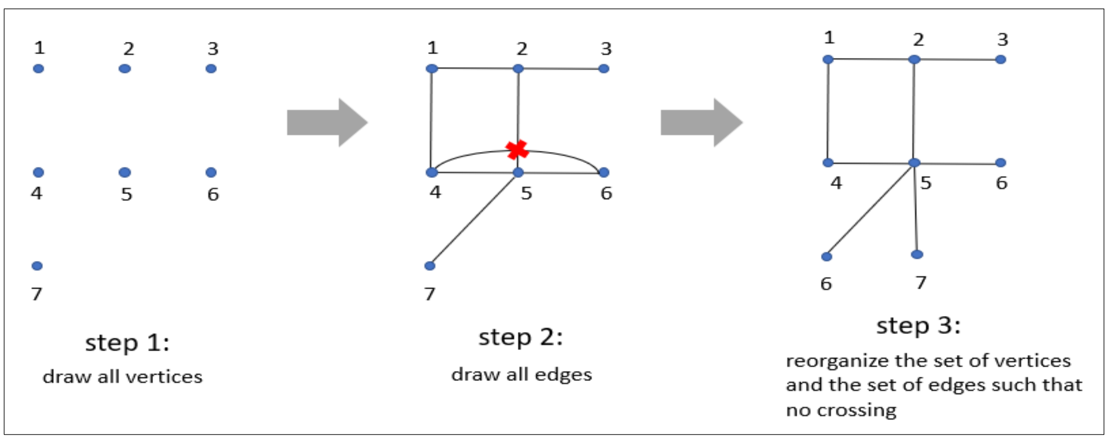


Figure 15. Mathematical investigation steps for the problem in Figure 6

Unfortunately, only a few students with good and excellent levels could complete each step perfectly. Most students who could organize and record data only reached the first two steps to conduct a simple graph by drawing all vertices and all edges without considering whether the resulting graph contains crossing edges or not, as presented in Figures 11(a) and 11(b). On the other hand, Figure 11(c) stands for the answers of students who have been able to organize and record data up to the third step such that it yields a simple graph without crossing edges. However, the student did not explain further whether the graph is simple or not, as asked in the question. It suggests that the student did not fully capture the instructions in the problem. Consequently, the mathematical investigation process was not completed well. From a conceptual point of view, students' answers in Figures 16(a) and 16(b) do not violate the definition of a simple graph. However, in procedural terms, certainly, the student's answer in Figure 16(c) is more appropriate for drawing a graph. Upon tracing and observation, it turns out that Figures 16(a) and 16(b) are answers from students with poor prior knowledge levels, while Figure 16(c) is an answer from a student with good prior knowledge.

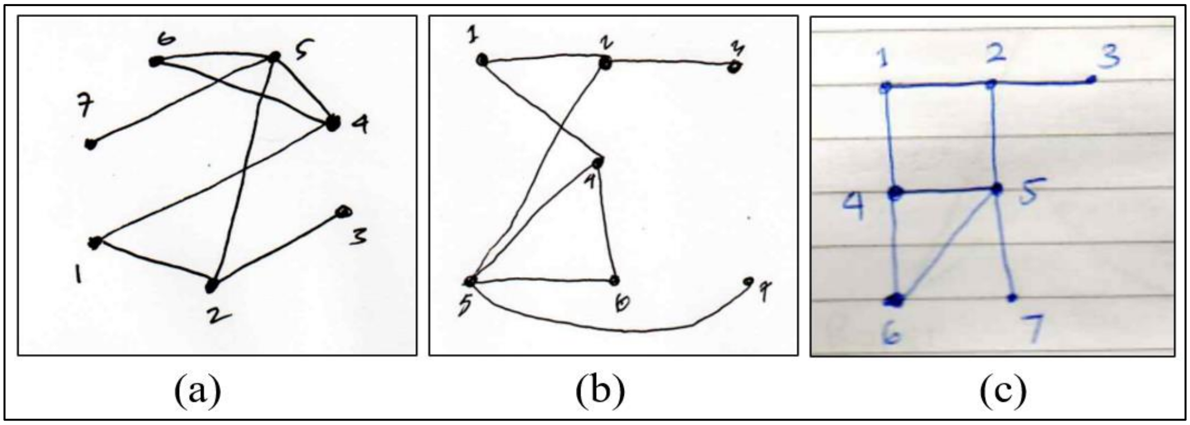


Figure 16. Example of student answers for mathematical investigation test on organizing and recording data

Furthermore, according to the problem in Figure 7, to specify a Hamiltonian graph, the investigation begins by searching for a walk with no repeated vertices or a walk of positive length that starts and ends at the same vertex and covers each vertex. Some possible mathematical investigation steps are illustrated in Figure 17.

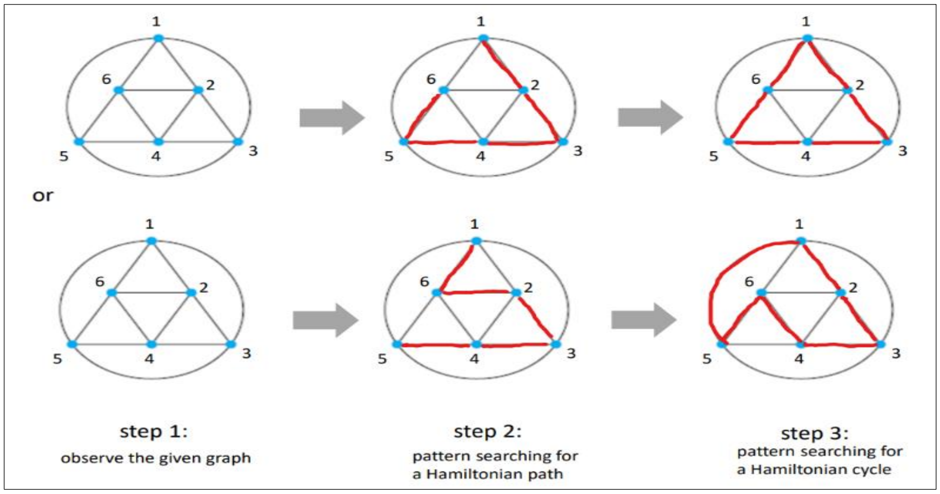


Figure 17. Mathematical investigation steps for the problem in Figure 7

Most of the students performed the mathematical investigation steps in an ordinary. Students tend to search for path or circle patterns with ordered vertices, as the first solving step in Figure 17. Only excellent students can search other patterns randomly to find Hamilton paths or cycles from the graph, as the last answer in Figure 17.

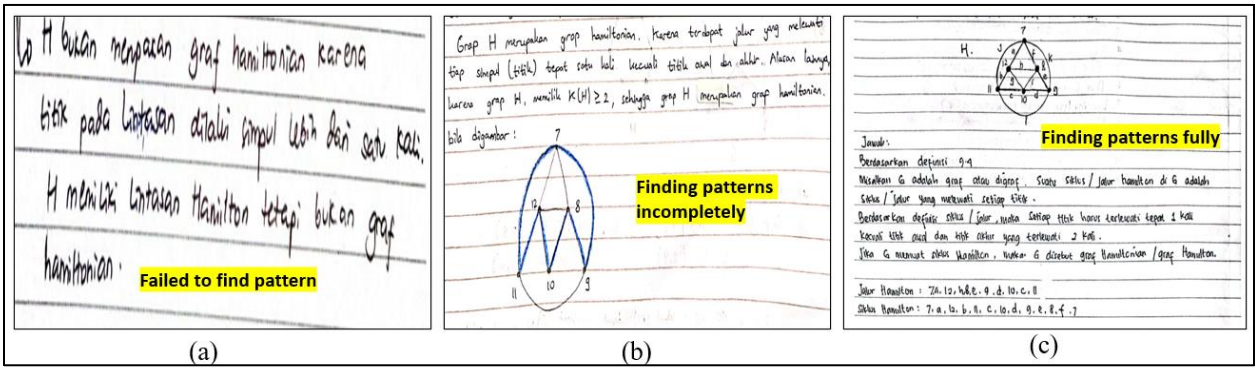
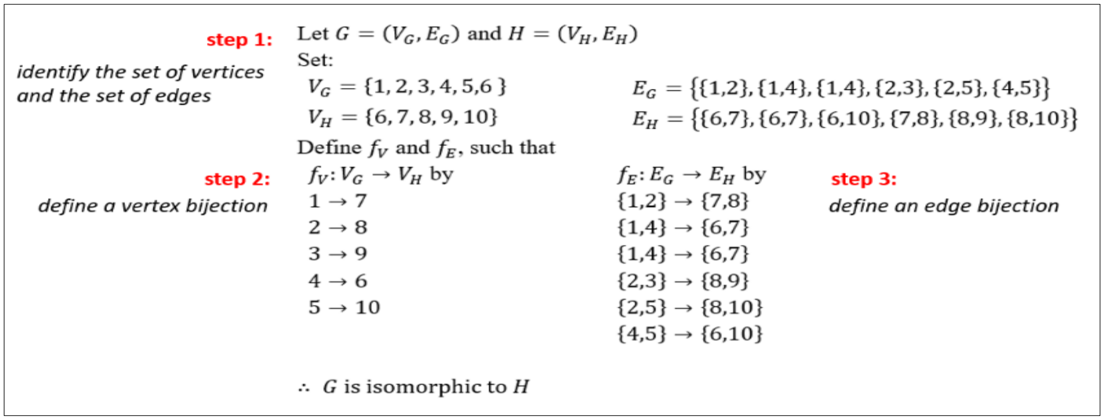


Figure 18. Example of student answers for mathematical investigation test on pattern searching

Figure 18(a) displayed the student answer who failed to find the Hamiltonian cycle or path pattern. The student states that the given graph is not a Hamiltonian graph since the vertex on the path is traversed more than once but does not mention a specific vertex. The student added the argument that the given graph has a Hamiltonian path, but it is not a Hamiltonian graph. Since the student does not specify such a path, it can be identified that the student already knew the definition of the Hamiltonian graph but did not clearly understand how to apply it. Meanwhile, Figure 18(b) presents the student answer who did an incomplete pattern search. The student successfully investigates the given graph to find the Hamiltonian cycle in the graph with an acceptable explanation but is slightly less careful in reading the instruction, hence missing answering the question regarding the Hamiltonian path. Likewise, Figure 18(c) exhibits the student's answer, who finds the pattern of both Hamiltonian path and cycle fully with a proper explanation.

The next problem in Figure 8, given two graphs, G and H. Such a problem asks the student to prove that they are isomorphic. By definition, two graphs are isomorphic if a graph isomorphism exists from one to another. Hence, this problem leads students to make conjectures by defining two bijective functions, f_V and f_E, such that any two vertices of G are adjacent in G if and only if assigned to two adjacent vertices in H.


step 1: *identify the set of vertices and the set of edges*

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$

Set:

$V_G = \{1, 2, 3, 4, 5, 6\}$ $\quad E_G = \{\{1,2\}, \{1,4\}, \{1,4\}, \{2,3\}, \{2,5\}, \{4,5\}\}$

$V_H = \{6, 7, 8, 9, 10\}$ $\quad E_H = \{\{6,7\}, \{6,7\}, \{6,10\}, \{7,8\}, \{8,9\}, \{8,10\}\}$

Define f_V and f_E, such that

step 2: *define a vertex bijection*

$f_V: V_G \to V_H$ by

$1 \to 7$
$2 \to 8$
$3 \to 9$
$4 \to 6$
$5 \to 10$

step 3: *define an edge bijection*

$f_E: E_G \to E_H$ by

$\{1,2\} \to \{7,8\}$
$\{1,4\} \to \{6,7\}$
$\{1,4\} \to \{6,7\}$
$\{2,3\} \to \{8,9\}$
$\{2,5\} \to \{8,10\}$
$\{4,5\} \to \{6,10\}$

$\therefore$ G is isomorphic to H

Figure 19. Mathematical investigation steps for the problem in Figure 8

Figure 19 provides a possible mathematical investigation step to solve the problem. The assessment and evaluation result suggests that students with borderline and poor prior knowledge could not make conjectures as asked. Some of them fail to define a vertex bijection and an edge bijection to prove the two graphs are isomorphic.

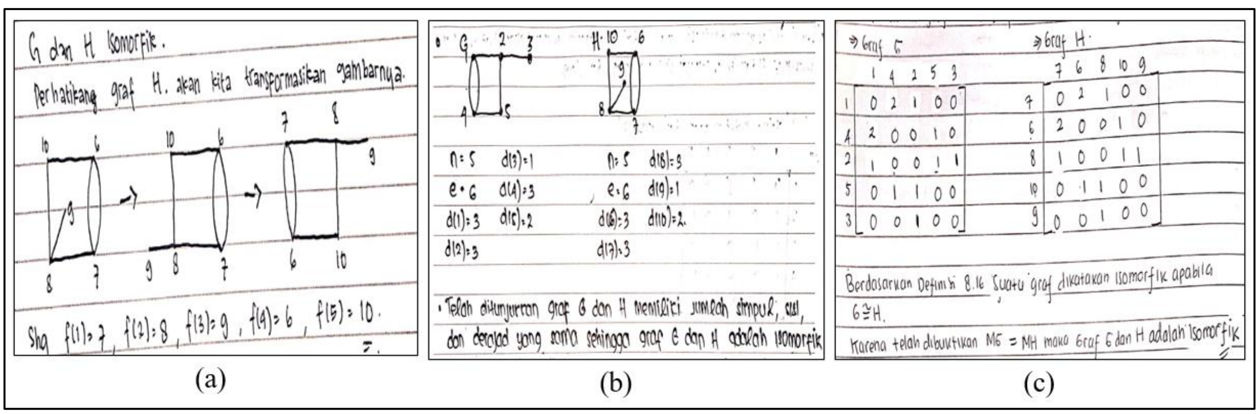


Figure 20. Example of student answers for mathematical investigation test on conjecturing

Another interesting point is that some students make the conjecture in a different way than what is directed by the question. In proving two given graphs are isomorphic, some students made conjectures by performing a transformation process on one of the graphs to obtain the other graph and then defined the mapping vertices, as shown in Figure 20(a). Some others make conjectures by applying graph invariant properties such as degree invariant and having a common adjacency matrix, as presented in Figures 20(b) and 20(c). It suggests that students' constraints are not in making conjectures but restricted to defining a pair of vertex and edge bijections from graph G to H. In the conceptual framework, these ways of constructing conjectures are acceptable.

The next investigation problem is regarding inferring, as in Figure 9. In this problem, students were required to connect their prior knowledge about the weighted tree and investigate the minimum weight. The left hand in Figure 21 shows the step after all the edges of weight 1 or 2 have been chosen. At that point, the edges of weights 3 and 4 cannot be chosen since they yield a cycle. Consequently, the edge of weight 5 is chosen next. After that, note that although there are two edges of weight 6, only one of them is a possible addition to the tree. In this step, we have the minimum spanning tree. Such investigation leads the student to infer that the cheapest power grid connecting all apartment buildings is reflected by a minimum spanning tree (Ferland, 2019).

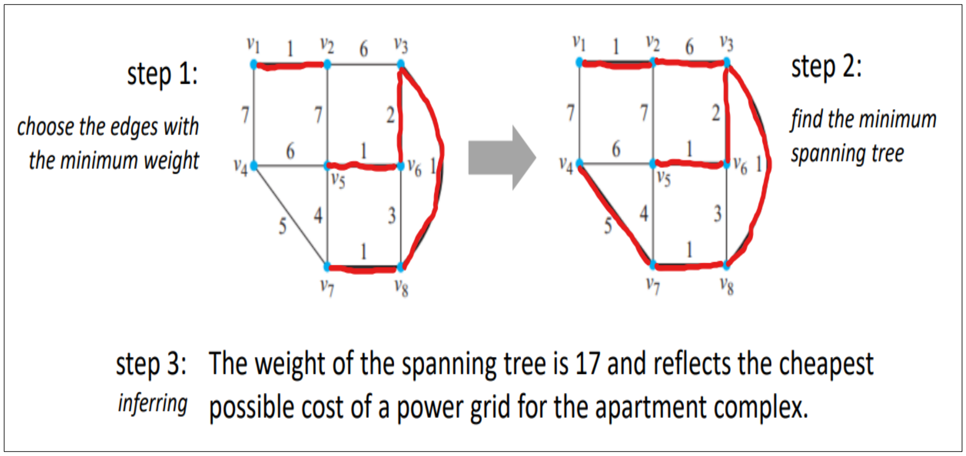


Figure 21. Mathematical investigation steps for the problem in Figure 9

The analysis of student answers shows that only good and excellent students can infer it. Figure 17 shows the answers of students who are misleading in inferring the solution of their mathematical investigation. In Figure 22 (a), the student only focuses on finding a spanning tree of the given graph and ignores choosing the minimum weight. In Figure 2 (b), the student focuses on taking the edge with small weights in the earlier steps and is trapped in executing the final step, hence not considering other possibilities for a minimum spanning tree. An essential point in inferring is validation or cross-checking the solution before drawing a conclusion. However, it is a common misstep for students to omit.

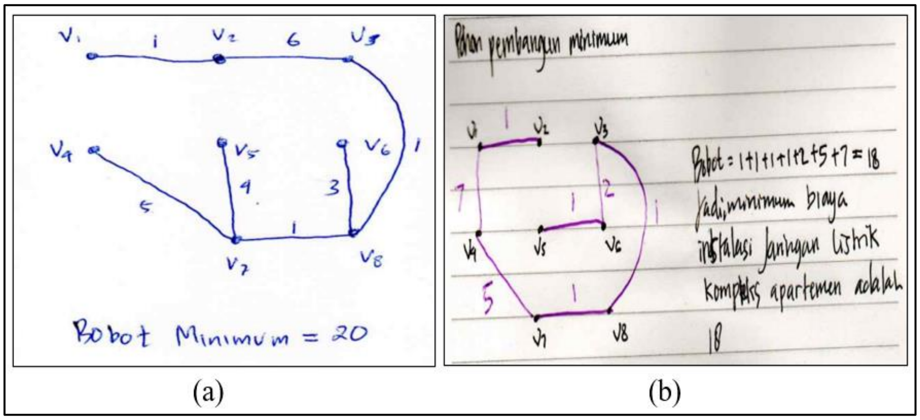


Figure 22. Example of student answers for mathematical investigation test on inferring

For the last problem, as in Figure 10, students should do an investigation to find a Hamiltonian path through the tournament. A tournament is a directed graph whose underlying graph is complete. In this case, the determination of the tournament is conjecture for justifying whether a ranking of its competitors is clear. After making a justification, the student should give the reason for their justification. See Figure 23.

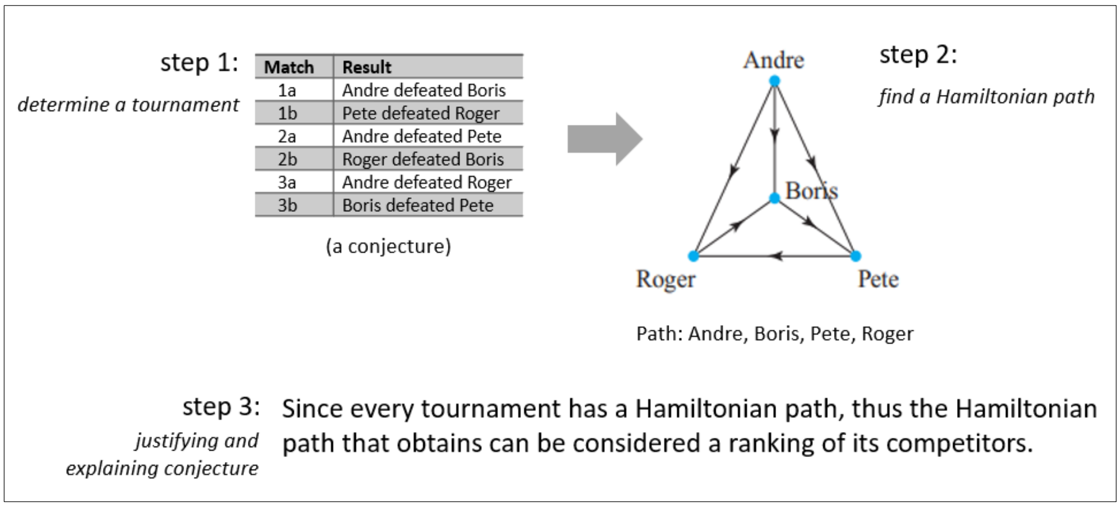
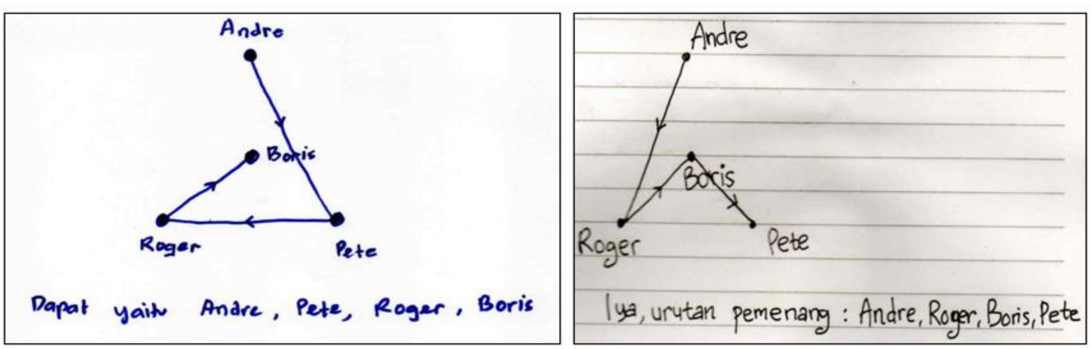


Figure 23. Mathematical investigation steps for the problem in Figure 10

The underlying graph is complete graph K_4. In this case, the tournament reflects the results in a table at the first step. For instance, the edge from Andre to Boris represents the victory of Andre over Boris in their match (1a). Here K_4 reflects the results of a tournament in which every possible pair of players competed in a match an edge (u, v) would be present if and only if player u defeated player v. Since every tournament has a Hamiltonian path, at the end of the tournament, a way to rank the players could be provided by a Hamiltonian path (Ferland, 2019).



(a) (b)

Figure 24. Example of student answers for mathematical investigation test on justifying and explaining conjecture

Unfortunately, none of the students with borderline, poor, and average prior knowledge did this investigation. In addition, only a few students with good and excellent levels can justify and explain the conjecture to solve this problem. In summary, this analysis shows that differences in the students' prior knowledge level yield differences in performing the mathematical investigation step for each aspect. Figure 24 shows a common mistake made by students in justifying and explaining conjecture whether the tournament results can be used to determine the winner ranking. Instead of explaining a conjecture, almost all students did not make a conjecture or a roadmap for the tournament. As a result, the ranking of competitors determined by the students was incorrect.

CONCLUSIONS

Some findings lead to the conclusion that (1) there is evidence of a statistically significant association between prior knowledge and mathematical investigation skill; (2) a high level of prior knowledge allows the student to reach the mathematical investigation aspect completely, and vice versa; and (3) all categories of prior knowledge levels and mathematical investigation aspects statistically significantly contribute to the association structure between variables. Furthermore, the general tendency that can be concluded about students' thinking behavior on graph theory in terms of mathematical investigation skills and prior knowledge level is described as follows.

Students at the borderline level are adequate in organizing and recording data but mostly fail to find a pattern and tend to give up on problems that require higher mathematical investigation skills, such as conjecturing, inferring, justifying, and explaining conjecture. Students at the poor level can organize and record the data, sometimes fail to find a pattern, and may struggle to solve problems that require higher mathematical investigation skills, such as conjecturing, inferring, justifying, and explaining conjecture. Students at the average level can organize and record the data, tend to find a pattern incompletely, are limited of conjecturing, and may struggle in inferring, justifying, and explaining conjecture. Students at a good level understand how to organize and record data, tend to find a pattern fully, adequately in conjecturing and inferring, but mostly misleading and inconsistent in justifying and explaining conjecture. Students at the excellent level are experts in organizing and recording data, quickly capturing patterns, appropriately in conjecturing and inferring, and adequately in justifying, but sometimes reluctantly in explaining conjecture.

This study concludes some findings that provide novelty and open issues for future research, our recommendation is to develop a learning environment supporting mathematical investigation activities that involve (1) reinforcement of prior knowledge; (2) teaching multidimensional mathematical investigation problems; (3) teaching and providing a variety of problem-solving strategies including open-ended problems; (4) extending knowledge or applying knowledge in new contexts; (5) promoting a conjecturing atmosphere; and (6) encouraging students to work systematically. Particularly, in our department, we expect certain prior skills; thus, the correspondence plot obtained could be used as self-reflection for our department, whether we have gained the results as expected to strengthen certain investigation approaches.

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Investigating Pre-Service Primary School Teachers' Difficulties in Solving Context-Based Mathematics Problems: An Error Analysis

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***Abstract**: The issue of students' difficulties in solving context-based mathematics problems has been extensively investigated by numerous studies. However, limited study focus on how pre-service primary school teacher (PSPSTs) encounter difficulties in solving context-based fundamental mathematics problems. To fill this gap, this study aims to investigate PSPSTs' difficulties encountered while solving context-based mathematics tasks by identifying the error type they made based on the error classification proposed in Newman Error Analysis (NEA). This is an error analysis study with a summative qualitative content analysis approach involving 87 PSPSTs in an Indonesian Islamic University. Data were collected through a test, in-depth interviews and document analysis of the PSPSTs' responses. Data were analyzed qualitatively and quantitatively. Qualitative content analysis was performed using Atlas.ti software. The findings revealed that many PSPSTs encounter difficulties in solving context-based problems. Approximately 22.1% of PSPSTs committed errors in comprehension, 17.5% each in reading and encoding, 14.7% in transformation, and 8.7 % in process skill. Furthermore, the findings indicated a hierarchical structure in the occurrence of errors. Errors in the early stages have a high potential to cause errors in subsequent problem-solving stages. All the results are discussed, along with their implications for practice and suggestions for future research.*

Keywords: solving context-based problems, pre-service primary school teachers, error analysis

INTRODUCTION

Context-based mathematics problems have become increasingly popular in recent decades as they represent the importance of learning mathematics in a real-world context (Barcelos-Amaral & Hollebrands, 2017, Widjaja, 2013). A context-based problem refers to a mathematical task that employs situations derived either from real-life or non-real-world settings, providing related and meaningful connections with the learner's daily life experience (Ilhan & Akin, 2022, Kohar et al., 2019). These contextual problems might present the relevance of mathematical concepts in students' everyday lives, which allows them to see how the concepts they are learning can be applied in practical situations. However, while context-based problem is essential in mathematics instruction, it also remains many challenges. Several studies have indicated that contexts can also be problematic when they are used in mathematics tasks (Greer et al., 2007, Hoogland et al., 2018, Wijaya et al., 2014). In another study, Can and Yetkin Özdemir (2020) revealed that students were more successful on non-context-based tests than context-based mathematics tests.

Many students encountered difficulties in solving context-based mathematics tasks (Fonteles Furtado et al., 2019; Wijaya et al., 2014). To address this issue, teachers undoubtedly play a significant role in enhancing students' understanding of mathematics concepts and problem-solving skills. Unfortunately, one of the common problems is that many teachers are also struggling with context-based mathematics problems (CbMP), either in creating problems or even solving CbMP. This will surely be a serious problem as how could teachers employ student problem-solving improvement strategies if they themselves lack of that skill.

The use of context-based problems is not restricted to the topics associated with the advanced-level mathematics. Ideally, fundamental mathematics concepts should be represented in CbMP to create more meaningful learning. Arithmetic, including numbers and their properties and operations (Guberman, 2016), is one of fundamental mathematics concepts (Chin & Zakaria, 2015) that is taught in primary school. A deep conceptual understanding of numbers is arguably one of the most foundational mathematics learning goals for students at all levels of schooling and beyond (Elias et al., 2020). Primary teachers must have good content knowledge and pedagogical skills to teach them accurately. Despite the perceived simplicity of this mathematical concept, Butterworth et al. (2011) argued that arithmetic difficulty is a common problem for children and adults. It is crucial to investigate whether pre-service primary school teachers still encounter difficulties in solving arithmetic problems.

It is widely emphasized that teachers' mathematical knowledge is crucial in enhancing teaching competencies related to mathematical content organization (Burgos & Godino, 2022). PSPSTs are individuals who are prepared to become teachers, those who are prepared to teach mathematics to primary school children in the future. However, several prior studies have indicated that a significant number of pre-service teachers possess inadequate mathematics content knowledge (Blömeke et al., 2013, Guberman, 2016). If they themselves experience misconceptions or struggle in solving context-based fundamental mathematics concepts, this raises the concern of how these challenges may trickle down to their future students. Therefore, it is essential to examine the accuracy of PTPSs' solutions and identify their errors in solving context-based arithmetic problems. Analyzing test takers' errors can help identify the difficulties they encounter in solving challenging mathematics problems (Abdullah et al., 2015, Hadi et al., 2018). Identifying difficulties as well as the underlying factor structure of these errors will be valuable for a better understanding of future educators when teaching students in real school settings (Khalo et al., 2015).

Newman's Error Analysis (NEA) is one of the most commonly used methods for identifying errors when solving mathematical problems. This framework includes five cognitive processes involved in solving mathematical problems: reading, comprehension, transformation, ability, and encoding (White, 2010). These five processes are also required to solve context-based mathematical problems. In addition, NEA is an appropriate approach for scrutinizing errors in written sentence problems (Clements & Ellerton, 1996, Prakitipong & Nakamura, 2006), including word problem and context-based problem. Therefore, this study uses NEA to identify PSPSTs' errors and consider the reasons that underlie the difficulties experienced with context-based arithmetic problems. Each type of PSPSTs' error is then classified based on its type, such as reading, comprehension, transformation, process skills, and encoding errors.

Numerous studies have undertaken error analysis on wide-ranging mathematical concepts, such as students' errors in geometry (Chiphambo & Mtsi, 2021, Sumule et al., 2018, Zamzam & Patricia, 2018), algebraic (Fitriani et al., 2018), set (Noutsara et al., 2021), integral calculus (Angco, 2021), and several error analysis studies related to the various aspects of students' mathematical ability, such as modelling (Kotze, 2018) and evaluating and creative thinking skills (Alhassora et al., 2017). In particular, several studies have also pointed out the students' difficulties in solving context-based mathematical problems (Wijaya et al., 2014). However, limited studies have investigated PSPSTs' difficulty in solving context-based arithmetic problems as fundamental mathematical concepts. Therefore, this study aims to address this research gap.

This study aims to investigate the potential difficulties that PSPSTs may encounter while solving context-based fundamental mathematics problems. This article identifies the most common misconceptions and errors made by PSPSTs when solving context-based mathematics problems, which can then be addressed to improve their understanding of the subject. This study is essential to ensure that future mathematics educators in primary schools have a strong foundation in content knowledge mastery. The results of this study provide insights into improvement interventions for pre-service primary school teachers that would be beneficial for enhancing their professionalism when they become teachers. It would also contribute to the theoretical knowledge about the teaching and learning of mathematics in context.

METHOD

Research Design

This is an error analysis study with a summative qualitative content analysis approach that focuses on investigating the PSPSTs' solution accuracy in solving context-based mathematics problems. The PSPSTs' error indicates the difficulty they encounter in solving problems. In this summative content analysis, researchers quantify the PSPSTs' documents and interview data and then delve into their underlying significance to uncover latent meanings using qualitative content analysis (Hsieh & Shannon, 2005, Setiawan, 2023).

Participants

This study involved 87 pre-service primary teachers (N = 87; 73 females and 14 males) as participants. They were studying in the 3rd year of the Primary School Teaching Department of an Islamic University in South Sulawesi, Indonesia. Participants were purposively selected from students enrolled in a 'mathematics for primary school' course. Ten of the 87 participants, representing the high, middle, and low cognitive groups, were interviewed.

Instrument and Procedures

This study investigated the PSPSTs' difficulties encountered while solving context-based mathematics tasks. A test with five context-based mathematics problems about arithmetic, including numbers and their properties and operations, as fundamental mathematics concepts at the primary level, was employed to explore all 87 PSPTSs' problem-solving strategies and solution accuracy while solving such problems. A semi-structured interview approach was also adopted to collect useful qualitative information that would provide in-depth information about PSTSs' ways of thinking, rationale behind their answers, and type of error they made based on the error classification proposed in the Newman Error Procedure.

The five context-based arithmetic problems used in this study are as follows:

1. My mother has $3\frac{3}{4}$ kg of sago flour. $\frac{1}{3}$ part of the flour is used to make sago cheese cake. My mother then buys an additional 2 1/2 kg of flour. How many kilograms of flour does my mother have now?
2. During winter, the temperature in Alaska City experienced extreme changes. Today's weather report shows that the temperature in the afternoon reaches -2°C and drops dramatically to -19°C in the evening. What is the temperature difference (in °C) between the afternoon and evening in Alaska City?
3. Mrs. Rissa paid the price of three fans using 10 pieces of one hundred thousand rupiah, two pieces of fifty thousand rupiah, and one piece of two thousand rupiah without any change. Mrs. Rissa asked the cashier for a purchase receipt. Write the number representing the price of the fans that the cashier should write on the receipt!

Stimulus for questions 4 and 5

Mrs. Dika and Mrs. Jeni went to the traditional market to buy daily necessities. In the market, they checked the prices of several items they wanted to buy, and the results are as follows (see Table 1).

Table 1. Price of Item(s).

No	Item(s)	Price (in Rupiah)
1	Rice “Super Enak” (5 kg)	69.700
2	Rice “Putih Premium” (2 kg)	28.800
3	Egg (1 kg)	32.000
4	Sugar (1 kg)	13.000
5	Apple (1 kg)	42.000
6	Wheat Flour (1 kg)	13.500
7	Chicken (1 kg)	57.000

4. Mrs. Dika brought shopping money amounting to Rp 220,000.00 (two pieces of one hundred thousand rupiah and two piece of ten thousand rupiah). The first store Mrs. Dika visited was a store that sold sugar and wheat flour. If Mrs. Dika buys 1 kg of sugar and 2 kg of wheat flour, how much change will she receive from the cashier?
5. Mrs. Jeni only brought money amounting to Rp 100,000.00, and she intends to spend that money shopping. Mrs. Jeni decides to buy 1 kg of Rice ‘Super Enak’, 1 kg of sugar, and 1 kg of wheat flour. In your opinion, is Mrs. Jeni's decision correct to buy these items? Explain.

Data Analysis

The data were analyzed quantitatively and qualitatively using summative qualitative content analysis. Quantitative analysis was performed to determine the percentage of each type of error encountered by the PSPSTs in solving arithmetic problems. Newman (1977) identified five common types of errors made by students in solving written mathematics test: reading errors, comprehension errors, transformation errors, process skills errors, and encoding errors. Analyzing these errors comprehensively is known as Newman Error Analysis (NEA).

Reading errors occur when students misread or fail to accurately understand important information from a question. For example, students are wrong reading specific terms, symbols, numbers, words, or important information in the question, resulting in inappropriate answers. Comprehension error is the second type of error in which students have read the problem well, but do not understand the meaning of the question. Comprehension errors occur when students do not grasp the concept or intent of the question or material (e.g., they cannot identify known and asked about the problem), even after reading it correctly. In other words, reading errors are associated with mistakes in reading or understanding information accurately, while comprehension errors involve difficulties in understanding the concepts or meanings of the information, even after reading it correctly. Reading errors are more related to the accuracy of understanding words or phrases, while comprehension errors are more related to the overall understanding of concepts (Clemen, 1980).

Transformation error is the third type of error that occurs when students make mistakes in changing a problem into a mathematical model such as equations, drawings, graphics, or tables. Process skill error is a student's mistake in choosing rules/procedures or students already using correct procedures/rules, but errors occur in the calculation or computation. Encoding error is the fifth kind of mistake that students in this case make in writing the answer correctly, cannot show the truth of the answer, or do not write the conclusion of the answer (Clements & Ellerton, 1996).

The data analysis procedures in this study included several stages: analyzing all PSPSTs' answers, classification of error types based on NEP, calculating the percentage of each error type, qualitatively analyzing the error made by PSPSTs based on written responses, and analyzing the interview analysis to acquire in-depth information about PSPSTs' difficulty in solving problems. Data from in-depth interviews were qualitatively analyzed to obtain detailed information about the rationale behind PSPSTs' answers and the type of error they made based on the error classification proposed in NEP.

In the initial phase, the analysis focused on evaluating the errors made by PSPSTs by examining their responses in the test answer documents. These errors were independently identified, suggesting that in this situation, it is likely that PSPSTs may made multiple mistakes on different occasions. In every item of the problem, they could perform more than one error. For example, in Problem 1, a participant made errors in comprehension and transformation. Then, for every kind of error in NEA, the researchers calculated the percentage to identify the number of PSPTs who committed each type of error, dividing it by the total number of research participants (N = 87).

Furthermore, the data were analyzed qualitatively. Data on PSPSTs' written responses in solving context-based arithmetic problems were analyzed using the document content analysis method. Qualitative analysis was performed to analyze the error of PSPSTs based on NEP. Qualitative data analysis consists of three stages: data reduction, data display, and drawing conclusions (Huberman & Miles, 2019). The initial step of qualitative data analysis in this research involved analyzing the PSPSTs' written responses and in-depth interview results in solving context-based arithmetic problems. Coding was then conducted to ease the process of content analysis. During the data reduction phase, the researchers simplified the gathered information from the initial stage (comprising interview results and documents of students' written responses) by organizing and selecting essential data, recognizing themes and patterns, and excluding irrelevant information. Relevant information regarding PSPSTs' difficulties in solving context-based arithmetic problems was then grouped into themes or categories (i.e., categories of error types). Subsequently, researchers interpreted the reduced and displayed data. Finally, the researchers performed verification to guarantee the precision of the analysis outcomes, draw conclusions, formulate findings, and attribute significance derived from the data analysis. Atlas.ti (version 22.2.5) software was used to employ these qualitative analysis procedures, particularly in the coding, data reduction, and data display stages.

RESULTS AND DISCUSSION

Results

Written responses to the tasks were analyzed using the NEA framework. The analysis focuses on the error made by PSPSTs in answering context-based arithmetic problems. The representative answers indicate various types of errors from the 10 PSPSTs, which were further explored in-depth through the interview process. Using NEA, researchers identified the error type that PTPSs made when solving context-based arithmetic problems. The numbers of PSPSTs who gave incorrect answers, correct answers, and those who did not answer are provided in Table 2.

Table 2. Summary of the students' error analysis result (N = 87).

	Problem 1		**Problem 2**		**Problem 3**		**Problem 4**		**Problem 5**		**Average**
	n	%	n	%	n	%	n	%	n	%	%
Correct Answer	65	74.7	54	62.1	22	25.3	27	31.0	69	79.3	54.5
No Answer	0	0	0	0	3	3.4	2	2.3	0	0	1.1
Incorrect Answer	22	25.3	33	37.9	62	71.3	58	66.7	18	20.7	44.4
Reading Error	7	8.0	0	0	11	12.6	58	66.7	0	0	17.5
Comprehension Error	15	17.2	0	0	23	26.4	58	66.7	0	0	22.1
Transformation Error	0	0	6	6.9	58	66.7	0	0	0	0	14.7
Process Skill Error	0	0	27	31.0	4	0	0	0	11	12.6	8.7
Encode Error	0	0	0	0	0	0	58	66.7	18	20.7	17.5

A detailed explanation regarding examples and underlying reasons beyond the occurrence of the five types of errors is presented as follows, both in written and verbal forms.

Reading error. This type of error occurs when students make errors when reading important words or crucial information in a question. These errors occur because of missing or unused important information from the question in problem-solving. An example of a reading error committed by the PSPST is presented in Figure 1.

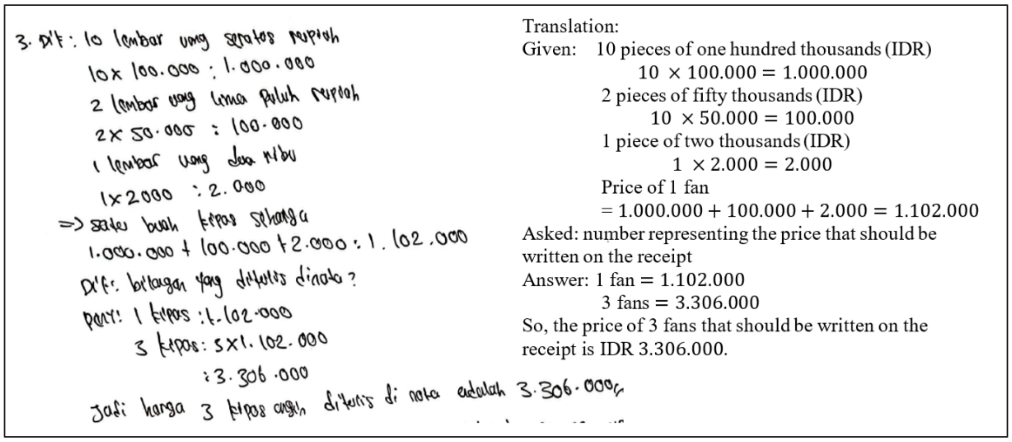


Figure 1. Written response of PSPST S-01 on problem 3

Figure 1 shows that there is an error of solution performed by S-01 on problem 3. By using the place value concept, S-01 was able to correctly calculate and write the number that represents the amount of money paid by Mrs. Rissa (IDR 1.102.000). From the information in figure 1, it is also apparent that S-01 is able to write what is asked in the question, which is the price of the three fans that should be written by the cashier on the receipt. However, an error occurred in the written problem solving strategy. Interviews were conducted to identify the cause of this error. The results of the interview between the researcher (R) and one of the PSPSTs (coded as S-01) are as follows:

R: Can you explain why it's 3×Rp 1.102.000,00

S-01: Because the question asked for the price of 3 fans. The price for one fan is Rp 1.102.000,00.

R: Oh, really? Can you show me where it states that the price of one fan is Rp 1.102.000,00?

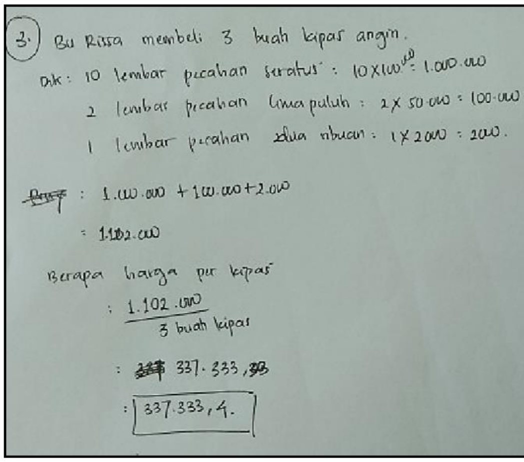
S-01: Carefully reading the question].

S-01: Oh, there's no information about the price of one fan. That means I didn't read it carefully. I thought the given information of Rp 1.102.000,00 was the price for one fan.

R: But do you understand the intention of the question? Where is the mistake?

S-01: Yes, I understand, Ma'am. Rp 1.102.000,00 is already the price for 3 fans. The question was asking for the price of 3 fans as well.

Comprehension error. This error type is related to the PSPSTs' understanding and interpretation of texts or questions. The inability of the test taker to understand accurately the given information and what is being asked in the question strongly indicates their difficulty in comprehending the problem. The following is an example of a comprehension error committed by PSPST S-02 (Figure 2).



Translation:
Mrs. Rissa bought 3 fans.
Given: 10 pieces of one hundred thousands (IDR)
$10 \times 100.000 = 1.000.000$
2 pieces of fifty thousands (IDR)
$10 \times 50.000 = 100.000$
1 piece of two thousands (IDR)
$1 \times 2.000 = 2.000$
$= 1.000.000 + 100.000 + 2.000 = 1.102.000$
Asked: how much is the price of one fan?
Answer: $= \frac{1.102.000}{3\ fans}$
$= 337.333{,}33$

Figure 2. Written response of PSPST S-02 on problem 3

Unlike S-01, who made a reading error in solving question 3 about a purchase receipt, another student, S-02, made a comprehension error in the same question. Figure 2 indicates that student S-02 failed to answer question 3 correctly. Looking at the given answers, S-02 was able to write down the information provided by the question. However, they made a mistake in writing down what was asked in the question. To investigate whether this error was due to a reading error or a comprehension error, an interview was conducted with the PSPST S-02, with the following results.

R: What is asked in problem 3?]

S-02: Price of one fan

R: What do you mean by the price of 1 fan? Can you explain?

S-02: Yes, Ma'am. 1 fan. Because Mrs. Rissa paid for 3 fans. The question asked for the price of 1 fan.

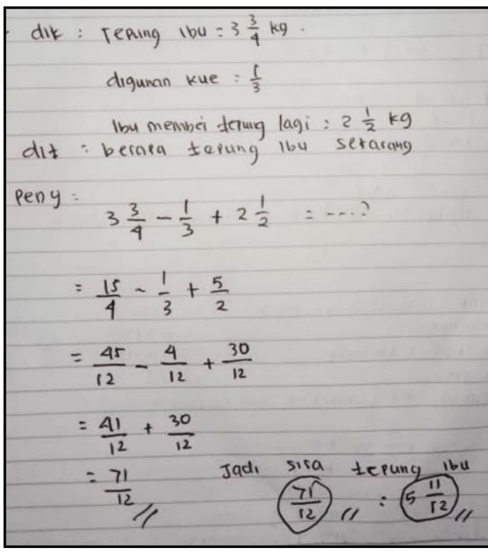
R: Is it written in the question that they are asking for the price of 1 fan?

S-02: No, it's not mentioned explicitly here about the price of 1 fan. Since it was stated that Mrs. Rissa paid for 3 fans, I assumed that the price for 3 fans was already known (more than 1 million). So, I thought they were asking for the price per fan.

The interview fragment indicated that S-02 was capable of reading the question but failed to comprehend it accurately. S-02 read all the information from the question but did not understand the meaning of the sentence "the price of the fan that should be written by the cashier on the receipt." S-02 interpreted this sentence as indicating that information about the prices of the three fans has already been given in the question, so it is unlikely that the same information is being asked again. This interpretation leads to the prediction that the question concerns the price per fan. Therefore, it can be concluded that S-02's comprehension error stems from an interpretation mistake in the given question sentence. S-02 had difficulty comprehending the meaning of the questions. Consequently, he was unable to identify accurately what was being asked. Therefore, the identification of problem-solving strategies does not proceed well, because the meaning of the question itself is not well understood.

Transformation error. This type of error occurs when test takers make mistakes in transforming a word problem into an appropriate mathematical model. Transformation errors will result in errors in selecting the appropriate formula or employing the calculation process, ultimately leading to students solving the problem incorrectly.

Figure 3 shows that S-03 committed a transformation error by being unable to convert or transform the information from Question 1 into a mathematical model. In the context of this question, a transformation error occurs because S-03 is unable to convert the narrative information in the contextual-based problem into a mathematical model. Upon further investigation through an interview, S-03 explained that: "I transformed the information from the question “My mother has $3\frac{3}{4}$ kg of sago flour. $\frac{1}{3}$ part of the flour is used to make sago cheese cake” into math sentence $3\frac{1}{3} - \frac{1}{3}$” From this statement, it can be perceived that the mathematical model created by S-03 is incorrect. This error occurred since S-03 did not thoroughly understand the meaning of “$\frac{1}{3}$ part of the flour”. In other words, the transformation error that occurs in this context is the inability of the test taker to accurately create a mathematical model because of a lack of understanding of the question (comprehension error). However, not all transformation errors caused by comprehension errors. There is no guarantee that students who do not have difficulty understanding the meaning of the question can perform the transformation process accurately.



Translation:

Given: Mother’s wheat flour = $3\frac{3}{4}$ kg

Used for making cookies = $\frac{1}{3}$ kg

Buying additional flour = $2\frac{1}{2}$ kg

Asked: how many kg of flour does my mother have now?

Answer $3\frac{3}{4} - \frac{1}{3} + 2\frac{1}{2} = \cdots ?$

$= \frac{15}{4} - \frac{1}{3} + \frac{5}{2}$

$= \frac{45}{12} - \frac{4}{12} + \frac{30}{12}$

$= \frac{41}{12} + \frac{30}{12} = \frac{71}{12}$

So, the remaining flour of my mother now is $\frac{71}{12} = 5\frac{11}{12}$.

Figure 3. Written response of PSPST S-03 on problem 1

Process skill error. This error type is characterized by test-taker-making errors when applying a formula, making mistakes when performing algorithms or mathematical calculations, encountering errors in algebraic manipulation, and struggling to apply mathematical processes when solving a problem. The following is an example of a process skill error committed by the PSPST S-04.

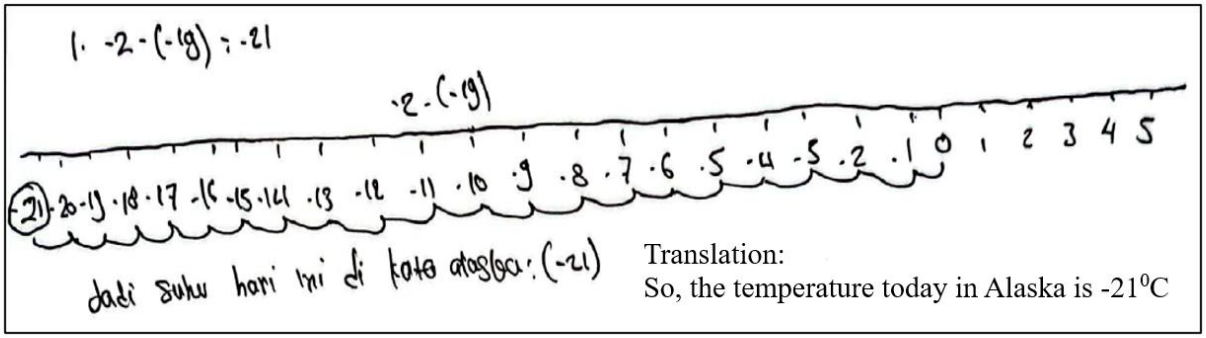
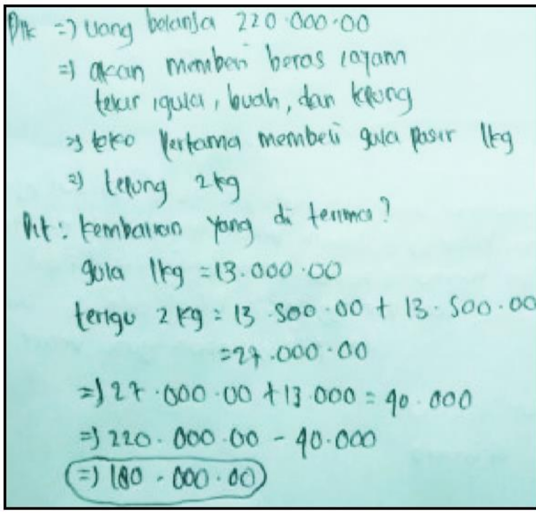


Figure 4. Written response of PSPST S-04 on problem 2

Figure 4 shows the process skill error. This written response was then confirmed through an interview. The interview results indicate that S-04 has good comprehension, being able to understand the meaning of the question well and accurately identify what is given and what is asked. Student S-04 stated, "The temperature in the afternoon is -2°C, and the temperature at night is -19°C. The question asks for the temperature difference, which means subtracting -2 from -19." This statement indicates that S-04 could interpret the words 'difference' and 'temperature decrease' accurately and can formulate them into the mathematical model $2-(-19)=\cdots$. S-04's verbal response indicates that he can explain the correct mathematical model and formulation to apply but fails to execute the procedure accurately. This error occurs because he does not understand the concept of integer operations that involve negative integers. The use of the number line does not help either because he does not understand the concept of subtracting negative integers.

Encoding error. This type of error is indicated by the test-taker's error in drawing the conclusion. Encoding error refers to the mistake in accurately expressing the mathematical solution in a written format in a real-life context. Encoding errors can also occur because of the inability to interpret or validate whether the generated mathematical solution is appropriate in accordance with the context of the problem. The PSPST S-04 exemplifies an encoding error, as shown in Figure 5.



Translation:
Given: Mshopping money IDR 220.000
willing to buy rice, chicken, egg, sugar, fruit, flour
buying sugar 1 kg and flour in the first store
Asked: how much changes she receive?
Answer: Sugar 1 kg = 13.000 (IDR)
Flour 2 kg = 13.500 + 13.500 = 27.000 (IDR)
= 27.000 + 13.000 = 40.000 (IDR)
= 220.000 – 40.000 = 180.000 (IDR)

Figure 5. Written response of PSPST S-07 on problem 4

Upon initial observation, there seems to be no error in the response of S-07, as shown in Figure 5. S-07 demonstrated the ability to perform process skills using the correct algorithm and calculation procedures. However, upon closer examination of the given context of the problem, it was stated that Mrs. Dika has two pieces of one hundred thousand rupiah (IDR) and two pieces of ten thousand rupiah (IDR), which are used to purchase sugar and flour with a total price of IDR 40.000. The question asks for the change Mrs. Dika receives from the cashier. If we connect this with the real-world context, it can be assumed that the most likely scenario is that Mrs. Dika will pay for sugar and flour with one piece of one hundred thousand rupiah money. To pay for an item priced at IDR 40.000, Mrs. Dika does not need to use all the money she has (two pieces of one hundred thousand rupiah and two pieces of ten thousand rupiah). Based on this assumption, it can be concluded that the change Mrs. Dika receives is IDR 60.000.

The error that occurred was S-07's inability to interpret a mathematical solution in terms of a real solution. He was unable to verify whether the mathematical solution was reasonable or aligned with the context of the problem. Based on the interview results, S-07 made an encoding error because of misinformation while reading the problem. They overlooked specific details about Mrs. Dika's money (two pieces of one hundred thousand rupiah and two pieces of ten thousand rupiah). S-07 only focused on the total amount of IDR 220.000, which led them to simply interpret how much money Mrs. Dika had left after paying for sugar and flour. The main mistake made by S-07 was the inability to connect the given problem to a real-life context. In a real-life scenario, when we want to pay for an item worth Rp 40.000 while having two pieces of one hundred thousand and two pieces of ten thousand, we would typically use one piece of one hundred thousand (without using the entire IDR 220.000). S-07 acknowledged this mistake during the interviews.

DISCUSSION

The analysis of PSPSTs' written discourse and verbal responses through interviews indicated that many of them made errors in answering the given questions. In questions 3 and 4, the percentage of students who answered incorrectly exceeded 65% (see Table 2). Similarly, for the other three questions, the error percentage ranged from 20 to 40%. This clearly indicates that many students still face difficulties in solving context-based mathematical problems.

The data in Table 2 also indicate that many prospective teachers have difficulties with the five NEA procedures. Process skill errors are the least common type of error. Conversely, the most frequent errors were comprehension (22.1%), followed by reading and encoding errors, each at 17.5%. These findings are quite surprising, especially regarding the high occurrence of reading and encoding errors. These findings contradict with several previous studies (Alhassora et al., 2017; Fitriani et al., 2018; Wijaya et al., 2014) which mostly showed that reading errors are rarely, even do not not at all, committed by test takers. Similarly, in terms of encoding, multiple studies have shown that encoding is the least common type of error committed by students (Fitriani et al., 2018; Hadi et al., 2018; Wijaya et al., 2014). Similar findings were also revealed in several other studies (Hadi et al., 2018; Shinariko et al., 2020), which revealed that no reading and encoding errors were identified for all recorded questions. Alhassora et al. (2017) also reported that students generally had good reading and encoding abilities in their study.

Specifically, students experienced reading errors when solving questions 1, 3, and 4. The students' answers and interviews for these three questions indicated that reading errors occurred because students made mistakes in reading important words in the questions or misunderstood the main information. As a result, they did not use this information to solve the problem. These errors were caused by students' lack of attentiveness or disregard for crucial words or sentences, leading to misunderstandings of the entire question. Reading errors have implications for transformation errors, which involve mistakes when modeling a situation into mathematical sentences. As revealed by Isik & Kar (2012) in their study, pre-service elementary mathematics teachers face linguistic challenges when transitioning from symbolic to verbal representation. Based on the students' answers and interview results, it was found that reading errors directly impact comprehension errors, and consequently, the transformation stage, ultimately leading to incorrect final answers. These findings support Newman’s (1977) argument regarding the hierarchical structure of error types. They suggested that when students encounter difficulties in a specific step of a task, it becomes a barrier that hinders their progress to the subsequent step.

The next type of error is comprehension error. Comprehending this question is an essential element of the problem-solving process. However, the findings of this study reveal that many PSPSTs still struggle to thoroughly understand the given information from context-based mathematics problems (more than 20% error). This finding reinforces the results of previous research (Burgos & Godino, 2022) that the difficulties most frequently identified by prospective teachers were those concerned with understanding the statement requirements from the problems.

The comprehension errors experienced by the PSPSTs in this study are caused by misinterpretation of the given question sentences. There is a misunderstanding about given keywords and instructions of the problem. The errors resulting from misunderstanding the instructions are closely related to reading errors. Due to the lengthy sentences and abundant information presented in context-based problem, PSPSTs experienced confusion to determine what should they do with the problems. Consequently, they struggled to extract the intended meaning from the problem, leading to potential misunderstandings that created uncertainty about the appropriate approach to solve the problems (Tambychik & Meerah, 2010).

However, it is crucial to highlight the fact that comprehension errors differ from reading errors. Reading errors are connected to inaccuracies in reading or understanding information, whereas comprehension errors involve difficulties in grasping the concepts or meanings of information even after reading it correctly (Clemen, 1980). As found in this study, the PSPSTs were able to correctly read information in questions, including words, sentences, and numbers. However, they struggled to comprehend the meaning of the sentences, failed to discern what the question was asking, and lacked a solid understanding of the concepts of the problem.

Furthermore, it should be emphasized that not all comprehension errors occur because of reading errors. Another case in this study is when students are able to read the problem comprehensively but fail to understand the meaning of the problem, resulting in an incorrect interpretation of what they are asked to do. In addition to misunderstanding the keywords of the problem, comprehension errors can also be caused by participants being unable to select relevant and irrelevant information given in the problem (Wijaya et al., 2014). Comprehension errors indicate mistakes in PSPSTs' understanding of the question when they have difficulty comprehending its meaning. Consequently, they are unable to identify what is being asked and what information is known from the given question. Errors in understanding the meaning of the question lead to students' inability to write down the important and relevant information required from the given question. The written answer provided by the PSPST did not represent what was asked or what was given in the test item, and it did not lead to the correct answer (Hadi et al., 2018). As observed in the error made by PSPST in answering questions 3 and 5 (see Figure 2 and 5), PSPST tends to utilize all the numbers provided in a task while neglecting the relevant information. These findings align with previous studies that indicate students face difficulties in understanding the wording of context-based tasks and identifying relevant information (Prakitipong & Nakamura, 2006; Wijaya et al., 2015).

Apart from understanding the problems, other difficulties encountered by PSPSTs is performing mathematics modeling. Often, PSPSTs understood the problems well, yet they still could not solve them. Multiple studies have demonstrated that transformation errors have become a major type of error experienced by test participants, particularly students, in solving mathematical problems (Shinariko et al., 2020). The findings of this study indicate that transformation errors occur because of students' inability to transform narrative information in contextual-based items into mathematical models. Students' failure to convert information from text to more abstract mental representations is attributed to their poor comprehension of the meaning of the problem. In other words, comprehending errors contribute to the occurrence of transformation errors. Transformation errors can indirectly result from reading errors, leading to comprehension errors. There is significant potential for misunderstanding the meaning of the problem, which can greatly contribute to errors in mathematical modeling or transformation errors.

However, it should be emphasized that not all transformation errors are caused by comprehension errors. The findings of this study also indicate that there are prospective teachers who make transformation errors even though they correctly understand the intent of the problem. This is because of the lack of ability of PSPSTs to model real-life situations into formal mathematical models. Transformation errors occur because participants are unable to construct a mathematical model by mathematizing a real-world problem that represents the given situation.

The next error type is the process skill error. Process skill errors were not major errors caused by the PSPSTs in this study. It appears that students do not struggle much to perform mathematical calculation procedures. Technical calculation errors were also minimal. This is because the calculation algorithms required to solve context-based mathematics problems in this study can be considered very simple for prospective mathematics teachers, involving basic arithmetic operations, such as addition, subtraction, and multiplication. The high occurrence of process skill errors in this study (see Figure 4) depicts, the failure to perform the procedure accurately, which is caused by PSPSTs' lack of understanding of the concept of integer operations involving negative numbers. Operations involving negative numbers are often considered challenging topic for many students.

The next error type is the process skill. However, process skill errors were not the major errors committed by the PSPSTs in this study. It appears that students did not encounter significant difficulties in performing mathematical calculation procedures. Technical calculation errors were also minimal. This is because the calculation algorithms required to solve context-based mathematics problems in this study can be considered quite simple for prospective mathematics teachers, involving basic arithmetic operations such as addition, subtraction, and multiplication. The process skill error that occurred most frequently in this study (see Figure 4) reflects the inability to accurately perform the procedure, which is caused by PSPSTs' lack of understanding of the concept of integer operations involving negative numbers. Operations involving negative numbers are often considered to be challenging for many mathematics learners. Previous research has also indicated challenges in the conceptual understanding of negative integers (Almeida & Bruno, 2014, Fuadiah et al., 2019) and their depiction of the number line among pre-service teachers (Widjaja et al., 2011).

The last type of error is encoding. Encoding errors can occur because of students' inability to interpret and evaluate mathematical solutions obtained in real-world applications. The students struggled to accurately understand and verify the mathematical solution in relation to real-life problems. This mistake is indicated by an impossible or unrealistic answer (Wijaya et al., 2014).

In this study, encoding errors were frequently found in the PSPSTs' responses in solving problems 4 and 5. Based on the analysis of students' written responses, encoding errors were not the least common type of error committed by the PSPSTs. More than 17% of participants had encoding errors. These findings tend to contradict previous research indicating that test takers made fewer errors in the interpretation of the mathematical solution in real-world situations (Alhassora et al., 2017, Fitriani et al., 2018, Wijaya et al., 2014).

The findings of this study revealed that the occurrence of errors committed by participants has a hierarchical structure. This means that when participants make errors at the initial stages (e.g., reading or comprehension errors), there is a high possibility that they will become stuck or unable to proceed with the problem-solving process in the subsequent stages (e.g., transformation, process skill, and encoding). Alternatively, under different conditions, there is a high probability that they will cause errors in subsequent stages. When PSPSTs are already stuck in the early stages (e.g., reading, comprehending, or modeling process), they will not arrive at the stage of carrying out mathematical procedures and performing encoding accurately. Additionally, the research findings also revealed that reading comprehension significantly influences the ability of PSPSTs to solve context-based mathematics problems correctly. Reading and comprehension errors are two types of errors that act as initial barriers to PSPSTs, preventing them from accurately solving problems.

CONCLUSIONS

The analysis of PSPSTs' written and verbal responses through interviews revealed that many encountered difficulties in solving context-based mathematics problems. This is indicated by the number of errors made. Based on the error types in the NEA framework, the most common error committed is comprehension error (22.1%), followed by reading and encoding errors, each accounting for 17.5%, transformation error (14.7%), and process skill error as the least common type among the others (8.7%). Furthermore, the study findings indicate a hierarchical structure in the occurrence of errors among test-takers. Errors in the initial stages significantly impact the progression of the problem-solving process in subsequent stages. Test takers who encounter challenges in the early stages (e.g., reading and comprehension) are highly likely to find difficulties and make errors when performing transformation, process skill, and encoding in the subsequent stages. The hierarchical identification of specific cause-and-effect relationships between error types could be a topic for further research.

In light of these findings, it is recommended to focus on addressing identified error types to improve the problem-solving abilities of PSPSTs. Emphasizing strategies and interventions to enhance reading, comprehension, transformation (modeling), processing skills, and encoding skills can significantly help PSPSTs to be more ready to teach mathematics in real class situations. Furthermore, future research should explore the hierarchical relationships among error types in greater depth. Investigating specific cause-and-effect relationships between error types can provide valuable insights into instructional design and tailored interventions to enhance problem-solving proficiency. Overall, these recommendations aim to enhance the overall performance of PSPSTs by addressing specific error types, fostering a comprehensive understanding of mathematical problems, and improving their ability to solve context-based problems.

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From Informal to Formal Proof in Geometry: a Preliminary Study of Scaffolding-based Interventions for Improving Preservice Teachers' Level of Proof

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***Abstract**: This is a preliminary study of design research that investigates preservice mathematics teachers' proof level and the possible task of scaffolding-based interventions in proving the triangle theorem. The research subjects consisted of 58 second-semester mathematics education students at Universitas Negeri Surabaya, Indonesia. This research is descriptive using quantitative and qualitative approaches. Data collection uses a test to determine the level of proof of prospective mathematics teachers based on Miyazaki's classification. This method classifies four levels in constructing a proof, mainly Proof A, Proof B (deductive), Proof C, and Proof D (inductive). The results showed that there were 38% of students' answers in constructing proof with level Proof A, 5% of students' answers in constructing proof with level Proof B, 15% of students' answers in constructing proof with level Proof C, and the remaining 42% of students' answers in constructing proof with level Proof D. Furthermore, the scaffolding-based intervention task refers to the preservice teacher's difficulties in proving the triangle theorem, including a lack of understanding of concepts, not understanding language and mathematical notation and difficulties in starting proofs.*

Keywords: Level of Proof, Scaffolding, Geometry

INTRODUCTION

Proof is at the heart of mathematical thinking and deductive reasoning (Cheng & Lin, 2009). Hernadi (2008) explains that proof is a series of logical arguments that explain the truth of a statement. Mingus and Grassl (1999) define proof as a collection of statements that are true and linked together in a logical way that serve as arguments to convince other of the truth of mathematical statements. Meanwhile, Griffiths (2000) states that mathematical proof is a formal and logical way of thinking that starts with axioms and moves forward through logical steps to a conclusion. In addition, proof is also a major component of understanding mathematics (Kogce et al, 2010). Proof is recognized as the core of mathematical thinking (Hanna et al, 2009). One cannot study mathematics without studying mathematical proofs and how to make them (Balancheff, 2010).

The role of proof for a mathematics learner is a determinant of the level of maturity in the process of thinking mathematics (Otten et al., 2014). This is because proof requires a person to use mathematical knowledge and write it down in a logical argument, so it requires a comprehensive mathematical thinking process (Cervantes-Barraza et al., 2020). Recently, several universities have begun to introduce lectures on the introduction of proof or mathematical reasoning programs (Epp, 2003, Selden & Selden, 2007), which aim to make it easier for students to understand the formal language of mathematics and its axiomatic structure. This can be seen in the first year students at Universitas Negeri Surabaya where this research took place, because the majority of students have been provided with the initial lecture program, namely in the fundamentals of Mathematics and number theory lectures. Clark and Lovric (2008) say that in the process of transitioning into constructing mathematical proofs for students there are many challenges to be faced. They suggest that this transition requires students to change the type of reasoning used, namely shifting from informal to formal language; for reasons of using mathematical definitions; to understand and apply theorems; and make connections between objects in mathematics.

Various research results have concluded that the learning process regarding proof of university students has not reached the optimal stage as expected (Azrou & Khelladi, 2019, Daguplo & Development, 2014, Jones, 2010, Michael et al., 2013). The research results of Reiss and Renkl (2002) revealed that there were still many student limitations in the proving process. Furthermore, Maarif et al. (2018) concluded from the results of their research that the limitations of student concepts in constructing geometric proofs included difficulties in sketching diagrams with proper geometric labels and difficulties in constructing conjectures in writing formal proofs. In addition, Moore (1994) also said that students were unable to understand and use language and mathematical notation in compiling proof. From this, it is necessary for us to optimize the process of exploring the ability to construct proof in order to improve preservice teachers' level of proof in geometry.

Proof in mathematics consists of several universally accepted methods. The methods used in the proof are divided into 2, namely the deduction method and the induction method (Siswono et al., 2020). Proof is recognized as the core of mathematical thinking and deductive reasoning (Cheng & Lin, 2009). In deductive proof, a conclusion must be true if the premises are true (Anderson, 1985). The deduction method involves several methods such as direct proof, proof with contraposition and proof with contradiction (Morali et al., 2006). Whereas in inductive proof, arguments whose conclusions are not necessarily true but are very likely to be valid (Anderson, 1985). Miyazaki (2000) classifies proof into four levels, namely Proof A, Proof B, Proof C, and Proof D. According to Miyazaki (2000), Proof A is a level of proof that involves deductive reasoning and functional language used in working on the proof, Proof B is a level of proof that involves deductive reasoning and does not use functional language, images, or manipulation of objects that can be used in the process of proving. Whereas Proof C is a level of proof that involves inductive reasoning and does not use functional language, images, or manipulation of objects that can be used in the process of making proofs, Proof D is a level of proof that involves inductive reasoning and functional language used in proving.

Miyazaki's (2000) research explains more about levels in algebra, but in this study the focus will be on geometry. Even though proof is very important, there are still many students who experience difficulties in proof (Stylianides & Philippou, 2007, Weber, 2001). Because students often show difficulty in proving, researchers submit assignments to students, and in addition, provide scaffolding through Hypothetical Learning Trajectory (HLT) as a strategy to help students' difficulties in proving so that students can increase their level from informal to formal (Rahayu & Cintamulya, 2021). Anghileri (2006) divides the scaffolding hierarchy into three levels in learning mathematics. In scaffolding Level 1 is the most basic level. At this level, a suitable learning environment is needed that can support the learning process. Level 2 in scaffolding is known for several types, namely explaining, reviewing, and restructuring. Assistance provided at that level is used by students to achieve understanding. Level 3 in scaffolding is conceptual development, namely the level of scaffolding that develops concepts students already understand to build connections between concepts.

Scaffolding is given to students who experience difficulty in proving through Hypothetical Learning Trajectory (HLT). HLT is a description of students' thinking during the learning process in the form of conjectures and hypotheses from a series of learning designs to encourage the development of students' thinking so that mathematics learning objectives can be achieved as expected (Afriansyah & Arwadi, 2021, Sarama & Clements, 2004). The term hypothetical learning trajectory (HLT) itself was first proposed and used by Simon (1995) who stated that hypothetical learning trajectory consists of three components in the form of learning objectives, learning activities, and alleged learning processes - predictions about how students' thinking and understanding will develop in the future context of learning activities. The aim intended in this research is to achieve an understanding of the concept of proof. The intended learning activity is a series of tasks to find out how students can prove. The intended hypothesis of students' way of thinking is students' flow of thinking in understanding the concept of proof with the help of scaffolding according to Anghileri (2006).

HLT is very necessary in designing learning that will suit students' thinking patterns and characteristics (Rezky, 2019). In this research, HLT is a learning tool that contains a series of instructional tasks in the form of scaffolding and anticipation of possible difficulties that may occur for students in proving in order to help students understand the concept of proof so that students can increase their level from informal to formal. HLT with scaffolding is very rarely used by teachers in designing lessons, especially about geometric proofs in class. With this HLT, researchers hope to help teachers when teaching the concept of proving geometric theorems in class. Based on the description above, this study aims to investigate pre-service mathematics teachers' proof level and the possible task of scaffolding-based interventions through HLT in proving the triangle theorem.

METHOD

Research Approach and Design

The method used in this research is a preliminary study of design research. Researchers followed three research phases: the initial design stage (preliminary design); design testing through preliminary teaching and teaching experiments; and the retrospective analysis stage (Gravemeijer & Cobb, 2006). In this article, the discussion focuses only on the initial design stage (preliminary design). To explain a preliminary study, the researcher uses descriptive research using quantitative and qualitative approaches. At the preliminary stage, the researcher wanted to look at preserving teachers' levels of understanding of proof and preserving teachers' learning trajectories. The participants involved in this study were 58 second-semester mathematics education students at Universitas Negeri Surabaya, Indonesia. There were 2 classes, in which each class consisted of 29 prospective teachers. The choice of research location was based on the curriculum structure of the research location. There is a Basic Geometry course that accommodates proving geometry as an outcome of the learning process. In addition, the selection of the research location was carried out at the author's institution on the grounds that from previous experience teaching geometry, there were still many students who had difficulty in constructing of proof.

Data Collection

The data collection technique to see teachers' levels of understanding of proof was carried out by giving a mathematical proof test to 58 students. The data was taken from the results of student work after the lecture process ended, then they were given a 15-minute mathematical proof test to construct geometric proofs. Afterwards, each prospective teacher's response was assessed to pre-service mathematics teachers' proof level of their deductive and inductive knowledge in constructing a proof. The present study tends to examine more on deductive and inductive proof without employing interviews like what Miyazaki (2000) did. The data were collected using a simple task of constructing one mathematical proof, namely to prove that the sum of the angles in a triangle is 180^0. Actually, the task type could be more than one, such as the sum of the three external angles of a triangle is 360^0, or prove the sum of the measures of the angles of a pentagon is 540^0. However, the main point of this study was a proof method whether using deductive proof or inductive proof at each level of proof.

Data Analysis

The process of assessing student answers is carried out by providing scoring coding following the level of proof of Miyazaki's classification (2000) in constructing a geometric proof. Since the subject of this study is the early mathematics education students who had received both methods in high school (Brady & Bowd, 2005) and these two methods have often been used by previous researchers in constructing a proof at the university level (Almeida, 2001). Furthermore, researchers try to make a student's learning trajectory (LT) for constructing a proof. This LT has yet to be tested on small-scale subjects; it was only made based on learning possibilities that can be used in constructing proof. See Table 1.

Table 1. Levels of proof in mathematics (Miyazaki, 2000).

Representation	Method	
	Deductive	**Inductive**
Using functional language according to the theorem	Proof A	Proof D
Do not use functional languages, use images, or manipulate objects	Proof B	Proof C

The process of assessing student answers is carried out by providing scoring coding following the level of proof of Miyazaki's classification (2000) in constructing a geometric proof. Since the subject of this study is the early mathematics education students who had received both methods in high school (Brady & Bowd, 2005) and these two methods have often been used by previous researchers in constructing a proof at the university level (Almeida, 2001). Furthermore, from students' answers that show the results of Proof B, C, and D (non-formal proof) they will be assisted with scaffolding via HLT to help students' difficulties in proving so that students can increase their level from informal to formal (Proof A). The scaffolding used in this research refers to Anghileri's (2006) theory, namely level 1 (environmental provisions), level 2 (Explaning, Reviewing, and Restructuring), and level 3 (Developing Conceptual Thinking). This scaffolding is carried out through HLT which will be prepared by researchers to increase student evidence from informal to formal. HLT can support students in their understanding and construction of a proof (Anwar et al, 2022). According to Anwar et al (2022), HLT activity is using reading and constructing proof through constructing a geometric figure. Meanwhile, according to Agustiani and Nursalim (2020) there are four activities in HLT for proof, namely reading proof, completing proof, examining proof, and constructing proof. So this research uses the four HLT activities used by Agustin and Nursalim (2020). In contrast to Agustin and Nursalim (2020), the topic used is algebra, in this research it will be related to geometry.

In the preliminary design, the researcher designs the Hypothetical Learning Trajectory (HLT) to help students' difficulties in proving so that students can increase their level from informal to formal. HLT contains learning objectives (mathematical goals), teaching and learning activities, and the conjecture of student thinking (Simon, 1995). Before HLT is used in design testing through preliminary teaching and teaching experiments (further research), an expert review activity is needed, the instrument was reviewed by 2 experts who were lecturers from two universities with relevant knowledge. The selection of experts considers the length of service as a lecturer, the level of education, and the quantity and quality of research that has been carried out.

RESULTS AND DISCUSSION

Based on data collecting technique, the findings of the research can be categorized according to the focus established at the beginning of the research, namely:

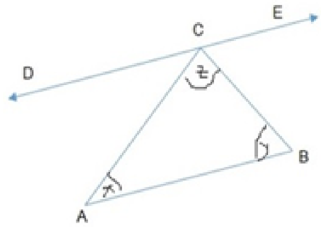
Preservice teachers' levels of understanding of proof

In this study, data were collected through a mathematical proof test to assess pre-service mathematics teachers' proof level in geometry based on Miyazaki's (2000) classification. The results of this mathematical proof test (see Table 2) will be explained as follows:

Table 2. Teachers' proof level in geometry (Miyazaki, 2000).

Level	Total of students	Percentage (%)
Proof A	22	38
Proof B	3	5
Proof C	9	15
Proof D	24	42

Based on Table 2, Proof D has the highest score for preserving teachers' answers, totaling 24 preserved teachers' responses. This indicates that many preserved teachers' answers still utilize non-formal proof. Furthermore, the table reveals that 38% of preserved teachers' demonstrated Proof A, which requires deductive reasoning and the use of functional language to construct proofs. Meanwhile, 5% of the preserved teachers presented Proof B, utilizing deductive reasoning and manipulating objects or using sentences without functional language in their proofs. Additionally, 15% of preserved teachers exhibited Proof C, employing inductive reasoning and various languages, images, and manipulated objects to construct proofs. Moreover, 42% of preserved teachers displayed Proof D, using inductive reasoning and functional language for constructing proofs. The following section will provide some examples of preserved teachers' answers. See Figure 1, Figure 2, Figure 3 and Figure 4.



Buat sebuah segitiga sebarang dan beri nama tiap titik sudutnya A, B, dan C. Buat garis yang sejajar sisi AB dan melalui C. dan beri nama garis tersebut. Dalam kasus ini diberi nama DE.

Sudut CAB bersebrangan dengan sudut ACD, sudut CAB = sudut ACD = x^0.

Sudut ABC bersebrangan dengan sudut BCE, sudut ABC = sudut BCE = y^0.

Dan besar sudut ACB yaitu z^0. Sehingga jumlah sudut ACD + ACB + BCE = $x^0 + y^0 + z^0 = 180^0$.

Translation

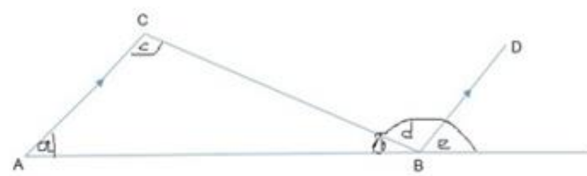
Create an arbitrary triangle and name each vertex A, B, and C. Draw a line parallel to side AB and through C, and name the line. In this case it is named DE.

Angle CAB is opposite angle ACD, angle CAB = angle ACD =x^0

Angle ABC is opposite angle BCE, angle ABC = angle BCE =y^0

And the size of the angle ACB is z^0. So the sum of the angles ACD +ABC+BCE = $x^0 + y^0 + z^0 = 180^0$.

Figure 1. Proof A



Jadi, karena ketiga sudut itu terletak pada garis lurus maka jumlahnya yaitu 180^0

Translation

So, because the three angles lie on a straight line, their sum is 180^0

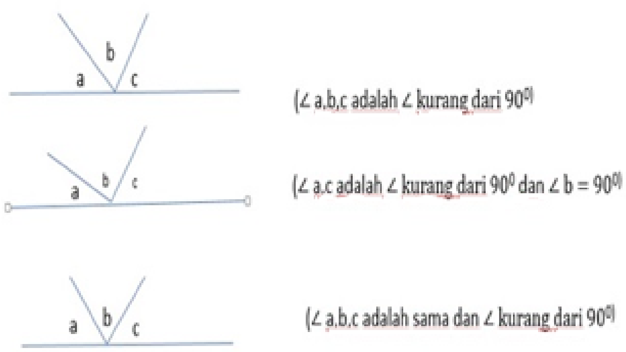
Figure 2. Proof B

$30^0 + 60^0 + 90^0 = 180^0$
$45^0 + 65^0 + 70^0 = 180^0$
$60^0 + 60^0 + 60^0 = 180^0$
Jadi, jumlah ketiga sudut dalam segitiga sama dengan 180^0

Translation

So, the sum of the three angles in a triangle is equal to 180^0

Figure 3. Proof C



Translation

(∠ a, b, c is ∠ less than 90^0)

(∠ a, c is ∠ less than 90^0 and ∠ b = 90^0)

(∠ a, b, c is equal and ∠ less than 90^0)

Figure 4. Proof D

The preservice teachers perform Proof A, Proof B, Proof D, and Proof C types with the percentage of 38%, 5%, 42%, and 15%, respectively. Therefore, it shows that Proof D is the most commonly found in the prospective teachers' answers than those of other types. It aligns with the results of Kögce et al. (2010), in which the study results report that the inductive method is performed by most students than the other types of proof (51.2%). Researchers found that many students' answers indicated informal proof (62%) with several difficulties in proving, namely starting the proof (12%), understanding the concept (40%), and using symbols or language in compiling the proof (10%). In line with Baker (1996), many students experience difficulties in using symbols in constructing a proof. Harel and Sowder (1998) also concluded that many students had difficulty coming up with invalid deductive arguments and inductive arguments. Based on these difficulties, a learning trajectory is needed in the form of scaffolding to help students' difficulties in proving so that students can increase their level from informal to formal.

There are three difficulties experienced by students in compiling the proof of this theorem, namely in the starting of the proof, understanding concepts, and using symbols or language in compiling the proof. Based on Agustiani and Nursalim (2020), there are four activities in HLT for proof, namely reading proof, completing proof, examining proof, and constructing proof. Difficulties in starting the proof can be helped by reading proof activities in HLT, difficulties in understanding concepts can be helped by examining proof, and constructing proof activities, and difficulties in using symbols or language in compiling the proof can be helped by completing proof activities. This is in line with Anwar et al (2022), that students who have difficulty in the concept of proof can be helped by constructing a geometric figure. Then Miyazaki et al (2017) explained that reading of proof activities will help students in compiling or starting the structure of deductive proof.

Preservice teachers' learning trajectories using scaffolding in Geometry

After the researcher obtains quantitative data, the researcher can continue by analyzing student answers which include informal proof, to arrange scaffolding in the HLT so that this HLT can help students' overcome difficulties in proving, enabling them to increase their level from informal to formal. Then it is given to experts to provide input. Based on expert comments, researchers arrange things necessary to be discussed with experts. In outline, two things are subject to discussion between researchers and experts: students' understanding of using four levels of proof (Proof A, B, C, and D) that will be used in help organize HLT activities and the need to make separate steps. HLT, arranged as an initial design, is called the initial prototype. The initial prototype HLT consisted of four teaching-learning activities: reading proof, completing proof, evaluating proof, and constructing proof. In the expert review activity, the researcher intends to obtain an expert judgment on the relevance of the activities to achieve the expected goals along with the researcher's hypothesis about the conjecture of students' thinking. After the discussion with the experts, the following revision materials for the initial prototype HLT are in Table 3.

Table 3. HLT using scaffolding in Geometry.

No	Activity	Goals	Students conjectured thinking	Type of scaffolding
1	Reading Proof	The purpose of the first activity "Reading Proof" is to introduce the parts that must be present in the sentence of proof and the levels of proof in constructing proof (deductive reasoning).	1. Read carefully the proof of the following basic geometry theorems (Students are given complete proof, inductive proof for answer a question with Proof C) 2. After reading the proof of the theorem, then write down the premises (statement / closed sentence) of each statement of proof! 3. After reading the proof of the theorem, then write down the things you have understood	With student has difficulty in starting the proof Level 2 (explaining, reviewing, and restructuring)

			(in a few points, if any) in the box below! 4. From the results of the class, discussion write in full the conclusions / new understanding that you get (If any)!	
2	Completing Proof	The purpose of the second activity "Completing Proof" is to train students to identify sentences/statements of proof that must be present in the proof sentence (incomplete), the use levels of proof in constructing the proof.	1. Read carefully the proof of the following basic geometry theorems! (Students are given incomplete proof, inductive proof for answer a question with Proof D). 2. After reading the proof of the theorem in point 1, then write down the things that you think are incomplete (if any) of the proof of the theorem! 3. Write the complete proof of the theorem on point 1!	With limitations of students do not understand and use language and mathematical notation Level 2 (explaining, reviewing, and restructuring.)
3	Examining Proof	The purpose of the third activity "Evaluating Proof" is to train students to evaluate the sentences/statements of proof presented by identifying errors	1. Read carefully the proof of the following basic geometry theorems! (Students are given proof by logic/ wrong correct concept, deductive proof for answer a question with Proof B) 2. After reading the proof of the theorem, then write the things that are FALSE in your opinion (if any) in the box below! 3. Write the right proof of the theorem on point 1!	With difficulty understanding the concept students Level 2 (explaining, reviewing, and restructuring)
4	Constructing proof	The purpose of the fourth activity "Constructing Proof" is to train students to construct their sentences/statements of proof from	1. Read carefully the proof of the following basic geometry theorems! (Students are given proof by logic/wrong concept, deductive proof for answer a question with Proof A)	With difficulty understanding the concept students Level 2 (explaining, reviewing, and

	several theorems provided with the correct sentence and proof of logic	2. In your opinion, the theorem in point 1 is more effectively proven using deductive proof or inductive proof? Explain your reasons! 3. Write the right proof of the theorem on point 1!	restructuring) or Level 3 (conceptual development)

Table 3 shows HLT activities starting from level C proof, namely, reading proof because preservice teachers need to introduce the parts that must be present in the sentence of proof. Then, it continues with the second activity, namely completing proof with level D, because preservice teachers to identify sentences/statements of proof that must be presented in the proof sentence (incomplete). The third activity is examining proof with level B, because preservice teachers to evaluate the sentences/statements of proof presented by identifying errors. Lastly, constructing proof is the last activity at level A because preservice teachers to train students to construct their sentences/statements of proof from several theorems provided with the correct sentence and proof of logic. Of these 4 activities, the learning trajectory used is from informal to formal proof. In line with Agustin and Nursalim (2020), constructing proof activities are activities with formal proof.

The scaffolding that will be used in HLT refers to the difficulties in starting the proof, understanding concepts, and using symbols or language in compiling the proof, namely level 2 (explaining, reviewing, and restructuring) and level 3 (conceptual development). Level 2 is used for all activities in HLT, and for level 3 only construction proof. According to Anghileri (2006), at level 3, there is making connections, namely making connections by encouraging students to use their mathematical knowledge in developing their own strategies in the problem-solving process so that they are suitable for use in constructing proof activities. This level really helps teachers in implementing HLT when applied in the classroom for learning proof.

CONCLUSIONS

Most preservice teachers' answers in constructing proofs use inductive methods (Proof D and Proof C) rather than deductive methods (Proof A and Proof B). Researchers found that 62% of preservice teachers' answers were informal proof, and Proof D was the most commonly found in the prospective teachers' answers than those of other types. Some of the difficulties that preservice teachers in proving are starting the proof, understanding the concept, and using symbols or language in compiling the proof. From these three difficulties, an HLT was prepared containing a scaffolding that could help students' overcome difficulties in proving, enabling them to increase their level from informal to formal. HLT activities consist of reading proof, completing proof, examining proof, and constructing the proof. Each activity contains level 2 scaffolding and only the constructing the proof activity also contains level 3 scaffolding.

This research was limited to one problem, aiming to identify the level of proof by adopting a single problem to explore and investigated preservice mathematics teachers' proof level and the possible task of scaffolding-based interventions in proving the triangle theorem. Apart from that, this research is still being carried out in the initial design stage (preliminary design) in design research. Expected future research will focus on the second phase, namely design testing through preliminary teaching and teaching experiments. Researchers can implement HLT with students who have difficulty proving the triangle theorem in class so that it can help increase their level from informal to formal.

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Exploring Learning Difficulties in Convergence of Numerical Sequences in Morocco: An Error Analysis Study

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Abstract: *Numerical sequences are one of the mathematical subjects linked to everyday life, and are taught at several levels in Morocco. However, many students still had difficulty teaching this subject, more specifically the limits of numerical sequences and the nature of convergence. The aim of this study was to analyses the learning difficulties of students in the 2nd year of the baccalaureate, experimental sciences series in Morocco. To this end, a written test consisting of 6 questions relating to the notion of convergence of a numerical sequence was administered to 60 students, followed by a questionnaire sent to mathematics teachers at the qualifying secondary level. The results show that the learning difficulties associated with this concept are of several kinds: didactic, pedagogical and epistemological. Using the "Teaching at the Right Level" approach (TARL), teachers can help learners overcome the learning difficulties of numerical sequence convergence by providing support tailored to their individual needs. This promotes progress that is more effective and a better understanding of mathematical concepts. This can help students overcome learning barriers by offering varied teaching aids and promoting a better understanding of the concepts of convergence of numerical sequences.*

Keywords: Learning difficulties, Number sequences, Convergence, Learning obstacles

INTRODUCTION

The concept of convergence of numerical sequences is fundamental in mathematics, particularly in analysis. It allows us to study the behavior of sequences and determine whether they tend towards a specific limit value as the index of the sequence increases.

Learning mathematics is a major challenge for many students around the world, and Morocco is no exception. Among the mathematical concepts that often pose difficulties for science baccalaureate students is the convergence of numerical sequences, which occupies a central place. Understanding this key concept is essential for developing mathematical skills and preparing students for higher education.

There is now a well-documented research literature, testifying to students' difficulties with understanding the concept of limit (Cornu, 1983, Cottrill et al., 1996, Williams, 1991). Here we set out some of the misconceptions reported: - The limit is the last term of an infinite sequence. - A limit is a boundary that cannot be crossed. - A limit is a number that can eventually, theoretically, be reached (Flores & Park, 2016, Swinyard & Larsen, 2012). Mamona-Downs (2001) has proposed three didactic steps for teaching the concept of the limit of a sequence as follows: (i) Initiate and develop intuition by raising problems in class discussion. (ii) Introduce the formal definition and analyze it in conjunction with (i). (iii) Endorse or revoke the opinions expressed in step (i) in relation to the formal definition, in particular via the representation in (ii) above. The difficulties associated with the notion of limit are more conceptual than computational. Current textbooks focus on techniques for calculating limits, and propose either a formal (rigorous) or an informal (intuitive) definition of the limit of a sequence. In reality, the debate between informal and formal teaching of Calculus concepts is "a problem that runs through the history of calculus" (Moreno-Armella, 2014). Students do not seek to understand independently, but are used to being consumers of the teacher's explanations. In this case, students' potential for reflection is not optimal, and their understanding is only partial. Aztekin (2020) aimed to investigate students understanding of the concept of limit in mathematics using the repertory grid technique. The repertory grid technique is a method that involves identifying and analyzing the constructs that individuals use to make sense of a particular subject. In the study, Aztekin applied this technique to a group of mathematics students to explore their conceptualizations and misconceptions related to the limit concept. The findings of the study provided insights into the students' understanding of limits and highlighted common misconceptions that need to be addressed in mathematics education. Arnal-Palacián et al. (2020) desired to find and characterize an infinite limit of sequences that is recognized by specialists in mathematics, and then examine the phenomenology of that limit. They consulted experts to select a suitable definition of the infinite limit of sequences and then explored its phenomenological aspects using intuitive and formal approaches. The study determined and defined phenomena including one-way and return infinite limits, infinite intuitive growth, and endless intuitive reduction that are associated with the infinite limit of sequences. The authors illustrated these phenomena with verbal, graphical, and tabular representation approaches. The study aimed to provide support for pre-university students in understanding and overcoming difficulties associated with the concept of limit in mathematics. Rachma and Rosjanuardi (2021) use an onto-semiotic method to investigate the challenges that students have when learning sequences and series. The study adopts a didactical and interpretive research approach to analyze the epistemological, ontogenic, and didactical obstacles that students encounter. The study findings elucidate that students encounter obstacles in accurately conceptualizing mathematical ideas within the framework of sequences and series. They are able to convert an issue into a mathematical model, but they struggle to apply the right process. The identified obstacles they categorized as didactical, ontogenic, and epistemological obstacles. The onto-semiotic approach used in this study helps better understand the specific difficulties that students face in learning sequences and series. The findings of this study hold practical implications for teachers, allowing them to develop tailored learning environments they better aligned with the needs of students in mastering the topic at hand.

Learning difficulties linked to the convergence of a numerical sequence can be of several kinds: pedagogical, didactical and epistemological. To verify or refute the validity of this conjecture, practical investigations will be carried out. This situation raises an interesting question: What are the difficulties encountered in learning about the convergence of numerical sequences? What are the causes of these difficulties? This article focuses on the explicit study of learning difficulties specific to the convergence of a numerical sequence in students in the 2nd year of the baccalaureate, experimental science series. We have used a mixed-methods approach, combining quantitative and qualitative methods. This will enable us, on the one hand, to determine the difficulties encountered, and on the other, to describe the reasons for these difficulties via a test focusing on different axes of the convergence of numerical sequences in the final year. Next, a didactical analysis of the content of the chapter dealing with the convergence of a numerical sequence will be conducted, and the corresponding interpretations and conclusions will constitute the final part of this article.

THEORETICAL FRAMEWORK

In this section, the theoretical framework based on the definition of the limit of a sequence, then the learning obstacles, the typology of errors and the difference between error, obstacle and difficulty.

The notion of limit

The notion of "limits" holds significant importance within the field of analysis due to its interconnectedness with various other fundamental concepts, including derivatives, integrals, and continuity. The comprehension of limits is frequently considered as challenging, primarily due to the involvement of abstract ideas such as infinity, the infinitesimal, and transfinite. Research suggests that students often acquire a superficial understanding of this subject matter, as evidenced by studies conducted by (Arnal et al., 2017, Chorlay, 2019).

The definition of a concept plays a crucial role in its technical application, distinct from its intuitive and colloquial interpretations. In the particular context of characterizing a sequence's infinite limit, as Arnal-Palacián et al. (2020), has already mentioned, this definition is commonly found in textbooks shortly before introducing the concept of the infinite limit of a function, and sequentially following the discussion on the finite limit of a sequence.

In light of the multitude of definitions pertaining to the infinite limit of a sequence, Arnal et al. (2017). Vinner (1991) undertook a consultation process with university professors and secondary education teachers. The objective was to identify a precise definition that would be universally accepted within the mathematical community across various educational levels, the resultant definition that was chosen is as follows:

> "Let F be an ordered field and $\{an\}$ a sequence of elements of F. The sequence $\{an\}$ has a "plus infinite limit", if for each E element of F, there exists a natural number v, so that an > E for all $n \geq v$" (Arnal-Palacián et al., 2020) .

> "Let F be an ordered field and $\{an\}$ a sequence of elements of F. The sequence $\{an\}$ has a "minus infinite limit", if for each E element of F, there exists a natural number v, so that an < E, every time $n \geq v$" (Arnal-Palacián et al., 2020).

The notion of obstacle

Brousseau has provided a definition of obstacles that arise from the interaction between students and a didactic situation during the process of acquiring knowledge. However, this interaction can sometimes result in the formation of misconceptions (Brousseau, 2002). The obstacles encountered during the learning process can be classified into three main categories according to Guy Brousseau. Firstly, epistemological obstacles refer to forms of knowledge that become ineffective in a new context and are inherent to the nature of the targeted concepts, as evidenced by the history of these concepts. Secondly, didactical obstacles are created by the teaching methods themselves, sometimes due to inappropriate approaches or deficiencies in educational resources. Lastly, ontogenical obstacles are related to the specific limitations of the learner at a given moment in their cognitive development, thus reflecting the challenges inherent to their individual progression (Brousseau, 1976).

According to Bachelard (2020), didactical obstacles differ from mathematical errors and can be classified into several categories based on their origins, as follows: linguistic obstacles, notation obstacles, task design obstacles, motivational obstacles, cultural obstacles, obstacles related to negative beliefs, and obstacles related to teachers' mathematical knowledge (Bachelard, 2020).

The typology of errors

Jean-Pierre Astolfi, in his book "L'erreur, un outil pour enseigner", "Mistakes as a teaching tool" (Astolfi, s. d.), analyzes the different natures of errors committed by students, such as comprehension errors, alternative conceptions, cognitive overload and discipline-specific comprehension. He proposes positive listening to what students express, and suggests that teachers should be aware of the clarity of their instructions and understand the difficulties encountered by their students. Constructivist models can be used to integrate errors into the learning process and help students acquire new knowledge. Astolfi has identified 8 types of errors made by students:

- Understanding instructions: The student does not understand the instructions and cannot execute the didactic contract, the problem may stem from the difficulty of the statement.
- Errors resulting from school habits or poor understanding of expectations: The student learns according to a mechanical principle (habitual pedagogy), then finds it difficult to respond to instructions that do not conform to his or her habits.
- Errors due to cognitive overload: memory limitations or inappropriate estimation of the cognitive load of the activity, when the student has to process several pieces of information at the same time.
- Students must systematically reinvest their knowledge in each subject. This requires constant gymnastics on the part of the brain, which sometimes leads to errors.
- Errors reflecting alternative conceptions: “The mind can only be formed by reforming itself”: Bachelard's idea of the obstacle (Bachelard, 2020).
- Errors concerning the approaches adopted: The methods used by students do not necessarily correspond to what the teacher expects. However, teachers, who expect very precise answers, often perceive this approach as a mistake. Pupils already have intellectual representations of the concepts they are studying, so they do not wait for the teacher's lesson to give them explanations.
- Errors caused by the inherent complexity of the content: the teacher has to explain the instructions to the students. If the student pretends not to understand the instruction or the context, an error will result.
- Errors linked to the intellectual operations involved: The student does not have the necessary skills to meet the teacher's expectations.

The difference of difficulty, error and obstacle

When comparing difficulty and obstacle, there are at least two main differences to be noted: the relationship to prior knowledge and the relationship to external factors. According to Brousseau, G (2002), difficulty can be measured as the difference between a correct response and an incorrect response (in relation to the previously treated probability of success), and there is no parasitic knowledge to contend with, meaning that it need not always be the outcome of prior knowledge. According to Bachelard, G (2020), however, difficulties might arise from prior knowledge rather than from a lack of knowledge or external impediments (complexity, fugacity, etc.). Additionally, the history of science and genetic epidemiology emphasize that errors and difficulties are inherent to every envisaged body of knowledge.

These three concepts (error, difficulty, and obstacle) have complementary meanings and connections despite their definitional differences. As an illustration, let us consider Brousseau (2002) definition of an obstacle, which emphasizes that an obstacle in mathematics manifests as a set of shared difficulties among numerous actors (individuals or institutions) who share “an inappropriate conception of a mathematical notion”. Similarly, Alibi and Boilevin (2021) defines difficulty as "something that is difficult, such as an obstacle or a barrier...". Then it was said that a difficulty might be overcome, circumvented, or redirected, something that is difficult.

It is noted that researchers express one idea by means of another (Alibi & Boilevin, 2021). In fact, Lajoie (2009) uses the word "obstacle" to define the concept of "difficulty", Vergnaud (1988) and Brousseau (2002) agree. Additionally, Brousseau defines a barrier as a particular group of challenges. Moreover, in addressing the first approach to hurdles, he states that even when difficulties seem to go away, they will reappear and lead to mistakes if the notion is resilient.

METHOD

For this study, we used a mixed methods approach, combining quantitative and qualitative methods. On the one hand, it will enable us to determine the difficulties encountered by students in learning to converge numerical sequences, and on the other, to describe the reasons for these difficulties by means of a test focusing on different aspects of the convergence of numerical sequences in the final year.

Choice of population

The test is designed for students aged between 17 and 18 years old, in the 2nd year of the baccalaureate in the experimental sciences, physical sciences stream, French option, who number 60 and represent two classes at the Lefkih Tetouani secondary school in the town of Salé in the Rabat-Salé-Kenitra regional education and training academy.

Test elaboration

The concept of the convergence of numerical sequences was chosen because of its importance in the mathematics curriculum in qualifying secondary education in Morocco. The test is used to identify the various errors made by students in the convergence of numerical sequences.

Another aspect revealing the relevance of our choice lies in the fact that the convergence of numerical sequences is of significant importance as a transition point between high school and university level in the context of mathematics teaching. This notion occupies a fundamental place in the continuity of students' mathematical learning during their transition from secondary education to higher education.

Numerical Sequences according to the official pedagogical orientations (OPO)

The content of the chapter on numerical sequences according to the school curriculum (MEN, 2007) for classes in the experimental science series. See Table 1.

Table 1. Pedagogical recommendations.

Skills required	Pedagogical recommendations
*Using the limits of reference sequences and convergence criteria is a methodological approach to determining the limits of numerical sequences. * The use of sequences enables us to solve problems from a variety of professional fields, by proposing methods and models for their solution. *Study the convergence of sequences of the form: $U_{n+1} = f(U_n)$ and $V_n = f(U_n)$. * The limit of geometric sequence (a^n).	*Based on the limits of certain reference functions, the limits of reference sequences will be admitted. *Operations on finite and infinite limits are considered valid, and it is essential that students acquire the habit of using them appropriately. * The criteria for convergence of a sequence are accepted, and their approach is based on the compatibility of operations on limits and order in a sequence. $\mathbb{R}$.

The questionnaire is composed of the following questions (Table 2).

Table 2. Test on the convergence of a numerical sequence.

Questions	Capabilities required
Q1. If you had to explain a convergent sequence to one of your classmates. What would you say?	*Recognize the definition of a convergent sequence.
Q2. Let's consider the sequence (w_n) defined by: $w_n = \ln(u_n)$ such that: $\lim\limits_{n\to+\infty} u_n = 1$ and $(\forall n \in \mathbb{N}): u_n > 0$. Determine $\lim\limits_{n\to+\infty} w_n$.	*Recognize a sequence of type: $v_n = f(u_n)$ and determine its limit.
Q3. Study the convergence of the sequence (u_n) defined by: $(\forall n \in \mathbb{N}): u_n = \dfrac{2n-3}{3n+2}$.	*Study the convergence of a numerical sequence.
Q4. Let's consider the sequence (w_n) defined by: $(\forall n \in \mathbb{N}^*): w_n = \dfrac{\sin(n)}{n}$.	*Use convergence criteria to determine the limit of a numerical sequence.

What can we say about the nature of convergence of the sequence (w_n).

Q5. . Let's consider the sequence (v_n) defined by :

$(\forall n \in \mathbb{N}): v_n = \frac{2^n - 4^n}{4^n + 3^n}$.

Calculate the limit: $\lim_{n \to +\infty} \frac{2^n - 4^n}{4^n + 3^n}$. What can we say about the convergence of the sequence (v_n).

*Limit of geometric sequence (q^n).

Q6. Let's consider the sequence (u_n) defined by: $u_0 \in]0;1]$;

$u_{n+1} = \frac{u_n}{2} + \frac{u_n^{\ 2}}{4}$.

a. Is (u_n) a convergent sequence?

b. Determine the limit of the sequence (u_n).

*Recognize a sequence of type: $u_{n+1} = f(u_n)$ and determine its limit.

RESULTS

There had been a Cronbach's test (Table 3), six questions were used as variables, and a binary approach was adopted for the analysis, where responses were classified as 'yes' or 'no'. According to Cronbach's reliability analysis, the test's alpha coefficient is equal to 0.778. This coefficient is used to assess the internal consistency of the test. An alpha value greater than 0.7 is generally considered acceptable in terms of reliability, although higher values are preferable. With an alpha coefficient of 0.778, our test demonstrates good internal consistency, although additional measures could be taken to further enhance its reliability.

Table 3. Reliability statistics.

Cronbach's Alpha	Number of elements
,778	6

Based on the learning obstacles described in the introduction and following analysis of the students' copies, the following results were revealed:

General results

Depending on the error situation for each question, the distribution of students tested is shown in the figure below (Figure 1):

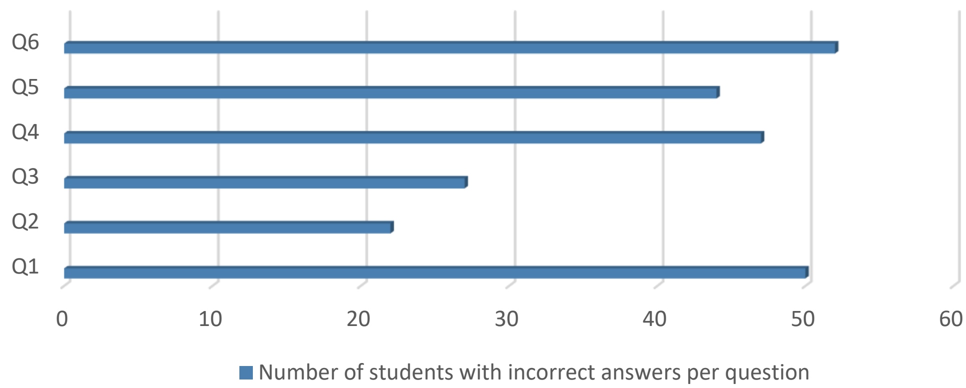


Figure 1. Distribution of students by number of errors made

Analysis of the data collected from 60 students reveals variations in the distribution of errors for the different questions studied in Figure 1. Of the six questions, question 1 shows the highest number of errors, with 50 errors made by the students. This observation suggests that this question posed significant difficulties for a considerable percentage of the students in the sample. Similarly, question 6 also stands out for its high number of errors, with 52 recorded. This finding indicates that this question was particularly difficult for a substantial number of students. Questions 4 and 5 also show relatively high error levels, with 47 and 44 errors recorded respectively. These results suggest that these questions presented significant challenges for students. In contrast, questions 2 and 3 generated a relatively lower number of errors, with 22 and 27 errors respectively. These figures indicate that these questions were relatively more accessible to students than the other questions. Analysis of the distribution of errors reveals significant differences in understanding of the notion of convergence of a numerical sequence for the different questions. Identifying these variations raises questions about the factors that led to these results, leading us to analyze the errors made by the students via an analysis of the obstacles linked to these errors and with the help of a questionnaire addressed to qualifying secondary school teachers.

Distribution of Obstacles by Student Errors: Insights from Teacher Surveys

The teacher survey revealed the following results concerning the most frequent errors linked to the convergence of numerical sequences:

1. Error 1: A significant majority of 95% of the surveyed teachers acknowledged the prevalence of this error in student work.
2. Error 2: The entirety of 100% of the participating teachers identified this error as a commonly occurring mistake.
3. Error 3: The vast majority of 90% of the educators acknowledged the frequent occurrence of this error.

4. Error 4: A consensus of 95% of the teachers concurred with the presence of this error.
5. Error 5: Additionally, 95% of the teachers recognized this error as commonly encountered.
6. Error 6: A substantial proportion of 86% of the surveyed teachers agreed on the recurring nature of this mistake.

These results indicate a high degree of consensus among teachers regarding the most common errors in teaching the convergence of numerical sequences. Errors 2, 4 and 5 were particularly recognized, with agreement from 100% or 95% of teachers. Further analysis of the reasons behind these errors could help develop effective teaching strategies to correct them. See Figure 2.

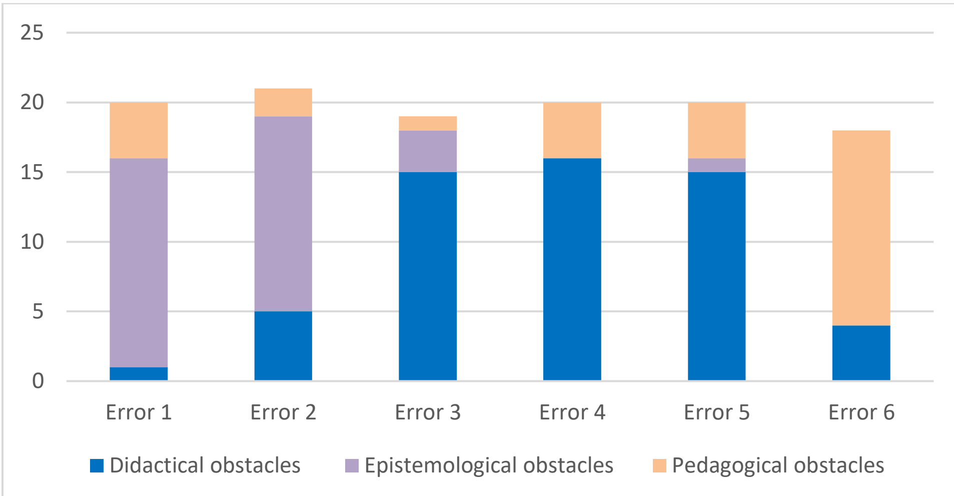


Figure 2. The distribution of obstacles according to student errors

The results reveal that errors 1 and 2 are associated with an epistemological obstacle, while errors 3, 4 and 5 are linked to a didactical obstacle. Error 6, on the other hand, is related to a pedagogical obstacle. Here is a more detailed explanation of these terms in a scientific context:

Epistemological obstacle: Errors 1 and 2, identified as being linked to an epistemological obstacle, indicate a profound conceptual difficulty on the part of learners. This suggests that some students are struggling to build a solid understanding of the fundamental concepts of numerical sequence convergence, perhaps due to their prior misconceptions or lack of understanding of theoretical principles.

Didactical obstacle: Errors 3, 4 and 5, classified as didactic obstacles, refer to difficulties linked to the specific teaching and learning of the convergence of numerical sequences. This may include issues such as inappropriate teaching methods, confusing conceptual presentations or gaps in available teaching resources. These obstacles can hinder the effective transmission of the knowledge and skills needed to correctly understand and apply the concepts of sequence convergence.

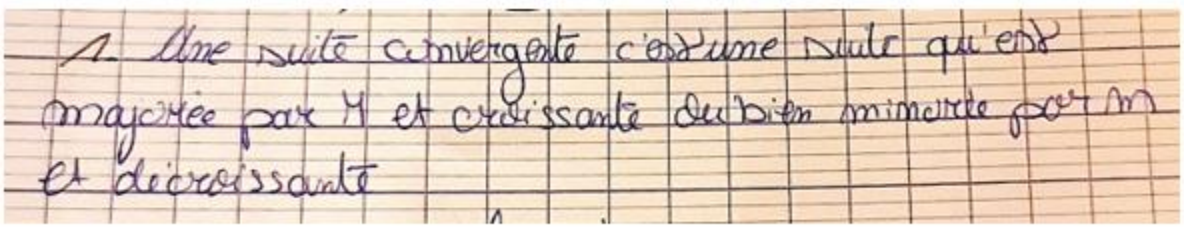
Pedagogical obstacle: Error 6 is associated with a pedagogical obstacle, implying a difficulty in the teaching process itself. This may be due to communication problems between teacher and learners, poor organization of teaching sessions, lack of clarity in explanations or other factors that hinder the effective transmission of knowledge.

The results of the survey confirmed the analysis of student errors in the next section, according to which errors 1 and 2 are linked to an epistemological barrier, errors 3, 4 and 5 are associated with a didactical obstacle and error 6 is related to a pedagogical obstacle. This consistency between the results of the survey and the previous analysis reinforces the validity of the scientific interpretation of the data collected, confirming our research hypotheses.

Individual student errors

From our analysis of the copies and answers to the student questionnaire, we have identified three types of difficulty: epistemological, didactical and pedagogical. We illustrate each type of difficulty with an extract from a copy or an answer to a question.

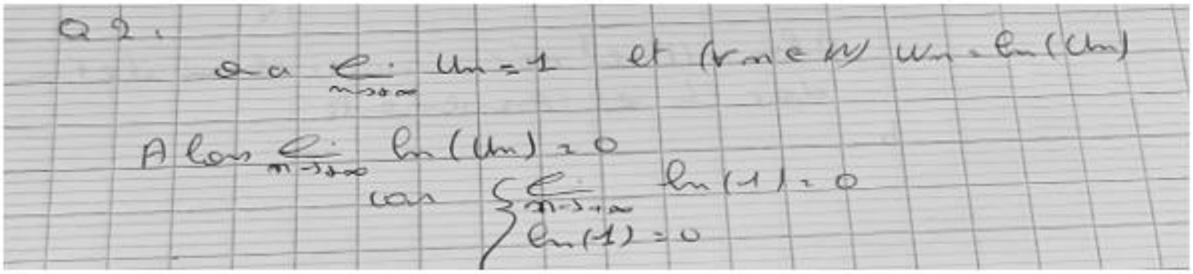
Epistemological obstacles



Translating in English:
A convergent sequence is a sequence that is bounded above by M and increasing, or bounded below by m and decreasing.

Figure 3. Student 1

In question 1, 50 out of 60 students answered in the same way as student 1 (Figure 3), using the characteristic property of a convergent sequence rather than the definition. In this situation, a potential epistemological obstacle is the tendency of students to focus on the characteristic properties of a convergent sequence rather than understanding and applying the formal definition of convergence. This may be the result of previous learning based on specific examples of convergent sequences, where students have associated these characteristic properties with the notion of convergence without being exposed to a deeper understanding of the definition. This can lead to a superficial understanding of the notion of convergence and to conceptual errors. Students may use only properties such as the finite limit or the progressive approach to the terms of the sequence to judge convergence, without fully grasping the idea of convergence to a specific limit value. They may also have difficulty differentiating between the notions of convergence and limit, using these terms interchangeably.



Translating in English:
We have: $\lim_{n\to+\infty} u_n = 1$ and $(\forall n \in \mathbb{N})$: $w_n = \ln(u_n)$. Then: $\lim_{n\to+\infty} \ln(u_n) = 0$ because: $\lim_{n\to+\infty} \ln(1) = 0$ and $\ln(1) = 0$.

Figure 4. Student 2

In question two, it was observed that 48 students gave a correct answer to the question posed but insufficient (Figure 4). However, closer analysis reveals that these students neglected or forgot to take into account the continuity of the function at 1 in their reasoning. Yet, despite the need to consider this continuity, all 48 students omitted or overlooked this key feature when solving the question. This may be due to a lack of understanding or attention to the notion of continuity, or confusion about its importance in the context of the question. This omission of the continuity of the function in 1 may lead to misinterpretation of the results or an incomplete answer. Students may find it difficult to grasp the influence of this continuity on the behavior of the function and to draw accurate conclusions.

Didactical obstacles

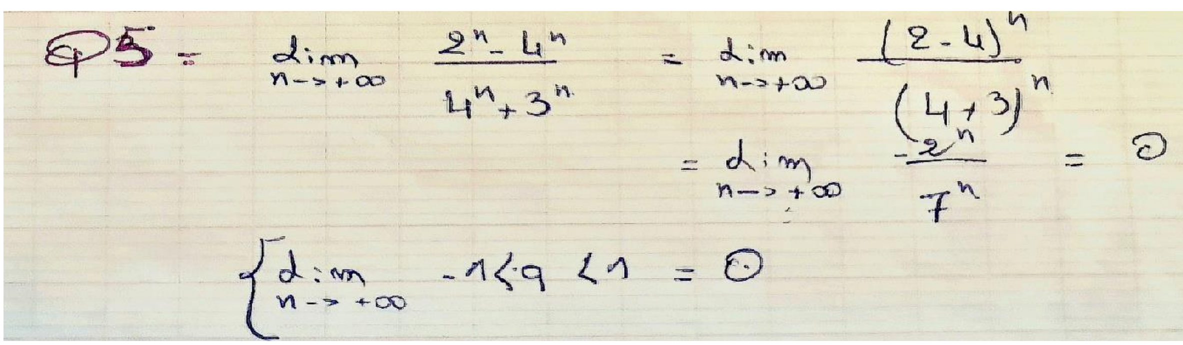


Figure 5. Student 3

In this context, student 3 (Figure 5) made a mistake by incorrectly applying the remarkable identities when calculating this limit. This leads us to a didactic obstacle, as the correct use of remarkable identities is essential for simplifying and solving algebraic expressions. Incorrect application of these identities can lead to errors in the calculations and results obtained. It is therefore important for students to understand and master these remarkable identities correctly, in order to apply them appropriately in their mathematical calculations.

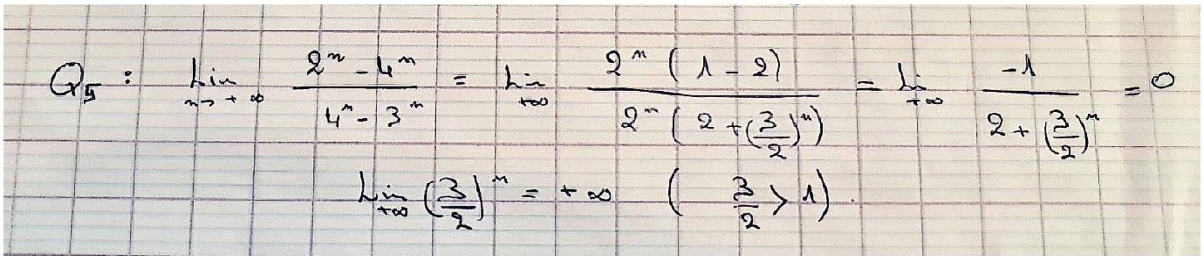


Figure 6. Student 4

Error in factoring can take different forms, as in Student 4 (Figure 6). The student may incorrectly apply these rules or choose an inappropriate method to factor the expression, leading to an incorrect answer. The lack of sufficient examples can be a challenge when it comes to understanding a concept or method. When the examples provided by the teacher are limited in number or variety, this can make it difficult for students to fully grasp the different applications of limit calculus.

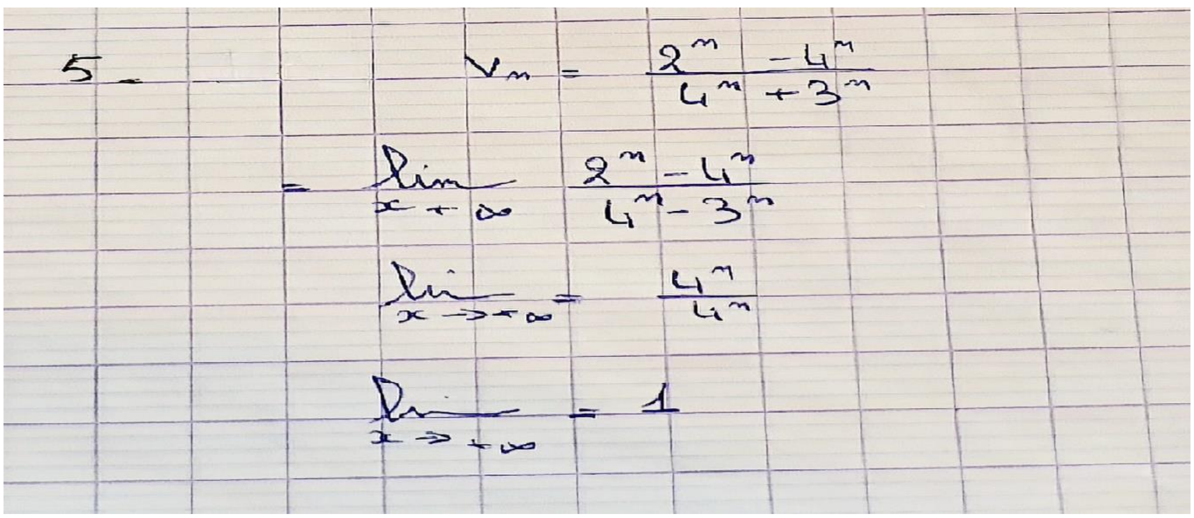


Figure 7. Student 5

Student 5's mistake (Figure 7) was to confuse two different concepts. He probably tried to calculate the limit of a rational function using traditional limit calculation techniques. However, these two concepts are fundamentally different and require distinct approaches. The didactic obstacle here lies in insufficient understanding of the difference between these two notions of limit. This confusion may stem from a misinterpretation of the concepts taught, an incorrect assimilation of limit calculation methods or a lack of practice and understanding of the underlying concepts. To remedy this didactic obstacle, it is important to clarify and clearly distinguish between the concepts of calculating the limit of a geometric sequence and the limit of a rational function. Teachers should provide precise explanations and relevant examples to help students understand the distinct nature of these two notions.

Pedagogical obstacles

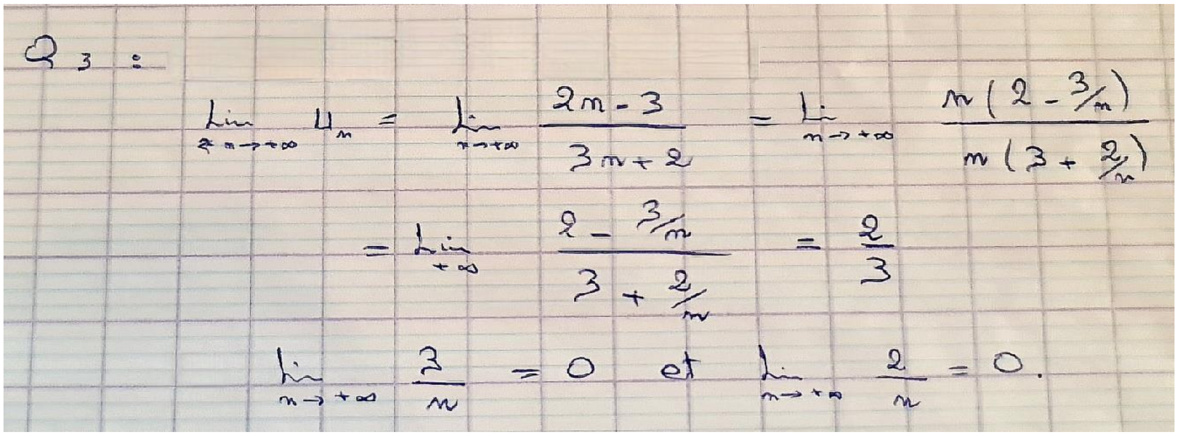


Figure 8. Student 6

Student 6 (Figure 8) provided a correct answer, but his method of arriving at this answer was long and detailed. He demonstrated a precise understanding of the calculation, but might have arrived at the same result by simplifying his method and taking solutions that are more efficient. Although the answer is technically correct, a more concise approach could have been used to solve the problem more efficiently and save time. 99% of the students with a correct answer answered in the same way as student 6 (Figure 8), which led us to orally question a few teachers in this way, and then the majority answered that you have to factor to calculate this limit, and not used the limit of a homographic sequence. After some analysis of the General Pedagogical Guidelines (MEN, 2007), we found that there is no such thing as the limit of a homographic sequence, and that you have to use the usual limits in the form : $\frac{1}{n};\frac{1}{n^2};\frac{1}{n^3};\frac{1}{n^p}$ such that $p \geq 4$. In this context, the general pedagogical guidelines require revision to simplify the task of calculating a homographic sequence.

The pedagogical obstacle identified in question 6 arises from the students' lack of familiarity with the type of question asked. It is possible that they have not encountered this type of question previously, or they may not have sufficiently developed the skills needed to answer it adequately.

CONCLUSIONS

In conclusion, this article confirms the hypotheses put forward previously. The results show that difficulties in learning to converge a numerical sequence in students in the 2nd year of the baccalaureate, experimental science series, are associated with a few obstacles: didactic, epistemological and pedagogical. It is important to recognize these obstacles and implement appropriate pedagogical strategies to overcome them. This can include clear, structured explanations, the use of concrete, relevant examples, hands-on, interactive activities, and pedagogical approaches that help students build links between the convergence of numerical sequences and other previously acquired mathematical concepts. However, this research has certain limitations, such as the complexity of the proofs, the dependence on assumptions, the rigidity of the methods, the limited practical applicability and the sensitivity to round-off errors.

In view of the above, using the TARL (Teaching At the Right Level) approach will be very efficient in addressing the diverse learning needs of students in MOROCCO (Binaoui, A, Moubtassime, M, & Belfakir, L, s. d.). By implementing TARL, teachers can assess the individual competency levels of students and group them accordingly. This allows for tailored instruction that meets each student's specific needs and provides appropriate learning challenges. The TARL approach ensures that students are neither overwhelmed by material beyond their current understanding nor bored by content that is too easy for them. Instead, it promotes an optimal learning environment where students are actively engaged and motivated to learn. By utilizing differentiated teaching strategies, such as targeted resources and interactive activities, teachers can effectively support students at their respective levels and foster continuous progress.

This research will provide crucial information for tailoring future interventions to students' needs in learning to converge numerical sequences. It will enable teachers to design more effective teaching strategies, develop appropriate resources and offer personalized support to help students overcome difficulties and succeed in this specific area of mathematics. Future research should examine the transition from high school to university in the case of the convergence of a numerical sequence.

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Analysis of the Strategies Used by High School Students in Solving Area Problems: A Case Study

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Abstract: *The purpose of this study was to analyze the activities in the Problem-Based Learning (PBL) methodology among high school students when solving problems related to the field of plane figures, both in extra-mathematical and intra-mathematical environments. The research was qualitative and constituted a case study. The technique used to select the research subjects was intentional sampling, which involved the selection of a group of sixth-grade students. The data were collected through tests and interviews. The data analysis techniques employed included data reduction, data presentation, and drawing conclusions. The results showed that the students used perimeter calculation as an erroneous representation of area calculation, and they exhibited a lack of argumentation during problem-solving. Furthermore, they expressed a lack of recognition of certain plane figures and their properties, a characteristic of students at the visualization level, also known as the first level of the Van Hiele levels. Another significant finding in this study was the use of problems involving extra-mathematical contexts, which had a greater impact on the problem-solving process and the understanding of mathematical concepts.*

Keywords: problem-solving, didactic unit, Problem-Based Learning, area of plane figure.

INTRODUCTION

Problem-solving is considered by various authors to be an important part of mathematical activity (Sintema & Mosimege 2023), as it addresses diverse phenomena in the real world and the mathematical realm. Furthermore, "the formulation and resolution of problems enable students to implement strategies for tackling problems and formulating questions, in order to increasingly strengthen their analytical capacity" (Sanabria, 2019, p. 16). In this way, students play an active role, fostering significant learning and promoting the perception of mathematics as a discipline with utilitarian value from a perspective that goes beyond arithmetic. Consequently, it can contribute to improving students' performance (Alifiani 2023, Malvasi & Gil-Quintana, 2022, Maphutha et al., 2023). In the case of geometry, students are more likely to develop their visual perception, intuition, critical analysis, problem-solving skills, and use of arguments and logical evidence (Caviedes et al., 2023, Jupri, 2017, Seah, 2015). Due to its applicability to everyday life issues, the field of plane figures is a fundamental concept that students must master (Herawati et al., 2022, Winarti et al., 2012). Furthermore, the best way for students to achieve good results in the topic of three-dimensional space is to have a solid understanding of the area of plane figures (Battista et al., 1998) and to understand the above, it is important to resort to multiplicative reasoning (Jain et al., 2022).

Given the relationship between problem-solving, geometric reasoning, and understanding of the area of plane figures, an exploration into teaching methodologies and strategies becomes essential. The following research question guides the study: How does the utilization of Problem-Based Learning strategies influence the understanding and application of the concept of area among secondary school students across different Van Hiele geometric reasoning levels?

This question arises from a convergence of observational and theoretical insights. A diagnostic test and a semi-structured interview conducted with a focus group revealed weaknesses in calculating the area of plane figures and their application in problem-solving, prompting an investigation into the depth of these challenges. The student's conceptions did not align with any of the area manifestations outlined by Corberán (1997). They also demonstrated a lack of recognition of certain plane figures and their properties, which is characteristic of students at the visualization level, also known as the first level of the Van Hiele hierarchy (Gutiérrez & Jaime, 1991, Vargas & Gamboa, 2013).

Furthermore, to investigate the different methods, strategies, and techniques used for the development of studies on the learning of mathematical or scientific objects, the categorization taxonomy proposed by Rodríguez and Arias (2020) was used as a basis. This analysis showcases the most commonly used methods, among which Problem-Based Learning (PBL) (Arnal-Bailera & Vera, 2021, Artés et al., 2015, Cruz & Puentes, 2012, Zumbado-Castro, 2019, De Jesus, 2020, Endah et al., 2017, Galviz et al., 2016, Khalid et al., 2020, Salcedo & Ortiz, 2018, Setyaningrum et al., 2018) and Cooperative Learning (Cruz & Puentes, 2012, Fortes & Márquez, 2010, Galviz et al., 2016, Khalid et al., 2020, Villada, 2013) stand out for their extensive utilization. Based on this literature review, the most commonly used strategy is PBL, which will be employed for the development of the mathematical object specific to this study.

Van Hiele Levels

Both Gutierrez & Jaime (1991) and Vargas & Gamboa (2013) agree that the Van Hiele model of geometric reasoning explains how students acquire skills to the point of developing competencies in geometric reasoning. This model divides the process into five levels: visualization, analysis, informal deduction, formal deduction, and rigor. Each level is further subdivided into five phases: information, directed orientation, explication, free orientation, and integration. Upon completing

Phase 5 of a particular level, the student progresses to the next level. The Van Hiele reasoning model demonstrates the interdependence of the different levels, emphasizing that an individual cannot skip any level of reasoning.

Area of Plane Figures

Regarding the concept of area, Freudenthal (1983) states that it is a magnitude used to measure multiple objects, and he identifies three phenomena inherent in the learning of the area concept: (1) area as equal distribution, (2) area as comparison and reproduction of shapes, and (3) area as measurement. In line with this, Corberán (1997) proposes that the concept of area is not limited to a single concept, but is discerned through four manifestations, which are: (1) area as the amount of plane occupied by the surface, (2) area as an autonomous magnitude, (3) area as the number of units that cover the surface, and (4) area as the product of two linear dimensions. According to the author, it is essential that all four manifestations be present in the teaching and learning process. Therefore, this conception of the area was considered for the planning, design, and implementation of this intervention. Each of these manifestations is associated with different actions or procedures developed by the students.

Table 1. Manifestations of area and associated actions (Caviedes et al., 2019).

Manifestations of area (M)	Actions/Procedures (P)
M1. Area as the amount of plane occupied by a surface.	P1. Geometric procedures: P1.1. Comparing surface areas without using numbers. P1.2. Direct comparison of areas by superposition; indirect comparison of areas by cutting and pasting, decomposing the surface.
M2. Area as an autonomous magnitude	P2. Two-dimensional geometric and numerical procedures: P2.1. Decomposing surfaces into equal parts; comparing areas of surfaces and recognizing that different-shaped surfaces can have the same area. P2.2. Measuring the area of the same surface using different units of measurement.
M3. Area as the number of units covering a surface.	P3. Two-dimensional numerical procedures: P3.1. Fractionating the area of a surface and/or counting the number of units covering a surface using a two-dimensional unit of measurement. P3.2. Comparing the area of a surface with the two-dimensional unit that measures that surface.

M4. Area as the product of two linear dimensions.	P4. One-dimensional and two-dimensional numerical procedures: P4.1. Calculating the area of polygonal surfaces that can be decomposed into triangles or rectangles. P4.2. Apply formulas for calculating the area of a rectangle or square to determine the areas of triangles and parallelograms.

METHOD

The didactic unit was implemented taking into account the class moments and planning proposed by Diaz-Barriga (2013) and Ártes et al. (2015), considering the action research design presented by Hernández et al. (2010). For this reason, the design of the activities or learning sessions takes as reference those aspects of mathematical knowledge in which students encounter the greatest difficulties, specifically those related to problem-solving in metric thinking. The activities are structured in three stages, namely: the opening stage, which aims to activate students' attention towards the new learning, establish the purpose of the activity, increase interest and motivation for the new learning, provide a preliminary overview of what will be developed, and activate prior knowledge; the development stage, where the focus is on processing new information, directing attention to the new learning, implementing teaching and learning strategies to facilitate the acquisition of new knowledge, and putting the new learning into practice; and finally, the closing stage, where formative assessment takes place. In this stage, it is necessary to review and summarize what has been learned, transfer the new learning, motivate once again, draw conclusions, and conclude the session. Throughout this process, assessment is continuous and occurs in each of the class moments.

After the first activity was implemented, it was shared and evaluated using a checklist, and then the following activities were carried out, repeating the same cycle. Next, the participants and their context are presented, followed by the procedure used for implementing the activities and the sequence of activities in the didactic unit for this project.

Participants

This study was initially conducted in a sixth-grade course at an official educational institution. However, due to situations related to the COVID-19 pandemic, the implementation of the activities in the didactic unit was carried out with the same group of students (37 students aged between 11-14 years) who were part of the diagnostic phase but were in seventh grade (7th) at the time of the activities.

Procedures

This study was conducted over the course of seven class sessions, each lasting 80 minutes. Activity 1 was presented over four sessions, Activity 2 over two sessions, and Activity 3 in one session. These activities were presented in three stages: (1) exploration of prior ideas, where everyday problems were used to activate students' previous knowledge of the concept of polygonal area and to motivate the acquisition of new learning; (2) development, where new ideas about the concept of area were constructed through problem-solving, structuring, and practicing the new learning; and (3) closure, where the aim was to formalize the concept of area.

Didactic Unit

Activity 1. Exploring a Space.

Marcel wishes to replace the entire flooring of his living room and kitchen. He is currently heading to the flooring and tile store. He wants to know how much money he should bring to carry out the renovation. If he has a floor plan of the house (see Figure 1), how would you help him estimate how much money he will spend on the renovation? Justify your answer.

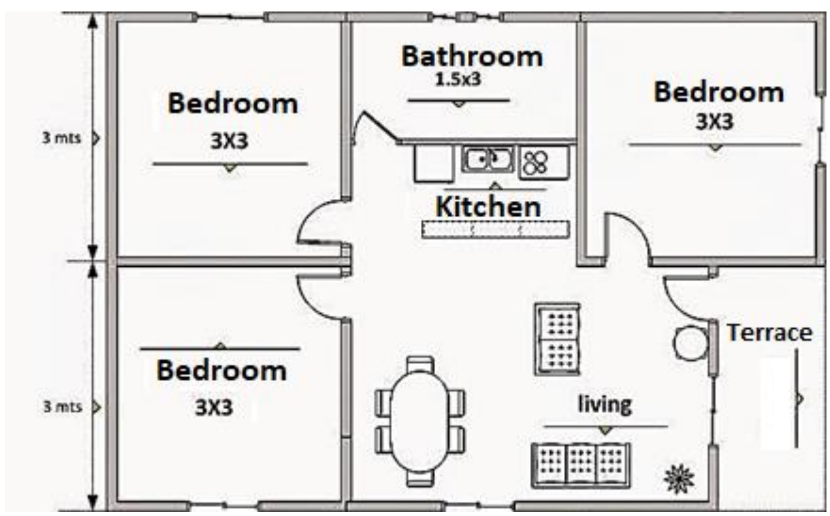


Figure 1. Top plan of Marcel's house. Translated

After the time elapsed, a group discussion was held to identify difficulties in solving the problem, and some previous concepts were activated that could help solve the problem through the following questions: What can you see in the image? What mathematical elements can you find embedded in the image? Why? Do you recognize any flat figures? Which ones? Why do you consider them flat figures? What data do you consider necessary to know the cost of the renovation? Justify your answer. Following this exploratory moment, 20 minutes were given to the students to explore different heuristics to solve the problem on their own, and a tour was made of each of the worktables to investigate how they had approached the problem. After the tour, each worktable was interrupted with thought-provoking questions to help the students reflect. Questions were asked, such as: What mathematical concepts do you think would help you develop the problem? Explain your answer. How would you use the measurements in the plan to help Marcel? Is it possible to calculate the area by adding the measurements? Is it feasible to buy the exact amount you calculated? Why? Does the seller sell the tile in boxes or by square meters? Does Marcel only need to buy the floor to do the renovation? Why?

Activity 2. Building My Home

Considering the information obtained in the previous activity, develop a floor plan of your home, and estimate how much money you would spend to change the flooring in some parts of your house. Four students were asked to share and describe the strategy they used to determine how much money they would spend to change the flooring in some part of their house. Each of the students' interventions was facilitated by the teacher in the same way as Activity 1, using the same guiding questions as a guide.

Activity 3. Approach to the Formal Concept of Area

Considering the following figure (Figure 2), how could we calculate the area of the rectangle and the triangle without counting the squares? (Justify your answer). Initially, the problem was presented, and students were given 10 minutes to independently identify the relationship between the two surfaces. After this period, we addressed students' errors or difficulties with probing questions: How many square units cover the rectangle? Can you explain your calculation? How many square units cover the central triangle? Can you explain your approach? These questions facilitated a group discussion in which we evaluated the strategies used by the students to calculate the number of square units constituting both figures. After the group discussion and debate about the number of square units covering both the rectangle and the triangle, students were given another 5 minutes to establish a relationship between the areas of the two figures. Subsequently, the students were asked to answer the: How could you determine the area without counting each individual square unit? To conclude the exercise and reinforce the lessons learned, the students' responses were supplemented with the following questions: How would you define the 'area' of a flat figure? Can the concept of area be applied to solve other everyday problems? Could you mention or create a problem where this concept could be utilized? Through this structured approach, students were guided to develop a deep understanding of the concept of area and its practical applications.

RESULTS

Activity 1

It was observed that the students approached the teacher with phrases such as, "Teacher, what do I have to do?" and "I don't understand the problem," as seen in the following classroom situation.

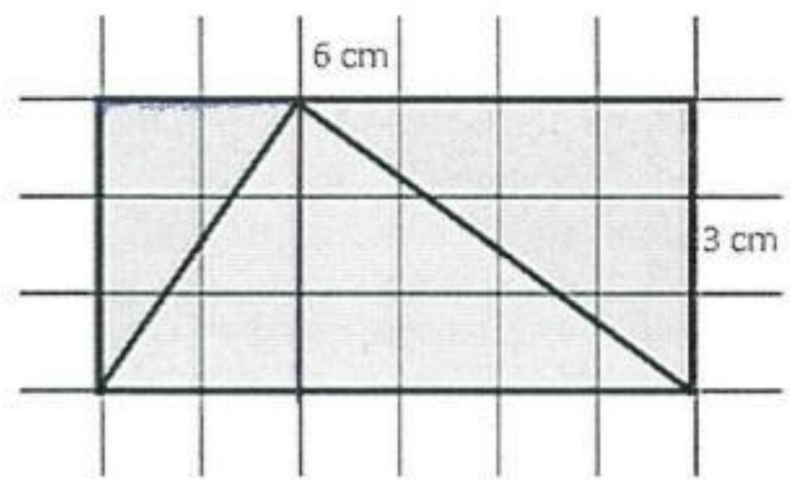


Figure 2. Relationship between areas of two flat figures (Caviedes et al., 2019)

Transcript 1: Initial difficulties in problem development.

Teacher: How are we doing?

Student: we don't understand what else to do.

Teacher: Let's read the problem

Student: Marcel wants to change all the floor...

Teacher: what do we need?

Student: the tile thing, teacher, if we don't know how much we are going to spend.

Teacher: okay, the price of the tile, but what is initially asked in the problem?

Student: how much he is going to spend.

Teacher: and to know how much we are going to spend, what do we need?

Student: the price of the tile.

Student: the perimeter.

Student: but we don't know how many tiles we are going to use.

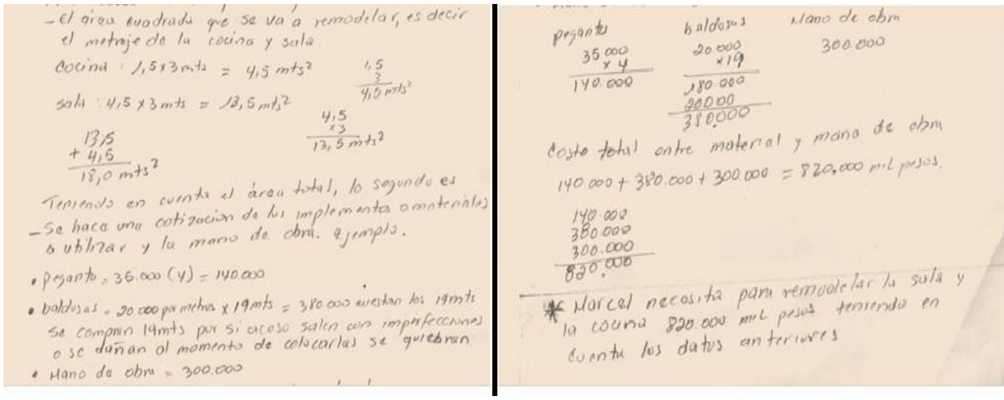
Teacher: and how do I know how many tiles we are going to use?

Student: Measuring. Teacher: and what do we measure?

Student: the area.

Table 2. Results of the checklist activity 1.

Objectives	Descriptors	Assessment		
		Correct	Acceptable	Incorrect
Propose and develop strategies for estimating, measuring, and calculating areas to solve problems.	1. Recognizes, in different contexts or in problem situations, the calculation of the area of plane figures, and represents them in their different manifestations.	18	19	0
	2. Explain with arguments, verbally or in writing, his/her concept of plane figures.	17	10	10
	3. Explain with arguments, verbally or in writing, his/her concept of area of plane figures.	21	16	0
	4. Calculates and uses the area of plane figures in their different manifestations when solving a problem and concludes.	33	4	0
	5. Solves mathematical problems and tasks involving the calculation of the area of plane figures.	37	0	0
	6. Communicates the processes developed to arrive at the solution to the problem.	21	16	0



- El área cuadrada que se va a remodelar, es decir el metraje de la cocina y sala
Cocina: 1,5 x 3 mts = 4,5 mts²
sala: 4,5 x 3 mts = 13,5 mts²
13,5 + 4,5 = 18,0 mts²
Teniendo en cuenta el área total, lo segundo es
- Se hace una cotización de los implementos o materiales a utilizar y la mano de obra. ejemplo.
• pegante = 35.000 (4) = 140.000
• baldosas = 20.000 por metros x 19 mts = 380.000 cuestan los 19 mts. Se compran 19 mts por si acaso salen con imperfecciones o se dañan al momento de colocarlas se quiebran
• Mano de obra = 300.000

pegante 35.000 x 4 = 140.000; baldosas 20.000 x 19 = 380.000; Mano de obra 300.000
Costo total entre material y mano de obra
140.000 + 380.000 + 300.000 = 820.000 mil pesos.
Marcel necesita para remodelar la sala y la cocina 820.000 mil pesos teniendo en cuenta los datos anteriores

The square area to be remodeled is the size of the and living room.
Kitchen: 1.5 x 3 mts = 4,5 mts ²
Living room: 4,5 x 3 mts = 13,5 mts ²

13,5	4,5	1,5
4,5	3	3
18,0 mts ²	13,5 mts ²	4,5 mts ²

Taking into account the total area, the second is:

- A quotation is made for the implements or materials to be used and the labor. For example:
  - Glue: 35000 (4) = 140000
  - Tiles: 20000 per meter x 19 meters = 380000 for the 19 meters. 19 meters are bought in case the come out with imperfections or are damaged at the moment of placing them, they break.
  - Labor= 300000

Glue	tiles	labor
35000	20000	300000
x 4	x19	
140000	180000	
	20000	
	380000	

total cost between material and labor
140000+380000+300000=820000 pesos

140000
380000
300000
820000

Marcel needs to remodel the living room and kitchen 820000 thousand pesos taking into account the above data.

Figure 3. Solution to Problem 1 Example 1

In the previous solution (Figure 3), the student identifies the use of the area of flat figures and represents it as a product of two magnitudes according to Corberan (1997). Additionally, considering the representation, the student identifies the related flat figures (rectangle) and some of their properties, which, according to Vargas & Gamboa (2013), is characteristic of a student at the Van Hiele level of informal deduction. In contrast, in the following solution (Figure 4), the student identifies the importance of the area, but none of the manifestations of the concept of area proposed by Corberan (1997) are visualized. Likewise, although the problem is solved, there is no evidence of the use of area calculation to make decisions, nor does the student communicate all the processes used to solve the problem. This is an example of a solution that, in general terms, is in level of regular compliance with the descriptors.

Activity 2

No initial blocks in the problem approach were observed during its development. This was evidenced by the absence of phrases such as: "I don't understand, teacher", "I don't know how to start". After the development and evaluation of Activity 2, the following results were obtained (Table 3).

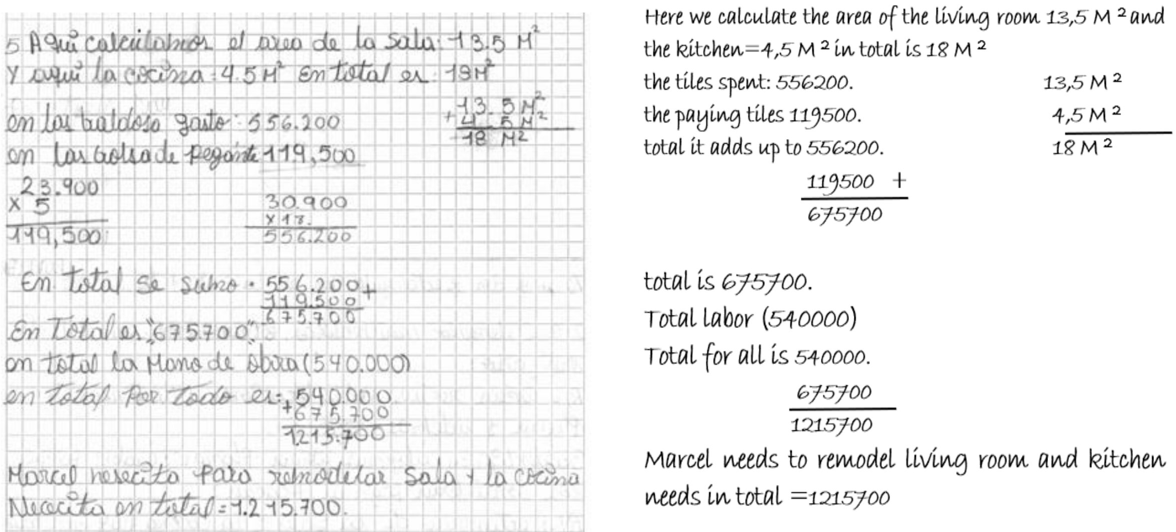


Figure 4. Solution of problem 1 example 2

Table 3. Result of checklist activity 2.

Objectives	Descriptors	Assessment		
		Correct	Acceptable	Incorrect
Propose and develop strategies for estimating, measuring, and calculating areas to solve problems.	1. Recognizes, in different contexts or in problem situations, the calculation of the area of plane figures, and represents them in their different manifestations.	18	9	8
	2. Explain with arguments, verbally or in writing, his/her concept of plane figures.	14	20	3
	3. Explain with arguments, verbally or in writing, his/her concept of area of plane figures.	18	9	10
	4. Calculates and uses the area of plane figures in their different manifestations when solving a problem and concludes.	14	19	4

	5. Solves mathematical problems and tasks involving the calculation of the area of plane figures.	21	16	0
	6. Communicates the processes developed to arrive at the solution to the problem.	15	16	6

Quantitatively, the results show that Activity 2 constituted a greater cognitive effort for the students, as in most of the descriptors, between 3 and 10 students were placed in the non-compliance (NC) box. The students did not require class management to communicate the decisions and procedures for problem development; therefore, they were more autonomous in the process in general, which is also understood as an indicator of improvement, as seen in the following class examples (Figure 5 and 6).

In the previous example (Figure 5), the student identifies the area to work on and the flat figure that composes it, which according to Vargas & amp; Gamboa (2013) is characteristic of a student at the Van Hiele level of informal deduction. The student also identifies the importance of area calculation and represents it with one of the manifestations proposed by Corberan (1997), which is understood as an empirical argument for the concept of area. Furthermore, the student uses the concept of area in conjunction with strategies based on the heuristics of the Anglo-Saxon school of problem-solving (Rodriguez & amp; Marino, 2009) to respond to the situation and always argues in a written and procedural manner about the decisions made and the results obtained. Therefore, this is an example of compliance with the descriptors on the checklist and thus meets the objective of Activity 2 of the didactic unit.

Contrary to this, in the following example (Figure 6), the absence of representation and application of the calculation of the area of flat figures (Corberan, 1997) for problem development is evident. The student does not respond to the problem question, and although they communicate the area in which the remodeling will be carried out, they do not argue the decisions and procedures that lead to a subsequent response. This is an example of a resolution that, in general terms, is at a level of non-compliance (NC) with the descriptors.

Compared to the results of the previous activity, it was observed that most students did not confuse the concept of area with that of perimeter when solving the problem. On the other hand, based on student perceptions, this problem reinforced the learnings worked on in activity 1, and generated greater motivation since, when asked "how did you find activity 2?", situations such as those observed in Transcript 2 emerged.

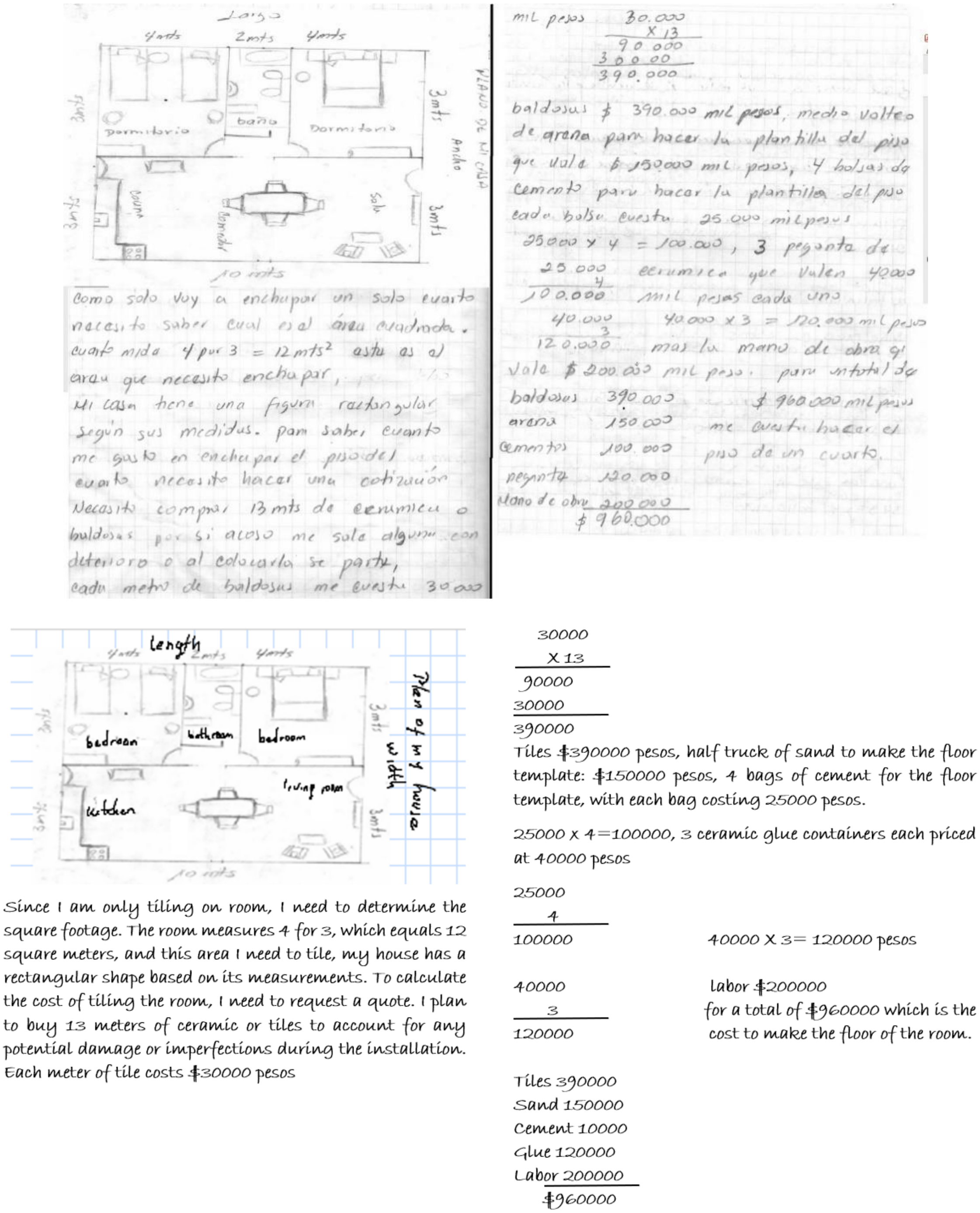


Figure 5. Solution to Problem 2 Example 1

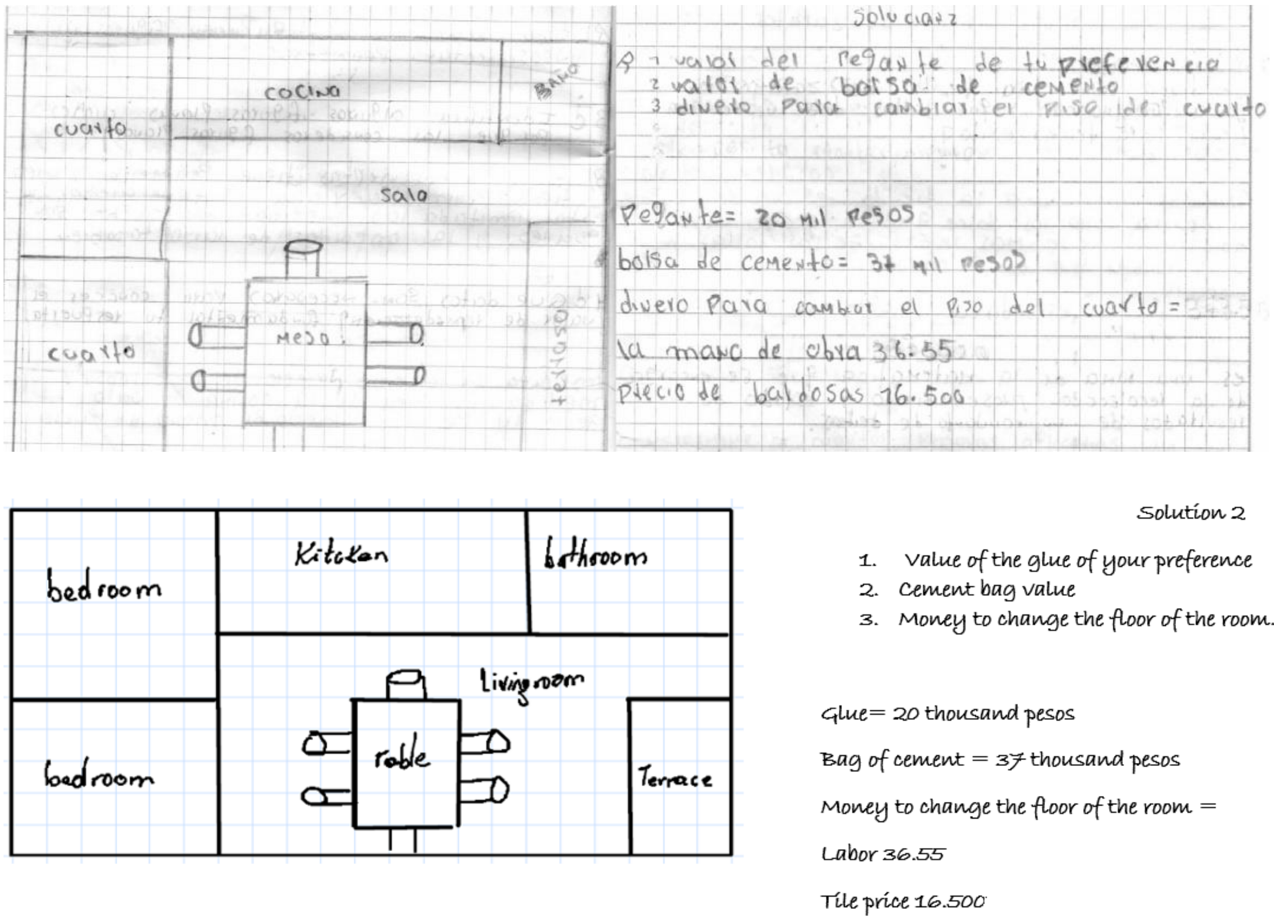


Figure 6. Solution to problem 2 example 2

Transcript 2: Student Perceptions of Activity 2

Teacher: When you were doing this activity, what did you feel?

Students: I felt happy 6 S what the area of plane figures was for.

Teacher: Why?

Students: Teacher because I understood.

Teacher: What did you understand?

Students: what the area of plane figures was for.

Students: that was precisely the intention of this activity, to show you that math is not only on the board, but also outside, for example, the person who laid this floor, went through the same process that you did.

Activity 3

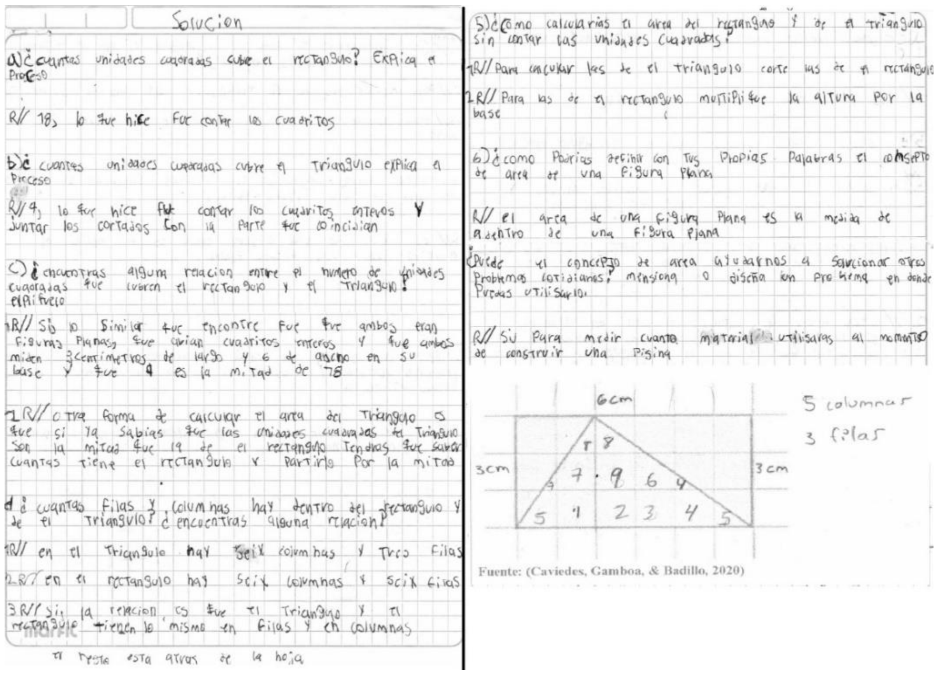
After the development and assessment of Activity 3, the following results were obtained (See Table 4).

Table 4. Result of checklist activity 3.

Objectives	Descriptors	Assessment		
		Correct	Acceptable	Incorrect
Propose and develop strategies for estimating, measuring, and calculating areas to solve problems.	1. Recognizes, in different contexts or in problem situations, the calculation of the area of plane figures, and represents them in their different manifestations.	12	22	3
	2. Explain with arguments, verbally or in writing, his/her concept of plane figures.	4	8	25
	3. Explain with arguments, verbally or in writing, his/her concept of area of plane figures.	9	25	3
	4. Calculates and uses the area of plane figures in their different manifestations when solving a problem and concludes.	8	25	4
	5. Solves mathematical problems and tasks involving the calculation of the area of plane figures.	23	12	2
	6. Communicates the processes developed to arrive at the solution to the problem.	7	12	17

For the students, recognizing, representing in one of its manifestations, and calculating the area of plane figures (Corberan, 1997) for the problem's development was a strength, as most of the students were able to calculate the area of at least one of the two proposed figures. In addition, for the students, communicating the processes developed in this activity did not pose a challenge, as 35 students communicated some or all of the processes developed to answer the problem.

However, describing the concept of the area of a plane figure from a theoretical perspective was a challenge, as only four students were able to fulfill it. Yet, in most cases, the lack of knowledge of the theoretical concept did not have a direct relationship with the development of the concept from a practical point of view, that is, most students did not comply with the explanation of the definition of the area, but they did apply the practical part when calculating the area of the rectangle or the triangle (Figures 7 and 8).



a) how many square units does the rectangle cover? Explain the process.

A// 18, what I did was to count the squares.

b) how many square units does the triangle cover? explain the process

A// 4, what I did was to count the whole squares and join the cut ones with the part that matched.

c) do you find any relation between the number of square units that cover the rectangle and the triangle? explain it

1A// yes, the similarity I found was that both were flat figures that had squares and that both were 3 centimeters long and 6 centimeters wide at their base and that 9 is half of 18.

2A// another way to calculate the area of the triangle is that if you already knew that the square units of the triangle are half that of the rectangle you will have to know how many square units the rectangle has and divide them in half.

d) How many rows and columns are there inside the rectangle and the triangle? do you find any relationship?

1A// in the triangle there are six columns and three rows.

2A// in the rectangle there are six columns and six rows.

3A// Yes, the relationship is that the triangle and the rectangle have the same number of rows and columns.

The rest is at the back of the sheet.

5) How would you calculate the area of the rectangle and the triangle without counting the square units?

1A// to calculate the triangle cut the rectangle.

2A// For those of the rectangle, multiply the height by the base.

6) How could you define in your own words the concept of area of a plane figure?

A// the area of a plane figure is the measurement of the inside of a plane figure.

¿ Can the concept of area help us solve other everyday problems? mention or design a problem where you can use it.

A// yes to measure how much material you will use when building a pool

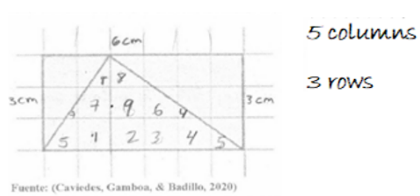
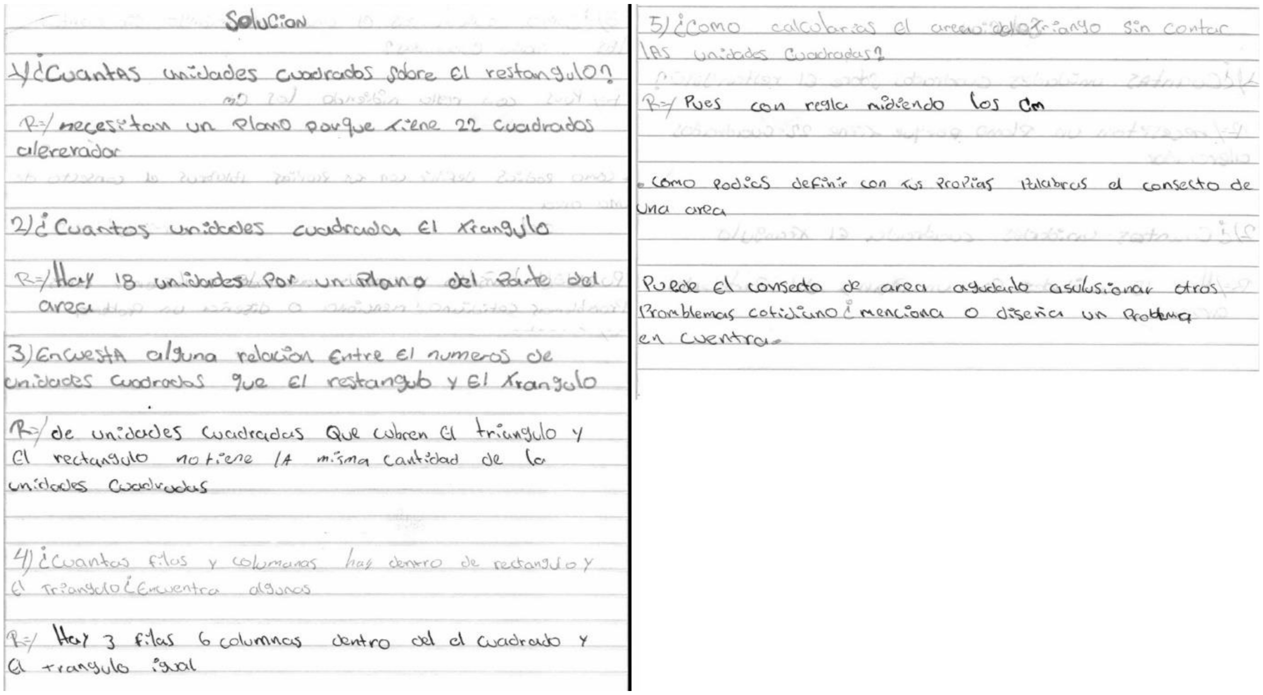


Figure 7. Solution to problem 3 example 1

In the previous image, the student initially identifies the use of the concept of area to respond and represents it in writing. The manifestation of the area as the amount of plane occupied by a surface (Corberan, 1997) is materialized when the student communicates the process carried out to count the square units of both the rectangle and the triangle. Additionally, the student uses the image of the problem to uniquely count the square units, which demonstrates the use of the resource to identify and represent the concept of area. The student also makes this concrete by defining the process and the concept in question considering some mathematical signs and the communicated arguments, which is an example of a resolution that, in general terms, is at a level of compliance (C) with the descriptors inherent to the problem.

In contrast to this, in the example shown in Figure 7, it can be observed that the student attempts to represent the area by means of the manifestation of the area as the number of square units of a surface but does so incorrectly. The student is also not capable of defining the concept of the area of plane figures and does not communicate the procedures developed to arrive at the solution to the problem. This is an example of a resolution that, in general terms, is at a level of non-compliance (NC) with the descriptors.

It was evidenced that the use of mathematical symbols for the generalization of information about the concept and calculation of the area of plane figures represented a weakness for the students, as 17 students did not meet this criterion. This was because when they were asked, "How would you calculate the area of the triangle without counting the square units?" it was expected that they would relate the number of square units used to cover the rectangle with those of the triangle, to conclude that the area of the triangle is half that of the circumscribed rectangle. Taking into account the results of the previous activities, there is an improvement in recognizing the area of plane figures and calculating the area of quadrilaterals, as well as in communicating the processes to solve the problem. However, a weakness in the formal conceptualization of the area of plane figures is observed. See Figure 8.



Solución

1) ¿Cuantas unidades cuadrados sobre el restangulo?
R=/ necesitan un plano porque tiene 22 cuadrados alrerador

2) ¿Cuantos unidades cuadrada el triangulo
R=/ Hay 18 unidades por un plano del parte del area

3) Encuesta alguna relación entre el numeros de unidades cuadrados que el restangulo y el triangulo
R=/ de unidades cuadradas que cubren el triangulo y el rectangulo no tiene la misma cantidad de unidades cuadradas

4) ¿Cuantas filas y columnas hay dentro de rectangulo y el triangulo ¿Encuentra algunos
R=/ Hay 3 filas 6 columnas dentro del el cuadrado y el triangulo igual

5) ¿Como calcularas el area del triangulo sin contar las unidades cuadradas?
R=/ Pues con regla midiendo los cm

Como podrías definir con tus propias palabras el consepto de una area

Puede el consepto de area ayudarlo a solucionar otros problemas cotidiano ¿menciona o diseña un problema en cuentra

1) How many square units over the rectangle?
A/ they need a plan because it has 22 squares around it.

2) How many units square the triangle
A/ there are 18 units for a plan of part of the area.

3) find some relationship between the number of units that the rectangle and the triangle
A/ of square units that cover the triangle and the rectangle do not have the same number of square units.

4) How many rows and columns are there inside the rectangle and triangle? Can you find any?
A/ there are 3 rows 6 columns inside the square and the same triangle.

5) How would you calculate the area of the triangle without counting the square units?
A/ Well, with a ruler measuring the cm.

How could you define the concept of an area in your own words?

Can the concept of area help you solve other everyday problems? Mention or design a problem.

Figure 8. Solution to problem 3 example 2

DISCUSSION

The activities in this study were carefully planned to cover theoretical and practical aspects to unravel the complexity surrounding student involvement and comprehension of the idea of area. The exercises aimed to promote students' holistic, metric thinking by using methods of estimate, measurement, and area computation together with the recognition of flat figures. Looking more closely at the results, a trend emerged showing that students, especially in the early going, mistakenly used the perimeter calculation as a stand-in for the area calculation, demonstrating a concrete misunderstanding. This did not translate to other tasks, suggesting that this misunderstanding may be cleared up with repeated completion of the tasks. But even as practical comprehension appeared to be expanding, there was still a noticeable gap in the area concept's formalization and generalization.

These results are consistent with the research of Winarti et al. (2012), which suggests that students' comprehension of area and perimeter may be significantly enhanced by activities connected to their immediate environments. There is, however, a difference in the application of the contextual activities' focus: whereas Winarti et al. investigated the connection between perimeter and area, the current study directed the contextual activities toward comprehending the notion of area, first from a utilitarian and then from a theoretical perspective. Furthermore, the modification of Activity 3, one of the area activities suggested by Caviedes et al. (2023), was helpful in recognizing various approaches or ways to compute the area. Compared to Caviedes et al., the current study significantly increased the use of geometric processes, which may have been caused by the extra-mathematical context in which the issues were implemented (Activities 1 and 2).

In contrast to the methods suggested by Gutierrez & Jaime (1991), the study proceeded through the first three levels of Van Hiele, guaranteeing a comprehensive investigation of extra- and intra-mathematical activities. Students' development and critical evaluation of the first three Van Hiele levels were greatly aided by this two-pronged approach. Parallel to this, the study showed that the suggested activities strongly emphasized the mathematical process of problem-solving even if they were designed to encourage the learning of the area idea. Thus, as transcript 1 illustrates, the exercises followed Polya's (1945) definition of issues, which calls for investigation and contemplation before coming up with answers. This supports the claims made by Ortiz & Salcedo (2018) about the critical role didactic issues play in mathematical learning, which are further supported by the observable advantages that arise when activities are used in an extra-mathematical setting in this study. However, in order to improve the effectiveness of this approach, a second phase of research can be considered that takes into account the weaknesses of the students. This second phase has been designed in detail to allow teachers to carry it out in different classes.

This guideline integrates theoretical foundations and practical applications to create an instructional manual that addresses student misconceptions, improves plane figure recognition, builds argumentative competencies, and efficiently uses PBL paradigms.

Objective: Address student misconceptions and errors in solving area problems through structured teaching and learning activities.

- Step 1: Directly address misconceptions. Misconceptions about perimeter: Develop activities that clearly differentiate between the concepts of perimeter and area, ensuring that students do not conflate the two.
- Step 2: Facilitate argumentation skills. Encourage verbalization: In problem-solving sessions, encourage students to verbalize their thought processes and reasoning. Promote logical reasoning: Engage students in activities that enhance their logical reasoning and argumentation skills in mathematical problem-solving.
- Step 3: Enhance recognition and understanding of plane figures. Visual Recognition: Use visual aids and physical models to enhance recognition and understanding of different plane figures and their properties. Explore Properties: Engage students in activities that allow them to explore and understand the properties of different plane figures.
- Step 4: Leverage PBL with real-world contexts. Use of real-world problems: Engage students with problems that have tangible, real-world contexts, which, according to the research, enhances understanding. Collaborative problem-solving: Use PBL in a group setting to allow students to share and discuss various strategies for problem-solving.
- Step 5: Integrate continuous assessment. Feedback on misconceptions: Provide continuous feedback, especially targeting misconceptions about area and perimeter, and argumentation during problem-solving. Reflection on Mistakes: Engage students in reflective activities where they analyze and learn from their mistakes and misconceptions.
- Step 6: Explore extra-mathematical and intra-mathematical problems. Diverse problem contexts: Ensure that problems presented to students are diverse, including both extra-mathematical and intra-mathematical contexts, to enhance applicability and theoretical understanding.

CONCLUSIONS

The conclusion drawn from the application of activities in the didactic unit enabled the accurate representation of various manifestations of area and its utilitarian character. Nevertheless, the implementation of these activities did not significantly contribute to the generalization and formalization of the concept of the area of flat figures. Furthermore, these activities facilitated the utilization of various heuristics proposed by the Anglo-Saxon school of problem-solving. The employment of problems involving extra-mathematical contexts can have a pronounced impact on enhancing the process of problem-solving and the appropriation of mathematical objects, as evidenced by the results of activities 1 and 2 (extra-mathematical context), contrasted with the outcomes of activity 3 (intra-mathematical context). Problem-Based Learning, in tandem with the manifestations of the area proposed by Corberan and the levels of Van Hiele, emerges as a viable strategy to promote the problem-solving process related to the area of flat figures.

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Direct and Indirect Effect of Self-Efficacy, Anxiety and Interest on Algebraic Problem-Solving Achievement

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Abstract*: This study examines the direct and indirect effects of some affective constructs, such as self-efficacy (SE), mathematics anxiety (MA) and mathematics interest (MI) on algebraic problem solving achievement (PSA). The sample of the study consists of 400 class IX secondary school students in Morigaon district of Assam, India. The instruments employed in this investigation were the SE scale, the MA scale, the MI scale and the algebraic PSA test. The relationship between affective constructs and PSA in algebra is investigated using two-stage structural equation modeling. The results reveal that SE is the only affective construct that had a direct effect on algebraic PSA, while MI has an indirect effect on algebraic PSA through SE. Also, MA has an indirect effect on algebraic PSA through MI and SE. The findings suggested that mathematics educator should adopt innovative strategies to make the subject matter more interesting and increase students' self-efficacy, which may reduce math anxiety. These will assist the students in enhancing their problem-solving abilities and their achievement.*

Keywords: Algebraic Problem-Solving; Mathematics Anxiety; Mathematics Interest; Self-Efficacy; Structural equation modeling.

INTRODUCTION

Algebra is regarded as a gateway to the elevated mathematical achievement and future opportunities (Foegen, 2008). Learning of algebra improves students' reasoning ability, critical thoughts and problem solving ability (Bell, 1996). Algebra is introduced informally at the elementary level when learning arithmetic, but it is explicitly used at the secondary level. In the Indian education system, algebra is introduced in grade 6 (age 11+). At the beginning stage, the main approach is conventional and places a strong emphasis on symbol manipulation. At the secondary level, algebra is widely discussed and at this stage, symbolic algebra is used for solving daily life problems through the development of algebraic expressions, polynomials, linear and quadratic equations, and their solutions (NCERT, 2005). Algebra occupies a major part of secondary mathematics (NCERT, 2005) and many students face difficulties due to its abstract nature at this stage (Kieran, 2006). Algebraic problem solving is one of the most challenging aspects of mathematics at the secondary level. Most of the nation puts ample stress on the application of mathematics in daily life and promote mathematical reasoning as well as problem-solving techniques in their mathematics curricula (Faulkner et al., 2020). In spite of efforts made to provide mathematical problem-solving strategies, it is found that student achievement is not satisfactory (Holton et al., 1999, Wicaksono & Korom, 2022).

Achievements in mathematics lead to a world of abundant opportunities. But, unfortunately it has been witnessed that both developing and developed countries are plagued by the issue of poor achievement in mathematics (Ozcan & Gumus, 2019). Consciousness on the cognitive and affective constructs of the students would bring greater achievement in mathematical problem solving (Furinghetti & Morselli, 2009, Guenyari et al., 2022, Hoffman, 2010). The understanding of learning in the cognitive domain is thought to be aided by affective factors, which also significantly influence learning outcomes (Furinghetti & Morselli, 2009). Empirically, it has been established that several factors affect students' problem-solving achievement (Guven & Cabakar, 2013, Zan et al., 2006). Among these, the most influential affective factors connected to mathematical problem solving that are given the greatest importance and relevance in the mathematical domain are self-concept, self-efficacy, mathematics anxiety, mathematics interest and attitudes. Guven and Cabakor (2013) investigated the relationship between affecting factors and problem-solving achievements (PSA) of 7th standard Turkish students. Their study reveals a moderate correlation between PSA and problem-solving beliefs, self-efficacy, and anxiety. Hoffman (2010) highlighted that mathematics anxiety (MA) and self-efficacy (SE) are significant factors in mathematical problem-solving. Several studies have shown an association between mathematical problem-solving and SE (Karaoglan Yilmaz, 2022; Kohen et al., 2019). Other studies also signify that anxiety have major effect on their mathematical achievement (Vukovic et al., 2013). Moreover, SE and MI have been recognized as the most influential factors in problem-solving (Niemivirta & Tapola, 2007). According to Heinz et al. (2005) mathematics interest (MI) is one of the most influential factors in mathematical achievement. In support, Köller et al. (2001) mentioned that interest influences mathematical achievement both directly and indirectly. They also point out that increasing interest in mathematics also increases its achievements. Students increasing interests typically result in increased SE (Bandura & Schunk, 1981) and SE has been found to be positively related to mathematics achievement (Zhang & Wang, 2020). It has also been established that while SE has been controlled, there is no incremental variance in MI (Lent et al., 1991). So SE is an important factor between MI and achievement. Moreover, Bandura (1977) and Bandura (1986) mentioned that students' MA can be controlled by higher SE. Therefore, there may be a relationship between MI and SE, which is related to PSA, as well as a relationship between MA and SE.

On the other hand, Gupta and Maji (2022) examined the relationship of students' MA, SE and mathematics performance in Indian context. They found that anxiety have indirect effect on performance through SE. Moreover, Jasani (2022) carried out a study to determine the relationship between MA, SE, and algebraic performance. He found the indirect relationship between MA and PSA through SE. The relationship between attitude and algebraic PSA among secondary school students in the Morigaon district of Assam was studied by Das and Ali (2023). Their investigation highlighted the positive relationship between attitude and algebraic PSA.

The research findings discussed above predict the relationship between affective factors and PSA. Majority of the studies related to some of the factors mentioned above have been done outside India. However, it is less clear how these constructs might interact in a model that includes both direct and indirect effects on each other. It is also noteworthy to mention that no research is done to establish the direct and indirect effect of these affective factors on algebraic problem-solving achievement in Indian context. Therefore, in order to bridge the relationship this investigation is conducted to offer both theoretical and empirical proof of the relationship between SE, MI, MA, and algebraic PSA.

LITERAURE REVIEW

Self-Efficacy and Problem Solving

Self-efficacy (SE) describes a person's confidence in their capacity to carry out the behaviours required to achieve specific problem (Bandura, 1997). This confidence has a significant impact on one's activity, effort, perseverance, determination, learning, and achievement (Bandura, 1986). Research demonstrated that SE significantly correlates with cognitive and affective constructs, together with learning achievement (Schöber et al., 2018). SE has generally been recognised as an influential factor of academic achievement (Hayat et al., 2020), particularly in mathematical problem solving (Pajares & Graham, 1999). Several researches unveil that mathematics SE and students' problem solving at high school standard is closely interrelated (Pajares & Miller, 1994). Students with higher self-efficacy (SE) put in greater effort, make more attempts at cognitively difficult tasks, have greater persistence in solving difficult problems and achieve better performance (Bandura, 1977, Karaoglan Yilmaz, 2022). A number of studies have established that SE is positively associated with mathematical problem-solving achievement (Lopez et al., 1997, Shimizu, 2022). Moreover, SE has been identified as the strongest predictor of problem-solving due to its substantial connection with problem-solving achievement (PSA) in mathematics (Pajares & Miller, 1994). Therefore, SE is considered as the most influential affective construct in the present study.

Mathematics Anxiety, Self-Efficacy and Problem-Solving

Mathematics Anxiety (MA) is defined as "the panic, helplessness, paralysis, and mental disorganization that arises among some people when they are required to solve a mathematical problem" (Tobias & Weissbrod, 1980). In such a psychological ambience, students may have negative thoughts towards the subject. This is one of the prominent reasons why students can't perform to the best of their ability. Thus, one of the affective factors of mathematical understanding is mathematics anxiety which also hinders students' problem-solving abilities (Guven & Cabakcor, 2013). Numerous empirical studies have revealed a negative relationship between math anxiety and performance, indicating that people with math anxiety would struggle to cope with it or solve problems (Ashcraft & Kirk, 2001, Namkung et al., 2019). Hoffman (2010) pointed out that higher MA is correlated with ineffective problem solving. So, to curb MA is a pre-requisite for inculcating problem-solving skills. In this context, SE may be the most relevant among the affective variables. The social learning hypothesis suggests that lack of SE may be the cause of MA (Bandura, 1977). On the other hand, anxiety levels can be controlled with the help of strong SE beliefs (Bandura, 1986). Therefore, we assumed that SE mediates the link between MA and PSA in the study. Therefore, in this investigation we considered MA to have a direct effect on PSA as well as an indirect effect through SE.

Mathematics Interest, Self-Efficacy and Problem-Solving

According to Ainley (2006), interest is defined as an affective situation that corresponds to the individual's perception of learning. Interest is the most important aspect of learning mathematics (Yu & Singh, 2018), which psychologically boosts students attention, cognitive processes, perseverance and affective participation. Research shows that students who are more interested in mathematics are able to employ effective mathematical techniques and methods (Fisher et al., 2012). Students' lack of interest in mathematics may hinder their learning ability, particularly in problem solving ability. One of the reasons for their poor achievement and discontinuation of learning is lack of mathematical interest (MI) (Schraw et al., 2001). It has been established that MI and achievement are positively associated at the secondary stage (Zhang & Wang, 2020). Also, interest in mathematics is a predictor of mathematical and academic achievement. Empirically, it has been revealed that high MI may lead to high SE and self-efficacy also has a positive connection with mathematical performance (Kohen et al., 2019). Moreover, students who have very high MI and SE they can perform well, when they faced difficulties in mathematical activities like reasoning or problem solving (Huang et al., 2019). Therefore, in this investigation we considered MI to have a direct effect on PSA as well as an indirect effect through SE.

The present study

In light of the existing literature, the purpose of this study is to establish the relationships between SE, MA, MI, and algebraic problem-solving achievement (PSA) by adding both direct and indirect effects to the hypothesized models. Going through the outcomes of various investigations, while it is endeavored to design a model in which these variables are taken collectively, it is hypothesized that SE has a direct impact on problem solving (Pajares & Miller, 1995). In addition to that, SE can act as a mediating variable between affective constructs and problem solving efficiency (Hayat et al., 2020, Ozcan & Gumus, 2019). According to Lent et al. (1991), SE can serve as a mediator between MI and mathematics achievement. Also, the structural modeling study of Zhang and Wang (2020) found MI has a direct effect on SE and an indirect effect on problem solving.

On the other hand, MA has been recognized as one of the most significant affective variables that directly affect on interest (Huang et al., 2019) and a direct effect on SE (Bandura, 1977), as well as PSA in mathematics (Ashcraft & Kirk, 2001, Guven, & Cabakcor, 2013). The correlation between MA and SE has been demonstrated using both theoretical (Bandura, 1977) and empirical evidence (Griggs et al., 2018, Huang et al., 2019, Pajares & Miller, 1994). MA also has indirect effects through SE on PSA (Gupta & Maji, 2022, Ozcan & Gumus, 2019). Based on the findings, we therefore hypothesized the direct and indirect effects of affective factors on algebraic problem solving achievement (PSA). The hypothesized theoretical model is shown in Figure1.

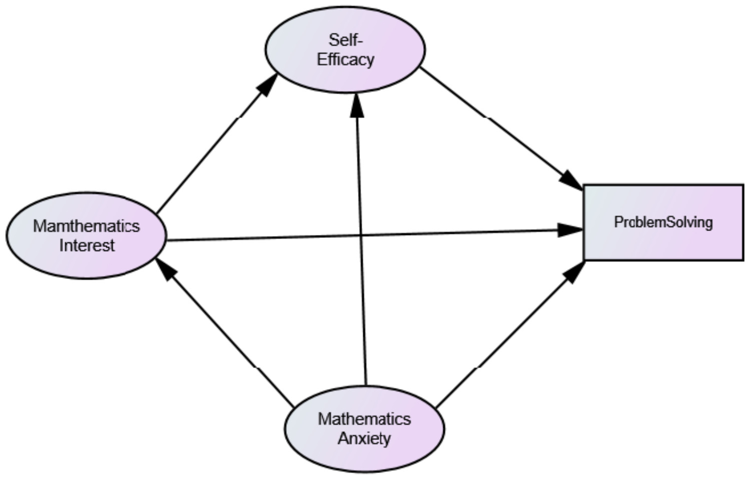

Figure 1. Hypothesized theoretical model

METHODS

Sample

The sample consists of 400 class IX students in the Morigaon district of Assam, India. The sample is randomly selected from the secondary schools in the district.

Instruments

In the study, the SE scale, MI scale, MA scale, and the algebraic problem solving achievement (PSA) test are used to gather the data. The scales MI, MA, and SE are adopted from PISA 2012 (OCED, 2013).

Self-Efficacy

The mathematics SE scale consists of five items and was adopted from PISA 2012.The items are, "I feel confident enough to solve algebraic word problems in my mathematics class"; "I am confident in calculating P(x) =x^2-4x+3 at x=1"; "I believe I can solve equations easily"; "I believe I am the type of person who can do mathematics" and "I believe I can understand the content in a mathematics course". These five items are given a 5-point rating. The scale of the Likert type ranges from "5-strongly agree" to "1-strongly disagree". Results of Confirmatory factor analysis (CFA) had acceptable fit indices (Chi-squared (χ2)/degree of freedom (df) =2.33, $p< 0.001$, Goodness of Fit Index (GFI) =0.99, Adjusted Goodness of Fit Index (AGFI) =0.97, Root Mean Square Error of Approximation (RMSEA) =0.06, Comparative Fit Index (CFI) =0.99), and Tucker-Lewis index (TLI) =0.98. The internal consistency (Cronbach's alpha) of the scale at 0.86 is acceptable.

Mathematics Interest

The MI scale was assessed with five items and adopted from PISA 2012. The scales are namely, "I do mathematics because I enjoy it"; "I would like solve algebraic problem"; "I enjoy reading about mathematics"; "I am always interested to solve algebraic word problems"; and "I am interested in the things I learn in mathematics". The MI is a 5-point Likert-type scale ranging from "5-strongly agree" to "1-strongly disagree". CFA had acceptable fit indices (χ2 /df =3.57, $p< 0.001$, GFI = 0.98, AGFI=0.95, RMSEA=0.08, CFI=0.98, and TLI= 0.96). The internal consistency of the scale at 0.82 is acceptable.

Mathematics Anxiety

The mathematics anxiety scale is also consisting of five items and adopted from PISA. "I get nervous when taking a mathematics test"; "I worry that I will not be able to get a good grade in my mathematics course"; "I feel helpless when doing a mathematics problem"; "I become very nervous doing mathematics problems" and "I often worry that it will be difficult for me in mathematics classes". The MA is a 5-point Likert-type scale ranging from "5-strongly agree" to "1-strongly disagree". CFA had acceptable fit indices (χ2 /df =6.5, $p< 0.001$, GFI = 0.97, AGFI=0.91, RMSEA=0.11, CFI=0.94, and TLI= 0.89). The internal consistency (Cronbach's alpha) of the scale at 0.78 is acceptable.

Algebraic Problem -Solving achievement

Students' algebraic problem solving achievement (PSA) is measured by a test. The test consists of 20 multiple choice items based on the course syllabus of class IX standard SEBA (Secondary Education Board of Assam) text book. The test items were validated with the help of an expert mathematics teacher and two other educators with more than 10 years of experience. The reliability Cronbach alpha is 0.89 acceptable. One example of the sample question is-

"The taxi fare in a city is as follows: For the first kilometer, the fare is Rs 8 and for the subsequent distance it is Rs 5 per km. Taking the distance covered as x km and total fare as Rs y, write a linear equation for this information".

Procedure

We began gathering the data with the principals' and class teachers' consent. Students who expressed a willingness to participate in the study were approached. They were required to complete the SE, MI, and MA scales and then an algebraic PSA test. All participants were made aware that the information we gather would be kept confidential. The participants completed the whole data collection process within 50 minutes. The SPSS 26.0 version was used to calculate the descriptive statistics and AMOS 18 for the hypothesized effects.

Data analysis and validation

The collected data has been arranged for analysis, the descriptive statistics are used. The Pearson correlation was computed to establish the relationship between SE, MI, MA, and algebraic PSA. After that the hypothesized relationship between the construct (Figure 1) has been tested with the help of two-stage structural equation modeling (SEM), as recommended by Anderson and Gerbin (1998). At first examine the model fit indices whether it is acceptable or not. Secondly, the direct and indirect effects are analyzed using AMOS.23 for the casual relationship. Moreover, parsimonious model is also used for the least number of paths. The model fit indices ($\chi 2$ /df, GFI, AGFI, RMSEA, CFI and TLI) are adopted to assess the Hypothesized model. The values of GFI, AGFI, RMSEA, CFI, and TLI greater than 0.90 are often considered acceptable, and RMSEA values less than 0.08 indicate a reasonable fit (Byrne, 2016, Hu & Bentler, 1999, Kline, 2016).

RESULTS

Preliminary analysis

In Table 1, the descriptive statistics and correlations of the studied variables are shown. The correlation results of the variables (MI, MA, SE, and PSA) are significantly correlated at the 0.01 level, as predicted theoretically. From Table 1, it is seen that algebraic PSA has a significant positive correlation with SE ($r= 0.81, p < 0.01$), MI ($r= 0.74, p < 0.01$), and a negative correlation with MA ($r= -0.65, p < 0.01$). Also, SE has a positive correlation with MI ($r= 0.78, < 0.01$) and a negative correlation with MA ($r= -0.61, p< 0.01$), There is a negative correlation of MI with MA ($r=- 0.62, p< 0.01$).

Table 1. Descriptive statistics and Pearson correlation of latent constructs.

	SE	MI	MA	PSA
SE	1	0.795**	-0.561**	0.799**
MI		1	-0.600**	0.763**
MA			1	-0.619--
PSA				1
Mean	3.5	3.37	2.9	11.6
Standard Deviation	0.75	0.64	0.55	5.39

** p< 0.01

Measurement model

In the measurement model, three latent constructs are formed by the 15 observed constructs (the SE, MI, and MA scales). To assess the validity of the constructs of the three related scales, CFA has been used. The CFA results indicated the fit indices are in the acceptable range of the two-stage measurement model ($\chi 2$ /df =2.09, p< 0.001, GFI=0.95, AGFI=0.93, RMSEA=0.05, CFI=0.96, and TLI=96) (Byrne, 2016; Hu & Bentler, 1999; Kline, 2016).

Structural model

The initial model has been examined (Figure 2) with and found acceptable fit indices (Table 2). Though it has been found acceptable fit, $\chi 2$ /df =2.07, p< 0.001, GFI=0.94, AGFI=0.92, RMSEA=0.05, CFI=0.96, and TLI= 0.96 (Hu & Bentler, 1999), but some path of the model is not significant. The insignificant path from MA to SE and MI to problem solving achievement (PSA) has been removed from the initial model and the revised model was rested. The revised model (Table 2 and Figure 3) also found acceptable fit, ($\chi 2$/df =2.04, p< 0.001, GFI=0.94, AGFI=0.92, RMSEA=0.05, CFI=0.96, and TLI=0.96) (Hu & Bentler, 1999).

Table 2. Goodness of fit indices of initial, revised and final models.

Goodness of fit	$\chi 2$ /df	GFI	AGFI	CFI	RMSEA	TLI
Initial model	2.07	0.94	0.92	0.96	0.05	0.96
Revised model	2.04	0.94	0.92	0.96	0.05	0.96
Final model	2.10	0.94	0.92	0.96	0.05	0.96

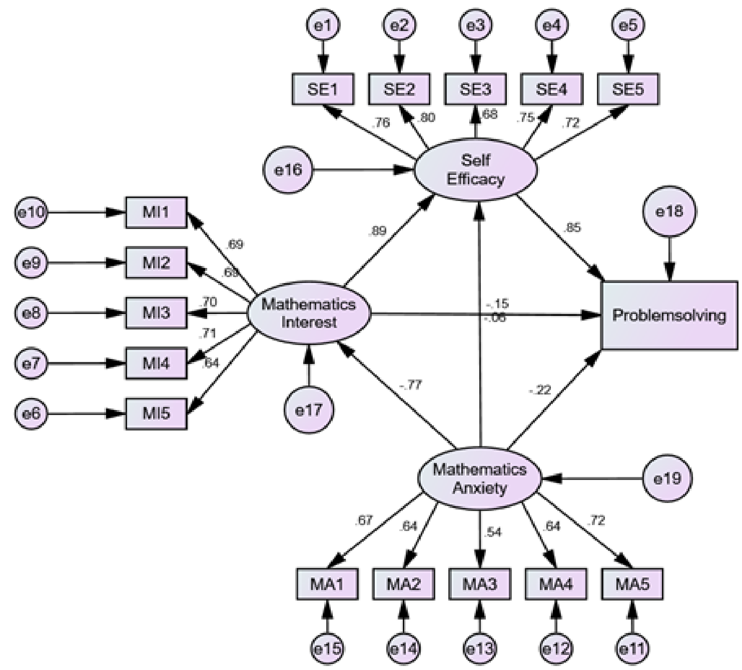

Figure 2. Standardized parameter estimates for initial model of algebraic PSA

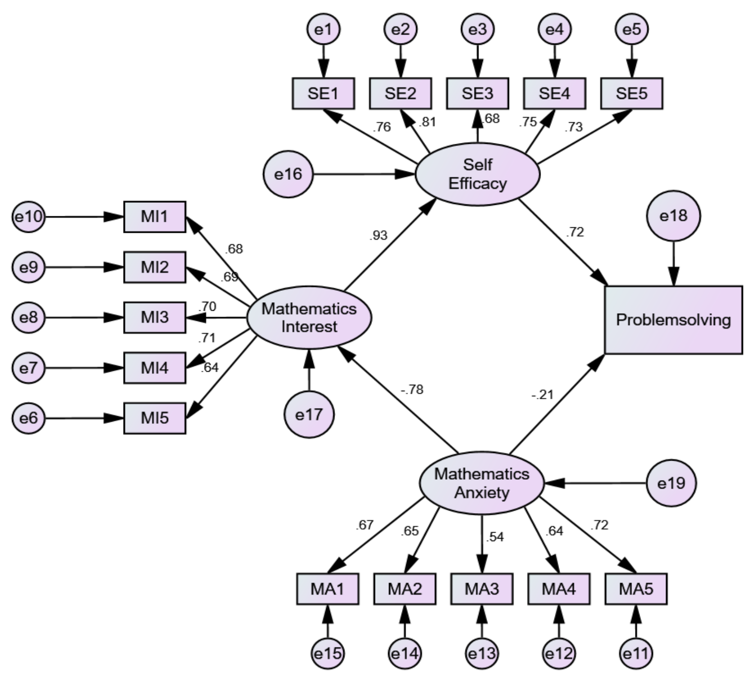

Figure 3. Standardized parameter estimates for revised model of algebraic PSA

Again for the parsimonious fit the model is preferred with less parameters or pathways. So, the path from MA to algebraic PSA is removed and the model is again tested. The final (simplified) model fit indices ($\chi2$/df =2.1, p<0.001, GFI=0.94, AGFI=0.92, RMSEA=0.05, CFI=0.96, and TLI=0.96) revealed the same fit indices, with the exception of a minor chi-square change. This final model (Table 2 and Figure 4), indicated that SE has a direct and positive effect on algebraic PSA (β=0.88, p< 0.001), MI has a direct and positive effect on SE (β=0.96, p< 0.001) and MA has a direct and negative effect on MI (β=-0.76, p< 0.001).

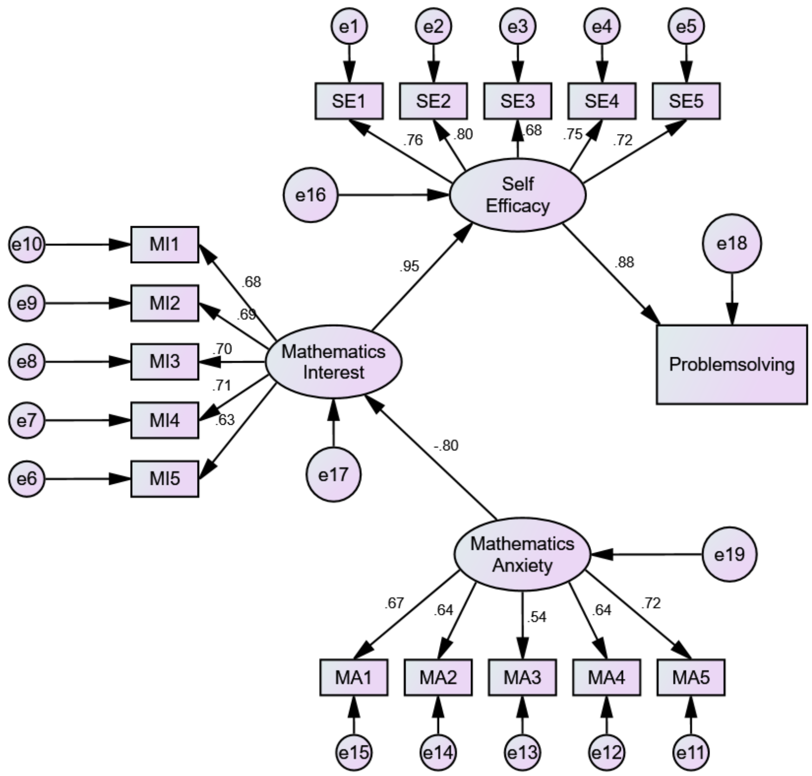

Figure 4. Standardized parameter estimates for final model of algebraic PSA

DISCUSSION

From the final model, only SE has a direct positive effect on algebraic PSA and mediates the effects of MI as well as MA. This result reveals that only SE is the most influential variable among the affective constructs. This result supports the theoretical justifications and empirical research on SE. Higher self-efficacious students do better because they put in more effort, make more attempts at cognitively difficult tasks, have greater persistence in solving difficult problems, and achieve better performance (Bandura, 1977, Karaoglan Yilmaz, 2022, Pajares & Graham, 1999). According to Collins (1982), students with higher SE can solve more mathematical problems and are preserved until a problem is solved. Therefore, it can be explained that students with higher SE always have a positive relationship between problem solving and learning achievement. Numerous studies have highlighted the positive relationship between SE and PSA (Lopez et al., 1997, Shimizu, 2022). In a structural modelling study, Zarch and Kadivar (2006) also found that SE has direct effects on mathematics achievement. Moreover, the positive correlation between SE and algebraic performance has also been established in India among high school students by Jasani (2022), who mentioned that higher SE has higher algebraic achievement. Thus, SE serves as the most influential factor in mathematical achievement, especially in algebraic problem solving.

Though among the non-cognitive constructs, MI is the most important predictor of mathematics achievement, the study reveals that it has an indirect effect on algebraic PSA through SE. In the structural model, MI also mediates MA. Many empirical studies have explored the effects of students' MI on achievement, both in math problem-solving and other domains (Huang et al., 2019, Schraw et al., 2001). But the role of MI as a mediating variable and the other variables that mediate it are rarely studied. The study of Zhang and Wang (2020) mentioned that the positive link between MI and mathematical achievement is mediated by SE. Though, MI has no direct effect on PSA but the positive and direct effect of MI on SE and the direct positive effect of SE on PSA indicates that students with more MI tend to feel motivated to get engaged in an algebraic task, boost their capacity, and be able to use effective learning strategies, which may likely lead to good performance (Fisher et al., 2012, Yu & Singh, 2018). Thus, SE is the most influential predictor of algebraic PSA among the affective variables; it also serves as an important mediator. This result is consistent with the hypothesized role of SE theory (Bandura, 1986) and other recent studies (Zhang & Wang, 2020). The results indicate that students who are more interested in learning mathematics may eventually increase their SE, which may lead to improved problem solving.

In the final SEM model, MA is the only construct found to have both direct and indirect effects on algebraic PSA. The direct effect of MA on PSA is found to have a weak association in the revised model; the model fit indices are almost identical after eliminating it. This explains that the large effect of MI on the algebraic PSA is mediated by the other non-cognitive constructs in the model. In the final model, the result reveals that the effect of MA on algebraic PSA is mediated by MI and SE. Many investigations have shown that MA has a direct effect on SE (Guven & Cabakcor, 2013, Hayat et al., 2020), but no direct effect is found in this study. In contrast, Gupta and Maji (2022) found direct effects of MA on both SE and mathematical achievement in Indian school students. Literature has established a consistent and inverse association between problem solving and anxiety, highlighting the fact that this relationship is more complicated than it first appears to be (Ashcraft & Moore, 2009). This finding highlights the significance of examining variables in models where algebraic PSA is a dependent variable that mediates the effect of MA as an indirect predictor. The fact that MA is the sole variable in our model can offer valuable information about how non-cognitive constructs should be incorporated. The model's casual association shows that decreasing MA raises MI and increased MI positively affects SE is a significant. Thus, increasing secondary school students' MI and SE can significantly improve their algebraic PSA, while increasing MA can have the opposite effect (Guven & Cabakcor, 2013, Hoffman, 2010).

CONCLUSIONS

The present study investigates the direct and indirect effects of affective constructs on algebraic PSA among secondary school students. The findings revealed that SE was the only non-cognitive construct that had a direct effect on algebraic PSA. The significant direct effect of SE indicates that students with higher SE have a significant contribution to algebraic PSA. Apart from this, MI and MA have indirect effects on PSA through SE in the model. The mediating role of SE indicates that reducing students' anxiety might increase their interest and self-efficacy, which might lead to better achievement in problem solving. In order to reduce anxiety and promote students' active participation in classroom transactions, teacher should substitute formative evaluation for high-stakes examinations. Teachers should be provided in-service training for the argumentation of problem-solving strategies by taking into account the affective factors during classroom interaction. Moreover, stakeholders and curriculum planners should design programs that will increase students' interest and self-efficacy.

The current study offers very helpful information and implications for mathematics educators to make the subject matter more interesting, increase students' efficacy and reduce math anxiety. These will assist the students in enhancing their problem-solving abilities and their achievement. The study contributes to the development of more literature by highlighting how SE, MI and MA predict algebraic PSA and how SE plays a major role in this process.

The findings of the study cannot be generalized to nationwide due to the constraint of small data sample. In the present study students self-reported data were used which might raise response bias. But, this study can be further investigated using larger and more diverse samples.

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Disclosure statement

No potential conflict of interest was reported by the author(s).

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A case study of proving by students with different levels of mathematical giftedness

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Abstract: *We present a case study of proving by three 12–13-year-old students with different levels of mathematical giftedness. After analysing students' proofs, we conclude that: there was a relation on the consistency and the students' levels of mathematically giftedness, being the least consistent the student not mathematically gifted and the most consistent the student with the highest level of mathematical giftedness; the variability was greater in the arithmetical problems; the quality of the proofs produced increased as the level of mathematical giftedness did; the two students with a lower level did better proofs in the arithmetical than in the geometrical problems, while the student with the highest level did not show significant differences between the two areas.*

Keywords: mathematical giftedness; deductive proofs; empirical proofs; arithmetical problems; geometrical problems.

INTRODUCTION

Many authors consider proving a fundamental aspect of mathematical education and emphasise its importance to promote mathematical understanding (Hanna, 2000). In fact, national curricula from various countries, researchers, and educators have made a call to pay more attention to proving (Campbell et al., 2020, NCTM, 2000).

Mathematical proving is recognized to be difficult to teach and learn, and different studies concluded that only a minority of students can build consistent deductive proofs at the end of high school (Gueudet, 2008). However, with instruction, mathematically gifted students (m-gifted students hereafter) may master inference principles, which are precursors to proof, as early as the fifth grade and, in grade nine, their proof processes and intuitive notions of proof show similarities to those of mathematicians (Sriraman, 2004).

M-gifted students are those who show 'a unique aggregate of mathematical abilities that opens up the possibility of successful performance in mathematical activity' (Krutetskii, 1976, p. 77). Their problem solving abilities are unusually higher than those of their peers. Their identification is highly controversial, and we can find in the literature intelligence tests, creativity tests, mathematical achievement and ability tests, although research indicates that multidimensional approaches are the most adequate to identify them (Pitta-Pantazi et al., 2011; Dündar et al., 2016; Yazgan-Sağ, 2022).

A part of the literature on mathematical giftedness (m-giftedness hereafter) has paid attention to the differences between ordinary and m-gifted students, but, to our knowledge, there are few studies analysing differences between the problem-solving processes carried out by students with different levels of m-giftedness.

Some studies analysing such differences are those by Fritzlar and Karpinski-Siebold (2012) in the context of linear patterns and generalization, showing differences in algebraic thinking between m-gifted students and others not m-gifted but with good grades in school mathematics, and those by Leikin et al. (2014, 2017) and Paz-Baruch et al. (2022) on cognitive attributes. However, there are no studies focusing on differences when learning to prove.

Housman and Porter (2003) observed that undergraduate female students with good college mathematics marks produced different types of proofs even during one short time span, exhibiting even the least sophisticated types. In our paper we explore the types of proofs produced by students considering different grades of mathematical talent as a variable of the study. We consider as proofs any mathematical argument raised to justify the truth of a mathematical statement, not only the formal proofs made by mathematicians, in the line of authors like Balacheff (1988), Harel and Sowder (2007), and Fiallo and Gutiérrez (2017).

There is no agreement regarding possible differences on student's performance in making proofs in different areas of mathematics: some works revealed differences between arithmetic and geometry (Healy & Hoyles, 2000, Hoyles & Küchemann, 2000), while others (Buchbinder & Zaslavsky, 2018, Recio & Godino, 2001) did not. There are also few studies on mathematical content that might more likely promote students' proofs (Lin, 2016). In this paper we continue exploring this issue.

We present a part of a larger research aimed to analyse the proving abilities of students with different levels of m-giftedness. It consists of a case study with three 12–13-year-old students who excel in school mathematics (grades higher than 90%) and are nominated by their teachers as students with a very good competence in mathematics, but have different levels of m-giftedness. Our specific research objectives are:

1. To analyse the consistency of the types of proofs produced by each student across the experiment and relate it to the student's level of m-giftedness.
2. To analyse possible differences in the types of proofs produced by each student in arithmetical and geometrical problems.
3. To compare the proofs produced by the three students and relate them to students' levels of m-giftedness.

The results of our study can contribute to the description of characteristics of m-giftedness associated with proving and help teachers and researchers when designing educational interventions that adjust to the diversity of their pupils, allowing to develop the proving competence of m-gifted students according to their mathematical ability.

LITERATURE REVIEW

Mathematical giftedness

There is not a commonly accepted definition of m-giftedness (Paz-Baruch et al., 2022; Yazgan-Sağ, 2022) and since the study by Krutetskii (1976), researchers have deepened in its characteristics. Jaime and Gutiérrez (2014) described various of them based on the characteristics proposed by Freiman (2006), Greenes (1981), Krutetskii (1976) and Miller (1990). These characteristics often differentiate between mathematical abilities, such as high mathematical memory, or atypical problem solving, and general personal traits, like perseverance or interest in challenging tasks (Singer et al., 2016). Related to the abilities to prove, researchers recognise as indicators of m-giftedness the abilities to abstract and generalise (Greenes, 1981, Krutetskii, 1976; Sriraman, 2004), to see mathematical patterns and relationships (Miller, 1990), to use heuristic thinking, and to appreciate mathematical proofs (Sriraman, 2004).

Mathematical abilities relate to the potential to do mathematics (Leikin, 2018). So, excellent grades in school mathematics are not an indicator of m-giftedness, since they may not reflect students' independent mathematical reasoning (Leikin, 2018; Paz-Baruch et al., 2022). Nor does m-giftedness imply excellent grades (Juter & Sriraman, 2011). Leikin (2018, p. 3) includes these ideas in her definition of m-giftedness, indicating that a student is m-gifted 'if s/he exhibits a high level of mathematical performance within the reference group and is able to create mathematical ideas which are new with respect to his/her educational history'. This definition takes into account creativity, considered by many authors as a component of m-giftedness (see also Dündar et al., 2016, and the references therein).

Mathematical proofs

There is a general agreement that the formal texts produced by mathematicians to communicate their results are proofs, but there is an open discussion about the texts produced by students which do not fit the requirement of mathematicians' proofs (Stylianides et al., 2017). We align with Balacheff (1988), Harel and Sowder (2007), Fiallo and Gutiérrez (2017), and other authors in considering as proofs any mathematical argumentation raised to justify the truth of a mathematical statement, not only the formal proofs made by mathematicians.

Since Polya (1945), problems that ask to prove a conjecture are named proof problems. The conjecture may be given in the statement or may have to be found by the solver as part of the solution.

Some researchers described students' work when solving proof problems and identified several types of empirical proofs –using examples as the main element of conviction– and deductive proofs –based on abstract properties and logical deductions– (Asghari et al., 2018, Balacheff, 1988, Harel & Sowder, 1998, Marrades & Gutiérrez, 2000). In many studies, secondary school students were not successful completing deductive proofs, even with instruction (Clements & Battista, 1992). Even when students seem to understand the function of proofs and to recognize that they must be general, they prefer to rely on empirical methods (Hoyles & Küchemann, 2002) and on a few examples for proving a general claim (e.g., Balacheff, 1988, Harel & Sowder, 2007, Healy & Hoyles, 2000).

Mathematical giftedness and proving

The learning and understanding of proofs by m-gifted students has attracted attention of researchers (Housman & Porter, 2003). Sriraman (2004) suggested that m-gifted students may have an intuitive notion of proof and its role in mathematics, even without instruction, and that their processes to construct a proof show similarities to those of mathematicians.

However, Kwon and Song (2007) showed that, although m-gifted students in their study were confident in their abilities and had a balanced idea about the role of proofs, they failed to distinguish between empirical evidence and deductive proofs. Moreover, Housman and Porter (2003) observed that above-average mathematics students produced different types of proofs even during one short time span and even students with prior experience exhibited the least sophisticated types, in line with the results obtained by Harel and Sowder (1998). In our work we want to deepen into this aspect by exploring possible differences in the consistency in the types of proofs produced by students with different levels of m-giftedness.

Solving arithmetical and geometrical proof problems

An issue present when investigating students learning of transversal topics, like proof, generalisation, etc., is whether the specific mathematical content used in the teaching experiments influences students' outcomes and performance. Some studies on students' conceptions of proof revealed differences in performance and treatment of (counter-)examples between arithmetic and geometric contexts (Arcavi, 2003, Healy & Hoyles, 2000, Hoyles & Küchemann, 2000, Zodik & Zaslavsky, 2008), others did not observe differences (Buchbinder & Zaslavsky, 2018). On the other hand, Recio and Godino (2001) suggested that the arithmetical and geometrical content of problems had little influence on mathematical proof capacity (Harel & Sowder, 1998).

Related to students' ability of proving, Lin (2016) observed that there are insufficient studies on the mathematical contents that might more likely promote students' argumentation. In our study we deepen into this issue in the context of students with different levels of mathematical giftedness.

THEORETICAL FRAMEWORK

Levels of mathematical giftedness

We are interested in exploring the context of proving on the three levels of mathematical giftedness considered by Leikin et al. (2014) on their studies on cognitive attributes: super-mathematically gifted students (super m-gifted students hereafter), m-gifted students, and students with expertise in mathematics not generally gifted.

They consider m-giftedness as a combination of general giftedness (IQ>130) and expertise in mathematics (Leikin et al., 2017, Paz-Baruch et al., 2022), where expertise is determined by student's scores in school mathematics and in an assessment test in mathematics. At an operational level, they define expertise in mathematics as high performance in school mathematics, so in our study we consider it as excellence in school mathematics (grades higher than 90%) and being nominated by the teacher as a student with very good competence in mathematics. M-giftedness, giftedness, and expertise in mathematics are interrelated but different constructs, and not any gifted is an expert in mathematics and not any expert in mathematics is gifted (Leikin et al., 2017; Paz-Baruch et al., 2022). Super m-gifted students are defined as m-gifted students nominated by their teachers as having exceptional talent in mathematics, and who displayed exceptional achievements such as membership in national Olympiad teams.

Student's proofs classification

Considering our specific context, we distinguish the following constructs: a *conjecture* is a mathematical statement the veracity of which is doubtful; an *argument* is a verbalization aimed to explain how a conjecture was identified, to convince that it is plausible, or to be part of a proof; a *proof* is a mathematical argumentation, not necessarily formal, produced to justify the truth or untruth of a conjecture (Balacheff, 1988; Harel & Sowder, 2007; and Fiallo & Gutiérrez, 2017); *proof problems* are problems asking to prove a conjecture which may be given in the statement or may have to be found by the solver as part of the solution (Polya, 1945).

Marrades and Gutiérrez (2000) proposed a framework to classify students' proofs focusing on their production processes. The framework is based on the integration of previous categories by Balacheff (1988) and Harel and Sowder (1998), and it is adequate to evaluate students' proving skills along a learning period, although in the context of our experiment, it is necessary to clarify some aspects. We have made a refinement of Marrades and Gutiérrez's framework to correct a gap and some imprecisions:

- Crucial experiment proofs consist of just checking the conjecture in some examples, so we do not consider Marrades and Gutiérrez subtypes analytical and intellectual, since they include the production of abstract statements of definitions, properties, etc.
- Generic example proofs include abstract statements of properties identified after having manipulated some examples, so we do not consider the subtypes of example-based and constructive proofs, as they do not require to do any abstraction.
- Students do not jump from thought experiment proofs (based on getting information from examples) to formal proofs in mathematical language. Instead, they progress by doing deductive proofs that are not fully detached from examples and lack the detail and rigour of formal proofs. To fill this gap in, we have defined the *informal deductive* proofs as deductive proofs expressed in informal, mostly verbal, ways.
- We have reworded some Marrades and Gutiérrez's definitions to include proofs of the untruth of conjectures based on counter-examples.

Taking these points into account, we define the following categories of proofs (Fig. 1):

- *Empirical proofs*: are 'characterised by the use of examples as the main element of conviction' (Marrades & Gutiérrez, 2000, p. 91). Students use examples or relationships observed in them to prove the (un)truth of a conjecture. Depending on the ways examples are selected, there are several types of empirical proofs and, depending on the way the examples are used, the types have some subtypes:
    - *Naive empiricism*: proofs where one or more examples, selected without any particular criterion, are used to show that the conjecture is true or false in them.
        - Perceptual proofs: the examples are used to check the conjecture only by visual or tactile perception.
        - Inductive proofs: the checking of the conjecture includes the use of mathematical elements or relationships identified in the examples.
    - *Crucial experiment*: conjectures are proved (refuted) by showing that they are true (do not work) in a specific, carefully selected, (counter-)example or sequence of examples. Students are aware of the need for generalisation, so they choose the examples as non-particular as possible, although these are not considered as representatives of their family of examples. Students assume that, if the conjecture is true in the chosen example, then it is universally true.
        - *Example-based proofs*: the proof only consists of showing the existence of an example or the lack of counter-examples (for positive proofs), or the existence of a counter-example (for negative proofs).

- *Constructive proofs*: the proof focuses on the way of getting the examples or counter-examples.

- *Generic example*: students select one or more specific examples, seen as representatives of their class, and the proof includes abstract statements of definitions, properties or relationships of the family empirically observed after operations or transformations on the examples.
  - *Analytical proofs*: the properties used are stated in a general and abstract way but remaining linked to the specific examples.
  - *Intellectual proofs*: the properties used in the proofs are, at least partially, decontextualized from the examples, since the proofs include some deductive parts in addition to statements based on the examples.
- *Failed*: proofs lacking the coherence or the detail necessary to assign them to the previous categories but showing signs of an empirical reasoning.

- *Deductive proofs* are 'characterised by the decontextualization of the arguments used, [and] are based on generic aspects of the problem, mental operations, and logical deductions, … to validate [or refute] the conjecture in a general way' (Marrades & Gutiérrez, 2000, p. 93). Examples, when used, are a help to organise the proofs, but the particular characteristics of an example are not considered in the proof. Depending on the level of formalisation, we differentiate three types of deductive proofs:
  - *Thought experiment*: abstract deductive processes supported by previous observations on specific examples used to organise the proof. The examples are not part of the proof, but an aid to find properties and relationships to construct the proof. Depending on the way the proof is organised, we consider:
    - *Transformative proofs*: based on mental operations producing a transformation of the initial problem into another equivalent one. The role of examples is to help foresee convenient transformations.
    - *Structural proofs*: sequences of logical deductions derived from the data of the problem, axioms, definitions, etc. The role of examples is to help organize the steps in the sequence of deductions.
  - *Informal deductive*: abstract deductive processes expressed informally, combining verbal expressions with mathematical language, using statements assumed to be obviously true, etc., and based on mental operations that can be carried out with the help of specific examples or not. The informal deductive proofs may be *transformative* and *structural proofs*, defined like for the thought experiment proofs, but with the examples having a more limited role or even, not being used.
  - *Formal deductive*: abstract deductive processes expressed in a formal way (using mathematical language, justifying all the steps, etc.) and based on mental operations performed without the help of examples. The formal deductive proofs may be

transformative and *structural proofs*, defined like for the thought experiment proofs, but without the presence of examples.
   - *Failed*: proofs lacking the coherence or the detail necessary to assign them to the previous categories but showing signs of a deductive reasoning.
- *Non-codifiable proofs*: when there is no answer, or the answer does not allow characterising it as an empirical or deductive proof.

Failed proofs do not necessarily refer to incorrect proofs since some incorrect proofs can be classified in an adequate type if they show the reasoning characteristics of such type.

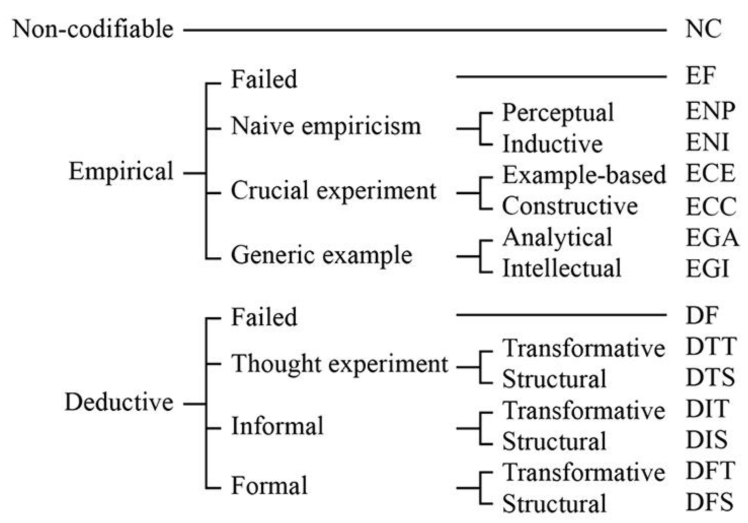


Figure 1. Scheme of the categories of proofs integrating our theoretical framework

METHOD

Description of the sample and the experiment

We are interested in exploring the context of proving in the three levels of m-giftedness considered in the study by Leikin et al. (2014) and defined in the theoretical framework: super m-gifted students, m-gifted students, and non-gifted students with expertise in mathematics. To this aim, we designed a case study of three students, each of which belonging to one of the three groups above, in order to analyse their performance when solving arithmetical and geometrical proof problems. None of the three students had received previous formation in proving.

To preserve the anonymity of the students in the experiment, we refer to them as S1 (or 'the non-gifted student'), S2 (or 'the m-gifted student'), and S3 (or 'the super m-gifted student'). All of them were selected because they excel in school mathematics and are nominated by their mathematics teachers as students with very good competence in mathematics. Student S1 was 13 years old and studied grade 7 (the first grade in Spanish secondary school); he is not generally gifted. Student S2 was 12 years old and studied grade 7, and student S3 was 13 years old and studied grade 8. Both S2 and S3 had been identified as gifted students (IQ>130), had been advanced one academic year, and attended out-of-school programs for general gifted students. These programs consisted of workshops devoted to problem-solving and to introduce non-curricular mathematical topics, such as strategy and logic games, and mathematical activities of various kinds, not focused on teaching to prove. Although both S2 and S3 participated in mathematical competitions some years after the workshop, S2 did not achieve relevant positions, while S3 ranked high in the national mathematics Olympics competitions and was nominated by their teachers as having exceptional talent. So, S3 meets the requirements indicated by Leikin et al. (2014) to be considered a super m-gifted student and S2 a m-gifted student.

The experiment was conducted by using a videoconference platform that offered group and private chats, and a whiteboard with pointing, writing, and drawing tools, available simultaneously to all participants. The students used computers to connect to the video-conference sessions, and smartphones, to share written answers through an instant messaging network, photos of their drawings, etc. One of the researchers acted as the teacher who conducted the experiment, proposed the proof problems and led the discussion, and the other researchers acted as observers. The teacher also used the private chats and the instant messaging network to communicate with each student and know the progress of their solutions. He also used the group chat to promote general discussions or to clarify students' doubts. GeoGebra was available and all students had the necessary expertise to use it when they considered it useful.

The experiment consisted of 6 sessions of about 100 minutes each. Each problem was stated verbally while displayed on the shared whiteboard. Depending on its complexity, the teacher could give a short explanation to promote understanding. After posing a problem, a time slot was left for the students to work individually; during this time, chat messages were exchanged between students and the teacher so that students showed him privately their solutions to the problem. Later, students were asked to explain their solutions to the group, by using the whiteboard if convenient to give graphical or textual support to their explanations. The teacher encouraged students to discuss the others' solutions and, when an answer was incorrect, to check whether it was correct or incorrect and justify it. Finally, the teacher institutionalised the correct answers by using adequate mathematical language. After each session, the researchers discussed the performance of the students and decided how to proceed in the next session.

The sequence of proof problems

We took the problems posed in the experiment from mathematics education literature and mathematical Olympiads training courses, and we adapted some of them to fit the aims of the research and the characteristics of our students. Most problems (Table 1) have several parts, consisting of different related questions. As for the codes, e.g., problem 1.2 was the second problem of session 1 and it has two parts, with part 1.2B also having two parts.

Table 1. Structure of the sessions of the workshop. Problems in grey cells are arithmetical, and those in white cells are geometrical.

Session 1	Session 2	Session 3	Session 4	Session 5	Session 6
1.2A 1.2B(a, b)	2.1(A-C)	3.1(A-C)	4.1(A, B)	5.1	6.1
1.3(A-D)	2.2(A, B)		4.2(A, B)	5.2(A-C)	6.2(A-D)

The sequence of problems (see the Appendix) alternated a balanced number of arithmetical and geometrical problems. Problem 1.1 was aimed to introduce the students to the sessions and the video-conference platform and has no interest in this paper, so we ignore it.

The several questions of the problems go from specific situations, linked to particular examples, to general contexts, to guide the students in the acquisition of processes of generalisation. They admit both empirical and deductive proofs, to identify different students' behaviours. The statements of some geometrical problems include figures and students were provided with GeoGebra files with dynamic versions of the figures, that they could drag to get more examples and generate conjectures. The GeoGebra measurement tools to calculate lengths, areas or angles allowed students to approach the solutions empirically and check properties. The arithmetical problems allowed the students to easily find particular examples to base their arguments, generate conjectures or counter-examples, and produce solutions as algebraic or verbal generalisations that they had to prove. These problems were ordered by their difficulty to generate examples and the need of deductive proofs.

Analysis of student's outcomes

The data for the analysis were obtained from the video recordings, students' written responses provided in the chats or through photos, and the field notes by the researchers. For each student, we took as units of analysis all their interventions in each problem part and assigned each students' productions to the types of proofs described in the theoretical framework. When a student showed different proofs in the same problem part, we registered the one in the highest type. We made a triangulation of our analysis. Each session was analysed independently by two researchers, focusing on the types of proofs provided by each student. Then, both analyses were reviewed by a third researcher. Finally, a consensus was reached when different types of proofs were assigned to the same student's solution.

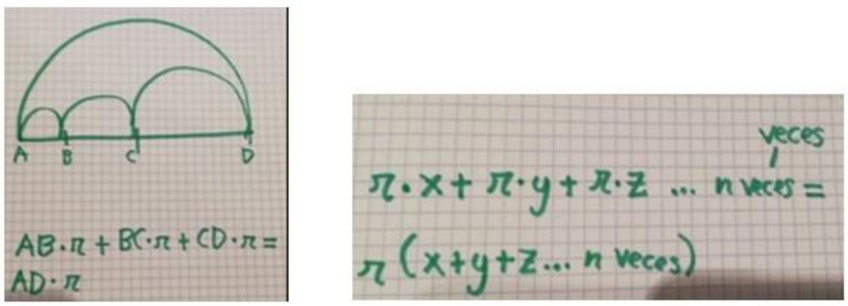
RESULTS

In this section we analyse the types of proofs produced by the students in the different problems.

Analysis of the geometrical problems

As examples of the proofs produced in the geometrical problems, we show the answers to problem 5.1. This problem induced rich answers representative of the types of proofs produced by each student in the geometrical problems.

Fig. 2 shows S3's answers. To answer the first question (Fig. 2a), he took mentally out the common factor π and obtained the obvious equality *AB*+*BC*+*CD* = *AD*. In the second question (Fig. 2b), S3 used a decontextualized argument since he was able to generalise the problem to *n* points in the diameter without the need of drawing particular points. Therefore, this is a deductive informal structural proof (DIS).



a b

Figure 2. S3’s answers to problem 5.1

S1 and S2 gave naive empirical proofs. S1 made visual comparisons by dragging points in the GeoGebra construction, providing empirical naive empiricism perceptual proofs (ENP):

> S1 [chat response]: I think that the red path is longer because if you move one of the points, like E, to A with GeoGebra, the red line occupies more.

> S1 [verbal answer]: I think that both are the same [length] because if, for example, you move all the points with GeoGebra to A or B, the red line is always just on [superposes] the blue one, or vice versa.

S2 measured the semicircles with the Length tool of GeoGebra, so he made an empirical naive empiricism inductive proof (NEI):

> S2: I think that they are the same, but because there is a tool [in GeoGebra] that is the length, I have used it to measure [...] so I have added everything [...]

Analysis of the arithmetical problems

To exemplify the proofs produced by the students in the arithmetical problems, we show the solutions to several problems. Each student produced a quite large diversity of types of proofs in the arithmetical problems, although no problem induced by itself such a diversity, so we present the proofs produced in several problems by each student.

S1 and S2 made deductive proofs in problem 4.2A:

> S1: It is odd because it would be like adding that odd number the same number of times, so it is odd. If it were to add it the number of times of an even number, it would be even: 9 times 9 = 81 but 9 times 8 = 72.

S1 made a transformative thought experiment (DTT) since he transformed n^2 into the sum of n times n and used general properties and deductions based on specific examples.

> S2: Odd, because when you multiply odd numbers, it gives an odd number. If n is odd, then n^2 is an odd number since if you multiply an odd number by another odd it gives odd.

Student S2 made a deductive informal structural proof (DIS) based on known general properties of odd numbers.

In the other arithmetical problems, S1 and S2 only made empirical proofs, although S2 produced more elaborated types. Student S3 only produced three empirical proofs, the others being deductive. So, there is a difference in students' styles of proofs, which may be well exemplified by comparing their answers to problem 4.1A:

> S1: They would be all multiples of three. 0 + 1 + 2 = 3, 1 + 2 + 3 = 6, 2 + 3 + 4 = 9. If we start with the first example, we could follow the sequence and each result would be the previous addition plus 3.
>
> Teacher: Very well. So, which numbers are the sum of three consecutive natural numbers?
>
> S1: All multiples of 3.

Student S1 began by checking a sequence of examples, that let him generalise the results to all multiples of 3. He based his argument on the recursive relationship he had observed (each addition is the previous addition plus 3) and generalised it to give a general answer. S1 did not prove the recursive relationship, so this is an empirical generic example intellectual proof (EGI).

Student S2 began solving the problem by trying several sets of consecutive numbers, all their sums, by chance, being multiple of 6. Then, he verbalised his answer:

> S2: They are multiples of 6, from what I am seeing.
>
> Teacher: Are you sure? If I give you a multiple of 6, can you tell us which are the three consecutive numbers?

> S2: The result of the addition is the same if you multiply the number in the middle by 3. For example, 2 + 3 + 4 = 9.
>
> Teacher: Note that the question I asked is the reverse. Which numbers can be written as [the sum of] three consecutive [numbers]?
>
> S2: Aaahhhh. So, they are multiples of 3.

Student S2, reacting to the teacher's comment, noticed that the decomposition not only worked for multiples of 6, but also for multiples of 3, based on the example shown (2+3+4=9). Therefore, S2 considered some specific examples and induced the property, stating it in an abstract way but without any argument to prove it apart from the examples themselves. So, S2 produced an empirical crucial experiment example-based proof (ECE).

Student S3 found a solution and a procedure to calculate the addends:

> S3 [chat response]: Given three consecutive numbers, the middle one multiplied by 3 gives the sum of these three numbers. So, given a natural number, if it is divisible by 3, it can be decomposed into three numbers like this: n/3, the previous and the next. Given that number [a multiple of 3], dividing it by three, it will give us, of those three consecutive numbers, let's say the middle one, if we put them in order, because [...] in one session we saw that (n - 1) + n + (n + 1) was 3n. So, just divide by three, the previous, and the next.

S3 proved his solution by resorting to previous knowledge and following a deductive informal structural proof (DIS).

The most elaborated level of S3's proofs (DIS in most cases) may also be observed in his answers to problem 6.2, which are representative of his way of reasoning. In part 6.2A, all students did empirical crucial experiment example-based proofs (ECE), since each of them looked for a pyramid whose apex was not multiple of 4, presenting it as a counter-example.

S3 started his solution to part 6.2B making a wrong proof, but then he changed his approach and produced a deductive informal structural proof (DIS):

> S3: If the numbers in the base row are, if the first number is n, if the second, let's say, is n+2, then the third should be n+4. Why? Because if from the first to the second [number] there are 2, that difference has to be the same to the third [number]. [He wrote in the blackboard of the video-conference]

4n+8

2n+2 2n+6

n n+2 n+4

> Teacher: What does 4n+8 mean?

S3: That it is multiple of 4.

Teacher: What do n, n+2, n+4 mean?

S3: That there must be the same difference between the first and second numbers and between the second and third numbers.

Teacher: What did you like to write a while ago?

S3: That, if they are n and n+x, the third [number] should be n+2x.

Teacher: So, would this be true for any difference?

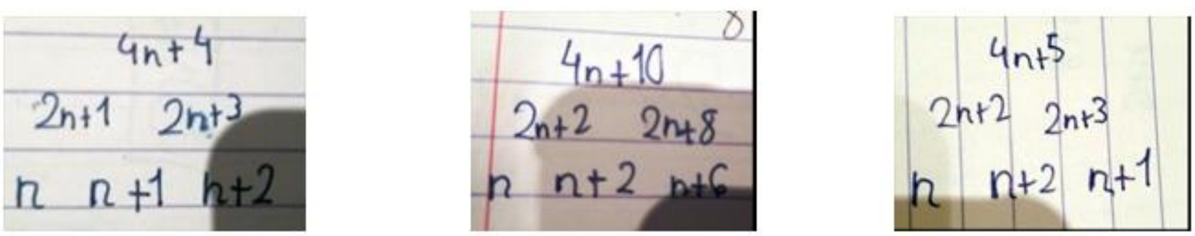
S3: Yes.

Teacher: What is in the other rows?

S3: 2n+x and 2n+3x in the second, and 4n+4x in the third.

Before listening to S3's answer to part 6.2B, S1 had provided an empirical crucial experiment example-based (ECE) by showing two specific numeric pyramids as an example and a counter-example, and S2 had produced an empirical failed proof (EF). They did not answer parts 6.2C and 6.2D.

In part 6.2C, S3 wrote a deductive informal structural (DIS) solution, including an example (Fig. 3a) and two counter-examples to show that, in the base row, the difference between the numbers must be constant (Fig. 3b) and that the numbers must be ordered (Fig. 3c). In 6.2D, S3 used algebraic language to prove that the relationship is true also in the case of integer numbers and he generalised the solution to a pyramid with four numbers in the base row (Fig. 4), thus providing another deductive informal structural (DIS) proof.



a b c

Figure 3. Student S3's answers to parts 6.2C

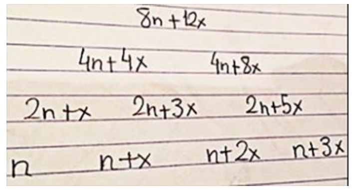


Figure 4. Student S3's answers to parts 6.2C

DISCUSSION

The research objectives stated include to analyse the consistency of the types of proofs produced by the students across the experiment and possible differences between the types produced in the arithmetical and geometrical problems. To answer these objectives, we summarise the types of proofs produced by each student in the geometrical (Fig. 5) and arithmetical problems (Fig. 6) and the frequencies of each type (Table 2).

Fig. 5 shows that student S1 did not produce any deductive proof in the geometrical proof problems, S2 did one, and S3 did eight of them. Three main types of proofs became apparent: empirical naive empiricism perceptual (ENP), based on visual arguments; empirical naive empiricism inductive (ENI), based on measures made with the GeoGebra tools; and deductive informal structural proofs (DIS), consisting of abstract deductive processes expressed as combinations of verbal and algebraic expressions. There was a considerable amount of non-codifiable answers (NC), corresponding to inconsistent responses. The students did not produce proofs in the types between empirical generic example analytical (EGA) and deductive informal structural (DIS). Student S3 was clearly the one producing the best types of proofs, and S1 was the student who did the worst types.

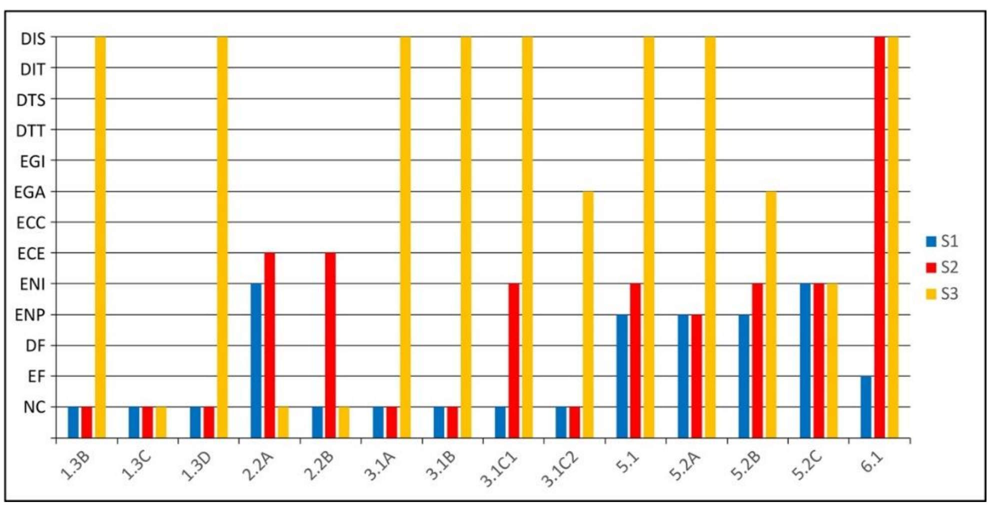


Figure 5. Types of proofs produced in the geometrical problems

All S1's proofs and all but one S2's proofs were empirical. Except on two problems where S2 used empirical crucial experiment example-based proofs (ECE), the other S1 and S2's answers were perceptual or inductive naive empiricism proofs (ENP or ENI). They did not provide codifiable answers in problem 1.3 and some parts of problem 3. In contrast, S3 produced empirical proofs only three times; in particular, he used the inductive naive empiricism (ENI) once and the analytical generic example (EGA) twice. We have differentiated part 3.1c (as 3.1c1 and 3.1c2) because S3 used first a deductive informal structural proof (DIS) to generalise the problem to simple quadrilaterals and next an empirical generic example analytical proof (EGA) for complex quadrilaterals.

Concerning the arithmetical problems, Fig. 6 shows that almost all types of proofs were produced at least by one student, in contrast with the geometrical problems, where the types of proofs were not so many. All three students produced deductive proofs in the arithmetical problems (D-- codes), but while S1 and S2 did it only once, S3 showed it on ten proofs, nine structural informal proofs (DIS) and one structural thought experiment proof (DTS). Student S1 did proofs in almost all parts of the arithmetical problems, most of them being the type of empirical crucial experiment (EC-) and S2 only produced six valid proofs, which were of very diverse types, ranging from two empirical example-based crucial experiments (ECE) to one deductive informal structural (DIS) proof. Again, S3 was clearly the one producing the best types of proofs, and now S2 was the student who got the worst results in terms of the number of proofs produced, although there is not a difference between the quality of the proofs done by S1 and S2.

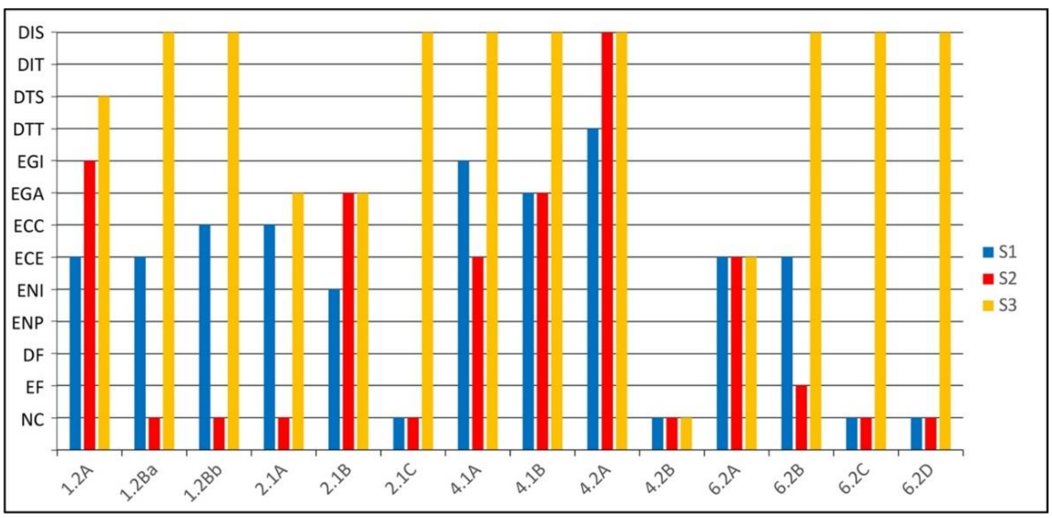


Figure 6. Types of proofs used in the arithmetical problems

Table 2 summarises the number (and percentage) of proofs of each type produced by each student in the arithmetical (A) and geometrical (G) problems. The results show inconsistencies in the types of proofs used, even in the same problem. We observe more variability in the arithmetical problems, where S1, S2 and S3 showed 6, 5, and 4 different types of codifiable proofs, respectively, while in the geometrical problems, S1 and S3 showed 3 different types, and S2 showed 4 types.

Table 2. Number (and %) of proofs of each type produced.

	S1		S2		S3		S1+S2+S3	
	A	G	A	G	A	G	A	G
NC	4 (28,6)	8 (57,1)	7 (50)	6 (42,9)	1 (7,1)	3 (21,4)	12 (28,6)	17 (40,5)
EF	0 (0)	1 (7,1)	1 (7,1)	0 (0)	0 (0)	0 (0)	1 (2,4)	1 (2,4)
DF	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)
ENP	0 (0)	3 (21,4)	0 (0)	1 (7,1)	0 (0)	0 (0)	0 (0)	4 (9,5)
ENI	1 (7,1)	2 (14,3)	0 (0)	4 (28,6)	0 (0)	1 (7,1)	1 (2,4)	7 (16,7)
ECE	4 (28,6)	0 (0)	2 (14,3)	2 (14,3)	1 (7,1)	0 (0)	7 (16,7)	2 (4,8)
ECC	2 (14,3)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	2 (4,8)	0 (0)
EGA	1 (7,1)	0 (0)	2 (14,3)	0 (0)	2 (14,3)	2 (14,3)	5 (11,9)	2 (4,8)
EGI	1 (7,1)	0 (0)	1 (7,1)	0 (0)	0 (0)	0 (0)	2 (4,8)	0 (0)
DTT	1 (7,1)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	1 (2,4)	0 (0)
DTS	0 (0)	0 (0)	0 (0)	0 (0)	1 (7,1)	0 (0)	1 (2,4)	0 (0)
DIT	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)
DIS	0 (0)	0 (0)	1 (7,1)	1 (7,1)	9 (64,3)	8 (57,1)	10 (23,8)	9 (21,4)
DFT	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)
DFS	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)
Total	14 (100)	14 (100)	14 (100)	14 (100)	14 (100)	14 (100)	42 (100)	42 (100)

The most inconsistent student was S1, who showed a great difference in performance depending of the types of problems, his proofs being more sophisticated in the arithmetical ones: in the arithmetical problems S1 mainly produced empirical proofs, of the types crucial experiments (EC-) and analytical generic examples (EGA), and also a deductive thought experiment transformative proof (DTT), but in the geometrical problems he only produced empirical failed (EF) and naive empiricism proofs (EN-).

S2 produced two deductive informal structural proofs (DIS), one in each kind of problem, and the amount of non-codifiable answers was similar in both types, but he produced more elaborated proofs in the arithmetical problems, with empirical generic example proofs (EG-) being the most used (28,5%) and empirical naive empiricism proofs (EN-) being the most frequent in the geometrical problems (35,7%).

S3 was the most consistent student since he usually did structural informal proofs (DIS) on both kinds of problems. However, his global performance is better on arithmetic problems, with only one non-codifiable answer, in contrast to the three of the geometric problems. His difficulties are related to a lack of knowledge of algebraic language. S3's variability in the geometrical problems was mainly in the last sessions, where an increment in the difficulty made him move from deductive to empirical proofs.

The global data in Table 2 show that geometrical problems were more difficult for the students than arithmetical ones, in which the three students produced better types of proofs, this difference being more noticeable in S1 and S2.

These results support the conclusion that, although the three students show inconsistencies in the types of proofs produced, there is a relation on the grade of consistency and the students' levels of mathematically giftedness, being the less consistent the non-gifted student and the most consistent the super m-gifted student. The case study also seems to indicate that the dependence of the outcomes on the content area (geometry and arithmetic in our experiments) decreases as the level of giftedness increases.

CONCLUSIONS

In this article, we have presented the results of a case study designed to analyse the behaviour of three students with different levels of m-giftedness –S1 being non-gifted and with expertise in mathematics, S2 m-gifted, and S3 super m-gifted–, when solving arithmetical and geometrical proof problems. None of them had received previous formation in proving.

A contribution of the paper is the framework of our study, a refinement of the classification of proofs by Balacheff (1988), Harel and Sowder (1998), and Marrades and Gutiérrez (2000), including the new category of *deductive informal proofs* for decontextualized deductive proofs which are not expressed with formal mathematical language. This category proved to be important because it is the highest type of proof produced by the participants in our study.

The first research objective was to analyse the consistency of the types of proofs produced along the experiment and relate it with student's level of m-giftedness. In the line of results obtained by other researchers (Harel & Sowder, 1998; Housman & Porter, 2003) we observed that the three students were not consistent, producing different types of proofs even in different parts of the same problem. The super m-gifted student (S3) was the most consistent, producing most times deductive informal structural proofs (DIS), both in the arithmetical and geometrical problems. His empirical proofs can be associated to an increment in the difficulty. The most inconsistent student was the non-gifted student (S1), showing up to six different types of proofs in the arithmetical problems (only one deductive). These results seem to indicate that as the students' level of m-giftedness increases, the more consistent their types of proofs are both on arithmetical and geometrical problems.

The second research objective was to identify possible differences between the types of proofs produced in arithmetical and geometrical problems. The data point to that the dependence of the types on the area decreases as the level of m-giftedness increases. The super m-gifted student (S3) had a similar behaviour on both types of problems, but the non-gifted student (S1) and the m-gifted student (S2) made more sophisticated proofs in the arithmetical problems. In the geometrical problems, the non-gifted student (S1) only was able to produce empirical naive empiricism proofs (EN-), whereas in the arithmetical ones he mainly did the more elaborated empirical crucial experiment proofs (EC-). Most of the m-gifted student (S2)'s solutions in the geometrical problems were also empirical naive empiricism proofs (EN-), while most of his proofs in the arithmetical problems were empirical generic example proofs (EG-). The differences observed between arithmetical and geometrical problems are consistent with the results by Healy and Hoyles (2000) and Hoyles and Küchemann (2000), but differ from those by Buchbinder and Zaslavsky (2018).

Our results also suggest that students who are able to produce deductive proofs produce the same types across mathematical areas, and that arithmetic proof problems seem to promote more sophisticated proofs on students who produce mainly empirical proofs, although more research on this issue is necessary. Some reasons for it may be: (i) the students were more familiar with arithmetic than geometry, since teachers usually devote much more time to teach this area; (ii) the dragging and measurement tools of the GeoGebra may have influenced students to make more empirical naive empiricism (EN-) and crucial experiment (EC-) proofs in the geometrical problems, as pointed by Healy (2000), Mariotti (2002) and Komatsu and Jones (2019). We noticed that some types of proofs were more related to some type of problem: the empirical naive empiricism perceptual proofs (ENP) were only produced in geometrical problems, and the empirical crucial experiment (EC-) and generic example (EG-) proofs were more frequent in the arithmetical problems. Respect to deductive proofs, the thought experiment (DT-) was produced only in arithmetical problems.

Regarding the third research objective, focused on comparing the proofs produced by the students and relating them to their levels of m-giftedness, we observed different levels of proficiency, since most proofs by the non-gifted student (S1) and the m-gifted student (S2) were empirical, while the super m-gifted student (S3) showed from the very beginning his capacity to produce deductive proofs (64% of his proofs). The less able student was the one not gifted (S1). In general, he was the slower student completing the solutions and, on many occasions, he did not give coherent proofs. He was the one who showed more variability in the types of proofs produced, both in the arithmetical and in the geometrical problems. According to theories of creativity, a variety of types of proofs is a symptom of highly creative minds. However, throughout the article, when we refer to variability, we refer to the use of different types of proofs in terms of Marrades and Gutiérrez's (2000) framework, not to different ways to approach a problem. His proofs were more sophisticated in the arithmetical problems although most of the times were of empirical type. The m-gifted student (S2) also did better proofs in the arithmetical problems, so arithmetic may be more adequate to promote the improvement of students' proving abilities. The ablest student was the super m-gifted student (S3) and had a similar behaviour on arithmetic and geometric problems, using mostly deductive informal structural proofs (DIS). He did not reach the deductive formal type (DF-) because of a lack of instruction on mathematical language and the verbal character of most responses. He was the fastest in giving solutions and he used to generalise from the very beginning, even when we did not ask for it.

Our conclusions are based on a case study with three students solving specific proof problems, so the results presented do not pretend to be generalisable. However, they contribute to the development of instructional practices to create opportunities for m-gifted students to improve their abilities to do proofs. The analysis of the types of proofs produced allows designing interventions adjusted to the diversity of these students and may help them develop their proving competence. The study also contributes to the description of characteristics of m-giftedness associated with proving, based on the types of proofs produced by students in arithmetic and geometric contexts and the consistency of the types along a sequence of proof problems.

The sequence of problems presented, aimed to introduce secondary schools students to the learning of proving and proofs, can be used by teachers of m-gifted students, but it can also be used in their ordinary classrooms, granting that the teachers guide their students, at least in the first problems, to make them aware of the kind of arguments they are supposed to produce. The arithmetic problems allow students to easily produce examples that can direct their attention to discover a general property of numbers. The geometric problems are adequate to be approached with the support of a dynamic geometry software, which will help students to identify properties of the figures evident when dragging, that students would convert into conjectures and then try to prove.

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APPENDIX

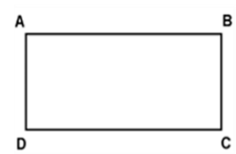
SESSION 1

1.2A. Calculate some products of three consecutive natural numbers. Is the product always a multiple of 6?
If the answer is yes, do you think this property holds for all products of three consecutive natural numbers? Why? Justify your answers mathematically.
If the answer is no, explain why not. Find a condition that three consecutive natural numbers must hold for their product to be a multiple of 6. Justify your answers.

1.2B. Calculate some sums of three consecutive natural numbers.
a. Is the sum always a multiple of 6?
- If the answer is yes, do you think this property holds for all products of three consecutive natural numbers? Why? Justify your answers mathematically.
- If the answer is no, explain why not.

b. Find a condition that three consecutive natural numbers must hold for their sum to be a multiple of 6. Justify your answers.

1.3A. The polygon ABCD is a rectangle. Draw it in your notebook and draw its two diagonals.

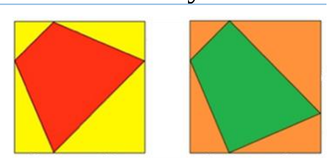


1.3B. What is the relation between the lengths of the diagonals of rectangle ABCD? Justify your answer.
1.3C. Draw another rectangle. Is there the same relation between its diagonals? Justify your answer.
1.3D. Is there the same relation between the diagonals of any rectangle? Justify mathematically that your answer is correct.

SESSION 2

2.1A. How many numbers in the form 2n are there between 100 and 150? Justify your answers mathematically.
2.1B. How many numbers of the form 3n + 1 are there between 100 and 150? Justify your answers mathematically.
2.1C. If a is a fixed number between 2 and 49, How many numbers of the form a ·n + 1 are there between 100 and 150? Justify your answers mathematically.

2.2. Consider the following squares.
2.2A. Is the yellow area the same as the red one? Why?
2.2B. Is the orange area the same as the green one? Why?



SESSION 3

3.1. Let us consider a rhombus ABCD whose diagonals measure AC = 6 cm and BD = 11 cm.
3.1A. Find the area of the rhombus.

3.1B. Let us consider a quadrilateral EFGH, which is not a rhombus, whose diagonals are perpendicular and measure EG = 6 cm and FH = 11 cm. Find the area of this quadrilateral.

3.1C. What can you say about the previous results? Generalise and prove that property.

SESSION 4

4.1A. Find numbers that can be decomposed into the sum of 3 consecutive natural numbers.

- Deduce a general procedure to find numbers of this type. Justify the procedure.
- Have you found all the numbers that can be decomposed as a sum of 3 consecutive natural numbers? Prove that your answer is correct.

4.1B. Consider problem 4.1A with 2 consecutive natural numbers instead of 3.

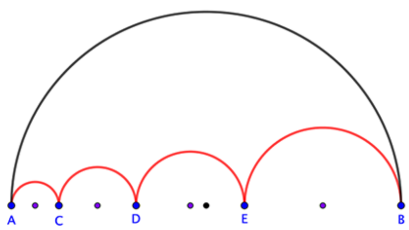
4.2A. If a number n is odd, is its square n^2 even or odd? Prove your answer.

4.2B. If a number n is even, is its square n^2 even or odd? Prove your answer.

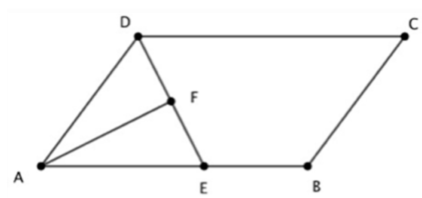
SESSION 5

5.1. The blue path goes directly from A to B and the red path does it through partial paths (from A to C, from C to D, from D to E and finally, from E to B).

- If all paths are semicircles, which path is longer, the blue one or the red one?
- What would happen if there were more points between point A and B? Explain it.



5.2. Build a parallelogram ABCD in GeoGebra. Draw the bisector of <D and denote E the point where it cuts the side AB. Draw the bisector of <A and denote F the point where it cuts the segment DE.



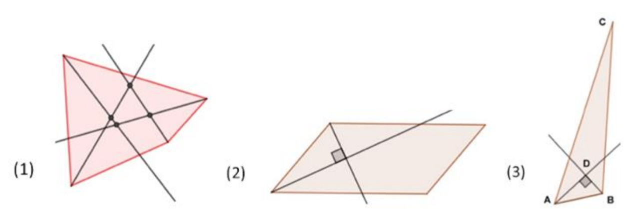
5.2A. What kind of triangle is ADF? Why?

5.2B. What kind of triangle is AEF? What is the relation between triangles ADF and AEF?

5.2C. Modify the parallelogram by dragging its vertices. Do you observe any particularity?

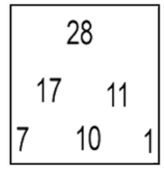
SESSION 6

6.1. In general, the bisectors of two consecutive angles of the quadrilaterals are not perpendicular (1). But, in last session we saw that the bisectors of two consecutive angles of particular quadrilaterals, the parallelograms, are always perpendicular (2).



What can you say about the bisectors of two consecutive angles of a triangle? Can they also be perpendicular? (3).

6.2. Observe the pyramid of numbers. Each number in the middle and top rows is the addition of the two numbers under it. Note that the number in the apex (28) is a multiple of 4.



6.2A. Make another pyramid with the same structure but with other numbers in the bottom row. Is the apex always a multiple of 4?

6.2B. Can you find a rule for the numbers in the bottom to get in the apex a multiple of 4?

6.2C. What if the numbers in the bottom row are natural numbers having a constant difference between them, for example 127, 134, 141?

6.2D. If the numbers in the bottom row are whole numbers, is the apex a multiple of 4?

The use of technology and academic performance in the teaching of Mathematics in secondary education

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Abstract*: Technological resources in secondary education have had problems in their implementation, especially in public institutions, this has caused teachers to keep traditional learning methodologies, with minimal alternatives to use active methodologies due to the limited access to Information and Communication Technologies (ICT). The present research aims to analyze the use of technology and academic performance in the teaching of Mathematics in the Joaquín Lalama Educational Unit. The methodology applied had a quantitative approach, with a pre-experimental experimental design and a pretest and post-test level of analysis, with the participation of 58 first and second year high school students; direct observation and inquiry were used as techniques. The results obtained indicated that 86.2% of the students consider that they learn better when the teacher uses a traditional class; 93.1% stated that they would like to learn Mathematics with the use of technology; on the other hand, 24% of the students mentioned that teachers never use a scientific calculator in their classes and 71% and 79% stated that teachers rarely or never use a projector or software, respectively. For the application of inferential statistics according to the characteristics of the variables and the hypotheses proposed, Student's t-test for related samples was used, obtaining p-values of less than 0.05, which means that the use of technology through the application of Inquiry Based Learning (IBL) establishes significant differences in the academic performance of students. The IBL has generated an increase in academic performance at the time of the pretest and posttest analysis; 0.696 points in the average.*

Keywords: Technology, Academic Performance, Inquiry-Based Learning.

INTRODUCTION

Education has had a continuous evolution in methodological and pedagogical aspects through the incorporation of technological resources or tools that have improved educational quality standards in secondary education institutions. In addition, the incorporation of new methodological strategies by teachers has improved the teaching and learning processes of students according to the skills with performance criteria and evaluation indicators located in the micro-curricular planning.

Education at the national level is structured by fiscal, private, fiscal-commissioned and municipal educational systems, each of which has different realities. The fiscal institutions are those where there are deficiencies in the teaching and learning process of students and that directly influence their academic performance. Each subject that is taught has different strategies, but those that are part of the exact sciences such as mathematics are the most disadvantaged due to the complexity and level of understanding by students.

Mathematics is a difficult subject for students to understand, in many cases they do not have affinity due to the frustration that arises at the time of the development of the exercises or problems posed. By keeping a traditional class, teachers maintain the students' lack of interest in understanding and comprehending mathematics; therefore, it is important to use strategies that adapt to the students' cognitive abilities. Currently, students have an affinity with the use of technology, therefore, incorporating the progressive use of Information and Communication Technologies (ICT) would capture their attention.

The methodological processes currently used in the Ministry of Education have generated controversy in the educational field due to the lack of resources needed to turn the teaching and learning process around, with the aim of gradually improving the performance of teachers and, above all, educational quality standards.

LITERATURE REVIEW

Education has evolved in technological and pedagogical aspects in different areas of knowledge. Talking about exact sciences such as Mathematics is essential to improve cognitive aspects with emphasis on logical reasoning and critical thinking. Currently, the use of technology as an intermediary in the teaching and learning process has become indispensable to improve the general knowledge of young people, who seek to understand and comprehend why it is important to have skills in this type of subjects. For Calcines et al. (2016) mention that "the integration of ICT plays a dual role: on the one hand, it contributes to the improvement of results and, on the other hand, it has a decisive influence on them, because it substantially stimulates the motivation of students towards learning" (p. 43). The use of technology can interfere in the educational process to reinforce students' knowledge in a playful, interactive and efficient way.

Technology must be properly addressed in a classroom and depends on the objectives that the teacher wishes to achieve, therefore, a structured planning with adaptation of ICT and active methodologies according to the needs, benefits the teaching and learning process, and the academic performance of students. For James et al. (2022) “the use of multimedia tools is considered an important contribution to teaching and a highly influential instrument for acquiring knowledge, in which students actively contribute to the elaboration of knowledge and value their own learning” (p. 18). Any technological tool, through its proper use, can be applied to generate significant learning, in addition, fostering a culture with ICT criteria causes young people to incorporate investigative and inquiring aspects that benefit their academic training.

Teachers are familiar with technological tools or environments regarding the preparation of annual and micro-curricular plans, so that the teaching role involves the constant use of ICT for professional development and fulfilment of obligations. The continuous training of teachers in technological topics can enrich their personal knowledge and the students they are in charge of. On the other hand, Falcó et al. (2016) are of the opinion that “teachers turn to the Internet mainly in search of resources and use technologies to design activities, attend to diversity, collaborate with other teachers and manage classroom work” (p. 84).

Specific characteristics of ICT with a focus on Mathematics

The use of ICTs in Mathematics has particular characteristics in the teaching and learning process, which must be well defined for its application. It is advisable that the teacher who teaches this type of subjects masters the technological resources that are adapted to the needs of the students and are articulated to the skills established in the class planning. According to Espinoza and Rodriguez (2021) establish several reasons why ICTs should be considered at the time of their application:

- It improves the development of critical thinking.
- It improves problem-solving skills.
- It benefits the materialization of theoretical and practical knowledge.
- The application of software in Mathematics helps in the analysis and interpretation of the problems posed.
- It improves the interaction between teacher and students in exploratory and experimental aspects.
- It promotes the use of active methodologies directed to inquiry and research.

ICT and academic performance

The use of ICT does not ensure a total improvement in the academic performance of students in the subject of Mathematics, but it implies a development in the growth of skills for the understanding of knowledge. However, interactivity through computational aspects captures the attention of students, with much more reason, when gamification is used, it can generate significant learning that improves the participatory and reflective conditions of the student. It is important to have concentration in a class, therefore, the methodology is fundamental, although for Miguel-Revilla (2020) there are important aspects to consider and he mentions that "introducing educational technologies in schools has to do with the degree of motivation and involvement with which students face the classes, which may lead to wonder about the impact of digital tools" (p. 1130).

From another perspective, the use of ICT allows the student to inquire and have a better vision of the resource or technological tools, in addition, they have the ability to acquire skills that adapt to their learning and diversification in educational development. For their part, Pardo-Cueva et al. (2020) have a different conception and mention that "when different technological tools are incorporated into educational training, students have the ability to learn in different ways and at unequal rates" (p. 936).

Active methodologies in education

Active methodologies are alternative processes used in education to improve the teaching and learning process of the student, in most cases, various technological resources are used as a means of communication that allow the information to be received in an interactive and dynamic way, in addition, this type of methodologies has a systemic approach, that is, they are structured through sequential parameters with the purpose of benefiting the student in the construction of his own knowledge.

Inquiry-Based Learning

Inquiry Based Learning (IBL) is one of the active methodologies that should be applied in the teaching of Mathematics, for the simple fact that students must learn to generate an investigative culture in their training process. This type of methodology is adapted to the context of the exact sciences because it strengthens critical thinking, fosters the investigative culture, creativity, deepens experimentation, allows the establishment of hypotheses, adaptability to technology and the refutation of criteria in collaborative work, furthermore, for Sala and Font (2019) mention that IBL "highlights the importance of establishing a commitment to solve a problem, working collaboratively, discussing and dialoguing, considering alternative approaches with critical thinking and reflection on learning and communication" (p. 75).

Mathematics is a subject that students have long had complications in understanding, in several cases, causing frustration when analysing a specific topic. It is essential to apply learning methodologies that involve student autonomy with research criteria, and to accomplish this, inquiry is essential because it outlines the role of the student in cognitive development, reasoning and perception in knowledge. Using IBL is the starting point to change the students' ideology, therefore, for Huang et al. (2021) indicate that this methodology "is a pedagogical approach that emphasizes learning through experience and construction, in addition, it fosters students' autonomy in the learning process and involves learner-centered activities" (p. 1506).

IBL, when incorporated as a methodology in a classroom, creates basic skills in the student in order to construct his or her own knowledge. There are specific IBL criteria related to transmission, discovery and challenge throughout the teaching and learning process. The in-depth analysis of the criteria or descriptors is detailed in Table 1 and Figure 1 according to the different approaches of the methodology.

Table 1. IBL specific criteria.

Approach	Transmission	Discovery	Challenge
Viewed from the subject.	A given set of standardized knowledge and procedures. A set of universal truths and rules to be transmitted to learners.	A creative subject in which the teacher adopts a passive and facilitating role, expecting students to create their own concepts and methods.	A set of interconnected ideas that the teacher and student create together through discussion.
Viewed from a learning perspective.	An individual activity based on observing, listening and imitating until fluency is achieved.	An individual activity based on practice, exploration and reflection.	An interpersonal activity in which students are challenged.
Viewed from the teaching perspective.	Structure a linear curriculum for students; give verbal explanations and check for understanding through practice questions; correct misunderstandings when students do not grasp what has been taught.	Assess when a learner is ready to learn; provide a stimulating environment that facilitates exploration; avoid misunderstandings by careful sequencing of experiences.	A non-linear dialogue between teacher and students in which meanings and connections are verbally explored. Misunderstandings are made explicit and worked through.

Source: Adapted from García (2011).

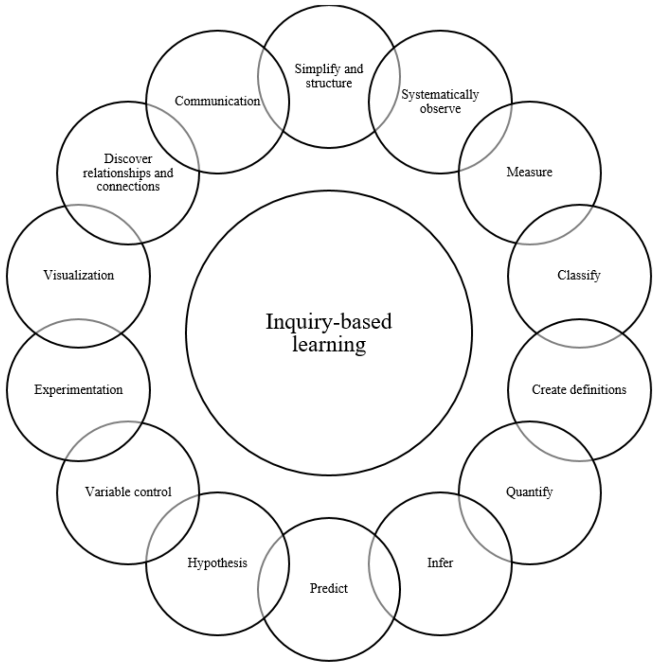


Figure 1. IBL Process. Source: Adapted from García (2011)

METHOD

Research approach

In the research topic, the quantitative approach was established, because the study variables use of technology (through the application of IBL) and academic performance are measurable and have their respective measurement scales, therefore, descriptive and inferential statistics were used for the analysis of the data collected, in addition, it is important to mention that when using statistical inference, hypotheses (null and research) should be raised in order to verify or contrast them.

Research design

It is important to define the research design to systematize the process to be followed in the methodology, for this reason, once the study variables were defined, an experimental design was established, this is because there is minimal control or manipulation of the variables. It is important to point out what was established by Palella and Martins (2012) where they state that "Not every educational situation is feasible or convenient to be treated experimentally, but when the circumstances are propitious and allow it, the experimental study should be applied" (p.85), therefore, being a research work related to the teaching of Mathematics as an exact science, applying an experimental design is feasible and beneficial for scientific development.

Type of research

In the educational field there are several cases where the study variables can be measured subjectively, and this depends on the form and context established by the researcher, having an experimental design it is important to minimize subjectivity in the study when analysing educational aspects, for this reason, the type of research was defined as pre-experimental based on the minimum control of the study variables.

Research level

Regarding the research level, a pretest and posttest analysis was established with a single group of students, the objective was to compare the traditional methodology currently used with respect to IBL as an active learning methodology. In this type of level, it is important to diagnose the current situation of the variables where there is an initial reference level before the intervention or treatment, after which a comparative analysis was performed in order to verify whether or not there is an improvement in the behavior of the variables.

Population and sample

In order to define the population with its respective representative sample, it is important to analyze the context or environment where the research is applied, and to consider the various resources that may be involved in the study. The researcher designs with inclusion and exclusion criteria the population and sample according to the ease of collecting the information. The current study, which was directed at students of the Joaquín Lalama Educational Unit, assumed that all members of the target population have the same characteristics based on the variables of analysis, that is, the principle of homogeneity.

As previously stated, a non-probabilistic sampling of census type and convenience was used, with all students considered in the study serving as a sample. For Hernández and Escobar (2019) on the intentional or convenience sample, they point out that “individuals are selected intentionally from the population to whom there is generally easy access or through open calls, in which people come voluntarily to participate in the study” (p.78).

The students included through a free and voluntary call to participate in the study were those who are part of the high school level in the first and second courses from the morning and afternoon sections, offered by the Educational Unit with a total of 58 students, who collaborated to perform the analysis of the current situation and verify the behavior of the defined study variables.

Data collection techniques

The techniques generally used in the research were direct observation and inquiry; the former was applied at the time of collecting the information because the facts were directly perceived by the researcher; and the latter was used when applying the IBL with the selected sample when executing the agenda established in the two subjects for the post-test analysis.

In addition, the survey and the evaluation test were considered as specific techniques with greater depth analysis for obtaining the data of the current situation; the first one had the purpose of measuring and correlating the items on the study variables, it had to be structured through the phases of preparation, design, analysis and presentation of results. This technique was applied with the purpose of knowing the level of usefulness of ICT (prospective studies) at the time of teaching Mathematics classes by the teachers of the institution; the second was used to verify the behavior of the students' academic performance at the time of the intervention with the application of the IBL in the post-test analysis.

Data collection instruments

Each of the specific techniques established in the previous section must be validated with its respective information collection instrument, for the execution of the survey, the questionnaire called "the use of technology and academic performance in the teaching of mathematics" was used as an instrument or tool, structured with two questions related to sociodemographic data and divided into three dimensions; The first dimension was designed with four questions about the previous inquiry on the use of technology; the second was structured with three items focused on academic skills in reference subjects; and finally, the third dimension consists of seven questions directed to the use of technology in the teaching process. It is worth mentioning that the different items were elaborated with dichotomous response alternatives and a Likert-type scale.

Hypothesis statement

H0: The use of technology through the application of Inquiry-Based Learning does not establish significant differences in students' academic performance (H1 = H0).

H1: The use of technology through the application of Inquiry-Based Learning establishes significant differences in students' academic performance (H1 ≠ H0).

RESULTS

Survey results

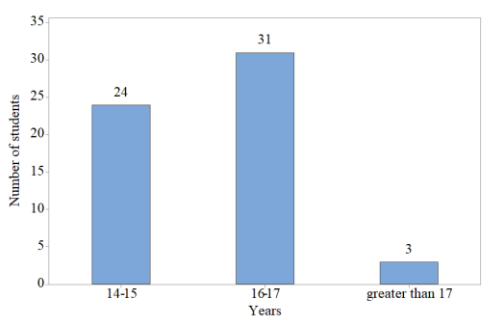


Figure 2. Age

Figure 2 shows that most of the students who took part in the study were between 16 and 17 years of age, with 31 cases specifically. In general, this age is in line with the educational level, this is due to the fact that the study was conducted in the first and second levels of high school. Knowing the age in an investigation is fundamental to establish the characteristics of the sample and to categorize the information according to the study variables.

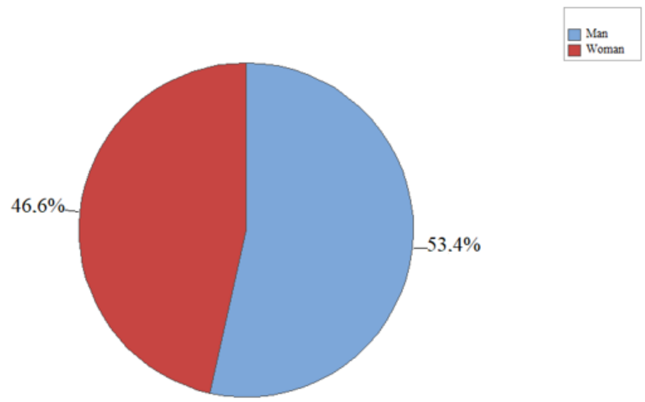


Figure 3. Genre

Figure 3 shows that the participants were selected equally, with 53.4% (31 cases) representing males and 46.6% (27 cases) females. Inductive and deductive thinking in adolescents are different according to the sex of the individual, from the point of view of Morales-Bautista and Díaz-Barriga (2021) refer that "critical thinking requires problematization, in which skills such as analysis, evaluation, and argumentation are involved" (p. 3).

Dimension 1: Prior inquiry into the use of the technology

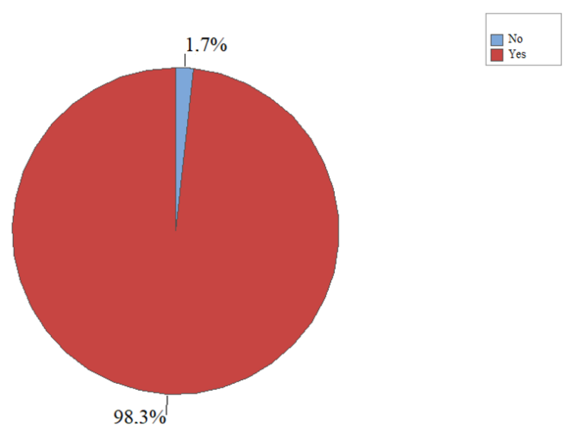


Figure 4. Taste for technology

In Figure 4, it is observed that students are attracted by the use of technology, this represents 98.3% (57 cases) of the respondents, while 1.7% (1 case) consider the use of technology unnecessary.

González-Ramírez and López-Gracia (2018) ratify that "psychological and personal well-being is the main reason why adolescents interact and manage the use of the Internet and social networks" (p. 74).

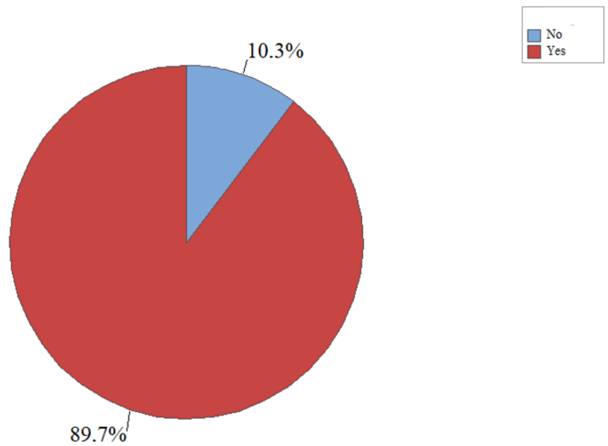


Figure 5. Internet availability

In Figure 5, 89.7% (52 cases) have internet at home, on the other hand, 10.3% (6 cases) do not have this resource. Madrigal and Contreras (2016) argue that "the internet continues to occupy one of the highest positions within the sources that adolescents access to search for information they find interesting" (p. 15).

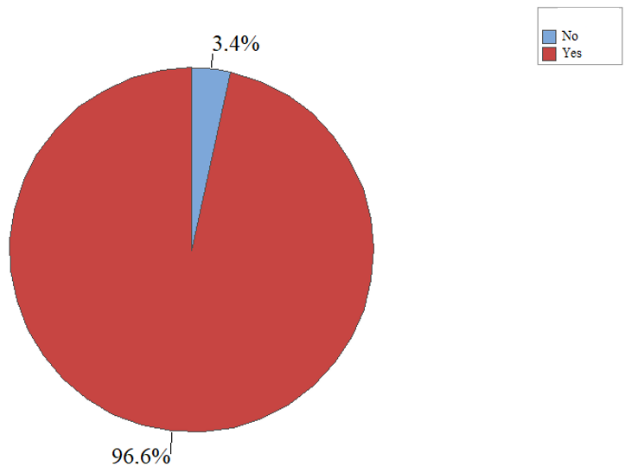


Figure 6. Knowledge of social networks

In Figure 6, 96.6% (56 cases) of the students know about social networks at present, while 3.4% (2 cases) lack information. The registration of users in social networks has had a considerable growth and especially in the adolescent stage (Rojas-Jara et al., 2018).

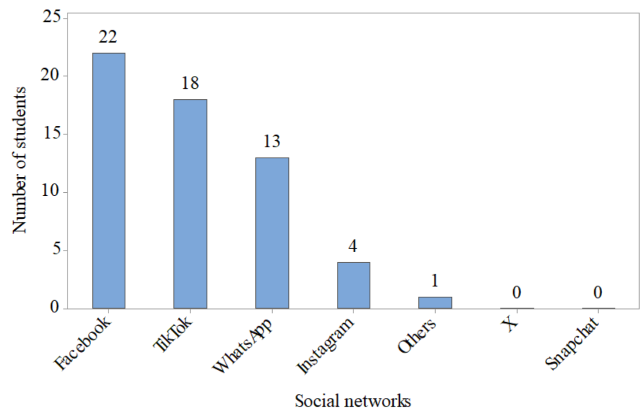


Figure 7. Most used social network.

Figure 7 shows that the social networks most used by students are Facebook, TikTok and WhatsApp with 22, 18 and 13 cases respectively. For Santillán-Lima et al. (2019) "social networks are a medium that propitiates interpersonal communication among students immediately and positively influences their academic performance" (p. 27).

Dimension 2: Academic skills in reference subjects

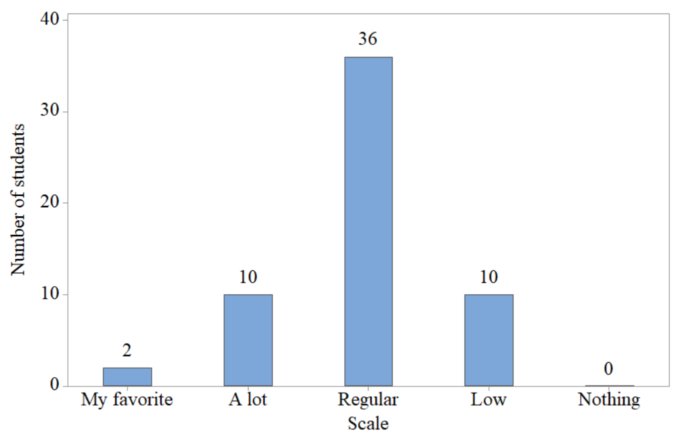


Figure 8. Interest in Mathematics subjects

Figure 8 shows that the students' interest in the subject of Mathematics is on a scale of regular with 36 cases, in the scales of very much and little there are 10 cases for each, and only on 2 occasions is it mentioned as a subject of preference. It can be concluded that the degree of interest in the mentioned subjects is minimal with approximately 21%, a worrisome case that should be analyzed in depth with methodological intervention.

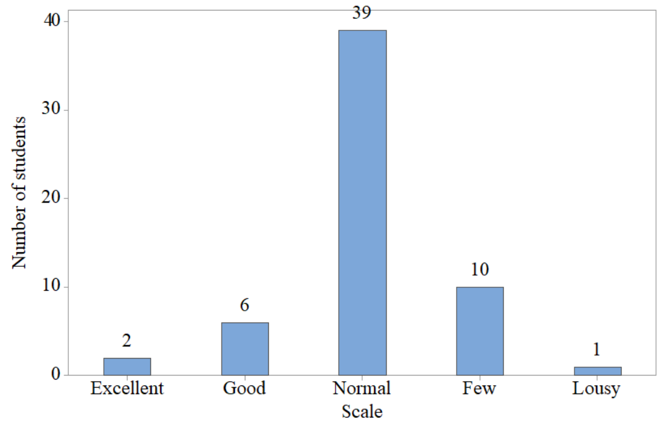


Figure 9. Mathematics Subject Matter Skills

Figure 9 shows that the skills in the analysis subjects are considered as normal with 39 cases, in addition, it is indicated that there are 11 cases distributed in the scales of few and very bad, and only 8 cases consider that they have excellent and good skills. In percentage terms, approximately 14% of the students state that they have acquired sufficient skills in the subject of mathematics.

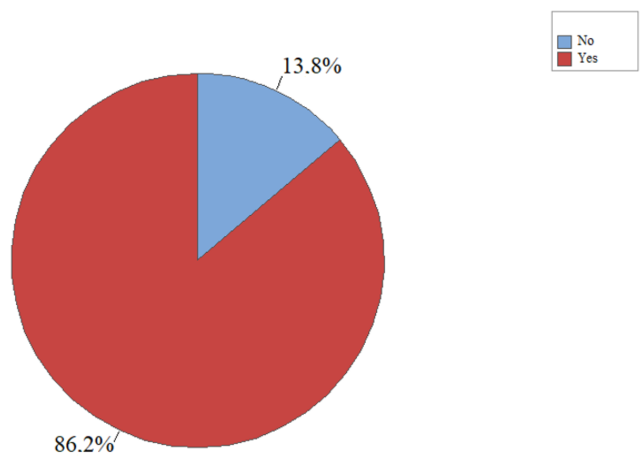


Figure 10. Criteria for a traditional classroom

In Figure 10, 86.2% (50 cases) of the students consider that they learn in a better way when the teacher uses a traditional class, on the other hand, 13.8% (8 cases) indicate that a traditional class has generated a delay in knowledge and especially in subjects that belong to the exact and experimental sciences. The use of ICT has become indispensable in the teaching and learning process, from the point of view of Velasco and Vizcaíno (2020) describe that "the use of ICT will make learning activities more dynamic, meeting the needs of forging innovative people" (p. 160).

Dimension 3: Use of technology in the teaching process

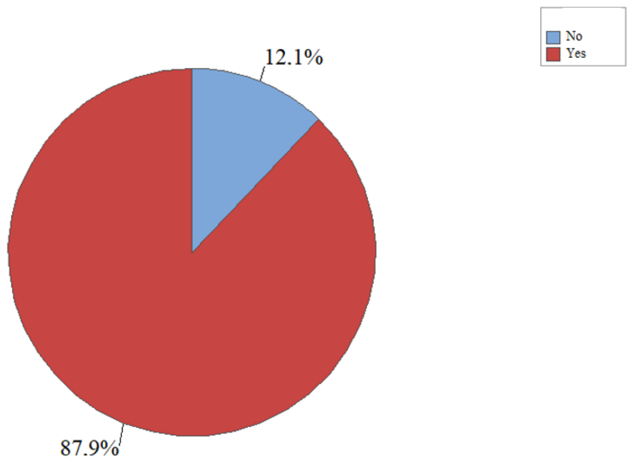


Figure 11. Criteria for a traditional classroom

In Figure 11, 87.9% (51 cases) of the students indicate that it is important for Mathematics teachers to use technological tools to improve their learning, while 12.1% (7 cases) mention that the use of ICT is not indispensable when receiving classes. According to Coloma-Andrade et al. (2020), they state that for teachers "the assessment and evaluation of these tools becomes extremely essential, since they do not have sufficiently solid criteria to encourage their use, so they are not motivated to bring these elements to their classrooms" (p. 205).

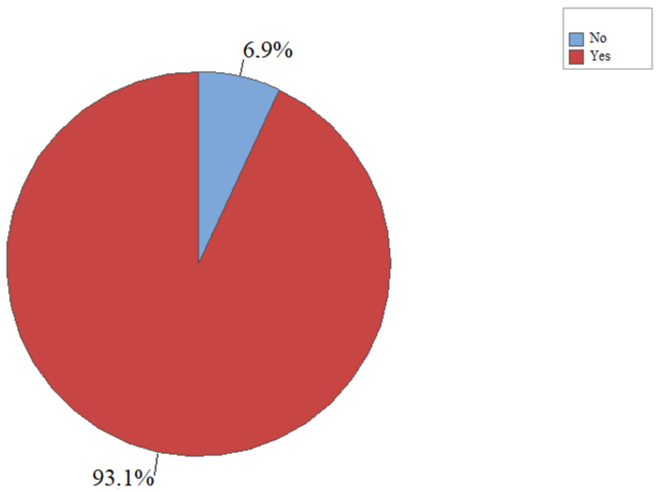


Figure 12. Interest in learning with the use of technology

In Figure 12, it can be observed that 93.1% (54 cases) of the surveyed students indicate that they would like to learn Mathematics with the use of technology, on the other hand, 6.9% (4 cases) do not agree with acquiring knowledge through the application of technological resources. Based on

Uvidia-Rodríguez (2019), the use of ICT "originates a high percentage of motivation in the learning of mathematics in students and provides the necessary elements for them to develop in this virtual environment" (p. 239).

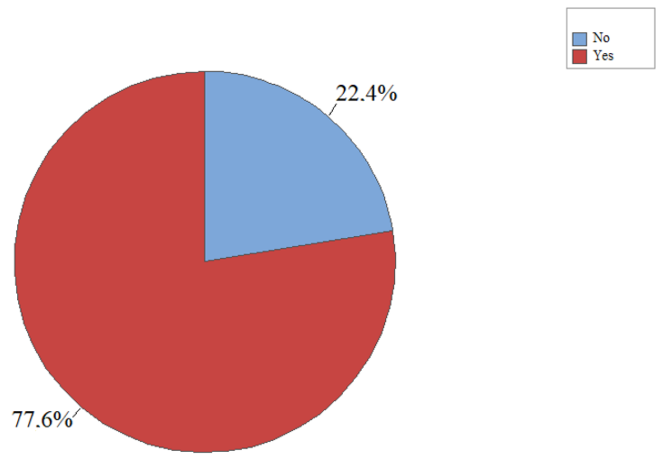


Figure 13. Teachers' use of technological tools

Figure 13 shows that 77.6% (45 cases) of the students express that Mathematics teachers have used technology in their classes; on the other hand, 22.4% (13 cases) indicate that teachers have not incorporated any technological resource at the time of teaching classes.

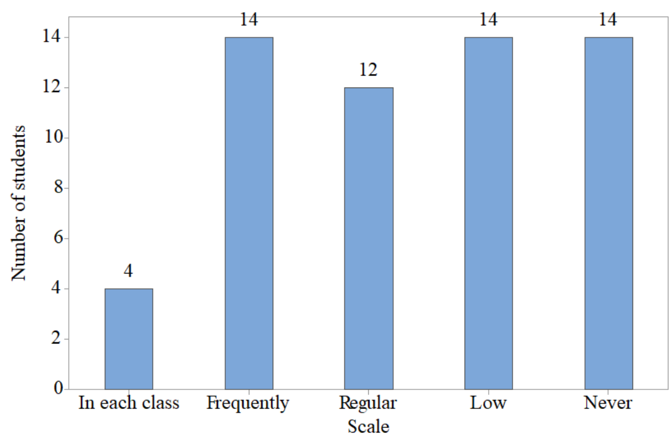


Figure 14. Use of the scientific calculator

The use of the scientific calculator has become a fundamental tool for verifying the results obtained in the resolution of mathematical exercises or problems. In Figure 14, it can be identified that the highest concentration of data is found in the scales of frequently, rarely and never, each with 14 cases. In percentage values, 7% of the surveyed students mention that the teacher has used a scientific calculator in each class, while 24% indicate that this tool has never been used. Mendoza-Alonzo (2019) refers that using a calculator "makes it possible to develop and strengthen important general skills such as estimation, mental calculation, the search for regularities, creativity, spatial vision and mastery of basic operations" (p. 50).

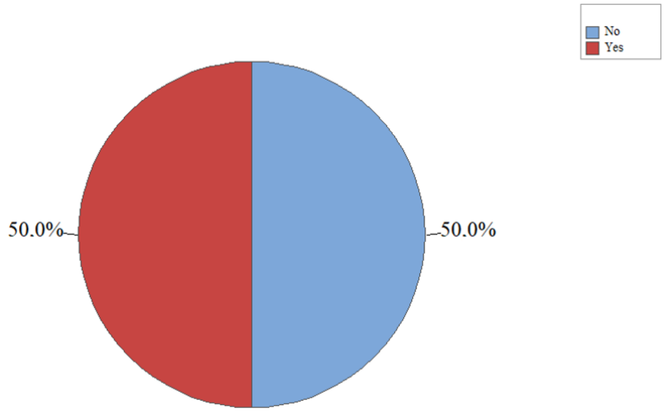


Figure 15. Explanation to students on the use of the scientific calculator

At the high school level, the use of the calculator becomes an indispensable technological tool, with more emphasis on the subject of Mathematics. In Figure 15, it can be observed that 50% (29 cases) of the students state that the teachers have explained the use of the scientific calculator; on the other hand, the same percentage indicates that the explanation on the operation of this tool has not been presented in the classes. According to Segarra (2022) "the calculator is shown as a useful tool in the teaching and learning process, especially as a support for independent work and that allows developing skills independently and creatively" (p. 3).

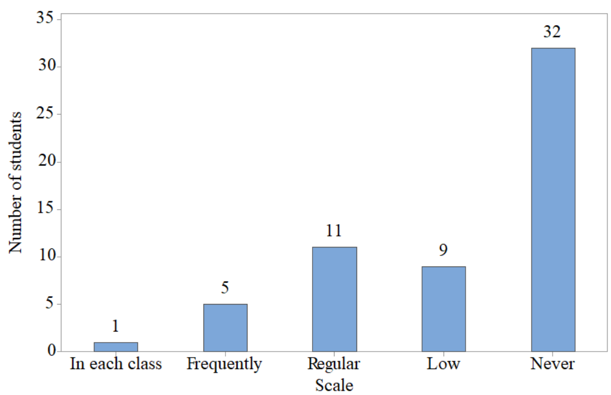


Figure 16. Use of the projector in Mathematics classes

The use of ICT in the teaching and learning process has become an indispensable and intermediary means for the understanding of a subject that needs to strengthen knowledge; in general, in subjects with high levels of difficulty, technological tools offer their potential and quality in the visualization of information. In Figure 16, the use of a projector has become a deficient aspect in the institution, where 9 and 32 cases indicate that its usefulness is in a scale of little and never respectively, this represents 71% approximately.

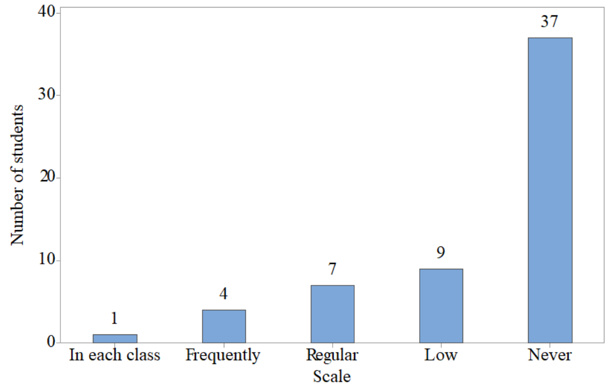


Figure 17. Use of the projector in Mathematics classes

There are topics in Mathematics such as equations, functions, descriptive statistics, among others, where the use of software should be aligned in the teachers' planning. Its importance refers to the interactivity and dynamism of incorporating these resources in classes so that students can generate greater adaptability, inquiry, creativity and innovation in the teaching and learning process (Morales-Olivera and Blanco-Sánchez, 2019). In Figure 17, it can be observed that there is a lack of applicability of software focused on Mathematics, where 9 and 37 cases indicate that its usefulness is on a scale of little and never respectively, this represents 79% approximately.

Pre-test result

In the application of the pretest, the teaching methodology used was a traditional class based on the ERCA technique (Experience, Reflection, Conceptualization and Application) where the only technological resource used was the scientific calculator. The topics selected for Mathematics were: systems of equations with the application of the methods of substitution, reduction, equalization and determinants; and descriptive statistics in relation to the calculation of the arithmetic mean for non-grouped data. After the contextualization of the theoretical foundations and the resolution of exercises or problems, we proceeded with the application of an evaluation of knowledge, where an exercise was established for each subject treated, four in total. The results obtained are detailed in Table 2.

Table 2. Averages through the application of the traditional class.

	Averages (points) without intervention			
	Student	Mathematics	Student	Mathematics
Experimental group	1	5.2	30	7.4
	2	7.4	31	9.1
	3	9.1	32	8.4
	4	4.7	33	6.4
	5	8.1	34	5.1
	6	7.5	35	6.2
	7	6.7	36	6.4
	8	8.4	37	8.8
	9	7.4	38	6.2
	10	6.1	39	6.8
	11	8.7	40	6.7
	12	8.1	41	6.4
	13	6.1	42	6.3
	14	6.1	43	6.7
	15	5.8	44	7.9
	16	9.6	45	6.8
	17	7.7	46	8.0
	18	9.4	47	7.2
	19	6.7	48	7.1
	20	9.1	49	7.0
	21	8.3	50	6.4
	22	6.4	51	4.8
	23	6.9	52	6.6
	24	7.9	53	6.9
	25	9.0	54	6.4
	26	9.7	55	6.3
	27	7.2	56	7.9
	28	8.5	57	6.5
	29	7.9	58	6.0

Post-test result

In the application of the post-test, the IBL was used as a teaching methodology, where the students received the classes in the Computer Science laboratory. The turning point of the IBL was the inquiry and its first step was the use of the YouTube platform as an online resource in the teaching and learning process. The functionality of YouTube was to inquire into the established topics in order to contextualize the theoretical and practical foundations in a generalized manner. The didactic platform Khan Academy was used to deepen the knowledge and materialize specific aspects of the topics, where the teacher helped to deepen the students' concerns using the role of mediator.

The functionality of the virtual simulators in the IBL was indispensable for the visualization of the information in an interactive way, the inquiry in this type of environments improved the understanding of knowledge in the students. For the verification of the results obtained in the resolution of exercises or problems posed, the online software Symbolab and Mathway were applied, in addition, Photomath as a mobile application in an alternative way for the same purpose in the comparison of results. At all times, inquiry was the intermediary technique for the materialization of information and discovery of analytical, deductive, inductive and investigative aspects by the students. See Table 3.

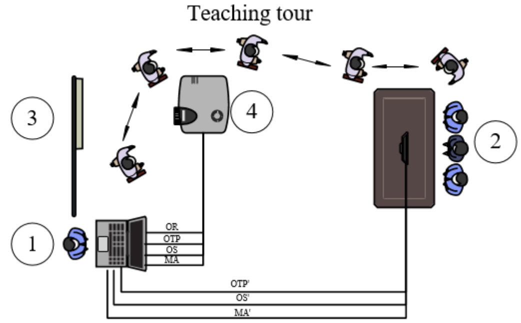
Table 3. Online elements used in the IBL application.

Element	Links
Online resource	• https://www.youtube.com/watch?v=LTfv1G2iYuQ&ab_channel=Matem%C3%A1ticasprofeAlex • https://www.youtube.com/watch?v=apPXOlZnRhg&ab_channel=Matem%C3%A1ticasprofeAlex • https://www.youtube.com/watch?v=0ilTVp5uRz8&ab_channel=Matem%C3%A1ticasprofeAlex • https://www.youtube.com/watch?v=jZIk90KQo6s&ab_channel=Matem%C3%A1ticasprofeAlex • https://www.youtube.com/watch?v=JwsfkIy6B_o&ab_channel=Matem%C3%A1ticasprofeAlex
Online software	• https://es.symbolab.com/solver/system-of-equations-calculator • https://es.symbolab.com/solver/arithmetic-mean-calculator
Online didactic platforms	• https://es.khanacademy.org/math/algebra-basics/alg-basics-systems-of-equations • https://es.khanacademy.org/math/cc-sixth-grade-math/cc-6th-data-statistics/mean-and-median/a/calculating-the-mean
Virtual simulators	• https://phet.colorado.edu/sims/html/equality-explorer-two-variables/latest/equality-explorer-two-variables_all.html?locale=es
Mobile applications	• https://photomath.com/en

Didactic component

For the understanding of IBL as an active methodology that adapts to the teaching and learning process of Mathematics, the following systemic process was applied for the established topics (Figure 18):

1. The teacher summoned the students of the experimental group to the computer lab, the students were distributed in groups of 3 per available machine (15 computers in total working properly).
2. The teacher conducted a discussion on the topics with the use of inquiry as a technique for the collection of prior knowledge provided by the students, the objective was to know the level of elementary knowledge and its prerequisites for the understanding of the class.
3. Once the teacher had conducted the prior knowledge inquiry, he incorporated and contextualized the topics of analysis in a generalized manner.
4. The teacher used the YouTube platform as an online resource in order students can watch the theoretical and practical explanation of the Edutubers, then a discussion was held with questions and answers of what was learned, where the teacher obtained the role of mediator.
5. The teacher shared with the students the Khan Academy page as an online didactic platform to deepen the theoretical and practical knowledge for the resolution of the exercises, each group made the respective inquiry of the platform where the participation of the students was the turning point. Collaborative work was indispensable in the process. Once the students had analyzed and investigated the information on the platform, the teacher provided feedback to materialize the knowledge.
6. The teacher presented and solved the exercises according to the established topics. At the time of the resolution, the teacher used the online software Symbolab and Wolfram Alpha in the free version to analyze and compare the procedure, as well as to indicate to the students the different resolution alternatives to obtain the same result.
7. Then, the teacher proposed the exercises so that the groups could develop and interact with the software shared in the classroom. The inquiry in the interactivity between computer and student groups remained latent at all times, where attention, observation and direct communication were indispensable in the teaching and learning process.
8. Subsequently, the teacher used Photomath as a mobile application to verify the result; only a photo of the exercise was taken and the result was automatically displayed on the mobile device, together with the resolution process. The groups carried out the same process with the exercises proposed by the teacher to verify the results.



Description
1. Teacher and computer.
2. Group of students and computer.
3. Blackboard.
4. Projector.

Meaning
(OR) = Online Resource - Teacher.
(OTP) = Online Teaching Platform - Teacher.
(OS) = Online Software - Teacher.
(MA) = Mobile Application - Teacher.
(OTP') = Online Teaching Platform - Students.
(OS') = Online Software - Students.
(MA') = Mobile Application - Students.

Figure 18. Didactic component of IBL

After explaining the topics through the IBL, the same knowledge assessment was applied again to determine the impact of the active methodology on the academic performance of the experimental group. The results obtained are detailed in Table 4.

Table 4. Averages through the application of the IBL.

	Averages (points) with intervention			
	Student	**Mathematics**	**Student**	**Mathematics**
Experimental group	1	5.9	30	7.9
	2	7.9	31	9.4
	3	9.2	32	8.6
	4	5.0	33	7.8
	5	8.6	34	7.2
	6	8.1	35	7.0
	7	7.1	36	7.5
	8	8.9	37	9.7
	9	7.8	38	7.4
	10	6.8	39	7.0
	11	9.2	40	8.5
	12	8.8	41	7.2
	13	6.7	42	7.9
	14	6.5	43	7.9
	15	6.5	44	8.0
	16	9.8	45	8.8
	17	8.2	46	10.0

18	8.7	47	7.9
19	7.9	48	7.5
20	9.8	49	7.5
21	8.8	50	8.3
22	7.1	51	7.1
23	6.3	52	7.4
24	8.4	53	7.2
25	10.0	54	7.6
26	10.0	55	7.9
27	7.0	56	6.4
28	9.0	57	6.8
29	9.1	58	6.3

Result of hypothesis testing

Since academic performance is a numerical variable that complies with a normal distribution, statistical inference was applied using Student's t-test for related samples based on the decision criteria of parametric statistics and for each subject established. The results are shown in Table 5.

Table 5. Student's t-test result for related samples.

Academic performance	**t**	**df**	**p-value (bilateral)**
Pre-test and post-test for Mathematics	-7.7	57	0.000

Note: t is Student's t-test for related samples; df is the degrees of freedom (n-1) and p is the significance level or bilateral or two-tailed error.

According to the results shown in Table 5, the application of the pretest and posttest in the academic performance variable with the IBL intervention through the use of technology has generated a p value of less than 0.05 in the subject of Mathematics, also with a t value of -7.7 with 57 degrees of freedom respectively. This indicates that according to the hypotheses stated, H0 is rejected and H1 is accepted where the use of technology through the application of IBL establishes significant differences in the academic performance of students.

DISCUSSION

The application of inferential statistics made it possible to contrast the hypotheses proposed by means of criteria and decision rules, and also helped to identify large-scale problems, especially in pedagogical aspects in the subjects that were part of the research before and after the experimentation. The usefulness and applicability of an active methodology such as IBL benefited students in the understanding and materialization of knowledge, where inquiry became the main axis at the time of analyzing the topics in the classroom. Students strengthened mathematical skills with the use of ICT and changed their role from direct observers to active observers (focused on inquiry) and participatory (focused on collaborative work).

The incorporation of any type of active methodology is fundamental to improve the teaching and learning process, it is a matter of the teachers of the different subjects, to investigate, analyze and understand each one of them for their correct implementation according to the requirements or needs in the class planning.

There are different didactic components that can be used in the teaching and learning process to improve students' academic performance and make the learning experience more attractive and effective. The following can be used as alternatives: feedback mechanisms (Forums using AI) to implement systems for students to receive instant feedback on their performance, allowing them to track their progress and make necessary adjustments; access to resources (Cloud Resource Sharing) with the aim of providing access to a digital library of mathematical resources, such as textbooks, articles and educational videos; assessment tools (Moodle) with the aim of developing online quizzes and assessments to evaluate student understanding and adjust instruction accordingly; teacher training (Technologies in Education Courses) to provide training modules and support for teachers to effectively integrate technology into their mathematics lessons, ensuring that they can maximize its educational benefits.

The applicability of IBL through the use of technology can be affected especially in rural areas, where connectivity is a drawback for both teachers and students in the execution of academic activities when using technological resources. According to the above, it is important to look for alternatives so that the core of IBL is reflected in the teaching and learning process. An IBL could be designed according to the available resources and needs of the educational institution, that is. a conventional IBL without losing the essence of using enquiry as a technique throughout the teaching process.

When designing a conventional IBL, the use of technology would cease to be the mediator in the entire teaching and learning process, but technological resources such as a scientific calculator or mobile devices (for the use of the calculator built into the device or another application that does not require connectivity) could be used to complement the conventional IBL process and are available to teachers and students who carry out academic activities, especially in rural areas. It is worth mentioning that the use of a scientific calculator and calculator as an application installed on the mobile device does not require internet access.

In subjects such as Mathematics, it is essential for ICT to be present at some point in the teaching and learning process, the purpose being that the student, apart from acquiring enquiry as a technique in the process of materialising knowledge, has the opportunity to obtain technological skills and become familiar with computational thinking.

CONCLUSIONS

The literature review has demonstrated the applicability of ICT in all educational fields, regardless of the subjects, technological resources or tools can be incorporated in the teaching planning at all levels. For exact sciences such as Mathematics, the bibliographic sources consulted have recognized the impact of ICT in the teaching and learning process, with an improvement in the understanding and comprehension of knowledge in the students of the different subjects applied and which has generated an increase in their academic performance.

In the previous analysis on ICT and academic performance through the application of the questionnaire, 87.9% of the students mentioned that it is important that Mathematics teachers use technological resources when teaching classes, 93.1% indicated that they would like to learn these subjects with the use of ICT, but 24% of the respondents mentioned that teachers never use a scientific calculator in their classes and 71% and 79% stated that teachers rarely or never use a projector or software, respectively. From the above, students are predisposed to use ICT in Mathematics classes, but by preserving traditional methodologies on the part of teachers, academic performance in these subjects has not improved in their averages.

The application of an active methodology such as IBL mediated by the use of ICT has improved study techniques in students, especially in aspects of inquiry, research and innovation, in addition, IBL has generated an increase in academic performance at the time of pretest and posttest analysis; 0.696 points. In the statistical inference analysis, it was verified by means of Student's t-test that further use of technology through the application of Inquiry-Based Learning establishes significant differences in the academic performance of students ($H1 \neq H0$), with a p value of less than 0.05.

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Characterization of Primary School Students' Perceptions in Understanding Negative Integer

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Abstract*: Students' understanding of negative integers is an important aspect of learning mathematics as it is a requirement for learning broader mathematical concepts. The purpose of this research is to explore students' initial perceptions in understanding negative integers on a number line. This descriptive exploratory qualitative research was conducted on fifth grade elementary school students. The subjects were selected based on students' initial perceptions of writing negative integers on a number line before the teacher presented the material. Students' initial perceptions can be seen from the exploration results of their thinking abilities. The data of this research was collected through written tests and interviews. From the exploration results of research subjects, it was found that there were three characteristics of students' thinking perceptions in understanding the concept of negative integers. These three characteristics are: assume there are no negative numbers; unary understanding; and pseudo understanding. From these findings, it is necessary to design learning which provide scaffolding for students to avoid errors in understanding whole numbers.*

Keywords: Characteristics, Negative Integers, Perception, Students

INTRODUCTION

Students' understanding of negative integers is very important to be studied as it is the basis for understanding broader mathematical concepts. This is in line with the results of previous research which states that students' understanding of the concept of numbers is very necessary in solving proportional problems (Parameswari et al., 2023). Furthermore, students' inability to understand integer can be an obstacle in completing tests related to fractional numbers (Susiswo et al., 2021).

The presence of these obstacles resulted in a perception (Ridwan et al., 2022) related to the concept of numbers.

Bishop et al., (2015) revealed several challenges experienced by students in understanding negative integers: First, students develop habit based on initial mathematical experience where integers are used to calculate real objects which made them see numbers only from the perspective of quantity. Second, students assume that the nature of cardinal numbers corresponds to the representation of natural and real numbers which is in contrast to negative integers that cannot be represented by something real or certain objects. Third, during learning, it is impossible for the teacher to give instructions to reduce or delete more numbers than what is available. Furthermore, there are three types of negative integer; unary (structural signifier), binary (operational signifier) and symmetric functions (opposite the positive numbers) (Bofferding, 2014; Vlassis, 2004).

According to Gallardo (2002), the existence of particular problem related to negative integers is very important for the success of students' understanding. He gave the example that it would be easier for students to understand the value of "50 debt" rather than saying "minus 50". The reason for students' difficulties in understanding negative integers is because negative integers cannot be represented concretely or negative integers are "opposed" to real objects (Bishop et al., 2015). In fact, for this reason, negative integers are often considered as fictitious number by students (Vlassis, 2004).

There are several empirical study results from researchers that can be used to convey negative integer material making it easier for students to understand (Fischer, 2005; Beswick, 2011; Bishop et al., 2014; Enzinger, 2015). Beswick (2011) revealed that the use of number lines is still the main tool in giving students experience related to negative integers, however he also believes that there are still many students who represent number lines with objects around them. Meanwhile, Bishop et al., (2014) explained how second grade elementary school students successfully involved the display of number ordering as a reason for solving integer. Enzinger (2015) documented the results of his case study on a student using single and double set drawings (strip drawing) as a medium for solving negative integer problems.

Even though many empirical solutions are offered, there are many students are still facing difficulty in understanding negative integers. Fuadiah & Suryadi (2017) revealed that students are still facing difficulties in dealing with negative integer problems, both at pre-learning and at the formal level. Among many factors, one that can prevent students from understanding negative integers is the minus (negative) sign, which can make it difficult for students to find solutions because they do not understand the meaning of negative integers (Vlassis, 2008). Some students think that subtraction is more difficult than addition (Karantzis, 2010). Therefore, this research aims to explore elementary school students' perceptions in facing negative integer problems before the material is taught by their teacher. The exploration results will be classified in three characters.

LITERATURE REVIEW

Perception and experience are two interrelated things. A new experience from time to time can be built through conceptual embodiment by combining perceptions and actions that develop through the mental world (Sa'adah et al., 2023). In this case, perception is the student's initial understanding of negative integers which can be depicted on a number line. If we look at it from a cognitive development perspective, students of a certain age need concrete examples as a medium for understanding the concept of integers. Likewise, in introducing negative integers to students, teachers must provide relevant examples so that students can easily understand them.

Fischer (2005) suggests that students who are faced with negative integers problems will have varying results. This can happen because students use their cognitive abilities to solve these problems using the number line. Furthermore, Gersten et al., (2009) also revealed that a very powerful tool in supporting students' understanding of various mathematical concepts is the number line. Siegler (2009) also explains that introducing integer through a number line is a very useful way to test students' understanding of numbers because this task requires students to be able to estimate the exact location of numbers on each number line scale. According to Stephan & Akyuz (2012) Students can use their experience with assets (money), debt, and net wealth to find positive and negative integer results. In this concept, number line is used to explore students understanding about negative integers. Earnest (2007) argued that the use of number line is important to help elementary students in identifying numbers.

Students who have obstacles in understanding the concept of negative integers resulted in different perceptions. The existence of these differences can be influenced by each student's thinking ability. According to Faizah et al. (2022a) thinking is a tool for constructing knowledge in mathematics learning. Students can create varied perceptions as they construct knowledge based on their experiences in everyday life. Therefore, the results of their construction are different.

Students' initial perceptions in solving negative integer problems can be categorized into several characteristics based on the location of writing negative symbols on the number line. This categorization can be classified into three main characteristics: assuming that negative sign has no meaning; unary understanding; pseudo understanding. Students assume that negative sign has the same meaning as minus sign in subtraction operation (Vlassis, 2004). Students assume that negative sign is located on the left side on zero (Bofferding, 2014). On the other hand, pseudo understanding is the result of thinking process in which the result is not the real result as the given answer could not represent the thinking result (Subanji, 2016).

The differences in students' perceptions about the location of negative integers on the number line can be overcome by providing scaffolding. Students with little experience of numbers will need more scaffolding from experts than those with more experience. Providing this scaffolding can be done through action from teachers or more expert colleagues (Faizah et al., 2022b). Scaffolding is given to overcome students' thinking obstacle in understanding negative integers concept. Scaffolding can be given based on students' characteristic results.

METHOD

This research is included in exploratory research with a qualitative approach. This research was conducted on 29 fifth grade elementary school students in Nganjuk district, East Java, Indonesia. Researchers selected students who had never received learning material about negative integers because the researchers wanted to explore students' initial thinking abilities regarding negative integers. Subject selection is also based on students' experience of using the concept of debts and the number line in completing test. Students who do not submit the results of their test work cannot be used as research subjects.

Written tests and interviews were used to collect the data of this research. The test is used to determine students' initial abilities regarding negative integers before students receive the material from the teacher. Students can complete tests based on their knowledge and experience without any interference from researchers or teachers while taking the test. Meanwhile, interviews were used to explore the subject's ability to think about negative integers. Researchers recorded the interview process using a video tape recorder to facilitate the data transcription process. The test instrument used in this research is adopted from Schindler et al. (2016) which is then validated. The written test can be seen in Figure 1.

There are six people talking about their financial problems. They have debt and assets. Randi is known to have 200 in debt while Ayu has 400 in debt. Based on this information, who has more debt?

a. Draw a number line from -500 to +500, then place the number that show Randi's and Ayu's debt in the number line!
b. If the financial situation of these six people are illustrated as follow:

Randi = 200 in debt	Ana = -300	Toni = 300
Ayu = 400 in debt	Tika = -200	Nina = 100

Please write the six numbers above on the number line based on the debts and assets that they have!
c. Please explain the meaning of "minus" sign in front of Ana's and Tika's numbers! Why aren't there any "minus" sign in front of Toni's and Nina's numbers?

Figure 1. Test Instrument

This research uses qualitative data analysis which refers to the results of written tests and task-based interviews. Qualitative data analysis includes: data observation, transcribing interview results, data reduction, data validation, data categorization, data interpretation, and drawing conclusions (Creswell, 2014). Researchers observed the results of written tests to determine the characteristics of students' thinking perceptions related to negative integers. The researcher transcribed the results of interviews with selected subjects. Data reduction was used to select and focus the data appropriate to the research objectives. Meanwhile, data validity in qualitative research is performed through triangulation. This research uses a triangulation method to determine the suitability of the data obtained from the results of written tests and interviews. Researchers categorized the perceptual characteristics that emerged in subjects when understanding and completing tests about negative integers. Then the researcher interprets the data based on the characterization results for the process of drawing conclusions.

RESULTS AND DISCUSSION

The results show that only 12 students were able to complete the written test. The twelve students had different perceptual tendencies in understanding the concept of negative integers. These differences can be classified into three types of perception, namely: (1) there are three students who take the test by accumulating debts and assets on the right side of the number line, (2) six students repeat the numbers on the number line symmetrically positive-negative, and (3) three students repeat the numbers on the number line in a positive-positive symmetrical manner. Of the three types, they were chosen randomly to continue with the interview process. Researchers coded the first type of subjects with S1, the second type with S2, and the third type with S3.

From each perception characterization, it can be seen that each subject is able to differentiate between debts and assets. The researcher did not provide any interference when the subject completed the test, but during the interview, the researcher gave the subject simple trigger questions to explore their understanding of negative integers in the context of the difference between debts and assets (money). The subjects said that when they chose debt, the money had to be returned. However, when subjects were asked to mark the location of debt and assets on a number line, they were still confused. From this confusion, students' initial perceptions are obtained when understanding the concept of integers because this material has not been taught by the teacher. The results of the exploration of each subject are as follows:

Subject 1

Subjects with the first type of perception work on questions about negative integers by stacking or placing debt and assets at the same point to the right of the number line. The subject put Tika and Randi at point 200, even though Tika owed Randi 200. The subject also placed Ana and Toni at the same point, even though Ana owed Toni 300. This can be observed from the results of the subject's work in Figure 2.

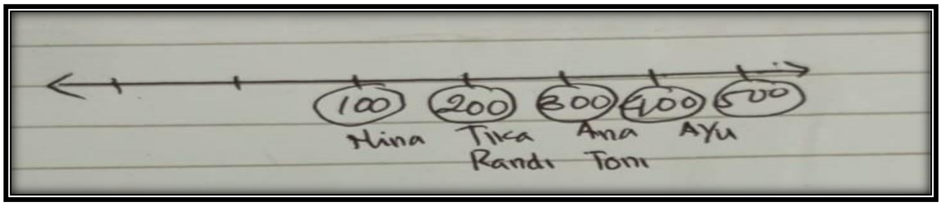

Figure 2. Answer from subjects 1

R: why does Nina occupy that place on the number line?

S1: because Nina has 100 Rupiah

R: Then, why do Tika and Randi occupy the same place?

S1: Because Tika has 200 in cash and Randi has 200 in debt

…

R: try counting backwards starting from 5

S1: Five, four, three, two, one

R: and after that?

S1: zero

R: after zero?

S1: nothing

Perception of Subject 1

Even though the subjects did not give a clear reason why after zero there were no more numbers, the answers they wrote showed that the subjects only knew positive numbers, because they did not pay attention to debts which should be negative numbers. This can be seen in the number line they have drawn, they only use the right side of the number line to place the position of debts and assets. They did not place a negative sign to the left of the number line to indicate debt as an embodiment of a negative integer. To find out whether or not the subject understands negative integers, the researcher provides scaffolding by asking the subject to count backwards starting from 5, when he reaches the number one, the subject stops and does not continue to the zero number.

Initially the subject did not mention zero, but after receiving a question from the researcher, the subject then mentioned zero. But after saying zero, the subject stopped and said that there were no more numbers before zero. One of the factors that causes subjects did not mention zero is because subjects were not used to counting things that are not in that number (Utami et al., 2018). Bofferding (2014) also explains that students will treat negative integer values like positive integers, because students view the minus sign only as a subtraction operation. As a result, the subject did not differentiate between the location of numbers as assets and debts on the number line, so they accumulated at the same point on one of the line segments.

When the subject said there are no more numbers before zero, this showed that the subject only understands positive integers. Woods et al., (2017) said that indirectly students have developed positive numbers quickly which make the understanding embedded in their mind that there are no more numbers other than positive integers. At that time, students begin to develop abstract mental representations that can generalize numbers from calculation routines (Wynn, 1990).

Subject 2

The second type of subjects completed the test in a different way than the first type of subjects. These differences can be seen in Figure 3.

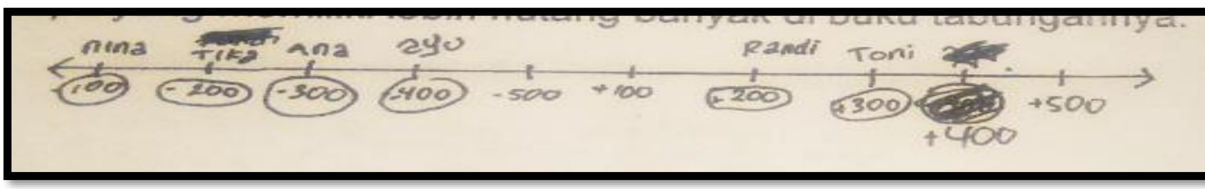

Figure 3. Answer from subjects 2

Based on the subject's answer in Figure 3, it shows that the subject drew a number line and wrote it repeatedly in the form of positive – negative numbers. Then the subject placed the debt and assets numbers correctly, but the position of the numbers was still incorrect. Subjects made errors because they did not write the number zero between the positive and negative number lines. The subject also made errors in writing -100, -200, -300, -400, and -500 starting from the left side, the subject should have started from the middle after zero. Researchers conducted interviews with subjects to determine their initial understanding of negative integers as in the following transcript:

R: try counting from two onwards!

S2: three, four, five, six…

R: try counting backwards now

S2: five, four, three, two, one

R: try continuing again?

S2: min five, min four, min three, min two, min one

R: hmmm...why is that?

S2: yes because this is five hundred plus (while pointing to the right-hand number line) and this is minus five hundred (while pointing to the left-hand number line).

R: So, does that mean before one is minus five?

S2: oh yeah yeah... (starting to feel confused)

Perception of Subject 2

Based on Subject 2's answer, it can be seen that the right side of the number line is the place for positive numbers, and the left side of the number line is the place for negative numbers. However, the subject did not write down number zero to differentiate it, so the subject made mistake when writing the numbers on the number line. According to Fischer (2003), students who understand negative integers on the number line tend to be from an ontogenetic perspective, that is, students understand negative integers by assuming that these numbers tend to be on the left side of the number line. Likewise, the subject's understanding of negative signs. Subject 2 has understood the meaning of negative signs as unary, namely negative signs that follow numbers (Bofferding, 2009; Stephan & Akyuz, 2012).

Tzelgov et al. (2009) stated that students have long-term memory, which states that the numbers 1 to 9 are the only digits that can be represented, while other numbers are produced from nine digits. Therefore, it is very possible that subjects will find and understand negative integers from giving tasks in certain contexts, such as in this study.

Subject 3

Type 3 subjects provided different solutions to type 1 and type 2 subjects. This can be seen in Figure 4.

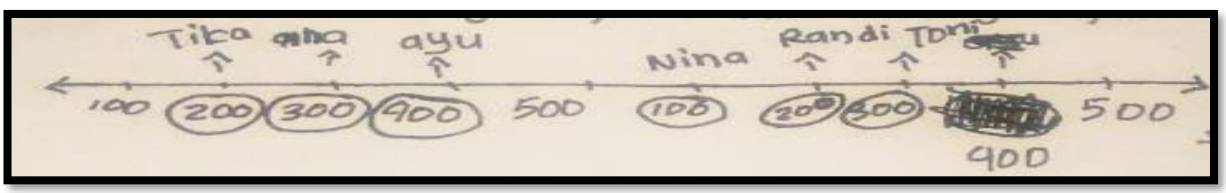

Figure 4. Answer of subjects 3

Figure 4 shows that the subject drew a number line and wrote positive-positive repeatedly. Subjects placed debt on the left and assets on the right of the number line. Even though it was correct in placing debt and assets, the subject still writing negative numbers incorrectly because the subject did not put a negative symbol (-) in front of the number written on the left of the number line, and the subject also did not write the number zero on the number line.

Perception of subjects 3

The subject appeared to have understood the concept of negative integers in its implementation in debts and assets because the subject places the number that indicates debt on the left on the number line. But the subject did not write the number of zero (0) in the middle of the number line, the subject also did not put a negative sign (-) in front of the number written on the left of the number line, and the order of the numbers written was also reversed because the subject should have written -100 to the left of 0 instead of 500. From the subject's answer, it is known that he has a false understanding of negative integers when depicted on a number line. The subject knew the position of debt as a negative number which is located on the left but did not write a negative sign (-) in front of the number, and the subject also knew that money or assets are positive numbers which are located on the right on the number line. This error occurred because the subject wrote numbers on the number line repeatedly, positive-positive, even though the subject should have written negative sign in front of the number on the left.

During interviews, subjects gave unclear reasons as to why they wrote numbers repeatedly. Type 3 subjects tend to lead to pseudo or apparent understanding, according to Subanji & Nusantara (2016) who say that pseudo understanding is the condition of students when they understand a concept but do not know the continuation of the concept. This is in line with the result of the research conducted by Young & Booth (2015), which revealed that knowledge of the magnitude of negative integers follows a similar pattern to that of positive numbers, but negative estimation performance is far behind positive. Subject 3's answer is also similar to the number line anticipation proposed by Bofferding (2014) in that students repeat numbers in positive-positive form when understanding the concept of whole numbers. Even though integers contain negative and positive numbers.

Based on the exploration results of each subject, characteristics of students' perceptions in understanding negative integers when depicted on a number line was found. The characterization is presented in Table 1.

Table 1. Characterization of Students' Perceptions of Negative Integers.

Characteristics	Description
Assume there are no negative numbers	• Interpret debt and assets at the same point on the number line. • The negative sign is only present in the subtraction operation.
Unary understanding	• Assuming that negative integers tend to be on the left of the number line. • Assuming that negative signs are signs that only follow positive numbers. • Placing negative numbers on the number line in a positive-negative symmetrical manner.
Pseudo understanding	• Knowledge of negative numbers following the pattern of positive numbers. • Do not give a negative sign because the numbers to the left of the number line are definitely negative. • Place negative numbers on the number line in a positive-positive symmetrical manner.

Table 1 shows that the 1st type of student did not assume that there are no negative numbers. Therefore, the student did not write the "-" symbol before the negative numbers as they assumed that the symbol of "-" is the same as the symbol used in subtraction. On the other hand, unary type of student only understood that the negative numbers are located on the left side of zero on the number line but do not understand the meaning. Students with unary type of thinking face difficulty in implementing the negative integers in debt-asset concept. Last, pseudo type of students understood the existence of negative integers but placed them in the same position with the positive integers. This means that pseudo type of students consider both negative and positive integers are the same, so they did not write negative "-" symbol before the numbers although they write the numbers on the left side of zero on the number line. There are differences from each type of students during the thinking process in understanding the concept of negative integers (Bofferding, 2014, Stephan & Akyuz, 2012, Subanji & Nusantara, 2016, Vlassis, 2004).

Students' inability to understand negative integers can occur due to thinking disorders. This disturbance is caused by one conceptual error which then has an impact on other conceptual errors (Sukoriyanto et al., 2016). Students can experience thinking disorders because of the cognitive conflicts they experience in understanding a certain concept, for example students' errors in understanding the concept of integers which have an impact on conceptual errors regarding fractional numbers. Thinking in mathematics learning can be designed to provide positive encouragement (Faizah & Sudirman, 2022), to help in developing students' confidence in the results of their solutions.

Students can understand the concept of fractional numbers through cognitive that encourage changes in students' way of thinking in defining words or identifying numbers (Pratiwi et al., 2020). To ensure that students do not experience thinking disorders, there needs to be scaffolding from teachers or experts (Anghileri, 2006, Faizah et al., 2022b), so in this case the researcher gives trigger questions to direct the subject to understand negative integers on the number line but the subject cannot accept the scaffolding. Therefore, it is necessary to carry out further research that discusses strategies for providing scaffolding to students who have difficulty in understanding the concept of negative integers.

CONCLUSIONS

The results of the research show that there are three characteristics of students' perceptions in understanding negative integers. The first characteristic is that students assume there are no negative integers, so they put negative integers at the same point with the positive integers on the number line. The second characteristic is that students have a unary understanding because they think that the negative sign in a number is just a meaningless sign that is in a positive integer and is located to the left of the number line. The third characteristic is pseudo understanding where students seem to understand negative integers because they know that they are on the left but when writing on the number line, they do not put a negative sign. The third type of student does not write the number "0" on the number line which resulted in an error when they write negative numbers on the number line.

This research was conducted when the teacher had not taught negative integer material to students which means that students took the test only based on their initial perception of numbers. Therefore, the results of this research provide an opportunity for other researchers to conduct further research on students' cognitive development in understanding the concept of negative integers when teachers teach this material. If there are students who experiencing difficulties, scaffolding needs to be provided so that different finding can be obtained.

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Development of a Traditional Game-Based Computational Thinking Supplementary Textbook for Elementary School Students

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Abstract: *This study aimed to develop a supplementary mathematics textbook on traditional game-based computational thinking for elementary school students. This textbook was developed using a research and development model consisting of five stages: analysis, design, development, implementation, and evaluation (ADDIE). The mathematical contexts of the textbook were based on four traditional Indonesian games integrated into four computational thinking skills: decomposition, pattern recognition, abstraction, and algorithms. The supplementary mathematics textbook was developed and validated by experts in mathematics and culture. The textbook was tested with 20 fourth-grade students. The findings of this study indicated that the supplementary mathematics textbook of traditional game-based computational thinking has high usability as rated by students. The students gave positive feedback for use of the textbook in the classroom because of its connection to culture and games. Therefore, the study enriches learning materials on culturally integrated computational thinking skills for elementary school students.*

Keywords: computational thinking, research and development, supplementary mathematics textbook, traditional games

INTRODUCTION

Computational thinking is one of the important skills that should be cuntivated in our students in today's globalized and dramatically changing world (Yadav et al., 2016). It was first introduced by Seymour Papert (Papert, 1980) and then reintroduced by Jeanette Wing (Wing, 2006) who mentioned it as one of the basic competences that complement literacy and numeracy skills (Lodi & Martini, 2021). It is defined as the ability to analyze a problem and present the solution (Wing, 2017). Computational thinking can also be understood as students' ability to solve a problem, understanding behavior by describing concepts and designing systems so that it can be solved properly (Weintrop et al., 2016). Therefore, teachers need to have a better understanding of how computational thinking can be integrated into their students' learning by promoting problem solving and critical thinking skills.

Computational thinking may be seen as a more general problem-solving strategy, which can be applied to different domains beyond computer science (Kale & Yuan, 2021). Therefore, computational thinking has been suggested to be a fundamental cognitive ability that should be acquired in education - comparable to literacy and numeracy (Tsarava et al., 2018). The rapid development of science and technology requires an effective strategy to develop computational thinking, one of which is through existing culture, so that students remain rooted in their own culture (Dousay, 2021, Putra et al., 2022, Vieira & Hai, 2022). A large global culture is currently needed; education that relates to culture that prepares students to have a local identity and recognize the identity of the nation with the aim of improving the world together and introducing existing cultures to others (Owens, 2014). In Indonesian contexts, as of 2017 there is a law (number 5) concerning the promotion of culture in order to protect, utilize, and develop Indonesian culture. Within education, we can integrate culture into learning using traditional games (Republic of Indonesia, 2017).

Indonesia is rich in traditional games that have been passed down from generation to generation (Jabar et al., 2022). Therefore, teachers can use traditional games in learning activities. Through games, students can express their thoughts and feelings, in turn making students' cognitive abilities increase because learning can be well received (Wulandari et al., 2022). Traditional games can support various aspects of a child's development, including motor, social, cognitive, emotional, language, spiritual, moral and environmental (Misbach, 2006). Traditional games can foster cognitive intelligence, emotional abilities, and creativity (Nurhayati, 2012); children become more dynamic and innovative. Traditional games can also be used as therapy for children. They foster children's pluralistic intelligence including nurturing children's insights, fostering children's relational abilities to understand individuals at a deeper level as well as fostering children's thinking, spatial and spiritual skills.

Although there have been many studies on traditional games, we find a lack of study connecting traditional games to computational thinking skills. This is significant because traditional games are one of the activities that have developed in society that are still played in order to preserve the richness of culture (Suteja et al., 2022), whereas computational thinking skills are thinking skills that lead students to more complex thinking in solving problems while playing the games (Lin et al., 2020). Furthermore, traditional games are still popular and played by many children, which motivates us to build computational thinking skills through traditional games. A study conducted by Putra et al. (2022) has developed computational thinking tasks based on Riau Malay culture, and we found that they have developed some tasks based on traditional games. Thus, there is a potential to develop computational thinking learning resources based on traditional games. The present study concerns developing a traditional game-based computational thinking supplementary textbook for elementary schools. The textbook is expected to be an additional learning reference for teachers in supporting students' computational thinking skills. Thus, this study specifically seeks answers to the following questions:

(1) How is a traditional game-based computational thinking supplementary textbook developed?

(2) How beneficial is the traditional game-based computational thinking supplementary textbook for classroom use in elementary schools as evaluated by experts?

(3) What are students' views on the traditional game-based computational thinking supplementary textbook?

Computational Thinking Skills

Computational thinking is a term that refers to the fundamental thoughts and ideas that exist in the field of software engineering and informatics (Bocconi et al., 2016). According to Wing (2006), computational thinking includes the capacity to handle problems, plan frameworks, and understand how humans behave by drawing on important ideas for software engineering. For reading, writing, and arithmetic, we must add computational thinking to the scientific capacity of every child (Wing, 2006). Computational thinking is a way of thinking that is needed in formulating problems and solutions, so that these solutions can become effective information processing agents in dealing with problems(Putra et al., 2022). Although computational thinking is a way of solving problems and finding solutions using computer concepts, it is also an important skill in the field of mathematics.

Computational thinking has four basic skills consisting of decomposition, pattern recognition, abstraction, and algorithm (Gunawan et al., 2023; Putra et al., 2022; Safitri et al., 2023). These four skills need to be cultivated in our elementary school students to promote their ability to deal with complex mathematical problems. First, decomposition is a skill used in breaking down problems into simpler forms so that they can be solved, developed, and evaluated to understand the complexity of a problem. A complexproblem will be easier to solve if someone breaks the problem into smaller parts. Second, pattern recognition relates to a skill in finding the correct pattern. This skill is needed for some problem-solving tasks. Recognizing recurring patterns or characteristics of the problem is a strategy that will make it easier to find solutions or find similarities between various problems. Third, abstraction is a skill to identify and recognize relationships, similarities, and differences that are most important to a problem, while ignoring information that is considered irrelevant to finding a solution. Finally, algorithm is the skill of getting to a solution through a clear set of steps.

Traditional Games

Games are a means of expressing one's emotions, involvement, experiences, hopes and self-satisfaction (Fitri et al., 2020). Games are also a tool for someone to keep their body and mind healthy and develop their character. Character development occurs because of changes in the nature of the game and changes in the social environment (Varzani, 2013). A game is an activity that is supportive of personality of elementary school students. The game is an act to entertain the heart using tools or without tools, and it is fun exercise for students.

Traditional games are hereditary practices that develop and cultivate in certain areas, which contain social characteristics; the advantages of local life are taught from one era to another. Traditional games have playing conditions that are associated with certain principles or goals (Misbach, 2006). Associating children with traditional games can nurture the qualities of creative thinking, responsibility, and cooperation.

Traditional games are exercises that have been carried on for a long time and have determined rules that must be followed and passed down from one generation to another using basic tools and according to their traditions (Achroni, 2012). There are several characteristics that must be considered to make a game 'traditional': (1) when playing games you must use simple tools which are easy to obtain and materials that can easily be made yourself; (2) traditional games are played by several people because they are concerned with shared joy; (3) traditional games can invite players to have social interactions (Wijayanti, 2018).

The various advantages of traditional games as expressed by Misbach (2006) are that games can strengthen various aspects of a child's development including motor, social, cognitive, emotional, linguistic, spiritual, moral, and environmental. Nurhayati (2012) argues that the advantages of traditional games are that they can foster intellectual intelligence, develop emotional abilities, foster creativity, help children become more dynamic and innovative, can be used as therapy for children and can foster children's multiple intelligences, including: creating children's insights, cultivate children's relational abilities to understand individuals at a deeper level, cultivate children's thinking, spatial awareness and spirituality.

Traditional Games-Based Computational Thinking

Game-based learning is a learning approach that is able to support students' computational thinking skills. A computer game-based learning is commonly developede to support students' computational thinking skills. For example, Hooshyar et al. (2021) developed an adaptive educational computer game, called AutoThinking, to support elementary school students' computational thinking skills and conceptual knowledge. The results of this study show that learning based on adaptive educational computer games significantly improves students' computational thinking skills in both conceptual and skill aspects. In addition, students using the adaptive educational computer game showed a significantly higher level of interest, satisfaction, flow state, and technology acceptance in learning computational thinking. The same results were also shown by Yang et al. (2023), namely that a role-play-based micro-game strategy had a significant influence on students' computational thinking skills, especially in the dimensions of computational concepts and computational practices.

A lack of digital equipment is a major obstacle to the implementation of computational thinking in education, particularly for rural schools. Therefore, some studies have tried to develop computational thinking learning instruction and media without the use of digital technology, but integrating traditional games (Putra et al., 2022; Zhang et al., 2023). Zhang et al. (2023) in his study developed Game-Based Learning (GBL) using a newly designed board game and GBL with parental involvement. The results of the study show that GBL approaches (i.e., with and without parents) significantly enhanced the students' CT skills compared to the traditional approach. Meanwhile, Putra et al. (2022) have developed computational thinking tasks based on Riau Malay culture. The tasks were developed in various contexts, including the context of traditional games such as *galah panjang*. The tasks have been used to measure elementary school students' computational thinking skills (Gunawan et al., 2023; Safitri et al., 2023). Therefore, traditional games-based computational thinking instructional learning has a potential to be developed to support students' computational thinking skills.

METHOD

The type of research used in this study is research and development (R&D) (Branch, 2009; Pratiwi et al., 2022). This research aims to develop a product that can support students in learning computational thinking. The product here is a traditional game-based computational thinking supplementary textbook. In addition, the ADDIE research model (analyze, design, development, implementation, and evaluation) (Branch, 2009) was used to develop the traditional game-based computational thinking supplementary textbook (Figure 1). The reason for choosing the ADDIE model is because the concept of product development is applied in performance-based learning (Branch, 2009).

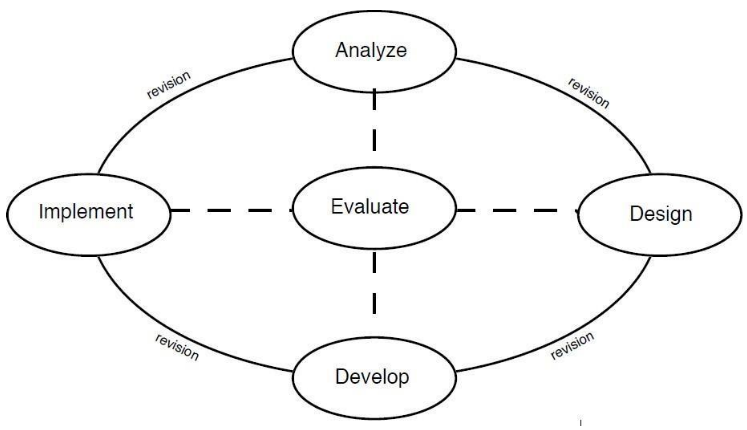

Figure 1. The ADDIE model (Branch, 2009)

The ADDIE model consists of five-phases (Branch, 2009; Mamolo, 2019). The first phase is analysis, which aims to determine whether the problem needs to be investigated and whether effort should be given toward resolution. The design phase follows to emphasize the description of the product that will be produced in the final stage of development. The purpose of the development phase is to have an expert review regarding the quality of the product. The implementation phase deals with real situations with actual learning, while the evaluation phase deals with reflecting on and revising upon the pilot stage to assess the quality of the product and process (Branch, 2009).

The first step taken by the researchers was to collect as much information as possible related to students' computational thinking abilities by conducting interviews with homeroom teachers regarding the learning process in the classroom, textbook references used, computational thinking of fourth grade students, and traditional games played by the students. At this stage, the researchers also carried out several processes including analysis of curriculum and learning resources, student characteristics and material analysis.

The second step was designing. This stage served a very important role because researchers began to organize the procedures to develop the product. These included: 1) searching for various supporting reference sources related to enrichment books and computational thinking skills for elementary school students; 2) making instrument grids as a follow-up step from observations and interviews that had been conducted previously; 3) planning the creation of the textbook by dividing the learning objectives according to each traditional game; 4) developing a supplementary textbook map; 5) creating a book structure consisting of an arrangement of parts which are then combined, so that it becomes a whole that is worthy of being called the supplementary textbook; 6) creating the supplementary textbook design using the *Canva* application and *Microsoft Word* application.

The third stage was development. Researchers conducted validation of the product by experts. Expert validation was carried out to determine the feasibility of a traditional game-based computational thinking supplementary textbook to be used as a learning reference. With the help of experts, book improvements were made appropriately. Expert validators for this study consisted of three experts; one mathematics expert, one cultural expert and one experienced elementary school Grade Four teacher.

After the process of validation of the product (the textbook) was completed and was declared to be feasible to be tested, the next stage was the implementation stage (sometimes called the trial stage). This is the stage of testing the book in real situations (Cahyadi, 2019). The researchers conducted the test of the supplementary textbooks twice. First, it was tested with 6 fourth grade students, focusing on two traditional games, *Congklak* and *Setatak*. The second test was with 12 fourth grade students, focusing on two other traditional games, *Galah Panjang* and *Yeye*. In the first test, the researcher (first author), explained the procedures of the game and the aspects of computational thinking in the game. While in the second test, the researcher divided the students into four groups to have them work collaboratively to play the games and to solve the computational thinking tasks in the supplementary textbook. After that, students were asked to fill out a questionnaire to share their thinking. All students were able to read the questions, but the researcher aided students who had difficulty with comprehension. After that, the researchers conducted an evaluation with the teacher and students regarding their learning experiences when using the supplementary textbooks.

The evaluation stage was carried out in each stage. Mean score from the experts was analyzed and interpreted as presented in table 1.

Table 1. Interpretation of validator to the computational thinking book.

Score	Category
V > 0.8	Very Valid/Very feasible
$0.4 \leq V \leq 0.8$	Valid/ feasible
< 0.4	Moderately Valid/ Not feasible

RESULTS

The traditional game-based computational thinking supplementary textbook was developed using the ADDIE model which consists of five stages. Therefore, the researchers present the results from each step as shown in the following subsections.

The analysis stage is divided into three stages, namely analysis of curriculum and learning resources, analysis of student character, and analysis of learning materials. The following is an explanation of the stages of analysis.

Analysis of Curriculum and Learning Resources

The curriculum is a set of experiences in the learning process that will be obtained by students while following an educational process (Fujiawati, 2016). The existence of a curriculum will make it easier for educators to achieve learning objectives. The curriculum will also make it uncomplicated for teachers to create concepts, methods and learning strategies for students.

Based on the information obtained when researchers were conducting interviews and observations, fourth grade students at the elementary school were still using the 2013 curriculum. Previously this school had implemented a new curriculum, called the independent curriculum, but the school returned to the 2013 curriculum because data from the national education center suggested that the school should continue using the 2013 curriculum. From the curriculum, the researchers formulated the objectives of mathematics learning and developed the learning resources for creating the supplementary textbook. The results of the analysis of the curriculum are presented in table 2.

Table 2. Indicators and Objectives.

No	Name of Game	Learning Materials	Indicators	Objectives
1.	*Congklak*	1. Operation of whole numbers	1. Students can perform addition, subtraction, multiplication, and	1. Students can perform arithmetic operations of addition, subtraction,

		2. Great common divisor (GCD) and least common multiple (LCM)	division of whole numbers. 2. Students can describe the difference between GCD and LCM. 3. Students can solve problems related to GCD and LCM	multiplication and division. 2. Students can find the difference between GCD and LCM. 3. Students can solve problems related to GCD and LCM with existing computational thinking skills.
2.	*Setatak*	1. Flat Shapes including square, rectangle, trapezoid, and circle. 2. Figures and number patterns.	1. Students can identify various kinds of flat shapes. 2. Students can explain the concept of perimeter and area of flat shapes. 3. Students can determine the pattern of figures and numbers.	1. Students can identify various kinds of flat shapes (square, rectangle, trapezoid, and circle) correctly. 2. Students can explain the concept of perimeter and area. 3. Students can determine patterns of figures and numbers using computational thinking skills.
3.	*Galah Panjang*	1. Relations between lines. 2. Perpendicular lines. 3. Parallel lines. 4. Intersecting lines.	1. Students can show the relationship between lines. 2. Students can draw lines based on their relationship. 3. Students can determine the relationship between lines.	1. Students can show the relationship between lines (perpendicular, parallel, and intersecting) correctly. 2. Students can draw lines based on their relationships during the learning process. 3. Students can determine the arrangement of perpendicular, parallel and intersecting lines with computational thinking skills.
4.	*Yeye*	1. Multiplication as Repeated Addition	1. Students understand the concept of multiplication as repeated addition. 2. Students can solve multiplication tasks	1. Students can explain the concept of multiplication as repeated addition. 2. Students can complete multiplication tasks with computational thinking skills.

	with repeated addition.	

Analysis of Student Characters

The research aimed to develop a computational thinking enrichment book for elementary school students. The research was conducted at a public elementary school in Pekanbaru, Indonesia. The fourth-grade students ranged from 9–10-year-olds which means that students were still in the concrete operational stage (Piaget, 1976). Piaget suggested that children during this period are less egocentric; they display the ability to understand concepts and they can solve complex problems (Bjorklund, 2022). In addition, classification is another important characteristic of the concrete operational stage and children at this stage can classify objects into different types using different attributes such as shape, value, and size; children can also consider their associations. Therefore, in organizing learning activities, teachers should consider the level of students' thinking characteristics. Students should learn within a memorable personal experience process through activities provided by the teacher (Webb, 1980).

To be able to facilitate students to develop their computational thinking abilities through memorable experiences, teachers can use traditional games. Traditional games are games that have existed for generations which are part of the regional cultural heritage that must be sustained. In the interview, the teacher stated that students still play traditional games during recess such as playing a game called *Yeye*. With traditional games, students can learn by engaging in direct experience while feeling happy in the learning process. This is in line with traditional games for elementary school students, especially grade four elementary school students who have an age range of 9-10 years. These students are at the concrete operational level of thinking and they need visualizations in order to achieve the expected competencies.

Analysis of Learning Materials

The analysis of learning materials was carried out in order to ensure that the supplementary textbook was aligned with the desired learning objectives. The material was selected in accordance with the analysis previously carried out by researchers, namely by dividing the four traditional games, *Congklak, Setatak, Galah Panjang and Yeye* in accordance with the computational thinking skills, namely decomposition, pattern recognition, abstraction and algorithm thinking. This will be used as a reference by researchers in developing traditional game-based computational thinking supplementary textbooks for elementary school students. The mathematical concepts which will be introduced to the children through traditional games include number operations, great common divisor and least common multiple, flat shapes, and relationships between lines.

Design

Before starting the design stage, researchers collected materials related to mathematics learning that could be developed through the traditional games of *Congklak, Setatak, Galah Panjang* and *Yeye*. The researchers looked for various references related to possible activities connected to computational thinking that were appropriate for elementary school students. The product design was carried out by researchers starting with compiling a map of supplementary textbooks according to the needs of teachers and students. The design of the product began by drafting an initial version of the supplementary textbook manually (written) and then finalizing it using *Canva* and *Microsoft Word* applications. The supplementary textbook comprises 3 parts, namely the beginning, the core section and the final section. The initial part of the traditional game-based computational thinking supplementary textbook contains a cover, preface, table of contents, and introduction to computational thinking. In the core section, the researchers presented the computational thinking skills aligned with the four traditional games; *Congklak*, *Setatak*, *Galah Panjang*, and *Yeye*. The final part of the supplementary textbook contains exercises, references, and authors' information. The evaluation at the design stage carried out by the researchers through self-evaluation and a focus group discussion focused on several activities that needed improvement in order to address computational thinking skills.

Development

At this stage, the researchers started to finalize the supplementary textbook into a real product, which was validated by experts. The following sections present the content of the supplementary textbook and the results of the validation.

Contents of the Supplementary Textbook

The supplementary textbook begins by giving a brief introduction to computational thinking skills. It also contains some explanations of the four computational thinking skills with a link to traditional games (Figure 2).

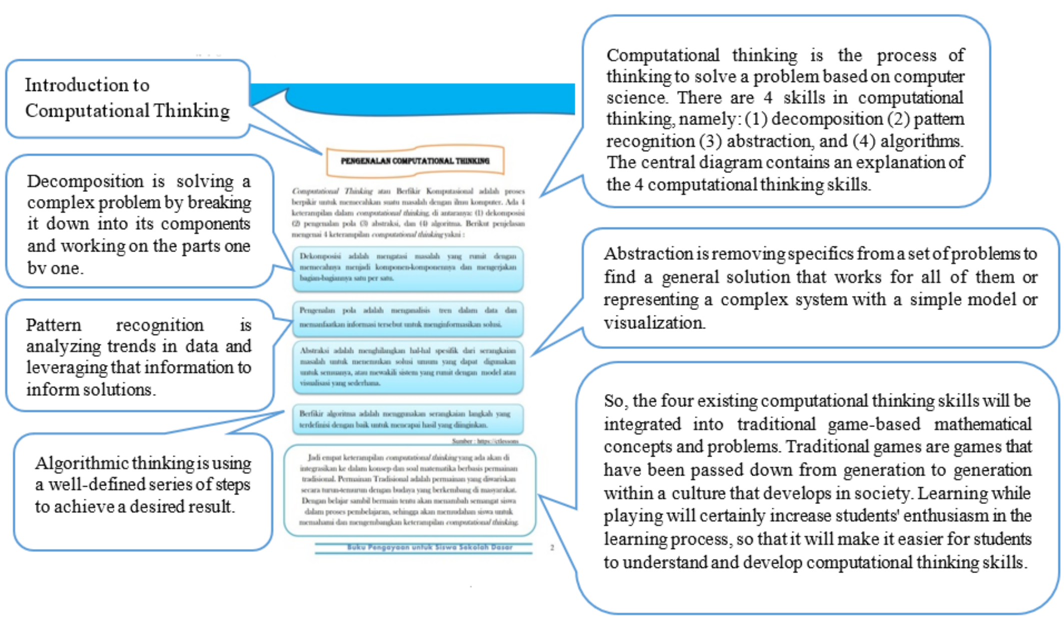

Figure 2. Introduction to Computational Thinking

The core part of the supplementary textbook presents four sections, and each section focuses on one specific computational thinking skill and traditional game. The section is supported by pictures and texts. For example, the first section discusses the traditional game of *Congklak*, so the illustrations and text content presented are also related to this game (Figure 3). The text content presents the history, procedures, and benefits of the game. The subsection contains some material that can be integrated into (introduced through) the traditional game. This includes problems and solutions using computational thinking skills. At the end of each section, there are computational thinking challenges in the form of traditional game-based computational thinking problems that students can solve to promote their problem solving abilities.

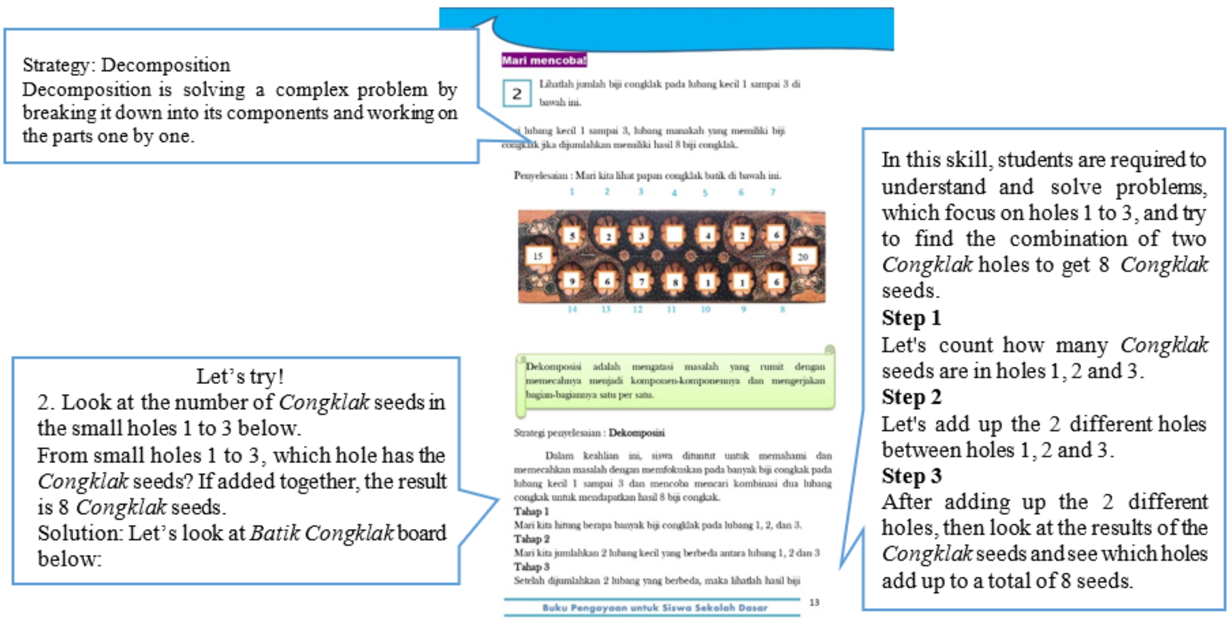

Figure 3. The Core Section of the Supplementary Textbook

At the end, there is an explanation of each computational thinking skill developed in each activity. The exercise at the end of the supplementary textbook presents some challenging tasks that can combine two or more computational thinking skills. Figure 3 presents an example of the task.

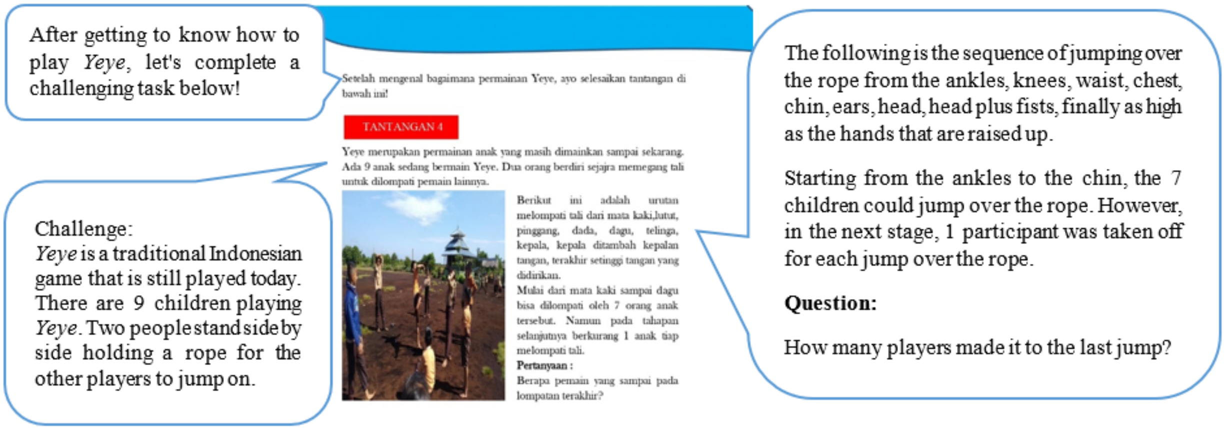

Figure 4. An Example of Tasks in the Exercise Section

Product Validation

Expert validation aims to determine the feasibility of the supplementary textbook developed before testing it on students. Expert validation was conducted by three experts. They evaluated the design and content of the book (Table 3). Overall, the design and content of the traditional game-based computational thinking supplementary textbook achieved the category of highly valid. This means that the supplementary textbook is suitable to be tested on elementary school students.

Table 3. The Results of Validation.

Variables	Score	Criteria
Product Design	0.92	Very Valid
Content of the Book	0.91	Very Valid
Average	0.91	Very Valid

Implementation

The traditional game-based computational thinking supplementary textbook was validated by experts and then tested with 20 fourth grade elementary school students. The first researcher conducted the test with the students. Before conducting the test, the researcher asked permission from the teacher, collected information regarding student background, divided students intogroups and prepared instruments for the data collection. The researcher selected 3 students for the *Congklak* game and 3 students for the *Setatak* game. Then, the researcher selected 8 students for the *Galah Panjang* game and 4 students for the *Yeye* game. At the beginning of the meeting, the researcher provided an explanation of the activities, including a brief explanation about computational thinking skills and how they could be learned through traditional games.

Learning computational thinking skills began with the delivery of material related to computational thinking in line with the supplementary textbook that was developed. The first researcher, acting as the teacher, explained and discussed several traditional games and prompted students to see the connection between these games and computational thinking skills. Next, students played these traditional games in groups in the school field (Figures 5, 6, and 7), except for the *Congklak* game, which was played in the classroom. The first researcher noticed a high level of engagement and enthusiasm between students while they were playing the games.

The *Congklak* game aims to build students' computational thinking skills about abstraction and decomposition. When a student moves *Congklak* seeds (Figure 5), she has to focus on filling her holes and ignore her opponent's holes. In filling her holes, she has to predict which seeds she should carry so that her holes are filled more and so that she does not stop at an empty hole. This is related to abstraction because it requires a focus only on important factors while ignoring other factors.

Figure 5. Students while playing *Congklak*

The first researcher, then linked the game to the computational thinking skills of decomposition by asking students to look at holes 1 to 3 on the *Congklak* and determine the two *Congklak* holes that add up to a certain number. Students started counting and adding. In this case, students were prompted to understand and solve problems by looking for holes that add up to certain results. The researcher explained to students that the activities they carried out were related to decomposition, namely the process of solving complex problems by breaking them down into small components.

The second game was *Setatak* (Figure 6), which included two computational thinking skills; pattern recognition and algorithm thinking. While the students were forming a pattern of shapes, the researcher asked them questions about the pattern that they were forming (one by one) and related them to various flat shapes. The researcher asked the students about the number of different flat shapes they drew, and the students answered that there were three types of shapes. When the researcher asked the students to explain how they knew that there were three types of flat shapes, the students explained that they saw from their drawings. Based on students' actions and responses, the researcher concluded that students recognized the different types of flat shapes based on some properties of the shapes. This is one aspect of computational thinking, namely pattern recognition.

Figure 6. Students playing *Setatak*

After the students had finished forming the *Setatak* pattern, the researcher explained how to play the *Setatak* game for some of them who could not yet form/notice the pattern. Teaching students how to play the game step by step is related to another aspect of computational thinking, namely algorithmic thinking; the ability to use a series of well-defined steps to achieve the desired result. After all participating students understood the game, the researcher conducted a discussion by asking them to count the number of shapes that they had to jump when the *gaco* (a stone used by each player to be put on different *Setatak* shapes) was placed in different positions. In this case, students were invited to explore how many shapes they had to go through from the start to the end of the game.

In the *Yeye* game (Figure 7), the researcher began by demonstrating two computational thinking skills; algorithmic thinking and pattern recognition. After the students took turns playing, the researcher asked the students a question about the activities they had done from start to finish. The students explained the steps they followed in playing the game. The researcher explained that the initial activities they carried out until they reached the highest peak and accordingly won the game, represent an aspect of computational thinking, namely algorithmic thinking. The researcher then asked how many rope jumps had to be passed to get to the highest level. Students started counting and sharing their responses enthusiastically. The researcher then gave a different problem by asking: 4 players were jumping such that, the first player could jump up to their knees, the second player could jump up to their waist, and the fourth player could jump up to the span of one hand. If the third player could jump up to any part, what would it be? Students began to think and search, but experienced confusion. The researcher then intervened by relating each jump to a number. This part of the activity was related to the pattern recognition aspect of computational thinking, where the jumps are up to ankles, up to knees, up to waist, up to chest, up to chin, up to head, up to one-handed span, and up to two-handed span. By giving each jump a code: ankle (1), knee (2), waist (3), chest (4), chin (5), head (6), one-handed span (7), and two-hand span (8), the pattern of the 4 players in the problem is 2,3,...,7. The students had to guess a possible number between 3 and 7.

Figure 7. Students playing *Yeye.*

In the *Galah Panjang* game, the researcher began to demonstrate two computational thinking skills; abstraction and pattern recognition. The activity began with the researcher explaining the game and asking students to work together to draw a playing field using the remaining bricks. Before drawing the field pattern, the researcher asked what tools were needed for the game. Students replied that they would be using the remaining bricks to draw the playing field pattern. The researcher then asked students to look for bricks in the schoolyards. After bricks were collected by the students, the researcher focused students' attention on the "remaining bricks" that would be used to create the playing field. This demonstrated the computational thinking skill of abstraction, which is finding the thing to focus on by ignoring other things. The researcher continued by asking the students to draw a pattern (playing field) using the remaining bricks they collected. While the students formed a *Galah Panjang* pattern, the researcher asked about the shapes that the students were creating and related them to the relationships between the lines that they formed. The researcher also asked how many rectangle shapes should be formed when each team consists of four people and how many straight and intersecting lines are formed. After students finished drawing the playing field, the researcher explained the game (the long pole game) again for those who were not able to participate effectively the first time.

The researcher asked each group to start thinking of playing strategies to win the game because students only knew they could run without looking at their opponents. In this case, the researcher taught students aspects of computational thinking, where building strategic plans, seeking support from teammates, and creating a winning strategy are all activities related to decomposition skills.

After playing, students were asked to complete computational thinking tasks that were in line with the game they had played. Students were able to understand the instructions presented in the supplementary mathematics textbook and were able to solve the tasks presented. The students were also interviewed regarding their experiences learning computational thinking using traditional games. Following is an interview excerpt with student A:

Resercher: How do you feel after learning while playing?

Student A: Very happy!

Resercher: Does learning through playing make you more interested in learning mathematics?

Student A: Yes, I am interested because learning mathematics is very fun.

Resercher: Do traditional games make you understand the mathematics content better?

Student A: The first time I was confused and after playing traditional games I was able to understand it.

Resercher: Can you understand after playing traditional games?

Student A: Yes, because traditional games need to be understood carefully in order to play.

Resercher: What is the most memorable thing you got from learning this subject by using traditional games?

Student A: Through playing *Galah Panjang*, I was happy when playing with friends but none of us won.

Student A was very happy playing the traditional game. She could understand how the game was played, and she was able to understand mathematical knowledge to be learned in the supplementary mathematics textbook. However, we noticed that students did not talk very much regarding the mathematical content that they had learned through traditional games.

Evaluation

Evaluation in this research was carried out formatively. This means that every step of research and development is evaluated. The evaluation carried out was related to evaluation from experts and evaluation of the results of trials with the students.

DISCUSSION

Computational thinking is a skill that is needed in line with current developments in science and technology. On the other hand, local culture must still be maintained so that it does not vanish over time. Therefore, education today requires ideas that integrate knowledge with culture. This research addressed this need by developing a traditional game-based computational thinking supplementary textbook for elementary school students. From the experts point of view, the supplementary mathematics textbook developed has fully met criteria. Therefore, this media can continue to the practicality testing stage with elementary school students.

The supplementary mathematics textbook developed contains four traditional games that are usually played by Indonesian children with four computational thinking skills. The integration of traditional games with computational thinking skills is carried out simultaneously so that students can enjoy the game without realizing that they are also developing their computational thinking skills. The games require prompt problem solving, collaboration, and creative thinking abilities from students (Espigares-Gámez, 2020, van den Heuvel-Panhuizen et al., 2013).

The development of the supplementary mathematics textbook used a research and development approach with the ADDIE model. The first stage was analysis where the researcher analyzed the curriculum, student character, and also learning materials (content). The mathematics curriculum in elementary schools in Indonesia does not yet explicitly contain computational thinking. This is very different when compared with other countries' curricula. For example, in the UK computational thinking has been included in the curriculum since 2013 (Larke, 2019, Manches & Plowman, 2017, Willianson, 2016). However, some of the mathematics content in the 2013 curriculum in Indonesia can be linked to computational thinking and also to traditional children's games in Indonesia. This is in line with the analysis of student development where children at elementary school age still think concretely and semi-concretely (Piaget, 1976). They display the ability to understand conceptual things and they can solve complex problems (Bjorklund, 2022). Additionally, classification is another important characteristic of the concrete operational stage and children at this stage can classify objects into different types using different attributes such as shape, value, and size. Children can also consider their associations. Therefore, in organizing learning activities, teachers should consider the level of students' thinking and its different characteristics (Fatmanissa et al., 2023).

To facilitate students developing computational thinking skills through memorable experiences, teachers can use traditional games. Traditional games are games that have been played for generations and are one of the cultural elements that must be developed and sustained. Through traditional games, students can learn by engaging in direct experience and feeling happy during the learning process. Traditional games for elementary school students, especially those aged 9-11 years, should be adapted to the level of concrete operational thinking. Therefore, students need visualization to achieve the expected competencies.

The design and development stages are two stages that focused on creating and developing the traditional game-based computational thinking supplementary textbook for elementary school students. During these stages, researchers played an important role in integrating computational thinking, traditional games, and mathematical content into the curriculum. The product that has been designed was then evaluated for suitability by experts. This is in line with existing procedures in research and development (Mamolo, 2019). The expert assessment regarding the supplementary mathematics textbook obtained a very strong score so that the product developed has considered to be very effective. Based on the experts' evaluation of the product (the textbook), it was considered ready to be tested with students to examine its practicality.

The supplementary mathematics textbook was tested with elementary school students. The test results show that students enjoy playing traditional games and could understand the mathematical content during the game. This is in line with Sitanggang et al. (2020) who argue that learning while playing will give rise to feelings of joy, and from these feelings, students are interested in understanding the learning being delivered. In our study, students were engaged with a game that required them to develop a strategy to win the game so that some computational thinking skills were developed in solving the problems they faced (Kazimoglu, 2012).

The integration of computational thinking in traditional games is in line with Law Number 5 of 2017 concerning the promotion of culture which aims to protect, utilize and develop Indonesian culture (Republic of Indonesia, 2017). This integration can increase students' knowledge regarding Indonesian culture by utilizing traditional games in learning. In addition, the integration of traditional games promoted students’ understand of the mathematical material. Kurniati (2017) argued that, through games, students can express thoughts and feelings which improve students' cognitive aspects because learning can be well received. We concluded that computational thinking can be introduced not only through technology but also through traditional games.

CONCLUSIONS

The traditional game-based computational thinking supplementary textbooks are feasible to be used as a reference for mathematics learning to develop students' computational thinking skills. Based on our findings, we encourage elementary school teachers to use this supplementary textbook and traditional games to develop students' computational thinking. The validation test results showed that the traditional game-based computational thinking supplementary textbook for elementary school students was very valid. The students gave a positive response by mentioning that they liked the learning process with traditional games. Thus, the traditional game-based computational thinking supplementary textbook is considered to be feasible to use and effective for teachers and students.

This supplementary textbook can help teachers incorporate traditional games into mathematics learning by utilizing it to foster a fun and engaging learning environment in the classroom. Therefore, the researchers believe that teachers’ utilization of the textbook can promote their students' computational thinking skills using traditional games. As for students, this book can help them in developing some computational thinking with(in) a relevant context in the form of traditional games. For further investigation, researchers can refine this traditional game-based computational thinking textbook and can test it at other grade levels. In addition, it can also aid development of other books with variations of computational thinking questions and other traditional games to add more book references for teachers and students.

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Teachers' Efforts to Promote Students' Mathematical Thinking Using Ethnomathematics Approach

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Abstract: *Mathematical thinking is closely related to students' success in solving mathematical problems. Local culture can be a resource to teach mathematical thinking skills. Nonetheless, a lot of teachers still struggle to help students develop their mathematical thinking abilities. This study aims to investigate the ethnomathematics approach applied by the teacher and the types of mathematical thinking that can be promoted through this approach. Case studies are used in this study with three junior high school mathematics teachers as key informants. They were selected based on their experience using the ethnomathematics approach in the classroom. Non-participant observation and in-depth interviews were used as data collection techniques. The findings show that different ethnomathematics approaches, with the use of different local cultures, will lead to different types of mathematical thinking skills. Based on these findings, conceptual and empirical implications are discussed for further recommendations for teachers and school stakeholders.*

Keywords: etnomathematics, mathematical thinking, problem-solving, high school

INTRODUCTION

Teaching mathematical thinking is a challenge for teachers in Indonesia (Marsigit, 2007, Sari et al, 2020, Rohati, et al, 2022). The 2019 and 2023 Program for International Students Assessment (PISA) study results indicate that Indonesian students perform worse than the global average. In mathematics, only 18% of Indonesian students reach Level 2, compared to the OECD average of 69%. Students can understand and identify the mathematical representation of a basic situation at this level. The inability to comprehend context-based tasks and convert them into mathematical problems (Wijaya et al., 2014), the inability to reason and communicate mathematically (Mahdiansyah & Rahmawati, 2014, Rahmawati, et al., 2021), and the challenge of developing mathematical models using symbols (Al Jupri & Drijvers, 2016) are a few factors that are believed to be the cause. Among these are mathematical thinking skills.

LITERATURE REVIEW

There is a strong correlation between students' success in solving mathematical problems and their mathematical thinking (Schoenfeld, 1992, Mason et al., 2010, Ellenberg, 2014). Diverse interpretations exist regarding mathematical thinking. According to Adam (2004), mathematical thinking is a process of reasoning that entails abstraction, generalization, symbolization, and logical reasoning. Mathematization, abstraction and its application, and mathematical sense-making are the processes involved in mathematical thinking, according to Schoenfeld (1992). According to Mason, Burton, and Stacey (2010), the process of mathematical thinking involves specializing, generalizing, conjecturing, justifying, and persuading. Several of these definitions identify mathematical thinking as the mental process that comes into play when applying mathematics to solve problems.

By dividing mathematical thinking into three categories—mathematical thinking attitudes (mindset), mathematical thinking methods, and mathematical thinking content—Isoda and Katagiri (2012) provide a more thorough explanation of the concept. When students think mathematically, they approach problems with an attitude of attempting to develop perspective, ask questions, think with data that can be applied, and strive to convey ideas succinctly and clearly. Abstract thinking, simplification, generalization, specific thinking, symbolic thinking, and thinking expressed in terms of numbers, quantities, and figures are a few examples of mathematical thinking techniques. Examples of mathematical thinking content are the idea of operation, the idea of approximation, the focus on rules and nature, finding the rules of the relationship between variables, and the idea of formulas.

Mathematical thinking activities can be promoted in several ways. It can be characterized as a reflective process that reorganizes the real world using mathematical concepts in an idealized setting (Marsigit, 2007). Mason et al. (2010) suggest that teachers can foster mathematical thinking by creating an environment that involves questioning, posing challenges, and encouraging students to think critically. According to Tall (2008), three main factors lead to the way we think mathematically: the ability to recognize patterns in data, such as similarities and differences; the ability to repeat actions in a sequence until they become automatic; and the ability to use language to explain to enhance our thought processes. Mathematics relies heavily on pattern recognition, which includes number and shape pattern recognition. Learning mathematical procedures requires repetition as well, but it is best if teachers help students understand the reasoning behind the procedures. Students have to understand the rationale behind this initial step as well as the obvious course of action. Students must apply mathematical terms and symbols correctly, concentrate on the important details, and utilize them when explaining.

Using an ethnomathematics approach can help develop mathematical thinking abilities. Students have the opportunity to study mathematics in diverse cultural contexts through the use of ethnomathematics, which broadens their comprehension of the subject's applicability (D'ambrosio, 2006). Empirical data from Utami et al. (2019) study demonstrates that the ethnomathematics approach improves students' critical thinking abilities. Through contextual learning with ethnomathematics, even students with different cognitive levels can be encouraged to develop mathematical problem-solving abilities (Nur, et al., 2020). According to Maasarwe et al. (2012), using artifacts in geometry instruction also improves students' creative problem-solving abilities.

One advantage of utilizing non-routine problem-solving concepts or procedures, similar to those taught in school, to solve mathematical problems in the context of students' socio-cultural environment is that the problems have meaning (Masingila, 2002). This demonstrates how teaching students to apply mathematical reasoning to real-world problems enhances their ability to think mathematically. Teachers must be familiar with the culture to help students understand the connection between mathematics and culture when integrating ethnomathematics into the classroom (Risdiyanti & Prahmana, 2020). The teacher incorporates issues from the students' everyday lives into the development of learning strategies. By applying pertinent cultural activities, these issues are transformed into a deeper understanding of real-life situations (Rosa & Orey, 2010).

Mathematical thinking connects the application of mathematics in culture with traditional mathematical systems (Adam, 2004). To help students better understand the meaning of mathematical symbols used in the classroom, local culture can provide a context for interpretation (Abas, 2001, Gerdes 2011). To further encourage students to actively develop their thinking skills, teachers can also use local culture as a subject of study through hands-on and investigative activities (Maasarwe et al., 2012). Studies on the integration of realistic mathematics instruction with an emphasis on regional culture also revealed that students were able to investigate and expand their understanding of the region, which affected the variety of approaches they employed for solving geometric problems (Nirawati et.al., 2021).

Teachers have to encourage mathematical thinking exercises that connect students' early intuitive understanding of everyday situations with their formal education in mathematics. Despite the favorable opinion that Indonesian teachers have of the ethnomathematics approach (Mania & Alam, 2021), a considerable number of teachers lack the necessary resources and knowledge to plan mathematical thinking activities for their students' math classes (Marsigit, 2007). Teachers control the learning process, which hinders students' development of reasoning, critical, and creative thinking skills (Putra et al., 2020).

Several previous studies (Utami et al., 2019, Nur et al., 2020, Maasarwe et al., 2012) employed the researcher's design to develop learning employing an ethnomathematics approach. We have to understand the attempts made by teachers to teach mathematical thinking skills through the application of the ethnomathematics approach. To enhance learning and help teachers better teach mathematical thinking, knowledge about the efforts of the teachers can be a good place to start. Based on teachers' personal experiences, we can also explore different ways of thinking about mathematics that can be further developed through the use of an ethnomathematics approach.

Therefore, we concern on the following research questions: 1) How do teachers apply the ethnomathematics approach in learning mathematics? and 2) What types of mathematical thinking can be promoted from this approach?

METHOD

This study utilizes a qualitative design or a naturalistic research paradigm, allowing the researcher to apply an interpretive theoretical attitude to concentrate on findings, insights, and understandings (Merriam, 1998).

Design and Procedure

In Indonesia, ethnomathematics is a relatively recent phenomenon and is rarely utilized in the classroom by teachers. Case studies are suitable for analyzing modern phenomena, claims Yin (2014). Teachers who have implemented the ethnomathematics methodology can serve as case studies. Research questions can be addressed by examining what renders teachers who utilize an ethnomathematics approach special or different (Stake, 2009). Every informant experiences the same process. The researcher observed classroom teachers who employ an ethnomathematics approach. The researcher then conducted an in-depth interview with the teacher to gain further insight.

Research Context and Key Informant

Two junior high schools in Indonesia's Yogyakarta Special Province and one junior high school in the Central Java Province served as the research locations. Javanese culture is the prevalent cultural context at the research site. Yogyakarta is classified as a special region because it is a royal province governed by the Sultan. A governor's decree on culture-based education was released by the Sultan of Keraton, Jogjakarta.

Three teachers of mathematics served as the key informants. By selecting junior high school math teachers who were considered to have sufficient training and experience in applying the ethnomathematics approach, the informants for the research were deliberately selected as the sources of the data. The criteria for interview subjects involve junior high school math teachers who have been teaching for at least five years and who have employed the ethnomathematics approach for a minimum of one academic year.

The term “ethnomathematics” was introduced to the first informant during the 2015 National Seminar. She was motivated to utilize the internet to research ethnomathematics further and attempt to further refine her theories for use in the classroom. Since 2014, the second informant has been drawn to the ethnomathematics approach and has felt compelled to contribute to the preservation of Javanese culture. Subsequently, she performed an ethnomathematics investigation by mapping out the locations of cultural buildings or community centers, researching Javanese culture online or consulting with more knowledgeable colleagues, and creating her educational resources. The supervisor socialized the third informant about cultural values in education following Yogyakarta Province Governor Regulation Number 66 of 2013, and the informant then engaged in culture-based learning. The school also designated the development of local culture as one of its featured programs. Every Friday, mandatory Javanese language classes are held. Other activities include traditional dances, music, and crafts, as well as batik and crafts. Inspired by these developments, she attempted to understand how to implement culture-based mathematics education, including conducting online research for resources. She learned about the ethnomathematics approach through this search.

Data Collection

The objective of observation is to gather data on the application of the ethnomathematics approach for encouraging mathematical thinking in the classroom. Through non-participant observations, the researcher investigates three areas: 1) how local culture is used to teach mathematics, 2) how the learning flow works, and 3) what challenges teachers have when teaching mathematical thinking. Video is employed for recording the learning process so that it can be viewed again. Field notes are used to document the observations' findings.

In-depth interviews are conducted by the researcher with the following questions in consideration: 1) Why is this method of integrating ethnomathematics into learning utilized? 2) How does this method encourage the development of the mathematical thinking process? 3) What challenges do teachers have when it comes to imparting mathematical thinking? To obtain more detailed information, questions are created. Tape recorders are employed to record interviews, which occur in two or three sessions lasting at least sixty minutes each. See Table 1.

Table 1. Instrument Details.

Instrument	When	Why
Field Notes	1. While observing the teacher teach 2. While watching a video recorded	1. To formulate intriguing interview questions 2. To describe how teachers implement the ethnomathematics approach
Unstructured interview	After observing the teaching session and after watching video-recorded	To understand teacher perceptions and ideas about teaching mathematical thinking

The researcher verifies the validity of the research data by comparing the information discovered from the observations and the interviews, interpreting the results following pertinent theories, and then presenting the findings to the informants for responses and clarifications.

Data Analysis

Qualitative data were analyzed using the model developed by Miles and Huberman (1994) encompassing data reduction, data presentation, and conclusion. To draw conclusions and answer research questions, the data collected from field notes and interviews were evaluated in three phases.

Phase one involves reading field notes to identify key learning moments and connecting them to pertinent theories to extract keywords. Every keyword has a code. To identify patterns or relationships, similar codes are grouped. Information not relevant to the study objectives is eliminated. The interview data undergoes the same process. For instance, the teacher could provide students instructions to search for traditional food examples that feature specific geometric shapes in their written field notes. Subsequently, the teacher creates a table on the whiteboard to record the outcomes of the class discussion. Finding examples of traditional foods is a crucial step at this point. The analysis produces the keyword "find", which is then assigned a learning flow code. Since they have no bearing on the primary subject of the research, details about the activities teachers engage in when explaining tables on the whiteboard are deleted.

Phase two involves verifying the field note analysis results with the interview analysis results. To enable the drawing and verification of conclusions, the information will be refined, arranged in a sequential manner, and concentrated on the research questions. To obtain a more detailed description of the learning flow and to understand its reasoning, for instance, the results of the interview analysis about the reason the teacher placed the activity at the end of the stage corroborate the field notes analysis results with the keyword "find" in the learning path.

We developed a code matrix (Table 2) to categorize the various forms of mathematical thinking. The Tall (2008) foundational set of mathematical thinking and the Ishoda and Katagiri (2012) type of mathematical thinking are the foundations upon which the matrix is constructed.

Table 2. Code Matrix for Type of Mathematical Thinking.

Mathematical Thinking Related to Attitude	Mathematical Thinking Related to Mathematical Method	Mathematical Thinking Related to Mathematical Content
1. Attempting to grasp one's problems or objectives or substance clearly, by oneself (A1) 2. Attempting to take logical actions (A2) 3. Attempting to express matters clearly and succinctly (A3) 4. Attempting to seek better things (A4) 5. Recognizing patterns, similarities, and differences (A5) 6. Repetition the action sequence (A6) 7. Language to describe (A7)	1. Inductive thinking (M1) 2. Analogical thinking (M2) 3. Deductive thinking (M3) 4. Integrative thinking (M4) 5. Developmental thinking (M5) 6. Abstract thinking (M6) 7. Thinking that simplifies (M7) 8. Thinking that generalizes (M8) 9. Thinking that specializes (M9) 10. Thinking that symbolizes (M10) 11. Thinking that express with numbers, quantifies, and figures (M11)	1. Idea of sets (C1) 2. Idea of units (C2) 3. Idea of expression (C3) 4. Idea of operation (C4) 5. Idea of algorithm (C5) 6. Idea of approximation (C6) 7. Idea of fundamental properties (C7) 8. Functional Thinking (C8) 9. Idea of formulas (C9)

In one learning exercise, for instance, students are asked to investigate an image of an artifact and then identify the artifact's nets to calculate its surface area. After reviewing Tall (2008) or Isoda and Katagiri (2012) mathematical thinking attributes, we ascertain which of these attributes are fulfilled by the thinking exercises. A code for the task was provided by each researcher. For instance, regarding the concept of units, Isoda and Katagiri (2012:71, author's highlight)

... figures are comprised of points (vertices), lines (straight lines, sides, circles, and so on), and surfaces (bases, sides, and so on). For this reason, thinking that concerns on these constituents, unit sizes, numbers, and interrelationships, is important.

The researcher provided the code C2 for the learning activities. We discussed to agree on the appropriate type of mathematical thinking.

During the third phase, we present the verified data for each case in an understandable table based on the research focus. The study's conclusions are derived through a cross-case synthesis since each case is independent of the others or does not impact others. Yin (2014) argues that we utilize argumentative interpretation to conclude after comparing and observing each profile to determine if there is a difference. Finally, thorough activity descriptions and quotes from teacher-student discussions are provided to further corroborate the findings.

RESULTS AND DISCUSSION

The results are presented initially for each informant teacher individually, denoted by the initials Case 1, Case 2, and Case 3. To address the research questions, a summary that compares and contrasts the informant teachers is then provided.

Case 1. Ethnomathematics Approach for Teaching Three-dimensional Shape

First, the teacher employs real objects, such as cans, plastic balls, and cones, to assist the ninth-graders in learning about three-dimensional geometric shapes. Students are first reminded of the prior subject by the teacher.

Teacher: How many three-dimensional shapes did we learn this semester?

Students answer: Three, ma'am. Cone, cylinder, and sphere.

Teacher: That's what we will learn today. Do you think there's any benefit in learning three-dimensional shapes?

Student: Yes, ma'am

Teacher: If there is, what are the benefits? (Students are silent).

This exchange demonstrates that the teacher's initial objective for the class was for them to comprehend the advantages of learning about three-dimensional shapes. Students struggle to provide examples of the advantages that three-dimensional shapes offer, despite their awareness of these advantages. Then, using questions to elicit responses, the teacher engaged the class by outlining the significance of learning about this subject.

Teacher: What is an example of a cylindrical object?

Student: Water container, ma'am...

Teacher: Well, if we want to know how much water is in a container, how do we do it?

Students: Calculate the volume of water in the container, ma'am.

Teacher: Well, we can calculate the volume of water using the volume of the container. That's one of the benefits of learning to the cylinder, what else?

Students: Calculate the surface area, ma'am.

Teacher: Have you ever seen the label on canned milk? If you remove the label, you can calculate the area.

Utilizing examples from daily life, Case 1 inspired students to learn about the advantages of studying three-dimensional shapes. Questions and answers based on students' experiences explore what aspects of mathematics are present in the objects around them.

Afterward, the teacher divides the class into groups of three or four people. To avoid confusing the class, the teacher told them not to open their books. Students were instructed to concentrate on group-based activities. By accomplishing this, students will be able to articulate their thoughts based on the tasks completed rather than the theory covered in the text.

Teacher: Do you know any examples of three-dimensional shapes around you?

Students: Examples of cylinders are milk cans, pipes... Examples of cones are caps, steamers, tumpeng, boiled peanut wrappers... Examples of spheres are basketballs.

Teacher: I have prepared three objects of different shapes. Your task is to observe the objects, then draw and fill in a table.

To determine how many sides, vertices, and edges an object has, the teacher assigns the students to discuss in groups. They are required to sketch the item as well. The teacher then asks the class to name the traditional dish that is in the same shape.

Each group received one item from the teacher. After about five minutes of observation, the second object changes places with the other group. After five minutes of observation, the second object was traded for the third object with the other group. This was accomplished to provide every group with an opportunity to observe three different kinds of objects.

The teacher designates a group representative for the presentation once the group discussion is considered to be finished. The teacher pays attention to mistakes made in the abstraction of geometric images throughout the presentation. The use of curved lines on an abstract image of a sphere is corrected by the teacher. Dotted curved lines should be used to depict the inside of the sphere to highlight invisible areas and denote a three-dimensional abstraction. Furthermore, the cone abstraction drawing's use of curved lines is corrected by the teacher. The definition of sides, vertices, and edges is subsequently employed by the teacher to explain the components of a cylinder, sphere, and cone. The teacher seemed to place a high priority on learning mathematics as a symbolic language, particularly when it came to writing.

Case 1 does not utilize local culture as a starting point for teaching, or not as a content base to explore its mathematical aspects. The teaching flow is presented in Figure 1.

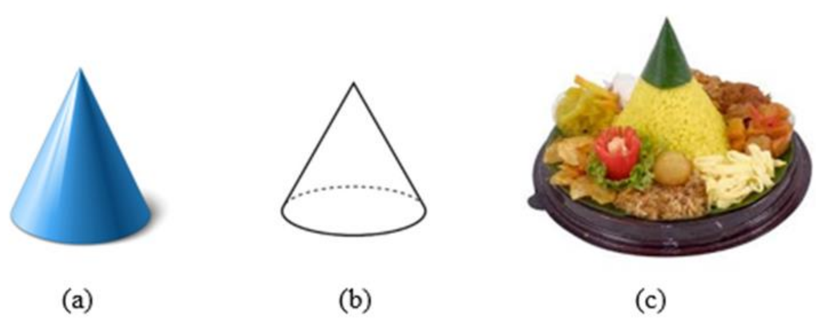


Figure 1. Teaching flow of Three-dimensional Shape

Props serve as a mathematical model and are the first teaching tool (a). Utilizing props instead of bringing in traditional food is more cost-effective for the teacher. The teacher's job is to explain mathematics in the context of symbols while assisting students in actively developing their understanding through discussion (b). Finally, the students are instructed to locate traditional foods that share the same shape (c).

Promoted Type of Mathematical Thinking

Since she believes that students learn mathematics by comprehending symbolic language, Case 1 does not examine any cultural aspects of mathematics. As a result, the teacher begins by outlining the components of three-dimensional shapes before connecting them to customary foods that share that shape. Using props, the teacher encourages students to think mathematically. The teacher assists the students in abstracting from actual objects to image forms or from three-dimensional to two-dimensional forms. Moreover, the abstraction is restored to its original form as an artifact.

Case 1 illustrates the application of abstract thinking to mathematical problem-solving. Students use concrete objects to think at first, then they abstract the objects' physical forms to create figures. Students employ recognition strategies to search for shape similarities between props and traditional foods when they recognize those foods in the same form.

A wrong answer could start a traditional discussion; thus, the teacher selected a group whose answers were incorrect to present. Other students disputed the presenters' understanding, which believed that cones are not symmetrical.

Questioning student 1: Why the cone does not have an edge? I think there is one.

Presenter: Wrong! ... the cone has no edges.

The student's explanation of why the cone lacks edges was not logical, and the presenter's explanation failed to make use of mathematical arguments. The same situation happens as in the following dialogue example.

Questioning students 2: In my opinion, there are two faces, how come only one?

Teacher: How are the other groups? Agree if the cone side is two? The presenter group accepts?

Presenter: Oh right ... I forgot to count the base.

The teacher did not direct attention toward correcting the presenter's reasoning during the presentation or the student-led question and answer sessions. The presenters' justifications have not been supported by mathematical evidence. The teacher did not refute the students' nonsensical claims when they clarified and addressed queries. Students may be inspired to think mathematically by logical arguments.

Based on the interview, the teacher realizes that for students to be able to argue logically they also require an understanding of language logic "*......sometimes there are but only one or two students who are quite special and can solve problems in a way other than mine, but it's very rare. When* *it comes to reasoning models, it is connected to language skills, for example, the result of subtracting this from this, but they are used to front minus back*"

An interesting case appears in the cone presentation dialog. There is a question from a student triggering a polemic in classical discussions.

Student: I am still confused about the vertex point. Is this the vertex point, ma'am?

Teacher: In the book, there are differences about cone vertices. Some books mention that the cone has one vertex, but some books say that the cone has no vertices, but edges.

According to Abrahamson (2009a), there needs to be a bridge between formal mathematical knowledge and intuitive knowledge. Liaisons are necessary, from an epistemological perspective, to resolve cognitive conflicts resulting from disparate perspectives on the same object. Regarding the cone's vertices, one intuitive perception is that it encompasses a single point, which is the pointed part at the top. However, another view maintains that the cone has no vertices at all since no angle is formed when two lines intersect. This other view is less intuitive but more formal. Students who experience cognitive conflict are more likely to develop mathematical thinking attitudes.

In this instance, the student demonstrates the mathematical thinking attitude of attempting to completely understand the subject matter. To elevate his thinking from a concrete to an abstract level, the student assesses and enhances his thinking. The portion of the cone that appears pointed is the vertex, as the student understands it intuitively. Formally speaking, though, a vertex point is where two edges converge. While the teacher's explanation is grounded in formal mathematical knowledge, the student's perception is based on his observations. To comprehend the vertices of the cone, the teacher and student engage in a negotiation-based educational dialogue.

Case 2. Ethnomathematics Approach for Teaching Surface Areas of Prisms

In the teaching of surface areas of prisms for grade 8, the teacher uses a picture of a building in the Yogyakarta Palace called *Kedhaton*, as shown in Figure 2.

Figure 2. Kedhaton Building of Sultan Palace

The teacher identified the lesson's objectives, which are to compute the surface area of the prism and determine the formula for a vertical prism's surface area. She then reviewed the names of the different three-dimensional shape forms through questions and answers.

Teacher: Do you know what shape the building in the palace garden is?

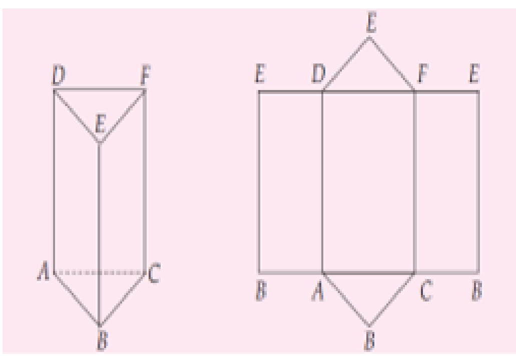
Student: Octagonal prism, ma'am.

Teacher: How did our ancestors know about octagonal prisms?. Students do not respond, they just listen.

The teacher illustrates the building's mathematical features, which include an octagonal prism without a base or roof. She provides that The King meets with royal employees in this building, which serves that purpose. The teacher then poses a problem,

Teacher: If the length of the base edge is 1.2 meters and the height of the building is 2.8 meters, how do you calculate the surface area of the building? To answer the problem posed, you need to understand the concept of the surface area of a prism.

The teacher utilizes a picture of a triangular prism and explains the formula analytically, as demonstrated in Figure 3.



The surface area of Prism = area of base plane + area of top plane + area of the lateral surface
= 2 × area of two planes + area of lateral surface
Area of lateral surface = AB × t + BC × t + AC × t = (AB + BC + AC) × t = (circumstance of base) × t
Surface area of Prism = (2 × area of base) + (circumstance of base × height)

Figure 3. Analytical Explanation of Surface area of Triangular Prism

The teacher elaborates the explanation into manageable steps, beginning with the composition of the prism's surface area, which is comprised of the total area of the sides. Afterward, the formula for the area of each side, expressed as a rectangle or triangle, is discussed to arrive at the prism's surface area total. To make sure the students comprehend the explanation, the teacher asks questions and provides answers. The teacher then provides an example of the problem of calculating the surface area of a prism and demonstrates how to solve it after the students have been assessed to ensure they understand the formula. The problem is that "*a prism whose base is in the form of an isosceles triangle has two equal sides of 10 cm and another side that is 12 cm. If the height of the prism is 15 cm, without drawing first, determine the surface area of the prism*".

The questions are displayed first, and then the teacher encompasses the students to solve the problem by asking questions to make students think, for instance "*Which one has a length of* 15?"; "*Where did the number* 8 *come from*?". The students answer these questions with tremendous enthusiasm; in addition to answering, they can also justify their answers with arguments. The teacher assigns a worksheet for group discussion. Students are provided with word problems regarding the prism's surface area to solve in the context of the local culture. The teaching flow is presented in Figure 4.

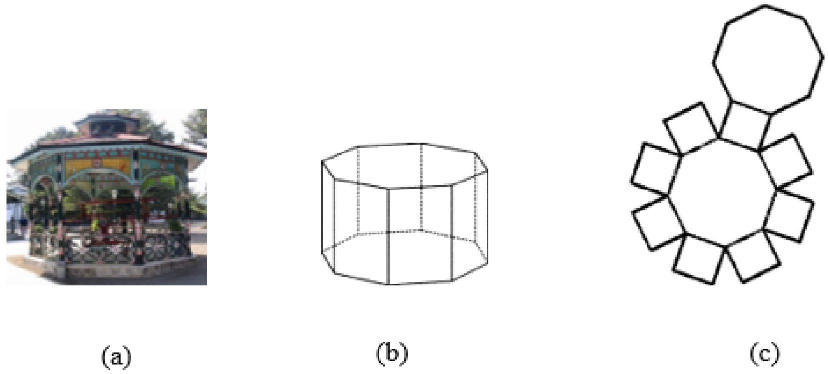


Figure 4. Teaching flow of Surface Area of Prisms

The teacher presents an octagonal prism-displaying cultural artifact (a). Based on the interview, we discovered that she generates students' attention to or interest in learning with a picture of the Kedhaton building. The geometry abstraction model is presented by the teacher as an illustration (b). The assignment for the students is to recognize the nets of an octagonal prism (c).

Promoted Type of Mathematical Thinking

Using images of artifacts, the teacher encourages students to employ mathematical skills. The teacher then assists students in identifying the elements and nets of the artifact. Students can comprehend the idea of surface area because artifact photos are utilized early in the lesson. Subsequently, the teacher uses an abstract model to explain the surface area formula. The teacher-student conversation is necessary to help the students comprehend the meaning of symbols.

There is a "leap" from the octagonal prism that is the Kedhaton building's physical form to the abstraction of a triangle prism and the formula for its surface area. The teacher no longer discusses the Kedhaton building's physical form; instead, she demonstrates how to obtain the triangular prism formula's surface area analytically. According to Abrahamson (2009b), it's a semiotic leap. The inquiry process is started when the perspective of the Kedhaton building transforms to an analytical justification of the triangular prism's surface area formula. Students investigate triangular prisms after attempting to understand octagonal prisms intuitively. These are inquiry activities, encompassing intimations, and implementations, which are critical (Sfard, 2002).

The teacher can use another artifact in the form of a triangular prism but she doesn't. Based on the interview, the teacher does not realize the need for interrelated learning flows so that there is no leap. The inquiry thinking process that occurs because of the leap is not designed by the teacher. In addition to not using the Kedhaton building to obtain the surface area formula, the teacher also does not discuss the Kedhaton's building surface area. This is a result of the teacher providing the formula explanation priority because she believes that her time is limited.

Because the teacher uses a question-and-answer format in her explanation, the students do not seem to have any difficulty understanding it and can complete the worksheet's questions. Questions and answers encourage students' attitudes to think mathematically. When responding to the teacher's questions, they attempt to base their answers on evidence or presumptions. They also use abstract thinking, which is a type of mathematical thinking. Students first use the artifact's image to guide their thinking before abstracting the wall's shape to create an octagonal prism model. Students then consider describing an octagonal prism net using an octagonal prism abstraction. Students are encouraged to think in terms of numbers and figures when the teacher presents a contextual problem.

The following is a conversation excerpt that illustrates the question-and-answer technique used by the teacher to encourage students to think mathematically.

Teacher: What is the formula for the surface area of a prism?

Many students raised their hands, and the ones the teacher pointed at were able to provide the right response.

Teacher: Do you remember, yesterday I showed you a picture of the building in the Yogyakarta Palace Garden? What shape is the building?

Student: Octagonal prism.

Teacher: If the length of the base edge is 1.2 meters and the height of the Kedhaton building is 2.8 meters. How to calculate the surface area of a Kedhaton building?

One of the students said the surface area of an octagonal prism without a base and a roof is the length of the base times the height,

Student: Because the length of the base edge is 1.2 meters and the octagonal prism has 8 base edges, the length of the base is equal to the sum of each base edge length. So the surface area of the Kedhaton building is obtained from the length of the building base times the height of the building, namely $(1.2m+1.2m+1.2m+1.2m+1.2m+1.2m+1.2m+1.2m) \times 2.8m$.

Teacher: There is another way, isn't there?

The teacher proceeded to guide the students by demonstrating that the Kedhaton building's upright side has a rectangular shape (Figure 5). The form of the Kedhaton building's upright side was described by her as follows on the whiteboard.

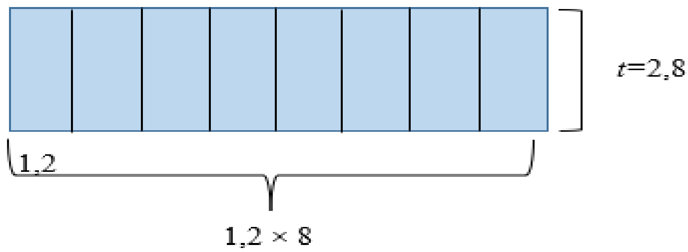


Figure 5. Calculation the surface area of a Kedhaton building

> Teacher: Instead of counting one by one. Take a look, the length of the base is 1.2 meters. The Kedhaton building is shaped like an octagonal prism without a base and a roof, so the length of the edges around the building is 1.2 *m* × 8 = 9.6 *m*. The surface area of the Kedhaton building is 9.6*m* × 2.8*m* = 26.88 m^2. So, the surface area of a prism without a base and a roof is equal to the perimeter of the base times the height.

Prism abstraction is the method the teacher employs to ensure that the students comprehend the symbols for the length of the base, the length of the edge, the relationship between symbols when they are presented with multiplication signs, the relationship between the surface area of the prism and the concept of the base's circumference, and so forth. Ishoda and Katagiri (2012) state that thinking symbolically entails a student's attempt to use symbols to express problems and make references to objects that have symbolic meaning. This occurs when learning how to comprehend the teacher's explanation of the formula for a prism's surface area when height is represented by the letter *t* (*tinggi* in Indonesian).

The teacher utilized dialogues and questions to motivate the class to argue with one another throughout the lesson. This indicates that analytical, problem-solving, and mathematical communication exercises—both oral and written—develop students' thinking skills. Based on prior mathematical knowledge—for instance, the surface area of a prism—the students addressed the teacher's problems by applying what they believed about Pythagorean triples and the areas of squares, rectangles, and triangles. In addition to teaching mathematics, the teacher includes the class in problem-solving exercises. To solve problems in the local cultural context, students are guided to be able to apply previously understood formulas through analytical explanations using geometric abstraction models.

When experienced with a shortage of teaching time, Case 2 prioritizes teaching mathematical structures and concentrates on content-based instruction. Early on in the learning process, local culture is used to inspire students and provide the mathematics material they will study later context and significance. Then, mathematical concepts are examined via their symbols. The teacher wants each student to become an expert in mathematical symbols utilizing arithmetic skills, calculation procedures, formula memorization, and symbolic reasoning. She explains while guiding students through interactive question-and-answer sessions that actively develop understanding. In this instance, the teacher introduces the symbols to the class first, and they jointly develop the meaning through conversation. Teachers' teaching practices remain focused on content to achieve material targets by the planned schedule and demands on the curriculum.

Case 3. Ethnomathematics Approach for Teaching Circle Intersections

The kawung batik motif (refer to Figure 6) is a widely recognized teaching tool for eighth-grade students when learning about circle intersections.

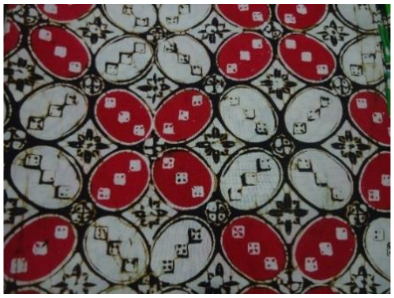


Figure 6. Batik motif Kawung

The teacher employed online resources to explain the origin of the kawung motif, stating that it was inspired by the shapes of four blossoming lotus petals or the kolang-kaling fruit, additionally referred to as coconut fruit. Then, a geometric arrangement of the motifs is established. The teacher displays a drawing of a batik kawung, similar to Figure 7.

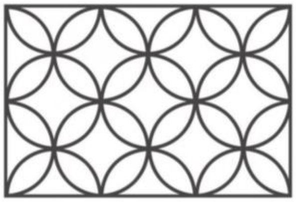


Figure 7. Batik motif kawung sketch

Teacher: How many circle patterns are there in the motif?

Student: Seven, ma'am.... four, ma'am.... (answer).

Teacher: Yes, ...let's see how the kawung motif can be produced from circle intersections.

The teacher elaborates on the blackboard on how the intersections of the circles produce the kawung motif, as presented in Figure 8.

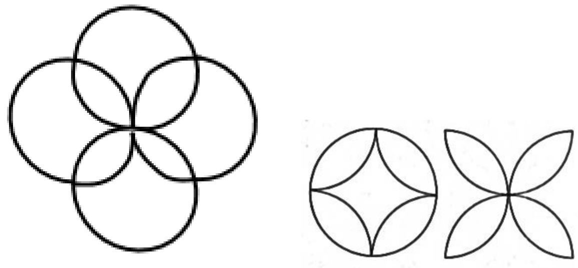


Figure 8. Batik Kawung as the intersection of the circle pattern

Teacher: So, how many circle patterns are there for the correct answer?

Student: Seven, Ma'am (in unison).

Drawing a kawung batik motif by joining circle patterns was the unstructured independent task assigned by the teacher. Students can design their own motifs. In addition to using the standard kawung pattern and color, students are free to add additional ornaments and colors to the pattern. Figure 9 displays a selection of student work.

Figure 9. Sample of Student work

The way that the teacher teaches students to sketch the kawung batik motif differs from how batik artisans do it. An alternative method for drawing a kawung batik motif involves drawing a square and a diagonal line, followed by the petals on the diagonal line (refer to Figure 10).

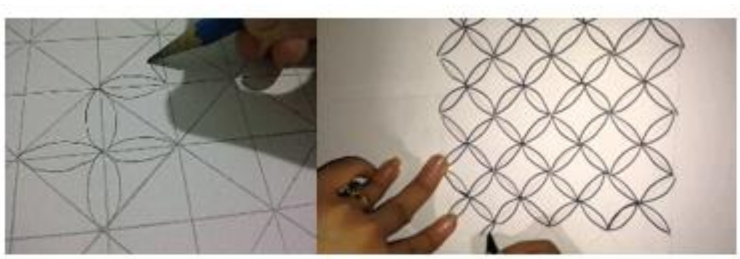


Figure 10. How Batik Artisan makes sketches of Kawung Motif

Through batik projects in after-school activities, students have acquired this technique. The teacher wants to convey to the students that a kawung motif shape can be generated when two circles intersect.

The teaching flow is presented in Figure 11.

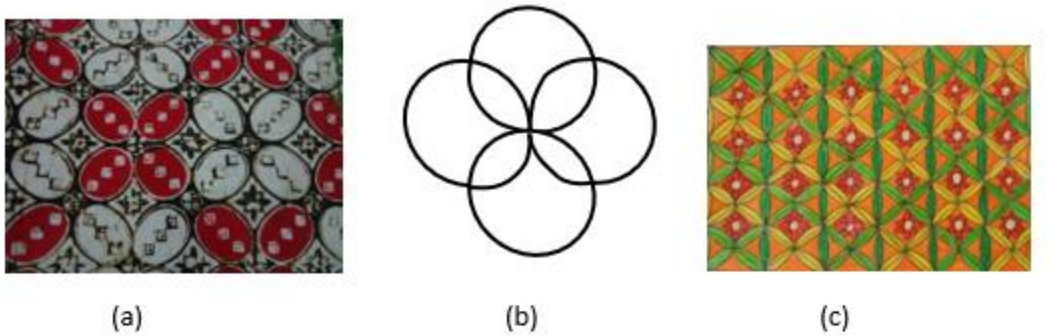


Figure 11. Teaching flow of Circle Intersection

Students are asked to examine how these motifs are applied mathematically, starting with cultural artifacts like batik Kawung (a). The teacher points out to the class that the kawung batik motif features a pattern of circles that are cut into or intersected by one another. After learning this pattern, students should understand that the ellipse, which serves as the kawung motif's fundamental shape, is the result of the intersection of several circle patterns (b). Students are then allowed to practice creating their kawung batik motifs, which further solidifies their understanding (c).

Promoted Type of Mathematical Thinking

Based on the interview, students understand the concept when using the local cultural context, but if the problem context is changed, students experience difficulties. The teacher recognized and comprehended this circumstance since the students were learning mathematical formulas through a direct approach, and their thinking abilities were at an intermediate level. Instead of teaching students how to discover formulas, teachers provide them to them along with examples of how to use them to solve mathematical problems. The teacher gives priority to the material that is simple for her students for them to master mathematics. The teacher also discussed how learning mathematics requires students to use not only their minds but also their emotions and will, which are expressed through action as a manifestation of their ideas and will.

The process of creating crafts involves mathematical thinking and activity (Gerdes, 2014). Teachers use cultural artifacts as a stepping stone for learning to help students develop their mathematical thinking skills. By repeating the steps involved in creating their batik, students can identify patterns on objects and practice creating them themselves. To create a kawung batik motif, they repeatedly create many circles intersection patterns. These activities are completed in the order listed. Tall (2008) identifies pattern recognition and sequence repetition as mathematical thinking.

Students do not independently research the integrated or intersecting circle pattern on the batik kawung; instead, the teacher demonstrates it to them. Students are not encouraged to refine their mathematical thought processes with this approach. The teacher said that, "*The students are* *enthusiastic when learning mathematics through practice. For example, on cube learning I invited them lèsèhan, while decorating the cubes they made, the opposite sides are given the same motif. But when the prism is pyramidal, they are confused. I've also used batik to teach the elements of a circle, I let them make other kawung batik motifs, but what the teacher teaches is what they imitate, the only difference is the color and decoration*."

Note: *Lèsèhan* (in the Javanese language) means sitting on the carpet or not employing a chair and for Javanese people, this way of sitting feels relaxing.

When given the challenge of creating kawung motifs out of circles, students will be motivated to expand their mathematical thought processes. By independent research, students can attempt to identify a solution. After that, students are free to share their ideas and solutions. Students can develop their mathematical understandings and be able to explain their work by engaging in investigation activities (Adam, 2004), which can foster mathematical thinking that is facilitated by the cultural context (D'Ambrosio, 2001).

A comparison of these overall findings is demonstrated in Table 3 as a summary of the three cases, based on research questions.

Table 3. Summary of The Three Cases.

Case	Teaching Flow with Ethnomathematics Approach	Type of Mathematical Thinking
Case 1	The teacher asked students to: 1. Observe Props (three-dimensional shapes) 2. Identify the number of edges, the number of vertex points, and the number of sides 3. Draw abstraction of the props 4. Discover traditional food with the same shape	Student attitude: ▪ Attempt to understand about substance clearly ▪ Strive to escalate thinking from a concrete level to an abstract level and evaluate their thinking to refine ▪ Recognize to identify the similarities in shape between props and traditional food Mathematical thinking related to the mathematical method: ▪ abstract thinking Mathematical thinking related to mathematical content: ▪ Focusing on constituent elements (units) and their sizes and relationships (Idea of units) ▪ Focusing on basic properties
Case 2	Teacher: 1. Display a picture of the Kedhaton Building as an	Student attitude: ▪ attempt to think based on the data or assumption

	Octagonal Prism Model 2. Ask students to identify the Kedhaton building elements 3. Ask students to identify nets of an octagonal prism 4. Explain the surface area of the prism by using an analytical approach 5. Explain an example of a problem and how to solve it 6. Apply the question-and-answer technique 7. Ask students to discuss word problems with cultural context	▪ describe the way they think about Kedhaton Mathematical thinking associated with the mathematical method: ▪ abstract thinking ▪ thinking that express with numbers and figures ▪ thinking that symbolizes Mathematical thinking related to mathematical content: ▪ The idea of a unit when identifying artifact elements ▪ Functional thinking when determining the relationship between octagonal prism and triangular prism ▪ Idea of operation when solving word problems with cultural context
Case 3	The teacher asked students to: 1. Recognize patterns in batik motif 2. Observe how to make the motif from the intersection of the circle 3. Draw batik by applying a circle intersection	Student attitude: ▪ recognizing pattern ▪ repetition of sequences of action Mathematical thinking associated with mathematical content: ▪ The idea of a unit when identifying the component of batik (circle, line, etc.) ▪ Functional thinking when finding the relationship between the batik motif and the intersection of the circle

Through the use of artifacts, the three teachers employ an ethnomathematics approach. Case 1 encourages students to identify cultural practices by using their understanding of the components of three-dimensional shapes. To help students better understand how mathematics is employed in their culture, teachers should be able to investigate more topics related to traditional foods and mathematics.

Gaining knowledge of ethnomathematics serves as a foundation for improving comprehension of school mathematics concepts (Gerdes, 1996). For various reasons, Cases 2 and 3 employ this strategy. Case 2 draws students' attention to the local culture at the outset of the lesson and uses it as a stepping stone to formal mathematics.

The teacher utilizes a question-and-answer format in addition to providing extensive explanations of mathematical concepts to encourage students to share their understanding. This finding is consistent with research by Kurniasih and Hidayanto (2022), which discovered that studentcentered activities like asking for an explanation of their thought processes and a justification for their reasoning demonstrate a teacher's efforts to encourage mathematical thinking skills.

Case 3 presents mathematics as a product of learning while accounting for the cognitive capacities of lower-middle-class students. Practices involving student creation replicate the process of mathematical discovery. However, because they are still developing their reflective thinking skills, students should only learn inductive mathematical concepts or principles.

The problems that Case 3 encountered demonstrate that while motivating students through familiar local culture can be effective, motivation is insufficient to help them grasp mathematical concepts. Students must possess mathematical qualities that call for the use of deductive reasoning. They must be able to accurately represent the knowledge acquired under somewhat different circumstances. In addition, basic mathematical abilities like counting have an impact on their capacity to resolve the presented issues.

CONCLUSIONS

According to the study, teachers implement the ethnomathematics approach to teach mathematics in a variety of ways. This includes: 1) utilizing the local culture to motivate students to learn the subject and create a sense of purpose in their learning; and 2) utilizing the local culture as a mathematical object, the context for mathematical problems, and real-world examples of mathematics.

The teachers' ethnomathematics approach, which incorporates local culture and learning flow, fosters the development of diverse mathematical thinking styles. The application of investigation and problem-solving in the ethnomathematics approach stimulates mathematical thinking more. Encouraging students to think mathematically through similarity recognition is an ethnomathematics approach with a learning path that initiates with the understanding of school mathematics knowledge and then looks for examples in cultural practice. The action sequence is repeated by the students with the assistance of an ethnomathematics approach and practice in batik making. It helps to promote meaningful learning and practical thinking to introduce local culture at the beginning of a lesson. It encourages students to participate in inquiry-based learning even though local culture serves as a stepping stone for formal mathematics. Using artifacts to teach mathematics will encourage mathematical thinking related to the concept of a unit.

The design of instruction using an ethnomathematics approach can benefit from further development of these findings. Teachers must examine mathematical concepts in the context of the local culture to help students connect with their culture on a deeper level. Mathematical expression in culture can take many forms, encompassing concepts and customs that permeate everyday life in addition to tangible objects. Considering this, the teacher will be able to facilitate learning mathematics on other topics in addition to teaching geometry using the ethnomathematics approach. Teachers can use projects and teaching aids in the ethnomathematics approach to allow students to engage in practical, investigative learning in a mathematics classroom. Then, the teacher encourages the class to think mathematically by using abstraction, generalization, symbolization, and logical reasoning.

The quantity of cases examined in this study places limitations on it. It is advised that more research be conducted and that the mathematics topics be changed or added to, incorporating algebra and other subjects in addition to geometry. Therefore, it may provide a more profound comprehension of how to encourage mathematical thinking using an ethnomathematics approach.

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The Problem Corner

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The Purpose of **The Problem Corner** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem and its solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Greetings, fellow problem solvers!

As the editor of The Problem Corner, I am delighted to announce that we have received accurate solutions for both Problem 24 and Problem 25. These submissions not only met the criteria for correctness but also exemplified effective strategic application. Our primary aim is to showcase what we believe are the best solutions to foster and elevate mathematical knowledge within our global community.

Solutions to **Problems** from the Previous Issue.

Problem 24. Deciphering the Puzzle of Lamp Height.

Proposed by Ivan Retamoso, BMCC, USA.

The diagram illustrates a lamp positioned three units to the right of the y-axis and casting a shadow due to the elliptical region defined by the equation $x^2 + 4y^2 \leq 5$. Given that the point $(-5, 0)$ lies on the shadow's edge, how far above the x-axis is the lamp located?

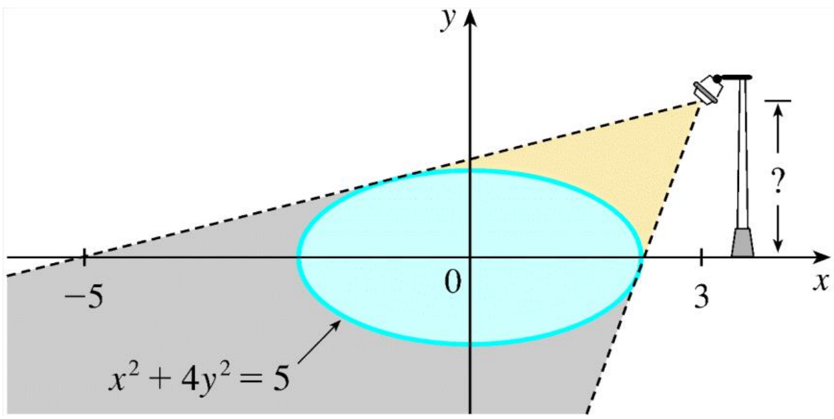

Figure 1. Deciphering the Puzzle of Lamp Height

Solution to problem 24 by Dr. Abdullah Kurudirek (Mathematics Ed. Department, Tishk International University)

This efficient solution combines the power of implicit differentiation from Calculus with Analytic Geometry to compute the slope of a line tangent to the ellipse in two different but equivalent ways. The solution to the problem then smoothly follows.

As can be easily seen in the given figure, there are two tangent lines shown with dashed lines to the elliptical area. Let's denote them as L_1 and L_2. Additionally, let's call the point tangent to the ellipse on the left side by $A(x, y)$. We can find the slopes of these lines with the help of the implicit function's differentiation and two points that lie on a line.

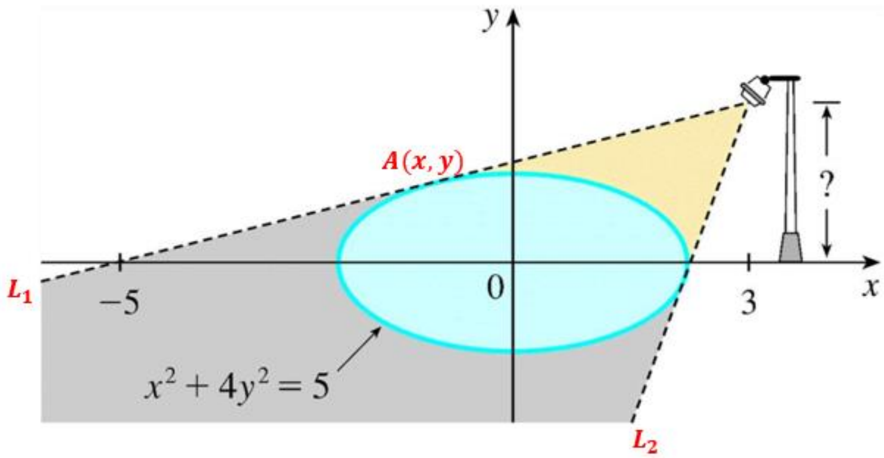

Figure 2. Kurudirek solution to problem 24

Slope of L_2 can be calculated as $2x + 8y\frac{dy}{dx} = 0$, by using implicit differentiation. So, $m_{L2} = -\frac{x}{4y}$

Slope of L_1 can be calculated by using the points $A(x, y)$ and $(-5,0)$ on this line. So, $m_{L1} = \frac{y}{x+5}$

These slopes have both x and y so they can be equated as follows:

$$\frac{y}{x+5} = -\frac{x}{4y}$$

After cross-product we get $4y^2 = -x^2 - 5x$ and also knowing that $x^2 + 4y^2 = 5$. After substituting the variable y for x we get $x = -1$ and then $y = 1$. Then calculate the $m_{L2} = -\frac{x}{4y} = \frac{1}{4}$ and the equation of line L_1 will be $y - 1 = \frac{1}{4}(x + 1)$. The location of the lamp $(3, h)$ must satisfy the line L_1.

Therefore, $h - 1 = \frac{1}{4}(3 + 1)$ and $h = 2$.

Problem 25. The Challenge of Finding Coordinates for a Point Outside a Circle.

Proposed by Ivan Retamoso, BMCC, USA.

The blue circle $x^2 + y^2 = 25$ has tangent lines at the points A and B.

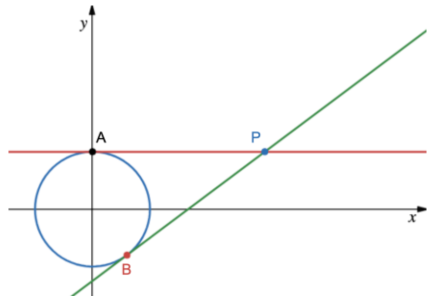

Figure 3. Finding Coordinates for a Point Outside a Circle

The point B has x-coordinate 3.

The tangent lines meet at the point P.

Find the coordinates of the point P.

Solutions to problem 25 by Dr. Hosseinali Gholami (University Putra Malaysia, Serdang, Malaysia)

Our solver presents two different approaches to solve this problem: one using the distance formula and the other using derivatives from Calculus. Labels and equations are added to facilitate the explanations.

Solution 1:

We know the point $B(3, m)$ is on the graph of the circle $x^2 + y^2 = 25$. Therefore, the equation $3^2 + m^2 = 25$ gives $m = \pm 4$. Based on the following figure, only $m = -4$ is acceptable and the coordinates of B are $(3, -4)$.

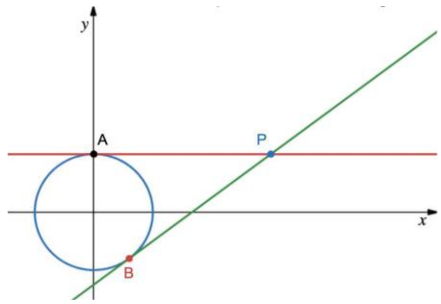

Figure 4. Gholami solution 1 to problem 25

Using the points $A(0,5)$, $P(n,5)$ and the relation $PA = PB$ we obtain the following equation:

$n = \sqrt{(n-3)^2 + 9^2} \Rightarrow n^2 = (n-3)^2 + 9^2 \Rightarrow 6\text{n} = 90 \Rightarrow \text{n} = 15.$

Therefore, the coordinates of point P are $(15,5)$.

Solution 2:

$x^2 + y^2 = 25 \Rightarrow \text{y} = \pm\sqrt{25 - x^2}$. We consider two functions $y = \sqrt{25 - x^2}$ and $y = -\sqrt{25 - x^2}$ separately. The point B is on the graph of the function $y = -\sqrt{25 - x^2}$.

$\frac{\text{dy}}{\text{dx}} = \frac{\text{x}}{\sqrt{25 - \text{x}^2}} \Rightarrow a_{PB} = \frac{3}{\sqrt{25-9}} = \frac{3}{4}.$

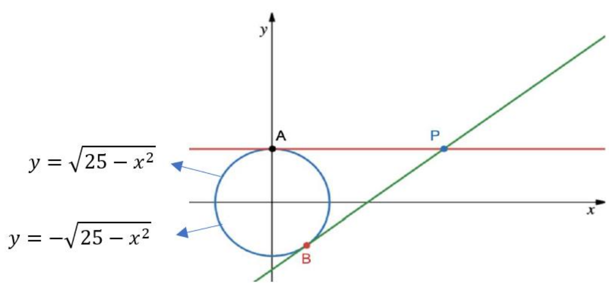

Figure 5. Gholami solution 2 to problem 25

It is clear that the coordinates of B are $(3,-4)$. The equation of tangent line PB is as below.

$y = ax + b \Rightarrow \text{y} = \frac{3}{4}x + b \Rightarrow -4 = \frac{3}{4} \times 3 + b \Rightarrow \text{b} = \frac{-25}{4} \Rightarrow \text{y} = \frac{3}{4}x - \frac{25}{4}.$

The point $P(m,5)$ is on the tangent line PB. Thus, we have:

$y = \frac{3}{4}x - \frac{25}{4} \Rightarrow 5 = \frac{3}{4}m - \frac{25}{4} \Rightarrow \text{m} = 15.$

It means the coordinates of point P are $(15,5)$.

Solution to problem 25 by Dr. Abdullah Kurudirek (Mathematics Ed. Department, Tishk International University)

This alternative solution combines differentiation from Calculus with the slope formula from Analytic Geometry.

We can easily determine the coordinates of the points $A(0,5)$, $B(3,-4)$, and $P(x,5)$ in the given figure below.

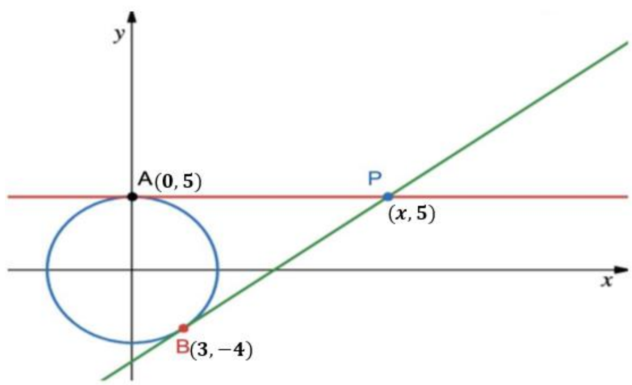

Figure 6. Kurudirek solution to problem 25

By using the points $B(3,-4)$, and $P(x,5)$ the slope of the green line can be calculated as $m_{green} = \frac{9}{x-3}$.

The green line is also a tangent line to the circle at the point B. The slope of this line can be calculated by differentiation as $\frac{dy}{dx} = m_{green} = -\frac{x}{y} = \frac{3}{4}$.

These slopes must be the same so, $\frac{9}{x-3} = \frac{3}{4}$ then $x = 15$ and the coordinates of the point $P(15,5)$.

Solution to problem 25 by Dr. Aradhana Kumari (Borough of Manhattan Community College, USA)

This solution meticulously details the process of finding the coordinates of point B through the equation of a circle in Cartesian geometry. It then employs implicit differentiation from calculus to pinpoint the coordinates of point P. If you enjoy thorough explanations and a methodical approach, you'll find this solution quite rewarding.

Consider the below Circle:

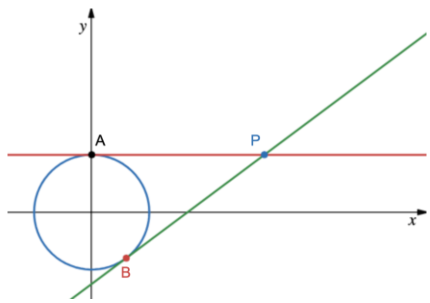

Figure 7. Kumari solution to problem 25

From the figure above, A lies on the y- axis. To find the coordinate of A which lies on y-axis we substitute $x = 0$ in the equation given by (1) we get:

$$y^2 = 25$$

Hence $y = 5$ *or* $y = -5$. As from the figure y is positive hence y is 5.

Therefore, the coordinate of A is (0,5).

The equation of line passing through A which is tangent to the circle at A is

y = 5.

From the figure above the y-coordinate of the point P is 5.

From the above figure x- coordinate of point B is 3 and point B lies on the circle hence it satisfies the equation of circle therefore

$$3^2 + y^2 = 25$$

Therefore

$$y^2 = 25 - 9 = 16$$

$$y^2 = 16$$

Hence $y = 4$ *or* $y = -4$. Since B lies on the fourth quadrant hence the y- coordinate of B is -4. Therefore, the coordinate of B is $(3, -4)$.

To find equation of tangent line at B. We differentiate the equation given by (1). We get:

$\frac{d}{dx}(x^2 + (y))^2 = \frac{d}{dx}(25)$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

Hence $\frac{dy}{dx}$ at $(3, -4)$ is $\frac{-2(3)}{2(-4)} = \frac{3}{4}$

The equation of line which is tangent to the circle at the point B with slope $\frac{3}{4}$ and passes through the point B $(3, -4)$ is $y = \frac{3}{4}x + b$

Substituting the coordinate of point B $(3, -4)$ we get $-4 = \frac{3}{4}(3) + b$

Hence b = $\frac{-25}{4}$. Hence the equation of the line which is tangent to the circle at the point B is the line $y = \frac{3}{4}x - \frac{25}{4}$ ....(2)

To find the x-coordinate of the point P we substitute $y = 5$ we get:

$5 = \frac{3}{4}x - \frac{25}{4}$

Hence $x = 15$.

Hence the coordinate of the point P is (15, 5).

Dear fellow problem solvers,

I'm pleased you enjoyed solving problems 24 and 25 and have gained valuable new strategies for your mathematical toolkit. Let's proceed to the next set of problems to further sharpen your skills.

Problem 26

Proposed by Ivan Retamoso, BMCC, USA.

The diagram illustrates a circle with a radius of 6 inches and center O, with UB as its diameter. Points P and Q are positioned on the circle so that OP and OQ are arcs of circles with a radius of 6 inches and centers at U and B, respectively. Determine, in exact form, the area of the "blue" region OPQ.

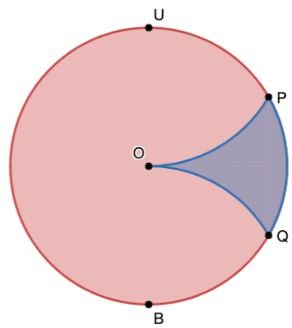

Figure 8. Problem 26

Problem 27

Proposed by Ivan Retamoso, BMCC, USA.

Consider the lemniscate curve $2(x^2+y^2)^2 = 25(x^2-y^2)$.

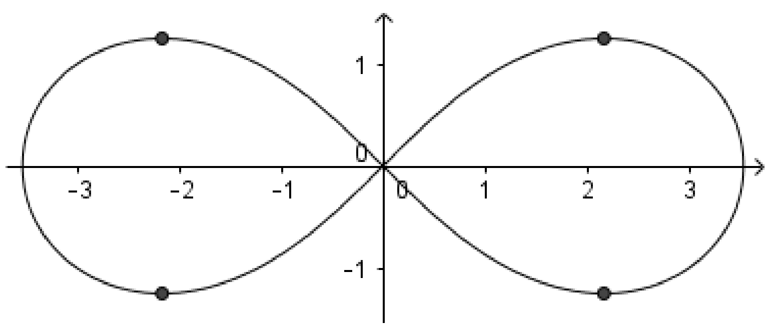

Figure 9. Problem 27

a. Find the slope of the tangent line to the lemniscate in terms of the variables x and y.
b. The four points on the lemniscate where the tangent line is horizontal are all on the intersection of the lemniscate with circle $x^2 + y^2 = k$, find the value of k.