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Editorial from Malgorzata Marciniak, Managing Editor of MTRJ

The occurrence of the solar eclipse overlaps with the publication of the current journal issue and is equally exciting among certain enthusiasts. Hopefully this temporary yet relatively frequent alignment of the Earth, the Moon, and the Sun brings alignment to nature around us and within us. The trajectory of the papers begins with problem solving, then shifts to thinking, cognition and later to challenges of the pre-service teachers. Papers about thinking and cognition revolve around various aspects of the mind, and include hermeneutics phenomenological aspects, triad learning or analyzing students' errors in integration categorized according to the level of students understanding.

The Impact of Project-Based Learning on the Development of Statistical and Scientific Skills: A Study with Chilean University Students from the Faculty of Health Sciences

Chuan Chih, Hsu, Chia Shih, Su, Kua I, Su, Chia Li, Su

The first paper, by Chuan Chih et al was written collaboratively by authors from Chile and Argentina, who discuss Project-Based Learning. Students of kinesiology in their statements praise PBL for the improvement in academic performance. The development of statistical and scientific skills among participants is confirmed in quantitative data analysis.

On a typology of errors in integral calculus in secondary school related to algebraic and graphical frames

Anass El Guenyari, Mohamed Chergui, Bouazza El Wahbi

Repetitive errors in integration performed by Moroccan students are analyzed by El Guyenari et al. The authors create multiple subcategories for the three known categories of errors: conceptual, procedural, and technical. It seems that technical subcategories represent a major source of the erroneous productions of the students.

Investigation of Students' Mathematical Thinking Processes in Solving Non-routine Number Pattern Problems: A Hermeneutics Phenomenological Study

Aiyub Aiyub, Didi Suryadi, Siti Fatimah, Kusnandi Kusnandi, Zainal Abidin

A hermeneutics phenomenological study is performed by Aiyub et al from Indonesia. The authors observe patterns in students' performance classifying and analyzing each category of performance. In addition, suggestions for supporting students' learning in each category are provided.

Socio-mathematical Norms Related to Problem Solving in a Gifted and Talented Mathematics Classroom

Aslı Çakır, Hatice Akkoç

Aslı Çakır, Hatice Akkoç from Turkey discuss how gifted students create social norms for discussing different solutions. The authors give practical guidelines for developing classroom culture related to such discussions. Their work is motivated by the benefits of such discussions for both mathematical and human reasons.

The Thinking Process of Children in Algebra Problems: A Case Study in Junior High School Students

Wa Ode Dahiana, Tatang Herman, Elah Nurlaelah

Dahiana Wa Ode, et al from Indonesia use the Harel's model to analyze thinking characteristics of students while solving algebra problems After analyzing the data the authors conclude that only a few students display algebraic invariance thinking or proportional and deductive thinking while majority use non-referential symbolic thinking.

Overview of Student's Mathematics Reasoning Ability Based on Social Cognitive Learning and Mathematical Self-efficacy

Habibi Ratu Perwira Negara, Farah Heniati Santosa, Muhammad Daut Siagian

Mathematical self-efficacy is discussed in the context of problem-based learning and social cognitive learning by Muhammad Daut Siagian, et al from Indonesia. According to the authors' work, in certain circumstances, one of those two kinds of learning is more impactful on students' mathematical reasoning abilities.

Designing Model of Mathematics Instruction Based on Computational Thinking and Mathematical Thinking for Elementary School Student

Rina Dyah Rahmawati, Sugiman, Muhammad Nur Wangid, Yoppy Wahyu Purnomo

Rina Dyah Rahmawati, et al from Indonesia perform experiments to use integrated computational thinking and mathematical practical design thinking to enhance students' computational thinking skills. The authors analyze the validation process, practicality, and effectiveness of such an approach.

Schema development in solving systems of linear equations using the triad framework

Benjamin Tatira

The Triad Framework was used by Benjamin Tatira from South Africa for studying his students progress in learning linear equations. The author points out that identifying the challenges encountered by the students provides sufficient insight to the instructors for overcoming them.

Assessing the understanding of the slope concept in high school students

José David Morante-Rodríguez, Martha Patricia Velasco-Romero, Geovani Daniel Nolasco-Negrete, María Eugenia Martínez-Merino, José Antonio Juárez-López

The concept of slope is the topic of research for José David Morante-Rodríguez, et al from Mexico. They analyze it within four dimensions: Skills, Properties, Uses and Representations using the SPUR model. And based on three conceptualizations: constant ratio, behavior indicator and trigonometric conception. According to the authors, most students master the procedural tasks of the slope: the constant ratio and trigonometric conception but miss the conceptual aspects.

Effects of Differentiated Instruction in Flipped Classrooms on Students' Mastery Level and Performance in Quadratic Equations

Gilbert G. Baybayon, Minie Rose C. Lapinid

Gilbert G. Baybayon and Minie Rose C. Lapinid from the Philippines discuss methods of addressing students' non-compliance with assignments in a flipped classroom. They administer tiered worksheets based on students′ readiness as reflected in pre-assessment results.

Case Studies: Pre-Service Mathematics Teachers' Integration of Technology into Instructional Activities Using a Cognitive Demand Perspective

Ahmet Oğuz Akçay

Ahmet Oğuz Akçay from Turkey test pre-service teachers on classroom technology. In addition, the authors study cognitive demands of mathematical tasks when technology is used.

Examining Pre-service Mathematics Teachers' Pedagogical Content Knowledge (PCK) during a Professional Development Course: A Case Study

Sunzuma Gladys, Zezekwa Nicholas, Chagwiza Conillius, Mutambara Tendai, L.

In another paper about pre-service teachers, Sunzuma Gladys et al, from Zimbabwe assess pre-service teacher's pedagogical content knowledge as observed in their teaching. Lesson plans and class videos from four subjects are analyzed from the perspective of the knowledge about the subject content matter, knowledge of learners, understanding of instructional strategies and familiarity of context.

The Problem Corner

Ivan Retamoso, BMCC, USA

The Problem Corner edited by Ivan Retamoso from BMCC, USA contains math problems and their solutions. This time solutions were submitted by Abdullah Kurudirek from Iraq and by Hosseinali Gholami from Malaysia who previously authored papers in MTRJ.

The Impact of Project-Based Learning on the Development of Statistical and Scientific Skills: A Study with Chilean University Students from the Faculty of Health Sciences

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Abstract: This study investigates the impact of Project-Based Learning (PBL) with an emphasis on statistics on 26 Kinesiology students from a prominent Chilean university. A mixed-methodological approach was employed for the qualitative and quantitative analysis of data collected through surveys, supplementary interviews, and performance evaluations of these students. Furthermore, group grades during the project execution were examined. The correlation between academic performance and the perception of learning through this method was explored. The results indicate a generally favorable assessment of PBL, emphasizing its contribution to the development of statistical and scientific skills, as well as improvement in academic performance, *with the option to incorporate additional methods to cater to different student needs. It is concluded that PBL is a potential pedagogical strategy that promotes active engagement in learning and the development of practical skills relevant to health sciences students in Chile.*

Keywords: Project-Based Learning, Statistics, University Student, Kinesiology, Student Perceptions, correlation

INTRODUCTION

In recent years, the higher education system in Chile has experienced significant growth, reflected in both the diversity of its academic offerings and the number of students seeking a university education (López et al., 2020). It is particularly noteworthy that there has been an increase in students choosing medicine, dentistry, and health science careers, making these fields some of the most sought-after in Chilean universities. According to recent data, a considerable percentage of aspiring higher education students in Chile opt for health-related careers as their first choice (Bordón et al., 2020).

This remarkable increase in the preference for health-related careers underscores the urgency of comprehensive and up-to-date academic training. This training should not only encompass the specific knowledge of each discipline but also equip students with the necessary skills to handle and analyze complex data. In this regard, statistical literacy becomes a crucial element of higher education, especially in health-related fields (De la Hoz et al., 2021). This need is further emphasized by the challenges posed by the interpretation and management of data in an increasingly information-driven world that relies on evidence-based decision-making.

Recognizing the importance of statistical literacy in health disciplines, we are faced with a contemporary challenge in higher education: ensuring that this training is effective and relevant. In the current context, competence in these areas is not only essential for academic development but also for practical application in various professional scenarios. It enables students and professionals to structure, interpret, evaluate, and communicate vital information related to these concepts and address complex and uncertain situations (Gal, 2002, 2005). However, current educational reforms have shortcomings in the teaching of these areas due to the lack of consensus on the appropriate pedagogical approach (Danä & Taniåžli, 2018; English, 2013; Kaplan & Thorpe, 2010; Sharma, 2017; Shaughnessy, 2007). To overcome this challenge, it is crucial to adopt a multidimensional perspective on statistical literacy, linking it to contemporary sociocultural and environmental issues, integrating data, and using methodologies such as project-based learning to promote interdisciplinary approaches.

Project-Based Learning (PBL) has established itself as a valuable and increasingly popular educational approach in various academic fields, including health sciences (Sáiz-Manzanares et al., 2022). This pedagogical method, which emphasizes active and student-centered learning, offers a unique opportunity to integrate theory and practice through real and relevant projects (Guo et al., 2020). In the context of teaching and learning statistics, PBL not only facilitates the understanding of complex statistical concepts but also promotes critical skills such as analytical thinking, problem-solving, and teamwork (Farrell & Carr, 2019).

The importance of PBL in teaching applied statistics to health sciences is gaining growing recognition. This recognition stems from PBL's effectiveness in integrating theoretical learning with concrete practical applications, a crucial competence for future healthcare professionals (Davidson et al., 2019; White, 2019). Through PBL, students have the opportunity to analyze real data, address contemporary public health issues, and develop evidence-based solutions. This approach not only enriches the educational experience with relevant practical applications but also equips students with the necessary skills to understand and process health data. These competencies are essential for making informed decisions in their future professional practice (Dierker et al., 2018).

The relevance of this study lies in several key aspects. First, as healthcare sciences evolve rapidly, future professionals must be equipped with robust practical and analytical skills, something that PBL can significantly facilitate (Elkhamisy et al., 2022). Second, understanding students' perceptions of PBL can provide valuable information for educators and curriculum designers, enabling adjustments that enhance teaching effectiveness (Elsamanoudy et al., 2021). Furthermore, studying these perceptions helps identify potential barriers and facilitators in the implementation of PBL, ensuring that the approach is as beneficial as possible for student learning (Mitchell & Rogers, 2020).

This work is also justified by the need to align teaching methods with the demands of the healthcare sector, which increasingly requires professionals capable of interpreting and applying statistical data in clinical and research settings (MacDougall, 2020; Larson, 2023). By understanding how Chilean university students perceive and engage with PBL in health-related statistics, this study aims to contribute to the development of educational practices that are not only theoretically sound but also relevant and applicable to the current professional context (Davidson et al., 2019).

The present study aims to investigate the impact of PBL on students, with a focus on statistics in Kinesiology. To achieve this objective, two specific objectives (SOs) have been established:

- SO1: Evaluate the perceptions of Kinesiology students regarding PBL.
- SO2: Assess the impact of PBL on the academic performance of Kinesiology students.

Literature Review

PBL is an innovative pedagogical approach that has gained ground in various academic areas, including statistics and health sciences (Davidson et al., 2019; White, 2019). This method focuses on active and collaborative learning, where students engage in projects that require the application of knowledge and skills in real or simulated contexts (Lai, 2021).

In the field of applied statistics in health sciences, PBL emerges as an effective tool to foster a deep understanding of statistical concepts and their practical application. According to Mujumdar et al. (2021), PBL facilitates the integration of theoretical learning with practical applications, which is essential in disciplines like statistics where the abstraction of concepts can be challenging for students. Additionally, Huang et al. (2023) emphasize that PBL in statistics not only enhances conceptual understanding but also develops critical skills such as data analysis, problem-solving, and teamwork.

In the health sciences, PBL provides an opportunity for students to experientially grasp the importance of statistics in their future professional practice. Dierker et al. (2018) underline that through PBL, students can analyse real health data, tackle contemporary issues, and develop evidence-based solutions. This approach is crucial in preparing future healthcare professionals for making informed decisions based on data analysis. Pilot et al. (2023) assert that PBL enriches the educational experience and equips students with essential skills for interpreting and applying statistical data in clinical and research contexts.

Furthermore, the literature suggests that PBL can significantly enhance student engagement and motivation. Xiao et al. (2019) and Pilot et al. (2023) indicate that PBL engages students more meaningfully than traditional teaching methods, as it allows them to see the direct relevance of what they learn in real-world scenarios. Davidson et al. (2019) add that this methodology promotes better knowledge retention, and a deeper understanding of how statistical concepts are applied in the field of health.

Previous Studies on Student Perceptions in Similar Contexts

Understanding student perceptions regarding PBL applied to statistics in health sciences is crucial for evaluating and refining this pedagogical method. Research conducted over a decade ago, such as the studies by Freeman et al. (2008) and Dierker et al. (2018), already delved into how health sciences students experienced and valued PBL in statistics, linking it to their disciplinary education. These studies have provided valuable insights for educators and curriculum designers. However, scientific exploration on this topic continues to evolve, expanding our understanding of the effectiveness and implications of PBL in health sciences education.

A key study by Davidson et al. (2019) focused on how PBL influenced the motivation of undergraduate students in the medical faculty. They discovered that PBL increased students' interest in statistics by providing them with a clear and applicable context, thereby demonstrating their commitment and motivation towards learning.

On the other hand, Si (2020) centered on the perception of 40 second-year pre-medical students regarding PBL as a tool to develop analytical and problem-solving skills. They found that students highly valued PBL for its ability to simulate real-world professional challenges, allowing them to better understand the applicability of statistics in the field of health.

Furthermore, the work of Elsamanoudy et al. (2021) highlighted students' perception of PBL as a means to enhance collaboration and communication within multidisciplinary teams. Students indicated that PBL provided them with a platform to collaboratively discuss and analyse data, emphasizing the relevance of teamwork – an essential element in health sciences.

Another significant study conducted by Pilot et al. (2023) explored the attitudes of health sciences students towards PBL in statistics. The results revealed that, although students were initially sceptical, over time, they recognized the value of PBL in applying theoretical knowledge, using statistics in real-life situations. This finding underscores the importance of students gradually adapting to innovative pedagogical methods.

Collectively, these studies emphasize that student perceptions of PBL in statistics applied to health sciences are generally positive, highlighting its utility in developing practical, analytical, and teamwork skills. However, they also point out the need for an adaptation period for students and the importance of a pedagogical approach that effectively integrates theory and practice.

On the other hand, despite significant advances in understanding PBL in statistics applied to health sciences, there are limitations in previous studies that this manuscript seeks to address. While previous studies, such as those by Davidson et al. (2019) and Si (2020), provided a fundamental understanding of how students perceive PBL, these investigations have mainly focused on specific contexts of medical students, which may not fully reflect the diversity of experiences and perceptions in different educational settings within health sciences.

Additionally, the adaptation and response of students to PBL in the Chilean context, a scenario with distinctive educational, cultural, and social characteristics, have been explored to a lesser extent. Despite research like that of Elsamanoudy et al. (2021) emphasizing the importance of teamwork and collaboration in PBL, there remains a lack of studies focused on how these aspects are experienced and perceived by students in Chile.

In response to this gap, this study seeks to investigate the perceptions of Chilean university students in Kinesiology regarding PBL from the perspective of statistics. It focuses on exploring how Kinesiology students, distinct from medical students, perceive and adapt to PBL, as well as how these perceptions influence their motivation and approach to learning.

Method

In this study, a mixed-methods approach is adopted, which combines quantitative and qualitative methods. This methodological choice aims to deepen the understanding of the phenomena under investigation. As Almeida (2018) points out, mixed-methods methodology is particularly effective in gathering essential data and enhancing thematic analysis, thus providing a more comprehensive and detailed insight. This approach is suitable for the context of this study, which focuses on Chilean university students in the health sciences and their experience with PBL. The integration of these mixed methods holds the promise of significantly enriching our understanding of the topic and supporting the development of future educational interventions tailored to their needs and specific contexts.

Context and Participants

In this research, we had the participation of 26 Kinesiology students from a university located in the central region of Chile. These participants, with an average age of approximately 19.7 years, were enrolled in a mathematics course that is a mandatory part of their professional curriculum. This course required students to carry out a research project related to issues in their field of study, using statistical techniques to analyse the collected data, develop evidence-supported conclusions, and suggest theoretically backed solutions.

On the other hand, an interview was also conducted with the teacher in charge of the mathematics course to inquire about her experience with PBL-based teaching. This teacher has 20 years of teaching experience in mathematics, and statistics and probability. In this context, her professional perspective offers a broader understanding of PBL implementation.

It is worth noting that, although this study was a common activity for all students in the mathematics courses, we only included in our sample those participants who gave their explicit authorization to disclose this information. Participation in the research was entirely voluntary and was conducted after obtaining their authorization and consent.

Teaching and Learning Strategy Used in the Course

During the first semester of the year 2023, which spans from March to July according to the Chilean educational system, the responsible teacher implemented the PBL method in the mathematics course. This approach aimed to make the teaching and learning process of mathematics, which includes the use of statistics, more closely connected with the topics of kinesiology. To achieve this, one hour per week was dedicated to providing essential guidance and instructions for each stage of the project (see Table 1). During these sessions, the teacher explained the rubric used to assess both the reports and oral presentations of the students, and carefully reviewed the reports submitted by each group.

Table 1. *Week and phase distribution of the PBL.*

Week	Phase	Detail	Assessment
1 2 3	*Phase 1: Statement of the problem and choice of physical activity to analyze.*	Students should choose a physical activity they want to analyze (for example, running, jumping, throwing a ball, etc.) and ask a research question that allows them to analyze human movement in that activity.	Progress monitoring 1
4 5 6	*Phase 2: Data collection.*	Students must collect data on the chosen physical activity. They could do this by filming a video of someone doing the activity or by directly observing someone doing it.	Progress monitoring 2
7 8 9	*Phase 3: Data analysis.*	Students will analyze data using vectors and mathematical functions. For example, they could use vectors to analyze the direction and magnitude of motion, and functions to analyze velocity and acceleration.	Progress monitoring 3
10 11 12 13	*Phase 4: Presentation of results and conclusions.*	Students must present the results and conclusions of their research. They could do this through a written report and an oral presentation.	Oral presentation and final PBL report
14 15	*Phase 5: Reflection on the research process.*	Students will be asked to reflect on the research process and share what they have learned about the use of mathematics in kinesiology.	--

This project is designed for students to apply the mathematical-statistical concepts they have acquired during the course to real and relevant situations related to their careers. Furthermore, it provides them with an opportunity to develop skills in research, data analysis, and presentation of results. Students have the option to choose from the topics suggested by the teacher, which can be found in Table 2, or alternatively, propose their own topic that aligns with the established guidelines and characteristics.

Table 2. *Topic proposals for the PBL.*

Topic	Detail
1	Analysis of the movement of specific athletes, such as marathon runners, basketball players, gymnasts, etc.
2	Study of movement in different stages of human development: infancy, childhood, adolescence and old age.
3	Research on movement in people with particular physical conditions: injuries, disabilities, chronic diseases, etc.
4	Comparative analysis of movement in various sports, including football, basketball, swimming, volleyball, among others.
5	Examination of movement in daily activities such as walking, running, lifting objects, climbing stairs, etc.
6	Study of movement in different physical therapy approaches: occupational therapy, physiotherapy, rehabilitation, etc.
7	Analysis of movement in cultural and traditional practices: folk dance, martial arts, yoga, among others.

Techniques and Instruments

To achieve the research objective of this study, various data collection techniques and tools were implemented. Firstly, a survey was administered to all participants to gain a general understanding of their perspectives. Additionally, supplementary interviews (with some students and the teacher in charge of the mathematics course) were conducted to delve deeper into the survey findings and obtain more specific details. Secondly, a quantitative analysis of grades obtained from group projects, carried out by the students, was also performed. The use of these two analyses allowed for a more comprehensive and evidence-based view of the impact of PBL on students' acquisition of statistical knowledge.

Data Analysis

To achieve the objectives of this research, a data analysis was conducted, taking into account its specific nature.

Regarding the qualitative analysis, the focus was on the data collected through surveys and supplementary interviews. During this process, thematic coding of the students' responses was performed to identify patterns and recurring themes related to their perceptions of the impact of PBL on applied statistics in health sciences. This approach facilitated a detailed and in-depth interpretation of the students' experiences and opinions, based on the methodology of content qualitative analysis (Alhussain et al., 2020).

On the other hand, the quantitative analysis centered on examining the grades obtained in group statistics projects to establish correlations between these grades and the students' perceptions and comments. This analysis involved the use of descriptive statistics to gain an overall view of academic performance associated with the use of PBL, following the methodology proposed by Jaiswal et al. (2021).

Finally, the integration of findings derived from both quantitative and qualitative analyses was carried out to achieve a comprehensive understanding of the impact of PBL on statistical education and its influence on academic performance. The integration of this data aimed to confirm, complement, or contrast the results obtained through different methods, thus providing a more complete and nuanced perspective on the research topic, following the guidelines established by Clark (2019).

Results

In this section, the survey and supplementary interview results are presented according to the SOs outlined in the study.

For SO1, a survey consisting of 18 open-ended questions was designed by a team of three professors and a healthcare professional. This survey was based on a study conducted by Pradanti and Muqtada in 2023 and addressed various aspects related to satisfaction and goal achievement, the development of statistical skills, the use of resources and tools, communication of results, teamwork, critical thinking, evaluation, challenges and difficulties, suggestions for improvement, as well as the influence on perception and academic progress. The main purpose of this survey was twofold. First, it aimed to collect valuable information that would significantly contribute to the improvement of future implementations of PBL in the field of applied statistics in health sciences. Second, it intended to understand students' perceptions of this teaching approach and evaluate the influence that this project has had on their learning process and the development of their statistical skills.

Subsequently, the results obtained are presented, approached from two perspectives: qualitative and quantitative. From a qualitative perspective, a detailed analysis was conducted of the responses collected through students' opinions from the open-ended questions in the survey (questions 1 - 18) with twenty-six students and in-depth interviews. This process allowed for the organization of information into four main categories, which emerged from both the collected data and the analysis conducted by the team. These categories are: 1) Development of skills and application of knowledge; 2) Communication and feedback; 3) Collaboration and teamwork; and 4) Overall experience and personal reflection, including representative examples for each category. On a quantitative level, a five-level rating technique was implemented to categorize each of the comments provided by the 26 students in relation to the 18 survey questions. This association of comments with their respective rating levels facilitates quantitative analysis, providing an overall view of the students' learning in the study with the PBL approach.

From the qualitative perspective, the analysis condenses student opinions from the open-ended questions in the survey (questions 1 - 18) with twenty-six students and in-depth interviews, classifying the responses into the corresponding categories. Below are the categories and some student responses.

Category 1) Development of skills and application of knowledge

In this category, we focus on the use of PBL for the development of critical skills. The trends indicate that a considerable number of students viewed the research and data analysis activities positively. This aspect encompasses not only data collection and management but also the ability to efficiently use search tools and relevant technology, suggesting that PBL has promoted a practical and systematic approach to statistics. For example:

> "Learning to research better, I was able to find more search tools as well as more sites to gather information." (Student 6)
> "I think it's something that still needs reinforcement, but I consider data collection and being organized in tables as a great achievement since I don't have much knowledge in Excel, which is the app commonly used for this." (Student 25)

Additionally, there are responses highlighting a significant improvement in creating and understanding graphs and tables, especially through the use of software like Excel. This demonstrates proficiency in data visualization, which is essential for interpreting and effectively presenting research findings. For example:

> "PBL really highlighted the importance of visual representation in statistics. Learning to create graphs not only made the data come to life but also allowed me to see the stories that those numbers tell. This skill is something I know I will use a lot in my future as a healthcare professional." (Student 8)
> "Mastering Excel through the project was eye-opening. The ability to organize and present data in tables is fundamental, and now I feel like I have a powerful tool at my disposal for future research and statistical analysis in my career." (Student 21)

Furthermore, some comments mentioned regarding interpretation and statistical analysis show that in some students, the project fostered a competence specific to statistics: the ability to handle statistical concepts for critically evaluating scientific literature. This aspect reflects a notable improvement in students' discernment when selecting reliable academic sources and their understanding of how data supports scientific conclusions. For example:

> "At first, reading and interpreting graphs was like deciphering a foreign language, but with the constant practice that PBL encouraged, I now feel like I can understand and communicate statistical findings with much more confidence." (Student 4)
> "The project took us beyond just using charts; it taught us to extract meaning from them. Learning to calculate the mean, median, and mode was like getting the keys to data, opening up a new level of critical analysis that I didn't know I could reach." (Student 17)

However, there is also a minority of students who did not identify the acquired skills or reported little change in their analytical abilities. This might indicate opportunities to optimize how PBL is communicated and integrated into the curriculum. For example:

> "PBL was a part of the course, but I didn't feel like it significantly transformed my learning process. While I recognize its value, I believe it would need to be more concretely integrated with our lab activities and class discussions to truly resonate with my learning process." (Student 3)

> "I confess that statistics has always been intimidating to me, and PBL adds a level of pressure that makes me anxious. Analyzing data under this methodology makes me feel out of my comfort zone, which can be stressful, although I understand it's an important skill to develop in my field." (Student 7)

Category 2) Communication and feedback

In this category, the majority of students mentioned receiving positive and constructive feedback, which is crucial for improving their ability to effectively communicate scientific findings. This feedback has become a valuable tool, assisting students in identifying their strengths and areas for improvement, both in written composition and oral presentations. For example:

> "The detailed review that the teacher provided for our project was key. It not only helped us organize our ideas but also taught me to structure information better, which is crucial for a coherent report." (Student 5)
> "The teacher's feedback was eye-opening; it allowed us to identify and work on our weaknesses. Without that guidance, many errors would have gone unnoticed." (Student 11)

Additionally, a segment of students recognized themselves in a developmental phase, expressing that they were improving and that the PBL process had helped them refine their communication skills. This suggests that PBL not only serves the transmission of technical knowledge but also fosters essential communication competencies. For example:

> "During the presentation, I felt like I could really connect with the audience. Using concrete examples helped me clarify complex points and ensure that the findings were understandable to everyone." (Student 10)
> "I feel confident in communication; presentations and reports are something I enjoy and handle with confidence, which I believe is reflected in the quality of my work." (Student 22)

There were also students who faced challenges, particularly in handling nervousness during public presentations. Practice and relaxation strategies recommended by the teachers have been essential in overcoming these obstacles, allowing students to present with more confidence. For example:

> "The standards for presentations were high, and despite following all the guidelines, I feel that we still haven't reached the expected level. It's an area where we definitely need to improve." (Student 16)
> "Public speaking is still a challenge for me. Although I'm working on it, I still feel that I need to overcome the nervousness barrier to communicate my ideas clearly." (Student 20)

On the other hand, some students mentioned specific difficulties in conveying information, highlighting the importance of personalized feedback to support individualized learning. Additionally, there was a minority that did not provide details about their communication process or did not face significant challenges, which may indicate variability in individual experiences or a possible reluctance to share difficulties. For example:

> "Personally, I didn't find significant obstacles in communicating our findings. I feel that our team did a good job conveying the information clearly and accurately." (Student 12)
> "Collective feedback was very beneficial, but I think we also need individualized feedback to address our areas of improvement more effectively." (Student 13)

Category 3) Collaboration and teamwork

In this category, the majority of students reflected on positive and collaborative experiences, highlighting the effectiveness of working together and sharing credit for team achievements. This positive perception emphasizes that collaboration not only enhances the quality of work but also distributes the workload more equitably. For example:

> "At the beginning, forming the team felt like rolling the dice; we were all new acquaintances and were unsure how to choose partners. However, we overcame the initial uncertainty and successfully completed the project." (Student 10)
> "Our start as a team was complicated; there was an imbalance in contribution. After an open conversation with the teammate in question and with the support of our teacher, we found a work rhythm where everyone could contribute equally." (Student 17)

However, some students reported mixed or varied experiences, suggesting that group dynamics and the team formation stage can significantly influence the collaboration experience. In some cases, individual differences and adapting to new teams presented initial challenges that eventually turned into valuable learning experiences.

> "Frankly, teamwork was not very good, mainly because we were just getting to know each other. Eventually, one person is left advancing the work, and the rest of the team, almost nothing." (Student 12)
> "Personally, I enjoyed the team experience. Sharing and expanding perspectives with others enriched our work and the learning process." (Student 26)

Furthermore, in this category, students also mentioned topics related to planning, communication, and collaborative work. Planning emerged as a fundamental pillar for effective teamwork. Students recognized the importance of setting schedules and dividing tasks, which facilitated a more structured and organized progression of the project. For example:

> "We discovered that the key to team success was solid organization, setting a clear order, and having a person to lead the group." (Student 17)
> "I was fortunate to be part of a team where transparency and cooperation were the norm. This collaborative environment brought us closer with each project milestone." (Student 22)

Regarding effective communication, it was highlighted as an essential strategy for maintaining clear and constant collaboration. The use of various means of communication reflects adaptability and openness to the communication needs of each team member. For example:

> "From day one, we established a communication channel via WhatsApp. This was crucial for exchanging ideas and monitoring our progress constantly." (Student 4)
> "Everyone contributed, but it's inevitable to notice that sometimes the effort is not uniform; some teammates seemed to be taking the lead." (Student 25)

In terms of collaborative work and mutual support, they also identified these as crucial aspects of the process, where equity in task distribution and readiness to help each other contributed to the harmonious development of the project. For example:

> "We did all the work together; we used to meet in the afternoon at the library or the autonomous study room to collect data and gather information." (Student 19)
> "Companionship was the driving force that propelled our project forward, culminating in work we all feel proud of." (Student 24)

Within collaborative work, some students emphasized the value of collaboration as key elements for success and project efficiency. For example:

> "The need to work as a team was evident from the beginning. Sharing the workload not only made the process more manageable but also enriched the content of our report." (Student 8)
> "Throughout the project, I learned the immense value of collaboration. Assigning tasks based on individual strengths was not only efficient but also allowed each person to shine in their specialty." (Student 11)

Category 4) Overall experience and personal reflection

In this category, students expressed positive feelings both about the overall satisfaction with the program and specific achievements. This includes the joy of accomplishing set objectives, gaining new knowledge, and seeing tangible and useful outcomes in their learning. This category encompasses both general satisfaction and specific satisfaction related to the achievement of results and acquired knowledge. For example:

> "Participating in the project has given me valuable insights into how health topics integrate into our field. I genuinely feel it has enriched my understanding of the health field." (Student 2)
> "It was an intense semester, focused on learning to analyze data and tables for health sciences research. I found it challenging, but incredibly useful in understanding how information is processed in our field." (Student 15)

Many students emphasized the importance of developing analytical and collaborative skills, as well as the value of understanding and applying the research process. These aspects are directly related to the gratification found in the learning process and the development of practical skills. For example:

> "Working as a team was an enriching experience. We learned not only to collaborate efficiently but also how to combine our skills to achieve a common goal, which was very rewarding." (Student 19)
> "Analyzing graphs and data related to health sciences was perhaps the most significant achievement for me. It gave me a real sense of contributing to the field with practical and relevant work." (Student 21)

Additionally, they identified challenges in information management and methodology, as well as suggestions for improving feedback and clarity of instructions. This may reflect the difficulties encountered in handling information properly and the need for more effective communication and clear guidelines. For example:

> "The process of conducting tests and collecting health data involved talking to relatives, friends, and neighbors. This interaction made me realize the importance of communication and community outreach in our field." (Student 22)
> "Handling project variables was perhaps the most complex challenge. I faced situations I had never considered before, which pushed me to think more critically and analytically." (Student 25)

Furthermore, there was a segment of students who recognized the value of the PBL experience in their academic and future professional development, viewing it as a solid foundation for future research and thesis projects. For example:

> "The most valuable aspect was learning to analyze and extract data from scientific texts. This practical and research-based approach has provided me with fundamental tools for my career." (Student 15)
> "This project has given us a research foundation that will surely be useful in the future. Now, when we are asked to conduct research in other subjects, we will have an initial advantage." (Student 25)

However, there were also students who did not provide specific information, did not identify challenges, or did not reflect on the program's influence on their academic and professional development. This suggests a variety of reasons, such as lack of interest, unmet expectations, or a disconnect between the learning experience and their career planning. For example:

> "Honestly, I haven't thought much about how this project affects my academic development. Maybe I need more time to appreciate its impact on my learning and professional development." (Student 12)
> "At this point, I don't feel that the project has had a positive impact on me. In fact, it has made me somewhat reluctant to research due to the workload. I prefer a more traditional approach to learning, where information is presented to us and then we study it." (Student 14)

On the other hand, we categorized the comments of each student in this study, adopting a five-level rating system. Each level corresponds to a specific score, with 0 indicating "Does not Value" and 4 indicating "Highly Values," as detailed below:

- *Does not Value* (Score 0): The student shows complete indifference or disinterest in the topic. Responses may be non-existent, vague, or irrelevant to the subject at hand.
- *Values Slightly* (Score 1): There is minimal or superficial appreciation of the topic by the student, with slight recognition of its value or importance but lacking depth or significant details.
- *Values Moderately* (Score 2): The student recognizes and values the topic with a basic understanding. Responses show some commitment and reflection, though they are not comprehensive or deep.
- *Values Noticeably* (Score 3): A good understanding and appreciation of the topic are reflected, with responses including details and specific examples, demonstrating well-developed commitment and reflection.
- *Highly Values* (Score 4): The student exhibits an excellent understanding and high appreciation of the topic, with detailed, deep, and articulated responses, showing exceptional commitment and reflection.

This rating system allows for the quantification of qualitative statements from students in surveys and interviews, facilitating a clear and structured evaluation of their perceptions and opinions. It is worth mentioning that this adopted technique is based on well-established qualitative and quantitative assessment approaches in the field of educational research. Studies such as those by Keiper et al. (2021) and Mertens (2023) emphasize the relevance of mixed methods for gaining a deep understanding of students' perceptions. Table 3 records the mean score and standard deviation (SD) of the 26 students for each survey question.

Table 3. *Mean score and SD of the 26 students for the survey questions.*

Question	Mean score	SD
1	2.38	0.85
2	2.04	0.96
3	1.73	0.92
4	2.15	0.88
5	2.31	0.84
6	1.96	1.11
7	1.96	1.08
8	1.69	1.16
9	1.81	0.94
10	2.19	0.69
11	1.92	1.02
12	1.92	0.84
13	1.92	0.80
14	1.88	1.07
15	1.58	1.03
16	1.50	0.81
17	2.04	1.04
18	1.77	1.11

The table above displays a variety of student opinions, reflecting both positive aspects and areas for improvement. Higher mean scores in questions such as 1, 5, and 10 suggest that certain elements of ABP are well-regarded, possibly due to their effectiveness in improving understanding and practical skills in statistics. However, lower mean scores in questions like 3, 8, 9, 15, and 16 indicate less favorable aspects of ABP, which may point to deficiencies in the methodology or its implementation. The variability in responses, observed through the standard deviations, reveals consensus in certain areas (low SD in questions like 10 and 13) as well as a wide range of opinions in others (high SD in questions like 6, 7, 8, 14, and 18), highlighting the heterogeneity of students' experiences and perceptions. This set of results underscores the need for ongoing evaluation and adaptation of ABP, considering both positively valued elements and those requiring improvements, to effectively meet the varied needs and expectations of students in their statistics learning.

Using this technique, the comments of each student were analyzed and rated on a scale from 0 to 4. This scale allowed for the assignment of a specific mean score to each student. Table 4 presents the 26 participating students based on their mean scores obtained from their responses to the 18 survey questions. Upon careful analysis of this table, it can be observed that the mean scores have grouped students into five distinct categories. Students are classified as "Highly Values" when their mean score reaches the maximum value of 4. Those who scored between 3 and 4 fall into the category of "Values Noticeably." On the other hand, students with mean scores between 2 and 3 are considered "Values Moderately," while those with mean scores between 1 and 2 are in the category of "Values Slightly." Finally, the category of "Does not Value" is for students whose mean score does not exceed the value of 1.

Table 4. Distribution of students by mean score.

Rating level	Mean score (x)	Number of students
Highly values	$x = 4$	0
Values noticeably	$3 \leq x < 4$	1
Values moderately	$2 \leq x < 3$	8
Vaules slightly	$1 \leq x < 2$	16
Does not value	$0 \leq x < 1$	1
	Total	26

According to the table, there is an interesting distribution of student mean score in the survey. It is noteworthy that no student reached the highest level of assessment, "Highly Values," implying that no participant achieved a perfect mean score of 4. On the other hand, only one student fell into the category of "Values Noticeably," with a mean score between 3 and 4, suggesting it was an isolated case of relatively high performance. Most students concentrated in the middle and lower levels of assessment. Specifically, eight students obtained a mean score that placed them in the "Values Moderately" category, meaning between 2 and 3. However, the most populated category was "Values Slightly," with 16 students, indicating that the majority of participants had mean score between 1 and 2. This suggests a generally low level of response on the survey. Finally, only one student was in the lowest category, "Does not Value," with a mean score below 1, indicating that almost everyone surpassed the lowest mean score. In summary, this distribution reflects that while most students did not achieve high levels of mean score on the survey, they also did not concentrate at the lower end of the scale, with the majority scoring in the intermediate to lower range.

During the interview with the teacher, she emphasized that throughout her years of experience teaching mathematics and statistics, she has always believed her knowledge could enhance the learning of higher education students. However, since she began implementing PBL in her course, she realized that her disciplinary and pedagogical knowledge was not sufficient to integrate relevant aspects from other disciplines, which led her to recognize the need for further training. To address this need, the research team provided her with information about the importance of consulting the literature, challenging her previous notion that educational guidelines come solely from the education ministry or the curriculum, and that knowledge is acquired exclusively through books.

By exploring the literature, she discovered more concrete and up-to-date strategies and actions that she could apply to her course. During the implementation of PBL, when faced with the task of reviewing each part of the students' work, she was forced to conduct thorough research and carefully study every aspect of the projects presented. This involved assessing the feasibility of applying certain mathematical and statistical concepts in the specific context of kinesiology chosen by the students. This experience represented a deep dive into the application of mathematics, thus demonstrating that this discipline serves as a tool in other areas of study. Although she claimed to have known this premise previously, it was confirmed through the implementation of PBL in her course. She felt that this approach not only allowed her students to apply statistical knowledge and model professional practice situations but also motivated her to deepen her mastery of these concepts and to understand the kinesiology fields of interest to her students, as well as how to link them with mathematics and statistics.

Regarding SO2, we present the grades obtained for the group project carried out during the first semester of 2023. During this period, students were divided into teams of five members each, with the aim of conducting a statistical study. This study focused on proposing solutions to specific problems in their respective professional areas. A notable example was the analysis of heart rate in children with obesity. This project was structured in several phases, starting with the preparation of a detailed and rigorous report. This report was required to include an introduction, theoretical framework, methodology, formulation and analysis of results, followed by discussions and relevant conclusions. This structure ensured comprehensive coverage of all stages of the project.

Continuing with the report preparation process, the supervision of the professor played a crucial role. She meticulously supervised the progress of each group, using evaluation rubrics previously defined by the research and teaching team. This thorough monitoring process paid special attention to key sections such as the introduction, theoretical framework, methodology, conclusion, and recommendations. Once the written report stage was completed, students moved on to the oral presentation. In this phase, a specific evaluation rubric was applied to assess their performance.

It is important to note that, beyond the report progress monitoring, students received individual grades for each monitoring phase, as well as for the oral presentation. This was done during interactions with the professor, who asked each student questions related to the work they had done. These grades were assigned on a scale from 1 to 7, following the Chilean assessment system, where 4.0 represents the minimum passing grade, and 7.0 is the maximum.

To provide a broader and more detailed view of the evaluation process, Table 5 presents the grades achieved by students in each stage of the project. These data are presented alongside the respective assessment levels determined based on the survey comments obtained. This table not only reflects numerical results but also provides a qualitative perspective on student learning and performance.

Table 5. *Grade of progress monitoring and oral presentation.*

Student	Progress monitoring 1	Progress monitoring 2	Progress monitoring 3	Oral presentation and final PBL report	Survey mean score	Rating level
1	3.2	3.2	4.5	5.4	1.50	Values Slightly
2	5.5	3.5	2.1	4.3	1.78	Values Slightly
3	5.3	3.0	3.2	4.6	1.72	Values Slightly
4	3.7	4.3	4.9	5.5	1.78	Values Slightly
5	6.7	5.9	5.5	6.5	3.06	Values Noticeably

6	4.0	3.8	4.2	5.5	1.89	Values Slightly
7	4.4	4.3	3.2	3.2	0.67	Does not Value
8	3.6	3.6	4.1	4.3	1.94	Values Slightly
9	4.4	5.7	3.5	4.6	1.72	Values Slightly
10	3.1	3.9	4.3	5.8	1.83	Values Slightly
11	4.1	3.5	4.3	4.8	1.89	Values Slightly
12	3.7	4.3	5.1	5.4	1.72	Values Slightly
13	4.0	3.3	3.5	4.5	1.39	Values Slightly
14	5.6	4.7	5.1	5.7	1.39	Values Slightly
15	4.5	4.1	5.3	4.6	2.56	Values Moderately
16	4.4	5.2	4.3	4.8	2.06	Values Moderately
17	4.4	4.3	4.6	4.8	2.39	Values Moderately
18	5.2	4.5	3.9	4.8	1.83	Values Slightly
19	3.7	5.2	4.1	3.8	1.67	Values Slightly
20	4.5	3.4	4.1	4.5	1.94	Values Slightly
21	4.9	5.4	5.1	5.1	2.06	Values Moderately
22	4.6	4.7	4.6	4.5	2.33	Values Moderately
23	4.9	4.5	4.5	5.4	2.17	Values Moderately
24	5.5	5.1	4.1	5.4	2.72	Values Moderately
25	4.5	4.3	4.4	4.1	2.28	Values Moderately
26	3.5	4.2	3.3	4.4	1.94	Values Slightly

According to the table data, it is evident that students who significantly value the project, as reflected in their survey mean scores and rating level, tend to achieve higher grades in all stages of the project. An example of this is Student 5, who places high value on the project and has consistently demonstrated outstanding performance in all progress monitoring stages and the oral presentation. On the other hand, those who show lower valuation or do not attach importance to the project tend to receive lower grades. Finally, this grade distribution suggests the relevance of valuation and commitment to the project in academic performance.

In order to determine if there is a statistically significant relationship between the grads obtained in the project and the perceptions expressed by the students, a bivariate correlation analysis was conducted. The results of this analysis are presented in Table 6.

Table 6. *Bivariate correlation result.*

Correlations		Oral presentation and final PBL report	Survey mean score
Oral presentation and final PBL report	Pearson correlation	1	0.435*
	Sig. (bilateral *p-value*)		0.026

	N	26	26
Survey mean score	Pearson correlation	0.435*	1
	Sig. (bilateral *p-value*)	0.026	
	N	26	26

* The correlation is significant with bilateral *p-value* = 0.05.

According to the data from the analysis, there is a moderate and statistically significant correlation (bilateral *p*-value = 0.435) between the grade obtained for the oral presentation and the project report and the score from the survey about perceptions of the impact of PBL from a statistical approach. This correlation in the educational context indicates that students who received higher grades in their oral presentation and report tend to have a more positive valuation of the impact of PBL, as reflected in the survey score, as illustrated in Figure 6.

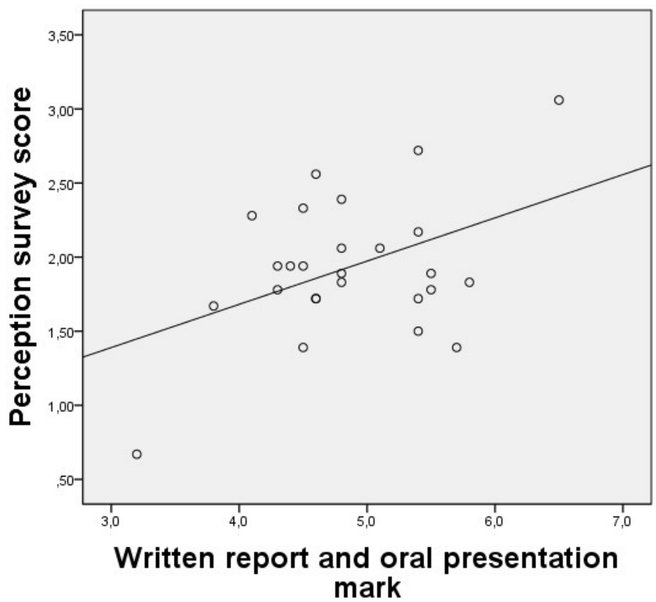


Figure 6. Relationship between score obtained from the project and perception survey.

However, since the correlation is moderate, this relationship is not strong or direct. This means that there are other factors that could also be influencing students' perceptions of PBL, such as personal interest in the topic, group dynamics during the project, or teaching style, which might be contributing to how students value their experience with PBL.

The statistical significance (bilateral *p*-value = 0.026) confirms that this correlation is unlikely to be random. This implies that in this educational context, the way students perceive and value their experience with ABP is somewhat related to their performance in the oral presentation. It could be interpreted that students who perform better in the oral presentation, perhaps due to a better understanding or greater commitment to PBL, tend to have a more favorable perception of this teaching methodology.

Additionally, we present an example of a project carried out by a group of students titled "Analysis of Heart Rate Variation in 19-Year-Old Individuals, Before and After Performing Specific Exercise Circuits, in Relation to Their Level of Physical Activity." This project corresponds to topic number 5 (see Table 2). It is important to note that the original work was conducted in Spanish, and we have translated procedural samples and Excel graphs from the report to facilitate its dissemination in the scientific community.

In the Introduction section, the students stated that their PBL project involves a study investigating how heart rate, assessed under various conditions, can indicate the state of cardiovascular health and overall physical fitness. This approach is supported by bibliographic references emphasizing the importance of cardiovascular monitoring. Furthermore, in this section, the students clearly defined their main objective and detailed the specific goals of their research.

In the Methodology section, the students provided a detailed description of their study's approach, specifying that it is quantitative and cross-sectional in nature. They also outlined the context and participants involved in the research. Additionally, they explained the techniques used for data collection (50-meter race, 1-minute squats, and 1-minute mountain climbers) and the procedures employed for analysis.

Figure 1 illustrates that the students conducted an experiment to recollect the heart rate responses of two different groups: 5 sedentary individuals and 5 athletic individuals. They measured the heart rate of each participant in both groups under three different conditions: before and after a 50-meter run, before and after a 1-minute squat exercise, and before and after a 1-minute mountain climbing exercise.

Heart rate in sedentary people

	Before 50m Race	After 50m Race	Before 1min Squats	After 1min Squats	Before 1min Mountain climber	After 1min Mountain climber
SedP1	76 bpm	162 bpm	76 bpm	150 bpm	76bpm	156 bpm
SedP2	84 bpm	174 bpm	84 bpm	156 bpm	84 bpm	158 bpm
SedP3	78 bpm	172 bpm	78 bpm	162 bpm	78 bpm	160 bpm
SedP4	72 bpm	152 bpm	72 bpm	144 bpm	72 bpm	162 bpm
SedP5	96 bpm	170 bpm	96 bpm	152 bpm	96 bpm	164 bpm

Heart rate of athletic people

	Before 50m Race	After 50m Race	Before 1min Squats	After 1min Squats	Before 1min Mountain climber	After 1min Mountain climber
AthP1	74 bpm	134 bpm	74 bpm	130 bpm	74 bpm	122 bpm
AthP2	70 bpm	140 bpm	70 bpm	135 bpm	70 bpm	138 bpm
AthP3	66 bpm	156 bpm	66 bpm	136 bpm	66 bpm	132 bpm
AthP4	58 bpm	108 bpm	58 bpm	124 bpm	58 bpm	114 bpm
AthP5	84 bpm	138 bpm	84 bpm	128 bpm	84 bpm	130 bpm

Figure 1. Work sample 1.

In Figure 2, the students conducted a comparison and analysis of the difference in the mean number heart rate of both groups, both before and after the 50-meter run. Similar analyses were also performed for the other two physical activities.

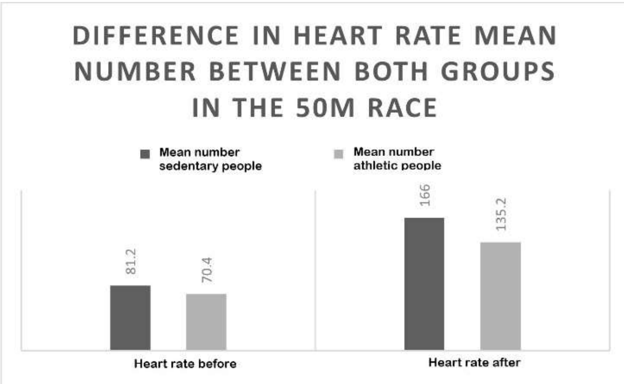


Figure 2. Work sample 2.

In Figures 3 and 4, the students selected one person from each group and conducted seven consecutive measurements, with one-minute intervals between each testing. This testing methodology was applied seven times in a row, first during the 50-meter race, followed by one minute of squats, and finally one minute of mountain climbers. Gathering these data allowed the students to develop a mathematical model based on their findings.

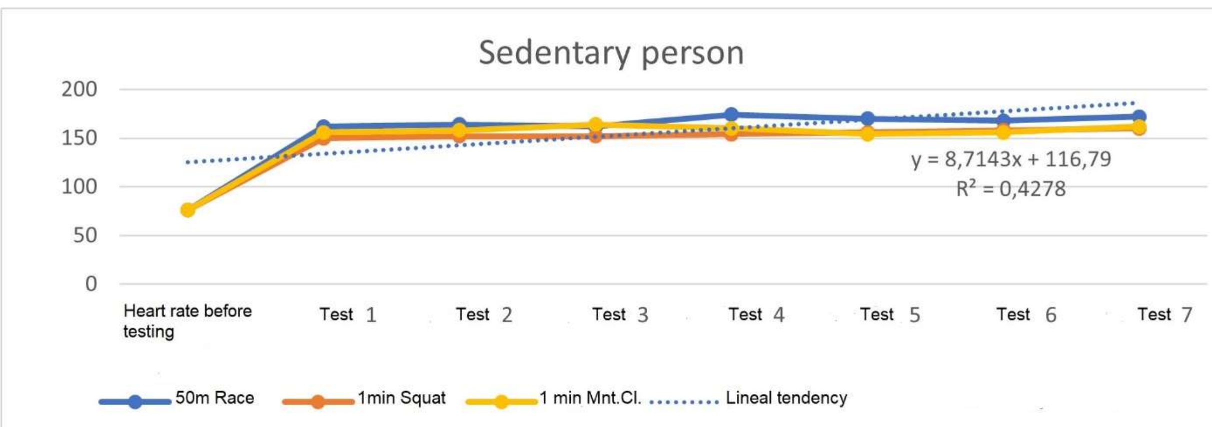


Figure 3. Work sample 3.

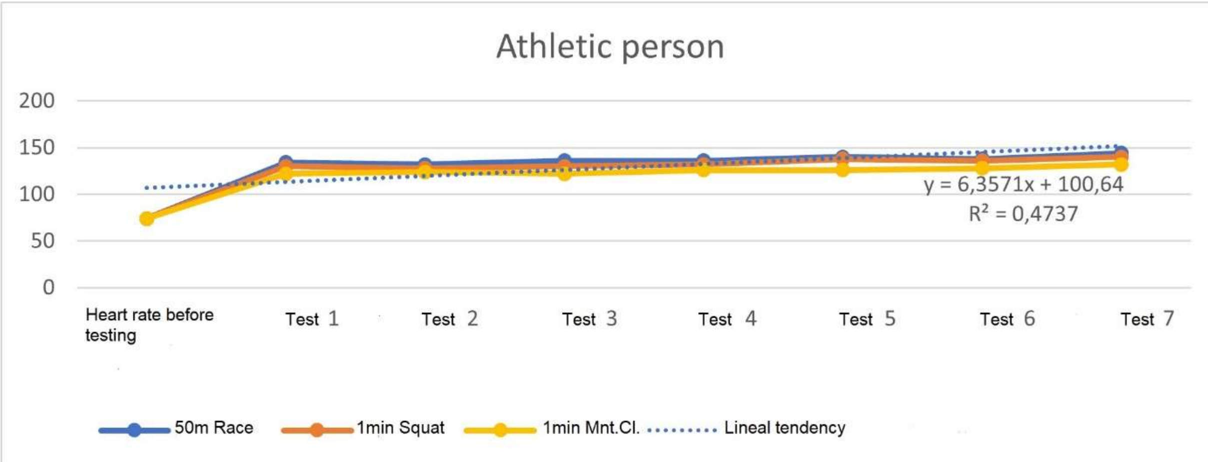
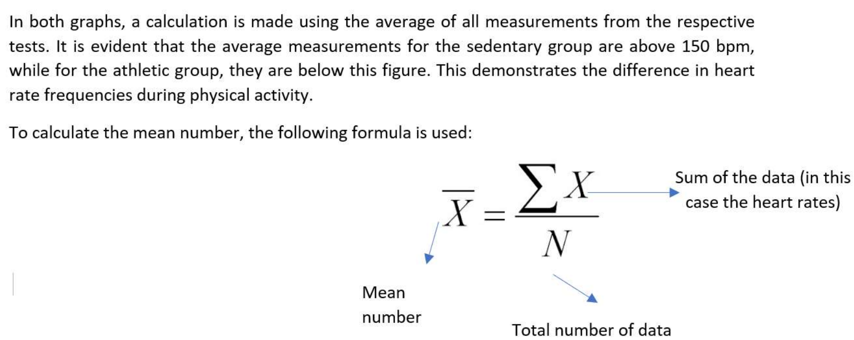


Figure 4. Work sample 4.

Then, in Figure 5, the students presented their formula for calculating the mean number heart rate and stated that this comparison demonstrates the variation in heart rate frequencies among individuals with different levels of physical fitness when they engage in the same physical activities.



In both graphs, a calculation is made using the average of all measurements from the respective tests. It is evident that the average measurements for the sedentary group are above 150 bpm, while for the athletic group, they are below this figure. This demonstrates the difference in heart rate frequencies during physical activity.

To calculate the mean number, the following formula is used:

$$\overline{X} = \frac{\sum X}{N}$$

Figure 5. Work sample 5.

Finally, in their conclusions, the students highlighted a limitation of the study, which was the need to tailor the exercises to assess their impact on heart rate in both athletic and sedentary individuals. They confirmed that the research validates the hypothesis that the level of physical activity significantly influences heart rate, demonstrating notable differences among the studied groups. Additionally, they included the consulted bibliographic references.

DISCUSSION & CONCLUSIONS

Analyse the overall perspective of the development of the activity. Share the difficulties in the course of implementation. Analyse each variable taken into account in the methodology. Share the results of your work and relate them to the literature review. Review the objectives set up at the beginning of the study. Make attempts to reflect on topics interesting to teachers, for example: how various aspects of the study may influence the results.

The implementation of PBL in applied statistics for health sciences in the Chilean university context has yielded significant results, as reflected in students' perceptions and performance. This study reveals that, overall, students have a positive view of PBL, although with variations in their rating level. When correlating students' valuation with their scores in progress monitoring and oral presentation, a clear trend emerges: those with a higher valuation tend to achieve better grades.

This trend aligns with existing literature highlighting the importance of student engagement and motivation in learning (Davidson et al., 2019; Pilot et al., 2023). The study by Pradanti and Muqtada (2023) suggests that when students recognize the relevance and applicability of what they are learning, their performance significantly improves. This is evident in the case of Student 5, whose high valuation of the project translated into outstanding grades.

The relationship between students' valuation of PBL and their academic performance (see Table 4) supports the theory emphasizing the relevance of active and team-based learning for the development of practical and analytical skills, as indicated by Huang et al. (2023). It is observed that students who exhibit higher appreciation for PBL tend to be more engaged in their learning process. This increased commitment is reflected in the advanced development of statistical competencies and their effective application in practical contexts.

On the other hand, the existence of students who slightly value PBL or do not value it at all, and who receive lower grades, indicates potential areas for improvement in the implementation of PBL. This finding suggests that some students may need more support to fully value the benefits of PBL, which is crucial for their engagement and performance. As highlighted by Elsamanoudy et al. (2021), it is essential to consider individual needs and differences in students' learning experiences.

The experience of the mathematics teacher highlights educational enrichment through PBL, emphasizing not only the importance of integrating various disciplines such as kinesiology into the teaching of statistics but also the need for continuous evolution in her field (Maass et al., 2019). This outcome has led to recommendations for the dissemination of PBL among teachers and students at all educational levels. Furthermore, it advocates for a pedagogy that values bidirectional communication and collaborative learning, allowing for the inversion of roles between teachers and students for mutual enrichment (Xie & Derakhshan, 2021). The implementation of PBL in statistics has shifted teaching towards practical applications, enhancing statistical understanding and contributing to patient quality of life, thereby demonstrating the relevance of statistics in professional contexts (Guo et al., 2020; Rajula et al., 2020).

This study on PBL in applied statistics for health sciences among Chilean university students highlights significant implications for statistics teaching in the field of health.

Pedagogical Implications: PBL has been shown to improve analytical and practical skills in statistics. Therefore, educational programs must integrate theory with applied practice, as suggested by Dawadi et al. (2021) and Mertens (2023). Constructive feedback and the development of communication skills are crucial to prepare students for professional communication (Ridlo, 2020). Additionally, effective collaboration, vital for success in group projects, should be encouraged in curricula (Hussein, 2021). Revelle et al. (2020) emphasize the importance of spaces for personal reflection and the analysis of learning experiences.

Implications for Academic Assessment: The correlation between the appreciation of PBL and grades indicates the need to adopt assessment methods that go beyond traditional approaches, recognizing active participation and the development of practical skills (Birdman et al., 2022).

Although the findings suggest that PBL enhances analytical and practical skills and fosters motivation (Dawadi et al., 2021; Mertens, 2023), one of the main limitations is the specific focus on the Chilean context. While it provides deep insights into students' perceptions and adaptation to this pedagogical approach in a particular setting (Huang et al., 2023), it may not fully capture the varied experiences and perceptions in different educational environments within health sciences. This highlights the need for comparative studies in other geographical and cultural contexts to enrich the global understanding of PBL in statistics.

Furthermore, future research should delve into the nuances of how PBL in statistics is received and implemented across different cultural contexts and educational systems. Investigating the variations and parallels in the implementation of PBL and its perception among various countries and academic disciplines may provide a richer and more nuanced understanding of its effectiveness and adaptability to diverse learning environments (Elsamanoudy et al., 2021; Pilot et al., 2023). Additionally, conducting longitudinal studies that track students over significant periods (from lower secondary education to higher education) emerges as a particularly valuable recommendation. Such research is poised to shed light on the sustained effects of PBL on students' academic and professional growth, offering insights into the lasting influence of PBL on career paths and academic advancement, especially within the realm of health sciences (Davidson et al., 2019). Adopting a comprehensive longitudinal approach, as suggested, would significantly enhance our understanding of the long-term impact of PBL, from secondary education to tertiary levels, thereby providing a more holistic view of its benefits and challenges across the educational spectrum.

In conclusion, this study illuminates the implementation of PBL within the Chilean educational system, providing valuable insights. However, for a more comprehensive and globally informed perspective on this educational approach, future research should expand its focus both geographically and temporally. To maximize the potential of PBL, it is essential for educators and curriculum designers to effectively communicate its goals and advantages (Ngereja et al., 2020), incorporate hands-on activities like laboratory work to enhance its impact (Domenici, 2022), and ensure ongoing support for students who may find PBL challenging (Boss & Krauss, 2022). Crucially, efforts should be made to facilitate students' exploration of learning opportunities that extend beyond the traditional curriculum.

Therefore, it becomes imperative to not only spread knowledge about PBL among secondary and higher education teachers and students but also to engage policymakers who may not be familiar with educational methodologies. By broadening the awareness and understanding of PBL among these key stakeholders, the groundwork can be laid for more informed decisions that support the integration of PBL into educational systems, thereby fostering an environment where learners can thrive both within and beyond the confines of their school curriculum (Authors, XXXX).

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On a typology of errors in integral calculus in secondary school related to algebraic and graphical frames.

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Abstract: The present study falls into the efforts to improve practices for addressing errors produced by learners in various situations involving the calculation of integrals. We attempt to clarify as precisely as possible the types of errors that secondary school students produce when using integrals in algebraic and graphical frames. Based on the synthesis of several works dealing with errors specific to integral calculus, we have been able to outline a typology of possible errors that can be produced by students in secondary school. We determine some subcategories for the three known categories of errors: conceptual, procedural, and technical.
After administering a test to a random sample of secondary school students and conducting a principal component analysis, we were able to deduce that in the algebraic frame, certain conceptual and procedural subcategories dominate, with a notable advance for errors due to failure to recognize the integrand function. In the graphical frame, errors related to technical subcategories represent a major source of the erroneous productions of the students tested.

Keywords: Teaching integrals, errors, misconceptions, interplay of frames.

INTRODUCTION

The concept of definite integral, like most concepts in real analysis, is polysemous. It can be interpreted in terms of area, primitives, or the limit of a sum. This diversity is also due to the fact that it is used in several disciplines. For example, it is used to calculate the mean value of a given quantity over a bounded interval. In many countries, it is taught in secondary schools and continues to be taught in higher education. It is part of what is known as modern analysis.
However, several dysfunctions have been pointed out in teaching and learning practices for integrals, as revealed by the authors (El Guenyari, Chergui, & El Wahbi, 2022). They concluded

that the lack of implementation of frame-changing activities and the conversion of registers of semiotic representations negatively impacts the learning of integrals. Here, a frame and a semiotic representation register are used with the same senses as stated by Douady (1986) and Duval (1993) respectively. Based on Donaldson's (1963) classification of errors, Orton (1983) observed difficulties with algebraic and graphical frames in high school and university learners on questions concerning integration and limits. The symbols used to write and calculate integrals were also found to be a source of difficulty.

In an attempt to provide more operationality to the analysis of student integral errors, Seah (2005) conducted research from which he was able to draw the conclusion that students had difficulty with problems involving the integration of trigonometric functions and the use of integration to calculate the areas of specific regions of the plane. When it came to activities dealing with the conceptual elements of integration, students paid less attention than they did with those dealing with the procedural aspects. When asked to determine the area of a surface defined by a function's curve, the x-axis, and two vertical lines, for instance, students failed to take the position of the curve with respect to the x-axis into consideration.

According to the research done by Muzangwa and Chifamba (2012) basic algebra knowledge deficiencies are a significant cause of errors in integral and differential calculus. Additionally, errors and misunderstandings are related to a lack of advanced mathematical thinking, which can be remedied by using a variety of processing frames for mathematical concepts.

The lack of a deeper understanding of the integral was also reported by Ely (2017) after finding that students were unable to cope with situations slightly modified from those with which they were familiar. Ely (2017) explained this vulnerability by pointing out that students had just acquired procedural knowledge of integration in terms of techniques without achieving adequate conceptual knowledge of the underlying structures. As an example, Darvishzadeh et al. (2019) observed that procedural errors are due to confusion between the processes of differentiation and integration.

V. L. Li et al. (2017) deduced, through a study carried out with higher education students, that conceptual errors have many consequences, such as the erroneous use of symbols like dx, the implementation of integration techniques by parts or by change of variables, and the inability to recognize the determination of primitives of usual functions. For this last reason, some students gave the following incorrect answer: $\int -\frac{2}{x}+\frac{3}{x+2}+\frac{2}{(x+2)^2}dx = -2ln|x| + 3\,ln(x+2) + 2ln|(x+2)^2| + c$. In fact, both primitives $ln(x+2)$ and $2ln|(x+2)^2|$ suggested by students are not correct.

Errors related to the manipulation of the bounds of the integral and the variable indicator dx have also been observed by Khanh (2006) with Vietnamese students who cannot understand that one can talk about an interval associated with a primitive, and they consider dx to be a useless factor and omit it from their productions.

It is interesting to note that, in parallel with these attempts to delimit as far as possible the sources of errors produced by learners in calculating integrals, work was also underway to develop the field of didactics dealing with the study of errors and misconceptions. In this context, we refer the interested reader to (Rushton, 2018; Ahuja, 2018), for example. This work covers both didactic and cognitive aspects (Porth, Mattes, & Stahl, 2022). Thus, several error typologies have been developed (Rong & Mononen, 2022) to better understand the nature of these errors and to set up effective remedial processes.

In this work, we are interested in classifying the errors produced by secondary school students in activities involving integrals. This classification has a cognitivist focus. More specifically, we focus on the types of errors made by secondary school students when dealing with integrals in both algebraic and graphical frames. Thus, we attempt to answer the following two main research questions:

- What types of errors are produced by high school students when dealing with the concept of integrals algebraically and graphically?
- What are the main factors that explain the types of errors that can be observed?

CONCEPTUAL FRAME

According to Descomps (1999), an error is a process that marks a difference between the reference point fixed by the didactic contract and the erroneous production. It should be recalled that the didactic contract is defined by criteria set by the teacher, based on the prescriptions of the curriculum and teaching resources. So, by retaining from this reminder that the various actors responsible of the contract are the ones who found the pedagogical practices, we can affirm that the error strongly depends on the context in which it appears. In other words, a statement that may be considered true in one situation may no longer be so in another. Errors are part of learning and provide information for both teacher and learner.

According to Fiard and Auriac (2005), error is essentially the product of a difference between what is produced and what a subject was expected to produce, in view of what he or she was assumed to know how to do. For these two authors, a student's error reflects his procedures, conceptions, or representations that are erroneous and not adapted to the context. However, the student who makes an error is not aware of it because he thinks that he is reasoning adequately.

The advantage of these characterizations of error is that they exclude any moral judgment on students' productions and place the responsibility for error on the student. On the contrary, Fiard and Auriac believe that error is useful for both teacher and learner, as it indicates the mental processes involved in learning.

In the literature, a clear distinction is made between the three concepts of error, difficulty, and obstacle. Difficulty refers to any condition in a situation that increases the probability of producing errors. Language difficulties and disturbances in the development of certain academic skills are examples that may well illustrate the meaning of a difficulty (Chergui, Zraoula, & Amal, 2019).

The obstacle is a witness to the slowness, regressions, and analogies that emerge during the thought formation process (Astolfi, 2015, p. 44). Obstacles encountered in the learning process manifest themselves materially in the production of observable errors. So, the two concepts, errors and obstacles, are complementary. Errors may be due to limitations in the student's intellectual capacities. In this case, we speak of an ontogenetic obstacle. An obstacle is described as epistemological when the knowledge acquired by the student does not enable him or her to carry out a new task proposed by the teacher. The third type of obstacle is called didactic, and includes everything to do with the didactic system put in place by the teacher: poorly formulated instructions, problems relating to the organization of the lesson, interpersonal relations, didactic transposition, and so on.

Given the importance of errors in teaching and learning processes, a number of studies have focused on classifying them. Donaldson (1963, pp. 183-185) identifies three types of error: structural, arbitrary, and executive. The first is due to the inability to appreciate the relationships involved in the problem, the second is due to the student's failure to take account of the constraints established in what is given; and the final one is caused by the inability to perform manipulations while understanding the underlying ideas.

Other studies have used the stages of problem solving established by (Newman, 1977) to identify student errors. These are: reading the statement, comprehension, transformation, process skills, and coding. Based on these procedures, the Australian Ivan Watson (1980) identified eight types of error as follows:

- Inability to read the statement of the situation. For example, he does not recognize words or symbols.
- Inability to understand the situation. This refers to general comprehension. For instance, the meaning of certain terms or symbols.
- Difficulty in identifying the mathematical processes required to obtain a solution.
- Technical difficulties manifest themselves in the inability to perform the mathematical operations required for the task.
- Coding problems are reflected in the inability to write the answer in an acceptable form.
- Motivational problems. The student would have solved the problem correctly if he had tried.
- Errors due to carelessness. These are inattentional errors that are unlikely to be repeated.
- Errors caused by the inappropriate way in which the problem was presented.

It should be noted that categories 2 and 8 are not identical. The first refers to the student who may misunderstand the statement, while the second indicates that the statement or instruction is inadequately formulated.

Based on Donalson's (1963) typology, Orton (1983) conducted a study of students' performance in calculus. Student responses to tasks concerning integration and limits indicated that students had difficulty understanding that integration is the limit of a sum and that there is a relationship between a definite integral and areas under the curve. According to him, many teachers accepted the fact that integration could not be made easy and reacted in various ways. In order to examine students' thinking and misconceptions in dealing with the Riemann integral. An investigation conducted by Thomas and Ye (1996) indicated that students' adherence to an instrumental and procedurally oriented way of thinking, which obstructed them from grasping crucial concepts, resulted in a lack of conceptual knowledge on their part.

In light of the work of Donaldson (1963) and Orton (1983), Seah (2005) has developed a conceptual framework for classifying the various errors and misconceptions that students may encounter when solving integration problems. The errors that students may make have been classified into the following three categories:

- Conceptual errors manifested by the failure to grasp the concepts in the problem or to appreciate the relationships in the problem.

- Procedural errors are attributed to failure to carry out manipulations or algorithms, although concepts in the problem are understood.
- Technical errors are due to a lack of mathematical content knowledge in other topics or to carelessness.

It is in the light of these three complementary aspects that our exploration of the types of errors committed by learners in integral calculus will be undertaken.

METHODOLOGY

To provide answers to the questions posed in this study and with reference to the literature review outlined above, we will use a test to explore the errors made by secondary school pupils. The results obtained will be analyzed both quantitatively and qualitatively.

Data collection

The Riemann integral is part of the course for the final year of secondary school in Morocco (MEN, 2007). The course begins with a presentation of the definition of the Riemann integral over an interval [a,b] using the Newton-Leibniz formula, followed by a statement of the computational properties and the technique of integration by parts. As applications of definite integrals, the program (MEN, 2007) stipulates applications to the calculation of the area of a part of the plane or of a volume.

Our investigation will be undertaken via the test, which is made up of 11 questions, divided into algebraic questions from Q1 to Q6 and graphical ones from Q7 to Q11. A statement of questions and possible answers is presented in Table 1.

Questions	Response strategies
Q.1 Let f be a continuous numerical function on $[1, 5]$ such that $\int_1^5 f(x)\,dx = 10$. F denotes a primitive function of f on [1,5]. Evaluate $F(5) - F(1)$.	Implementing the relation $\int_a^b f(x)dx = F(b) - F(a)$ where F is a primitive of f.
Q.2 f is a continuous function defined on $[a, b]$. Is there any relationship between the following two integrals: $\int_a^b f(x)dx$ and $\int_b^a f(z)dz$?	Using the notation $\int_a^b f(x)dx = -\int_b^a f(z)dz$.
Q.3 Let f be the numerical function defined on $\mathbb{R}$ by $f(x) = (x-3)^2$. Show that $\int_1^3 f(t)\,dt = \frac{8}{3}$.	Employing Newton-Leibniz formula after determination of a primitive function.
Q.4 Calculate the integral: $I = \int_1^3 (e^x + x\,lnx)dx$.	The use of the integral linearity and an integration by parts.
Q.5 Calculate $\int_1^3 \lvert e^x - e^2 \rvert\,dx$.	Application of Charles's relation to remove the absolute value.
Q.6 Calculate $\int_0^1 (e^{2y} + x)\,dy$.	Recognition of the variable to be considered in the integration.

Q.7 What is the sign of the following integral, $J = \int_2^1 (x-3)\ln(x)\, dx$? Justify the answer.

Determining the sign of the integrand function and compare the bounds of the integral.

Q.8 Let f be the numerical function defined on $\mathbb{R}$ by $f(x) = x - 1$ and represented in an orthonormal coordinate system $(o; \vec{\imath}; \vec{\jmath})$ by the curve (C_f) in Figure 1. Calculate by two methods $\int_2^4 f(x)dx$.

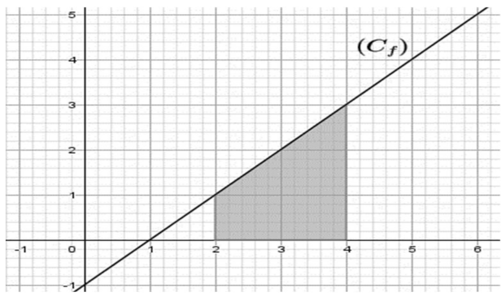


Figure 1

Method 1: Investing the integrals.
Method 2: recognizing the geometric figure concerned by the area calculation.

Q.9 Using integrals, express the area A of the domain of the plane colored in gray in Figure 2 below. (C_f) and (C_g) denote the respective curves of two functions f and g.

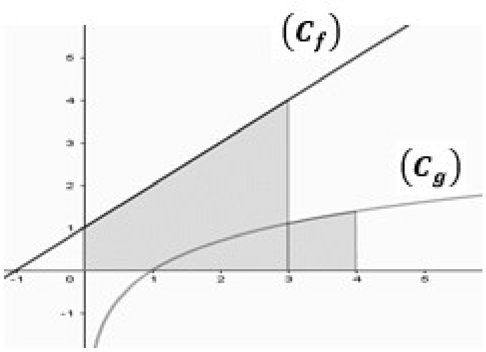


Figure 2

Using one of the two following formulas:

$A = \int_0^3 f(x)dx + \int_3^4 g(x)dx$ or

$A = \int_0^3 |f(x) - g(x)|dx + \int_1^4 g(x)dx.$

Q.10 Let f be the numerical function defined on $\mathbb{R}$ by $f(x) = 3x^2 - 4$ and (C_f) its representative curve in an orthonormal coordinate system $(o; \vec{\imath}; \vec{\jmath})$ (Figure 3). Calculate the area of the part of the plane colored in grey.

Employing the relation $A = \int_{-1}^{1} |f(x)| dx$ and take into account the position of the curve with respect to the abscissa axis.

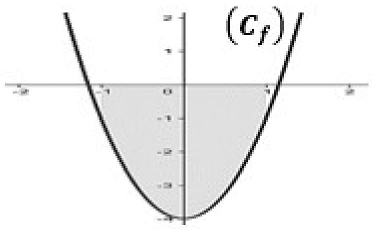
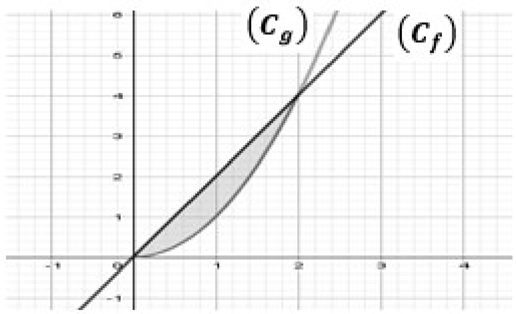
 Figure 3 Q.11 Let f and g be two continuous functions on $[0; 2]$ (Figure 4). Express the area of the domain colored in gray by an integral.  Figure 4	 Using the relation $A = \int_0^2 \lvert f(x) - g(x) \rvert dx$.

Table 1 : Test administered

The test was administered in April 2023 to 43 students in their final year of high school (17-18 years old) in the experimental sciences after they had taken the calculus course on integrals. The test participants were from various secondary schools in the Rabat-Salé-Kenitra Regional Education and Training Academy.

Results analysis tools

Based on the conceptual framework developed in the previous section, we set up the grid presented in Table 2, which presents a categorization of integral calculation errors according to algebraic and graphical frames.

Error category	Subcategories related to the algebraic frame	Subcategories related to the graphic frame
Conceptual	Ca1: Combine or confuse primitive and derivation. Ca2: Failure to recognize the integrand function. Ca3: Failure to master the importance of bounds.	Cg1: Lack of understanding of the link between integral and area or volume. Cg2: Inability to recognize the part of the plane concerned by the area calculation.

Procedural	Pa1: Inappropriate choice of operation or property. Pa2: Incorrect implementation of integral calculation by direct determination of a primitive. Pa3: Incorrect implementation of integration by parts.	Pg1: Inappropriate choice of formula for calculating the requested dimension. Pg2: Failure to take into account the position of the curve in relation to the x-axis.
Technical	Ta1: Errors in algebraic calculation Ta2: Errors in applying algebraic properties of common functions. Ta3: Errors in formulating the answer (e.g., forgetting dx, bounds, not placing bounds correctly).	Tg1: Failure to cut out correctly the part of the plan concerned by the surface measurement. Tg2: Failure to read point coordinates correctly.

Tableau 2: Grid for categorizing integral errors according to algebraic and graphical frames

Errors in students' responses to the test questions are classified according to the subcategories shown in Table 2. Each subcategory was encoded to simplify the treatment of the collected data. An enumeration of the numbers in each subcategory is carried out in order to perform an advanced statistical study. For this purpose, we opt for data processing using SPSS software.

RESULTS

Referring to the research questions posed, we will be mainly interested by analyzing the results obtained according to each frame separately, namely the algebraic and graphical frames. The cross-study of these two frames is not the object of this work, nor is it a question of re-exploring the importance of the complementarity between these two frameworks in learning the notion of integrals. But first, let us take a look at a sample of the errors made by the students tested.

Incorrect student productions

After examining the copies of the students tested, we identified the errors listed in Table 4.

Questions	Number of false answers	Errors in learners' productions
Q_1	17	• Since $\int_1^5 f(x)dx = 10$, $f(x) = 10$. So, $F(x) = 10x$, whence $F(5) - F(1) = 40$. • $f(5) - f(1) = 5x - x = 4x$.
Q_2	19	• The answer is yes without giving the relationship. • There is no relationship between the two expressions. • $\int_a^b f(x)dx = [F(x)]_a^b$ and $\int_b^a f(z)dz = [F(z)]_b^a$ (no comparison is given).
Q_3	5	• $\int_1^3 f(t)dt = \frac{1}{2}\int_1^3 (x-3)^2 dt = \frac{1}{2}\,[(x-3)^2]_1^3$.
Q_4	31	• $\int_1^4 xlnxdx = \left[\frac{x^2}{2} lnx\right]_1^4$.

		• Errors in implementing integration by parts.
		• Linearity of the integral not used
Q_5	39	• $\int_1^3 \lvert e^x - e^2 \rvert \, dx = \int_1^3 e^x - e^2 \, dx = [e^x - e^2 x]_1^3$ • $\int_1^3 \lvert e^x - e^2 \rvert dx = \left\lvert \int_1^3 e^x - e^2 dx \right\rvert$
Q_6	35	• $K = \int_0^1 (e^{2y} + x) \, dy = \left[\frac{e^{2y}}{2}\right]_0^1$ • $K = \int_0^1 (e^{2y} + x) \, dy = \int_0^1 (e^{2y} + y) \, dy$ • $K = \int_0^1 (e^{2y} + x) \, dy = \int_0^1 x dy = \left[\frac{x^2}{2}\right]_0^1$
Q_7	26	• J is negative because $(3 - x)$ is negative on $[1; 2]$.
Q_8	12	• No student was able to calculate the integral by recognizing the figure (trapezoid). • $\int_2^4 x - 1 = \left[\frac{x^2}{2} - x\right]_2^4$ (The absolute value and dx are missing)
Q_9	19	• $A = \int_0^3 f(x)dx + \int_1^4 g(x)dx$ • $S = \int_0^4 f(x) - g(x) \, dx$ • $A = \int_0^3 f(x)dx + \int_3^4 f(x)dx$
Q_{10}	16	• $\int_{-1}^1 3x^2 - 4 \, dx = [x^3 - 4x]_{-1}^1$ • $\int_{-1}^1 -f(x) \, dx = [-3x^2 - 4]_{-1}^1 = -8$
Q_{11}	9	• $A = \int_0^1 f(x) - g(x) \, dx.$ • $A = \int_0^2 f(x)dx + \int_0^2 g(x)dx$

Table 3: List of errors made by students

In addition to these errors, numerous algebraic calculation and notation errors were observed. A sample of the students' erroneous productions is provided in the Appendix.

Univariate analysis

Each of the errors listed in Table 3 was classified using the grid in Table 2, taking into account the frame used in the question and the corresponding aspect. To illustrate this, we take the example of the first error in Table 3. The answer given falls within the algebraic frame, and it is clear in this case that the student has not yet acquired that calculating the integral involves a primitive function. Consequently, this error falls into subcategory Ca1.

We used the straightforward descriptive statistics exhibited in Table 4 to analyze the responses in order to get a preliminary overview of the respondent population.

Subcategories	Scores	Mean	Std. Deviation
Ca1	11	,2558	,62079
Ca2	91	2,1163	1,69325
Ca3	26	,6047	,54070
Pa1	86	2,0000	1,19523
Pa2	42	,9767	,59715
Pa3	2	,0465	,21308
Ta1	41	,9535	,89850
Ta2	17	,3953	,54070
Ta3	64	1,4884	1,16235
Cg1	14	,3256	,47414
Cg2	1	,0233	,15250
Pg1	28	,6512	,78327
Pg2	9	,2093	,41163
Tg1	12	,2791	,54883
Tg2	9	,2093	,51446

Table 4 : Descriptive data

Scores were determined by counting the number of occurrences for each subcategory across all student productions. For the averages, indicated in Table 4, they are calculated by considering the total number of students tested. Thus, the first value 0.2558 represents the average of the subcategory Ca1 in the sample studied.
It is interesting to note similarities in certain averages. This concerns pairs of subcategories (Ca2, Pa1), (Pa2, Ta1), and (Pg2, Tg2). To confirm or refute this point, it is convenient to carry out a test of the averages using the t-test of two independent samples. If the p-value is less than the significance level ($p < 0.05$), the difference does not equal zero.

		Paired Samples Correlations		Paired Differences					t	df	Sig. (2- tailed)
							95% Confidence Interval of the Difference				
		Correlation	Sig.	Mean	Std. Deviation	Std. Error Mean	Lower	Upper			
Pair 1	Ca2 - Pa1	,424	,005	,1162	1,606	,2450	-,3781	,6107	,475	42	,638
Pair 2	Pa2 - Ta1	-,046	,767	,0232	1,101	,1680	-,3158	,3623	,138	42	,891
Pair 3	Pg2 -Tg2	,126	,423	,0000	,6172	,0941	-,18995	,1899	,000	42	1

Table 5 : Paired Samples Test

We observe that the variables Ca2 and Pa1 are moderately and positively correlated (r = 0,424, p = 0,005). This situation is no longer statistically true for the pairs (Pa2, Ta1) and (Pg2, Tg2) since the significance level exceeds the accepted value. Furthermore, there is no significant average difference between the three evoked pairs. This statement is also confirmed by the fact that the mean of the differences for each pair of variables lies within the confidence interval.

Bivariate analysis

The cross-tabulation of the variables indicating the different subcategories of errors that were identified during the processing of the activities on integrals enabled us to highlight some significant correlations at the 0.05 level (2-tailed), as shown in Table 6.

		Ca1	Ca2	Ca3	Pa1	Pa2	Pa3	Ta1	Ta2	Ta3	Cg1	Cg2	Pg1	Pg2	Tg1	Tg2
Ca1	Correlation	1														
	Sig.															
Ca2	Correlation	,537	1													
	Sig.	,000														
Ca3	Correlation	,308	,285	1												
	Sig.	,044	,064													
Pa1	Correlation	,385	,424	,147	1											
	Sig.	,011	,005	,346												
Pa2	Correlation	,402	,450	,266	,600	1										
	Sig.	,008	,002	,085	,000											
Pa3	Correlation	,088	-,279	-,250	,187	,009	1									
	Sig.	,575	,070	,106	,230	,956										
Ta1	Correlation	-,106	,113	-,039	,089	-,046	-,113	1								
	Sig.	,498	,470	,805	,572	,767	,471									
Ta2	Correlation	,330	,261	-,023	,074	,029	,043	-,010	1							
	Sig.	,031	,091	,885	,639	,853	,783	,948								
Ta3	Correlation	-,078	,140	-,064	,171	-,018	,002	,045	,064	1						
	Sig.	,618	,371	,682	,272	,911	,989	,774	,682							
Cg1	Correlation	,277	,337	,050	,126	,196	,082	-,019	,229	,223	1					
	Sig.	,073	,027	,752	,421	,209	,600	,901	,140	,151						
Cg2	Correlation	-,064	-,011	,114	-,131	,006	-,034	,356	,175	-,200	,222	1				
	Sig.	,682	,946	,466	,404	,969	,828	,019	,263	,199	,152					
Pg1	Correlation	,090	-,023	,116	,229	,033	,100	,010	,109	-,122	,441	,269	1			
	Sig.	,566	,886	,457	,140	,833	,525	,948	,489	,435	,003	,081				
Pg2	Correlation	-,121	-,207	,274	,145	,117	,158	,027	-,060	,080	,253	,300	,601	1		
	Sig.	,438	,184	,076	,353	,454	,312	,864	,704	,611	,102	,051	,000			
Tg1	Correlation	-,145	-,318	-,341	,073	-,198	,294	,172	-,060	,080	,374	,205	,620	,368	1	
	Sig.	,355	,038	,025	,644	,204	,056	,271	,704	,611	,013	,187	,000	,015		
Tg2	Correlation	-,023	,081	,048	,039	-,216	-,091	,279	,123	,024	,104	-,064	,422	,126	,378	1
	Sig.	,886	,607	,761	,805	,164	,562	,070	,430	,878	,505	,686	,005	,423	,012	

Table 6: Pearson correlations between subcategories of errors

In light of these results, which highlight a number of correlations between the variables studied, we feel it would be interesting to make further progress in processing the results obtained. To this end, we will conduct a principal component analysis (PCA).

Principal component analysis

In order to highlight the different subcategories of errors made by the students in their answers to the test on integrals, we carried out a PCA, which allows multivariate analysis of all the variables. PCA is administered with quantitative variables or with measured hierarchical variables. The principle of PCA is to minimize the number of variables. The new variables are called factors and represent linear functions of the initial variables.

The adequacy of the sample must be examined first in PCA (Johnson & Wichern, 2002). To achieve this, two tests can be administered: the Kaiser-Meyer-Olkin (KMO) test and Bartlett's sphericity test. The first gives a proportion of the variance between variables that could be a common variance. It is scored from zero to one, with zero being inappropriate and a value close to one being appropriate. For the Bartlett test, the observed correlation matrix is compared with the identity matrix. In general, KMO values of at least 0.50 and $p < 0.05$ for the Bartlett sphericity test are considered acceptable.

		Graphic frame	Algebraic frame
Number of items		6	9
Kaiser-Meyer-Olkin Measure of Sampling Adequacy		,714	,611
Bartlett's Test of Sphericity	Approx. Chi-Square	62,652	76,061
	df	15	36
	Sig.	,000	,000

Table 7: KMO and Bartlett's Test

From the values obtained, we can deduce that:

- Since the KMO index is sufficiently greater than 0.5, all items are factorable (2006);
- Bartlett's test revealed that the calculated p-value is below the 0.05 level of significance. It is therefore appropriate to reject the hypothesis that there is no correlation significantly different from 0 between the variables and to accept the fact that there are correlations that are not all equal to zero.

With regard to reliability, the Cronbach's coefficient was calculated for the items relating to each frame. The results are as follows:

Frame	Cronbach's Alpha	N of Items
Algebraic items	,721	9
Graphic items	,735	6
All items	,713	15

Table 8: Reliability Statistics

The reliability of our grid is therefore satisfactory. We can therefore conclude that all the items contribute to the reliability of the grid and that no purification is necessary.
To understand student performance in each frame, we carried out a PCA according to each frame.

- **PCA according to graphical frame**

By implementing the PCA on all the items in the graphical frame without previously fixing the number of factors requested, we obtained the results presented in Table 9.

Component	Initial Eigenvalues			Extraction Sums of Squared Loadings		
	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %
FG 1	2,668	44,459	44,459	2,668	44,459	44,459
FG 2	1,143	19,058	63,516	1,143	19,058	63,516
FG 3	,774	12,902	76,418			
FG 4	,663	11,042	87,460			
FG 5	,488	8,126	95,587			
FG 6	,265	4,413	100			

Table 9: Total variance explained by applying PCA for graphic frame

Using the Kaiser criterion, the components to be retained are those with an eigenvalue greater than 1. Consequently, the first two components explain more than 63% of the total variance, making a total of 44.45% for the first and 19.05% for the second. The sum of the corresponding eigenvalues is 3,8. This means that these two components can replace almost four items. Note also that the sum of the eigenvalues is equal to 6, which is the total number of items considered. The contribution of each subcategory of errors in forming the principal components is explicated in Table 10.

	FG1	FG2	FG3	FG4	FG5	FG6
Cg1	13,673	3,021	69,373	1,302	9,706	2,926
Cg2	6,855	41,098	3,979	46,931	1,095	0,041
Pg1	29,787	0,680	0,937	2,056	0,265	66,275
Pg2	18,259	5,451	22,565	31,578	3,629	18,519
Tg1	22,898	3,385	0,855	1,729	64,659	6,473
Tg2	8,528	46,365	2,291	16,404	20,647	5,766

Table 10: Contributions of graphic variables (%)

The Component plot of factors 1 and 2 on the F1 (component 1) and F2 (component 2) axes is shown in Figure 5. It corresponds to a projection of the initial variables onto a two-dimensional plane constituted by the two factors.

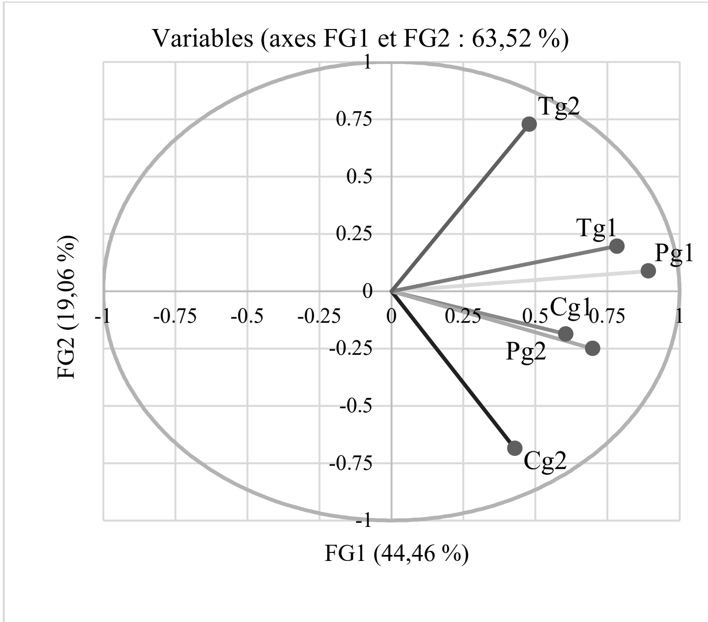


Figure 5 : Circle of correlations for graphic frame

- **PCA according to algebraic frame**

Taking into account the same considerations as in the previous case, we carried out a PCA, which gave the results listed in Table 11.

Component	Initial Eigenvalues			Extraction Sums of Squared Loadings		
	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %
FA1	2,636	29,292	29,292	2,636	29,292	29,292
FA 2	1,349	14,987	44,279	1,349	14,987	44,279
FA 3	1,201	13,339	57,618	1,201	13,339	57,618
FA 4	1,113	12,368	69,986	1,113	12,368	69,986
FA 5	,928	10,312	80,298			
FA 6	,662	7,355	87,653			
FA 7	,497	5,520	93,173			
FA 8	,327	3,635	96,808			
FA 9	,287	3,192	100,000			

Table 11: Total variance explained by applying PCA for algebraic frame

We note that four components have eigenvalues greater than 1. There are therefore four components that can be extracted from our grid, and the cumulative variance that they can explain is 70% of the total variance. The contribution of each subcategory of errors to forming the principal components is explained in Table 12.

	FA1	FA2	FA3	FA4	FA5	FA6	FA7	FA8	FA9
Ca1	21,900	1,383	5,794	6,277	0,903	2,256	30,608	1,180	29,698
Ca2	24,139	4,463	3,021	2,491	0,019	5,544	13,952	0,026	46,344
Ca3	8,448	18,189	9,269	1,543	2,906	51,296	6,179	0,087	2,084
Pa1	19,670	8,260	3,983	11,804	0,219	0,146	3,289	50,890	1,739
Pa2	21,379	0,829	0,655	13,185	0,019	10,453	15,618	37,210	0,653
Pa3	0,206	57,374	0,762	1,130	2,225	17,154	1,234	3,544	16,371
Ta1	0,012	5,368	33,851	1,166	47,665	8,010	0,030	2,959	0,938
Ta2	3,912	2,911	0,725	62,303	1,298	0,104	28,670	0,062	0,015
Ta3	0,333	1,223	41,941	0,101	44,746	5,037	0,421	4,042	2,158

Table 12: Contributions of variables (%)

DISCUSSION

From Table 4, we can clearly see that the errors that fall under the subcategories Ca2, Pa1, and Ta3 are the most frequent when dealing with integrals in the algebraic frame. Referring to Table 6, we note that the two subcategories failure to recognize the integrand function and inappropriate choice of operation or property are moderately and positively correlated with a fairly acceptable significance level (p-value = 0.005). But nothing can be confirmed with regard to the correlation between the subcategory Ca2 and errors in formulating the answer. This means that, for the student tested, not recognizing the integrand function has an impact on the choice of operations required to calculate integrals but not necessarily on the ability to formulate answers. This result is very interesting didactically. In fact, the calculation of integrals, whether directly by the Newton-Leibniz formula or by another technique, requires the determination of primitive functions. It is to this latter task that the teacher must then pay attention to mitigate the impact of the inability to recognize the functions to be integrated.

For the graphical frame, procedural errors are dominated by inappropriate choices of formula for calculating the requested dimension, followed by conceptual errors concerning a lack of understanding the link between integral and area or volume. Moreover, the correlation between these last two subcategories (Cg1 and Pg1) of errors is positively medium. This result seems quite logical to us, given that it is unlikely that a student who fails to understand the link between the integral and the geometric quantity to be measured will correctly choose the formula to use.

Note that these results are in harmony with those deduced by Seah (2005) in his study, where he observed difficulties in calculating integrals of trigonometric functions and in their applications in area calculations. We can also state that these preliminary results are aligned with those of Muzangwa and Chifamba (2012). This has motivated us to go further in our analysis of the results obtained, with the aim of better identifying the essential factors that explain the production of errors by students.

Within the same frame, several pairs of error subcategories are highly positively correlated. These include the couples of subcategories (Pa1, Pa2), (Pg1, Pg2), and (Pg1, Tg1). It is interesting to pay attention to the fact that these significant correlations relate to procedural issues in the majority of cases.

Other subcategory pairs are moderately positively correlated. For example, (Ca2, Pa1), (Ca2, Pa2), and (Pg1, Tg2). This last positive correlation between an algebraic subcategory and a graphical one is cognitively meaningful from the study performed by the authors (El Guenyari, Chergui, & El Wahbi, 2022). That is to say, the integral should be invested in various frames in the learning situations provided to students for good cognitive functioning when processing it.

However, it should also be noted that the subcategory Tg1 is negatively correlated with Ca2 and Ca3. This means that the inability to correctly cut out a part of the plane to calculate its area is negatively correlated with incompetence in recognizing the integrand function and with a lack of appreciation of the importance of the bounds of the integral. This result, which does not seem at all normal, questions the conditions for learning the integral among the students tested. More explicitly, why do students not succeed in practicing the change of frames easily?

In addition to this surprising result, which calls for greater precision, there is a lack of information on the correlation between several pairs of subcategories, as reported in Table 6. Remarkably, no statistical results were obtained concerning the correlation between the inability to recognize the part of the plane concerned by the area calculation and all the other error subcategories of the graphical frame. This situation also extends to the algebraic framework by observing, for example, the two subcategories failure to master the importance of bounds and errors in formulating the answers.

In order to clarify these points, we carried out a statistical analysis using PCA. Focusing on axis 1 (FG1) in Figure 5, we see that the first factor FG1 is positively correlated with all of the initial graphical subcategories. This correlation is quite strong with the subcategories Pg1 and Tg1. According to the results in Table 10, these two variables are the most important in forming the principal component FG1. This can be interpreted by the fact that the errors that fall into these two subcategories, which can be considered as forming the FG1 factor, are the main contributors to total variability.

The presence of an acute angle between two variables indicates that they are fairly well correlated. This is the case between several subcategories, as shown in Figure 5. But when the angle is almost right, the variables are rather uncorrelated. The inability to recognize the part of the plane concerned by the area calculation and the failure to read point coordinates correctly fall into this latter case.

With regard to the second principal component FG2, the subcategories Cg2 and Tg2 present a high correlation, which can be clearly visualized by the projection on the vertical axis in Figure 5. To identify this second principal component, it should be noted that the subcategories Cg2 and Tg2 make a major contribution to its formation. In Table 10, their contributions are 41.098% and 46.365%, respectively.

The analysis of the circle of correlations in Figure 5 shows that axis 2 highlights an opposition between subcategories with positive correlations (Pg1,Tg1,Tg2) and those with negative correlations (Cg1, Cg2, Pg2).

From the PCA carried out on the nine subcategories of errors on integrals related to the algebraic frame, it turned out, as shown in Table 10, that four principal components can be extracted. The first, which contributes to explaining over 29% of the variance, is mainly formed, according to Table 12, by the subcategories Ca1, Ca2, and Pa2. The second main component, which explains around 15% of the total variance, is made up mainly of the Pa3 and Ca3 subcategories. The component FA3 is formed by the subcategories Ta1 and Ta3. Finally, the last principal component, FA4, is essentially formed by the subcategory Ta2, with a proportion that exceeds 62%. It is important to note that the principal components FA3 and FA4 are formed by technical subcategories and together contribute in explaining almost 25% of the total variability. So, almost 75% of the types of errors committed by students in the algebraic frame are mainly due to conceptual reasons, followed by procedural ones. Errors due to an inappropriate choice of operation or property are not included in the latter type. The impact of a lack of conceptual knowledge about integrals was underlined by Ely (2017).

CONCLUSION

No one doubts the importance of learning integral calculus in secondary school. It is hard to conceive of a curriculum in which this notion is absent, given its usefulness in other disciplines. However, teaching it, and therefore learning it, poses problems that have a negative impact on the acquisition of other mathematical topics and other skills. Hence the need to find effective ways of attenuating the impact of these problems. One approach is to make the practice of dealing with errors produced by learners in various situations involving the calculation of integrals as reflexive as possible.

The present work fits into this context by attempting to elucidate as accurately as possible the types of errors that secondary school students make when using integrals in algebraic and graphical frames. The literature review we have carried out has enabled us to draw up a typology of errors specific to integral calculus. This typology is the synthesis of several works that have addressed the same theme. We were able to determine subcategories for the three categories of error: conceptual, procedural and technical.

On the basis of this typology, we examined the work of 43 secondary school students on a test involving algebraic and graphical questions. The examination consisted in classifying the errors identified according to 15 possible subcategories.

For the algebraic frame, conceptual errors were dominated by failure to recognize the integrand function, while procedural errors were caused by inappropriate choice of operation or property and incorrect implementation of integral calculation by direct determination of a primitive. The main technical errors are attributable to faulty algebraic calculations or incorrect formulations of the answer.

Principal component analysis showed that all the subcategories relating to conceptual and procedural aspects represent the essential factors responsible for the variability observed in the test results. Several of these subcategories are positively correlated. We cite the example of errors due to not recognizing the integrand function with those arising from the choice of operations required to calculate integrals. This conclusion has an important pedagogical character in teaching practice. It calls for sufficient attention to be paid to identifying the functions to be integrated. This is also justified from an epistemological point of view, as it is well known that for several classes of numerical functions, there are appropriate techniques for determining primitive functions. It is also

interesting to note that, in the algebraic context, technical errors do not play a significant role in students' erroneous productions.
For the graphical frame, it was concluded that errors attributed to a lack of understanding of the link between integral and area or volume, an inappropriate choice of formula for calculating the requested dimension, and failure to cut out correctly the part of the plane concerned by the surface are dominant. To overcome this conceptual problem, it is recommended to adopt teaching methods based on the employment of contextualized situations, given their effectiveness in acquiring the meaning of mathematical concepts, as deduced (Sijmkens, Scheerlinck, De Cock, & Deprez, 2022) or in (Naamaoui, Chergui, & El Wahbi, 2023). In addition, the PCA allowed us to extract two main factors that explain student errors. the technical subcategories contribute to the formation of these two main factors. In this regard, it should be mentioned that technical tasks in the graphic frame are not to be undervalued. In fact, as well as being tasks requiring meticulousness, they also demand well-developed cognitive and visual levels.

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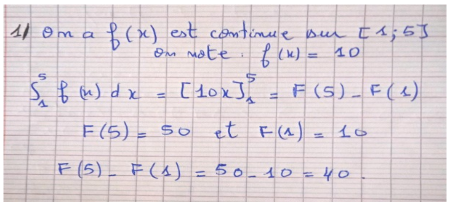
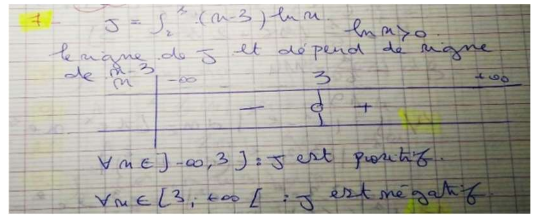
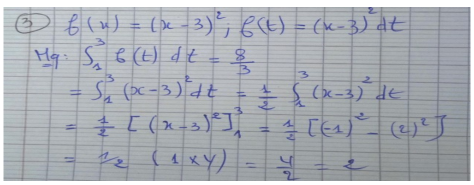
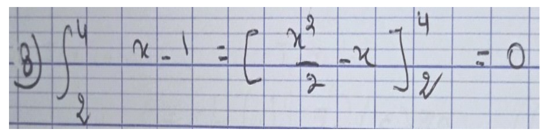
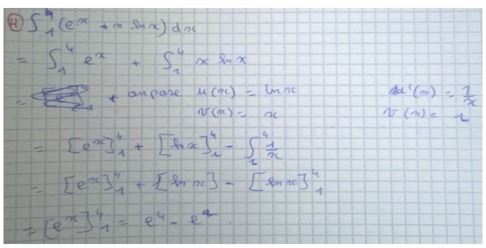
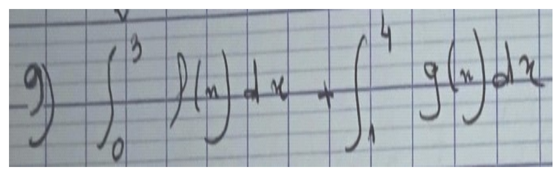
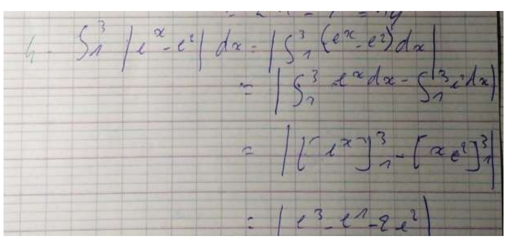
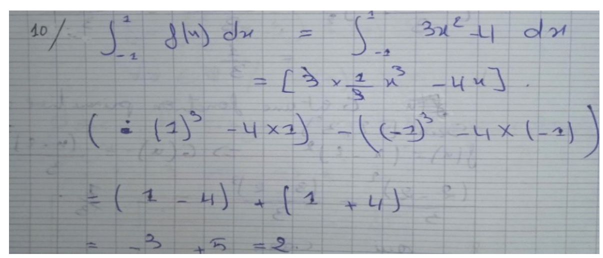
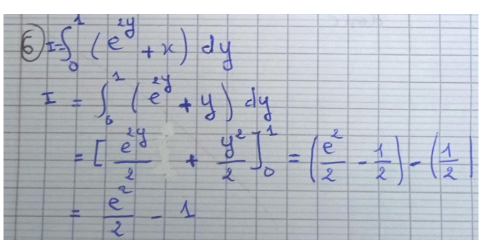
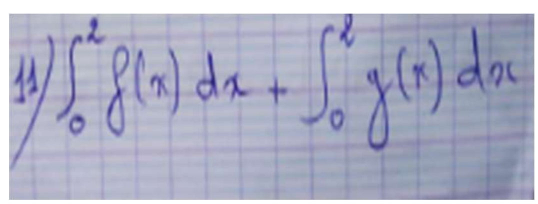
APPENDIX: Examples of Students' Erroneous Productions

Question	Example of students' productions	Question	Example of students' productions
Q1		Q7	
Q3		Q8	
Q4		Q9	
Q5		Q10	
Q6		Q11	

Investigation of Students' Mathematical Thinking Processes in Solving Non-routine Number Pattern Problems: A Hermeneutics Phenomenological Study

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Abstract: This study aimed to interpret and describe students' mathematical thinking processes of non-routine mathematical problems that were solved based on didactic situation theory. This study uses a qualitative method, a phenomenological hermeneutics study for grade 8 students at a junior high school in Banda Aceh in the 2021-2022 academic year. Research data obtained through data were collected using instruments, namely written tests based on the didactic mathematical situation theory framework, structured observation, documentation, and clinical interviews carried out after the action. The results of the study show that the students' mathematical thinking processes in the critical reflection category can reach the convincing stage with algebraic arguments in validation situations. Subjects in the explicit reflection category can reach the convincing stage by providing arithmetic arguments in validation situations. Meanwhile, the category of students who cannot solve problems and can only specialize by giving examples of what is being asked. Students in this category have difficulty identifying relevant patterns and formulating the mathematical models needed to solve the problems. To support students in developing the level of mathematical thinking, the teacher can present contextual problems that are in accordance with the level of student thinking, can predict possible responses or ways of thinking of students to the problems given and present problems according to the structure of the concept sequence and the functional order of students' thinking. To support students in algebraic thinking category, teachers can start learning by presenting contextual problems that are easily recognized by students, then expand that context in symbolic form.

Keywords: Mathematical Thinking, Didactical Situations, Problem Solving, Number Patterns

INTRODUCTION

To improve student learning activities, it is necessary to understand the status of developing their thinking and reasoning. The more information we have about what they know and think, the more opportunities we can provide for student success in the classroom (Pellegrino et al., 2001). To determine the level of students' thinking and reasoning in learning mathematics can be observed by examining the use of problem-solving strategies, display of mathematical domain knowledge, representation of the completion process, and justification of mathematical reasoning based on problem situations (Cai, 2003). Therefore, differences in students' thinking in solving mathematical problems are important to understand in order to help students learn mathematics.

Developing students' mathematical thinking has recently become a research focus (Carpenter et al., 2017; Breen & O'Shea, 2010; Fraivillig et al., 1999; Schoenfeld, 2016). Schoenfeld (2016) says that the mathematical thinking found in the process of learning mathematics means (a) developing a mathematical perspective, appreciating the process of mathematization and abstraction and having a tendency to apply it, and (b) developing competence with mathematical tools, and use them to understand the structure and build mathematical understanding. Schoenfeld says that mathematics learning needs to be directed to 1) find solutions, not just memorize procedures; 2) explore patterns, not just memorize formulas; and 3) formulate conjectures, not just do exercises. There are two references in mathematics education to define mathematical thinking. One perspective focuses on mathematical processes (Burton, 1984; Mason et al., 2010a; Polya, 1985; Schoenfeld, 1992). This perspective focuses on the problem of how mathematical thinking is realized. Another philosophy is based on conceptual improvement (Dreyfus, 1991; Freudenthal, 1973; Tall, 2002). This view is related to how individuals construct mathematical concepts in their minds.

There are three essential goals for thinking of mathematics taught in schools, first as an important goal of learning mathematics at school, second as a way of learning mathematics, and third for teaching mathematics (Stacey, 2006). Stacey said that thinking mathematically to solve problems is essential to learning mathematics at school. In a broader scope, mathematical thinking will support the development of science, technology, and economy in a country. Currently, more and more countries in the world are realizing that the economic welfare of a country is greatly influenced by a strong level of mathematical thinking, which is called mathematical literacy. Therefore, students' mathematical thinking in learning mathematics needs to get more attention from mathematics educators to equip students with the ability to think mathematically.

Number patterns are mathematical concepts that students need to solve broader problems. As a general skill, distinguishing a pattern is the foundation of the ability to generalize and abstract

(Burton, 1982; Threlfall, 1999). Threlfall (1999) says that often through the use of patterns, a teacher can unlock truths in mathematical theorems and proofs. With the experience of learning number patterns, students can explore new ideas for solving problems. This shows that the experience of identifying patterns in solving mathematical problems will enrich students' experiences in being able to solve broader problems that cannot be solved in the usual way but can be solved using patterns.

In order to obtain more complete data on mathematical thinking processes, this study will use the didactical theory framework of learning mathematics from (Brousseau, 2002), namely action situations, formulation situations and validation situations. This is as stated (Vygotsky, 1978) that learning can generate various stored mental processes that can only be operated when a person interacts with adults or collaborates with fellow friends. The interaction between students or students and teachers is expected to occur in the exchange of different learning experiences so that mental action can continue as expected. Meanwhile, scaffolding techniques can be used not only to direct the thinking process, but also to provide further challenges so that the desired mental action can occur properly. Nickels & Cullen (2017) reported on increasing the learning activities and mathematical thinking of critically ill children by using robotics within the framework of Brousseau's mathematical didactical situation theory.

Many previous researchers have examined students' mathematical thinking processes in solving mathematical problems (e.g., Gereti & Savioli, 2015; Lane & Harkness, 2012; L. Burton, 1984; Uyangör, 2019; Yıldırım & Köse, 2018). All the research that has been done only considers students' abilities in mathematical thinking that are mature in solving problems or actual development, without considering the skills of students who are still in the process of maturation or potential development. As stated by (Vygotsky, 1978), to see the effect of student learning outcomes in addition to seeing results that are already mature, actual developmental, it is also necessary to consider the abilities of students still in the process of maturing potential development.

This paper explores and makes sense of students' mathematical thinking processes in solving number pattern mathematical problems. Consistent with this aim, we will answer the following research questions. What are the students' mathematical thinking processes in solving non-routine number pattern problems within didactical situations?

Literature Review

The most common sense is that thinking mathematical can be defined as using mathematical techniques, concepts, and methods, directly or indirectly, in solving problems. Sumarmo (2010) Mathematical thinking is processing information in drawing specific conclusions based on arguments that can be justified based on mathematics. Mathematical thinking is a way of thinking about mathematical processes or methods of solving simple and complex mathematical tasks. Mason et al. (2010) say that mathematical thinking is a dynamic process that allows us to increase the complexity of ideas and broaden our understanding of mathematics. Burton (1984) argues that mathematics is not about the subject matter of mathematics but a style of thinking which is a function of certain operations, processes, and identifiable dynamics of mathematics.

Many researchers use indicators of mathematical thinking, namely specialization, generalization, conjecture, and convincing (such as Aiyub, 2023; Burton, 1984; Mason et al., 2010; Stacey, 2006; Uyangör, 2019). Tall (2002) states that mathematical thinking includes components such as abstraction, synthesis, generalization, modeling, problem-solving, and proof. (Uyangör, 2019) says specialization means choosing clear or systematic examples and testing examples of problems to understand and interpret the status of the problem. Arslan & Yildiz (2010) say specialization is completion, demonstration, explanation, and selecting one or more examples is relevant. Nihayatus et al. (2023) uses three steps in the mathematical thinking process, namely abstraction, representation, and verification.

Based on the definitions and components of the mathematical thinking process presented by the experts above, the indicators of students' mathematical thinking processes in solving number pattern problems in this study are 1) specializing; 2) making generalizations; 3) making conjectures, and 4) convincing a statement based on facts from general conclusions. The descriptions of the four indicators of mathematical thinking processes are listed in table 1 below:

Num	Mathematical Thinking process indicator	Description of Mathematical Thinking Process Indicator
1	Specialization	Choose clear or systematic examples and test samples of problems to understand and interpret problem status
2	Generalizations	Set examples, and define relationships in linguistics or mathematics
3	Conjectures	Predict relationships and outcomes in linguistic or mathematical terms
4	Convincing	Showing and communicating the reasons why something is true in arithmetic or algebraic form

Table 1 Description of the indicators of students' mathematical thinking processes in solving the problem of being patterns

METHODS

Types and Research Subjects

This type of qualitative research uses the Interpretive Phenomenological Analysis (IPA) approach, which aims to interpret and interpret a phenomenon based on human experience (Eatough & Smith, 2017). The study of meaning is closely related to phenomenology and hermeneutics, which focus on one's experience. As said (Ricoeur, 1986), it is necessary to combine the study of experience and the study of meaning and meaning with that experience because they complement each other. This was chosen to reveal the various meanings of students' mathematical thinking processes in solving non-routine mathematical problems in number pattern material. The framework for this study is based on the Theory of Didactical Situations in Mathematical (Brousseau, 2002). Hausberger (2020) combined a phenomenological-hermeneutic approach and didactic situation theory would result in a fruitful interaction between philosophy and mathematics education.

The research design used to reveal this phenomenon refers to the Indonesian Didactical Design Research (DDR), which contains three stages of analysis: prospective analysis, metapedadidactical analysis, and retrospective analysis (Suryadi, 2013). As for the study participants, there was 32 grade 8 students from a junior high school in Banda Aceh for the 2021/2022 academic year. In addition, two research subjects will be selected for each category of student groups to explore and interpret mathematical thinking processes in solving number pattern problems from the results of essay tests within the TDSM framework (Brousseau, 2002) and interviewed in depth.

Instruments and Materials

The instruments and materials used in data collection and analysis include (1) the instrument for testing the mathematical thinking process of the number pattern material, (2) the student's Didactical and Pedagogical Anticipation (DPA) instrument for learning obstacles in solving problems, (3) semi-structured interview guidelines, aims to find out students' mathematical thinking processes in the category of solving mathematical problems related to numbers, (4) a digital voice recorder used to record interviews. (5) Ethical considerations. The college, school, mathematics teacher, participant, and parent or guardian granted permission to conduct this research. Before the study, each participant signed a consent form, and after the interview transcripts were completed, each participant read and confirmed the accuracy of the results of each interview. In addition, the participant's initials were also used to disguise the student's identity.

Based on the results of data analysis on the results of research instrument trials on 16 participants who participated in this study, potential learning obstacles, scaffolding, and predictions of student responses in solving non-routine number pattern problems were found. The following is a summary of data on potential learning obstacles, scaffolding, and forecasts of student responses in solving non-routine number pattern problems.

Num	Potential Learning Obstacles	Provided scaffolding	Responses Given by Students
1	Do not know how to show proof of a statement.	1. Is it sufficient to show proof of all palindromes divisible by 11 by giving some examples of palindromes divisible by 11?	1. (Students did not answer immediately, still thinking about possible answers.) A few moments

Num	Potential Learning Obstacles	Provided scaffolding	Responses Given by Students
		2. If not enough, how can we show all 4-digit palindromes divisible by 11?	later, the students answered, "Not enough, sir!" 2. Students do not answer directly, still thinking about how). How many moments later, they replied, "Maybe by looking at the pattern, sir!"
2	Difficulty identifying relevant patterns or rules	3. What are the observable patterns of 4-number palindromes? 4. Yes, what other patterns can be observed? 5. OK, which is the smallest palindrome?	3. The difference between one palindrome and the next is 110 packs! 4. Silent students have not been able to identify other patterns. 5. 1001 sir!
3	Difficulty constructing number patterns/rules in mathematical sentences or statements	6. What is the relationship between the smallest palindrome 1001 and the smallest palindrome for the next thousand, namely 2002 or 3003? 7. Yes, that's right; then what is the relationship between 110 and 220, and 330? 8. Then how do we write the palindrome 3220 in forms 1001 and 110?	6. 2002 equals 2 x 1001 and 3003 equals 3 x 1001; 7. 220 equals 2 x 110, and 330 equals 3 x 110. 8. 3. 3223 = 3003 + 220 = 3(1001) + 2(110)
4	Do not have the awareness to examine the results of his work	9. 1. Is it necessary to re-examine the results of our work? 10. What is the need to re-examine our work?	9. Need sir! 10. Yes, you need to check, sir, because if you make a mistake, you can fix it again.
5	Difficulty identifying alternative settlement strategies	11. How can we check the results that have been obtained? 12. For example, ABBA is a 4-digit palindrome with the first and second A being thousands and units each and the first and second B being hundreds and tens each. Can it be shown that ABBA is divisible by 11?	11. Partisans are silent about what can be done to be able to check the results of the work. 12. Participants show ABBA = (10001) A + (110) B = (11.91) A + (11.10) B = 11(91A+ 10B)

Table 2: Didactical and Pedagogical Anticipation Instrument (DPA) for students in solving non-routine number pattern problems

Procedure

The research procedure refers to the research design used in the Indonesian Didactical Design Research (DDR), which contains three stages of analysis: prospective analysis, metapedadidactical analysis, and retrospective analysis (Suryadi, 2013). In the future analysis stage, analyzing the phenomena that underlie the didactic design process for solving hypothetical problems is found from the results of testing the test instrument on students in the research target schools. In particular, the phenomena disclosed include 1) the didactic situation presented in the implementation of research instrument trials from didactic designs; 2) learning obstacles experienced by students in solving given mathematical problems. 3) Students gave scaffolding and responses in solving mathematical problems.

Metapedadidactic analysis is an analysis that aims to see the ability of researchers to identify and analyze student responses as a result of the didactic and pedagogic actions taken and the ability of researchers to carry out further didactic and pedagogic actions based on the results of the response analysis towards achieving the problem-solving target (Suryadi, 2013). In the retrospective analysis stage, the Researcher reflects and evaluates the situation of the hypothetical problem-solving design by analyzing the relationship between the results of the prospective analysis and metapedadidactical analysis. More specifically, at this stage, the Researcher conducted a suitability analysis between the hypothetical didactical situation and the didactical situation during implementation. This reflection and evaluation suggest improvements to the design of didactic situations for solving hypothetical number pattern problems in students' mathematical thinking processes in solving non-routine number pattern problems.

Data analysis

The results of data collection in the form of recordings of the problem-solving process, answer sheets and student scratch paper, student and teacher interview documents, as well as observation data during the study, were analysed based on the stages developed by Creswell (2007), namely data managing, reading-memoing, describing-classifying-interpreting, and representing-visualizing. Data managing, namely organizing data into computer files for analysis, transcribing recorded data and student interviews, and typing observation notes. Reading-memoing is reading and interpreting the collected data and giving letters or memos in the margins of field notes, transcripts, or under photographs to assist in the initial data exploration process. Describing-classifying-interpreting, namely forming codes or categories representing the essence of data analysis. Researchers construct detailed descriptions, categorize themes, and provide interpretations based on their views or perspectives in the literature. Representing visualizing, namely representing the results of data analysis in text, tables, or images.

RESULTS

Students' Mathematical Thinking Process in Solving Nonroutine Number Pattern Problems

Based on the results of research data analysis of the 32 subjects who participated in this study, students' mathematical thinking processes in solving non-routine pattern problems can be grouped into three categories, namely the first category of subjects who solve problems with new strategies (critical reflection), the second category of subjects solving problems with the help of scaffolding (explicit reflection), and finally the category of subjects who cannot solve non-routine number pattern problems. The following is a description of the data on the mathematical thinking processes of the three subject categories in solving the problem of non-routine number patterns described based on indicators of mathematical thinking processes from Mason et al. (2010), namely specialization, generalizing, guessing, and convincing.

The non-routine problems of number patterns given to research subjects are as follows:

> "A number like 1221 is called a palindrome because it reads the same both forward and backward. A friend of yours says that all palindromes with four digits are divisible by 11. Is your friend's statement true?"

Specialization

In the number pattern problem above, the three categories of subjects in this study were able to specialize well in the action-situation phase. S1 subject (critical reflection category) specializes by showing four examples of 4-digit palindromes that are divisible by 11 independently. The following is S1's response when specializing in solving non-routine number pattern problems in action situations.

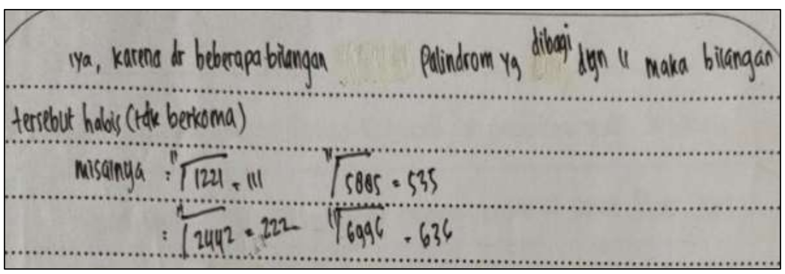

Figure 1: S1 response in specialization to solve number pattern problems in action situations

Based on the response given by S1 in Figure 1 above, it shows that subject S1 can specialize by choosing four examples of 4-digit palindromes, such as 1221, 2442, 5885, and 6996, to check whether the selected 4-digit palindrome is divisible by 11? Based on the results shown by S1, the selected examples show that the four palindromes are divisible by 11. The results of this specialization can be carried out by S1 subjects in action situations or working independently.

Like the S1 subject, the S2 subject (from the explicit reflection category) can also specialize independently or in an action situation. The following is the response given by S2 when specializing in solving non-routine number pattern problems in action situations.

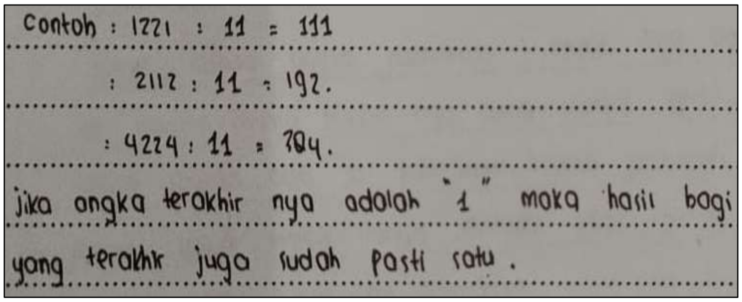

Figure 2: S2 response in specialization to solve number pattern problems in action situations

Based on the response given by S2 in figure 2 above, it shows that the subject can specialize by giving three examples of 4-digit palindromes, such as 1221, 2112, and 4224. Next, the subject tries to look at the four selected 3-digit palindromes. Is it divisible by 11? Based on the selected examples, S2 shows that the three palindromes are divisible by 11. This result can be given S2 in the action situation phase.

Like Subjects S1 and S2, Subject S3 (Subjects from the category that do not solve the 4-digit palindrome problem) can also specialize well independently or in action situations. The following is the response given by S3 when specializing in solving non-routine number pattern problems in action situations.

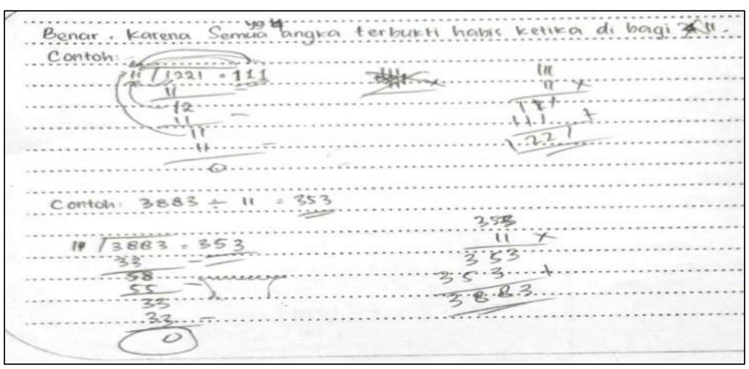

Figure 3*:* S3 response in specialization to solve number pattern problems in action situations

Based on the response given by S3 in Figure 3 above, it shows that the subject can specialize by choosing two examples of 4-digit palindromes, namely 1221 and 3883, to check whether the two selected 4-digit palindromes are divisible by 11. Based on the examples chosen, S2 shows that the two palindromes are divisible by 11. Subject S3 can carry out this result in the action situation phase.

The description of specialization data on the number pattern problem above shows that the three categories of subjects can carry out specialization properly in solving non-routine number pattern problems.

Recapitulation of the specialization phase of the mathematical thinking process of the three categories of students in solving non-routine number pattern problems is listed in Table 3 below.

Numb	Research Subject Categories	Specialized Response
1	Critical reflection subject	Give examples properly in action situations
2	Explicit reflection subject	Give examples properly in action situations
3	The subject cannot solve the problem	Give examples properly in action situations

Table 3 Data specialization of the three subject categories in solving number pattern problems

Generalization

Generalization is estimating a broader situation by acting on several examples, or it can be expressed as searching for patterns/relationships. Departing from a certain number of operations, a decision is attempted to be made about the claim, suggesting the specific procedures performed during the generalization (Arslan & Yildiz, 2010). Pilten (2008) said several strategies that students can use in this process could set more examples for determining relationships, collect as many samples as possible, and test conjectures (Uyangör, 2019). Student responses at this stage are categorized into three types: linguistic/mathematical expressions of exact relationships, linguistic/mathematical expression of the right relationship with the help of scaffolding, or the linguistic/mathematical term of the relationship is imprecise/ left blank.

S1 subjects can generalize in solving given problems by responding uniquely to linguistic and mathematical expressions or critically reflecting on formulation situations. The following is a generalization of the mathematical form provided by S1 subjects in solving number pattern problems in formulation situations.

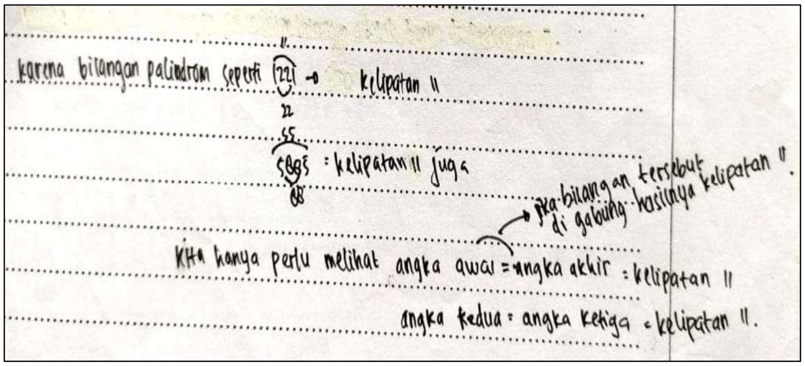

Figure 4: S1's generalization form response to the number pattern problem in a formulation situation

Based on the responses in Figure 4 above, subject S1 determines the generalization of the number palindrome problem in a mathematical form, namely two pairs of numbers multiples of 11. The first pair of numbers, namely the first number, is equal to the fourth number, and the second is the second number, the same as the third number, which is a multiple of 11. The results of this generalization can be given in S1 in the formulation situation.

The following is an excerpt from the interview of Researcher (R) with S1 (Subject from the critical reflection category) to gather information about generalizations on number pattern problems.

R : "What rules or patterns apply to the 4-digit palindromes you found?"

S1 : "Yes, here I see that in a 4-digit palindrome, the first and fourth numbers are the same, so this is a multiple of 11, so the second number is the same as the third number, so this is also a multiple of 11."

The responses and interview excerpts from S1 show that the subject can generalize to the problem of number patterns in the form of linguistic/mathematical expressions with precise and unique relationships.

While S2 can generalize from 4-digit palindromes by identifying patterns and relationships with the help of scaffolding from researchers, he can formulate patterns or rules from a 4-digit palindrome where the difference between a palindrome and the next palindrome is 110. Following is the response of subject S2 when generalizing in solving number pattern problems after being given a scaffolding or validation situation.

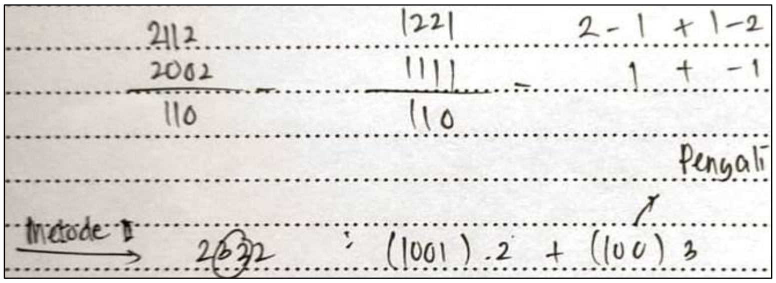

Figure 5: S2's generalization form response to the number pattern problem in a validation situation

Based on the response of subject S2, as listed in Figure 5 above, it shows that the subject can identify patterns or rules contained in a 4-digit palindrome, namely the difference between one palindrome and another palindrome is 110 so that the second and third digits of each number palindrome are multiples of 110. In addition, a palindrome's first and fourth digits are multiples of the smallest palindrome, namely 1001. S2 gives an example: the palindrome 2332 equals (1001)2 plus (110)3. S2 can carry out this form of mathematical generalization in validation situations.

The following is an excerpt of the Researcher's (R) interview with the subject of S2 to dig up complete information in generalizing the problem of number patterns.

R : "How can the relationship between the 2332 palindromes be arranged into (1001)2 and (110)3?"

S2 : "The distance of one palindrome from the next in the same thousand is 110, so the second and third digits of a 4-digit palindrome are multiples of 110". Then the first and fourth digits of a palindrome are multiples of the smallest palindrome, namely 1001, so 2002 can be written as (1001)2."

Based on the responses and excerpts from the S2 subject's interview, it was shown that the subject could generalize in solving number pattern problems in the form of mathematical expressions with the right relationships. S2 subjects can do this with the help of scaffolding from researchers or validation situations.

On the other hand, S3 has difficulty generalizing in identifying patterns or rules that apply to 4-digit palindromes independently or with the help of scaffolding. The following is an excerpt from the interaction between the Researcher and the S3 subjects in providing scaffolding assistance to generalize the 4-digit palindrome problem.

R : "What patterns or rules did S3 find?"

S3 : "The smallest palindrome is 1001, and the thousands and units of other palindromes can be added from this smallest palindrome."

R : "That's right... then which pattern can you see?"

S3 : "The difference of one palindrome from the next palindrome in the same thousand is 110, so the second and third digits of a palindrome can be added from 110."
R : "Do all palindromes have a difference of 110?"
S3 : S3: "It seems yes, sir, the difference is all 110."
R : "The correct palindrome in one thousand is 1991, and the smallest of the two thousand palindromes is 2002. What is the difference between the palindromes of 2002 and 1991?"
S3 : Subject S3 calculated the difference between the 2002 and 1991 palindromes, "the difference is 11, sir!"
R : "Yes, then how can we construct a 4-digit palindrome so that we can show it is divisible by 11?"
S3 : Stop thinking about the shapes that can be arranged. "I don't know, sir!"

Based on the quote from the Researcher's interaction with the S3 subject in providing scaffolding assistance, it shows that the subject has not been able to make generalizations in solving number pattern problems, whether in language or mathematical expressions with the right relationships.

Data recapitulation of students' mathematical thinking processes in generalizing to solve non-routine number pattern problems as shown in Table 4 below.

Numb	Research Subject Categories	Generalization Response
1	Critical reflection subject	Linguistic/mathematical expression of the exact relationship in the formulation situation
2	Explicit reflection subject	The mathematical expression of the appropriate relationship in a validation situation
3	The subject cannot solve the problem	Haven't been able to give a linguistic/mathematical expression of the right relationship.

Table 4: Recapitulation of data generalization of the three subject categories in solving the 4-digit palindrome problem

Conjecture

Conjecture arises in the process of specialization and generalization and is the process of researching the accuracy of a hypothesis by predicting that it is likely to be true. Mason et al., (2010b) said conjecture recognizes developing generalizations. Actions such as making linguistic or mathematical conjectures, formulating mathematical claims, generating results from hypotheses, and establishing and testing ideas can be relevant to this process (Arslan & Yildiz, 2010). Student responses at the guessing stage were categorized/grouped into three forms: linguistic conjecture, mathematical conjecture, and; incorrect math/linguistic assumption/left blank.

In the formulation situation phase, S1 can make linguistic and mathematical conjectures by using patterns or rules already identified at the generalization stage. The following is the response given by S1 in making mathematical conjectures based on the regulations or patterns specified in the previous generalizations.

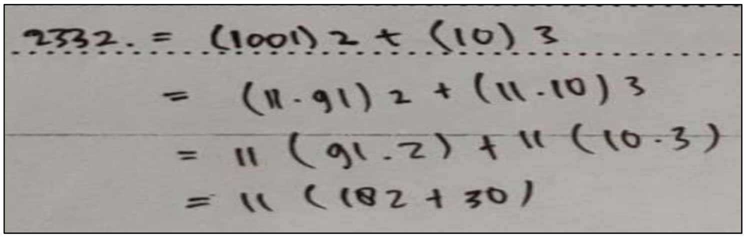

Figure 6: S1's response in making mathematical conjectures in solving problems in formulation situations

The response given in the figure above shows that subject S1 can make mathematical conjectures with one example of a 4-digit palindrome, namely 2332. S1 translates the form 2332 into 2002 + 330 and factors into (1001)2 + (110)3 so that the condition can be shown as a multiple of 11. Subject S1 can make this conjecture in the formulation situation phase.

The following is an excerpt from an interview with S1 to identify the assumptions made in solving the 4-digit palindrome problem.

R : "Why can the 2332 palindrome be translated into (1001)2 + (110)3?
S1 : "Yes sir 2332 can be written as 2002 + 330, so it can be translated into (1001)2 + (110)3
R : "Why does it need to be written as (1001)2 + (110)3?"
S1 : "Because it's easier to count multiples of 11."
R : "Why does using this form of proof conclude that all 4-digit palindromes are divisible by 11?"
S1 : "All 4-digit palindromes can be factored by 1001 and 110 packs, while 1001 and 110 can be divided by 11."

Based on the responses and excerpts from the results of the interviews, it was shown that S1 subjects could make mathematical conjectures in solving number pattern problems in formulation situations.

Meanwhile, S2 subjects can make conjectures using patterns or rules already identified at the generalization stage in validation situations. Based on the results of interviews with S2, it can be determined that the subject can make conjectures in mathematical form.

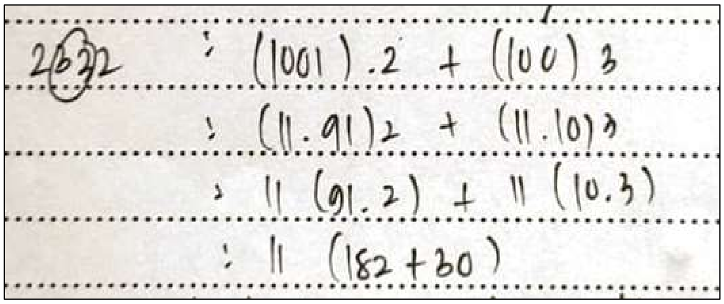

Figure 7: S2's response in making mathematical conjectures in solving problems in validation situations

Based on the response of subject S2, as listed in Figure 7 above, it shows that the subject can formulate mathematical conjectures through the example of the 2332 palindrome, which is translated into (1001) 2 plus (110)3. Using the 2332 palindrome, the S2 subject can show that the 4-digit palindrome is divisible by 11. This conjecture can be given S2 with the help of scaffolding from the Researcher or in a validation situation.

The following is an excerpt from an interview with S2 when exploring the conjecture stages carried out by S2 in solving the 4-digit palindrome problem.

R : "What is the meaning of the pattern or rule found? What can be concluded?"

S2 : "Because the smallest palindrome is 1001, and the distance of one palindrome from the next palindrome in the same thousand is 110, and the distance of the largest palindrome from a certain thousand to the smallest palindrome of the next thousand is 11. Because 1001,110 and 11 are divisible by 11, all 4-digit palindromes are divisible by 11."

The excerpt from the interview with the S2 subject above shows that the subject can guess the correct mathematical form in solving the 4-digit palindrome problem in validation situations.

Recapitulation of the conjectures stage data on the process of thinking mathematically in solving non-routine mathematical problems as listed in Table 5 below.

Numb	Research Subject Categories	Conjectures Response
1	Critical reflection subject	Make appropriate language predictions and relate the rules/patterns identified in the formulation situations
2	Explicit reflection subject	Perform appropriate language guesses and relate rules/patterns identified in the validation situation
3	The subject cannot solve the problem	Haven't been able to make precise language/math guesses yet

Table 5: Alleged data of the three subject categories in solving the 4-digit palindrome problem

Convincing

Convincing or giving evidence is an essential concept in learning mathematics (Knuth, 2002), it is also crucial for mathematical thinking. While creating evidence, actions such as explaining a hypothesis, saying why it is true or false, and selecting and using different ways of logical thinking (inductive and deductive) and types of proof become relevant (Uyangör, 2019). Student responses at this stage are categorized in code 3: arithmetic proof, algebraic proof, and left incomplete/incorrect/blank.

S1 subjects can provide proofs of algebraic proofs with the help of scaffolding from researchers or in validation situations. Evidence of the algebraic form of subject S1 can be done by assuming a 4-digit palindrome as naan, where the first and second n are the thousands and one's digits, and the first and second a are the hundreds and tens, respectively. The following is the response given by S1 at the convincing stage in solving the 4-digit palindrome problem in validation situations.

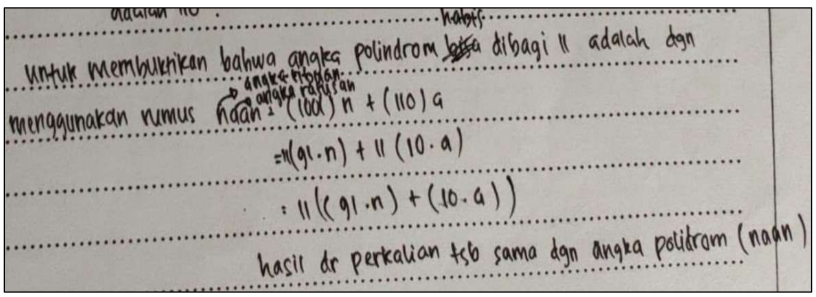

Figure 8: S1's response in convincing the 4-digit palindrome problem in a formulation situation

Based on the response given by the subject in the picture above, S1 proves the algebraic argument by assuming the general form of the palindrome, namely naan. The first letter n has the value as thousands, and n in the fourth digit has the value as the units digit. Then the first a is worth hundreds, and the second a is worth tens. The form of naan can be described according to the condition that has been identified so that it can be shown to be a multiple of 11. As for this algebraic proof, S1 can be carried out with the help of the Researcher's scaffolding.

The following is an excerpt from the Researcher's interaction with the S1 subject to provide scaffolding assistance in solving the 4-digit palindrome problem.

R : "Using this pattern, how do you show that all 4-digit palindromes are divisible by 11?"

S1 : S1 silently thinks about the strategies that can be used.

R : "Suppose anna is the general form of a 4-digit palindrome. Using the pattern above, can it be shown that anna is divisible by 11?"

S1 : "Yes sir, if naan is a 4-digit palindrome, then the first n has the value as thousands, and n in the fourth digit has the value as the units digit. Then the first a is worth hundreds, and the second a is worth tens".

R : "Yes, can the form of anna's palindrome be shown to have a factor of 11"

S1 : "Yes, sir, the result is 11 times 91a plus 10n."

R : "Then, what conclusions can be drawn using the general form of this palindrome?"

S1 : "Yes sir, by using this general form, it can be shown that the general form of the naan palindrome also has a factor of 11, so it can be concluded that all 4-digit palindromes are divisible by 11."

The responses and excerpts from the interview results show that S1 subjects can be convincing in solving number pattern problems in algebraic form. This algebraic proof shows that S1 is a new or unique way of validating situations.

While S2 can provide algebraic proof in solving the second problem with the help of scaffolding from researchers, S2 can show all 4-digit palindromes divisible by 11 using the place value of a 4-digit palindrome. The following is the response of the S2 subject in providing evidence for solving number pattern problems.

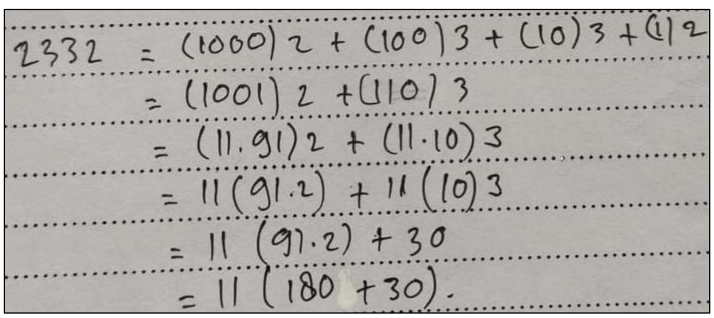

Figure 9: S2's response in convincing in arithmetic form using place value

Based on the response given by S2 in Figure 9 above, it shows that S2 can carry out the convincing stage by using inductive arguments or arithmetic proofs, namely by conducting all palindromes divisible by 11 using the place value of a 4-digit palindrome. Using the concept of place value, the subject can show the general form of a 4-digit palindrome having a factor of 11. S2 subjects, with the help of scaffolding from researchers or in validation situations, can carry out this solution.

The following is an excerpt of the Researcher's interaction with the subject S2 when providing scaffolding in the stage of compiling evidence in solving number pattern problems.

R : "Can you explain the value of each digit position of the 2332 palindrome?"

S2 : "The first 2nd position is worth thousands, the first 3's is worth hundreds, and the second 3's is tens, while the second 2nd is units."
R : "That's right... What can you write down if you describe it?"
S2 : "Trying to translate the 2332 palindrome into (1000)2 + (100)3 + (10)3 + (1)2
R : "OK.. Now how does it look if those with the same multiplication number are combined?"
S2 : Try concatenating to "(1001)2 + (110)3"
R : "Can you translate the numbers 1001 and 110 into multiples of 11?"
S2 : "Try dividing the numbers 1001 and 110 by 11 each".
R : Based on the results obtained, what conclusions can be drawn?
S2 : "All 4-digit palindromes can be shown to have a factor of 11, and this indicates that all 4-digit palindromes are divisible by 11

The responses and quotes from the interaction with the subject show that 'S2 can be convincing in solving number pattern problems with inductive arguments in arithmetic form. S2 subjects can do this through scaffolding assistance from researchers or in validation situations.

Recapitulation of students' mathematical thinking process data in convincing according to the research subject category in solving non-routine number pattern questions as listed in Table 6 below.

Numb	Research Subject Categories	Convincing Response
1	Critical reflection subject	Can reason algebraic forms in validation situations using proper rules
2	Explicit reflection subject	Can reason arithmetic forms in validation situations using place value concepts
3	The subject cannot solve the problem	Not yet able to give reasons to convince

Table 6: The response data convinced the three subject categories to solve the 4-digit palindrome problem

Discussion

Recapitulation of students' mathematical thinking process data for the three categories of research subjects in solving non-routine number pattern questions is listed in table 7 below.

Mathematical Thinking Process	Subject Categories Research Results		
	Critical Reflection Subject	**Explicit Reflection Subject**	**The subject cannot solve the problem**
Specialization	Shows some examples of 4-digit palindromes divisible by 11 in action situations	Shows some examples of 4-digit palindromes divisible by 11 in action situations	Shows some examples of 4-digit palindromes divisible by 11 in action situations
Generalizations	The linguistic and mathematical expression of the exact relationship in the formulation situation;	The mathematical expression of the exact relationship in the formulation situation;	Not yet able to provide linguistic/mathematica l expressions with proper relationships
Conjectures	Gives linguistic and mathematical conjectures of the exact relations in the formulation situation	It gives mathematical conjectures of the exact relations in the validation situation	Haven't been able to make precise language/math guesses yet
Convincing	Can provide reasons in an algebraic form in validation situations	Can reason in arithmetic with place value concepts in validation situations	Can't give a proper reason

Table 7: Recapitulation of mathematical thinking process data for the three subject categories in solving non-routine number pattern problems

The data from this study, as listed in table 7 above, shows that the three categories of student groups in solving number pattern problems can specialize well in action situations. This indicates that the three categories of research subjects can understand the problem using their knowledge or actual development. This result follows the results of previously reported studies (e.g., Arslan & Yildiz, 2010; Keskin et al., 2013; Uyangör, 2019; Yıldırım & Köse, 2018) that students can quickly fulfill the specialization process. Other studies report that specialization does not occur naturally (Lane & Harkness, 2012) unless given instructions. Based on research results, Lane & Harkness (2012) said that many students did not carry out the specialization process but immediately jumped to the guessing stage and even directly to the generalization stage.

In the generalization stage, the subject from the critical reflection category can correctly generalize with linguistic and mathematical expressions in formulation situations. This shows that the subject of the critical reflection category can be generalized in uniquely solving number patterns or achieving critical reflection (Suryadi, 2019b). At the same time, students in the explicit reflection category or solving problems with the help of scaffolding can generalize in the form of mathematical expressions using scaffolding in validation situations and achieve explicit reflection (Suryadi, 2019b). The results showed that students were able to write relationships linguistically in the process of generalizing problems but had difficulty writing them algebraically (Arslan &

Yildiz, 2010; Yıldırım & Köse, 2018; Keskin et al., 2013). However, it appears that students in this study, with the help of scaffolding provided in the form of questions, helped students make abstractions and reveal relationships between variables. This shows students' knowledge in generalizing in solving number pattern problems in the Zone of Proximal Development (ZPD). The ZPD area consists of actions that children can understand but are not capable of performing. In other words, the ZPD area is a zone where children act with understanding and awareness with the help of adults.

While students from the third category group have not been able to identify patterns and mathematical models in solving number pattern problems both in formulation situations and validation situations. This shows that the knowledge of this category of students in making generalizations on 4-digit palindrome problems experience technical difficulties or instrumental learning obstacles (Suryadi, 2019a) or are in the zone of student difficulties that cannot be overcome (Zaretskii, 2009). To be able to overcome this difficulty, it can be overcome by presenting the problem given to students in accordance with the student's level of thinking and predicting the possible responses given so that the scaffolding assistance provided can be useful for students. This is as stated by Suryadi (2019a) that students will experience learning obstacles due to too high or too low thinking demands they face. Apart from that, to overcome this problem it is necessary to pay attention that when presenting the problem you must pay attention to the structural order of the material, namely the relationship between concepts and functional order to see the continuity of the thinking process which has an impact on the student learning process. The presentation stages can also be interpreted based on certain theoretical perspectives, for example the theory of didactic situations in mathematics including action situations, formulation situations, validation situations, and institutional situations (Brousseau, 2002). The stages of presentation according to the philosophical-pedagogical view include a series of mental actions that form ways of thinking (WoT) and produce ways of understanding (WoU) (Harel, 2008).

Conjectures arise in the process of specialization and generalization by conjecturing that they may be true. Mason et al. (2010b) said conjecture is recognizing developing inferences. Actions such as making linguistic or mathematical conjectures, formulating mathematical claims, generating results from the thesis, and establishing and testing hypotheses can be relevant in this process (Arslan & Yildiz, 2010). The results of this study indicate that students from the category of being able to solve problems with a unique strategy or critical reflection can make predictions of linguistic and mathematical expressions in formulation situations. These results indicate that the subject of the critical reflection category can provide predictions in a new way or achieve critical reflection. Whereas students from the explicit reflection category can make conjectures with mathematical expressions with the help of scaffolding or validation situations. This suggests that the scaffolding assistance provided by the researchers from identifiable patterns can be used to denote all 4-digit palindromes divisible by 11. This shows that students in the explicit reflection category make conjectures with expressions or mathematical models in solving 4-digit palindrome problems using their potential knowledge or are in the Zone of Proximal Development (ZPD).

Convincing or proof is an important concept in learning mathematics (Knuth, 2002) and is also important for mathematical thinking. While creating evidence, actions such as explaining a hypothesis, saying why it is true or false, and choosing and using different ways of logical thinking (inductive and deductive thinking) and types of proof become relevant (Uyangör, 2019). Harel & Sowder (1998) define verification as the process used by a person to remove doubts about the truth of a statement. A distinction is made between confirming oneself and convincing others. One person's evidentiary scheme consists of what constitutes confirming and persuading others. The results of this study indicate that the subject of the critical reflection category can provide evidence or reasons in algebraic form by using rules or patterns from generalization forms that can be identified from a 4-digit palindrome. Meanwhile, the subject of the explicit reflection category can provide evidence or reasons in arithmetic form by using place value. This is by Harel & Sowder (1998) found that the most common proof scheme found by students is an inductive proof scheme, in which students ensure themselves and persuade others about the truth of the conjecture by direct measurement of quantities, numerical calculations, the substitution of specific numbers in algebraic expressions. And others. At the same time, Uyangör (2019), based on the results of his research, said that students preferred arithmetic proofs where of the five correct answers given at this stage, only one was an algebraic proof. The results of this study are also by the research of Lee et al. (2011) that proficiency in patterns predicts ability in algebra. Proficiency in patterning tasks is, in turn, expected to renew children's capacities. These findings suggest that providing algebraic proofs may be difficult for students who have difficulty recognizing and generalizing rules about patterns. To support students in algebraic thinking, teachers must design learning that begins by presenting real or contextual problem designs that are easily recognized by students, then expanding the context in symbolic form. This is as stated by (Tall, 2008) that the transition from arithmetic to formal axiomatic thinking can be built through concrete and symbolic experiences.

Future research is urgently needed to explore how mathematical thinking can be used to address modern problems in work and life. As stated by Goos & Kaya (2020a) that studying mathematical thinking in real-world contexts can produce insights into the nature of critical mathematical thinking in the workplace, the role of digital technology in providing problem-solving and reasoning strategies, and new approaches to dealing with interdisciplinary problems that require synthesis of mathematical thinking in knowledge domains in the fields of Science, Technology, Engineering, and Mathematics (STEM).

CONCLUSIONS

The results of the research show that the mathematical thinking process of students in the critical reflection category specializes by providing examples that are asked in action situations, making generalizations and conjectures in linguistic and mathematical form in formulation situations, and convincingly by giving reasons in algebraic form in validation situations. Meanwhile, students in the explicit reflection category specialize by providing specific examples that are asked about in action situations, generalize in linguistic form in formulation situations, make conjectures in mathematical form in validation situations, and convince by giving reasons in arithmetic using the concept of place value in validation situation. Meanwhile, students in the category who cannot solve problems can only specialize by providing specific examples of what is being asked in action situations. To support students in developing their level of mathematical thinking, teachers can present contextual problems that are appropriate to students' level of thinking, can predict possible responses or ways of thinking of students to problems given and present problems in accordance with the conceptual and functional sequence structure of students' thinking. To support students in algebraic thinking, teachers must design learning that begins by presenting real or contextual problem designs that are easily recognized by students, then expanding the context of the problem in symbolic form. Future research is urgently needed to explore how mathematical thinking can be used to address modern problems in work and life.

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Socio-mathematical Norms Related to Problem Solving in a Gifted and Talented Mathematics Classroom

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Abstract: This study explores problem solving practices in a gifted and talented mathematics classroom in response to the calls for investigating problem solving as a sociocultural cultural activity rather than a cognitive activity of individuals. Therefore, we used a socio-mathematical norm perspective for our investigation. Data consists of forty-three mathematics lessons in a gifted and talented classroom. We used the two dimensions of a socio-mathematical norm (student and teacher) to analyze the observational data. The findings revealed a social norm regarding different solutions that reflect the classroom's micro-culture in terms of problem solving and students offered mathematically different (especially easy, simple, or effective ones) and sophisticated solutions which pointed out a socio-mathematical norm about mathematically different solutions. We observed an explicit talk on different solutions. However, the classroom community lacked a socio-mathematical norm regarding evaluations of mathematically different solutions based on criteria such as easy, simple, effective, or sophisticated. A lack of such a norm resulted in low-level problem-solving practice which was not expected from gifted and talented students. We offer practical implications for the dynamics of a classroom where gifted and talented students engage in problem solving activities and theoretical implications regarding the two dimensions of a norm.

Keywords: socio-mathematical norms, problem solving, gifted students, talented students

INTRODUCTION

Mathematical problem solving (PS) has long been an important aspect of mathematics, its learning, and teaching. Therefore, the mathematics education community has been working on understanding the nature of PS for over four decades (Carlson & Bloom, 2005; Liljedahl et al., 2016). Most mathematics educators agree that the development of students' PS abilities should be the main purpose of teaching and that a wide variety of factors and decisions must be taken into account by curriculum developers and teachers to achieve this goal (Anderson et al., 2005; Lester, 1994, 2013).

In recent years, the complex nature of PS has been accepted by the mathematics education community, and the whole picture is even more complex when social aspects of mathematics education are considered (Koichu, 2019). After his book entitled "Mathematical Problem Solving" (Schoenfeld, 1985), which was published more than 25 years ago, Schoenfeld emphasized the social aspects of PS as follows: "Individuals do not work, or learn, in a vacuum …. characterizing productive learning environments – and the norms and interactions that typify them – is an essential endeavour, if we are to improve instruction. But learning environments are highly interactive, and the ideas that individuals construct are often built and refined in collaboration with others" (Schoenfeld, 2013, p.20). Similarly, according to Cobb and Yackel (2011), the classroom micro-culture (social context) emerges from the coordinated actions of its participants (Cobb & Yackel, 2011). In every classroom, there are implicit and explicit understandings, or a set of norms, that determine the behavior of teachers and students about what other members of the classroom do and value (Makar & Fielding-Wells, 2018). As the teacher and students interpret and respond to each other's actions, normative activities of the classroom community (social perspective) emerge and are continually regenerated by its members (Cobb et al., 2011). As a collective notion, a norm is related to what is taken-as-shared by a group. It refers to expectations and obligations that are negotiated among teachers and students (Yackel, 2004). Social norms (SNs) refer to "regularities in the interaction patterns that regulate social interactions in the classroom", while socio-mathematical norms (SMNs) are specific to mathematics (Yackel & Rasmussen 2002, p.315). Normative understandings of what counts as mathematically different, efficient, or elegant, and what counts as an acceptable mathematical solution or justification are examples of SMNs (Yackel, 2000). SNs and SMNs are related to beliefs (about one's own role, others' role, the nature of a specific mathematical activity, and mathematical beliefs) and values reflexively (Cobb et al., 2011, p. 125). Therefore, the notion of a norm could be a useful construct to explore the affective aspects of a classroom.

This study aims to explore SMNs related to PS in gifted students' mathematics classrooms. Gifted students' classrooms might have a different micro-culture than others. Researchers have reported that giftedness studies had not extensively approached the issue from a sociocultural perspective (Heyd-Metzuyanim & Hess-Green, 2019). Likewise, Singer et al. (2016) suggest investigating the nature of classroom culture and the role of the teacher in fostering mathematical expertise. Furthermore, Goldin (2017) recommends the examination of "effective sociocultural norms" in these classrooms.

Although the literature extensively investigated PS as a cognitive activity of individuals, there is a gap in the literature on the nature of PS as part of classroom culture. Considering this gap in the literature, an examination of the norms related to PS in gifted students' classes has come to the fore. Therefore, this study aims to answer the following research question: What are the existing SMNs related to PS in a gifted and talented students' mathematics classroom?

LITERATURE REVIEW

Problem solving

The issue of PS, which has attracted the attention of mathematics education researchers for more than fifty years, has been generally handled in three aspects of research: "as a cognitive enterprise, as something to be taught, and as something to teach through". However, this categorization does not fully explore the complexity of PS (Liljedahl & Cai, 2021, p.724). This approach ignores PS as a cultural activity of the classroom community and normative behaviours that support it.

Early studies focused on the PS process (Polya, 1957), while more recent studies focused on the characteristics of problem solvers that promote PS success (Carlson & Bloom, 2005). The framework proposed by Schoenfeld (2013) to explain students' PS behaviors consists of the use of basic mathematical knowledge, the use of cognitive or heuristic strategies, the use of metacognition or self-regulation strategies, and students' beliefs about mathematics and PS. The framework, which consists of four basic components, has been widely used not only to reveal how successful students are in their PS attempts but also to organize and support students' development of PS experiences in the classroom (Santos-Trigo, 2014). The development of students' PS abilities is not in isolation from the learning of other mathematical concepts. Furthermore, it should be considered as a system involving "the teacher's role" and "the classroom culture" (Lester & Cai, 2016, p.118).

Some of the beliefs and tendencies of students about PS, as revealed by Schoenfeld (1992), are as follows: there is only one correct solution to a problem, and the correct solution can only be reached by the solution shown by the teacher in the classroom, and PS is an activity that students carry out alone. Lesh and Zawojewski (2007) stated that little has changed in this regard since Schoenfeld's literature review. Many students' beliefs about PS are that problems given by their teachers should be solved expectedly (Cai, 2003; Lester & Cai, 2016). While it is considered important to reach the expected solution in a certain way in classroom environments (Mann et al., 2017), it has been reported that gifted students tend to solve problems in atypical ways (Singer et. al, 2016). In the context of PS, mathematical creativity is seen as a critical component associated with advanced mathematical thinking, which relates to one's ability to perceive original, non-algorithmic, and often insightful solutions (Leikin, 2021). It is stated that encouraging fluency, flexibility, and originality (components of creativity as reposted by Silver, 1997) can help eliminate students' wrong beliefs about PS as mentioned above (Levenson, 2022).

PS in mathematics classrooms is a sociocultural process (Koichu, 2019) and the development of PS ability is much more socially structured and contextually situated than traditional theories assume (Lesh & Zawojewski, 2007). According to sociocultural theories, SNs and SMNs are in relation to beliefs and values, and they evolve together in the classroom micro-culture (Yackel & Rasmussen, 2002). For example, the norm "The solution must correspond to the one that the teacher has in mind" (Bingolbali, 2011; Yenmez & Erbaş, 2022) may be concerned with students' beliefs that problems should be solved as their teacher expects (Cai, 2003; Lester & Cai, 2016). On the other hand, reported teacher behaviors that reflect their beliefs show that some teachers preferred their solutions to the problems and did not value students' ideas (Rott, 2020). Some other factors outside the classroom can also affect participants' beliefs and the classroom micro-culture. For instance, preparing for national student selection exams, which require solving problems fast, could promote a normative understanding in the classroom about solving problems quickly (Authors, 2021; Yenmez & Erbaş, 2022).

The notion of a norm, similar to beliefs and values, is a useful construct to explore the affective aspects of a classroom. Investigating the norms of the classroom have the potential for revealing the teacher's expectations of the students and their wrong beliefs about PS, if they have any.

Socio-mathematical norms

The argument that knowing and doing mathematics is a social and cultural activity by its nature is increasingly accepted by the mathematics education community (Yackel & Cobb, 1996; Cobb et al., 1997). According to the interpretative framework used to understand classroom practices, the social dimensions of classroom micro-culture and the psychological dimensions of students' activities in the classroom are in mutual interaction (Cobb et al., 2011).

The concept of SMN was introduced to separate the normative aspects of mathematical discussions specific to students' mathematical activities from general social norms (Yackel & Rasmussen, 2002). Since the teacher is at the center of the "mathematical meaning, interaction and negotiation process" in a mathematics classroom, SMNs that can encourage interaction are closely linked to the teacher's beliefs about learning, teaching, and mathematics (Kang & Kim, 2016). While teachers guide the establishment of SMNs, they support students' reorganization of their beliefs and values, which can be seen as their mathematical tendencies (Cobb et al., 2011). With the emergence of the notion of an SMN, student autonomy has begun to be seen as a feature of the way of participating in the community (Cobb et al., 2011). When students participate in the negotiation of SMNs, they develop mathematical beliefs and values that enable them to become increasingly autonomous members of the classroom mathematics community (Yackel & Cobb, 1996). SMNs not only determine the quality of teaching and learning activities in a mathematics classroom but also support and guide students' participation in mathematical activities (Kang & Kim, 2016).

Socio-mathematical norms related to problem solving

The norms reported in the context of PS (Lopez & Allal, 2007; Roy et al., 2014; Yackel & Cobb, 1996) mostly focused on the notion of a "different solution". The SMN regarding different solutions means reaching the same result in a differently and presenting the reasons for how the methods differ from each other (Roy et al., 2014). There are no predetermined criteria for what a mathematically different solution is (Lopez & Allal, 2007; Roy et al., 2014; Yackel & Cobb, 1996). As the teacher demands different solutions from the students, along the way they develop an understanding of which solutions count as mathematically different through interaction in the classroom. A teacher's answers and actions may limit students' understanding of what a different solution is (Lopez & Allal, 2007; Roy et al., 2014; Yackel & Cobb, 1996).

Another norm reported in the literature in the context of PS is about providing easy, simple, or effective solutions to problems. According to McClain and Cobb (2001), this norm is related to the SMN, offering different solutions to problems, because the evaluation of a solution as an easy, simple, or effective solution is closely related to the understanding of mathematical differences. Researchers (Roy et al., 2014; Yackel & Cobb, 1996) stated that in addition to these norms, a normative understanding of a sophisticated solution can develop in classrooms, but it is more difficult to develop this normative understanding than a different solution. When a teacher values particular solutions proposed by students, it becomes clear to other students what the teacher values. As the teacher asks whether there is a more sophisticated (advanced) solution to the problem or if there is one that effectively solves the problem, a normative understanding may start to evolve if students also develop an awareness in the same way. In this study, we will investigate SMNs related to PS in a mathematics classroom in the case of gifted and talented students.

Theoretical stance

Our theoretical stance is based on the notion of an SMN and has two distinctive features (Authors, 2020). The first one is about what is to be taken as evidence of a norm and the second is related to the two dimensions of norms.

A classroom norm defines “regularities in the interaction patterns that regulate social interactions” (in the case of an SN) and mathematical meanings (in the case of an SMN) in the classrooms (Yackel & Rasmussen, 2002, p. 315). As these interaction patterns emerge, the classroom community develops expectations and obligations (Yackel, 2004). Researchers investigating how negotiations on expectations shape classroom culture used sociological and psychological perspectives for defining a classroom community (Cobb et al., 2011). According to the psychological perspective teachers and students interpret and respond to each other’s actions in the classroom. As they interpret and respond to each other’s actions, normative activities emerge (sociological perspective) as a result. Conversely, established norms within a classroom (sociological perspective) show how teachers and students interpret and respond to each other’s actions (psychological perspective). Therefore, we will take both expectations and actions as evidence of a norm. Concerning expectations, an example would be a teacher’s or students’ expectations and awareness of these expectations about solving problems in differently. An action would be the teacher’s call for different solutions to a problem. Therefore, different solutions to a problem in a classroom could be considered evidence of SMNs related to PS.

The second feature of our theoretical stance is related to the two dimensions of norms: the student dimension and the teacher dimension (Authors, 2020). Since norms are negotiated among teachers and students (Yackel, 2004), we will focus on the student dimension as well as the teacher dimension of SMNs to be able to distinguish a two-way negotiation. The teacher dimension is concerned with teachers' expectations and actions, while the student dimension refers to actions, awareness of teachers' expectations, and the students' expectations of the teacher. Concerning actions, the teacher dimension could be asking for different solutions to the problem, and the student dimension could be students' initiation of solving problems in a different way.

METHODOLOGY

Context and participants

Norms, by their nature, cannot be studied independently of their context, as they are tied to the community from which they emerge (Klosterman, 2016). As the study of SNs and SMNs requires a long-term and in-depth analysis of the real classroom environment (Yackel & Cobb, 1996) qualitatively, we conducted a descriptive case study (Yin, 2009). A single classroom was chosen, which is a fifth-grade mathematics classroom with twelve (three girls and nine boys) gifted students, to be able to reflect the characteristics of the phenomenon (Bleijenbergh, 2010).

The teacher of the class was a special education specialist and had seven years of teaching experience. The school is a private secondary school supported by a foundation for gifted and talented students. The study was conducted in a class of fifth-grade students who were diagnosed as gifted according to WISC-IV which is not specific to mathematics. Although, the school accepts students with WISC IV, they did not share students' IQ's because of ethical principles. The teacher had been teaching the same class from the beginning of the first grade. We selected this teacher because our study aims to explore sustained SMNs rather than introduced norms (Roy et al., 2014). The teacher followed the standard mathematics curriculum and the textbooks which were also used in mainstream schools. He integrated the sixth and seventh-grade mathematics curriculum subjects into his lessons considering the level of his students.

Data collection

The main data source consisted of forty-three mathematics lessons (a total of 28 h and 40 min), that were video recorded during the 2018 autumn term. Since we aimed to investigate the SNs and SMNs existing in the classroom without any intervention, the first author observed the lessons as a non-participant observer.

An official permission letter was obtained from the school directorate. The school received permission from parents and the teacher for the video recording of the lessons. We started to video-record the lesson two weeks before data collection to help students get used to the camera.

Analysis of data

Transcripts of 43 lessons were analyzed using qualitative data analysis software. The content analysis method was used. We defined the first version of the descriptors for teacher and student dimensions based on the literature on SMNs related to PS. The descriptors referred to both dimensions (teacher and student) as evidence of an SMN because norms require a two-way negotiation between the teacher and students as described in the theoretical stance section.

We analyzed the transcripts of the lesson videos first by determining episodes in videos where the class was participating in PS activities. We then analyzed the transcripts of these episodes using the first version of the descriptors of teacher and student dimensions. As we conducted the content analyses, we revised the descriptors and coded the data using the revised version as shown below:

Different solutions should be offered (SN_DS) (Yackel & Cobb, 1996)

● *Teacher dimension:* The teacher offers different solutions to problems solved in the classroom and expects the students to offer different solutions.

● *Student dimension:* Students are aware of the teacher's expectation of different solutions to problems, and offer different solutions to problems.

Mathematically different solutions should be offered (SMN_MDS) (Lopez & Allal, 2007; Roy et al., 2014; Yackel & Cobb, 1996)

● *Teacher dimension:* The teacher solves the problems in mathematically different ways and expects students to solve the problems accordingly.

● *Student dimension:* Students are aware of the expectation of their teachers to solve the problems in mathematically different ways and to explain how the solutions differ from each other mathematically. They offer solutions that are mathematically different.

Mathematically easy, simple, or effective solutions should be offered (SMN_ESES) (McClain & Cobb, 2001)

● *Teacher dimension:* The teacher offers mathematically easy, simple, or effective solutions to the problems and expects students to develop such solutions.

● *Student dimension:* Students are aware that they are expected to develop mathematically easy, simple, or effective solutions and offer such solutions.

Mathematically sophisticated solutions should be offered (SMN_MSS) (Roy & et al., 2014; Yackel & Cobb, 1996)

● *Teacher dimension:* The teacher offers mathematically sophisticated solutions to problems and expects students to offer such solutions.

● *Student dimension:* Students are aware that they are expected to offer mathematically sophisticated solutions to problems and solve problems in mathematically sophisticated ways.

An example of coding an episode is presented in the Appendix. For validity concerns, the authors discussed the first version of the codes based on the norms related to PS reported in the literature. The two authors separately coded one episode concerning each norm and negotiated on a final code to increase the reliability of the findings. We will also present the frequencies of the codes for each norm for validity concerns. We consider a high frequency as evidence of norms since norms are defined as the regularities in the interaction patterns (Yackel & Rasmussen, 2002).

FINDINGS

Analysis of the data revealed normative aspects of PS activities. We will give illustrative examples of teacher and student dimensions of each norm (one SN and three SMNs) that we observed. In line with our theoretical stance considering teacher and student dimensions, we will refer to the teacher's and students' expectations, awareness of these expectations, and also their actions.

Mathematically different solutions should be offered

In this section, we will present our findings related to different solutions to problems that point out both an SN and an SMN. The data indicated that offering different solutions to problems has become a normative activity that reflects the social perspective of the classroom community. A total of thirty instances (ten for the SN and twenty for the SMN) were coded as the student dimension and twenty-nine instances (twelve for the SN and 17 for the SMN) as the teacher dimension. We observed that the teacher and students offered not only different solutions to problems but also mathematically different ones (psychological perspective).

The SN "Different solutions should be offered", and SMN, "Mathematically different solutions should be offered" are closely related norms. The teacher's statements such as "Does anyone have a different solution to a problem?", "I listen to those who do it differently" are clear indicators of his expectation about different solutions and reflect the teacher dimension of the SN. The dialogue that illustrates different solutions offered by two of the students in the classroom is presented below:

S1: Teacher, I did it differently.
T: I was wondering how he found it.
S5: Teacher, I did it in a different way. Shall I show you what I've done?
T: Show if you did it differently, if there is a difference, please show it.

In the dialog above, it is seen that the students are aware of the teacher's expectations and stated that they want to offer different solutions (psychological perspective). These actions show the student dimension of the SN. On the other hand, the teacher's reaction to his students (in the last line) indicates the SMN "Mathematically different solutions should be offered" because the teacher questioned if there was a difference. This questioning implies a mathematical difference because different solutions offered by students might not always be necessarily mathematically different. Below, we give further examples of this SMN which we also consider as evidence for the SN related to different solutions.

Through the SMN about mathematically different solutions, students realized that they were expected not only to offer different solutions but also mathematically different ones. The following dialogue illustrates the teacher's expectations for mathematically different solutions and students' awareness of the expectation, and different solutions offered by students (action).

> T: Are the fraction 25/100 and the fraction 3/12 equivalent?
> S5: 25 is not a multiple of 3. 100 is not a multiple of 12, but when you expand 3/12 by 100, it becomes 300/1200, and when you simplify this fraction by 12, it becomes 25/100 So it's equivalent.
> S3: I did like this...
> T: It is true, there may be another way, let your friend solve it...
> S3: I simplified 25/100 as 1/4. I divided 12 by 3 and we get 1/4. So these fractions are equivalent.
> T: One of your friends did it by expansion and the other by simplification…

The excerpt above illustrates a student's offer of a different solution. In other words, the SMN is evident beyond awareness and reveals itself as an act of offering a mathematically different solution (student dimension). After the student's proposal for a mathematically different solution, the teacher's explanation to the classroom community about the mathematical difference of solutions reflects the teacher dimension of the SMN. Therefore, the existence of both student and teacher dimensions points out a two-way negotiation.

Another episode illustrates that the act of offering a mathematically different solution was initiated by a student after the teacher asked them to solve a problem in the textbook.

> Problem: The hour and minute hands of a magic clock rotate in the opposite direction. The magic clock and a normal digital clock started together. If 40 minutes later the digital clock shows 20:17, what time does the magic clock show?
> T: Yes, then Ozan is coming, we are listening (Ozan is solving the problem on the board)...Your friend subtracted 1 hour and 20 minutes from 20:17.
> (When another student says he did it in a mathematically different way, the teacher goes to the student and looks at his solution.)
> T: Mete offers an alternative way to avoid the confusion, I think you shouldn't miss this idea... It's a great point of view... By combining the subject of hours with fractions... Bravo Mete, well done son...We applaud Mete. Mete used the following relationship: Can we express one minute as 1/60th of an hour? If the time is 20:17, then it is 20 and 17/60. It's correct. When I look at it, we can express one hour and twenty minutes as 1 and 20/60. I hadn't even thought of this solution.

In the dialogue given above, although there is no explicit demand for a mathematically different solution by the teacher, the presentation of a different solution (action) by the student points out the student dimension of the norm (psychological perspective). The solution offered by the student for this problem was considered an unexpected solution for this grade level. Although the mathematical difference was not communicated, it can be claimed that the classroom community reached an implicit agreement on this issue (sociological perspective). Expectations, awareness, and actions can be considered as evidence of the SMN in both student and teacher dimensions (psychological perspective). The absence of illustrative cases where the criteria for mathematically different solutions are explicitly discussed may be because the negotiation process of the SMN has not been observed. On the other hand, the existence of teacher and student dimensions of the SMN shows that there is a two-way negotiation.

Mathematically easy, simple, or effective solutions should be offered

This SMN is concerned with offering easy, simple, or effective solutions, and is closely related to the SMN "Mathematically different solutions should be offered". The data indicated that students were aware of the teacher's expectations and offered easy, simple, or effective solutions to the problems (psychological perspective). Ten instances were coded as the student dimension and seventeen instances as the teacher dimension of this SMN. An illustrative classroom dialog on how the teacher and students interpreted and responded to each other's actions is given below.

> T: Hande read 2/15 of a 600-page book on Monday and 7/30th on Tuesday. How many pages are left to read? Did everyone do this? Divide 600 by 15 and multiply by 2, divide 600 by 30 and multiply by 7, add the two and subtract from 600?
> S: Yes (Students responding altogether)
> T: Well that's right, how can we find a shortcut to this? She has read 2/15th and 7/30th. Guys, how can we add these fractions?
> S: The denominators must be the same (Students responding altogether)
> T: We expanded both sides, our new fraction…
> S6: Teacher, I did it in differently. Shall I show you how I did it?
> Teacher: Show if you did it differently, if there is a difference, show it.
> S6: I multiplied by 2 (expands the fraction 2/15 by 2), then I don't need to multiply with that (he says there is no need to expand the fraction 7/30).
> T: (The denominators) You made 30.
> S6: Yes, it's 30. Teacher, I multiplied by 40 and I multiplied it by 40 and found the results like that (he is trying to explain that he expanded the denominator of fractions to make 600)
> T: Yes, good.
> S1: Teacher, we can also do this with variables, right?
> T: Yes you can.

In the dialogue above, the teacher's prompt to find a shorter solution reflects the teacher dimension of the SMN. After the teacher's question, a student offered to equate the denominators of the fractions with 600 to find the parts of the whole (a 600-page book), which is an easier solution for her. This solution, which was presented by the student as an "easier solution", can also be considered more "effective", unlike the usually preferred way (equalizing the denominators at 30). The student's offer which is an easy, simple, or effective method for adding fractions without any request from the teacher reflects the student dimension of the norm. However, what is meant by a "mathematically different solution" was not clearly stated, and what an "easier, simpler, or more effective solution" was not explicitly discussed. Although there are no clearly stated criteria in this regard, student and teacher dimensions of the SMN are evident with expectations, awareness, and actions (two-way negotiation). Therefore we can claim that the classroom community has an implicit understanding of an "easy/simple or effective solution" (sociological perspective). Another episode that illustrates this finding is presented below:

T: Serkan suggests another way for 18x9
S5: I round off 18 to 20, then multiply 9 by 20 and subtract 9 twice.
T: It's a quite complicated method…

The easy, simple, or effective solution offered above by the student is an example of using the distributive property of multiplication over subtraction. For the student dimension, we can say that the student was aware of the teacher's expectations and offered a solution that could be easy, simple, or effective (action) (psychological perspective). A similar example where a student took the initiative to offer an easier solution is presented below.

S5: Can I explain to you how I did it?
T: Tell me Serkan.
S5: Teacher, I knew that 12x12 is 144. I changed one of the factors to 13 and added 12. Then I changed the other factor 13 and added 13.
T: Good…

As can be seen above, the student proposed an easier solution which is using the distributive property of multiplication over addition as a solution method, increasing each factor by one and transforming the operation into a simpler addition operation (action). We can claim that the classroom community agreed on offering easy, simple, or effective solutions to problems (sociological perspective), and both student and teacher dimensions of the SMN are evident with expectations, awareness, and actions.

Mathematically sophisticated solutions should be offered

This norm, which is related to the SMN "Mathematically different solutions should be offered", is concerned with offering more sophisticated solutions to problems instead of a standard solution. The data analysis revealed that the teacher expected mathematically sophisticated solutions to problems, the students were aware of the expectations, and they attempted to offer mathematically sophisticated solutions (psychological perspective). An episode that illustrates this SMN is presented below:

> T: The problem is a difficult one... How many numbers are there with two natural number divisors greater than 20 and less than 50? Serkan, how did you do this question?
> S5: Since he said two natural number divisors, I understood that he meant prime numbers. I looked for prime numbers between 20 and 50.
> T: Your friend has noticed a very good point... What does it mean to have two natural number divisors, its divisors will be 1 and itself... Serkan, it is very good... We applause.

It is understood by the teacher's reaction that the problem was difficult according to the grade level. For this kind of problem, students at this grade level will usually write down numbers one by one and their divisors and then look for natural number divisors. This case reflects the student dimension of the SMN. The teacher's reaction to the student's answer shows that he considered the student's solution mathematically valuable, and sophisticated. Although the teacher did not ask for a sophisticated solution, we can claim that the student was aware of the teacher's expectations and offered a solution (action) that could be considered sophisticated.
The first dialog, which is about finding a solution to the “magic clock problem” was presented as an illustrative example of the norm "Mathematically different solutions should be offered", can also be given as an example of the norm about sophisticated solutions. The teacher's reaction indicates that he considered the solution mathematically valuable. The teacher's statement "I hadn't even thought of this solution" implies that Mete's solution was a sophisticated one. The student's solution to the problem (action) indicates the student's awareness of the teacher's expectations of offering mathematically sophisticated solutions (psychological perspective). We can infer that the teacher and students mutually agreed on offering mathematically sophisticated solutions to problems (sociological perspective), and both dimensions of the SMN are evident with expectations, awareness, and actions. This finding implies a two-way negotiation.

DISCUSSION

In this study, we investigated the SMNs related to PS in a gifted and talented mathematics classroom micro-culture as a response to calls for investigating both the social aspects of PS (Lester & Cai, 2016) and “the nature of classroom culture and the role of the teacher in fostering mathematical expertise” (Singer et. al., 2016). We observed three SMNs related to PS that were also reported in the current literature but were not examined in gifted students' classroom micro-culture.

Regarding the first SMN, we observed that both the teacher and students offered mathematically different solutions. Although the evidence of the SN "Different solutions should be offered" is limited to only one classroom dialogue, we consider the existence of the SMN "Mathematically different solutions should be offered" as an indication of the fact that its related SN was a part of the classroom culture. As other researchers stated, SNs and SMNs are interdependent (Campbell & Yeo, 2021). For example, Roy, Tobias, Safi, and Dixon (2014) assert that the establishment of SNs fosters related SMNs in classrooms. In other words, the establishment of the SMN related to mathematically different solutions depends on its related SN regarding different solutions because the teacher's requests for different solutions initiate the process of comparing solutions mathematically (Yackel & Cobb, 1996). This SMN which is also associated with mathematical creativity (Leikin, 2018) allowed students to solve problems in different ways as suggested for gifted and talented students to deepen their understanding of mathematics and be experts in mathematics as much as possible (Singer et. al, 2016). This SMN has also the potential to support gifted students who might be iconoclastic problem solvers (who are apt to challenge the system in the classroom) in the context of affective aspects (Mann et al., 2017). However, for this observed norm, except for the basic ones, explicit evaluations of mathematically different solutions have not been made in the classroom. The reason for this finding might be that our study focused on established norms rather than the negotiation process of norm construction. Therefore, it was not possible to observe how the classroom members negotiated the criteria for mathematical differences. The other reason may be that the problems brought to the classroom by the teacher were not challenging enough for the students, because these kinds of problems have less potential for gifted students and their teachers to generate discussions on mathematical differences in solutions. The tasks selected by the teacher were from the textbooks that were used in mainstream schools, therefore not challenging or rich enough (Toprak & Özmantar, 2022). This might be due to his beliefs or his pedagogical content knowledge (Anderson et al., 2005; Ingram et al., 2020). Another reason may be that the teacher was not a mathematics specialist and did not tailor the tasks in the textbooks which lack the use of multiple solution strategies to meet his students' needs (Smedsrud et al., 2022).

Other observed SMNs, which are also associated with the SMN regarding mathematically different solutions are related to offering mathematically easy, simple, or effective solutions to problems. We presented the evidence of this SMN for both the teacher and student dimensions, which points out a two-way negotiation. Since the negotiations between the teacher and students were implicit, we could not explicitly distinguish solutions from each other based on the criteria such as easy, simple, or effective, but we interpreted the teacher's reactions such as his guidance and feedback as an implicit indicator (Yackel & Cobb, 1996). Since mathematically gifted children "strive for an economy of mental effort, rationality, ("elegance") in a solution" and "curtail the reasoning process, as shortening of its individual links" (Krutetskii, 1976), this SMN has the potential to support gifted students.

The third SMN observed in this study is called “mathematically sophisticated solutions should be offered”. The solutions offered by students were more sophisticated than usual solutions at this grade level. Offering mathematically sophisticated solutions to the problems was a taken-as-shared issue among the classroom participants. The teacher’s reinforcing stimulus to the student who solved the magic clock problem differently was interpreted as an implicit indicator. On the other hand, the teacher's comments on the student's solution: "It's a great point of view...combining hours with fractions...", also include an evaluation of why the solution was considered mathematically valuable. Although such evaluations of the teacher were limited, we can say that the classroom community has a taken-as-shared understanding of mathematically sophisticated solutions. Since unusual responses, thoughtful, flexible, or original solutions to a problem indicate mathematical creativity (Silver, 1997; Sriraman, 2005; Mann et al., 2017), this SMN has the potential to be supportive of gifted students.

CONCLUSION

In this study, we looked for answers to the research question “What are the existing SMNs related to PS in a gifted and talented students’ mathematics classroom?” Based on the findings, we came to some conclusions. The first conclusion is the determination of the SMN called “Mathematically different solutions should be offered”. This finding has some implications for classroom practice. More challenging problems (Leikin, 2018), “problems with multiple solutions” (Lev & Leikin, 2017), or “rich problems” that “can be solved in multiple ways and with different representations” (Ayalon et al., 2021, p.1701) could require different solution methods and strategies. Therefore, we suggest teachers select problems that may raise the need for more different solutions, and discuss clearly how the solutions differ from each other mathematically. The teacher's call for more than one different solution to such problems can support fluency in PS (Mann et al., 2017; Silver, 1997). As students offer more than one mathematically different solution to a problem, “developmentally sophisticated solutions” (Yackel & Cobb, 1996, p.466) can be seen in the classroom micro-culture.

We further suggest teachers make explicit evaluations so that students develop a clear understanding of solutions based on criteria such as easy, simple, effective, or sophisticated. Another study conducted in the same classroom revealed that problems posed by students were not evaluated explicitly by the teacher (Authors, 2020). The classroom community lacked explicit evaluations of both PS and problem posing activities, therefore an SMN regarding the evaluation of problems and their solutions. A lack of an SMN regarding evaluations of mathematically different solutions based on the aforementioned criteria resulted in low-level problem solving practice which was not expected from gifted and talented students. If a teacher endorses such an SMN to bring difficult and sophisticated problems to the classroom, it would also encourage students to offer different, effective, and sophisticated solutions. Revealing the mathematical differences between solutions is an important issue for gifted students. Therefore, we recommend their teachers include more complex and sophisticated solution processes and open-ended problems as part of more creative work (Zazkis, 2017). Furthermore, teachers should aim to develop an SMN regarding the evaluation of problems based on criteria to promote effective PS.

The second conclusion is related to our theoretical stance which deals with what is to be taken as evidence of a norm (expectations, awareness, and actions). It provided us with an operational data analysis framework with descriptors of teacher and student dimensions. The teacher's expectations and students' initiations (as actions) were considered as evidence of these SMNs. Unlike other SMNs in which we observed the teacher and student dimensions, for the SMN called "Sophisticated solutions should be offered", we did not observe the teacher's expectations regarding such solutions. Although the teacher did not explicitly ask for sophisticated solutions, the students offered these kinds of solutions. The teacher not asking for such solutions might be concerned with the lack of explicit evaluations of mathematically different solutions in the classroom because the SMN about mathematically different solutions "provide a basis for the subsequent emergence of the norms of what counted as a sophisticated solution" (McClain & Cobb, 2001, p.263). Therefore, we take students' solutions to the problems as the only evidence of taken-as-shared understanding of sophisticated solutions in the classroom micro-culture without the teacher's explicit talk about these solutions. Considering this finding, we suggest future research elaborate descriptors of teacher and student dimensions of SMNs or SNs.

The findings of this study should be considered with their limitations. The first limitation concerns the case selection. Since we aimed to examine existing (sustained) SMNs, we selected a classroom whose teacher had been teaching since the first grade. On the one hand, our choice has increased the possibility of observing existing SMNs, which were already established in the classroom. On the other hand, due to this preference, we could not observe the norm construction process. This was beyond our aim but one should consider this limitation when interpreting our findings. The second limitation, which is also connected with the first one, is about evaluations of mathematically different solutions. Explicit talks on the criteria such as easy, simple, effective, or sophisticated solutions to problems could be observed better during the negotiation process of SMNs. However, since this process is beyond our aim and our data are limited to the teacher's fifth year of teaching in the classroom, we could not observe the process of how the classroom members negotiated on the aforementioned criteria. The third limitation is that the teacher's approach was teacher-centered. Therefore, interaction among the classroom community was limited to teacher-student interaction and our observational data lacked student-student interaction because collaboration was not promoted.

We conclude with further suggestions for future research. First, since our findings are limited to one mathematics classroom, further studies can investigate SMNs regarding PS in different classrooms of gifted and talented students. Secondly, for an investigation of norm construction processes, researchers can observe the micro-culture of gifted classrooms from the first lesson which could guide researchers in SMNs' negotiation process with explicit talks of mathematical differences among solutions. A further investigation of SMNs regarding PS in other contexts could increase our understanding of the social aspects of, and beliefs about PS

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Appendix: Example of coding of an episode

Classroom episode and dialog	Discourse	SMN	Dimensions of the SMN
Problem: The hour and minute hands of a magic clock rotate in the opposite direction. The magic clock and a normal digital clock started together. If 40 minutes later the digital clock shows 20:17, what time does the magic clock show? T: Yes, then Ozan is coming, we are listening (Ozan is solving the problem on the board)...Your friend subtracted 1 hour and 20 minutes from 20:17. (When **another student says he did it in a mathematically different** way, the teacher goes to the student and looks at his solution.) T: **Mete offers an alternative way** to avoid the confusion, I think you shouldn't miss this idea... **It's a great point of view... By combining the subject of hours with fractions...** Bravo Mete, well done son...We applaud Mete. Mete used the following relationship: Can we express one minute as 1/60th of an hour? If the time is 20:17, then it is 20 and 17/60. It's correct. When I look at it, we can express one hour and twenty minutes as 1 and 20/60. **I hadn't even thought of this solution.**	(When **another student says he did it in a mathematically different** way, the teacher goes to the student and looks at his solution) **Mete offers an alternative way** to avoid confusion. **"It's a great point of view... By combining the subject of hours with fractions…"** **"I hadn't even thought of this solution."**	***Mathematically different solutions should be offered*** ***Mathematically sophisticated solutions should be offered***	The student told the teacher that he had an alternative way for solution (Student dimension of the SMN regarding different solution coded as SMN_MDS_S) The teacher brought the student's different solution suggested by the student to the classroom agenda (Teacher dimension of the SMN regarding different solution coded as SMN_MDS_T) The student used fractions for expressing the time in the problem (He expressed one minute as 1/60th of an hour. Then the time 20:17, turned to 20 and 17/60.) (Student dimension of SMNs regarding both different solutions and sophisticated solutions coded two times as SMN_MDS_S and SMN_MSS_S) The teacher gave feedback about the solution which showed he found it valuable (Teacher dimension of the SMN regarding sophisticated solutions coded as SMN_MSS_T)

The Thinking Process of Children in Algebra Problems: A Case Study in Junior High School Students

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Abstract: *Mathematics is a collection of cognitive products that have unique characteristics from other scientific disciplines. As cognitive products, mathematics, and thought processes are two things that cannot be separated. Although in the literature there have been many approaches proposed to support the analysis of students' thinking processes, not many have used Harel's theory of thinking to reveal the characteristics of students' mathematical thinking. Therefore, this research aims to describe the thinking process of class IX students in solving algebra problems based on Harel's thinking characteristics. To achieve this goal, qualitative research was conducted with a case study design. The questions on the topic of algebra were adapted and developed from questions from the National Examination (UN) and the International Program for Student Assessment (PISA). Next, a test was given to 30 class IX students followed by interviews. The results of the data analysis concluded that in general students still use non-referential symbolic thinking and only a few have algebraic invariance thinking or proportional and deductive thinking. This shows that there are still many students who have difficulty understanding algebra. In increasingly complex problems, students' thinking processes become less flexible, and concepts are understood separately (procedural).*

Keywords: *Mathematical thinking process, algebraic thinking, algebraic problems, and problem-solving*

INTRODUCTION

Thinking is a mental activity carried out by humans in every domain of their lives, including in the fields of education and learning. Specifically for learning mathematics, all activities or activities in mathematics are mental activities. In this regard, Suryadi (2012) states that mathematics is a way and tool of thinking. The way of thinking developed in mathematics uses consistent and accurate reasoning rules so that mathematics can be used as a very effective thinking tool for looking at various problems, including those outside mathematics.

Mathematics is a researched subject in cognitive psychology and cognitive science has brought many changes. Such changes are reflected in policy-level agendas such as in the National

Council of Teachers of Mathematics standards (NCTM, 2000). These standards call for dramatic changes in schools, and how mathematics is learned, understood, and taught. To study mathematics, you must know its characteristics. Mathematics has abstract concepts and relationships, and this is what differentiates it from other sciences. Perception of mathematical concepts or relationships occurs in the mind and their operations are only possible using certain signs and symbols (Duval, 2017; Uzun & Arslan, 2009). Thus, mathematics and mathematical thinking are two things that cannot be separated in learning. In this regard, Harel (2008b) explains that mathematics consists of two complementary subsets. The first part is a collection, or structure, of structures consisting of certain axioms, definitions, theorems, proofs, problems, and solutions. The second part consists of all the ways of thinking that are characteristic of the mental acts whose products comprise the first set. Therefore, developing mathematical thinking processes should be the focus of attention for classroom educators and mathematics observers in general.

The thinking process is also called the problem-solving process because thinking is related to problems (Sari et al., 2019). Meanwhile, according to Mayer (in Purwanto, 2019), students' thinking processes include three main components, namely: (1) thinking is an invisible cognitive activity, but can be inferred based on visible behavior, (2) thinking is a process that involves some manipulation of knowledge in the cognitive system, and (3) thinking activities directed at solving problems. This description indicates that problem-solving activities absolutely need to be given to students in order to train their thinking processes. A trained thought process produces skills that can be applied in a variety of situations.

Algebra is a subject that deals with the expression of symbols and extensions of numbers beyond whole numbers to solve equations, analyze functional relationships, and determine the representational structure of systems, consisting of expressions and relationships. In such activities, algebra is referred to as a tool for modeling real-world phenomena. Therefore, Algebra is said to be not only a set of knowledge and techniques but also a process or way of thinking (Kieran, 2004; Lew, 2004).

As a way of thinking, algebra refers to a way of producing meaning through activities, modeling situations, and manipulating those models in certain ways (Nottingham & User, 1992). Algebraic thinking is also the activity of doing, thinking, and talking about mathematics from a general and relational perspective (Kaput & Blanton, 2005; Mason, 1996; Windsor, 2010). Through activities like this, it is hoped that someone (student) can gain meaning from the object or concept they are studying.

Driscoll et al. (2001), explain algebraic thinking as the capacity to represent quantitative situations so that the relationships between variables become more visible. Some other opinions also define algebraic thinking as "habits of mind" that enable students to identify and express mathematical structures and relationships, such as structures in arithmetic and symbolic expressions, relationships in numerical and geometric patterns, and numerical and geometric structures in tables, graphs, and lines. numbers (Carraher et al., 2006; Kaput & Blanton, 2005; Mulligan & Mitchelmore, 2009; Novita et al., 2018; Radford, 2000; Warren & Cooper, 2008).

Manly & Ginsburg (2010), describe algebraic thinking into four thinking activities which involve (1) looking for structures (patterns and regularities), (2) making generalizations, using symbols for the number of variables, (3) representing relationships systematically with tables, graphs, and equations, (4) logical reasons to address/solve new problems. In this regard, van Amerom, (2002), Kieran, (2004), & Seeley (2004) stated that to develop algebraic thinking, generalization approaches, modeling, problem-solving, and function approaches need to be applied in algebra learning.

The various definitions of algebra and algebraic thinking explained above, ultimately algebraic thinking is based on basic mathematical ideas and concepts that involve various cognitive strategies and in turn these ideas are used to help understand mathematical concepts and solve increasingly complex and diverse problems (Permatasari & Harta, 2018; Windsor, 2010; Windsor & Norton, 2011). Basic mathematical ideas and concepts that are not optimal in their development affect students' ability to solve problems. This was expressed by Jupri & Drijvers (2016) in line with the findings of their research that several students' difficulties in algebra have been identified, including difficulty understanding words, phrases, or sentences and difficulty compiling equations or creating mathematical models. These difficulties result in errors in interpreting and in the problem-solving process resulting in students' unproductive (undesirable) way of thinking, namely non-referential algebraic thinking.

The difficulties experienced by students as described above do not only occur in Indonesia, but they also occur in several other countries, such as Thailand. One of the weaknesses in mathematics education in Thailand is that students lack thinking and problem-solving skills even though the main aim of organizing mathematics activities is to encourage students to reflect on their thinking and use their mathematical abilities to solve problems (Chimmalee & Anupan, 2022; Natcha & Yeah, 2010). Iji, Abakpa, & Takor (in Ojo, 2022) stated that although many students are proficient in mathematical operations related to symbols, they are less proficient in solving algebraic problems.

In connection with symbols in mathematics, Manly & Ginsburg, (2010) explained the results of their research that students had difficulty understanding symbols (letters) because: (1) letters are first found in formulas to determine parameters such as area or volume, (2) letters can also represent certain unknown numbers, and (3) letters can represent general numbers that are not a specific value. Meanwhile, Harel (2008b) states that algebraic symbols are often understood by students separately or are not understood as coherent entities, which represent quantities and have quantitative relationships.

Based on the description above, it is very important to help improve students' way of thinking gradually so that they have a correct and scientific way of understanding and are useful in solving problems. In this case, teachers need to identify their students' way of thinking in order to then take corrective action, for example by accustoming students from an early age to interpreting and solving problems in more than one way. Apart from that, the principle of iterative reasoning can be used as a solution to help students have the desired way of thinking, namely algebraic invariance thinking (Harel, 2008b; Harel & Sowder, 2013). Therefore, the aim of this research is to analyze and describe students' thinking processes in solving algebra problems with different levels of difficulty. Based on these objectives, the problem in this research can be formulated: (1) What are the characteristics of students' thinking in solving algebraic problems, (2) What is the students' thinking process in more complex algebraic problems and alternative solutions that can be provided to develop their algebraic thinking.

How do you know the process or way of thinking of students in solving algebra (mathematics) problems? Explicitly, Harel (2008b) explains this by starting from a way of understanding a certain cognitive product of certain mental actions carried out by an individual. These cognitive products are the results of interpreting a concept or algebraic symbols, proving mathematical statements, and solving problems. In contrast, the way of thinking is the cognitive characteristic of the mental actions to create the product (Harel, 2013). So, how to think can be known from conclusions after observing the results of work or behavior (cognitive products) from someone's (student's) mental actions. For example, a teacher who observes a student's work related to problem-solving may conclude that the student's interpretation of mathematical symbols is characteristically inflexible, lacking quantitative references, or, alternatively, flexible and connected to other concepts, and so on. From these characteristics, the teacher can conclude that the student's way of thinking includes algebraic invariance thinking, non-referential symbolism, or something else. Likewise, teacher observations in relation to evidence can conclude that students' justifications for mathematical statements are based on empirical evidence, inductive, or based on deduction rules (Harel, 2008a). This way of knowing the thought process is used as one of the author's references in the data analysis process of this research.

The expected way of thinking that is the focus of the study in this paper is (1) The algebraic invariance way of thinking with indicators that students can explain the algebraic symbols used and interpret the operations used (applied) in solving problems. (2) Proportional reasoning with indicators, students can explain relationships and changes in quantity from the operations used. (3) Deductive reasoning with indicators, students can determine the general form or deductive generalization of real-world problems logically. The term thinking process in this article is based on the meaning that the phrase "way" has the connotation of a kind of process (which produces a product) so to think means to apply a way of thinking (Harel, 2008b). Therefore, practically to explain students' thinking processes in this paper both terms (process and method) are used.

METHODOLOGY

This research uses a holistic type case study design (Yin, 2014). A holistic type case study design was used to describe various field findings related to the research question, namely how the thinking process of class IX students in solving algebra problems is based on Harel's thinking characteristics. The research was carried out in class IX, one of the state schools in the city of Bandung, Indonesia. Participants in this research were 30 students (aged 15-16 years) for the 2021/2022 academic year.

Data collection techniques used semi-structured tests and interviews. The test instrument consists of three essay questions and a non-test, namely an interview guide. The test questions used are in the form of problem-solving on algebra material which is compiled and adapted from national exam (UN) questions and from the International Program for Student Assessment (PISA). Before use, the test questions were first validated by 2 lecturers and 3 mathematics subject teachers and then tested on students in different classes. The results of the trial were revised again, after which they were given to the students who were participants in this research. The test questions are arranged in the form of story questions and are based on the student's type of thinking process, namely algebraic invariance thinking, proportional reasoning, and deductive reasoning. In connection with algebraic thinking, the activities for the three questions can be described: (1) identification activities are generally found in the three questions, (2) using and interpreting symbols, composing equations or mathematical modeling is more dominant in question number 1, (3) making a representation of relationships in tabular form, graphics are dominant in question 2, and (4) determining patterns, compiling logical reasons, making generalizations, found in question number 3.

The test takes approximately 60 minutes, students work on the questions on the sheet that has been prepared. After carrying out the test, students were asked to rest while the researcher corrected and grouped the results of their work to determine the participants who would be interviewed. Next, 13 students were interviewed based on the grouping of their work results. First, there were 6 people from the group of students who answered almost all of the questions given correctly. Second, there were 5 people from the group who answered the questions almost 50 percent correctly. Third, there were 2 people from the group who answered almost all wrong. Interviews were carried out over 2 days. During the interview, students' written answers as well as test questions are presented and they are encouraged to explain the results of their work. As a guide in conducting interviews, general initial questions and follow-up questions have been prepared. Common interview questions include: explaining the meaning of the question according to your understanding. What is your strategy/way to solve this problem? Explain the solution steps you made. What is the meaning of the symbols you use? Follow-up questions include, for example:

Why did you take this step? What do you mean by this step? What is the next step? What does it mean? Interviews lasted between 25 and 45 minutes, depending on the student's response.

Students' success in working on problems, whether using algebraic invariance thinking, proportional reasoning, or qualitative deductive reasoning, is considered to have the expected way of thinking (Harel, 2008b). On the other hand, students are said to have an unexpected way of thinking (symbolic non-referential), namely manipulating symbols without understanding their meaning. Students who have a non-referential symbolic way of thinking and students who do not provide answers (blank answer sheets) are considered to have difficulty solving algebra problems. The data analysis technique used in this research is qualitative analysis techniques, namely data reduction, data display, and conclusions (Miles & Huberman, 1994). The data analysis steps in this research are as follows.

1. Data Reduction

At this stage, the researcher summarizes the results of valid test and interview data, simplifies them, selects the main points, and focuses on things that are relevant to the research objectives. The researcher's activities at this stage are: (a) All recordings of student speech during interviews are opened and transcribed; (b) Select interview notes by deleting unnecessary parts; (c) Re-examine the correctness of the results of the transcript by playing back the recorded interview until it is completely clear what the student expressed in the interview; (d) Typing and compiling transcript results to facilitate the analysis process.

2. Data Display

At this stage, the researcher presents data, which is the result of data reduction, namely data on students' thinking processes in solving algebraic problems.

RESULTS AND DISCUSSION

General Findings

The following table presents the findings of students' work in solving algebra problems and their applications. In general, as expected, question number 3 is the most difficult for most students of the other two questions, and question number 2, including literacy questions, is relatively more difficult than question number 1. Even question number 1 is relatively easier and can be solved by some (50%) students. The remaining part (50%) contributed to making mistakes. Based on the expected competency indicators for question number 1, namely making mathematical modeling (algebra) with thinking process indicators, namely being able to solve problems using algebraic invariance thinking, and proportional reasoning, it seems that some (50%) of the students can do it. Apart from that, the procedure for solving question number 1 is immediately known to students using the procedures they usually use, namely elimination and substitution. Solving with the way of thinking that is often used, namely proportional reasoning, can make it easier for them to find a solution.

For question number 2, 6 students (20%) could solve it correctly. Likewise, for type number 3, only 3 out of 30 students were able to complete it correctly (10%). The rest of the students made mistakes, some even did not write anything on the answer sheet provided. Different from question number 1, question number 2 requires literacy skills to be able to understand the question correctly before taking the solving steps. Likewise, question number 3, which is a non-routine question, requires generalization and deductive reasoning skills. Students find it very difficult to solve them because they are not used to dealing with similar questions.

Table 1. Results of Analysis of Student Answers (N=30)

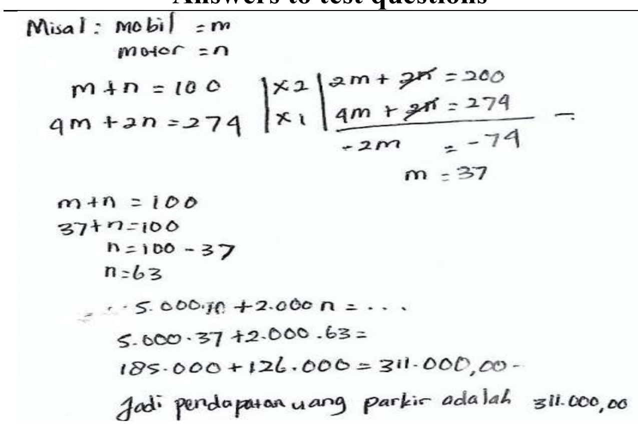
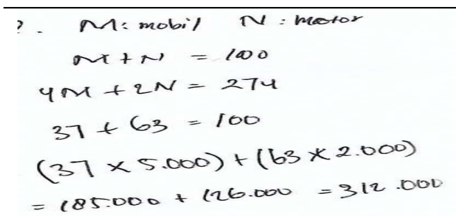
No.	**Question Characteristics**	**Category Thinking Process**			**Correct Answer (%)**
		Algebraic Invariance Thinking (%)	Proportional Reasoning (%)	Deductive Reasoning (%)	
1.	Demands students' ability to make examples, and construct mathematical equations or models. Use a solution method/strategy to determine the number of cars and motorbikes, and find the amount of income from a vehicle parking area.	15 (50)	15 (50)	-	15 (50)
2.	It is a matter of literacy, packaged in a verbal explanation as well as presented in a table in the form of information on the lengthening of the peanut tree each week. Students were asked to determine the height of the peanut tree on April 12, determine the time (date) when the peanut tree reached 50 cm high, determine the height of the peanut tree when it was harvested (3 months old), and make a graph of the growth of the peanut tree. using the calendar provided.	6(20)	6(20)	-	6(20)
3.	It is a matter of generalization that requires students to have the ability to think or	3(10)	-	3(10)	3(10)

inductive-deductive reasoning. Students are asked to determine the number of conifer trees and apple trees in the 8th row, find the general pattern or formula for the nth term from a number of rows of conifer and apple trees, and draw conclusions about the most planted trees.

Results and Discussion of Problem Number 1

In Table 1, it appears that 50% of students can solve problem number 1. The following is a representation of student answers.

Table 2. Example of Representation of Student Answers to Question Number 1

Answers to test questions	Translate
 (a)	for example: Car = m Motorcycle = n $m + n = 100$ | x2 | $2m + 2n = 200$ $4m + 2n = 274$ | x1 | $4m + 2n = 274$ - $-2m = -74$ $m = 37$ $m + n = 100$ $37 + n = 100$ $n = 100 – 37$ $n = 63$ 5000m + 2000n = … 5000.37 + 2000.63 = 185.000 + 126.000 = 311.000 So, parking revenue is 311.000,00 (a)
 (b)	M: Car N: Motorcycle M + N = 100 4M + 2N = 274 37 + 63 = 100 (37 x 5.000) + (63 x 2000) = 185.000 + 126.000 = 312.000 (b)

There were 14 students who answered correctly in part (a), while only 1 student answered correctly in part (b). The remaining 15 people gave wrong answers or provided solutions without understanding the meaning, both the meaning of the symbols used and the meaning and process of algebraic manipulation carried out. Judging from the thinking process, it is known that the 15 students who answered correctly knew the meaning of the symbols used and carried out the algebraic manipulation process correctly.

From the interview results, it was revealed that 14 students gave correct answers, they explained their answers based on the forms or solution strategies that had been given by the teacher. Only one person, namely student AS, has a different interpretation and solution method from the others (answer part (b)). Likewise, 15 people who gave wrong answers used the solution method given by the teacher but they did not understand the meaning of the algebraic symbols used nor did they understand the algebraic manipulation or the nature of the operations used. The results of interviews with RT and AS students related to answers (a) and (b) were asked (Q) how to answer problem question number 1, and the following explanation was given.

Interview with RT

Q : How do you answer the question in problem no.1?
RT : First make an example. For example, the car is m, and the motorcycle is n. Keep making the equations: m+n = 100 and 4m+2n = 274
Q : Next, what method is used to solve it?
RT : Using the elimination method. It is taken from the first and second equations. That's so that some can be eliminated, multiplied by 2 or 4. But here I'm multiplying by 2 so that they can be eliminated. [meaning RT elimination n]

RT students can explain the meaning of the symbols used, namely m and n as cars and motorbikes, and can construct equations or mathematical modeling of problem number 1, namely m+n = 100 and 4m+2n = 274. RT also understands the meaning of the operations used, for example, use the subtraction operation (-) to be able to eliminate n because if you use the addition operation (+) you cannot find any of the two values (m or n). Furthermore, RT can also determine the amount of income from parking using the substitution method after finding the values of m and n.

Interview With AS Students

AS : Cars are symbolized by M and motorbikes are symbolized by N. Then M+N= 100. 4M+2N= 274. Numbers 4 and 2 are the number of wheels. Then 37 + 63 = 100.
 : Where do the values 37 and 63 come from?
AS : I can just guess, mother, the number of cars and motorbikes. The number of cars is 37 and the number of motorbikes is 63.

Try to explain how to guess it.
First I divide by 2 the number of motorbikes and cars (50 each). Then I try multiplying by 2 and multiplying by 4 until it produces 274. If it is less or more than 274 then you have to keep looking until you find the right result (4m + 2n = 274)

Q : Try to explain how to guess it.

AS : First I divide by 2 the number of motorbikes and cars (50 each). Then I try multiplying by 2 and multiplying by 4 until it produces 274. If it is less or more than 274 then you have to keep looking until you find the right result (4m + 2n = 274)

Q : OK, what are the next steps?

AS : Next, the number of cars and motorbikes multiplied by the price of each parking, namely 37 x 5000 = 185,000 and 63 x 2000 = 126,000. So, the total parking revenue is 185,000 + 126,000 = 312,000.

Q : Is the calculation correct? Try counting again!

AS : The result is 311,000. More than 1000 bu. [US realizes his mistake]

AS students can also explain the meaning of symbols and the steps for solving operations using a trial and error strategy or what they call guessing.

Both respondents (RT and AS) gave correct interpretations of the algebraic symbols used and in solving problems they were able to interpret the operations applied even though they had different ways or methods of solving them. From observations of the mathematical statements and behavior of RT and AS students, it was concluded that they have an algebraic invariance way of thinking and proportional thinking.

On the other hand, there were some students who had difficulty solving problem number 1. NP students, for example, wrote the equations: x + y = 100 and 2x + 4y = 274 when using the elimination method, the equations changed to:

$$x + y = 100x2 \rightarrow 2x + 2y = 200$$

$$2x + 4y = 257x1 \rightarrow 2x + 4y = 257$$

Next, y (car) = 28.5 is obtained

From the description of the NP students' answers above, it is known that students have difficulty interpreting the questions. students do not understand objects (e.g. vehicles m = 28.5. Is it possible that the number of vehicles is in decimal form, not integers? Students do not examine the meaning of symbols of objects.

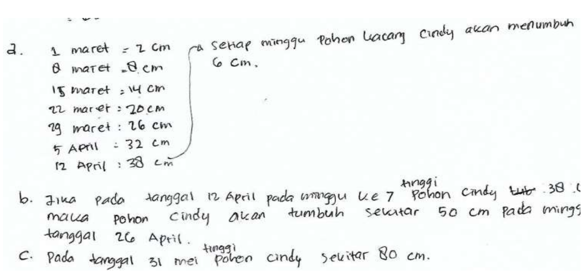
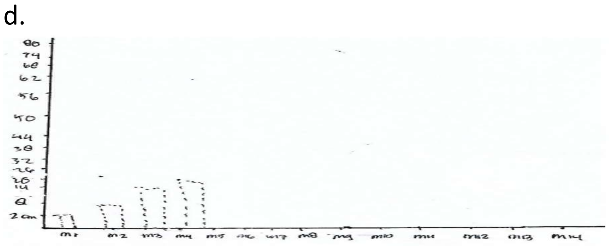
Another difficulty was also shown by MS students directly substituting values or prices for car and motorbike parking without understanding the objects and symbols used. MS does not interpret that 4m is a 4-wheeled car and 2n is a 2-wheeled motorcycle while 5000 and 2000 are the parking fees for each vehicle, not the number of vehicles, MS directly substitutes the value or price of car and motorcycle parking into the equation 4m + 2n = 100 substitutions parking fee: 4 (5000) + 2(2000) = 24,000.

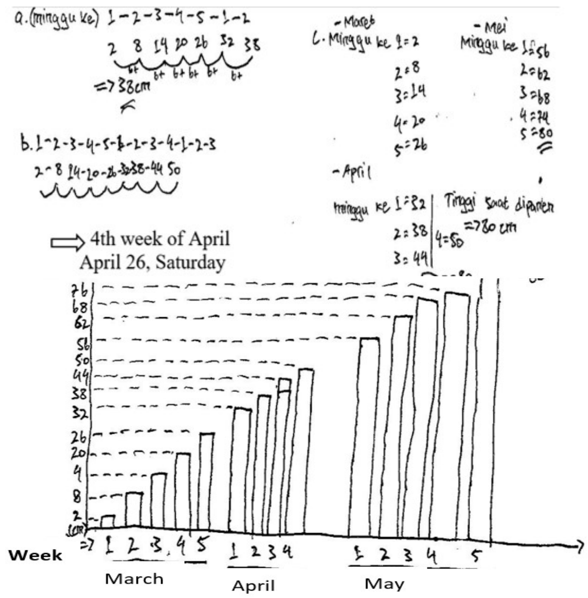
The difficulties experienced by students such as NP and MS are generally thought to be difficulties in understanding words, phrases, or sentences in word problems (such as problem number 1) which has become a difficulty for high school students around the world (Jupri & Drijvers, 2016). Difficulties like this result in errors in interpreting questions (concepts) and in the process of solving problems students cannot interpret the operations applied or manipulate without checking their meaning. Such a student's way of thinking is categorized as a non-referential symbolic way of thinking or an unwanted way of thinking.

Based on the description of the students' answers above, it appears that no one has used the substitution method to solve the system of linear equations or mathematical modeling that they have prepared. Even though this method is easier and more efficient than the elimination method. x + y = 100 as equation (1) can be changed to y = 100 – x then substituted into equation (2): 4x + 2y = 274 so that 4x + 2(100-x) = 274, 2x =74, x = 37. The value of x is substituted into equation (1), obtaining y = 63. Next, to find the value or income of the parking lot, the function approach can be used: f(x,y) = 5000x+2000y, so f(37,63) = 311,000. The function approach is intended to develop their algebraic thinking abilities (van Amerom, 2002; Kieran, 2004; Seeley, 2004).

Results and Discussion of Problem Number 2

Table 3. Representative Example of Student Answers to Question Number 2

Answers to test questions	Translate
 d.  (a)	a.1 March = 2 cm 8 March = 8 cm 15 March =14 cm 22 March = 20 cm 29 March = 26 cm 5 April = 32 cm 12 April = 38 cm → Every week Cindy's beanstalk will grow 6 cm b. If on April 12 in the seventh week the height of the Cindy tree is 38 cm, then on April 26 the Cindy tree will grow to around 50 cm. c. On May 31, Cindy's tree was about 80 cm tall d. (a)



(b)

a. (week) 1 – 2 – 3 – 4 – 5 – 1 – 2

2 8 14 20 26 32 38

6+ 6+ 6+ 6+ 6+ 6+

⟹ 38 cm

b. 1 – 2 – 3 – 4 – 5 – 1 – 2 – 3 – 4 - 1 – 2 – 3

2 8 14 20 26 32 38 44 50

c. March
Week 1 = 1
2 = 8
3 = 14
4 = 20
5 = 26

April
Week 1 = 32
2 = 38
3 = 44
4 = 50

May
Week 1 = 56
2 = 62
3 = 68
4 = 74
5 = 80

Height at harvest ⟹ 80 cm

(b)

Table 3 is a representative example of students' correct answers to problem number 2. It is known in Table 1 that only 6 out of 30 people (20%) were able to give the correct answer. Question number 2 has a higher level of difficulty than question number 1 and is a literacy question. Students need to read the questions carefully and thoroughly and follow the instructions in order to provide the correct interpretation and solution to the problem. From the results of the answers and interviews, it was discovered that the majority of students did not read the questions well and did not follow the instructions on the questions, so they gave wrong interpretations. The instructions in question are instructions for using the calendar in the problem to carry out the solution process. Completing the answer in part (a), students use procedural skills, follow the instructions, and continue the table in the question to determine the change in height (length) of the nut tree every week. The following is the explanation of ZP students in the interview regarding the answer to part (a).

Q : How did you finish number 2?

ZP : By looking at the question in the question, how tall is the peanut tree on Saturday, April 12? On what date does the height of green beans reach 50 cm? Mung bean height at harvest at the age of 3 months from the first measurement. So, in answer to the first question, we write down the weekly increase in the length of the beanstalk by 6 cm.

Q : Does that mean that on April 12 the height of the peanut tree reached 38 cm? [while pointing at the settlement table made by ZP]

ZP : Yes, ma'am.

Q : When solving this, did you use a calendar or something?

ZP : I'm using tables. I examined the increase in length every week and it turned out to be 6 cm. Then I looked around for a height of 50 cm and it turned out to be on the 26th. [Respondent using Table means continuing the existing table in the question and referring to the calendar]

The student's answer to part (b) by FR, is a unique (different) solution from all students who took the test. The following is a fragment of an interview regarding the strategy used by FR students in solving question number 2.

Q : What is your strategy to solve question number 2?
FR : First, you have to be based on the questions, the benchmark questions are every week on Saturday. The first and second weeks are looking for the difference, for example in the first week 2cm then in the 2nd week it increases to 8cm obtained from 2 + 6 then the height increase is 6 cm

Even though the two solutions by ZP and FR obtain the same answer, using the same difference value, namely 6, they both have different basic understandings. ZP in its calculations uses procedural knowledge by creating a tabular form to find the answer asked in the question. ZP describes changes in the height of nut trees every week by referring to the initial height, namely 2 cm on March 1, 8 cm on March 8, and so on increasing by 6 every 7 days. The answer to the question, "How tall will the beanstalk be on April 12?" ZP immediately looks in the table in the column for the date "April 12" and the resulting "height", as well as to determine the date that shows the height of the nut tree is 50cm. Meanwhile, FR in the process of solving the answer uses intuitive abilities, namely trying, guessing, guessing, then finding the answer. As FR explained, look for the difference, "for example, if in the first week, it was 2 cm, in the second week it was 8 cm and this is obtained from 2 + 6 then the increase in height is 6 cm". The explanation of the answer is in the form of a pattern, namely the height of the nut tree increases by 6 (6+) every week.

From the results of observations on student worksheets and interviews, it is known that ZP and FR students provide interpretations that have a quantitative relationship, for example predicting the height of the next week's nuts by referring to the initial height of the nut tree (2 cm). FR students provide flexible interpretations, connected to other concepts, for example, when asked about solving problem number 2 in another way, FR answered that he could use the concept of arithmetic series. ZP and FR students also solved the problem by interpreting the operations applied, namely always increasing by 6 every week (week), doing a counting jump (prediction) when determining the height of the beanstalk after three months and in what week the beanstalk reached a certain height (50cm). Based on the characteristics of this way of understanding, it is generally concluded that ZP and FR students have an algebraic invariance way of thinking.

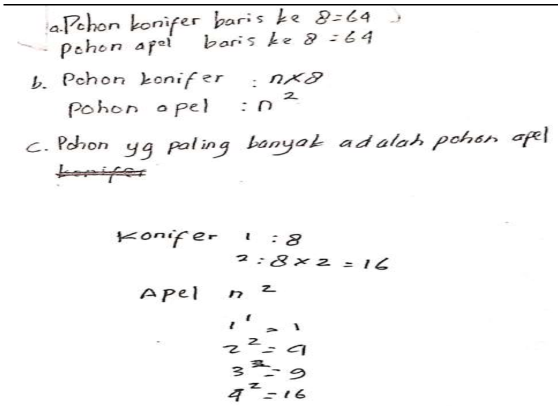
Some students encountered difficulty in solving question number 2, for example, student NS, namely on the question, "determine the height of the beanstalk at 3 months", NS gave the answer: 2 cm x 90 days x 6 cm = 1880 cm. He explained that 2 cm is the initial height, 90 days is 3 months of age and 6 cm is the increase in height every week, still 6 cm. This shows that NS's interpretation of problem number 2 has no quantitative relationship and problem-solving without interpreting operations or manipulating without checking the meaning. There is also another difficulty, namely the question "What date will the nut tree reach 50 cm in height?" PN students gave their answers, namely April 28 (Monday). Based on the question information, measurements are carried out every Saturday. These are some of the interpretation and problem-solving errors made by the majority (80%) of students.

Even though both students (ZP and FR) have the expected way of thinking (algebraic invariance) they both still use procedural abilities. Problem number 2 above can be solved using the concept of arithmetic sequences, to teach students that mathematics has more than one way of understanding (solution), namely using the formula for the nth number of rows: Un = a + (n-1)b with a = 2, b = 6, where Un is the amount of time (nth week). In this way, the answer to the question of how tall the tree will be on April 12 can be determined. This can be seen on the calendar, namely n = 7, so U_7 is 38. For the question, how tall will the tree be in three months? In this case n = 14, so U_{14}= 80. The next question is What date does the tree reach 50 cm in height? This section will be easier to complete with a functional approach. The basis for this understanding can be built from the formula for the number of nth terms: Un = a + (n-1)b, where Un is taken as the variable y which indicates the height of the tree, a and b are constants whose values are 2 and 6, and n can be taken as the variable x which indicates time. In this way, a function equation y = 2 + 6(x-1) can be formed, which is then simplified to y = 2(3x-2). When it reaches a height of 50cm (y = 50), then x = 9. In this case, the 9th week can be seen on the calendar, namely April 26. Through the function obtained, it will be easier to draw a graph of the growth of the beanstalk as requested in the problem.

Results and Discussion of Problem Number 3

Table 4. Example of Representation of Student Answers to Problem Number 3

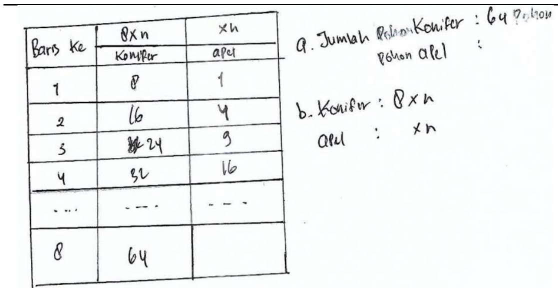
Answers to test questions	Translate



(a)

a. 8th row conifers = 64
   8th row apple tree = 64
b. Conifer Trees: n *x* 8
   Apple Tree: n^2
c. The most planted trees are apples
   Conifer 1 : 8
   2 : 8 *x* 2 = 16
   Apple n^2
   $1^2 = 1$
   $2^2 = 4$
   $3^2 = 9$
   $4^2 = 16$

(a)



(b)

Baris ke	8*x*n	*x*n
	Konifer	Apel
1	8	1
2	16	4
3	24	9
4	32	16
…	…	…
8	64	

a. Number of trees
   conifers: 64 trees.
   The Apple tree:

b. Conifers: 8*x*n
   Apple: *x*n

(b)

Table 4 is an example of a representation of students' correct answers and it is known based on Table 1 that only 3 out of 30 people (10%) can give the correct answer. Problem number 3 is a pattern generalization problem and has a higher level of difficulty than questions number 2 and 1. Based on the examples of answers to part (a), RT students and answers (b) by AS students provide interpretations of the images (patterns) and determine the number of each tree in each row using reasoning or inductive thinking. Next, students make deductive generalizations, namely for apple trees nxn or n^2 and for conifer trees 8n.
The following is the explanation of RT and AS students in the interview.

Interview with RT

Q	:	Tell me about the picture! [Points at question]
RT	:	The inside of the picture is an apple tree, and the outside is a conifer.
Q	:	: How do I find the answer to that question?
RT	:	To the question, "how many apple trees and conifers are in row 8?" I first make the formula, so it's easy to find the answer.
Q	:	What's the formula?

RT : Conifers = n x 8 (n is for the nth tribe or row, while 8 is the number of conifers in the first row)
Q : Before you conclude it is n x 8, what is the thought process to arrive at n x 8?
RT : Well Ma'am. Since the first equation has 8 conifers, 16 in the second, 24 in the third, and 32 in the fourth, that means nx 8.
Q : Okay for the apple tree how do I find it?
RT : Here I use the formula for apples=n^2
Q : How did you find the formula for n^2?
RT : Because first row 1 x1 =1, second row 2 x2=4, third row 3 x3=9 and fourth row 4x4=16
Q : OK, back to n, n can be any number or what does n mean?
RT : n can be any number because it is unknown.

RT students describe their understanding of problem number 3 and its solution clearly. RT can understand symbols and can also interpret the operations used. For example, he explains the symbol n as the nth term or line, making the correct guess by using reasoning or inductive thinking. For example, explain that in the first row of the Apple tree pattern: 1x1, second row: 2x2, third row: 3x3, and so on until the question the eighth row is 8x8 = 64. In the same way, for Conifer trees the first row is 8, the second row is 16, third row is 24. Next, RT can determine a formula to determine the number of apple trees in the nth row or term, namely nxn or n^2, and for conifer trees 8n. Forms n^2 and 8n are the result of a deductive process or way of thinking. The conclusion that apple trees are more numerous than conifer trees was obtained by RT after applying and comparing the n^2 and 8n formulas.

Q : Why are there so many apple trees?
RT : Well, ma'am, the conifer trees in the ninth row are 72, while the apple trees in the ninth row are 81.

Interview with AS

Q : What do you understand from question 3?
AS : Number 3 is that there are plants that have been planted in a rectangular area that continues to fold every few rows.
Q : Try to explain in the first line!
AS : For the first row there are 8 conifer trees and 1 apple tree. The second row of apple trees, it multiplies by 4 to 4 apple trees. For conifers 8 x 2 = 16 conifers. In the third row of conifers 8 x 3 = 24.
As for Apple, I don't know Mrs.

AS students appear to be trying to make a temporary conjecture that the number of trees in a row is obtained from the number of trees in the first row multiplied by that row. This is true for conifer trees but not for apple trees. At first, the US also made the conclusion that the number of coniferous and apple trees was the same. This conclusion was obtained because the US used the formula for determining the number of apple and conifer trees only in the eighth row. However, after being asked during the interview discussion, which one had more results between 8xn and the next? AS revised his answer, that there were more apple trees. As a result of the interview discussion with AS regarding question number 3, information was obtained that students need to be given indirect assistance in the form of encouragement to reflect on their thinking when they encounter difficulties in solving problems. This can help students develop their way of thinking about problems (Natcha Kamol & Yeah Ban Har, 2010).

From the description above, both statements on student worksheets and in interviews show that RT and AS students' interpretations of problem number 3 characteristically have a quantitative relationship, understand symbols, interpret the operations they apply, and can determine the general form (nth row). Based on this way of understanding, it is concluded that RT and AS students have an algebraic invariance way of thinking as well as a deductive way of thinking.

Based on Table 1, it is known that the majority of students (90%) made mistakes in solving problem number 3, including those who could not provide an answer (the answer sheet was blank). There are several types of errors made by students, for example, NA students try to give answers using the concept of arithmetic sequences but errors occur in understanding the concept. NA students write the formula for the nth term: $Un = (a+1n)b \rightarrow U_8=(8+1x8)8 \rightarrow U_8=128$. NA provides an explanation or statement that U_8: 8th term, a=8 because the first row of conifers totals 8, likewise for the difference (b)=8 because the first row totals 8. The students' interpretation and problem-solving do not have a quantitative relationship, do not understand symbols, not interpret operations applied and manipulated without checking their meaning. To the question, which tree has the most? 90% of students answered conifer trees with the reason that conifers protect apple trees so there should be more of them. Based on the characteristics of this way of understanding, it is concluded that students have a non-referential symbolic way of thinking.

Question number 3 is a pattern generalization question that is expected to develop students' algebraic thinking abilities. The thing that needs to be emphasized in this problem is the shape of the pattern and also the difference (difference) resulting from the existing number pattern or image. Based on known information or data, there are two types of trees, namely apples and conifers. If you look closely, the apple tree forms a perfectly square pattern, the number of sides is the same, so it can easily be determined that the first row is 1x1, the second row is 2x2, the third row is 3x3, and. From the shape of the pattern, it can be determined that the nth row is nxn. The difference (difference) produced varies and continues to increase, forming a series of odd numbers: 3, 5, 7, ... Meanwhile, conifer trees form a square image but only on the outside, forming a pattern with the number of first rows: 8, second row: 16, third row; 24, and so on with the resulting difference being fixed namely 8. From the shape of the resulting pattern it can be determined that the first row is 8x1, the second row is 8x2,... and so on. So, it is concluded that the number of conifer trees will increase by multiplying 8 by the corresponding row. Thus the formula for the nth term can be determined as 8xn where n is the number of rows. This description concludes that the most numerous trees if land continues to be expanded are apple trees. This is because the difference between apple trees continues to increase while conifers remain constant, and is based on the nth-term formula, where nxn is more than 8xn.

The results of the analysis of students' answers from both written answers and interviews, it is known that the majority of students have a characteristic way of thinking that is very far from the expected thinking, namely algebraic, proportional, and deductive invariance. Students perform manipulations without the ability to investigate the meaning of symbol relationships and any transformations involved in them. In other words, symbols are not understood as representations of a coherent mathematical reality. This shows that classroom learning does not pay attention to students' thinking processes in solving problems (Harel, 2008b).

CONCLUSIONS

In general, students have a non-referential symbolic way of thinking. This is based on the thinking characteristics of students, namely not understanding the symbols and operations involved, not being flexible or concepts being understood separately, and not connected to other concepts. This way of thinking illustrates that students' understanding is procedural in nature. There are also findings that the more complex the algebra problem given, the more students experience difficulty in solving it. In problem 1, which is relatively easier than problems 2 and 3, some students (50%) can solve it and have a way of thinking about algebraic invariance and proportional reasoning. In problem 2, which is relatively more difficult than problem 1, only 20% of students answered correctly and had an algebraic invariance way of thinking and proportional reasoning, while the rest (80%) of students had a non-referential symbolic way of thinking. Problem 3, which is a problem of pattern generalization that requires deductive thinking and algebraic invariance, only 10% of students answered correctly, and the rest (90%) of students had a non-referential symbolic way of thinking.

These findings can lead to further investigations, for example how to present material or an algebraic (mathematics) concept so that it can develop the desired way of thinking, namely algebraic invariance thinking, proportional reasoning, and deductive reasoning. To form the desired thinking habits, namely by providing opportunities for students to understand mathematical objects or problems in different ways or more than one way of understanding. This can be done since students are still in elementary school. In addition, generalization, modeling, functional, and problem solving approaches can be used to develop algebraic thinking.

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Overview of Student's Mathematics Reasoning Ability Based on Social Cognitive Learning and Mathematical Self-efficacy

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Abstract: Detailed mathematical reasoning abilities can help students to understand higher mathematical abilities, including proving, problem-solving, and critical thinking. However, based on surveys and research, students' mathematical reasoning abilities are low and require significant attention. Therefore, this study aims to gain a detailed picture of the mathematical reasoning abilities of students who got the social cognitive learning (SCL) model and students who got the problem-based learning (PBL) model by considering the students' mathematical self-efficacy (MSE). The study used a quasi-experimental Nonequivalent post-test-only group design with 70 students from class 11 SMA in one school in Bandung. The data collection used a Mathematical Reasoning Ability test and a mathematical Self-efficacy Questionnaire to classify MSE levels as low, moderate, or high. The data were analyzed using two-way ANOVA and a 3x2 factorial design. According to the study results, students taught using the SCL model had better mathematical reasoning abilities than those taught using the PBL model. Moreover, students with high MSE levels exceed low MSE levels in math abilities. The SCL model enhances students' mathematical reasoning abilities and expands the range of social cognitive theory's applicability of mathematics.

Keywords: Social Cognitive, Mathematical Reasoning Ability, Mathematical Self-efficacy, PBL

INTRODUCTION

The reasoning skills are needed to support higher mathematical abilities such as proof, problem-solving, and critical thinking (Krıstayulita et al., 2020; Öztürk & Sarikaya, 2021; Putrawangsa & Patahuddin, 2022). Mathematical reasoning allows students to acquire ideas, properties, and methods as logical, interconnected, and cohesive parts of mathematics, not just ordinary routines. Mathematical reasoning ability is an essential component of education, mainly needed to understand mathematics. According to NCTM (2014) and PISA (2018), mathematical reasoning ability is a way to evaluate and make arguments, evaluate interpretations and conclusions related to statements and problem solutions. Thus, mathematical reasoning skills are essential to developing at the primary, junior, and senior secondary education levels by placing students in situations where they can make, correct, and test their conjectures (Mansi, 2003; NCTM, 2020).

Students in high school are expected to be able to explain, verify, justify or validate to convince themselves, other students, and teachers about the truth of mathematical statements (Fiallo et al., 2021). If their mathematical reasoning abilities are not developed, they will see mathematics as a specialized set of rules, a collection of calculations and images produced without thinking (Payadnya, 2019). As a result, improving students' mathematical reasoning abilities in the classroom is an essential component of mathematics teaching and learning (Boaler, 2010; Mata-pereira & Ponte, 2017).

Research reports show that mathematical reasoning skills are essential in a deep understanding of mathematics (Herbert et al., 2022; Zhang & Qi, 2019). Promoting mathematical reasoning skills will also allow students to more easily understand math problems (Saleh et al., 2018; Supriadi et al., 2021) and geometric (Payadnya, 2019; Seah & Horne, 2021). Facts show that individuals with high mathematical reasoning abilities can more easily understand and solve mathematical problems (Erdem & GÜRBÜZ, 2015; Hasanah et al., 2019; Irawati & Hasanah, 2016; Rizqi & Surya, 2017). This research clearly demonstrates that mastery of mathematical reasoning abilities has far-reaching consequences for students.

Meanwhile, the facts show that the mathematical reasoning ability of Indonesian students is still low. Indonesian students are unable to compete on the international stage, as reflected in the evaluation results of TIMSS, which ranks 45 out of 50 countries (TIMSS, 2016) and the results of PISA which only ranked 63 out of 70 countries (OECD, 2018). A thorough evaluation shows that Indonesian students' mathematical reasoning abilities are low (Ayuningtyas et al., 2019; Mumu & Tanujaya, 2019; Sandy et al., 2019; Sumartini, 2015).

Developing students' mathematical reasoning abilities continues to be a current research priority. The study results explain that student activity-based learning models are the main thing in developing students' reasoning abilities (Erdogan, 2019; Masfingatin & Murtafiah, 2020; Ulya et al., 2017). However, this learning can be said to be effective when done offline. The change in the learning environment from offline-based to online-based due to Covid-19 is one factor that influences student learning success (Mukuka et al., 2021), especially students' mathematical reasoning abilities. Learning during the COVID-19 period has limited direct interaction between students and students and teachers. This change in the learning climate causes many physical, mental, and emotional responses compared to learning and teaching conditions in general (Ghazali et al., 2021). Adjusting learning models to support the learning process during the Covid-19 period is paramount. Changes in learning can be anticipated by still placing students to actively seek, listen and listen to various learning resources that can support the student's understanding process. One of the learning models that can accommodate this is the social cognitive learning model.

The social cognitive learning (SCL) model is based on social cognitive theory. In this model, students participate in learning, but they are involved in shaping themselves (Santrock, 2006). Although learning Classical and Operant Conditioning in specific ways is still a good learning pattern, most people learn about what they have learned from observation activities. (Santrock, 2006). Observational learning differs from classical and operant conditioning in that it does not involve direct personal experience with stimuli, reinforcement, or punishment. Learning through observation involves observing the behavior of others, called models, then imitating the model's behavior (Money, 2016; Nabi & Prestin, 2017). Children and adults alike learn many things from observation and imitation. For example, when children learn the language, social skills, habits, and many other behaviors, they observe their parents or older people.

Bandura (1977) states that learning through observation plays an essential role in developing a child's personality through observation. Humans acquire knowledge, rules, skills, strategies, beliefs, and attitudes by observing others. Individuals also look at models or examples to learn the usefulness and behavioral suitability of the modeled behavior; then, they act according to beliefs about their abilities and the expected results of their actions (Bandura, 1977). So that the SCL model is expected to be able to accommodate the online learning process during the Covid-19 period.

In previous studies, the application of social cognitive aspects in the field of Education (Ghazali et al., 2021), in the business field (Harinie et al., 2017; Healey et al., 2021; Kursan Milaković, 2021; Ng et al., 2021), in the criminal field (Proctor & Niemeyer, 2020), and the field of information and management (Lockwood & Klein-Flügge, 2020; Money, 2016; Pinho et al., 2020) have a positive impact. This study provides an overview of how social cognitive regulation can help convey positive attitudes and behaviors that researchers want to convey. However, the SCL model in mathematics learning has not been widely applied. So that researchers are interested in applying SCL in learning mathematics.

This research intends to expand on previous research (Ghazali et al., 2021; Mukuka et al., 2021), examining the SCL model's impact on students' mathematical reasoning abilities in online learning situations. Solid practical abilities in students must support this model. Self-efficacy is believed to play a vital role in student success, academic life, and career (Bandura, 1997; Kingston & Lyddy, 2013; Schunk & Dibenedetto, 2019). Self-efficacy in teaching and learning activities needs to be developed in students. The realization of this principle is to place the teacher in the leading role as a facilitator and motivator (Schunk & Dibenedetto, 2019). In detail, we expand the observational aspect in the study by (1) conducting comparisons involving the control class that applies problem-based learning (PBL); (2) including aspects of mathematical self-efficacy as aspects that influence mathematical reasoning abilities; and (3) conducting online-based learning using zoom meeting for interactive media and google classroom as a medium for organizing the results of the learning process.

This kind of research is essential in the hope that it can provide information in choosing the suitable model in responding to changes in the learning environment due to COVID-19. In addition, the results of this study are one way to introduce social cognitive concepts in mathematics learning. Therefore, this study aims to obtain a comprehensive picture of the mathematical reasoning ability of students who received the SCL model and students who obtained the PBL model by paying attention to the level of students' mathematical self-efficacy (MSE). The research questions posed are (1) Is there a difference in mathematical reasoning ability between students who get the SCL model and the PBL model? (2) Is there a difference in mathematical reasoning ability between students at different MSE levels? (3) Is there an interaction between model and MSE on students' mathematical reasoning ability (MRA)?

RESEARCH METHODOLOGY

This study uses a quantitative method with a quasi-experimental Nonequivalent post-test Control Group Design. Researchers were directly involved in treating both classes. The experimental class was given treatment by applying the SCL model, and the control class was offered treatment using the PBL model. The application of the SCL model syntax consists of (1) Attention, (2) Retention, (3) Production, and (4) Motivation (Bandura, 1977). The application of the syntax of the PBL model consists of (1) problem presentation; (2) Problem investigation; (3) Solution problems; and (4) Evaluation process (Awang & Ramly, 2008; Soden, 1994). In detail, a comparison of learning activities in the two classes is presented in Appendix 1. The fundamental difference between these two models occurs in the "retention and motivation" phase in the SCL model and the "Problem investigation and evaluation process" phase in the PBL model. The retention phase allows students to understand the problem and hear the teacher reinforce concepts. This experience was not obtained in the PBL model in the Problem investigation phase. In the PBL model, students are focused on understanding the problems given individually while the teacher only asks questions that can guide them. In the Motivation Phase in the SCL model, the teacher provides appreciation, praise, and reinforcement for what has been learned. Meanwhile, in the Evaluation process phase, the teacher asks students to reflect back on what they have learned and provides reinforcement at the end. Although the two groups of classes were given a different treatment, the material and number of meetings between the two groups of classes remained the same. The material provided is the function limit, with each session for six weeks. The learning process and the final test's implementation are conducted virtually through a zoom meeting. Student answer sheets are coordinated through Google Classroom. Before the experiment, the researcher identified the students' MSE level by distributing the MSE questionnaire through the Google form.

The research participants were 70 high school 11 students in one of the schools in Bandung, which consisted of two classes with a total of 35 each. The sampling technique used was Nonprobability Sampling with the type of Purposive Sampling. As for the considerations in choosing a class, the researcher determines the abilities possessed in the experimental and control classes in a balanced or equivalent state. The determination was strengthened by independent t-test analysis. The results of the study explained that there was no significant difference between the Experiment class and the control class ($t(68) = .837$ with $p > .05$), so the result can conclude that the two classes are in a balanced state.

They were collecting data using a mathematical reasoning ability test and MSE questionnaire. The mathematical reasoning ability (MRA) test is an essay consisting of 5 questions. Indicators of students' mathematical reasoning ability include (1) Memorized Reasoning, (2) Algorithmic Reasoning, (3) Novelty, (4) Plausible, and (5) Mathematical foundation (Jonsson et al., 2014; Lithner, 2008). The MRA test was tested for validity and reliability. To test the validity of using product-moment correlation with valid results. While the reliability test using Alpha-Cronbach with reliable results (Cronbach's Alpha reliability coefficient of 0.495). The MSE questionnaire uses 20 items, which are constructed from (1) the Magnitude dimension, (2) the Strength dimension, and (3) the Generality dimension (Bandura, 1997). Measurement of the MSE questionnaire using a Likert scale of 1-4. The questionnaire was tested for validity and reliability. The test results show that the 20 items are valid and reliable. The MSE levels in this study are grouped into three categories, namely low MSE levels, moderate MSE levels, and high MSE levels.

Descriptive statistical analysis includes frequency, mean and standard deviation used to describe the demographic features of students. Normality and homogeneity tests were performed as prerequisites before performing ANOVA analysis. The Shapiro-Wilk test assessed customarily distributed data (Ghasemi & Zahediasl, 2012). Observation of the normality of the data in this study, based on the standardized residual score and not normality for each data from the research variables (Everitt & Skrondal, 2010; Kozak & Piepho, 2018). Levene's test was conducted to assess the homogeneity of the research data. The analysis of the research hypotheses used a two-way ANOVA with a 3 x 2 factorial design. The post-ANOVA follow-up test used the Tukey test. The entire statistical calculation process uses SPSS 25.

RESULTS

The purpose of this study was to obtain a comprehensive picture of the mathematical reasoning ability of students who obtained the SCL model and students who obtained the PBL model by paying attention to the students' MSE level. A descriptive statistical analysis of the model's average MRA demographics based on the model is presented in Figure 1.

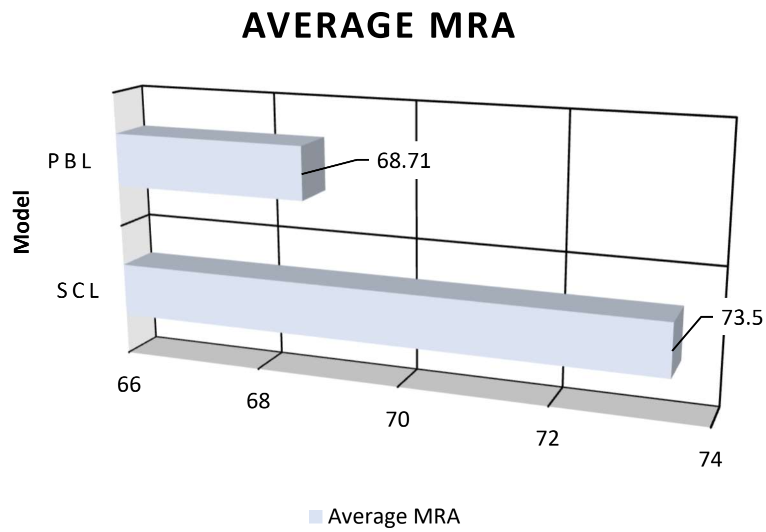


Figure 1: Mathematical Reasoning Ability Score

In Figure 1, it can be seen that the students in the SCL group ($\bar{x}_{SCL} = 73.50$) got a better average MRA than the students in the PBL group ($\bar{x}_{PBL} = 68.71$). Based on the standard deviation values in the two groups, the standard deviation of the SCL group (s = 13.03) was more significant than the standard deviation of the PBL group (s = 12.50). This indicates that students in the SCL group get different scores than students in the PBL group.

Furthermore, the demographics related to the MRA mean score based on the model and MSE are presented in Table 1, explaining that the high MSE level ($\bar{x}_{SCL} = 84.17, \bar{x}_{PBL} = 75.45$) obtained an MRA average that outperformed the MRA average at the other two MSE levels. Moderate MSE level students ($\bar{x}_{SCL} = 72.25, \bar{x}_{PBL} = 67.50$) obtained a higher MRA mean than the MRA mean of low MSE level students ($\bar{x}_{SCL} = 69.44, \bar{x}_{PBL} = 63{,}00$).

Model	MSE	Average	Std. Deviation	f
SCL	Low	69.44	11.30	9
	Moderate	72.25	13.23	20
	High	84.17	10.68	6
PBL	Low	63.00	9.77	10
	Moderate	67.50	14.77	14
	High	75.45	8.79	11
Total	Low	66.05	10.75	19
	Moderate	70.29	13.87	34
	High	78.53	10.12	17

Table 1: Descriptive statistics of MRA scores by Model and MSE

The results of the Shapiro-Wilk test show the Standardized Residual score of .968 with a significant level of .069, which is far above .05. This means that all data sets come from groups that are usually distributed. Meanwhile, in the Levene test, the Levene Statistic score was obtained at 1.282 with a significant level of .283, far above .05. Thus, the homogeneity condition is met. The hypothesis testing procedure can be carried out because the ANOVA statistical test requirements related to normality and homogeneity tests have been met. The results of the hypothesis are presented in Table 2.

Source	Type III Sum of Squares	df	Mean Square	F	sig.	Partial Eta Squared
Corrected Model	2121.539[a]	5	424.308	2.893	.020	.184
Intercept	315980.143	1	315980.143	2154.326	.000	.971
Model	671.515	1	671.515	4.578	.036	.067
Level_MSE	1678.438	2	839.219	5.722	.005	.152
Model * Level_MSE	42.035	2	21.018	.143	.867	.004
Error	9387.033	64	146.672			
Total	365800.000	70				

Table 2: Results of Analysis Tests of Between-Subjects Effects

Based on the results of data analysis in Table 2, several findings are produced. First, $F(1,64)$ = 4.578 with a significance level of .036, which is far below .05. This means that there is a significant effect between the application of the SCL model and the PBL model on mathematical reasoning abilities. Second, the score $F(2, 64)$ = 5.722 with a significance level of .005, far below .05. This means a significant effect between the MSE level on mathematical reasoning abilities. Based on the post hoc test in Table 3, the difference in mathematical reasoning ability occurs at the low MSE level and the high MSE level (significant level of .008, which is far below .05).

(I) MSE Level	(J) MSE Level	Mean Difference (I-J)	Std. Error	sig.
Low	Moderate	-4.2415	3.46893	.444
	High	-12.4768*	4.04319	.008
Moderate	High	-8.2353	3.59745	.065

Table 3: Multiple Comparisons of MSE Levels

Third, $F(2, 64)$ = .143 with a significant level of .867, which is far above .05. This condition explains that MSE interaction in SCL or PBL classes does not significantly affect students' MRA. As a result, students' success in MRA is only affected by their presence in SCL or PBL classes. Where the mathematical reasoning ability of high MSE level students in SCL and PBL classes outperformed low MSE level students' MRA, as illustrated in Figure 2, shows the effect of the model and MSE level on students' MRA.

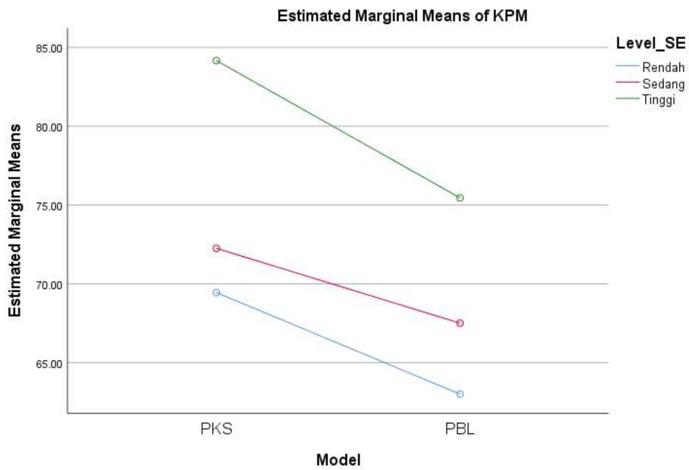


Figure 2: Graphical representation of the effect of model interaction and MSE level on MRA

Referring to the quantitative results described above, the researchers' findings were strengthened by differences in the results of the answers to students' mathematical reasoning abilities in the two classes. In the following, one of the results of students' responses in both classes is presented in solving the function limit questions in measuring students' creative reasoning, specifically on the Plausible indicator.

Indicator	Question
Indicator creative reasoning	Is known: $f(x) = \frac{bx^2+bx-21+b}{x^2+5x+6}$. If $\lim_{x\to -3} \frac{bx^2+bx-21+b}{x^2+5x+6}$ exists, then determine the limit value!

Table 4: Indicators and Problems of Mathematical Reasoning Ability

Based on the answers given in Table 5, it can be seen that there are differences in the responses provided by students in the two classes. On the plausible indicator, students studying with the SCL model can solve the questions. Students understand that the value $\lim_{x\to -3} \frac{bx^2+bx-21+b}{x^2+5x+6}$ exists, and for $x \to -3$ which will then be a Joint factor in the functions $bx^2 + bx - 21 + b$ and $x^2 + 5x + 6$. So, the student divides $(x + 3)$ by $bx^2 + bx - 21 + b$ and gets $b = 3$. Furthermore, the student determined the limit value of the requested function, which was 15. However, it was different for students who studied with the PBL model, he failed to understand the meaning of the problem, and even though he tried to solve the problem using the concept of derivative, he failed to solve the problem when the final result was is obtained in the form of "b."

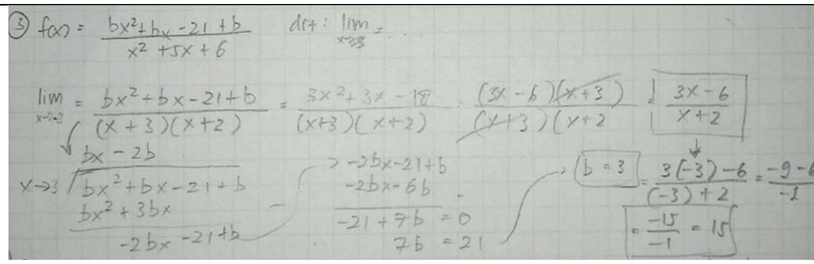
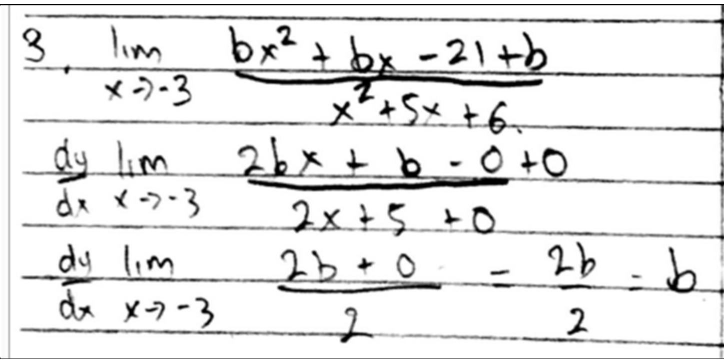
Model	Answer
SCL	
PBL	

Table 5: Students' answers in the two classes in solving creative reasoning questions on plausible indicators

This indicator requires students to solve problems by paying attention to mathematical characteristics or giving reasons acceptable (plausible), namely utilizing information from the problem which states that ***the limit value of the given function exists***. Based on the answers given, students who study with the SCL model are able to understand the meaning of the problem, and understand the ***concept of algebraic limits***, so that for x to approach a (a certain value) it is a common factor in the given algebraic function. Understanding this concept makes it easier for students to solve the problems given. The demonstrated concept of creative thinking refers to the development of adaptable ideas and task solutions based on sound arguments and inherent mathematical features. This creative reasoning does not refer to high or extraordinary intelligence, but rather to easy and new answers to mathematical problems (Lithner, 2006). so that students in the SCL class have creative reasoning abilities compared to students in the PBL class.

DISCUSSION

The results of the research related to students' mathematical reasoning abilities based on the learning model found that there were differences in MRA in the two groups. Based on the mean MRA scores of students presented in Figure 1, it can be seen that the MRA of students who received the SCL model (73.5) was higher than the MRA of students who received the PBL model (68.71). In the learning process, the SCL model provides opportunities or space for students to develop MRA compared to students who learn to use the PBL model, as shown in Figure 1. It can be seen that each student in each class is given worksheets that guide in conveying the one-sided limit concept. In the SCL model (figure a), students are given activities to determine the value of the function $f(x)$ with different domains. Next, students are led to try to describe the function $f(x)$ based on the value of the function that has been determined. At the end of the activity, students are asked to be able to draw conclusions based on the value $f(x)$ that has been obtained and an image of the function $f(x)$, whether the given function $f(x)$ has a limit value or not. Different from the PBL model (figure b), students are asked to be able to define the limit of a function based on the problem given. students are asked to observe the image of the function, the behavior of the function, and the value of the function at the given points. This situation requires students to understand the function in trying to make conclusions about the definition of the limit of the function.

In the SCL model, learning situations are carried out through observation. Students can develop their mathematical reasoning abilities by observing the teacher solve problems in this observation process. Through this observation, students freely determine whether the strategy can solve the following problem (Mata-pereira & Ponte, 2017). Even in this condition, students who want to confirm the truth of their reasons observe the collection of opinions from their friends. So that reasoning can be developed by asking students to explain the proof or justification for a mathematical concept. This is different from the PBL environment, where students solve problems independently without any previous experience. So that students' mathematical reasoning abilities are not well developed. In connection with these results, in the findings of previous studies, no research has examined the effect of these two learning models on mathematical reasoning abilities. However, specific findings regarding the application of the PBL model (Aslan, 2021; Bosica et al., 2021; Evendi et al., 2022; Ping et al., 2020) explain that this model can develop mathematical abilities, which lead to aspects of students' mathematical reasoning abilities. The problems' characteristics at the beginning of learning become a stimulus for students to link the problem with the mathematical concepts being studied. So that students' success depends on the ability of students to connect information that has been previously owned. The researchers' findings are slightly different, where the effect of the PBL model is not better than the SCL model in developing students' mathematical reasoning abilities.

MATHEMATICS TEACHING RESEARCH JOURNAL
EARLY SPRING 2024
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KEGIATAN 2

1. Lengkapilah dan tentukanlah nilai limit $f(x)$ berikut ini.

$f(x)\begin{cases}3x & untuk\ x \le 4\\ 3x+2 & untuk\ x > 4\end{cases}$

Arah Kiri	$x \le 4$	3	3,1	3,2	3,3	3,4	3,5	3,6	3,8	3,9	3,95	3,99
	$f(x) = 3x$	9	....	....	....	....	....	....	....	....	....	....
Arah Kanan	$x > 4$	4	4,01	4,02	4,05	4,1	4,2	4,3	4,5	4,6	4,9	5
	$f(x) = 3x + 2$	....	....	....	....	....	....	....	....	....	....	17

Dengan melengkapi tabel di atas, dapatkah kalian menentukan berapakah nilai $\lim_{x\to 4} f(x) =$ .....? sekarang coba kalian gambar grafik $f(x)$!

Perhatikan grafik yang telah kalian buat, jika x mendekati 4 dari kiri, maka nilai $f(x)$ mendekati .... jika nilai x mendekati 4 dari kanan maka nilai $f(x)$ .... hal ini dapat ditulis sebagai $\lim_{x\to 4^-} f(x) =$ ..... dan $\lim_{x\to 4^+} f(x) =$ .....

Grafik fungsi $f(x)$ untuk $x \le 4$ dan $x > 4$ tidak bersambung (ada lompatan di titik $x =$ .....). ini merupakan ciri grafik fungsi yang tidak mempunyai limit di satu titik.

Jadi, $f(x)\begin{cases}3x & untuk\ x \le 4\\ 3x+2 & untuk\ x > 4\end{cases}$ tidak memiliki limit untuk nilai x mendekati ......

Dari kegiatan di atas, apa yang dapat kalian simpulkan?

ACTIVITY 2

1. Complete and determine the limit value of $f(x)$ below.

$f(x)\begin{cases}3x & for\ x \le 4\\ 3x+2 & for\ x > 4\end{cases}$

Left Direction	$x \le 4$	3	3,1	3,2	3,3	3,4	3,5	3,6	3,8	3,9	3,95	3,99
	$f(x) = 3x$	9	....	....	....	....	....	....	....	....	....	....
Right Direction	$x > 4$	4	4,01	4,02	4,05	4,1	4,2	4,3	4,5	4,6	4,9	5
	$f(x) = 3x + 2$	....	....	....	....	....	....	....	....	....	....	17

By completing the table above, can you determine what the value of lim $\lim_{x\to 4} f(x) =$ .....? Try drawing a graph o $f(x)$!

Look at the graph you have made, if x approaches 4 from the left, then the value of $f(x)$ approaches .... if the value of x approaches 4 from the right, then the value of $f(x)$..... this can be written as $\lim_{x\to 4^-} f(x) = \cdots$ and $\lim_{x\to 4^+} f(x) = \cdots$

The graph of the function $f(x)$ for $x \le 4$ and $x > 4$ is not continuous (there is a jump at the point $x = \cdots$). This is a characteristic of function graphs that do not have limits at one point.

So, $f(x)\begin{cases}3x & for\ x \le 4\\ 3x+2 & for\ x > 4\end{cases}$ has no limit for x values approaching ......

From the activities above, what can you conclude?

(a)

MASALAH 3

Menghitung pendekatan dari nilai suatu fungsi

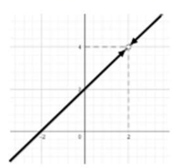
Di bawah ini disajikan salah satu alternatif penyajian limit dengan bantuan grafik fungsi.

Pandanglah fungsi $f(x) = \frac{x^2-4}{x-2}$

dengan domain $D_f = \{x|x \in R, x \neq 2\}$

Pada $x = 2$, nilai fungsi $f(2) = \frac{0}{0}$ (tidak tentu)

Carilah nilai-nilai $f(x)$ untuk x mendekati 2 dengan mengisi tabel berikut.



x	1,90	1,99	1,999	1,9999	....	2	2,001	2,001	2,01	2,1
$f(x) = \frac{x^2-4}{x-2}$	....	3,99	....	3,9999	....	....	....	....	....	4,1

Dari tabel di atas dapat disimpulkan bahwa untuk x mendekati 2 baik dari kiri maupun dari kanan, nilai fungsi tersebut makin mendekati ...., tetapi untuk $x = 2$ nilai $f(x)$ .... Dari sini dapat dikatakan bahwa limit $f(x)$ untuk x mendekati 2 sama dengan ..., dan ditulis dengan notasi

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{x^2-4}{x-2} = \ldots.$$

Pengertian limit yang seperti inilah yang disebut pengertian limit secara intuitif, yang secara umum dapat kita nyatakan sebagai berikut.

> Definisi limit secara intitif, bahwa $\lim_{x\to c} f(x) = L$ artinya bahwa bilamana x .... c, maka nilai $f(x)$ .... L.

PROBLEM 3

Calculating the approximation of the value of a function

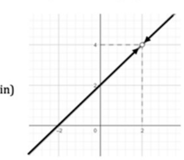
Below is one alternative for presenting limits with the help of function graphs.

Look at function $f(x) = \frac{x^2-4}{x-2}$

with domains $D_f = \{x|x \in R, x \neq 2\}$

At $x = 2$, the function value $f(2) = \frac{0}{0}$ (not certain)

Find values of $f(x)$ for x that are close to 2 by filling in the following table.



x	1,90	1,99	1,999	1,9999	....	2	2,001	2,001	2,01	2,1
$f(x) = \frac{x^2-4}{x-2}$	....	3,99	....	3,9999	....	....	....	....	....	4,1

From the table above, it can be concluded that as x approaches 2, both from the left and from the right, the value of the function gets closer to..., but for $x = 2$, the value of $f(x)$ gets closer to...

From here, it can be said that the limit of $f(x)$ for x approaching 2 is the same as ... and written in notation.

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{x^2-4}{x-2} = \ldots.$$

This kind of understanding of limits is called an intuitive understanding of limits, which we can generally state as follows:

> The intuitive definition of limit is that $\lim_{x\to c} f(x) = L$ means that if x .... c, then the value of $f(x)$ .... L.

(b)

Figure 3: (a) Student worksheet on the SCL model (b) Student worksheet on the PBL model

Although the application of the SCL model in mathematics learning has not been widely investigated, the application of the SCL model in the online learning environment in the COVID-19 situation in this study explains how the learning process occurs. The cognitive aspect filters students' thinking processes in observing cognitive behavior presented by students or teachers in online learning situations (Bandura, 1977). Cognitive engagement refers to the cognitive processes that enable students to absorb information. Model figures in this study, namely teachers, students, and exciting teaching materials, can arouse students' curiosity, can focus students' attention so that the learning process can flow, and students can concentrate (Kemp et al., 2019). In this research process, online presence is defined as the behavior of teachers with students in distance learning that intentionally involves, attends, or at least listens during online classes by utilizing the zoom meeting platform. Teacher behavior in explaining material concepts can contribute to student behavior to take action and be used as a problem-solving process (Tajudeen, Madarsha, Suryani, & Badariah, 2011). The research findings strengthen the research results conducted by Ghazali et al. (2021). His research findings explain the relationship of a social cognitive theory that consists of behavior, cognitive and situational factors described through social, cognitive, and teacher presence, as a case study framework on online and distance learning. In which it can support decision-making in education during the COVID-19 pandemic. Based on these results, the study concludes that applying the SCL model contributes to students' mathematical reasoning abilities and broadens the scope of the application of social cognitive theory in mathematics learning.

The following finding is related to students' mathematical reasoning abilities based on the MSE level. These results conclude that there are differences in students' mathematical reasoning abilities based on the MSE level. A significant difference occurred between the MRA of students who had a low MSE level and a high MSE level. Based on Table 1, it is known that students' MRA scores at the high MSE level (78.53) are higher than the MRA scores at the low MSE level (66.05). This result aligns with the research findings (Schöber et al., 2018; Yelorda et al., 2021), which states that higher academic achievement can be expected from students with high self-efficacy than students with low self-efficacy levels. Self-efficacy is based on social cognitive theory (Bandura, 1997), Individuals are viewed as proactive agents in controlling their cognitions, motives, behaviors, and emotions, according to this theory (Mayer, 2002). So self-efficacy becomes a factor in controlling perceived behavior (Patricia, 2020).

Meanwhile, the comparison between MRA for low MSE level students and moderate MSE level students has no difference. Likewise, with the MRA for students at the moderate MSE level and students at the high MSE level. This finding is slightly different from the research results (Ma, 2021; Schöber et al., 2018), where students with a high MSE level outperform those with a low MSE level. These findings can be used as a basis for further research studies; namely, online-based learning can accommodate students with low MSE levels to compete with students with high MSE levels.

Finally, the findings explain no interaction between model variables and MSE on MRA. From the fact that there is no such interaction, it can be concluded that the differences between the MSE levels (low, moderate, and high) for each learning model are the same. These characteristics are, of course, the same as the total mean characteristics at the MSE level. As seen from the total mean in Table 1, the MRA of students at the high MSE level is higher than the MRA of students at the low MSE level. Because there is no interaction, the same applies to the students who were given the SCL model and the PBL model. This means that in the SCL model, the MRA of students at the high MSE level is also higher than students at the low MSE level.

Similarly, it was concluded that in the PBL model, students at the high MSE level were also higher than students at the low MSE level. Furthermore, another implication of the absence of interaction is that the characteristics of the different learning models will be the same at each MSE level and will also be the same as the characteristics of the total mean. This means that, in general, the SCL model is better than the PBL model; if it is reviewed by students at low MSE level only, then the conclusion will apply that low MSE level students who receive the SCL model have a higher MRA than low MSE level students who receive the PBL model. Likewise, if viewed from students at the moderate MSE level and at the high MSE level. The findings of this study are in line with the results of research conducted by Jannah et al. (2019), which concluded that there is no interaction between the model and the classification of self-efficacy on understanding mathematical concepts, and there are differences in understanding of mathematical concepts based on the classification of self-efficacy. Likewise, Fajri et al. (2016) research show no interaction between the model and self-efficacy towards increasing spatial ability. A similar study conducted by Chotima et al. (2019) explained no interaction between the model and self-efficacy in solving mathematical problems. Differences in self-efficacy influence mathematical problem-solving. Based on the research conducted by several researchers above and associated with the findings of this study, it can be concluded that the research findings are in line with the findings of this study.

CONCLUSIONS

The results showed that students who received learning using the SCL model had a higher MRA than students who received learning using the PBL model. Furthermore, students with a high MSE level have higher mathematical reasoning abilities than those with a low MSE level. Meanwhile, the mathematical reasoning ability between students with high MSE level and moderate MSE level and students with moderate MSE level with low MSE level are not significantly different. The results of this study have limitations; namely, the research subjects are only 11 high school students, and these findings still need to be reviewed so that they can be generalized to lower or higher school levels. This study only focuses on mathematical reasoning abilities, so studies on other mathematical abilities of students need to be explored. As a suggestion for further research, the application of social cognitive learning models and observations of mathematical self-efficacy aspects in online learning situations can be used as alternative learning models in responding to changes in the learning environment today.

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Appendix. Comparison of learning activities in the two models.

Table 1. Learning Stages, Activity Descriptions, and Aspects of Capabilities expected in the SCL Model

Stage	Activity Description	Emerging aspect
	Initial activity	
Phase 1. Attention	1. The teacher conveys the learning model that will be used. 2. The teacher conveys the learning objectives. 3. The teacher gives appreciation in the form of limit terms that are commonly found in everyday life and asks students to give their opinions. Do you often hear/see the sentences "He almost fell", "His credit card has a limit" and "30 km/hour speed limit".	
	Core activities	
Phase 2. Retention	1. The teacher associates these everyday problems with the mathematical concepts that will be studied, namely the limits of algebraic functions. 2. The teacher displays an illustration of the value of a function at a point. 3. The teacher asks students to look at the ACTIVITIES contained in the Worksheet. 4. The teacher guides students by giving questions that can arouse students' reasoning in solving problems on student worksheets. 5. The teacher repeats the meaning of limits for each problem that has been solved by students.	*Memorized reasoning* *Algoritmic Reasoning*
Phase 3. Production	1. The teacher asks students individually or in groups to complete the exercises contained in the Worksheet. 2. The teacher asks one of the students to explain the answers they have found, 3. The teacher asks other students to pay attention and provide feedback.	Creative reasoning
Phase 4. Motivation	1. The teacher gives praise for each student's response 2. The teacher provides reinforcement of students' answers that are still lacking.	*Memorized reasoning* *Algoritmic Reasoning*

Closing Activities		
	1. The teacher directs students to draw conclusions about the activities that have taken place (attention and retention). 2. The teacher gives homework assignments a few questions regarding the material that has been studied (production). 3. The teacher ends the learning activity by delivering material to be discussed at the next meeting and giving a message to repeat the concepts that have been learned and always learn.	

Table 2. Learning Stages, Activity Descriptions, and Aspects of Capabilities expected in the PBL Model.

Stage	Activity Description	Emerging aspect
	Initial activity	
Phase 1. *Problem Presentation*	1. The teacher greets and opens the lesson. 2. The teacher conveys the learning model that will be used. 3. The teacher conveys the learning objectives. 4. The teacher gives appreciation in the form of limit terms that are commonly found in everyday life. 5. Do you often hear/see the sentences "He almost fell", "His credit card has a limit" and "30 km/hour speed limit". 6. Students are invited to examine the problem given and asked to express their opinion.	
	Core activities	
Phase 2. *Problem investigation*	1. The teacher associates these everyday problems with the mathematical concepts that will be studied, namely the limits of algebraic functions. 2. The teacher asks students to look at the PROBLEMS in the Worksheet. 3. The teacher guides students by giving questions that can arouse students' reasoning in solving problems on Worksheet.	*Memorized reasoning* *Algoritmic Reasoning*
Phase 3. *Solution problem*	1. The teacher asks students individually or in groups to complete the EXERCISES contained in the Worksheet.	Creative reasoning

	2. The teacher asks students to explain the answers they have found. 3. The teacher asks other students to pay attention and provide feedback.	
Phase 4. *Evaluation process*	1. The teacher helps students reflect on the process and results of their investigation 2. The teacher provides reinforcement of students' answers that are still lacking.	*Memorized reasoning* *Algoritmic Reasoning*
Closing Activities		
	1. The teacher directs students to draw conclusions about the activities that have taken place. 2. The teacher gives homework assignments a few questions regarding the material that has been studied. 3. The teacher ends the learning activity by delivering material to be discussed at the next meeting and giving a message to repeat the concepts that have been learned and always learn.	

Designing Model of Mathematics Instruction Based on Computational Thinking and Mathematical Thinking for Elementary School Student

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Abstract: In this new era, computational thinking is becoming increasingly intriguing for in-depth study. The 2022 PISA framework illustrates that computational thinking can play a significant role in solving real-world mathematical problems, both in formulating problems and in mathematical reasoning processes. Many countries have integrated computational thinking into their curricula, starting as early as elementary education. The 2022 PISA framework marks the first time that substantial attention has been given to the intersection between computational thinking and mathematical thinking. Consequently, research was conducted to analyze the validation process, practicality, and effectiveness of using integrated computational thinking and mathematical practical design thinking to enhance students' computational thinking skills. The research results demonstrate that the integrated computational thinking and mathematical thinking learning design are highly suitable for implementation. According to expert judgment, its validity rate is 95.5%, with a practicality score of 93.75%. The instructional design applied in this study has proven to be effective in enhancing the computational thinking skills of elementary school students. It's important to note that this research is limited to elementary school students, and further studies are needed to explore this topic at higher education levels.

Keywords: learning mathematics, computational thinking, mathematical thinking, elementary school

INTRODUCTION

One of the skills needed in the global era is mathematical literacy (Habibi & Suparman, 2020; OECD, 2018). Even learning mathematics starting from elementary school requires mathematical literacy skills (Lange, 2003), namely the ability to analyze, reason, convey ideas, and solve problems in various situations. Mathematical literacy is an important ability to support students' mathematical abilities (Lengnink, 2005; Yore et al., 2007). However, in reality, students' mathematical literacy in Indonesia is still low (Fatwa et al., 2019). The learning of elementary school (SD) mathematics applied in Indonesia has adapted the achievement of learning objectives in Indonesia, namely emphasizing modern pedagogic dimensions, so that the learning model used utilizes a scientific approach. This is also shown by the curriculum that has been prepared with attention to aspects of developing mathematical literacy, namely formulating, using, and interpreting mathematics in various contexts of everyday life (Afriansyah, 2016; Buyung, 2017). In order to prevent pupils from studying and interpreting mathematical concepts themselves, problems from real life are solely used as a source of inspiration for inventions or concept formulation. (Jeheman et al., 2019; Warmi, 2019). This causes the form of mathematics to tend to be rigid and far from the origins of the mathematical concepts construction, so that learning mathematics is only limited to the transfer of knowledge (Risdiyanti & Prahmana, 2021). Therefore a learning model is needed that integrates mathematical literacy, especially in learning in elementary school. Students in the age range in elementary school are in the concrete operational stage where the child's cognitive aspects will develop rapidly, especially those related to logical reasoning. The hope is that since the beginning of elementary school, learning activity programs in schools can stimulate and facilitate this aspect of logical reasoning (Rita Eka et al., 2017). This is relevant to mathematical literacy skills as an important skill (AACTE & P21, 2013).

Solutions to overcome low mathematical literacy have been carried out, but the results have not been maximized because they have not thought about the importance of a synergistic approach between computational thinking and mathematical thinking. The series of thought processes is still a new thing in the realm of learning development in Indonesia. This is due, in part, to the teacher's mindset which still leads to the development of learning materials that students must master. The implication is that learning tends to be associated with how students master the targeted material content, so that the target of achievement is limited to pursuing material achievements without regard to the competencies students acquire (Fajri et al., 2019). These unresolved problems show that teaching to cultivate mathematical thinking tends not to occur. Many teachers continue to deliver explicit mathematics knowledge using traditional approaches without connecting it to student life or daily activities (Risdiyanti & Prahmana, 2021). The first reason is that teachers do not appreciate the value of mathematical reasoning. Teachers cannot instruct something they do not comprehend, which is the second reason (Katagiri, 2004). Even though mathematics learning will be maximized if the teacher focuses on mathematical thinking and reasoning (Allen et al., 2020). Another study adds that teachers misunderstand computational thinking skills and lack knowledge of how to teach computational thinking skills in class (Sands et al., 2018). Teachers often neglect daily problem-based content to enhance computational thinking abilities, resulting in students' low computational thinking skills in math lessons (Munawarah et al., 2021).

Conceptually, solving long-term literacy problems is carried out by building a synergistic relationship between computational thinking and mathematical thinking, which is related to technology which is important in students' lives (OECD, 2018). This explanation is relevant to the PISA assessment which makes adjustments to the challenges of the times. It is in the PISA 2022 framework that, for the first time, more attention has been shown to the intersection between computational thinking and mathematical thinking, which gives rise to the same set of viewpoints, thought patterns, and mental models that pupils need to flourish in a world that is becoming more technical (OECD, 2019). Additionally, these advances give students a more accurate understanding of how mathematics is used in the real world and practiced in the professional sector, which better prepares students to pursue professions in related subjects.(Muhammad Zuhair, 2020). Therefore, this research will develop a mathematics learning design that integrates computational thinking and mathematical thinking in elementary schools and examine its effect on the computational thinking skills of elementary students.

LITERATURE REVIEW

Computational Thinking

Computational Thinking has become one of the most important abilities to be honed from an early age because in the information age, industrial era 4.0 or society 5.0. Humans live in the real world, and at the same time in the digital world surrounded by IoT (Internet of Things), Big Data, and Artificial Intelligence (ITB, 2020). One opinion even says that computational thinking is an ability worthy of being the "fifth C" in 21st Century Skills; the 4 C's include critical thinking, creativity, collaboration, and communication (Sung et al., 2017). Computational thinking is a critical component of problem-solving techniques (Al Farra et al., 2022). Computational thinking is important in learning in schools because it allows students to think in different ways, express themselves through various media, solve real-world problems, and analyze everyday problems from different perspectives (Bocconi et al., 2016). Computational thinking is a paradigm for changing patterns of access to knowledge (Ho et al., 2019), which includes high-level thinking processes involved in algorithmic thinking, creative thinking, solving problems, forming innovative solutions, and understanding human behavior based on the foundations of computer science (Barr et al., 2011; Kalelioglu et al., 2016).

Computational thinking, which describes fundamental computing ideas, functions as a strategy for problem-solving, system design, and understanding human behavior (Atmatzidou & Demetriadis, 2016; Maharani et al., 2020; Sneider et al., 2014; Zaharin et al., 2018). According to a study (Kalelioglu et al., 2016), computational thinking is a process for addressing problems that includes the following steps: problem identification, data collection, representation, and analysis, solution invention, selection, and planning, solution implementation, and solution evaluation. These stages of action are types of cognitive actions (Wing, 2008).

In this study, the components of computational thinking that will be used will adapt to learning in elementary school. The steps in this learning model are a combination of the opinions of two researchers and refer to the 2022 PISA (Ho et al., 2019; Shute et al., 2017) so that computational thinking aspects are obtained in this study, namely: abstraction, decomposition, algorithms, debugging, iteration, and generalization. The explanation of the aspects contained in computational thinking includes:

1) Decomposition: breaking the problem down into smaller bits and getting to the core of a problem, in order to address the problem one at a time and discover each component of where the problem originated.
2) Abstraction: Identify the broad principles that give rise to these regularities, trends, and patterns. Usually by examining the overall traits as well as modeling a remedy.
3) Algorithm: creating detailed instructions for resolving the same issue so that others may utilize the knowledge to resolve an identical issue.
4) Debugging: detect and identify errors, then fix errors, when the solution doesn't work.
5) Iterations: repeating the design process to refine the solution, until the ideal result is achieved.
6) Generalizations: formulate solutions in general so that they can be applied to different problems (Selby, 2013)

These aspects are used in complex tasks, when choosing the right representation of a problem, and when modeling the relevant aspects of a problem to make the problem easier to trace.

Mathematical Thinking

An important aim of education is to develop one's capacity for mathematical thought and the application of mathematics to problem-solving. In this situation, mathematical reasoning will assist science, technology, economic life, and economic development (R. Bybee et al., 2009). Mathematical thinking means a characteristic of thinking when carrying out activities or thinking mathematically related to content and solving mathematics (Vittayaboon et al., 2018).

Mathematical thinking is the knowledge and skills needed to solve every problem (Katagiri, 2004). Katagiri (2004) also added that mathematical thinking is an understanding of the importance of using mathematical knowledge and skills, learning how to learn independently, and achieving the abilities needed for independent learning, so mathematics is a complex activity. Mathematical thinking is very important to equip students with mathematical skills (Stacey, 2006).

Mathematical thinking can be demonstrated through two processes, namely: (1) Specialization and Generalization; and (2) Conjecturing and Convincing (Stacey, 2006). These two things are used by students to think and solve mathematical problems. Specialization ability is intended as the ability to solve various problems by looking at examples. Generalization ability is the ability to identify an issue, phenomenon, problem, or study based on patterns and connections. The conjecturing is intended as a form of ability to predict relationships and results. Meanwhile, the ability to convince is the ability of students to find and communicate the rationale for an issue, phenomenon, object, or problem that is considered true (Fajri et al., 2019). Students do mathematical thinking to achieve new knowledge or concepts (Sa et al., 2023). Mathematical thinking is a complex activity so it is important to equip students with these abilities since elementary education.

Stacey (2006) states in more detail that mathematical thinking includes mathematical knowledge, reasoning skills, the ability to use strategies, beliefs and attitudes, personal skills, and skills to communicate solutions. During math activities, mathematical thinking is applied and is strongly tied to content, arithmetic techniques, and mathematics. In order to be more exact, many techniques are utilised when arithmetic or mathematics is used to carry out mathematical operations and to provide various kinds of mathematical content. The substance of mathematical thinking influences the success of students' school mathematics learning.

Synergistic Relationship between Computational Thinking and Mathematical Thinking

The synergistic relationship between computational thinking and mathematical thinking was formed because of the shift in the PISA framework. The 2022 PISA framework when compared to the 2003 PISA and 2012 PISA framework, experienced several shifts in the assessment of mathematical literacy, according to what was conveyed by the PISA Governing Board (OECD, 2018, 2019). Of course, by not abandoning the basic ideas of mathematical literacy that were developed previously, this definition provides information about the meaning of mathematical literacy which indicates three main things. First, mathematical literacy refers to an individual's ability to reason mathematically when formulating, using, and interpreting mathematics in the real world. Second, the ability to describe, explain, and predict real-world phenomena using mathematical concepts, procedures, facts, and tools. Third, mathematical literacy helps individuals understand the active role of mathematics in the real world.

The process of interpretation of the 2022 PISA definition referred to (OECD, 2018) is applying and evaluating mathematical results by using computational thinking and mathematical thinking to make predictions, provide evidence for arguments, test, and compare proposed solutions. The trend is that the need to adapt to a rapidly changing world is driven by technology, where humans are more creative and involved, and make judgments for themselves and the communities in which they live (OECD, 2019). This explains why there is a recognition of the junction between computational thinking and mathematical thinking for the first time in the PISA 2022 framework (OECD, 2019).

The long-term trajectory of mathematical literacy includes a synergistic and reciprocal relationship between computational thinking and mathematical thinking (OECD, 2018). Computational thinking and mathematical thinking work well together to help students develop their conceptual understanding of the field of mathematics as well as their concepts and computational thinking abilities. This helps students develop a more realistic understanding of how mathematics is used and applied in the real world. Ultimately, this will improve students' readiness to pursue professions in relevant industries (Muhammad Zuhair, 2020). The process of computational thinking intersects with the ability to solve mathematical problems, where both have the same significant steps (Neneng Aminah, 2022). Computational thinking that is taught well is proven to foster a critical attitude of students (Surahman et al., 2020; Yadav et al., 2014, 2017), so that students will also be accustomed to thinking creatively and practically by looking for the most effective way to solve a problem (Ashish Aggarwal,Gardner-mccune & Touretzky, 2017; García-Peñalvo, F. J., & Mendes, 2018).

METHOD

The test subjects in this study consisted of 3 teachers and 82 fifth grade students at a private school favorite in Yogyakarta. The subjects were described as subjects for limited trials of 17 students, 31 students for the control class, and 34 students for the experimental class. Eight boys and nine girls made up the research participants in the limited trial class; sixteen boys and fifteen girls made up the control class; and twenty-one boys and thirteen girls made up the experimental class. The study participants, who ranged in age from nine to ten, had a variety of traits and varied origins in terms of culture and economic. The research sample used a purposive sampling technique by considering that 5th-grade elementary school students already can read, write, and count, which is considered sufficient to implement the development of mathematical literacy learning (Simarmata et al., 2020). The determination of the test subjects also took into account the heterogeneous backgrounds of the schools used so that the test was more comprehensive and representative of the research subjects. This study uses part of the steps in the ADDIE development model, with the steps illustrated in Figure 1 below.

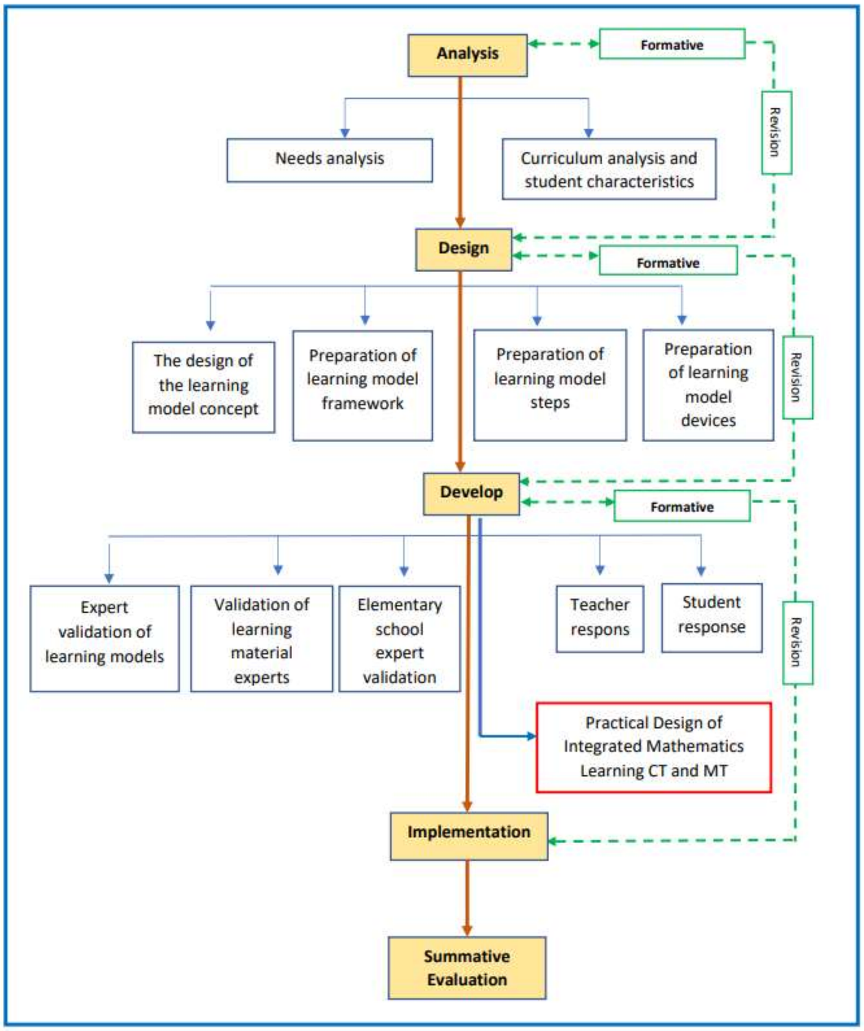


Figure 1: Adaptation of the ADDIE model development procedure (Branch, 2010)

The processes of creating this learning model start with the analysis stage, which includes analyzing the requirement to see existing circumstances, analyzing the appropriate curriculum, and analyzing student characteristics (Istikomah et al., 2020). In the initial identification process to obtain data on student needs, we observe learning in schools that have integrated aspects of computational thinking. Through interviews and field observations at schools, we generate data to identify learning models that are following the targets and ideas for practical learning models that are suitable for development. Interviews were conducted with the Principal and Teachers to explore information related to mathematics learning that has been implemented so far. The initial observation was carried out to see and record the integrated learning process of computational thinking that has been applied at school, so that the strengths and weaknesses were found as material for the initial analysis in this study. After conducting an analysis related to needs and performance, the researcher reflects on the results of the analysis which is part of the formative evaluation stage, before continuing the design process.

The next step is called the design stage, namely designing the concept of a practical learning model to be developed, compiling a learning model framework, compiling the learning model steps, and preparing learning tools. After carrying out the design related to the learning model concept, the researcher reflects on the results of the analysis which is part of the formative evaluation stage, before continuing the development process. The development stage, which is the process of turning the design into a product, comes after creating the concept model. Products that have been made will go through the next step, which is to implement test the products offered. For this reason, it is necessary to test its feasibility through the evaluation experts. Aside from that, rated practicality through analysis results, teacher responses, and student responses.

At the implementation stage, the teacher's teaching observation data was obtained in the form of written notes, recordings of interviews with researchers, and documentation in the form of photos and videos during learning. The research tool is a computational thinking test for students that has been validated through prior research (Angeli et al., 2016) and has been updated for the present environment. The interview guide is used to investigate the process, which calls for clarity from the outcomes of observations, and all instruments have been validated and approved as valid. The computational thinking skills test is given to students in the form of a test that integrates computational thinking. This test measures computational thinking with the aspects in it. This test is a high order thinking skills question that is suitable for grade 5 elementary school students and consists of a pretest and postest. Currently, not many have been found that can contain measurements of all aspects of computational thinking skills so that it becomes a note for further research to develop tests that measure computational thinking skills in other grades and materials in elementary school.

The evaluation phase includes formative and summative evaluations. According to Branson et al. (1975), the formative evaluation stage manifests itself in the form of expert validation and applicability at each point in this study. The activity concludes with the summative evaluation phase.

RESULTS

Preliminary Analysis. The development of integrated computational thinking learning design is limited to computational thinking oriented. The study begins with the **analysis stage**. By examining the circumstances that make learning primary mathematics necessary, examining the relevant curriculum, namely the 2013 curriculum, and examining student characteristics, the analysis's findings are presented as a needs analysis. We use the results of the analysis as a reference in preparing the design of this learning practice.

Needs Analysis. Based on the observations and interviews conducted, it can be concluded that there is a need for activities where students or teachers explain directly and in real terms the completion of a process using concrete media. In this case, it is needed because students need to see directly how the process, stages and sequences of solving a problem are real. It was also found that there was a presentation of search results by students from a given case, but there was no intense discussion between students, so only the collection of solutions from a given case was found. Another analysis obtained information that learning has not shown how the scheme has a clear causal relationship and mutual influence, and there are still many students who cannot understand how the process or sequence of problem-solving from start to finish is structured.

Analysis curriculum. In the analysis of the curriculum, information was obtained that in learning, namely still using the K-13 national curriculum as the main guideline for mathematics learning activities in class. The 2013 curriculum is the curriculum that applies in the Indonesian education system and aims to build students who are ready to face the future.

Analysis of student characteristics. Based on the analysis of student characteristics, information is obtained that 1) students are heterogeneous both in gender and background, 2) most students can operate computers. 3) students consist of various levels of cognitive abilities. 4) There are students with special needs who need teacher assistance.

The design stage is carried out to design a mathematics learning design integrated with computational thinking and mathematical thinking. The focus of model development is on the model of instruction approach. The model of instruction expresses an emphasis on the construction and application of a physical conceptual model of phenomena as an important aspect of learning and its implementation (Jackson et al., 2005). The model of instruction produces students who are intelligently involved in classroom discourse and scientific debate (Jackson et al., 2005). Modeling Instruction is an approach to inquiry-based learning. There are several interesting student activities in each stage of the Model of Instruction, namely small group discussions, class discussions (board meetings), and the use of whiteboards (whiteboarding) as a means of communicating the results of the learning process in the form of writing/pictures. This approach has the potential and flexibility to teach students 21st-century skills (Wicaksono, 2019). The design principles of the model of instruction are that one or more 21st-century skills must have explicit learning outcomes for the learning model (R. W. Bybee, 2009). The learning sequence of the model of instruction includes focused instruction, guided instruction, collaboratory learning, and independent learning

(Fisher & Frey, 2008) (Kylsyit, 2019). The following is a mapping of the model of instruction learning sequence and aspects of the computational thinking and mathematical thinking approaches.

Stages of problem-solving in CT	Aspect CT and MT	Stages of the Model of Instruction			
		Core Learning			output
		Focused Instruction	*Guided Instructions*	*Collaborative Instruction*	*Independent Learning*
Identification of problems	**Specialization**	1. Deliver learning objectives and learning scenarios 2. Split groups	Conduct discussions and ask questions about the material		
	Decomposition		Divide the problem into smaller parts		
	Abstraction		1. Analyze activity tasks and plan problem solving 2. Collect, organize, and analyze problems and interpret information from various sources		1. Analyze activity tasks and plan problem solving 2. Collect, organize, and analyze problems and interpret information from various sources
Solution creation, selection, planning	**Algorithm**			1. Exchange ideas to discuss finding solutions in the context of solving activity tasks 2. Make a series of sequential steps to solve a problem	Make a series of sequential steps to solve a problem
Assess solutions and achieve improvements	**Debugging**			Identify errors, then fix errors when solutions don't work as intended	Identify errors, then fix errors when solutions don't work as intended
	Conjecturing			Predict relationships and outcomes	
	Convincing			Finding and communicating the rationale for something that is considered true	
	Iteration			If there is an error, repeat the design process to refine the solution, until the ideal result is achieved	If there is an error, repeat the design process to refine the solution, until the ideal result is achieved
	Generalization			Presenting the results of completing the activity task	Formulate solutions so that they can be applied to different activity tasks

Table 1: The stages of solving CT problems adjust the model of instruction

The syntax in learning design refers to the overall flow or sequence of learning activities. Joyce et al. (Joyce et al., 2015) explained that "The syntax or phasing of the model describes the model in action". This shows that scenarios will be very useful for carrying out the learning process successively because each learning activity consists of several stages that become one in the learning process. Next, design this learning combines procedures for computational thinking and mathematical thinking so that the intersection is found, which then becomes a structured procedural framework that describes learning objectives and management.

Results of the development stage

Stage develop in this study were developed through several validation processes assessed by learning model experts, learning material experts, and elementary school experts, so that a revision process was carried out based on expert input. This stage applies the formative evaluation stage, namely the expert validation and practicality stages. Specific validation results can be seen in Table 2 below.

Aspect	**Validators**			**Information**
	1	**2**	**3**	
Completeness of Learning Design Structure	4	4	3	0 - 0.80 not feasible 0.81 - 1.60 less feasible 1.61 - 2.40 quite feasible 2.41 - 3.20 feasible 3.21 - 4.00 very feasible
Appropriateness of Supporting Theory	3	4	3,5	
Learning Design Focus	3,5	3,5	4	
Learning Design Syntactic	4	4	4	
Clarity of Learning Design Reaction Principles	4	4	4	
Clarity of Social Systems Learning Model	4	4	4	
Clarity of Instructional Impact and Accompanying Impact	3,5	3,5	4	
Clarity of Learning Design Tools	3,8	3,8	4	
Clarity of Learning Design Application Context	4	4	4	
Total	**33,8**	**34,8**	**34.5**	
Score average	**3.82 / very decent**			
Identified as	**95.5%/ very feasible**			

Table 2 : Results of validation by experts

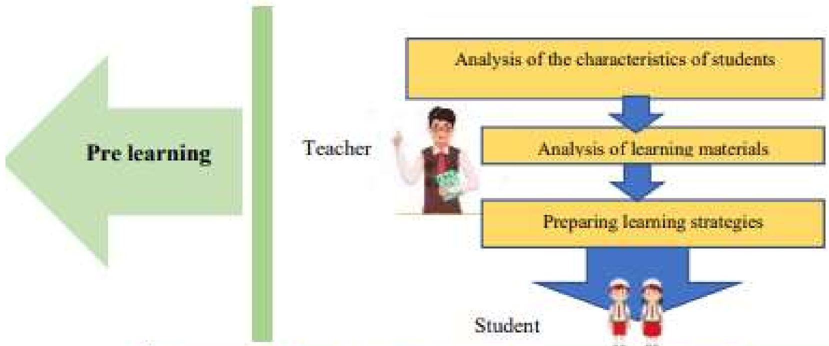
According to the experts' evaluation, the learning design falls into the 95.5% category. This claim demonstrates that the quality of the created learning design is regarded as valid and that the learning model can be further expanded through updates made in accordance with recommendations and advice from experts. Repairs are made in accordance with professional recommendations.

Expert advice. This information was gathered from specialists who were thought to be qualified to offer opinions and recommendations on the teaching practice model that was being created. suggestions or advice during the teaching expert validation phase to improve the learning design. Suggestions given by design and learning material experts can be seen in Table 3.

No	Aspect	Suggestion
1.	Learning stages	It is better to give an example of integrated learning of computational thinking and mathematical thinking in one learning material
2.	Details of activities	Arranged more specifically at each stage of the introduction, core activities, and closing
3.	Technology utilization	On worksheets, students are given a project that is directly related to computer operations in demonstrating broader CT integration
4	Learning design support	Revision and editing of learning design support devices so that they are more suitable for use in learning

Table 3 : Expert advice

The following figure illustrates how the researcher enhanced the learning design under the guidance of the learning expert validator (shown in Figure 2).



Model of Instruction	Teaching Activity	steps Learning model	Integration aspects of CT and MT	Tool
Focused Instruction	Submission of material	Deliver learning objectives and learning scenarios	Specialization	Teacher information and instructions
		Split groups		
		Conduct discussions and ask and answer questions regarding the material and solving story problems		
Guided Instructions	Analyze Activities Task and identify problems	Divide the problem into smaller parts	Decomposition	•*E-module* •*Concrete Media* •*Mathematics Activity Task Worksheet (MATW)*
		Analyze activity task and plan effective problem solving	Abstraction	
		Collect, organize, and analyze problems and interpret information from various sources		
Collaborative Learning	Guide the solution discovery process	Exchange ideas to discuss finding solutions in the context of solving activity tasks	At this stage apply aspects Algorithm debugging *Conjecturing* *Convincing* Iteration on an ongoing basis with the sequence of aspects that adjust to the needs	
		Make a series of sequential steps to solve a problem		
		Predict relationships and outcomes		
		Identify errors, then fix errors when solutions don't work as intended		
		find and communicate the rationale for something that is considered true		
		If there is an error, repeat the design process to refine the solution, until the ideal result is achieved		
	Review the completion of the activity task	Presenting the results of completing the activity task	At this stage the aspects of generalization, convincing, and debugging apply, continuous iteration according to needs	
		find and communicate the rationale for something that is considered true		
		Identify errors, then fix errors when solutions don't work as intended		
Independent Learning	Analyze other activity tasks as implementation of independent solutions	Formulate solutions so that they can be applied to different activity tasks	At this stage, aspects of decomposition, abstraction, algorithms, debugging, iteration, and generalization apply continuously according to needs.	Evaluation Tool (Test)

Mathematical Creative Self Efficacy and Computational Thinking Skills

Figure 2: Computational Thinking and Mathematical Thinking Integrated Learning Design Framework

The practicality of teaching practice models. Questionnaires were used in this study to state that the integrated learning design of computational thinking and mathematical thinking is practical to use. Data were taken from three teachers as practitioners. Apart from that, to see the practicality of our learning design, we also involved 17 students through a questionnaire. After designing, mathematics learning practices integrated with computational thinking and mathematical thinking are developed to obtain valid and practical results. Furthermore, the learning design is implemented to see its effectiveness, following the expectations of researchers. Based on the teacher's assessment, a value of 3.75, or 93.75%, is obtained, as shown in table 4, so that it can be declared feasible for the learning design.

Aspect	Subject Score			Information
	1	2	3	
The practicality of the instructional model guide book	3.75	3	4	0 - 0.80 is not feasible 0.81 - 1.60 is not feasible 1.61 - 2.40 is quite decent 2.41 - 3.20 decent 3.21 - 4.00 very decent
Lesson plan practicality	3.75	4	4	
The practicality of the learning model syntax	3.75	3.75	3.75	
Math Activities Task Worksheet practicality	4	4	3.75	
The practicality of the e-module	3.75	3.75	3.75	
Total	19	18.5	19.25	
Average	3,8	3,7	3.85	
score average	3.75/very practical			
Identified as	93.75/very practical			

Table 4: Practicality Based on Teacher Response

While practicality according to students, obtained an average total value of 3.79 with a percentage of 94.5%, which is included in the very practical category.

Results of Implementation Stage

The implementation stage is carried out after the design of this learning design has obtained valid and practical final results. Then the learning design was implemented in 5^{th} grade elementary school in February - March 2023.

Figure 3: Process Computational Thinking and Mathematical Thinking Integrated

Students in the control and experimental classes took the initial and final learning tests. The intervention in the experimental class was carried out five times, with each lesson taught at each meeting. Students receive different treatment according to the project that has been prepared at each meeting.

Meeting	Learning objectives	Integration of CT and MT
Meeting 1	Describe the volume of geometric shapes of cuboids and cubes	At this stage, CT and MT aspects have been involved with the integration of the model of instruction, especially on the aspects of focused instruction and guided instruction
	Determine the volume of cubical and cuboidal shapes	
Meeting 2	make nets of cubes and blocks according to their creativity.	At this stage, CT and MT aspects have been involved with the integration of the model of instruction, especially in the aspects of focused instruction, guided instruction, and collaborative learning
	solve problems related to nets of cubes and blocks.	
Meeting 3	solve problems about cubes in everyday life.	
	solve problems about blocks in everyday life.	
Meeting 4	solve a problem about cubes in everyday life.	
	solve a problem about blocks in everyday life.	
Meeting 5	learning evaluation related to blocks and cubes	At this stage, the evaluation given involves CT and MT aspects with the integration of the model of instruction, especially on the independent learning aspect

Table 5: Details of activities in each meeting

The project takes measurement and geometry content, with material on the volume of blocks and cubes and their nets, following the material that grade 5 students should acquire in that semester. Examples of tasks completed by students to explore their computational thinking skills are shown in figure 4 below.

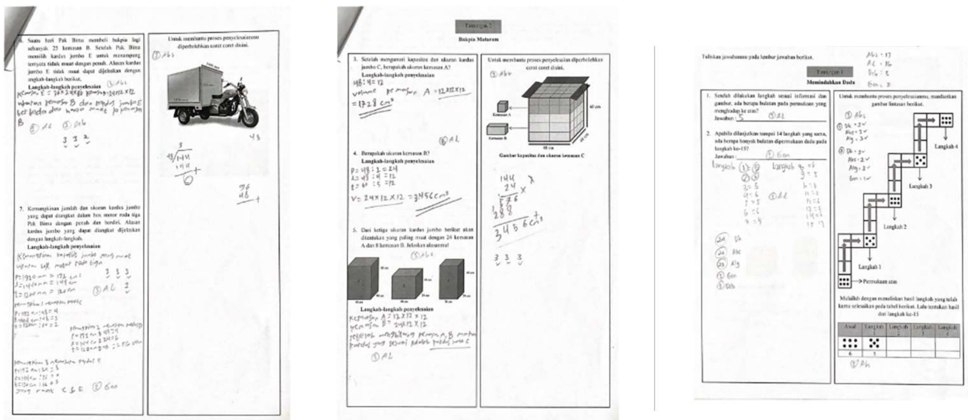


Figure 4: Examples of students' mathematical task completion results

After the intervention was carried out, the data was obtained, which was then processed. Analysis of the difference test data is carried out after the prerequisite tests, namely the normality test and homogeneity test. Based on the results of the analysis, it can be concluded that the data are normally distributed and homogeneous, then it can be continued with the parametric statistical method of the independent t-test.

Once it is established that the data are homogenous and normally distributed, the independent t-test is used to determine whether there is a difference between the means of the two groups. Following are the results of testing the effectiveness of computational thinking skills using independent samples t-test :

	Levene's test for equality of variances		**t-test for equality of means**					**95% confidence interval of the difference**	
	F	***Sig.***	***t***	***df***	***Sig (2-tailed)***	***Mean Differences***	***std. Error Difference***	***Lower***	***Upper***
Equal variances assumed	2,289	.135	9,367	63	.000	18.28264	1.95185	14.3822	22.18310
Equal variances not assumed			9,450	62,417	.000	18.28264	1.93463	14.4159	22.14939

Table 6: Test the effectiveness of computational thinking skills

The test was given to the experimental class to see the effectiveness of the learning design on computational thinking skills. The results of the difference test to determine differences in computational thinking skills that received direct learning and integrated learning of computational thinking and mathematical thinking obtained a significance of p value = 0.135 (p 0.05). Based on p=0.000, $p < 0.05$ we can be concluded that the skills of control and experimental class students have significant differences.

Results of Evaluation Stage

The numerous processes stated above, starting with the analysis stage and ending with the execution stage, have all been merged to carry out the evaluation stage. The evaluation conducted at the end of this activity is a summative evaluation.

The results of the research show that evaluations have been carried out by learning design experts, satisfaction from the practical use of computational thinking and mathematical thinking integrated learning models, and increased student competency mastery after implementation. Based on these results, an integrated learning model of computational thinking and mathematical thinking has been successfully designed for elementary school students.

DISCUSSION AND CONCLUSION

The integrated learning design of computational thinking and mathematical thinking can be used as an alternative to learning mathematics for elementary students in class. As stated in the PISA 2022 framework (OECD, 2019), more attention should be paid to the intersection between computational thinking and mathematical thinking. These advancements give students a more accurate understanding of how mathematics is used and practiced in the real world, thereby improving their readiness to pursue jobs in related subjects (Muhammad Zuhair, 2020). The integrated learning design of computational thinking and mathematical thinking in this study has a valid category, as indicated by the percentage score of 94.5% of SD learning design experts. This learning design is also stated to be practical based on the percentage of teacher responsesof 93.75% and student responses of 94.5%. Therefore, the integrated learning design of computational thinking and mathematical thinking can be declared valid and practical, so it is suitable for use as an alternative to elementary mathematics learning models. It is now commonly acknowledged that incorporating computational thinking into many topic areas of K–12 education will enhance student learning (Güven & Gulbahar, 2020). The integration of computational thinking into learning has been carried out in several countries, such as the UK, the European Union, America, Malaysia, and Thailand (Bocconi et al., 2016; Chongo et al., 2021; Threekunprapa & Yasri, 2020), also the Indonesian government in 2022 has made it a policy that computational thinking is integrated into several subjects, including mathematics, starting in elementary school (Kemendikbud, 2020). The things that need to be considered are that if the teacher wants to encourage mathematical thinking in students, then they need to be involved in mathematical thinking throughout the lesson (Stacey, 2006). Teachers should also allow students to gain more insight into mathematics. This is in line with the opinion (Isoda & Katagiri, 2012) that mathematical thinking can improve understanding, skills, and independent learning. As stated (Henderson et al., 2002) that mathematical thinking is a mathematical technique, concept, and method that is used directly or indirectly in the process of solving problems, so mathematical thinking is needed in learning mathematics.

It is appropriate that the reciprocal relationship between mathematical and computational thinking in math learning is a long-term trajectory in mathematical literacy, which in turn generates a similar set of perspectives, thought processes, and mental models that students need to succeed in a world that is becoming more technological (OECD, 2018). Of course, the teacher still has to know an effective way of motivating learners to represent and relate prior knowledge and understanding and effectively use them in depth and breadth during problem-solving (Ashish Aggarwal,Gardner-mccune & Touretzky, 2017).

Based on data analysis, the integrated learning design of computational thinking and mathematical thinking shows significant effectiveness in students' computational thinking skills. A study has been conducted by researchers on computational thinking skills, which states that computational thinking must be included as a mandatory skill in the 21st century (ITB, 2020; Wing, 2008). Computational thinking is a thinking process for formulating problems and strategies in determining effective, efficient, optimal solutions to be carried out by the information processing agent (solution) (Nurohman et al., 2022). Early instruction in computational thinking should equip kids with the abilities to: (i) prepare for the workforce and fill ICT job openings; and (ii) think creatively, express themselves through new media, and solve problems in the real world (OECD, 2019). Computational thinking also hones logical, mathematical, and mechanical knowledge, which is combined with modern knowledge regarding technology, digitalization, and computerization and even forms confident, open-minded, tolerant, and sensitive characters to the environment (Marifah et al., 2022).

Based on the research conducted on the integrated computational thinking and mathematical thinking learning design, it can be concluded that the design is suitable for use as an alternative model for learning mathematics for elementary school students in the classroom. This research is still limited to the elementary school level in grade 5. As a recommendation, this research can be extended to other mathematical contexts and content as well as to other classes both in elementary schools and schools above.

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Schema development in solving systems of linear equations using the triad framework

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Abstract: Solving systems of linear equations is a core concept in linear algebra and a wide variety of problems found in the sciences and engineering can be formulated as linear equations. This study sought to explore undergraduate students' development of the schema for solving systems of linear equations. The triad framework was used to describe the schema development in general and the system of linear equations was used as an example. A case study of an undergraduate class doing a linear algebra course in 2020 was considered in this study, where fifteen students participated in the study whereby they responded to a task with three question on solving a system of linear equations. The findings revealed that albeit minor manipulation errors, students were able to solve given systems of linear equations using the Cramer's rule and Gaussian elimination. However, students could not adequately go beyond the algorithmic computations to attain appropriate mathematical reasoning and establish underlying relations required to solve systems of linear equations, as no-one attained the trans-stage of conceptualization. It follows from this study that the identifications of the challenges that students encounter when solving systems of linear equations empowers course instructors on how to overcome the challenges.

Keywords: schema development; system of linear equations; triad theory; matrices; linear algebra

INTRODUCTION

Linear algebra is a branch of mathematics concerned with solving systems that are modelled with multiple linear functions. Modelling in linear algebra is essential in the study of systems of equations and is based on the description of the properties that characterize them and it is an activity that cuts across all disciplines (Barragan, Aya & Soro, 2023). The term systems indicates that the linear equations are not considered individually, but collectively. Many problems found in natural sciences and engineering can be formulated as multiple linear equations. Systems of linear equations (SLEs) play an important and motivating role in the subject of linear algebra. In the introductory stages of linear algebra, undergraduate students learn SLEs and how to solve them using each of the methods of Cramer, inverse, Gaussian elimination and Gaussian-Jordan elimination. However, Gauss-Jordan is an extension of Gaussian elimination, hence regarded as unitary. There is no guaranteed method which is regarded as the best in solving SLEs. Students are often left to determine for themselves the method which is efficient and easier. Efficiency in solving SLE entails determining the values of the unknowns in less time and complicated procedures (Gharib, Ali, Khan & Munir, 2015). For instance, the inverse and Cramer methods are not usable in cases where the coefficient matrix is non-square or the determinant is zero. In addition, the inverse and Cramer's methods cannot be used to evaluate solution beyond the unique solutions. By not relying on determinants, the Gaussian elimination method can be used to solve almost all types of SLEs.

Students' learning of linear algebra has not been without difficulties. Linear algebra is considered to be a difficult subject by students (Possani, Trigueros, Preciado & Lozano, 2009) and Kazunga and Bansilal (2017) note that the teaching and learning of undergraduate linear algebra is regarded as a daunting experience. This affirms that linear algebra is difficult for both instructors and students (Salgado & Trigeuros, 2015). Jupri and Gozali (2021) no doubt posit that prospective teachers encounter difficulties in mathematics concepts but these should be overcome. One way to overcome difficulties is to strengthen students' conceptual understanding and procedural fluency. In all the methods of solving SLEs mentioned above, memorizing algorithms without understanding the basic concepts of linear algebra is rife among students. Kazunga and Bansilal (2017) note that undergraduate students cope with procedural aspects of linear algebra but have problems of conceptual understanding of the same. Students' preferably procedural successes coming from repeating algorithms to some extent masks possible underlying conceptual gaps (de Villiers & Jugmohan, 2012).

The use of non-matrix methods is common in high schools mathematics but wanes as students transition to undergraduate mathematics. Notably, as the number of equations and unknowns increase, the foundational methods of substitution, elimination and graphical become cumbersome. Undergraduate students are introduced to matrices and matrix operations to develop procedures that are suitable for solving SLE of any size, that is, of order $m \times n$ where $m, n \in \mathbb{N}$ (Mandal, Cherukuru & Rani, 2021). Thus, the matrix method is useful and convenient to solve SLEs of many equations and unknowns effectively.

In Skemp's (1962) seminal work on schema, he defined it as an organized body of knowledge that students develop in the process of learning a particular concept. According to Skemp, a schema connects students' past learning to the current and future as follows: "more efficient current learning, preparation for future learning, and automatic revision of past learning" (p. 140). A schema is a mental construction that enable students to solve problems and this disposition plays a central role in formal learning and teaching (Kirschner & Hendrick, 2020). Students rely on developed schemas to organize knowledge and solve problems (Powell, 2011). If the schema is broad, it is more likely that students will recognize connections between strategies that have been taught and the implied problems that make use the same strategies (Fuchs, et al., 2006). According to Soderstrom and Bjork (2015), this forms part of knowledge transfer whose attainment is regarded as one of the most important goals of instruction. Success in solving problems creates full mental representations of the schema in students, which in turn facilitates the recall of information as needed to solve related problems (Skinner & Cuevas, 2023). Schemas are key determinants of the progress of students' learning of mathematics (Ndlovu & Brijlall, 2019) and having fully developed schemas would provide opportunities for students to make connections to mathematics concepts in the same and unfamiliar contexts.

Problem statement

Solving SLEs is a core concept in introductory linear algebra (Liu, 2015) and having a deep understanding of it assists in comprehending many applications and topics in linear algebra (Karunakaran & Higgins, 2021). The choice of the method to use in solving a SLE is dependent on the make-up of the system. As part of schema development in solving SLEs, undergraduate students ought to demonstrate knowledge to determine the appropriate method to use in the given circumstances. It is important to study how students learn solving SLEs since there are many subtleties involved in students' understanding of SLEs (Borji, Martínez-Planell & Trigueros, 2023). Hence, the purpose of this study was to explore the nature of students' development of the schema for solving SLEs. The objective of this study was the use of the Triad stages to assess undergraduate students' schema development in learning linear algebra, particularly in SLEs. By seeking to understand how students think when engaged in solving mathematics problems, course instructors might be able to improve the teaching and learning of solving SLEs by making it more meaningful so that the schema for the given concept is attained. Research on students' schema development of solving SLEs through the lens of the triad theory is scarce (Parraguez & Oktaç, 2010).

LITERATURE REVIEW AND THEORY

Students are initially introduced to matrix theory in linear algebra then to vector calculus in order to be able to solve SLEs of any order. Studies on students' understanding of matrices and solving SLEs mostly used the Action-Process-Object-Schema (APOS) theory. Kazunga and Bansilal (2017) explored the students' mental conceptions of matrix operations using the APOS theory. Students in that study managed to interiorise matrix addition, subtraction, multiplication and scalar multiplication concepts. But the same students did not develop the schema for multiplication of a row and column matrices. Students' lack of conceptual understanding of essential concepts in SLEs might negatively affect their achievement in linear algebra. Dis-fragmented knowledge and reliance on procedures is an indication of non-schematic learning according to Skemp (1962). To alleviate this, student-developed worksheets were used as a strategy to make sense of new concepts in linear algebra in the study by Arnawa, Yerizon, Nita and Putra (2019). The use of APOS-based approach improved students' achievements in learning matrices and vector spaces, particularly in SLEs. Modelling problems were also used in the worksheets, where modelling SLEs represents one of the applications of linear algebra to real life situations.

Without full schema development, students oftentimes have difficulties in understanding and integrating new mathematics knowledge. A study by Berger and Stewart (2020) used the concept of topology proofs to investigate how students' schema develop and the effect of interactions with peers and instructors on that development. They discussed the definition of schema according to Skemp (1962) and Dubinsky & McDonald (2001) and how these align to the Piaget and Garcia's (1989) triad framework. The content analysis of students' responses to a final examination revealed that the majority of students did not develop the full schema of topology proofs but operated in the initial stages by the end of the semester. Similarly, a study by Hannah et al. (2016) sought to develop students' conceptual understanding in linear algebra by combining the frameworks of Tall's (2004) three worlds and APOS theory to analyse students' resulting levels of cognition. After the analysis of students' tests and examination scripts and interview transcriptions, Hannah et al. found that students did not rely on rote to learn basic concepts in linear algebra. The students could explain precisely the concepts and were able to use the acquired knowledge to solve given problems. Moreover, a study by Stewart and Thomas (2010) also combined the APOS and Three worlds of mathematics thinking frameworks to describe all possibilities of understanding in linear algebra concepts of linear independence, basis and span. After analysing students' responses using this dual framework, Stewart and Thomas found that students in traditional courses were proficient in matrix manipulation as a process or symbolic conception but were inferior in establishing connections between matrix manipulations and the associated concepts. This normally happens when students regard linear algebra concepts as a collection of definitions and procedures to be learnt by rote (Martin et al., 2010). Kazunga and Bansilal (2020) assert that "mathematics instructors need to focus on their students' understanding of interrelationships between concepts, rather than carrying out procedures" (p. 341).

The choice of a method to solve SLEs depends on the spontaneity, speed and accuracy. Maharaj (2018) investigated students' choice of a method as a relation to the students' level of understanding; the majority chose to solve SLEs using the Cramer's rule and the Gauss elimination method. His results revealed that students prefer the Cramer's rule relative to the Gaussian elimination, even though it requires longer steps. The Gaussian method is a systematic elimination of variables using elementary row operations which should be spontaneous for systems involving large number of unknowns. Students require guidance in order to initiate the Gaussian elimination method. To solve large systems of equations, a good method hinges on precision and speed because the computations involved are sometimes immense (Mandal, Cherukuru & Rani, 2021; Gharib et al., 2015). The goal of solving SLEs using the Gaussian elimination is to reduce computational time and complexity.

The development of the level mathematics cognition of the students culminates in the schema for that mathematical concept. To determine the possible mathematical understanding of the students in this study, the Piaget and Garcia's (1989) triad framework was used. The triad theory focuses on the hierarchical mental constructions that goes on in the mind of students when trying to learn a mathematical concept. According to the triad theory, before a schema becomes coherent, it must go through the three stages; the intra-, inter- and trans-operational stages. The intra-stage is the preliminary level of conceptualisation whereby a concept is conceived in an isolated manner. The students' understanding is localised and relationships between processes are not perceived. At this lowest stage of schema development, students would have a collection of rules for solving SLEs using Gaussian elimination, inverse or Cramer's methods, but would not recognise the relationships between them. For example, in the schema development of solving SLEs by the Gaussian elimination, performing step-by-step elementary row operations is at the intra-stage. The student realises that some facts and principles in the schema are connected but is not able to justify and explain the connection (Borji & Martinez-Planell, 2020). Once connections are established between concepts and other previously-held schemas, the individual is at the inter-stage. The schema enters the inter-stage when students make connections between the nature of SLE (facts) and the method that corresponds to type of solution (principles). This includes more complex examples and links to the geometrical representation of solutions to SLEs. The student can explain and justify how changes in one structure leads to changes in the other.

The trans-stage outlines a coherent structure that underlie the relationships constructed in the inter-stage. It is further characterised by the construction of coherent structures underlying some of the connections discovered in the inter-stage of development (Clarke et al., 1997). Being able to take appropriate decisions to effectively solve SLEs depicts the trans-stage (Possani et al., 2009). At this stage, students can modify their knowledge to solve related situations. The triad framework is not linear but represents a continuous spectrum for developing a schema of a mathematics concept. The APOS theory's description of schema coincides with the trans-stage of triad since that is where a coherent structure emerge. The isolated (intra-) and connected (inter-) objects constitute a pre-schema. Students at trans-stage can coordinate two or more different interpretations of solving SLEs to mean the same thing.

Conceptual learning of mathematics should enable students to see relationships between facts and principles, instead of relying on procedures (Dewi et al., 2021). The triad theory is used in this study as a mechanism to describe schema development in general and the solution of SLEs is used as a particular instance. It provides the structure for interpreting the students' understanding of the SLEs and classifying their responses according to the three stages of the triad theory.

METHODOLOGY

To address the research problem on solving SLEs, a descriptive-exploratory study design was adopted on a case study of 160 B.Ed. degree students majoring in mathematics. After a traditional instructional in linear algebra, the author conducted semi-structured interviews with selected students to determine the extent of their schema development. This allowed the author to obtain information about students' understanding of solution of SLEs. The following task formed part of the interview:

1. Find the solution to the 3×3 system: $x + y - z = 6$; $3x - 2y + z = -5$; $x + 3y - 2z = 14$.
2. Solve this system of linear equations: $2x + 3y + 3z - u = 3$; $x + y - 2z + 3u = 4$; $5x + 7y + 4z + u = 5$

3. If $A = \begin{pmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{pmatrix}$, find the products AB and BA and hence solve $x - y + z = 4;\ x - 2y - 2z = 9;\ 2x + y + 3z = 1$.

In item 3, participants were expected to see the underlying role played by the identity matrix in showing that two matrices are inverses of each other. The relationship was further complicated by the presence of the constant 8, which led to the intricate relationship $B^{-1} = \frac{1}{8}A$. The second part of the question required participants to identify that matrix B was given, hence the need to solve the SLE using the inverse method. But in place of B^{-1} on the right-hand side, they need to pre-multiply by $\frac{1}{8}A$. The solution for this item represent a coherent connection among many concepts in matrix algebra, an attainment of such is evidence of full schema development in solving SLEs. The expected full solution for this question is given in Figure 1.

We find $$AB = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8I$$

and $$BA = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} = \begin{bmatrix} -4+7+5 & 4-1-3 & 4-3-1 \\ -4+14-10 & 4-2+6 & 4-6+2 \\ -8-7+15 & 8+1-9 & 8+3-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8I.$$

So, we get $AB = BA = 8I$. That is, $\left(\frac{1}{8}A\right)B = B\left(\frac{1}{8}A\right) = I$. Hence, $B^{-1} = \frac{1}{8}A$.

Writing the given system of equations in matrix form, we get

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}. \text{ That is, } B\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}.$$

So, $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} = \left(\frac{1}{8}A\right)\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

Hence, the solution is $x = 3, y = -2, z = -1$,

Figure 1: The expected solution for question 3

These items were carefully chosen to appeal to each of the cognitive operations identified in the triad theory.

The author used the purposive sampling technique to select 15 students to participate in semi-structured interviews covering solving of SLEs. To strike a balance in the distribution, five above average, five average and five below students made up the sample, based on their performance in final examination of the previous year. This enabled the researcher to observe a wide range of student responses according to the principle of maximum variation sampling technique (Doruk, 2019). The author conducted individual task-based interviews to each of the participants. The participants were assigned pseudonyms T1, T2 and so on up to T15 for ease of reference while maintaining the real identify of participants confidential. The limitation of this study was the impossibility to deduce the schema of another individual, thus, the best the researcher could do was to conjecture about participants' schema development based on analysis of their responses. Nonetheless, the task-based interviews provided data that acts as proxy of how students think and reason about the mathematical concept at hand (Plaxco & Wawro, 2015).

Data for this study were generated from the written and verbal responses to the three questions on the task on solving SLEs. The verbal responses were audio-recorded and transcribed. Data collection provided evidence of how students made the mental relationships when learning the matrix method to solving SLEs. Data analysis was two-fold: the author performed content analysis to the participants' responses to the three items and ascribed categories to describe observations according to the intra-, inter- and trans-stages for all questions, as well as to describe any unexpected observations (Borji, Martínez-Planell & Trigueros, 2023). The categories were introduced as observations were made and included re-analysing previously categorised data to make sure no prior occurrences of categories had been missed by mistake.

FINDINGS

The analysis of data was according to the intra-, inter- and trans-operation stages for all the questions in the interview.

Question 1 analysis

All the students' development of the solving SLEs schema commences at the intra-stage. Item 1 could be solved by any of the Cramer, inverse, Gaussian elimination methods. The frequencies of the method used are given in Table 1.

Method	Gaussian	Cramer	Inverse	Other	Total
Frequency	8	4	0	3	15
Frequency of correct solution	6	4	0	3	11

Table 1: The frequencies of the usage of methods to solve SLEs

The Gaussian elimination had the highest frequency as participants realised that this method can practically solve any given SLE, unlike the Cramer and inverse. The latter two only work for square matrices and are inconclusive if the solutions to the SLE are infinitely many or do not exist. Upon probing why they preferred the Gaussian elimination method, T1, T6 and T8 respectively said: "*because the question was not specific so I picked the easy one for me*", "*in most of the question papers I used to practice with Gaussian; it was mentioned that I should use the Cramer's rule on certain questions. So I thought if it wasn't specified that we use the Cramer's rule I should always use the Gaussian elimination*" and "*Yes, I can use the Cramer's rule it's fine but for* 3×3, *I can also use this one. It's the one that's much easier for me*". Figure 2 illustrates the correct solution by T12 using Gaussian elimination.

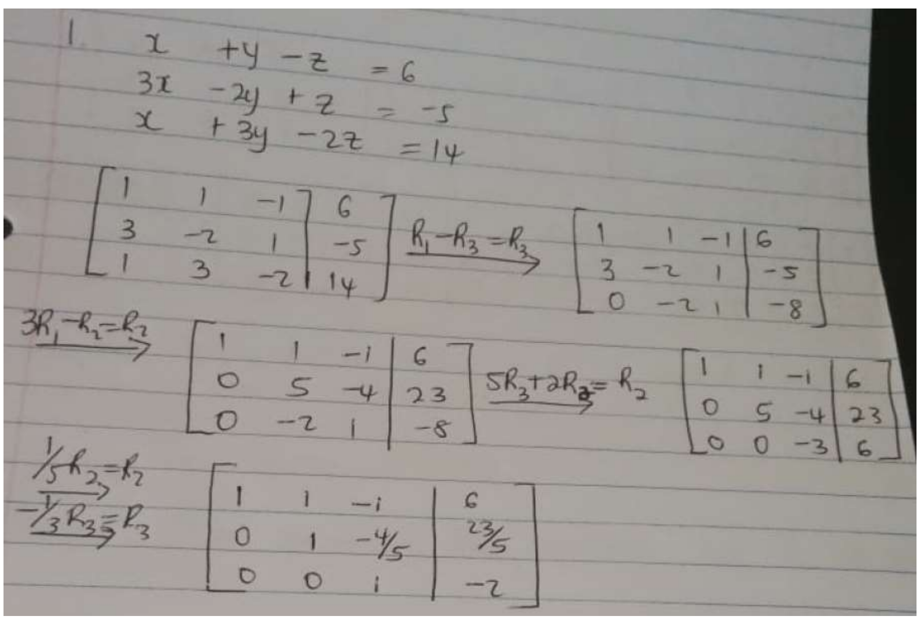

Figure 2: Correct steps in the Gaussian elimination method by T12

After T12 obtains $z = -2$, the rest of the values were obtained by back-substitution. T7 commented that, "*Since there were no instructions I thought it would be easier to use the Gauss-Jordan*" but did not get the correct final solutions due to multiplicity of steps involved. T2 also used the Gauss-Jordan method and managed to get the correct solutions despite doing a few more steps.

Three-quarters of those who chose to use the Gaussian got the correct solutions. The two who got incorrect solutions erred in executing elementary row operations. Four participants, T4, T9, T14 and T15 chose to use the Cramer's rule and they all solved it flawlessly. The four found the Cramer's rule easier to use, as shown in the dialogue with T9. R stands for the interviewer.

R: Why did you prefer the Cramer's rule in question 1?

T9: The question says find the solution; that's what makes me use Cramer.

R: Why didn't you use the inverse method in question 1?

T9: I dont know the inverse method; I only know the Cramer's rule. I don't know how to find the solution using any method other than the Cramer's rule.

None of the participants chose the inverse method, and T9 commented that he does not know it. After having been taught the topic of linear algebra, three participants did not use matrices to solve the problem in item 1. Instead, the three used the elimination of variables method to solve the SLE and obtained the expected solutions (shown in Figure 3).

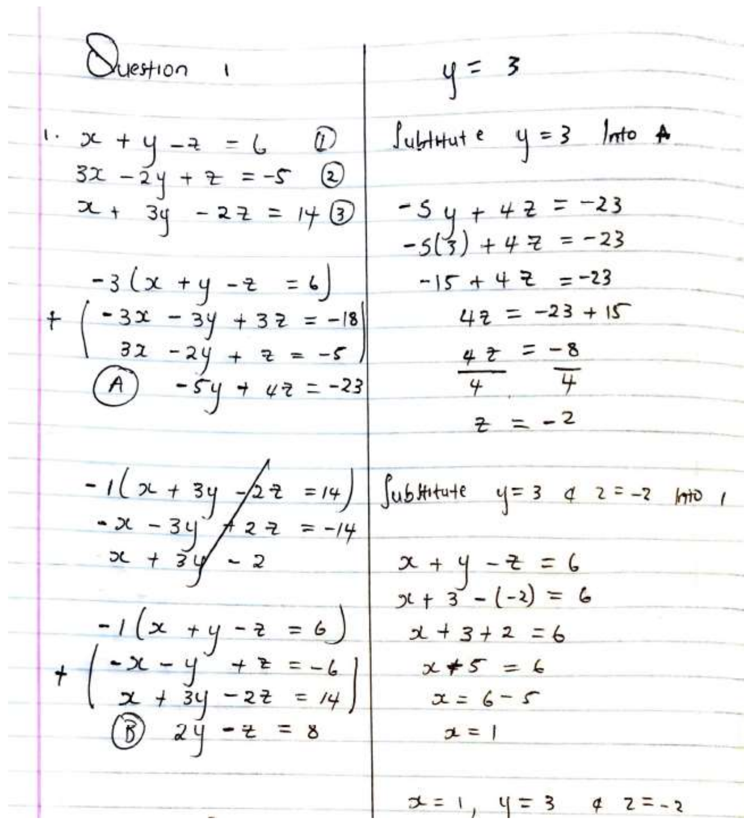

Figure 3: A correct elimination of variables method by T11

T3 also used the elimination of variables method, but due to the lengthy process, he made an error, which spoiled the entire solution. In the interview, T3 said the following concerning the elimination of variables.

R: I can see you used the substitution method where you made x the subject, y the subject and then you substituted into z to get your z. Is it the best method to use if we are given such a scenario?

T3: No Sir. But I tried.

R: You got all your answer correct. But can we rely on it to solve a system of linear equations in linear algebra?

T3: I don't want to say I can rely on it. But there are many ways to kill a cat.

R: Yes as long as the cat is dead. It's not important 'how'? But suppose I am going to give you a system with five equations and five unknowns. Do you think you can rely on the substitution method?

T3: No sir.

However, this represented a pre-intra-stage of solving SLEs in linear algebra. With an intra-conception, participants had high chances of doing the step-by-step processes to solve the SLE using the Gaussian elimination, Cramer's rule or inverse method but they do not perceive the connection of the relationships. For example, all the participants who used the Cramer's rule could not explain why it works.

Question 2 analysis

This item appealed to both the participants' intra- and inter-operational conceptions of solving SLEs. Participants were expected to establish the connection between the appropriate method to solve a SLE and the nature of the solutions. It was important for participants to identify that all SLEs with more variables than equations lead to free variables, hence yield either infinitely many solutions or no solution. The unique solution can never be possible when free variables are involved. In this item, only eight participants were able to apply the Gaussian elimination method, which was the only possible method in the given scenario, as shown in Figure 4.

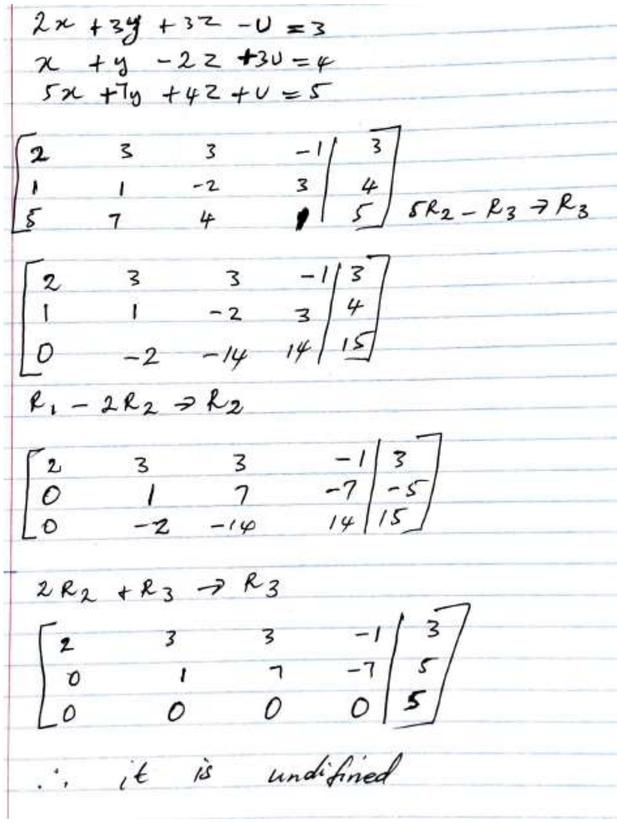

Figure 4: Correct use of the Gaussian elimination method on free variables by T4

Almost all the participants swopped row 1 and row 2 as the initial row operation in this question. Applying the rule to swop rows was correct, but some participants still had challenges to reduce the augmented matrix to upper triangle, as depicted in the dialogue below.

R: Why did you reverse row 1 and row 2?

T9: Because they say the leading number in row 1 must be 1, so I thought if I exchange the rows it makes things easier for me.

R: Perfect. However, why didn't you finish this question up to the point of finding the values of x, y, z and u?

T9: But didn't they say we must have that main diagonal composed of one's.

R: I get your point [interrupts]

T9: That's where I got stuck because I don't understand where to go to from here. So I just left it here.

Inadvertently, T4 still solved the problem without swopping rows. This implored the inter-operational skill. Five of the eight who chose the Gaussian elimination went ahead and performed the elementary row operations and obtained four zeros on the last row. This was obviously concluded as *no solution* since the element after the vertical line was a 5 or -5 in some cases. Performing the elementary row operations constitute inter-operational conception. The remaining three participants were lost in the row-reductions, culminating in incorrect conclusions.

T7 only managed to transform the equations to augmented form but aborted the solution as shown in Figure 5.

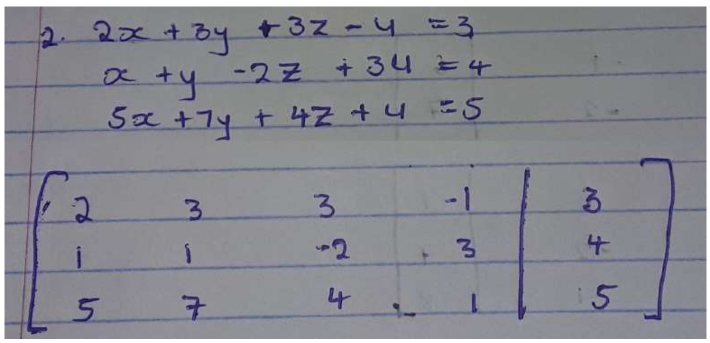

Figure 5: An incomplete solution by T7

On being asked why he didn't complete the solution, T7 replied "*Oh yes I was wondering how my entries would be there. I know that for a 3x3 matrix its 1-0-0, 0-1-0 then 0-0-1. I wasn't quite sure what I was going to do with the last row.*" He did not realise that reducing to upper triangle can be done for matrices of any order. For this item, six participants left it unanswered. T3 was one of them and when probed, he had this to say:

R: Why didn't you attempt this question?

T3: Because I don't know it Sir.

This was an evidence of lack of inter-operational conception of SLEs with more variables than equations. Also, none of the participants attempted to use the Cramer's or inverse methods as these were totally inapplicable. The dialogue below depicts T8's attempts to exclude the other methods.

R: You still prefer the Gaussian elimination method. But do you think the other two methods are applicable; the Cramer's rule and the inverse method in this particular case?

T8: It's going to be very difficult. Yes, they can be applicable, but to find the determinant of a 3x4 matrix is very difficult.

R: Is it difficult or impossible?

T8: No it's not impossible; you can find it. You can also find the determinant for this 3x4 matrix.

R: Is that so? We talk of determinant of a square matrix, don't we? And this is not a square matrix, is it?

T8: No we can't. I remember now. Because this is a 3x4. This one it's not possible.

Finally, one participant circumvented the matrix approach and pursued the elimination of variables method. Interestingly, he managed to get the *no solution* after all the variables were eliminated as shown in Figure 6.

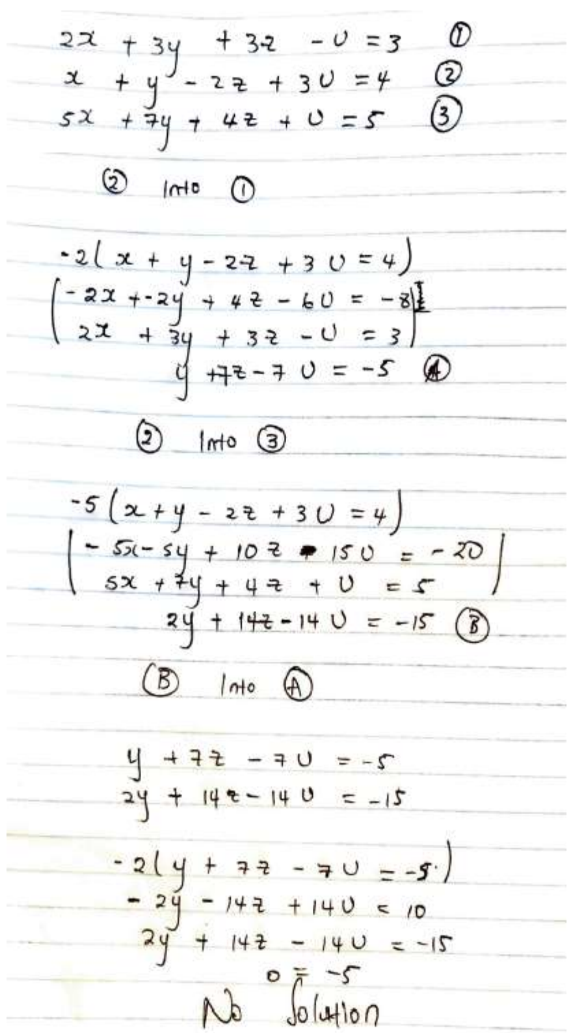

Figure 6: The correct use of the elimination of variables method by T11

However, these did not depict the schema development for linear algebra, even though the answer was correct.

Question 3 analysis

The analyses of data revealed that three participants, T3, T8 and T10, skipped the question altogether, which could be an indication of lack of inter- and trans-operational conceptions. Disregarding the products of the matrices A and B and B and A in the first part of the question, the rest of the participants attempted to solve the SLE in the second part using one of the methods they have learnt. Seven participants went straight ahead to solve the SLE using Gaussian elimination in the second part. These seven failed to use the fact that the question had two parts, whereby the first part imply the second. In-advertently, all of them were lost in the elementary row operations so that none of them got all the solutions right. T15 and T4 only managed to get the first solution $z = 1$. This on its own represent inadequate intra-conception of solving SLEs. Similarly, four students chose to use the inverse method of solving the SLE to the second part of the question. To find the inverse of the coefficient matrix, the participants were faced with monstrous task of computing the determinant, matrix of co-factors and the adjoint. Since finding the inverse using that way was involving, T13 aborted the process whilst T12 got lost in evaluating the adjoint and obtained the incorrect inverse. Finally, as was the case in question 1 and 2, T11 used the substitution and circumvented the matrix method to solve the SLE. This high school method is cumbersome to SLEs with three or more variables. Thus, his attempt did not yield correct solutions due minor manipulation errors. Figure 7 illustrates the simple error in T11's substitution method.

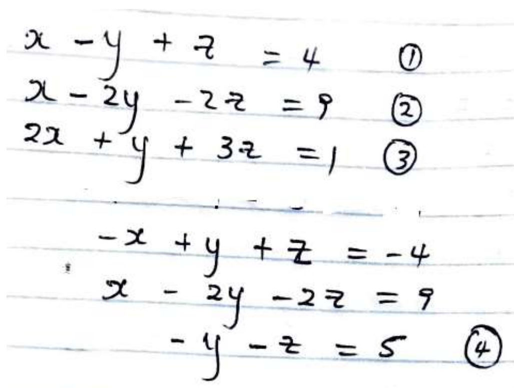

Figure 7: Minor error in the substitution method.

From Figure 7, equation 4 should be $-y - 3z = 5$, which caused the final result be wrong.

Ten participants followed the instructions to evaluate AB and BA but due to computational errors, they could not get $8I$ in either case. Figure 8 illustrates an attempt by T6. In the case of T1, he admitted to his errors by saying "*I was just lazy to multiply those two matrices*". This normally happens when participants feel that the steps are cumbersome.

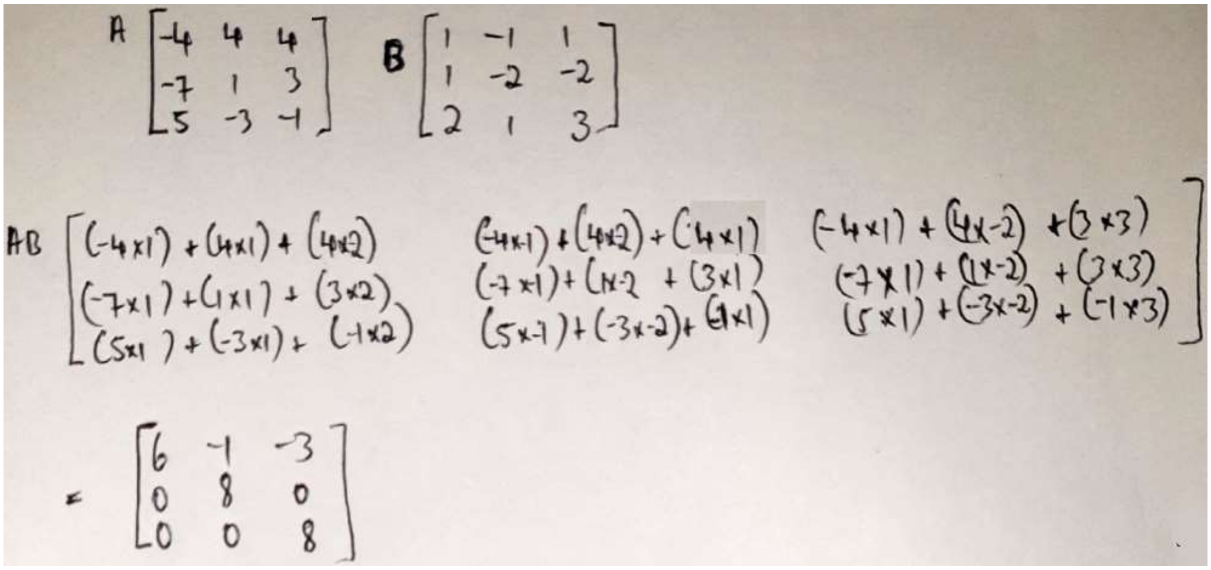

Figure 8: Computational error in matrix multiplication of A and B

The very essence of matrix multiplication was also a challenge to some participants. Figure 9 depicts an instance of incorrect multiplication. This direct multiplication of corresponding elements led to the unintended result $AB = BA$, but the participants did not take note of that.

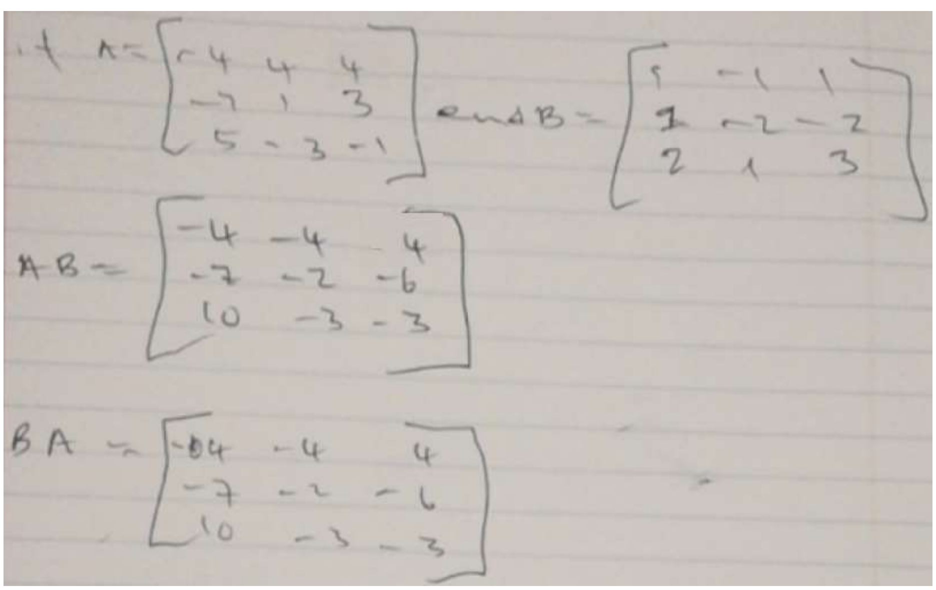

Figure 9: Wrong conception of matrix multiplication by T9

As such, the inverse relationship between A and B could not be established nor could they make use the fact that the coefficient matrix of the SLE is matrix B. Thus, the underlying connection of the inverse and identity matrices and inverse method could not be realised. On the other hand, some participants avoided the steps in matrix multiplication by skipping it. They went ahead to solve the SLE in the second part of the question using the other methods without using the concept of inverses arising from AB and BA. Many participants managed to get the correct solution using the other methods. T15 said, "*I used a different method. I used the Cramer's rule and got the correct answers as yours*". T12 seemingly used the inverse method since the question indicated the inverse arising from AB and BA (shown in the dialogue below).

R: Why did you the inverse method?

T12: It's in the question.

T5 also tried to use the inverse method but aborted it due to the cumbersomeness of the work (shown in Figure 10). She just stopped at the stage of computing the matrix of co-factors. A better way would have been to evaluate the inverse based on the elementary row operations on the augmented matrix $\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & -2 & -2 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right)$. Regardless, this solution would still be unacceptable as the question required a connection to the AB and BA relationship.

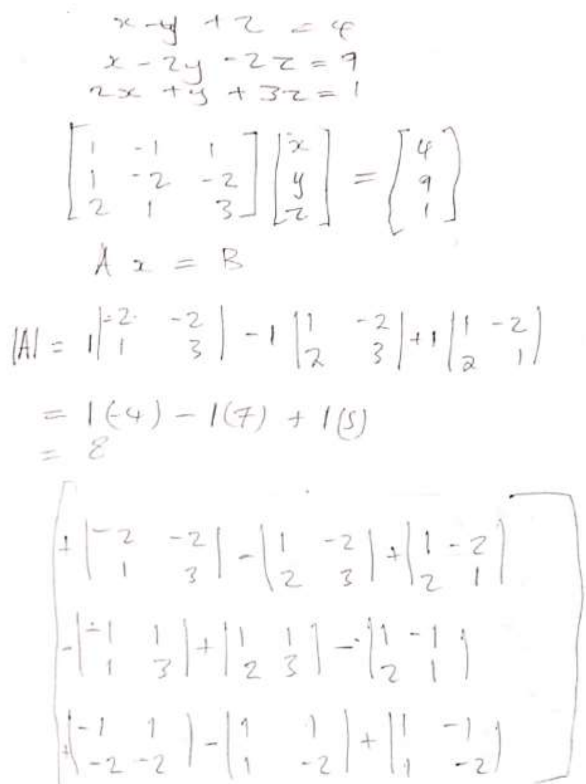

Figure 10: Attempt to solve the SLE using the inverse method by T5

DISCUSSION

Undergraduate students' level of understanding of linear algebra determines their universal approach to deal with situations in diverse mathematical tasks in linear algebra (Trigueros, 2019). In this study, comparison of students' written and verbal responses revealed critical differences in understanding that can be attributed to the stage schema development that they evidence (Arnon et al., 2014). Students have a tendency to go for an easy method for solving a SLE, thus the high number of students who used the Gaussian elimination and Cramer's rule. Mandal, Cherukuru and Rani (2021) concur that Gaussian elimination is the easy and best to solve an arbitrary SLE. Oftentimes, many students' approach to Cramer's rule is rule-bound and procedural (Ndlovu & Brijlall, 2019) thus in most cases, they cannot explain why the Cramer's rule works. This procedural approach indeed works better for the lowest level of conceptualisation, the intra-stage. Hence, the majority of the students who used the Gaussian elimination and Cramer's rule managed to attain the full solution. Hanna et al. (2016) argued that students can cope with the manipulation of numbers involved when solving systems of equations, but they do not develop a deep understanding of these concepts and cannot apply them in new contexts. Maullina and Setyaningsih (2023) posit that students' understanding of concepts should go beyond developing abilities for step-by-step routine computations. The only thing which students needs to strengthen is the manipulation of the each of the methods, which are the execution elementary row operations for Gaussian elimination and inverse methods, and computation of determinants for the Cramer's rule and inverse methods. Findings revealed that students are affected by manipulation errors in the process of solving SLEs. This was more pronounced on T7 who chose to use the Gauss-Jordan method in question 1; the extra steps of reducing the augmented to both upper and lower triangles led to many errors. Gaussian elimination is faster than the Gauss-Jordan method since the latter is an extension of the former (Gharib, Ali, Khan & Munor, 2015). The inverse method was not popular with students in question 1 and T9 confessed ignorance of this method. Moreover, when the inverse method was required as part of the solution in question 3, none of the students was able to apply it.

Oftentimes students focus on the methods which they find comfortable with at the detriment of alternative methods. At the inter-stage, students are expected to be decisive about the appropriate method to use based on the underlying relationships called upon in the question. Non-matrix method also featured amongst the methods used by students but the users admitted that for higher orders, these methods are cumbersome. That means that some students did not apply the Gaussian elimination spontaneously, even for systems with relatively large number of equations and unknowns (Harel, 2017). Though there is no best method yet proposed to solve SLEs, consensus point to the need for accuracy and speed as the determining factors (Gharib, Ali, Khan & Munor, 2015).

The findings also revealed that some students stick to a sole method of solving SLEs despite changes in problem-contexts. T1, T4, T6 and T9 used the Gaussian elimination to solve all the three items and T11 resorted to the elimination of variables technique for all the three questions. Students lack the flexibility in the application of methods of solving SLEs, leading to incorrect solutions in some cases. Especially the trans-stage require students to make decisions for the technique to solve SLEs by taking into consideration the implicit underlying structures. Learning occurs when experiences produce changes to connections in the brain (Gallistel & Matzel, 2013). Students had to recognise the structures in order to adequately respond to the questions (Maullina & Setyaningsih, 2023).

The schema development according to the Triad is hierarchical hence item 2 had both intra- and inter-conceptions and question 3 had all the three stages of intra-, inter- and trans-conceptions. In item 2, only 50 percent of the participants recognised the much-connection between the 3×5 augmented matrix and the Gaussian elimination method. If the matrix is non-square, the solution can only be either no solution or infinitely-many solutions and requires the Gaussian elimination method. The rest of the participants tried other unsuitable methods to this question and this represented inadequate inter-conception skills. As for intra-stage, based on the chosen approach to solving SLEs, most of the students proceeded to perform the correct procedures. However, this was only effective if the selected approach was the correct one.

Students are said to attain schema development in solving SLEs if they can demonstrate the trans-stage of understanding. In question 3, students were expected to establish the underlying relationship of the inverse of matrix A and B and its use in the solution of the SLE. To achieve this, students must have established that linkage of the identity matrix and A as a partial inverse of B. This pivotal connection represents the inter-conception of solving SLEs. The isolated relations of matrix multiplication and the use of the inverse method, Gaussian elimination or Cramer's rule to solve SLEs represent the intra-conception. According to Donevska-Todorova (2016), performing row reductions and computing determinants and inverses is regarded as procedural, whilst if an individual intends to coherently link row reductions, determinants and inverses to a particular method of solving SLEs, then such an understanding is regarded as trans-conceptual. It is important for students to be able to determine the relationship between facts and principles, instead of memorising algorithms (Dewi et al., 2021). In addition, students tend to perform better on questions that require procedural than conceptual understanding (Bouhjar et al., 2018), hence the frequency of correct responses honed as students progressed from question 1 to 3.

The coherence of these intra-, inter- and trans-conceptions typify a student who has attained the schema for solving SLEs. The intra-stage is necessary for inter-stage to take effect whilst the coordination of both the intra- and inter-stages precedes the trans-stage. Students' good decision-making skills are required to unpack some of the implicit and explicit global nuances of solving SLEs. However, the findings revealed that none of the participating students achieved the schema for solving SLEs, yet it is the climax of students' learning of linear algebra. The majority of students operated at the lower echelons of schema developments for solving SLEs, which is the intra-stage (Berger & Stewart, 2020). The nature of matrices and the thinking required to understand them pose challenges in linear algebra (Dorier & Sierpinska, 2001), most commonly to students who take linear algebra for the very first.

CONCLUSION AND IMPLICATIONS

The three questions in this study were carefully designed to bring out students' conceptualisations in solving SLE at the intra-, inter- and trans-stages. The final stage of trans-conceptualisation was not achieved due to manipulation errors, un-connected relations and bad decisions on the method to use. Students were limited in identifying and selecting the suitable methods that were applicable to the specific contexts of solving SLEs. An individual's level of cognition of a mathematical concept determines the individual's general tendency to deal with problems in diverse mathematical tasks of the concept. The complexity and depth of students' understanding of a topic depends on their ability to establish relations among the corresponding components of the schema. Instruction in linear algebra should appeal more to the trans-stage of understanding since the intra- and inter-stages represent pre-schema. Based on the findings, the majority of the students attained the intra-stage of conception as they used diverse methods to solve the SLE and no relationships were extrapolated therefrom. However, only fifty percent managed to reduce the 3×4 augmented matrix to echelon and conclude that there was no solution to the SLE. Furthermore, none of the students managed to establish the connection among matrix multiplication, the inverse relationship and the solution of the SLE. This classification of data according to the triad theory represents an incomplete development of the schema for solving SLEs. The study by Ndlovu and Brijlall (2019) also classified students' mental constructions in using the Cramer's rule to solve SLEs as predominantly at the action stage of the APOS theory. Using both the APOS and triad theories, Trigueros (2019) revealed the notion of schema development, which gives important information about students' progress in a course.

By patterning students' cognition of solving SLEs using the triad, this study fills the literature gap of improving the understanding of schema development (Borji & Martínez-Planell, 2020). The implication of this study is the need to develop activities that address higher stages of the triad in the students' minds. These activities foster coordination, accommodation and assimilation of new relationships into their developing schema and the construction of those relationships strengthens instruction of concepts through promoting coherence. It is necessary to continue studying the teaching and learning of linear algebra as this provides new insights into better ways for students to understand the subject (Stewart et al., 2022).

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Assessing the understanding of the slope concept in high school students

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Abstract: This research reports the implementation of an evaluation instrument of the slope concept in high school students. The design of this study was based on four dimensions: Skills, Properties, Uses and Representations (SPUR model; Thompson & Kaur, 2011) and on three conceptualizations: constant ratio, behavior indicator and trigonometric conception. This work adopts a qualitative approach to analyze the students' productions and a quantitative approach when obtaining the percentages of the student's responses. The general objective of the work is to evaluate the effect produced by a group of tasks designed with the SPUR model on the slope concept in high school students. The results show that students have traditional conceptualizations of the slope as a constant ratio and trigonometric conception. However, these conceptualizations emphasize more procedural aspects than conceptual ones. This finding could partially explain why students can solve certain tasks of procedural nature but not conceptual tasks that usually require a multifaceted view of the slope.

Keywords: slope, SPUR model, conceptions, assessment

INTRODUCTION

The slope is considered to be a key concept since its understanding is an essential requirement for the study of concepts associated with linear functions or rate of change in calculus, linear regression in statistics, uniform linear movement in physics and other topics of high-level mathematics (Clement, 1989; Nemirovsky, 1992, 1997; Simon & Blume, 1994; Nagle & Moore-Russo, 2014; Casey & Nagle, 2016).

Different studies report the challenges that students face with this concept. In particular, they highlight the lack of connection and interpretation of the slope in different contexts (Thompson, 1994; Stump, 1999, 2001; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Lobato & Siebert, 2002; Moore-Russo, Conner, & Rugg, 2011).

This notion makes sense when considering the existing diversity of conceptualizations associated with the slope. For example, the pioneer works of Stump (1999; 2001) offer nine conceptualizations of the slope concept according to the context where it can be used. Furthermore, Moore-Russo et al. (2011) expand this categorization to eleven conceptualizations. These works constitute the basis for further investigations addressing the study of conceptualizations of students and teachers from all educational levels (Stump, 1999; Moore-Russo et al., 2011; Nagle, Moore-Russo, Viglietti, & Martin, 2013; Nagle & Moore-Russo, 2013).

In the case of students, the difficulties connecting the different conceptualizations of the slope or the emphasis on one or a few aspects of it stand out. For example, Zaslavsky, Sela and Leron (2002); Nagle and Moore-Russo, (2013); Nagle et al. (2013) and Deniz and Uygur-Kabael, (2017) show that students associate the slope with its geometrical and algebraical characteristics but are usually not able to connect these features in higher demand tasks.

Regarding the case of teachers (professional teachers or student teachers), other research has shown that they possess a more varied range of conceptualizations of the slope than students (Stump, 1999; Nagle et al., 2013; Nagle & Moore-Russo, 2013) but seem to omit it in their teaching, concerning the educational level where the teacher performs (Nagle & Moore-Russo, 2013).

For example, in secondary school teachers, there is an absence of conceptualizations of the trigonometric type (Azcárate, 1992; Stump, 1999); a dominant conceptualization of the slope interpreted as a geometrical proportion (Stump, 1999; Moore-Russo et al., 2013) and difficulties to work with the average rate of change (Coe, 2007).

Another body of work proposes didactical approaches that highlight the relevance of working with different conceptualizations of the slope in the classroom to work and foster the development of this concept (Deniz & Uygur-Kabael, 2017), or from work with teachers or student teachers (Moore-Russo et al. 2011; Diamond, 2020).

Other approaches emphasize the role of variation and covariation as essential to understanding the linear function (Thompson, 1994; Carlson et al., 2002). For example, DeJarnette et al. (2020) adopt a social semiotic perspective to analyze the interactions between students and teachers, highlighting that students interpret the slope more as a quotient and less as a parameter of the linear function.

Lastly, other studies analyze textbooks at the undergraduate level. These studies report inaccuracies in describing geometric and algebraic connections when defining the slope (Zaslavsky et al., 2002). Studies have found that the lack of connection between different slope conceptualizations could be due to the frequent use of problems with low cognitive demands (Arnold & Hicks, 2011). Another research indicates frequent use of problems with procedural features and fewer conceptual concepts (Tuluk, 2020).

Theoretical Debate

Nagle and Moore-Russo (2013) establish that research regarding the slope concept has focused on the study of conceptualizations in teachers and students in an isolated manner, and little has been explored regarding its articulation, which could enable a 'conception network' to improve the understanding of this concept. There is general agreement regarding the existence of a marked difference between students' and teachers' conceptualizations. In order to avoid this, teachers should promote a conception network according to slope usage to obtain an integrated view of the slope (Moore-Russo et al.,2011; Stanton & Moore-Russo, 2012).

Regarding this point, Nagle and Moore-Russo (2013) propose an analysis of slope conceptualizations through a frame of reference that considers the procedural and conceptual dimensions to cluster and generate a network of conceptualizations.

The previous accounts exhibit the need for instruments that evaluate the student's understanding level of the slope concept, which can reflect the diversity of conceptualizations and their connections. This specific need leads to wondering about the appropriate instruments to perform this evaluation.

According to Popham (2000, as cited in Heuvel-Panhuizen, Kolovou & Peltenburg, 2011), the evaluation can be understood as a process where teachers use students' responses (generated through stimuli or natural conditions) to make inferences about the knowledge, skills or the affective state of students. In this context, teachers and researchers show concern about the problems that the systematic or traditional evaluation may raise in contrast to more integrative perspectives whose focus is not only on evaluating procedural fluency (Thompson & Kaur, 2011). For example, interpretative skills for problem-solving, skills in contextualized tasks, problem-solving processes, etc.

Regarding the evaluation aspect, Thompson and Kaur (2011) promote the idea that if teaching reflects a multidimensional approach, the evaluation should also be in line with this perspective; thus, the evaluation instruments must consider these considerations.

With this context in mind, this research focused on answering the following question: What is the effect of a group of tasks on understanding the slope concept in high school students when evaluated through a SPUR model?

Based on the above, the reference framework used in this research is discussed (framework proposed by Nagle & Russo, 2013; and the SPUR model), the method used and the description of the activities that make up the evaluation instrument are reported. Finally, the results are exhibited and discussed based on the productions of the students.

THEORETICAL FRAMEWORK

In this research, we adopt the theoretical framework proposed by Nagle and Moore-Russo (2013), which considers research regarding procedural and conceptual understanding and its relation with visual and analytic interpretations of the slope for each of the eleven slope conceptualizations identified in Moore-Russo et al. (2011).

The authors propose five key slope components whose connection (established through tasks or problems that demand some knowledge of the slope) promotes a network of conceptualizations that offers a more integrated characterization of the understanding of the slope. These components are (1) constant ratio, (2) determining property, (3) behavior indicator, (4) trigonometry and (5) calculus.

Given our study sample, this research only considers components (1), (3) and (4). The features of each component according to the procedural or conceptual dimension are summarized in tables 1, 2 and 3.

Table 1. Description of the conceptualization of the slope as a constant ratio

	Examples of the slope as a constant ratio	
	Procedural Emphasis	**Conceptual Emphasis**
Visual approach	$R_{v,p}$: rise/run or vertical change/ horizontal change.	$R_{v,c}$: similarity of slope triangles yields a constant ratio of rise/run regardless of the position on the graph.
Analytic approach	$R_{a,p}$: change in y over change in x; $\frac{y_2-y_1}{x_2-x_1}$.	$R_{a,c}$: constant rate of change between two covarying quantities; and equivalence class of ratios and hence a function.

Source: "SLOPE: A Network of Connected Components", by Nagle and Moore-Russo, 2013, *North American Chapter of the International Group for the Psychology of Mathematics Education*, p.130.

Table 2. Description of the conceptualization of the slope as a behavior indicator

	Examples of the slope as a behavior indicator	
	Procedural Emphasis	**Conceptual Emphasis**
Visual approach	$B_{v,p}$: increasing lines have a positive slope; decreasing lines have a	$B_{v,c}$: positive rise corresponds to a positive run for an increasing line,

	negative slope; horizontal lines have zero slope.	yielding a positive slope. For a decreasing line, a negative rise corresponds to a positive run, yielding a negative slope. A horizontal line has zero rise for any run, yielding a zero slope.
Analytic approach	$B_{a,p}$: value of m in the equation for a linear function (e.g.., in $y = mx + b$) indicates whether f is an increasing $(m > 0)$, decreasing $(m < 0)$ o constant $(m = 0)$ linear function [previously parametric coefficient].	$B_{a,c}$: application of the definition of increasing/decreasing/constant functions to explain positive/negative/zero slope respectively (e.g., f is increasing means that $f(x_1) < f(x_2)$ if $x_1 < x_2$, so $\frac{[f(x_2)-f(x_1)]}{x_2-x_1} > 0$.

Source: "SLOPE: A Network of Connected Components", by Nagle and Moore-Russo, 2013, *North American Chapter of the International Group for the Psychology of Mathematics Education*, p.131.

Table 3. Description of the conceptualization of the slope as a trigonometric conception

	Examples of the slope as a trigonometric conception	
	Procedural emphasis	**Conceptual emphasis**
Visual approach	$T_{v,p}$: steepness of a line; slope as the angle of inclination of the line with a horizontal; as a line is rotated about a point, the slope changes.	$T_{v,c}$: the angle of inclination determines the rise/run; a steeper line has a greater rise per given run than a less steep line
Analytic approach	$T_{a,p}$: slope is calculated as $tan\,\theta$, where θ is the angle formed by the graph of the linear equation and an intersecting horizontal line.	$T_{a,c}$: the angle of inclination determines the ratio of $(\frac{y_2-y_1}{x_2-x_1})$, which is equivalent to $tan\,\theta$.

Source: "SLOPE: A Network of Connected Components", by Nagle and Moore-Russo, 2013, *North American Chapter of the International Group for the Psychology of Mathematics Education*, p.132.

SPUR Model

In this research, the SPUR model (Skills, Properties, Uses and Representations) proposed by Thompson and Kaur (2011) is used as a theoretical framework. This approach promotes a multidimensional evaluation that reflects the student's knowledge more broadly in four key aspects, the same of which constitute the acronym of the model (SPUR). The following section describes the characteristics of each dimension.

Skills represent those procedures that students should master with fluency; they range from applications of standard algorithms to the selection and comparison of algorithms to the discovery or invention of algorithms, including procedures with technology.

Properties are the principles underlying the mathematics, ranging from the naming of properties used to justify conclusions to derivations and proofs.

Uses are the applications of the concepts to the real world or to other concepts in mathematics and range from routine "word problems" to the development and use of mathematical models.

Representations are graphs, pictures, and other visual depictions of the concepts, including standard representations of concepts and relations to the discovery of new ways to represent concepts" (Thompson & Senk, 2008, p.2)

According to this, the authors emphasize that if our educational aim is to create students with a solid and flexible understanding of mathematics, then it becomes essential to evaluate more than their skills' knowledge, as is traditionally done. The authors warn that if teachers focus their evaluation on a single dimension, they might obtain a wrong view of their students' understanding. In contrast, if teachers expand their evaluation dimensions, they might gain a better view of the strengths and weaknesses of their students.

The SPUR approach has been employed on an international project regarding mathematical achievement (IMPA) in a longitudinal survey on elementary education in different countries. Their results have highlighted the students' low performance on the dimensions evaluated by the model (Thompson & Kaur, 2011), offering a more integrated picture of the evaluation.

METHOD

The design of this research is of a qualitative and interpretative nature. The study was conducted in three stages: 1) a literature review of the slope concept, which included terms related to the evaluation of this concept; 2) a selection of the theoretical framework and task design and 3) instrument application and analysis.

The study sample comprised 35 high school students from a private school in Mexico. The selection was made following the curriculum of this educational level and the average performance of students. Only three (out of a total of five) of the slope characterizations proposed by Nagle and Moore-Russo (2013) were selected.

The test was directly applied as an evaluation instrument of the slope concept. The participating students had previously studied the slope concept on several occasions throughout their academic training, which included courses such as algebra, geometry, trigonometry and analytic geometry.

The design of the evaluation instrument incorporated the SPUR approach so that each of the model's dimensions was reflected in the tasks according to the selected conceptualizations.

The Tasks

In **task 1,** students only require one slope conceptualization of the *constant ratio* type. The approach can be analytic or visual, but with a procedural emphasis for both cases. The situation posed requires thinking analytically about the slope as a change in the variable y over a change in the variable x. Visually it implies identifying, once the line has been drawn, the vertical change between the horizontal change by means of the drawing of a reference right triangle. Regarding the SPUR model, the task corresponds to the *Skills* dimension since it requires the application of a standard algorithm. Additionally, it covers the *Representation* dimension should the student correctly make the required graphical scheme.

Task 2 can be characterized by the cognitive processes that the student employs in the resolution process. The task involves a *constant ratio* conceptualization through a visual or analytic approach that can lead to a procedural or conceptual emphasis. The task comprises the *Skills* and *Properties* dimensions of the SPUR model.

For example, for part (a) in the visual approach, a student who correctly relates the change in the variable y to the change in the variable x by means of an appropriate representation showing that the similarity of two right triangles at any locations of the lines shown maintain a constant relationship of elevation and run, it will show a visual approach with a conceptual emphasis compared to those students who could limit their reasoning to a procedural emphasis, if it only shows the value of the slope from a single right triangle of reference.

To elucidate this variant, subsection (b) is introduced, which questions the student about which of the two lines shown has a greater angle of inclination? This would show the conceptual or procedural emphasis of the individual since the answer demands to relate the conception of *constant ratio* with the *trigonometric conceptualization* (which corresponds to the dimensions of properties, representations and skills of the SPUR model). More specifically, if a student associates that the angle of inclination with the horizontal axis is greater for the line with the least slope, then his emphasis is procedural. In contrast, if in her answer she argues that the angle of inclination determines the rise-run which implies that a steeper line has a higher elevation then her approach is visual but maintaining a conceptual emphasis.

On the contrary, an analytical approach with procedural emphasis is associated with responses based solely on the calculation of the $\tan\theta$ without emphasizing its relationship with the slope (this situation is related to the dimensions of *skills* and *representations* of the SPUR model). Within the analytical approach, a conceptual emphasis (*properties* dimension) can be characterized based on whether the student argues that the relationship $\frac{y_2-y_1}{x_2-x_1}$ is equivalent to $\tan\theta$.

Task 3 involves different approaches. On the one hand, it involves the slope conceptualization as a *behavior indicator* since this task requires the interpretation of the slope as a change indicator: increase, decrease or constant. The approach is initially visual; subsection (a) and subsection (b) demand analytic work. The emphasis can either be procedural or conceptual, according to the student's development. Finally, the task can be associated with the four dimensions of the SPUR model: *Skills*, *Properties*, *Uses* and *Representations*.

In this way, a student who exhibits the years in which there were greater losses and gains, but his reasoning is limited to considering only the calculation of the slopes or the inclination of these will show an analytical or visual procedural emphasis, respectively. On the other hand, if your arguments, in part (b), focus on the constant rate of change between the two given covariant quantities or if you propose a similarity relationship between any two right triangles on the respective lines, your reasoning will correspond to a conceptual emphasis.

In **Task 4,** the *Uses* dimension from the SPUR model was included since its focus relies on applications of the slope concept. In this case, a verbal problem is presented and involves the visual and analytic approach with a conceptual emphasis. This requires that the student understand that the situation involves determining the vertical change from the horizontal change and consequently associating it with the slope.

The aim of **task 5** is to make the student use the *trigonometric* conceptualization of the slope. The approach is initially analytic, but the student can rely on a visual approach to solve the task. The emphasis is conceptual and results in a procedural emphasis. The dimensions of the SPUR model associated with this task are *Properties, Skills and Representations*.

In this task, a student associates the angle formed with the horizontal axis and a given line with that property that describes the elevation-range ($m = \frac{y_2-y_1}{x_2-x_1}$) equivalent to $\tan\theta$ will show a trigonometric conceptualization with a conceptual emphasis. In another case, when the student does not exhibit reasoning that allows elucidating the transit between the equivalences shown, we will say that his reasoning is of a procedural type.

Task 6 is an adaptation of the task proposed by Zaslavsky et al. (2002). Subsection (a) corresponds to the *constant ratio* conceptualization. The emphasis could be visual or analytic according to the student's path for solving the task. Any of these paths involve the use of the *Skills* and *Representations* dimensions of the SPUR model. Additionally, the combination of *Representations* and *Properties* can be established if the arguments correspond to a conceptual emphasis.

For example, locating the coordinates of any two points on the line and calculating its slope is associated with an analytical approach with a procedural emphasis. In another case, if it is determined through the outline of a right triangle that exhibits the change in height-travel, then the approach will be visual with procedural emphasis.

Subsections (b) and (c) seek to determine if the students manifest the trigonometric conceptualization of the slope. Part (b) is used as scaffolding to try to get the student to notice the change of scale in each displayed representation so that the question in part (c) makes sense. According to the student's response, a visual or analytical approach could be associated, but, in any case, although the task derives from a procedural emphasis, a conceptual emphasis is initially required to be able to justify whether the slopes are the same or different. The latter can be associated with the dimensions of *properties* and *representations* of the SPUR model.

Finally, **task 7** involves the *Uses* dimension of the SPUR model since the suggested problem is of a verbal nature and involves data associated with temperature measurement scales. The slope conceptualizations needed for this task could be various: *constant ratio*, and *behavior indicator*, according to the activities' subsections. In any case, it is necessary to start from a conceptual emphasis to be able to carry out the appropriate approaches that culminate in the generation of a covariational model and understand its parameters to respond to subsections (b) and (c).

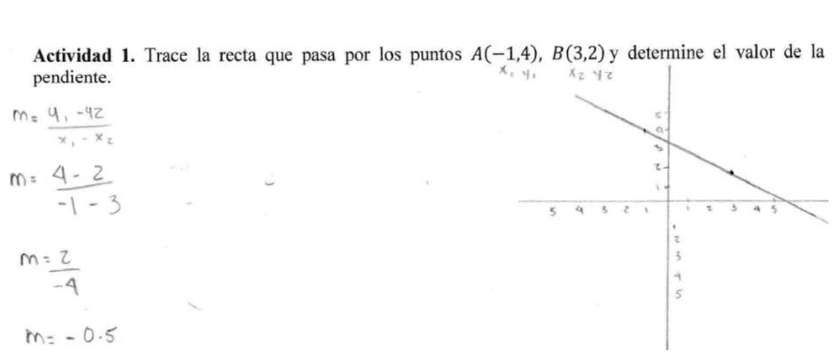
RESULTS

The theoretical framework of Nagle and Moore-Russo (2013) and the SPUR model were employed for the data analysis.

Below are examples of the conceptualizations of the slope of some students whose answers stood out for being more similar to the analysis framework.

The conceptualization of slope as a constant ratio

It was found that 67% of the total participating students show a conceptualization of slope as a constant ratio. From this percentage, 80% showed an analytic approach through a procedural emphasis associated with determining a change in variable y in relation to a change in variable x. This result is exemplified in the production of student E33, which inferred the corresponding slope value and presented the considered line's graphical representation but did not relate the two corresponding approaches (visual and analytic).

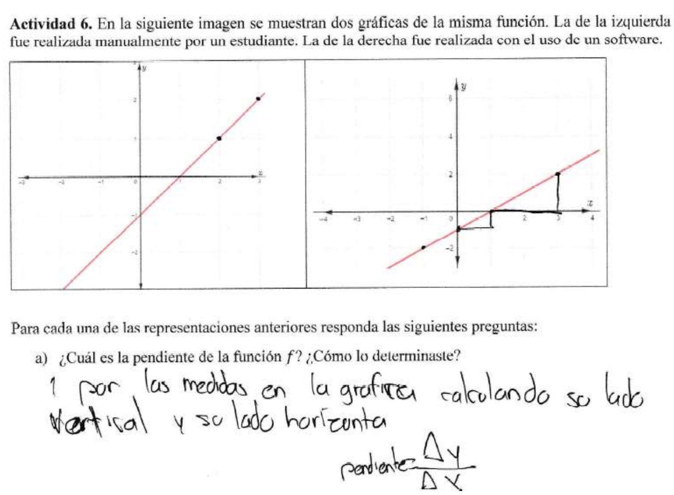


Task instruction: Draw the line that touches the coordinates A(-1,4), B(3,2) and find the slope value.

Figure 1. Response of participant E33 in Task 1.

This type of performance is reflected in most of the productions and is accentuated when observing the responses of Task 2, a task which demands the slope value by a graphical representation. The aim is to identify whether the student can solve this through a visual approach or needs to transform the task to an analytic approach.

On the other hand, student E11 (figure 2) is the only one showing a constant ratio conceptualization with a visual approach and conceptual emphasis, but only when making task 6 subsection (a).



Task 6. Instructions: The following image shows two graphs of the same function. The left-sided graph was manually drawn by a student. The right-sided graph was made employing software.

Answer the following questions for each of the previous representations:

a) Which is the slope of function *f*? How did you infer this?

Student's response: " 1^0 by measurements on the graph calculating its vertical side and its horizontal side".

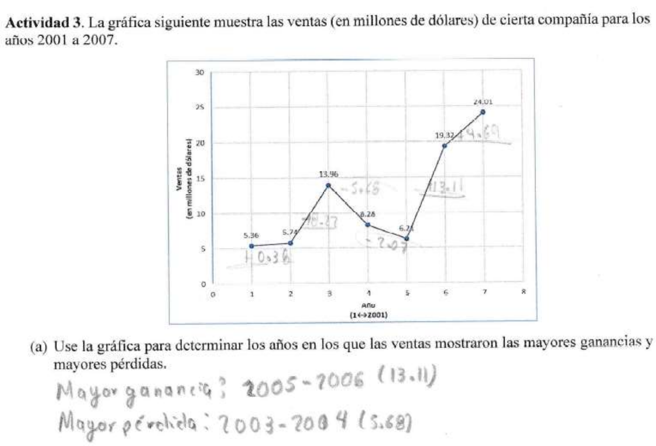
Figure 2. Response of student E11 in task 6(a)

As figure 2 shows, the student infers the slope value using a visual approach by drawing a pair of similar triangle rectangles that allow him to infer the slope value. His explanation even mentions vertical and horizontal change. Even though the student pointed out a pair of points in the left-sided graph, this relation was not displayed; in this case, this could be conditioned to the wording of the instruction when it mentions that it relates to the same function.

The conceptualization of the slope as a behavior indicator

The design of task 3 allowed the student to exhibit the conceptualization of the slope as a behavior indicator. According to our theoretical framework, this aspect relates the positive slope with the increment of the function, the negative slope with decrement and a zero slope with horizontal lines. According to the task, only 60% of the students showed part of this conceptualization, corresponding to a visual approach. Nevertheless, the remaining 40% of responses were possibly biased by the question's wording since it was stated in terms of higher and lower points of the graph and not in terms of the relation with the slope.

A standard answer made by the students can be seen in the case of student E7, which shows the correct answer (see figure 3). Also, the slope value can be observed for each pair of consecutive years.



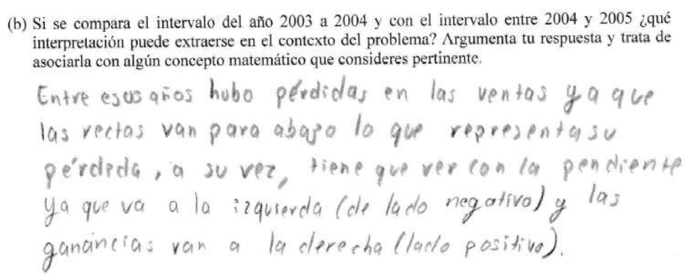
Task 3 Instructions: The following graph shows a company's sales (in million dollars) for 2001-2007.

(a) Use the graph to determine the years where the sales showed the highest earnings and the most significant losses.

Student's response: "Greater gain (2005-2006, slope value 13.11), greater loss (2003-2004, slope value 5.68)"

Figure 3. Response of student E7 in task 3(a).

These results show a conceptualization of the slope as a behavior indicator since its inference is not based on the highest or lowest points on the graph; instead, it is inferred from the slope value shown in the students' response. This claim can be reinforced with the student's E7 answer of subsection (b) of this task (see figure 4), where the answers are associated with the slope value.


(b) Si se compara el intervalo del año 2003 a 2004 y con el intervalo entre 2004 y 2005 ¿qué interpretación puede extraerse en el contexto del problema? Argumenta tu respuesta y trata de asociarla con algún concepto matemático que consideres pertinente.

Entre esos años hubo pérdidas en las ventas ya que las rectas van para abajo lo que representa su pérdida, a su vez, tiene que ver con la pendiente ya que va a la izquierda (de lado negativo) y las ganancias van a la derecha (lado positivo).

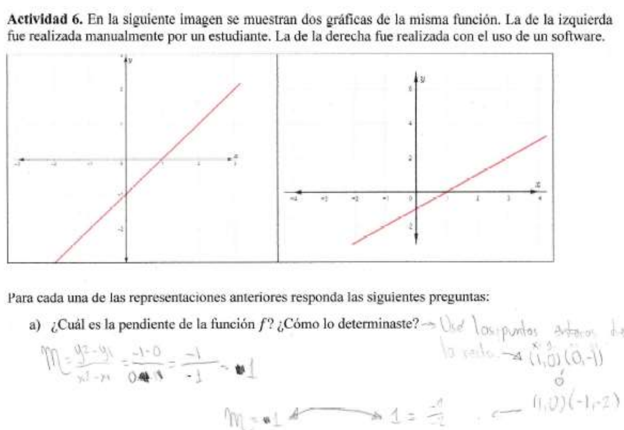
Task instructions: If you compare the 2003 - 2004 interval with the 2004 - 2005 interval, what interpretation can be drawn from the problem's context? Justify your answer and try to relate it to a mathematical concept of your consideration.

Student's response: "Between those years, there were losses in sales as the lines trend downwards, representing their decline. This is also related to the slope, as it goes to the left (negative side) for losses, and to the right (positive side) for profits."

Figure 4. The answer of student E7 in task 3(b).

The trigonometric conceptualization of the slope

This conceptualization was assessed in several stages of the evaluation: for example, in students' responses to subsection (b) of task 2 (22%), in task 5 (17%), or subsections (b) (28%) and (c) (32%) of task 6. In all of these tasks, the situation demands a trigonometric conceptualization associated with the slope, such as the angle's tangent of the line and the horizontal axis.



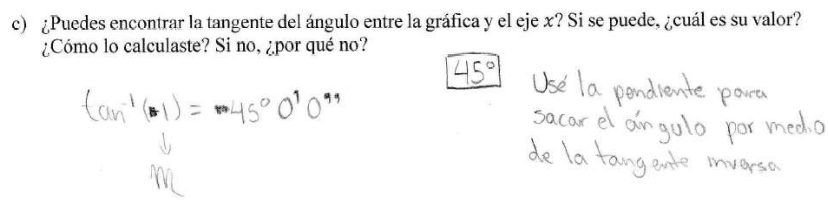
Task 6. Instructions: The following image shows two graphs of the same function. The left-sided graph was manually drawn by a student. The right-sided graph was made employing software.
Answer the following questions for each of the previous representations:

a) Which is the slope of function *f*? How did you infer this?

Student's response: "I used the integer points on the straight line".

Figure 5. Response of student E25 in task 6(a).

While, on average, 32% of the students showed some relations with this conceptualization, they could only present an analytic approach with a procedural emphasis rather than a conceptual one. A representative example of this type of response can be seen in student E25 (see figure 5). This student transformed the given task into an analytic approach by identifying a pair of points, in each case, to obtain the slope value. Nevertheless, when asking for the angle's inclination value in subsection (c), the student's response was restricted to an analytic one (see figure 6).

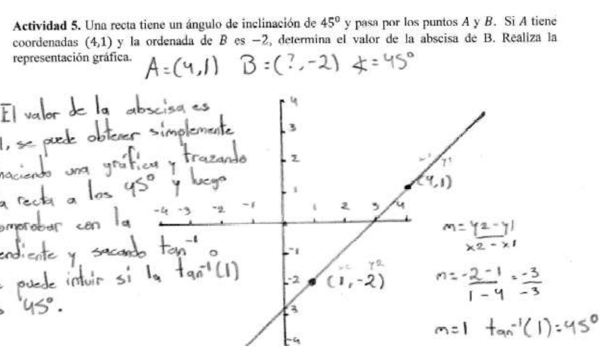


c) Can you find the angle's tangent between the graph and the *x*-axis? If it is possible to find it, which is the value? How did you calculate it? If it is not possible, state why.

Student's response: "I used the slope to calculate the angle using the inverse tangent".

Figure 6. Response of student E25 in task 6(c).

On the other hand, only 11% of the students showed the notion of trigonometric conceptualization from a visual approach with a conceptual emphasis. For example, in task 5 (see figure 7), E23 shows that the angle created by the line and the horizontal axis determines the sought relation and offers hints of the student's ability to infer the equivalence between the angle's tangent and the slope.



Task 5 Instructions: A line has an inclination angle of 45° and touches points *A* and *B*. If *A* has coordinates (4,1) and the ordinate of *B* is -2, infer the value of the abscissa of *B*. Draw the graphical representation.

Student's response: "The value of abscissa is 1, You can obtain it simply by graphing and drawing a line at a 45° angle and then you can verify it by checking the slope and calculating the tan to intuitively determine if the $\tan^{-1}$ (1) is 45°".

Figure 7. Response of student E23 in task 5

Considerations about the SPUR model's dimensions

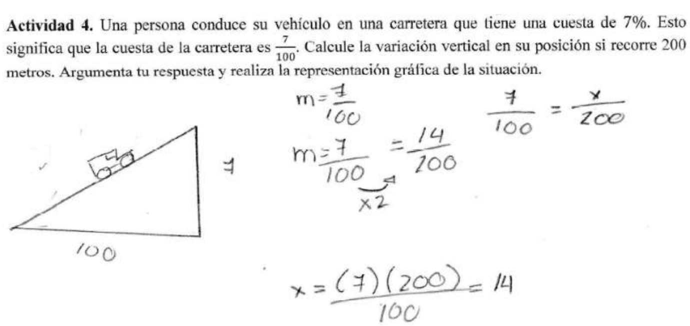
The SPUR model was included in this article as a theoretical framework that allowed us to observe a diversified evaluation of the slope conceptualizations. This multidimensional evaluation provided the conditions to show the connection level between the conceptualizations associated with the slope.

Another critical aspect is that these dimensions can suggest the type of activities that must be proposed in the classroom to encourage an articulated vision of the concept. For example, more than 90% of the participating students showed the *Skills* (S) and *Representations* (R) dimensions. However, in the case of tasks where more than one conceptualization was needed, like *Properties* (P) and *Uses* (U) dimensions, students had lower performance.

The evaluation of the *Uses* (U) dimension was categorized only for the constant ratio (tasks 4 and 7) and behavior indicator (task 3) conceptualizations. In the case of Tasks 4 and 7, their design did not focus on any visual or analytic approaches, and none of the procedural or conceptual emphasis was required. These tasks' resolution process requires an articulated vision of the concept in all its aspects, and its depth may be evaluated with the given responses.

The *Uses* (U) dimension of the SPUR model represents a vital referent to explore the level of integration of the different slope conceptualizations and the weaknesses or opportunity areas that can be developed with the students in the classroom.

For example, only 71% of the students showed a correct result in task 4. Nevertheless, their reasonings centered on a missing value task and not on visual or analytic reasoning of conceptual emphasis. It could be the case that the nature of the formulation of the problem may have caused this effect. The model's dimensions may contribute to the clarification of these aspects or also provide a broader picture of the event. A response example shown by most students can be seen in figure 8.



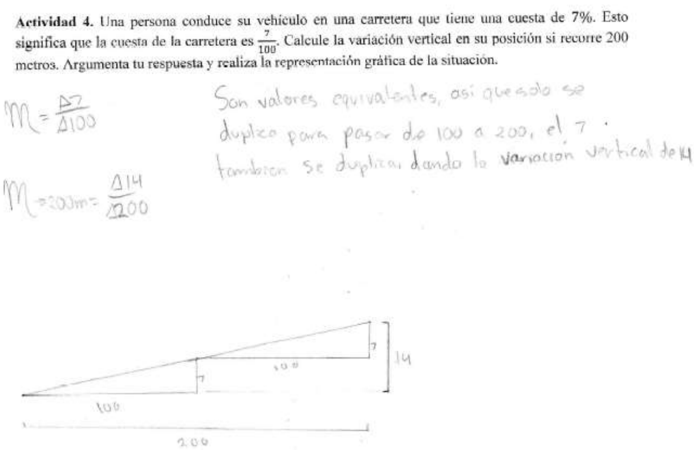
Task 4 Instructions: A person drives her vehicle on a highway that has a 7% slope. This means that the highway slope is $\frac{7}{100}$. Calculate the vertical variation in its position if she travels 200 meters. Justify your answer and draw a graphical representation of the situation.

Note. Obtention of the slope through the calculation of the missing value.

Figure 8. Response of student E15 in task 4

In the production of student E15, it is possible to observe that the student makes the corresponding calculation but does not show an articulation with the conceptual reasoning of the slope since the provided solution invokes an algebraic notion of a different nature that also allows the participant to express the correct answer.

Conversely, figure 9 illustrates a type of reasoning more related to the slope concept for the same task. In this case, student E25 alludes to the relation between the corresponding increments and shows a visual and conceptual approach in the answer.



Task 4 Instructions: A person drives her vehicle on a highway that has a 7% slope. This means that the highway slope is $\frac{7}{100}$. Calculate the vertical variation in its position if she travels 200 meters. Justify your answer and draw a graphical representation of the situation.

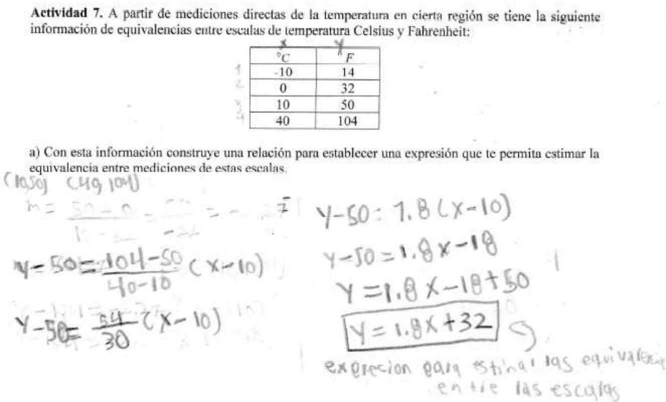
Student's response: “They are equivalent values, so it doubles to go from 100 to 200, the 7 also doubles resulting in a vertical variation of 14”.

Figure 9. Response of student E25 in task 4

The previous contrast allows us to highlight the strength of the SPUR model by offering a more detailed view of the skills and deficiencies with the shown dimensions. To reinforce this idea, one could reflect that, in a traditional evaluation test, any student showing the right numerical answer would have a correct answer, regardless of how the answer was obtained. On the opposite, according to the examples of the two previous cases, only student E25 would show the *Properties* dimension (in terms of the SPUR model), which can be associated with an understanding level of conceptual nature.

Finally, results show that, in the case of task 7, only 17% of the students outlined some reasonings associated with the task, but none of them effectively finished the task. The task problem focused on the SPUR model's *Uses* (U) dimension. The student had to connect the slope with its rate of change conceptualization between two covarying quantities. This task implied an analytic approach and conceptual emphasis that requires a global understanding of the slope.

Although student E28 was able to build the expression that models the situation, it is impossible to confirm that he conceptualizes the slope as a rate of change (see figure 10) of two covarying quantities since his developments do not make this relation explicit.


Actividad 7. A partir de mediciones directas de la temperatura en cierta región se tiene la siguiente información de equivalencias entre escalas de temperatura Celsius y Fahrenheit:

°C	F
-10	14
0	32
10	50
40	104

a) Con esta información construye una relación para establecer una expresión que te permita estimar la equivalencia entre mediciones de estas escalas.

Task 7 Instructions: given the direct measurements of the temperature of a particular region, the following information on equivalences between Celsius and Fahrenheit temperature scales was collected.

a) Consider the previous information to build a relation that defines an expression that allows you to estimate the equivalence between the measurements of these scales.

Student's response: "Expression to estimate equivalents between scales ($y = 1.8x + 32$)".

Figure 10. Response of student E28 in task 7(a).

DISCUSSION

The results found in this study coincide with those reported by Thompson (1994); Stump, (1999, 2001); Carlson et al., (2002); Lobato & Siebert, (2002); Moore-Russo, et al., (2011) regarding the lack of connection between the different conceptualizations of the slope, which is evident in the low percentages obtained by students in tasks of conceptual and analytical emphasis, as well as in the *Properties* dimension of the SPUR model.

Another element to highlight is that the students showed poor performance in tasks associated with the trigonometric conception of the slope. According to Azcárate (1992); Stump, (1999) secondary school teachers showed a poor trigonometric conceptualization of the slope, which could be associated with the weak mastery of this conceptualization by the students in this study.

Finally, our work allows us to show how the Nagle and Moore-Russo (2013) framework in conjunction with the dimensions of the SPUR model can help promote connections between the various conceptualizations of slope.

CONCLUSIONS AND RECOMMENDATIONS

The evaluation instrument, designed according to the SPUR model dimensions, gave us a more global view of the student's understanding of the slope conceptualizations and their articulations.

The implementation of this instrument revealed that students possess strong conceptualizations of the slope as a constant ratio and as a trigonometric conception but are more marked by a procedural emphasis than conceptual and correspond to the *Skills* (S) dimension.

While students' works showed a frequent use of the *Representations* (R) dimension, this dimension was not effectively employed in solving the tasks that promoted the *Uses* dimension (U). In particular, the *Uses* dimension requires integration with the other dimensions (*Skills, Representations and Properties*). Concerning this matter, if there is no promotion of activities related to the *Uses* dimension (U) in the classroom, students will have fewer opportunities to acquire a more integrated vision between the different conceptualizations of the slope.

In agreement with this, it is possible to highlight that the isolated conceptualizations of the slope allow the resolution of specific tasks of procedural nature to some extent. However, this is not the case for tasks of conceptual nature since they generally require a multifaceted view of the slope. This notion is in line with the claims by Nagle and Moore-Russo (2013), which emphasize that, in order to achieve a global view of the concept, the creation of a network of conceptualizations of the slope is required.

The results of this project represent a proposal for evaluating the slope through the use of instruments that demand an integrated understanding of the slope with all its conceptualizations. At the same time, it can also inform the teacher about the possible changes needed to reach this purpose. Based on the results obtained in this study and inspired by the reflections expressed by Thompson and Kaur (2011), we propose some recommendations and suggestions that may be useful to design the instruction and evaluation of the mathematics class in general, and in particular, when it comes to the concept of slope.

1) Teachers could use the multidimensional approach SPUR to design activities that allow them to identify students who do not yet adequately handle mathematical content. Knowing more about students' conceptual understanding can promote instructional design that is more aligned with content assessment.
2) It is also advisable to use diagnostic instruments based on the SPUR model, which allows mathematics teachers to know in more depth the knowledge, skills, and representation capabilities of their students.
3) We consider that SPUR dimensions can suggest the type of activities that must be proposed in the classroom to encourage an articulated vision of the concept.

In another aspect, some limitations of this study include:

1) Improve the design of the activities so that they more deeply reflect the conceptual and analytical emphasis of the students. This is because activities such as 1, 2 and 6 may not have been the most appropriate to promote thinking with a conceptual and analytical emphasis and, on the contrary, encourage procedural strategies.
2) In addition to the previous point, it should be mentioned that the responses shown by students may be conditioned and limited by the type of activities. Although they were

designed with opportunities to showcase the conceptual emphasis and *Properties* dimension of the SPUR model, a better design is needed to prevent students from seeking quick answers that do not allow for conceptual reflection.

3) The designed test was applied as a direct evaluation of the concept of slope. Although this way allows us to observe a general state of students' knowledge. It is necessary to complement it with other types of more diverse approaches that include iterative follow-ups, clinical interviews, periods of exploration, replication, etc., which together can reveal in a more profound way the state of knowledge of the students. However, it is emphasized that the use of the SPUR model allowed, in part, to better explore students' conceptualizations of the slope.
4) The study only covers three of the five conceptualizations of the slope concept proposed by Moore-Russo et al. (2011), although it is noted that these were selected according to the student level of the study group.

Finally, it is highlighted that this work constitutes a first approach to introduce these results in the classroom and that the proposal of activities must broaden its characterization with the aim of systematizing the elaboration of evaluation instruments that reflect the level of construction of the network of conceptualizations of the slope.

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APPENDIX

Assessment instrument

Name: ____

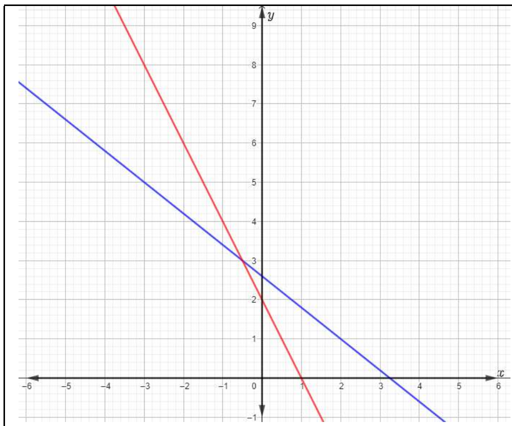
Task 1. Draw the line that touches the coordinates $A(-1,4)$, $B(3,2)$ and find the slope value.

Task 2. Use the following graph to determine

a) The slope value for each of the given lines

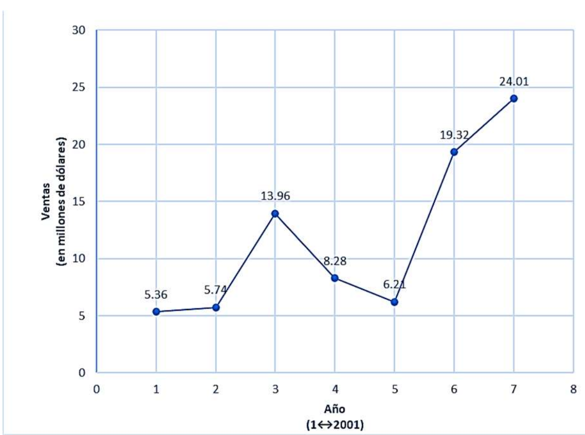
b) According to subsection (a), respond: which of the two lines has the largest inclination angle?

Justify your answer.



Work space

Task 3. The following graph shows a company's sales (in million dollars) for 2001-2007.



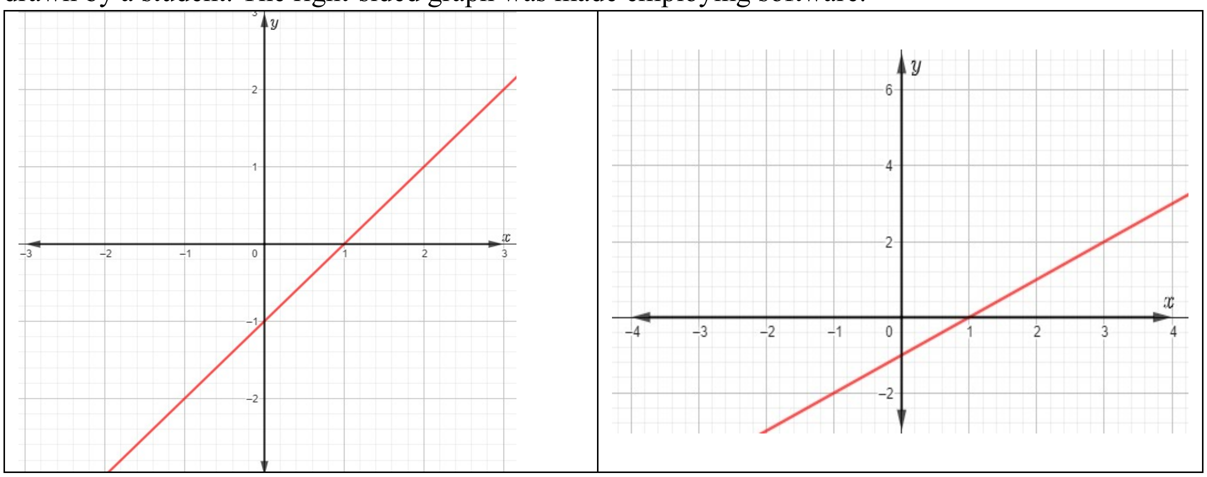
(a) Use the graph to determine the years where the sales showed the highest earnings and the most significant losses.

(b) If you compare the 2003 - 2004 interval with the 2004 - 2005 interval, what interpretation can be drawn from the problem's context? Justify your answer and try to relate it to a mathematical concept of your consideration.

Task 4. A person drives her vehicle on a highway that has a 7% slope. This means that the highway slope is $\frac{7}{100}$. Calculate the vertical variation in its position if she travels 200 meters. Justify your answer and draw a graphical representation of the situation.

Task 5. A line has an inclination angle of 45^0 and touches points A and B. If A has coordinates $(4,1)$ and the ordinate of B is -2, infer the value of the abscissa of B. Draw the graphical representation.

Task 6. The following image shows two graphs of the same function. The left-sided graph was manually drawn by a student. The right-sided graph was made employing software.



Answer the following questions for each of the previous representations:

a) Which is the slope of function *f*? How did you infer this?
b) Does the graph of *f* bisect the angle between the axes? How do you know?
c) Can you find the angle's tangent between the graph and the x-axis? If it is possible to find it, which is the value? How did you calculate it? If it is not possible, state why.

Task 7. Given the direct measurements of the temperature of a particular region, the following information on equivalences between Celsius and Fahrenheit temperature scales was collected.

^0C	^0F
-10	14
0	32
10	50
40	104

a) Consider the previous information to build a relation that defines an expression that allows you to estimate the equivalence between the measurements of these scales.
b) Determine the measurement at which the temperature is equal on both scales.
c) For which Fahrenheit scale measurement is twice the Celsius scale measurement.

Effects of Differentiated Instruction in Flipped Classrooms on Students' Mastery Level and Performance in Quadratic Equations

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Abstract: This study employed student choice and tiered worksheets as strategies of differentiated instruction on Quadratic Equations in addressing students' non-compliance with assignments in a flipped classroom. In each lesson, students choose among the instructional materials with guide questions to assist them in focusing on key areas during the asynchronous activities as homework. Tiered worksheets were administered in face-to-face classes based on students' readiness as reflected in pre-assessment results. Data from tiered worksheets show students' mastery levels and student engagement in online class instruction and in-class tasks. Additionally, there is a significant difference between pre-assessment and summative assessment percentage scores with a substantial effect size, implying improved student performance in solving quadratic equations.

Keywords: differentiated instruction, flipped classroom, tiered worksheets, quadratic equations

INTRODUCTION

With the formal education setup, most teachers resort to one-size-fits-all instruction in their heterogeneous classes. Some of the learning tasks and outputs tend to be too easy for exceptional students, while these can be challenging for academically struggling students. In a typical heterogeneous classroom, the teacher often instructs the intermediate learners, rather than the exceptional students (Newman, 2009) since students are treated by the teacher as if they are all the same (Valiendes & Neophytou, 2018). Consequently, students who excel academically may feel bored with easy tasks and be shortchanged of their appropriate learning competency. On the other hand, struggling students may find themselves lost if the learning tasks are difficult, leading to disadvantageous learning misconceptions. Differentiating instruction for struggling learners helps strengthen their basic skills, working memory, and fact retrieval. Similarly, differentiating instruction for high-performing students provides exemptions and cultivates their higher-order thinking skills (Laud et al., 2011). It is argued that the use of differentiated instruction as a teaching approach enables students to understand concepts efficiently by giving them appropriate learning tasks based on students' mastery level.

Differentiated instruction is an innovative and effective teaching approach (Tomlinson, 2000; Gregory & Chapman, 2013; Valiendes & Neophytou, 2018). Benjamin (2014) added that differentiated instruction refers to various classroom practices that aim to accommodate the learning differences among learners and involve balancing between content and learning competencies expected on assessments and varied activities to maximize learning.

Differentiated classrooms should be responsive to every student's learning style because students achieve better when instruction is appropriate to student's intelligence and learning preferences (Tomlinson et al., 2003). Identifying the individual student's mastery level can be applied so that the high achievers are not shortchanged by providing thought-provoking tasks. Similarly, the low-performing students should receive additional support and further remediation. Teachers are encouraged to support struggling learners through scaffolding to ensure that all students acquire the desired learning competency regardless of their cognitive skills. Especially since concepts in mathematics are built one after another; consequently, learning is more of a linear process, underscoring the importance of fundamental concepts.

Tomlinson (2000; 2005), Gregory and Chapman (2013), and Newman (2009) argue that not all students are the same and that they differ in terms of how they think and learn conceptual knowledge inside the classroom. Further, they asserted that according to existing literature, studies show heterogeneous classrooms are effective in social and academic aspects but have yet to prove the needs of all learners are being addressed. However, when teachers practice differentiation, heterogeneous classrooms are most effective (Cannon, 2017). Differentiated instruction is designed to assess students and tailor instruction to fit their varying needs, thus maximizing student's potential (Santangelo & Tomlinson, 2012). In addition, students' interest and motivation to learn are enhanced due to effective differentiation (Villamor & Lapinid, 2022), as this teaching approach enables teachers to engage all students in learning. Tomlinson (2000, p.25) states, "differentiated instruction is an approach by which teaching promotes high level and powerful curriculum for all students but varies the level of teacher support, task complexity, pacing and avenues to learning based on student readiness, interest, and learning profile."

Theoretical Basis of Differentiated Instruction

Differentiated Instruction as an approach to student diversity in the classroom is anchored on Vygotsky's Social Development Theory, specifically focused on the key concepts of Zone of Proximal Development and Scaffolding. Vygotsky's Zone of Proximal Development (ZPD) helps the students maximize their academic excellence through proper scaffolding of the "more-knowledgeable-others" (MKO). As mentioned by Eun (2017, p.3), Vygotsky (1978) defined "ZPD as the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem-solving under guidance or in collaboration with more capable peers." If the learning task is outside the student's comfort level but can be completed with the supervision of an adult, then it enhances student learning (Newman, 2009). An increase in knowledge and understanding to the fullest extent possible can be achieved if teachers allow students to work on appropriate tasks corresponding to their level within their proximal development (Morgan, 2014).

Conceptual Framework

In the study by He et al. (2016), the open-ended student responses revealed that one of the serious implementation issues of the flipped classroom model is the non-compliance of the students with the pre-class studying, which hinders the flipped classroom model from attaining its goals and objectives in learning. Hence, to address the aforementioned weakness of this instructional technique, this study incorporated "differentiated instruction" in conjunction with the "flipped classroom model" to ensure that activities are based on students' learning preferences and that students are accountable for their own learning. Furthermore, in order to promote active learning and productive use of knowledge, the instructional materials uploaded in the pre-class study or the online asynchronous class instruction and in-class activities such as worksheets and practice exercises must be integrated and complement each other (He et al., 2016). With this premise, we posit there is meaningful acquisition of learning mathematical concepts because students must do advanced preparations prior to in-class activities.

Studies which concern teacher's way of catering to students' individual needs in a blended learning environment are still scarce (Boelens et al., 2018). Further, Boelens et al. (2018) argued that it is vital for educators to have a clear stance as regards proactively planning differentiation strategies and simultaneously responding to students' diverse needs in the context of blended learning. In this study, the researchers differentiate the content and process of learning based on students' readiness and interest as they go through the pre-class readings of instructional materials, video watching, and tiered worksheets. The study includes ways to implement the differentiation process so that mathematics teachers are given ideas to operationalize this approach in their classes. Aside from the differentiation in allowing students to choose among the varied online class tasks, differentiation in content and process is also operationalized through in-class activities using the tiered worksheets. The recall part in the in-class tiered worksheets contains examples and essential concepts which scaffold students while answering the worksheets. All those aim to cement students' knowledge further, rectify their errors and misconceptions, and maximize student engagement.

Problem Statement

The study's main purpose is to investigate the effectiveness of differentiated instruction in a flipped mathematics classroom where students vary in cognitive abilities on student performance in Quadratic Equations. Specifically, the following questions guided the conduct of the study:

1. How do students' mastery levels differ
   a. between pre-assessments and final tiered worksheets in each lesson and
   b. across tiered worksheets within each lesson?
2. Is there a significant difference in students' performance before and after the intervention?

LITERATURE REVIEW

Forms of Differentiation

Content refers to the subject matter students must learn inside the classroom (Coubergs et al., 2017) or the knowledge and skills students need to obtain to reach mastery (Boelens et al., 2018). Instruction can be differentiated in the content based on the student's interests, readiness, or prior knowledge. The first step in differentiating content is deciding which learning competencies and standards are targeted (Gregory & Chapman, 2013). Differentiating content requires different means of presenting the lesson to the students. Teachers should plan instructional materials that are exciting but challenging and timely for learners to ensure that the targeted learning competencies and standards based on the curriculum are just right on the student's mastery level (Gregory & Chapman, 2013).

On the other hand, differentiation can also be planned according to the process, or the learning activities and tasks created based on the skills of the students, level of readiness, and preferred learning style (Taylor, 2015) or how students acquire new skills learned from the content (Boelens et al., 2018). Taylor (2015) further contends that when teachers use differentiation in the classroom, they can vary the level of complexity of the learning materials (e.g., below target, on target, and above target). Coubergs et al. (2017) define a *process* as the learning track of the students, which is associated with the learning tasks and activities aligned to the content objectives. It can be in the form of varied activities such as administering worksheets to students with different difficulty levels where students practice individually, by pairs, or by groups and make sense of what they learned in the content.

Lastly, instruction can also be differentiated according to the product, wherein teachers incorporate different assessments that allow students to showcase their creativity and manifest their learning in their own ways. "Product is the learning outcomes and achievements, based on which students can prove their accomplished goals" (Tomlinson & Imbeau, 2010, as cited by Coubergs et al., 2017). It refers to how students demonstrate their knowledge (Boelens et al., 2018), tantamount to the outcome of teaching and learning processes such as summative tests, research projects, topic presentations, simulations, and portfolios.

The Role of Flipped Classroom in Differentiated Instruction

With the aim to offer more flexibility in learning in terms of time and place to a group of diverse learners, blended learning is often integrated into the classroom (Boelens et al., 2018). *Blended learning* is an instructional approach that combines online and face-to-face or in-class instruction (Boelens et al., 2018). One such model of blended learning is the flipped classroom. As defined by Altemueller and Lindquist (2017), the flipped classroom reverses traditional teaching methods, aiming to deliver the instruction outside of the class, devoting the class time to other instructional tasks such as problem-solving, practice exercises, and hands-on activities.

Research says the flipped classroom model is a promising instructional technique because it frees up class time for more in-depth discussion through in-class activities such as supervised collaborative works and mastery exercises (He et al., 2016; Chen et al., 2018; Altemueller & Lindquist, 2017). This instructional model maximizes student learning through access to curated instructional materials anytime and anywhere, allowing the students more time to understand the content and more ways to support self-learning (Chen et al., 2018). The flipped classroom also enables students to have increased learning, motivation, and engagement (Rutkienė et al., 2022). The model benefits the struggling students the most because it allows them to pause, replay, and go back to the parts of the online learning videos. The same is true for reading instructional material or listening to a podcast. On the other hand, students who excel can quickly go to the challenging learning competencies and have mastery of the content knowledge.

He et al. (2016) mentioned in their study that the flipped classroom model consistently benefitted students with varied learning preferences. As a form of differentiated instruction, students may be provided online resources in different forms, such as videos, podcasts, and websites of instructional materials, and be given the liberty to choose which form of media they prefer based on their learning styles (visual, auditory, or kinesthetics).

The Role of Assessment

Effective and successful differentiation relies on the role of assessment. The data on different types of assessment employed by the teacher make informed decisions on how to help students succeed academically. Unfortunately, many teachers, students, and parents are more concerned about the scores obtained in the summative and achievement tests. Hence, students must memorize many facts and formulas such that authentic learning is often neglected. "These types of assessment teach students to memorize and encourage teachers to teach to the test, not for student understanding" (Newman, 2009, p.11). To further support this argument, Cannon (2017) said that the core of instruction must focus on meeting the needs of diverse learners in the classroom to reach for the standards and not teaching for the examination.

Tomlinson (2005) stressed that formative or ongoing assessments should not all be graded because their purpose is to help both the teacher and the student see how learning is developing and what adjustments are necessary in the instruction to ensure that learning stays on track. Assessment must be considered as a tool to underscore the strengths and weaknesses of the students so that proper remediation can be administered to rectify diagnosed misconceptions and help learners move to more advanced levels of understanding (Gregory & Chapman, 2013).

Cannon (2017) suggests determining the different mastery levels of students at the onset of differentiation to make quality instructional decisions. Particularly as a form of differentiated instruction in terms of process, a preassessment may be administered at the onset to gauge the existing knowledge and readiness of the students so that an appropriate corresponding instruction based on students' mastery level may be provided to students in a heterogeneous class. To support this, Connor et al. (2017) used assessment data to group students with common learning needs and provided individualized instruction to improve student achievement. Individualized instruction may be in tiered learning tasks administered as ongoing and formative assessments as differentiated instruction since continual assessment is an essential foundation for effective differentiation (Tomlinson, 2005; Santangelo & Tomlinson, 2012). Using tiered learning tasks is one way to support those who are low-achieving and simultaneously challenge those who are high-achieving (Laud et al., 2011).

Tiered worksheets refer to the varied learning modules with different levels of complexity to cater to the needs of the individual learners concerning their mastery levels. Using tiered worksheets conforms with Deunk et al.'s (2018) concept of mastery learning as one of the many differentiation strategies. Mastery learning uses regular assessment to check whether the students have reached a particular ability level (Deunk et al., 2018). Moreover, Huebner (2010) espoused that students must be taught at the proper instructional level responsive to their cognitive levels through different pathways to be appropriately engaged since students' attention is attracted to the idea that the learning task seems worthwhile (Cannon, 2017). Valiendes and Neophytou (2018) mentioned that tiered activities take three phases of implementation, which are (1) creating an "on-level" task based on the standards expected by the curriculum, (2) adjusting the task to create a "below-level" activity for struggling students and in the same manner, (3) adjusting the task to create an "above-level" task for advanced students.

The majority of the studies concerning differentiated instruction used flexible groupings in either homogeneous or heterogeneous ability groupings. However, according to the meta-analysis conducted by Deunk et al. (2018), simply grouping students and placing them physically together does not ensure effective differentiated teaching. Thus, individual differentiated work can be used as a strategy to differentiate instruction in a mathematics classroom as an alternative to ensure student performance improvement and progress (Nechifor & Purcaru, 2017). As an example of this practice, Hapsari et al. (2018) used tiered assignments in which the teacher created sets of questions to be answered by the students individually with varying levels of difficulty, and students liked the tiered task option because items were just appropriate to their capabilities.

METHOD

The study utilized a descriptive quasi-experimental quantitative research design. Participants comprise an intact class of 46 Grade 9 – Mathematics students from a public secondary school in the Philippines. The class was chosen using purposive sampling with an intact heterogeneous classroom as a criterion. The research participants were briefed on the nature of the research, highlighting the rationale and importance of conducting the study to obtain their full cooperation. The researchers also provided the student-respondents the extent of their participation as stipulated in the informed consent form, and they were asked to indicate their willingness, affixing their signatures, to participate in the study. All 46 students agreed to participate in the study by submitting their parent's consent form.

The intervention was implemented for almost five weeks, and the covered topics in Grade 9-Mathematics are limited to quadratic equations in the following lessons: identifying quadratic equations and solving quadratic equations by extracting the square roots, factoring, completing the square, and quadratic formula. Solving quadratic equations follows specific steps but the procedures require deep conceptual understanding. Hence, the researchers selected videos and instructional materials that ensure students' conceptual understanding of the procedures of solving quadratic equations. Some of the procedures warrant clear conceptual understanding – for example, the use of factoring as a method warrants understanding of its application of the zero-product property; the use of completing the square requires students to have a deep understanding of what makes a perfect square trinomial; and the correct numerical coefficients of a quadratic equation should be that which is written in standard form before they can use the quadratic formula because the quadratic formula was derived from the standard form. These were further processed by eliciting students' conceptual understanding in face-to-face class discussions.

We used several strategies to operate differentiated instruction as Newman (2009) suggested. We listed tasks aligned to the learning goals and competencies. Students were provided a list of activities and when each should be accomplished. Adhering to the flipped classroom model, students received pre-instruction video links and learning websites. The online materials were carefully and meticulously curated to avoid too many choices (Brandt & Columba, 2022). These materials and in-person class assessments were developed to ensure they are aligned with the intended learning competencies. To ensure students follow the given tasks, a Facebook group was set up as a communication channel where guide questions corresponding to each lesson were provided for a focused viewing and reading and for students to raise questions.

In-class activities include the administration of a pre-assessment to determine students' readiness to check whether students engaged with the given online tasks. The pre-assessment is a key tool in this study, administered at the onset of in-person class. The pre-assessment, comprising eight items, progresses in difficulty: items 1 to 3 are easy (1 point each), items 4 to 6 are of average difficulty (2 points each), and items 7 to 8 pose greater challenges (3 points each), yielding a total of 15 points. The students' mastery levels (Beginning Mastery, Approaching Mastery, High Mastery) coined by Gregory and Chapman (2013) were adopted so that appropriate tiered learning worksheets correspond to their existing learning competencies.

The tiered worksheets were designed to match varying initial mastery levels based on the students' pre-assessment scores. Focused on quadratic equations and their solution methods, the tiered worksheets tailored to the three levels of Mastery students offer scaffolding and challenge as needed. Each worksheet contains three parts of increasing complexity, contributing to a potential 15 points per student. By adhering to a passing rate of 60% of the school, students transitioned from lower to higher-tiered worksheets, fostering individual pacing while feedback and monitoring remained integral. Allotting a maximum of one hour, these worksheets reflecting the Zone of Proximal Development (ZPD) concept support students' progression toward higher cognitive abilities in quadratic equations. After addressing queries, reinforcing learning, and addressing students' common mistakes, the cycle goes on following the same process for subsequent lessons.

Following the intervention, we administered a summative assessment to measure student learning on the tiered worksheet lessons. This 40-item test mirrored the structure of the five pre-assessments, comprising 15 easy questions in Part I (1 point each), 15 average questions in Part II (2 points each), and ten difficult questions in Part III (3 points each), totaling 75 points.

RESULTS

Mastery Levels based on the Pre-Assessment and their Tiered Worksheet Progress

Figure 1 presents the changes in students' mastery levels based on the pre-assessment and the final tiered worksheet in each lesson and the students' progress in their tiered worksheets across the lessons. The pre-assessment served as the tool in differentiating students' mastery levels whether they are on the "Beginning Mastery Level" (scores from 0% to 59%), "Approaching Mastery Level" (scores from 60% to 79%), or "High Level of Mastery" (scores from 80% to 100%). Students who are at the BM level based on the pre-assessment were given T1 to work on in the class, while those at the AM level skipped working on T1 and went on to work on T2, and those at the HM level skipped working on T1 and T2 and went on ahead to work on T3. Within the given in-person class time, a student can progress from a lower-tiered worksheet to the next tiered worksheet if they get 60% of the total score in the tiered worksheet.

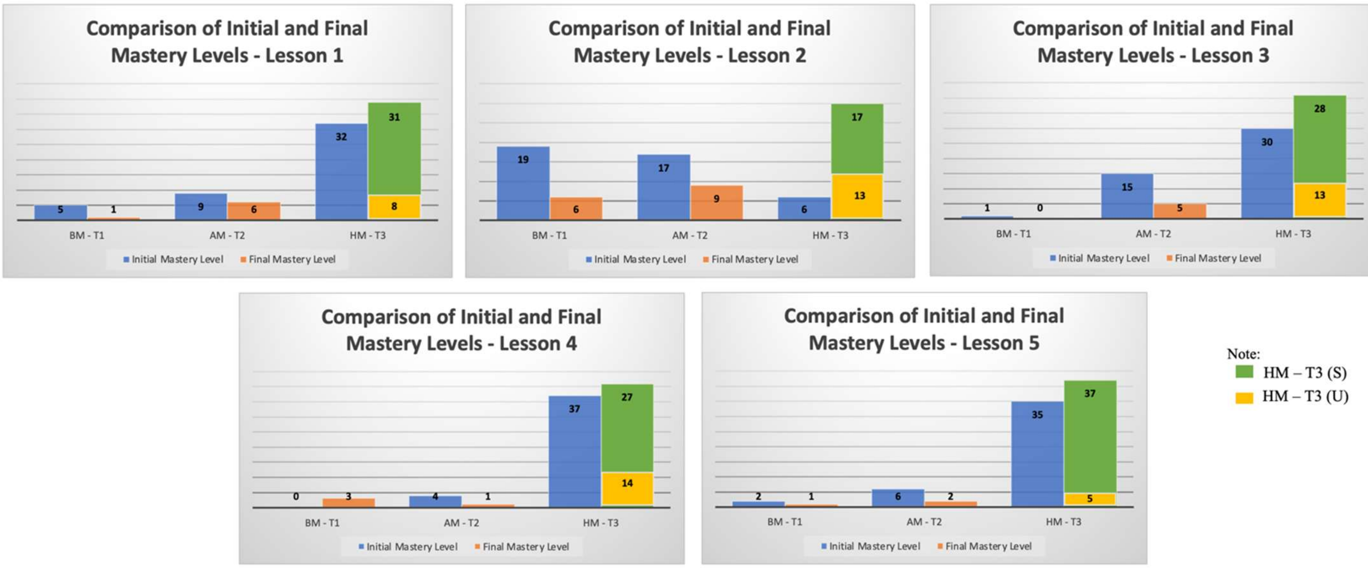

Figure 1. Graphical Representation of Students' Mastery Level at the Onset of In-Class Differentiated Instruction and their Final Tiered Worksheet Across Lessons

Legend:

BM – Beginning Mastery	T1 – Tiered Worksheet 1
AM – Approaching Mastery	T2 – Tiered Worksheet 2
HM – High Level of Mastery	T3 (U) – Tiered Worksheet 3 (Unsuccessful)
	T3 (S) – Tiered Worksheet 3 (Successful)

In Lesson 1 (Introduction to Quadratic Equations), out of 46 students, the majority of them (32 students) went straight to HM (T3) in their initial mastery level. Despite having some of the students in lower mastery levels (5 students in BM and 9 students in AM, respectively), they were able to move up to higher mastery levels in their tiered worksheet, which determined their final mastery level except for one student who remained in the same BM level. The increase of students who moved up to higher mastery levels shows that the options provided in the online class instruction and the recall part or the mini discussion contained on the front page of the tiered worksheet, which served as the scaffolding to the students really worked and were effective to increase student engagement and performance in tiered worksheets. The tiered worksheets helped students achieve optimal learning by immersing themselves in answering problems with varying difficulty levels. With this in mind, students did not get bored or frustrated because the worksheets were apt to their ability levels, and hence, their confidence in answering challenging questions but within their ZPD was enhanced as they went through the high-level worksheets. The scaffolding was paramount to increase their achievement in terms of tiered worksheet progress since they recalled concepts, which enabled them to cement their previous understanding gained from the online class instruction.

In Lesson 2 (Solving Quadratic Equations by Extracting the Square Roots), four students were unable to take the pre-assessment. Three of them came to the class the next day and were deemed regarded at the BM level and given T1 for the tiered worksheet. The majority of the students (19) were initially categorized under the BM compared with the AM (17) and HM (6) levels. Comparing it to the results in Lesson 1, the trend of the initial mastery levels of the students decreased.

When asked why they were unsuccessful in landing on an AM for T2 or HM for T3, compared to their initial mastery levels in the previous lesson, a student explained he needed help understanding the online mathematics videos in Extracting the Square Roots. Moreover, despite 45 students who saw (considered as “seen”) the list of the Online Class Instruction for Lesson 2 in the created Facebook group, some students still failed to participate in the given asynchronous tasks. Another student claimed that she prefers reading to watching videos. Students with lower mastery levels further clarified with the teacher if they could still watch the off-loaded mathematics video links and other online resource materials, albeit the concluded pre-assessment. According to them, they wanted to strive harder to land on a T3 level. The purpose of this was for them to still cope with the lesson's contents and consequently progress in their tiered worksheet level since they could not go through with the online class instruction, thus getting low scores in pre-assessment and being identified as T1 level. On the other hand, students who skipped lower-tiered worksheets also professed to go through the materials seriously more than once with the questions posted on Facebook as their guide.

Hence, in the final mastery level column for Lesson 2, the students who were initially classified in the lower levels were able to move up to higher mastery levels. From the initial 19 students, it went down to 6 students classified under BM as their final mastery level. Also, from the initial 17 students under the AM level, it decreased further to 9 students, while the initial six students under HM increased to 30 students constituting the T3 - U (13) and T3 – S levels (17). Since the teacher allowed the students to watch or read the online materials, the links of which were posted in the Facebook group, students were able to move up to higher levels. Students realized that they could still participate in the online class instruction intended and given the day before the administration of the tiered worksheet.

Likewise, looking at the results from the remaining Lessons 3, 4, and 5, the same increasing trend can be observed in the number of students who moved up to higher mastery levels, specifically in the T3 level. In Lesson 3, only one student was classified under BM, 15 in AM, and 30 under HM, respectively. For the final mastery levels under Lesson 3, the one student initially regarded as BM was able to move up to higher levels. Also, the original 15 students under AM became five in the final counting, and similarly, the initial 30 HM students increased further to 41 students, T3 - Unsuccessful Level (13) and T3 - Successful Level (28) combined.

The sudden increase in the frequency of higher mastery levels confirms that students studied harder for the pre-assessment because, according to them, landing on higher mastery levels encourages them to do better, and they feel that being classified as "High Level of Mastery" for them is some accomplishment. As one student shared in class,

> *"We do not want to be in low mastery levels because it will only mean we did not prepare or study for the in-class assessment. Besides, if we get lower scores, we have many worksheets ahead of us to answer. Actually, it motivates us when the teacher announces our mastery levels that's why we always look forward to answering the Tier 3 worksheet."*

The five and three absentees in Lessons 4 and 5 were still allowed to take their tiered worksheets for BM. The final mastery level column for Lessons 4 and 5 indicates that the majority of the students were already moving to higher mastery levels and were eager to be classified under HM so that they would be able to answer the T3. The number of HM in Lesson 4 increased from 37 students to 41 in the final counting, comprising the T3 - U (14) and T3 - S levels (27).

Correspondingly, the number of students under the HM in Lesson 5 also increased from 35 to 42 in the final counting, consisting of T3 - U level (5) and T3 - S level (37). Hence, the results showed that almost all students were getting used to this type of instruction under the online class and answering the tiered worksheets based on their ability level since the number of students classified under the T3 level increased throughout the intervention. This increasing trend confirms that students are engaged when they are given tasks appropriate to their mastery levels (Huebner, 2010; Laud et al., 2011). Additionally, students' academic progress improves if the individual learning tasks are personalized to their cognitive needs (Nechifor & Purcaru, 2017).

Students' Progress in Tiered Worksheets in each Lesson

Figure 2 illustrates the frequency distribution of the students in terms of their tiered worksheet progress. Specifically, this shows the students' particular movement with regard to their initial to final tiered worksheet level in each lesson throughout the intervention. There is a more significant percentage of the students who have movements (1-level, 2-level, or 3-level movements combined) than the students who did not have any movements or remained in their mastery level. Correspondingly, the percentages of the students who displayed movements from their initial mastery to higher mastery levels, 1-level and 2-level movement combined, throughout the intervention are 78.26%, 68.85%, 71.77%, 68.89%, and 93.29%. However, the percentages of the students who did not have any movement are 21.69%, 31.07%, 28.3%, 31.09%, and 6.66%.

Notice that only Lesson 5 (Solving Quadratic Equations by Quadratic Formula) has the lowest percentage of students who showed no movement in their mastery level. However, Lesson 2 (Solving Quadratic Equations by Extracting the Square Roots) and Lesson 4 (Solving Quadratic Equations by Completing the Square) have the highest percentage of students remaining in worksheet mastery levels. The reasons students gave why they remained in their initial mastery levels were that they did not have any internet loads, were busy, did not have a Wi-Fi connection, and needed help understanding the lesson from the online class instruction. Thus, the success of the flipped classroom depends on these factors, among others.

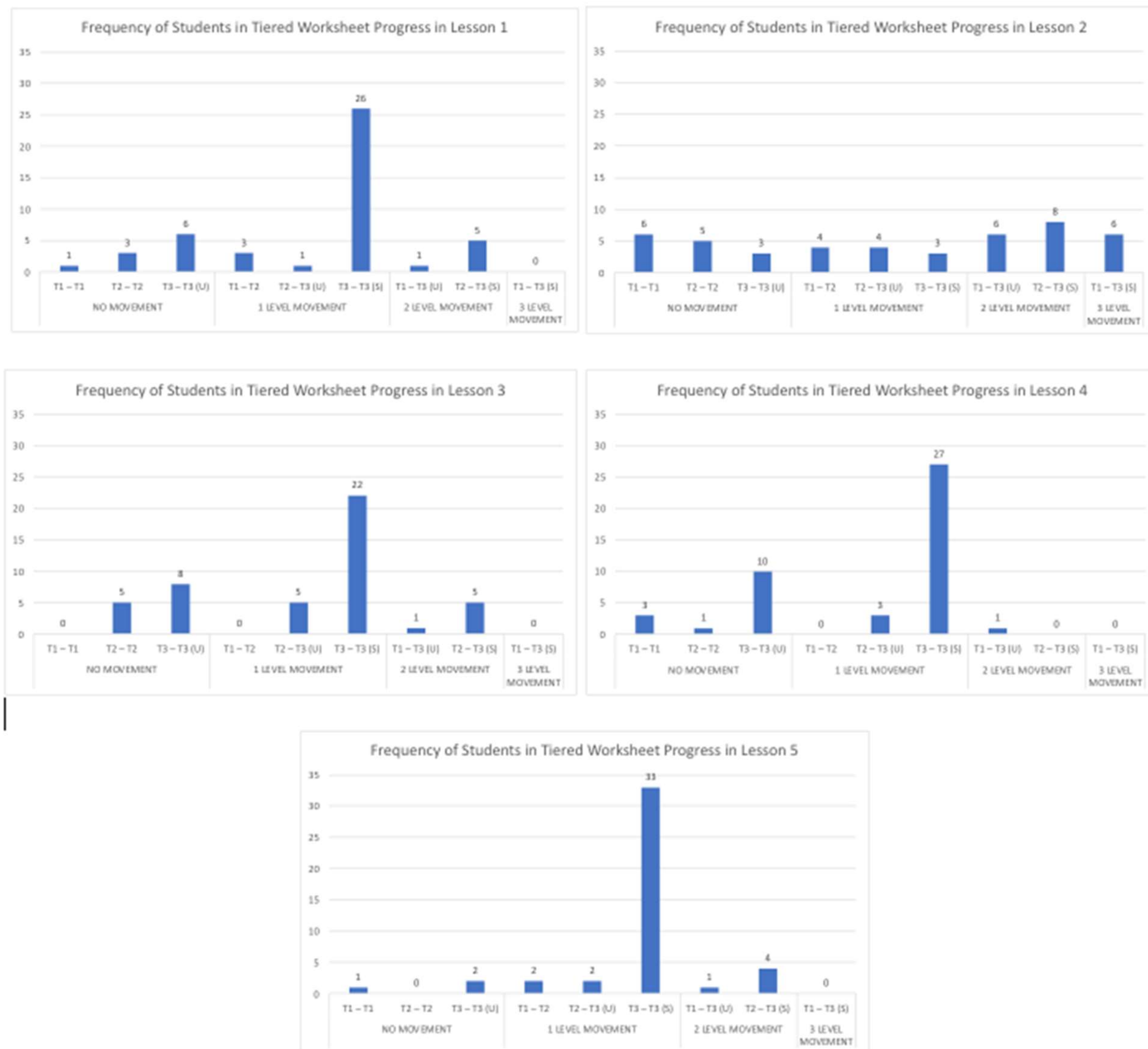

Figure 2. Frequency of Students in Tiered Worksheet Progress in each Lesson

Likewise, only Lesson 2 has the 3-level movement (13.3%). Students find it easier to extract the roots since the concepts regarding radicals were recently discussed prior to the conduct of the study based on the given curriculum. Further, it can be attributed to the nature of the topic since the process involved in Lesson 2 is more straightforward compared to the rest. Also, this can be due to time constraints since students were only given a one-hour period to accomplish the tiered worksheets. Since students under the T1 have three worksheets to answer for them to have 3-level movements, given the limited amount of time, they can only do at most 2-level movements. Hence, students initially in T1 rarely moved up to T3 – successful due to the lack of time, whereas, if the student was at T2 level at the onset, the student had a greater chance of reaching T3. Therefore, students were made to understand the importance of online class instruction to reach T3 - S. Moreover, it can be inferred that if students exert effort, they can progress to higher mastery levels and improve their performance.

In this regard, online class instruction was intended to address the limited face-to-face instruction time. Students can reach T3 (Unsuccessful or Successful), i.e., skip working on the lower-tiered worksheets if they get an AM or HM in their pre-assessment because they learn from working on the assigned asynchronous tasks during their independent learning preparation. Because of the limited time, rarely can one succeed, starting from the lower tier and achieving a higher tier level. Students can only successfully reach the higher-tiered worksheets if they study well in their online class instruction, focus on the learning materials, and follow the guide questions to answer the in-class worksheets quickly. Therefore, a flipped classroom was an excellent way to address the limited time for face-to-face instruction.

Additionally, some students struggle in Lesson 4 because they realize they must be proficient in completing the square. This process added layers of procedures, which consisted of applying the addition property of equality in coming up with a perfect square trinomial and factoring before extracting the roots to solve the quadratic equation. Most of the students were able to eventually move up from their initial tiered worksheet level because they were allowed to watch the online math videos in class, albeit reducing their time to accomplish their worksheets. With this in mind, students can comply with their assigned online tasks or risk not maximizing the in-class time because they must catch up with watching or reading the online resources.

Since learners inside the classroom were diverse, we also noticed some students relied on the scaffolding of the tiered worksheets since they preferred reading to watching videos. In contrast, others relied heavily on watching the learning videos. In a way, this validates the purpose of Differentiated Instruction in this study. Students who watched the videos because they disliked reading the printed detailed explanation of the worksheets. Others skipped the scaffolding part of the tiered worksheets since they had encountered it already from the online class instruction. At the same time, some opted to watch online mathematics videos to save time and directly answer the items in the tiered worksheets. Hence, this finding indicates that Differentiated Instruction in the flipped classroom context in this study effectively catered to students' preferred approach to learning mathematical concepts.

Students' Performance in the Pre-Assessment and Summative Assessment

Figure 3 shows the means of students' pre-assessment scores in each lesson. The highest possible score that a student can obtain from the pre-assessment is 15. Observe that there is no data for the BM – T1 for Lesson 3 because no student was categorized under that level in the final mastery level counting. On the other hand, there were no computed standard deviations (SD) for BM – T1 under Lessons 1 and 5 and AM – T2 under Lesson 4 since only one student was categorized under those levels in the final mastery level counting. Further, the means and standard deviations in Lesson 4 are 0 since the three students classified under the BM – T1 as their final mastery level were absent during the pre-assessment. Other standard deviation values range from 0 to 5.66. Also, observe that among the final mastery levels, the HM – T3 (S) always obtained higher means than the lower mastery levels except for Lesson 2, in which the HM – T3 (U) got the highest mean. This trend can be attributed to students' desire to obtain a high score in their pre-assessment so that they can skip answering T1 and T2 worksheets. One student stated during the in-class engagement:

> "*I always wanted to land on a High Level of Mastery, Sir, so that I will only answer one worksheet. Whenever I'm regarded in a lower mastery level, I feel that I wasn't able to give my best and also, I get sad and envious because my classmates are in Tier 3, while here I am, classified in Tier 1. Hence, I feel motivated to study so that I can be of equal footing with them in terms of ability level. So, I am delighted if I can land on high mastery level because it means that I learned from the videos*."

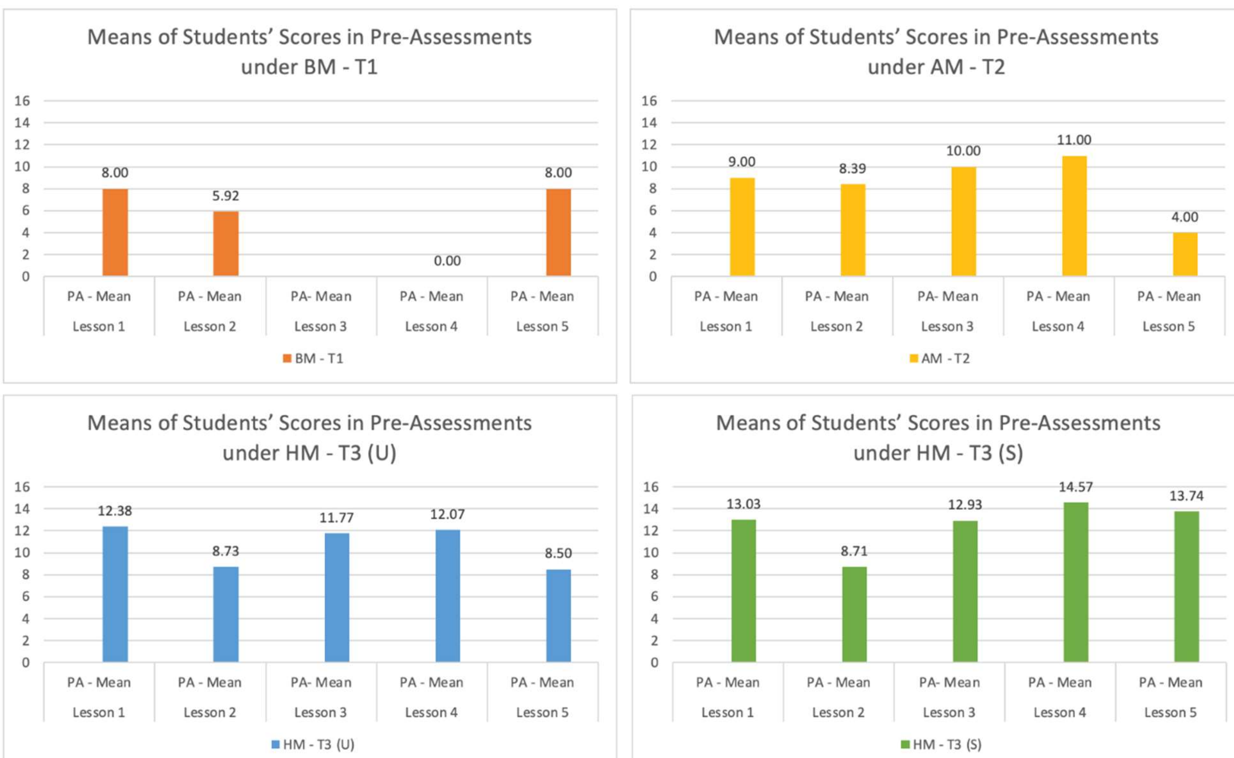

Figure 3. Means and Standard Deviations of Students' Scores in Pre-Assessments

During the intervention, students strive harder to land at higher mastery levels, reflecting how well they performed in their pre-assessment based on their understanding of the concepts gained from the online class instruction. Some students also realized that even though they were regarded at lower mastery levels in a particular lesson, they knew that they could still do better for the next lesson, where the idea of a second chance comes in. As one student said:

> "*Sir, it is okay for me if I am just in T1 level. Next time, I will surely do my best. Likewise, I can still answer the worksheets because of the mini lesson you provided there so that we can easily consult it whenever we answer the worksheets. Thus, it also helps us to familiarize ourselves in the items. Also, it's okay since I am given the chance to take three worksheets, hence, I see it as an opportunity to learn more*."

Tables 1, 2, and 3 present the Paired sample statistics: Correlation and t-test between the cumulative pre-assessment and summative assessment scores, respectively.

Percentage Scores	Mean	N	Std. Deviation	Std. Error Mean
Pre-Assessment	57.5435	46	11.09095	1.63527
Summative Assessment	62.6196	46	7.39007	1.08961

Table 1. Paired Samples Descriptive Statistics

The pre-assessment cumulative and summative assessment scores were converted to percentages. The increase in the mean percentage score indicates students' acquisition of required competency skills through the tiered worksheets and in-class remediation. As shown in Table 1, the summative percentage mean score is greater than that of the pre-assessment percentage mean score, implying an improvement in students' performance.

Percentage Scores	N	Correlation	Sig.
Pre-Assessment & Summative Assessment	46	.628	.000

Table 2. Paired Samples Correlation

In addition, as displayed in Table 2, there is a significantly strong positive correlation between the pre-assessment and the summative assessment percentage scores ($r = 0.628$) implying that students with higher pre-assessment tend to achieve more in their summative tests. This result is attributable to students' compliance with the online tasks prior to coming to class.

Based on Table 3, the negative mean difference between the pre-assessment and summative assessment scores signifies students performed better in the summative assessment than in the pre-assessment. The paired sample t-test revealed that this mean difference is significant (*p-value*=0.0002 and t=-3.982) at a 0.05 significant level. It shows that the use of the tiered worksheets was effective in increasing student achievement scores.

Paired Differences							
			95% Confidence Interval of the Difference				
Mean	Std. Deviation	Std. Error Mean	Lower	Upper	t	df	Sig. (2-tailed)
-5.07609	8.64611	1.27480	-7.64366	-2.50851	-3.982	45	.000

Table 3. Paired Samples Test

In order to check if there is a return on investment from this painstaking research process, we also computed the effect size to determine the impact of the intervention. The computed *Cohen's D* is 0.587 or approximately equal to 0.6, indicating a large effect size (Kneer, 2017). Thus, this further confirms the effectiveness of differentiated instruction, specifically the use of tiered worksheets as an intervention to boost student engagement and mathematics achievement.

DISCUSSION AND CONCLUSIONS

The results showed that there were significant changes in terms of the mastery levels of the students based on their pre-assessment and their tiered worksheet performance. Although some of the students were initially classified in lower mastery levels, most of them were able to move up to higher mastery levels in the lessons indicating that the options provided in the online class instruction and the tiered worksheets effectively addressed students' cognitive needs. Since the tiered worksheets were aligned with the online mathematics videos and reading materials, students were at ease and confident to answer the items in the tiered worksheets. Observations and interviews with students revealed that they have different learning preferences. Multiple options and opportunities for individual learning preferences were provided: online videos, online resource materials, and the mini-lessons in the worksheets. Students who missed doing their assigned online tasks were given a second chance to watch the videos or read online materials in the class to catch up. Even so, students realized the value of compliance with assigned tasks before the in-person class, that coming to class unprepared may give them limited time to move up or progress in learning due to the bounded contact time. Throughout the intervention, the number of students who were regarded in higher mastery levels increased, proving that they were intensely engaged in the online class instruction to get high scores in their pre-assessment. Likewise, the scaffolding part of the tiered worksheet (mini-lessons) also helped the students progress, especially those with lower mastery levels.

Receiving good grades can serve as an extrinsic motivation in Mathematics. However, it is also essential to balance external motivation with intrinsic motivation for a deeper understanding of concepts. Cook and Artino (2016) argued that individuals get smarter by constantly studying and practicing. In addition, even learners with low confidence and motivation in their existing abilities will choose challenging and thought-provoking tasks if they see that their "effort leads to mastery" (Cook & Artino, 2016). In this study, the researchers believe that considering students' preferences and adapting teaching methods based on their competency levels encourages and fosters an appreciation for learning mathematics. Hence, by utilizing tiered worksheets, students can experience the richness of learning mathematical concepts more deeply through problem-solving and strengthening their conceptual understanding.

The use of tiered worksheets can also be applied to lessons that require deeper and creative thinking. Nevertheless, the researchers included thought-provoking and problem-solving items throughout the intervention, specifically in Part III of each tiered worksheet. The intervention was also used to strengthen students' conceptual understanding and procedural fluency regarding quadratic equations.

Research says that when teachers respond to students' needs by readiness, like scaffolding and tiering, differentiated instruction is indeed effective (Abbati, 2012). The results of this study further validate this claim. Moreover, when students do the tasks in the class with immediate feedback from teachers, students become more focused, knowing that a more knowledgeable other is overseeing their activity and can readily provide feedback to ensure progress (National Research Council, 2004). Students are encouraged to perform well in class if the learning tasks are just right on their mastery levels. Despite learning differences especially in cognitive levels, tiered worksheets help students improve their acquisition of mathematics skills because this encourages them to have a deep understanding of mathematical concepts by going through the process from the lower level to a higher-level worksheet. Hence, alongside implementing the flipped classroom, tiered worksheets increase student participation and performance since the tasks are tailored to their mastery levels.

The pre-assessment results showing more students at a High Level of Mastery than those in the lower levels across lessons prior to tiered worksheet class activities show that the online materials have helped students learn. The significant positive correlation between students' pre-assessment and summative assessment percentage scores underscores the flipped classroom's online asynchronous tasks provision or opportunity for students to learn independently and the importance of coming to class prepared. Thus, class time is maximized. The significant difference in students' performance in pre-assessment and tiered worksheets and the computed effect size revealed the effective use of tiered worksheets. Thus, these results indicate that differentiated instruction in the flipped classroom context is effectively addresses students' diversity, specifically regarding learning interest and student readiness, in boosting student engagement and performance in the identified lessons on quadratic equations.

In this study, we tried to address the flipped classroom limitation, the students' non-compliance (He et al., 2016), by using differentiated instruction mainly by administering a pre-assessment and tiered worksheets. The paramount concern of this study is to increase student engagement and mathematics achievement by incorporating differentiated instruction strategies in the context of flipped classrooms, specifically the utilization of online class instruction. Given that students are diverse in terms of their learning preferences, learning interests, and learning readiness, the researcher addressed it by providing options to the students in the online class instruction and administering the pre-assessment for each assigned lesson to gauge student's understanding which they acquired from the online class instruction. Moreover, the pre-assessment results helped the teacher-researcher correctly identify students' mastery levels by giving their appropriate tiered worksheets commensurate to their individual readiness.

The options provided in the online class instruction effectively encouraged and engaged students to study in advance so that they could perform better in their pre-assessment. Since the initial mastery level of the students greatly depends on their score from the pre-assessment, they were encouraged and intrinsically motivated to participate in the online class instruction. The idea of having the pre-assessment excites the students because this determines whether they would land in higher mastery levels, and it was revealed that the intervention is effective in enhancing student engagement in learning. However, it was also shown that some students had inconsistent performance with regarding their initial and final mastery levels. This inconsistent performance can be attributed to the lesson's difficulty or the habitual absenteeism of some students during the administration of the pre-assessment. In general, the administration of the pre-assessment addressed the gap of the flipped classroom model since pre-assessment was used to determine whether students watched the online mathematics videos and whether they learned something from the online class instruction.

This study showed that the online class instruction maximized the time during the face-to-face instruction since the teacher only gave supplementary activities like the pre-assessment and the tiered worksheets, which strengthened the knowledge obtained by the students in their off-class engagement. The tiered worksheets were also effective in increasing student's ability in terms of critical thinking and problem-solving skills. In the study's experience, there was no idle or unused time, even for the high achievers. All students in the in-person classes were observed to spend the entire time answering their respective worksheets. The researchers noticed students who reached the highest-tiered worksheet double-checked their work and solutions. Additionally, the worksheets are challenging since the researchers ensured that the items specifically under the Tier 3 worksheets are thought-provoking and complex. For BM and AM students, since the items contained in the tiered worksheets were just right on their cognitive levels, it triggered them to be more confident in answering mathematical problems deemed challenging. Based on their pre-assessment scores, the tiered worksheets motivated students to be prepared to land at a higher mastery level. Further, the scaffolding component of the worksheets also increased their chances of getting the correct answers in the worksheet, thus helping them maximize their performance to move up to higher mastery levels.

Regarding student achievement, student absenteeism is still a perennial problem in public schools, and this immensely affected their performance in answering the tiered worksheets. Nevertheless, the intervention successfully and effectively increased students' performance for those who were at least present. Moreover, the lesson's difficulty affected whether students could have two or more movements. Hence, it was essential for the teacher to let the students have advanced readings at home since the contact hours in the face-to-face instruction were not enough to cover the topics provided the jampacked nature of the curriculum. Therefore, by looking at the data presented in this study, students must not rely exclusively on the discussion of the teacher if they want to master the lesson's contents. Considering the generation of the students, employing the responsible use of technology helped students achieve optimal learning. However, since many schools, especially public schools, cannot afford to establish their infrastructures, giving assignments and additional readings in a Facebook group and handheld devices can help students realize the flipped classroom and blended learning goals.

The researchers believe that the intervention is feasible. Nonetheless, factors such as the teacher's resources, pedagogical skills, and the specific classroom environment, including student-teacher ratio, classroom management, and, most importantly, time constraints, should be considered when planning differentiated instruction (Abbati, 2012). The intervention used in the study is challenging. However, it is doable, provided that teachers strategically group students based on the results of pre-assessments, provide well-guided instructions, and use technology effectively to assist them in differentiation. The benefits of differentiated instruction to students in this study outweighed the expended efforts. Nevertheless, we recommend varying strategies to foster student achievement and engagement, such as MyOpenMath, rotation station learning, and adaptive assessments for differentiating instruction (Benjamin, 2014; Gregory & Chapman, 2013). Nevertheless, the findings of this study have shown promise for students' mastery of learning.

Since studies concerning the effectiveness of Differentiated Instruction alongside the utilization of the flipped classroom model are scarce, this study serves as a springboard to encourage educators to adopt this teaching approach in dealing with student diversity. Considering that not all students are the same and differ in learning styles, preferences, interests, and readiness, the intervention carried out by the teacher-researcher may accommodate those needs, specifically focusing on student interest and readiness. In general, the intervention contributed significantly to the increase in student engagement both off-class and in-class, which resulted in an immense improvement in their achievement scores.

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APPENDICES

Appendix A Sample List of Online Class Instructions with Guide Questions and Pre-Assessment (Solving Quadratic Equations by Completing the Square)

LESSON: SOLVING QUADRATIC EQUATIONS BY COMPLETING THE SQUARE

A. ONLINE VIDEO INSTRUCTION:

1. https://www.youtube.com/watch?v=K_FahMr7V7k&list=PLp3WnNoPhT3KH4MUzLGWzsyR4Q33WrUj9&index=5 (URL)
   TITLE: Quadratic Equations and Inequalities - Solving Quadratic Equations by Completing the Square
   DURATION: 14 mins and 37 secs
   UPLOADER: Nick - Gurodawako – Agriam
   OTHER INFORMATION: The content of the video is based on Grade 9- Unit 1 Mathematics Module.

2. https://www.youtube.com/watch?v=QjME8FS2TtY (URL)
   TITLE: QE06 Solving Quadratic Equations - Completing the Square Part 1
   DURATION: 10 mins and 33 secs
   UPLOADER: Sipnayan
   OTHER INFORMATION: The content of the video is based on Grade 9- Unit 1 Mathematics Module and the uploader used Filipino language as his medium of instruction in the video.

3. https://www.youtube.com/watch?v=R5k96cNJXvE (URL)
   TITLE: QE07 Solving Quadratic Equations - Completing the Square Part 2
   DURATION: 6 mins and 26 secs
   UPLOADER: Sipnayan
   OTHER INFORMATION: The content of the video is based on Grade 9- Unit 1 Mathematics Module and the uploader used Filipino language as his medium of instruction in the video.

4. https://www.youtube.com/watch?v=oPE-THYrYAM (URL)
   TITLE: QE08 Solving Quadratic Equations - Completing the Square Part 3
   DURATION: 5 mins and 34 secs
   UPLOADER: Sipnayan
   OTHER INFORMATION: The content of the video is based on Grade 9- Unit 1 Mathematics Module and the uploader used Filipino language as his medium of instruction in the video.

5. https://www.youtube.com/watch?v=bNQY0z76M5A (URL)
   TITLE: Solving quadratic equations by completing the square | Algebra II | Khan Academy
   DURATION: 14 mins and 5 secs
   UPLOADER: Khan Academy

6. https://www.youtube.com/watch?v=prx_Bf2hakw (URL)
   TITLE: How to Solve by Completing the Square (NancyPi)
   DURATION: 17 mins and 32 secs
   UPLOADER: NancyPi

7. https://www.youtube.com/watch?v=C206SNAXDGE (URL)
   TITLE: Completing the Square Method and Solving Quadratic Equations - Algebra 2
   DURATION: 31 mins and 54 secs
   UPLOADER: The Organic Chemistry Tutor

B. MATHEMATICS LEARNING WEBSITE:

1. https://www.khanacademy.org/math/algebra/quadratics/solving-quadratics-by-completing-the-square/a/solving-quadratic-equations-by-completing-the-square?modal=1
2. https://www.khanacademy.org/math/algebra/quadratics/solving-quadratics-by-completing-the-square/v/ex1-completing-the-square?modal=1
3. https://www.khanacademy.org/math/algebra/quadratics/solving-quadratics-by-completing-the-square/v/rewriting-quadratics-as-perfect-squares?modal=1
4. https://www.khanacademy.org/math/algebra/quadratics/solving-quadratics-by-completing-the-square/v/solving-quadratics-by-completing-the-square?modal=1
5. https://www.khanacademy.org/math/algebra/quadratics/solving-quadratics-by-completing-the-square/v/completing-the-square-to-solve-quadratic-equations?modal=1
6. https://www.khanacademy.org/math/algebra/quadratics/solving-quadratics-by-completing-the-square/v/ex2-completing-the-square?modal=1
7. https://www.khanacademy.org/math/algebra/quadratics/solving-quadratics-by-completing-the-square/a/completing-the-square-review?modal=1

C. OTHER ONLINE RESOURCE MATERIALS:

1. https://www.mathsisfun.com/algebra/completing-square.html
2. https://www.purplemath.com/modules/solvquad3.htm
3. https://www.purplemath.com/modules/sqrquad.htm
4. https://www.brainfuse.com/jsp/alc/resource.jsp?s=gre&c=35261&cc=108824

Sample Guide Questions

**GUIDE QUESTIONS:
SOLVING QUADRATIC EQUATIONS BY COMPLETING THE SQUARE**

NAME: DATE:

GRADE AND SECTION: TEACHER:

1. How do you describe a perfect square trinomial?
2. How are you going to express a perfect square trinomial as the square of a binomial?
3. Solve the following quadratic equations by completing the square.
   a) $x^2 + 4x = 5$
   b) $-18 + 3x + x^2 = 0$
   c) $-2x^2 = 2 - 7x$

Sample Pre-Assessment

PRE-ASSESSMENT

LESSON: Solving Quadratic Equations by Completing the Square

NAME: DATE:
SECTION: SCORE:

Directions: Answer the following items correctly. Show your solutions for those items that need to be solved. Put a box in your final answer.

Part I: For nos. 1-3, determine a number that must be added to make each of the following expression a perfect square trinomial. Express it as a square of a binomial. An example below is provided to you. (1 point each).

Example:

$x^2 + 2x +$ ____

Answer: 1

$\therefore \mathbf{x^2 + 2x + 1 = (x + 1)^2}$

1. $x^2 + 4x +$ ____
2. $r^2 + 20r +$ ____
3. $m^2 + 24m +$ ____

Part II: For nos. 4-6, solve the following quadratic equation by completing the square (2 points each).

4. $x^2 - 2x = 3$
5. $-41 - 6x + x^2 = 0$
6. $3x^2 + 5x - 2 = 0$

Part III: For nos. 7-8, answer the following questions (3 points each).

7. Peter wants to use completing the square in solving the quadratic equation, $9x^2 - 16 = 0$. Can he use it in finding the solutions of the equation? Explain why or why not.
8. If you are to choose between completing the square and factoring in finding the solution of the quadratic equation, $x^2 + 9x + 18 = 0$, which would you choose? Explain and solve the equation using your preferred method.

ADDITIONAL QUESTION:

Did you watch the off-loaded online video links?

- ☐ If your answer is YES, were you able to follow or understand the contents of the video/s?
- ☐ If your answer is NO, state your reason for not watching the online videos.

Appendix B Sample Tiered Worksheets

TIERED MATHEMATICS LEARNING WORKSHEET 1
(BEGINNING MASTERY)

LESSON: Solving Quadratic Equations by Completing the Square

NAME: **DATE:**
SECTION: **SCORE:**

RECALL:

Hi! You are answering **TIERED WORKSHEET 1!** In this section, another way of **SOLVING QUADRATIC EQUATIONS** will be introduced and that is **BY COMPLETING THE SQUARE.**

To solve quadratic equation,

$ax^2 + bx + c = 0$ **by COMPLETING THE SQUARE**, the following procedure can be followed:

1) Divide both sides of the equation by a then simplify.
2) Write the equation such that the terms with variables are on the left side of the equation and the constant term is on the right side.
3) Add the square of one-half of the coefficient of x on both sides of the resulting equation. The left side of the equation becomes a perfect square trinomial; $c = \left(\frac{b}{2}\right)^2$.
4) Express the perfect square trinomial on the left side of the equation as a square of a binomial.
5) Solve the resulting quadratic equation by extracting the square root.
6) Solve the resulting linear equation.

EXAMPLE:
Solve $x^2 + 8x - 9 = 0$ by completing the square.

Step 1: The numerical coefficient of x is 1 so, there's no need to divide 1 to both sides of the equation.

$$x^2 + 8x - 9 = 0$$

Step 2: Add 9 to both sides of the equation then simplify.

$$x^2 + 8x - 9 + 9 = 0 + 9$$
$$x^2 + 8x = 9$$

Step 3: Add the square of one-half of 8 to both sides of the equation.

$$c = \left(\frac{b}{2}\right)^2 \to \left(\frac{8}{2}\right)^2 \to (4)^2 \to 16$$
$$\therefore x^2 + 8x + 16 = 9 + 16$$
$$x^2 + 8x + 16 = 25$$

Step 4: Express $x^2 + 8x + 16$ as a square of a binomial.

$$x^2 + 8x + 16 = 25$$
$$(x + 4)^2 = 25$$

Step 5: Solve $(x + 4)^2 = 25$ by extracting the square root.

$$(x + 4)^2 = 25$$
$$\sqrt{(x + 4)^2} = \pm\sqrt{25}$$
$$x + 4 = \pm 5$$

Step 6: Solve the resulting linear equation.

$$x + 4 = 5 \text{ and } x + 4 = -5$$
$$x = -4 + 5 \text{ and } x = -4 - 5$$
$$x_1 = 1 \quad \& \quad x_2 = -9$$

SOLUTION SET: $\{-9, 1\}$

Directions: Answer the following items correctly. Show your solutions for those items that need to be solved. Put a box in your final answer.

Part I: For nos. 1-3, determine a number that must be added to make each of the following expression a perfect square trinomial. Express it as a square of a binomial (1 point each).

1. $s^2 - 16s +$ _____
2. $x^2 - 20x +$ _____
3. $y^2 + 30y +$ _____

Part II: For nos. 4-6, solve the following quadratic equation by completing the square (2 points each).

4. $x^2 + 4x = 12$
5. $x^2 - 8 = 2x$
6. $-16 - 6x + x^2 = 0$

Part III: For nos. 7-8, answer the following question (3 points each).

7. Can Paul use the method of completing the square in finding the solutions of the quadratic equation, $2x^2 - 18 = 0$? Justify your answer.
8. Do you agree that any quadratic equation can be solved by completing the square? Explain your answer.

TIERED MATHEMATICS LEARNING WORKSHEET 2
(APPROACHING MASTERY)

LESSON: Solving Quadratic Equations by Completing the Square

NAME: **DATE:**
SECTION: **SCORE:**

RECALL:

Hi! You are answering **TIERED WORKSHEET 2!** In this section, another way of **SOLVING QUADRATIC EQUATIONS** will be introduced and that is **BY COMPLETING THE SQUARE.**

To solve quadratic equation,

$ax^2 + bx + c = 0$ **by COMPLETING THE SQUARE**, the following procedure can be followed:

1) Divide both sides of the equation by a then simplify.
2) Write the equation such that the terms with variables are on the left side of the equation and the constant term is on the right side.
3) Add the square of one-half of the coefficient of x on both sides of the resulting equation. The left side of the equation becomes a perfect square trinomial; $c = \left(\frac{b}{2}\right)^2$.
4) Express the perfect square trinomial on the left side of the equation as a square of a binomial.
5) Solve the resulting quadratic equation by extracting the square root.
6) Solve the resulting linear equation.

EXAMPLE:
Solve $x^2 - 5x + 2 = 0$ by completing the square.

Step 1: The numerical coefficient of x is 1 so, there's no need to divide 1 to both sides of the equation.

$$x^2 - 5x + 2 = 0$$

Step 2: Add -2 to both sides of the equation then simplify.

$$x^2 - 5x + 2 - 2 = 0 - 2$$
$$x^2 - 5x = -2$$

Step 3: Add the square of one-half of -5 to both sides of the equation.

$$c = \left(\frac{b}{2}\right)^2 \to \left(\frac{-5}{2}\right)^2 \to \frac{25}{4}$$
$$\therefore x^2 - 5x + \frac{25}{4} = -2 + \frac{25}{4}$$
$$x^2 - 5x + \frac{25}{4} = \frac{-8 + 25}{4}$$
$$x^2 - 5x + \frac{25}{4} = \frac{17}{4}$$

Step 4: Express $x^2 - 5x + \frac{25}{4}$ as a square of a binomial.

$$x^2 - 5x + \frac{25}{4} = \frac{17}{4}$$
$$\left(x - \frac{5}{2}\right)^2 = \frac{17}{4}$$

Step 5: Solve $\left(x - \frac{5}{2}\right)^2 = \frac{17}{4}$ by extracting the square root.

$$\left(x - \frac{5}{2}\right)^2 = \frac{17}{4}$$
$$\sqrt{\left(x - \frac{5}{2}\right)^2} = \pm\sqrt{\frac{17}{4}}$$
$$x - \frac{5}{2} = \pm\frac{\sqrt{17}}{2}$$

Step 6: Solve the resulting linear equation.

$$x - \frac{5}{2} = \frac{\sqrt{17}}{2} \text{ and } x - \frac{5}{2} = -\frac{\sqrt{17}}{2}$$
$$x = \frac{5}{2} + \frac{\sqrt{17}}{2} \text{ and } x = \frac{5}{2} - \frac{\sqrt{17}}{2}$$
$$x_1 = \frac{5 + \sqrt{17}}{2} \quad \& \quad x_2 = \frac{5 - \sqrt{17}}{2}$$

SOLUTION SET: $\{\frac{5-\sqrt{17}}{2}, \frac{5+\sqrt{17}}{2}\}$

Directions: Answer the following items correctly. Show your solutions for those items that need to be solved. Put a box in your final answer.

Part I: For nos. 1-3, determine a number that must be added to make each of the following expression a perfect square trinomial. Express it as a square of a binomial (1 point each).

1. $s^2 - 7s +$ ____
2. $x^2 + 11x +$ ____
3. $y^2 - 15y +$ ____

Part II: For nos. 4-6, solve the following quadratic equations by completing the square (2 points each).

4. $x^2 - 3x - 3 = 0$
5. $x^2 - 5x = 6$
6. $x^2 - 26 = 11x$

Part III: For nos. 7-8, answer the following question (3 points each).

7. If you are to choose between completing the square and factoring in finding the solution of the quadratic equation, $x^2 + 11x + 30 = 0$, which would you choose? Explain and solve the equation using your preferred method.
8. Luke tried to determine a number that must be added to make $x^2 - \frac{4}{5}x$ a perfect square trinomial. The parts of his solutions are shown below.

$$x^2 - \frac{4}{5}x \longrightarrow x^2 - \frac{4}{5}x + \_\_\_\_\_$$
$$c = \left(\frac{b}{2a}\right)^2 \longrightarrow \left(\frac{\frac{-4}{5}}{2(1)}\right)^2$$
$$\left(-\frac{4}{5(2)}\right)^2 \longrightarrow \left(-\frac{4}{10}\right)^2 \longrightarrow \left(-\frac{2}{5}\right)^2 \longrightarrow \frac{4}{25}$$
$$x^2 - \frac{4}{5}x + \frac{4}{25} \longrightarrow \left(x + \frac{2}{5}\right)^2$$

Do you think Luke arrived at the correct answer? Explain your answer.

TIERED MATHEMATICS LEARNING WORKSHEET 3
(HIGH LEVEL OF MASTERY)

LESSON: Solving Quadratic Equations by Completing the Square

NAME: **DATE:**
SECTION: **SCORE:**

RECALL:

Hi! You are answering **TIERED WORKSHEET 3**! In this section, another way of **SOLVING QUADRATIC EQUATIONS** will be introduced and that is **BY COMPLETING THE SQUARE.**

To **solve quadratic equation,**

$ax^2 + bx + c = 0$ **by COMPLETING THE SQUARE,** the following procedure can be followed:

1) Divide both sides of the equation by a then simplify.
2) Write the equation such that the terms with variables are on the left side of the equation and the constant term is on the right side.
3) Add the square of one-half of the coefficient of x on both sides of the resulting equation. The left side of the equation becomes a perfect square trinomial; $c = \left(\frac{b}{2}\right)^2$.
4) Express the perfect square trinomial on the left side of the equation as a square of a binomial.
5) Solve the resulting quadratic equation by extracting the square root.
6) Solve the resulting linear equation.

EXAMPLE:

Solve $2x^2 + 11x + 15 = 0$ by completing the square.

Step 1: Divide both sides of the equation by 2 then simplify.

$$\frac{2x^2+11x+15}{2} = \frac{0}{2}$$

$$x^2 + \frac{11}{2}x + \frac{15}{2} = 0$$

Step 2: Add $-\frac{15}{2}$ to both sides of the equation then simplify.

$$x^2 + \frac{11}{2}x + \frac{15}{2} - \frac{15}{2} = 0 - \frac{15}{2}$$

$$x^2 + \frac{11}{2}x = -\frac{15}{2}$$

Step 3: Add the square of one-half of $\frac{11}{2}$ to both sides of the equation.

$$c = \left(\frac{b}{2}\right)^2 \rightarrow \left(\frac{\frac{11}{2}}{2}\right)^2 \rightarrow \left(\left(\frac{11}{2}\right)\left(\frac{1}{2}\right)\right)^2 \rightarrow \left(\frac{11}{4}\right)^2 \rightarrow \frac{121}{16}$$

$$\therefore x^2 + \frac{11}{2}x + \frac{121}{16} = -\frac{15}{2} + \frac{121}{16}$$

$$x^2 + \frac{11}{2}x + \frac{121}{16} = \frac{-120+121}{16}$$

$$x^2 + \frac{11}{2}x + \frac{121}{16} = \frac{1}{16}$$

Step 4: Express $x^2 + \frac{11}{2}x + \frac{121}{16}$ as a square of a binomial.

$$x^2 + \frac{11}{2}x + \frac{121}{16} = \frac{1}{16}$$

$$\left(x + \frac{11}{4}\right)^2 = \frac{1}{16}$$

Step 5: Solve $\left(x + \frac{11}{4}\right)^2 = \frac{1}{16}$ by extracting the square root.

$$\left(x + \frac{11}{4}\right)^2 = \frac{1}{16}$$

$$\sqrt{\left(x + \frac{11}{4}\right)^2} = \pm\sqrt{\frac{1}{16}}$$

$$x + \frac{11}{4} = \pm\frac{1}{4}$$

Step 6: Solve the resulting linear equation.

$$x + \frac{11}{4} = \frac{1}{4} \text{ and } x + \frac{11}{4} = -\frac{1}{4}$$

$$x = -\frac{11}{4} + \frac{1}{4} \text{ and } x = -\frac{11}{4} - \frac{1}{4}$$

$$x = -\frac{11+1}{4} \text{ and } x = -\frac{11-1}{4}$$

$$x_1 = \frac{-10}{4} \ \& \ x_2 = \frac{-12}{4}$$

$$x_1 = -\frac{5}{2} \ \& \ x_2 = -3$$

SOLUTION SET: $\{-3, -\frac{5}{2}\}$

Directions: Answer the following items correctly. Show your solutions for those items that need to be solved. Put a box in your final answer.

Part I: For nos. 1-3, determine a number that must be added to make each of the following expression a perfect square trinomial. Express it as a square of a binomial (1 point each).

1. $w^2 - \frac{2}{3}w +$ ____
2. $r^2 + \frac{3}{4}r +$ ____
3. $x^2 - \frac{5}{2}x +$ ____

Part II: For nos. 4-6, solve the following quadratic equations by completing the square. (2 points each).

4. $x^2 + 7x = \frac{51}{4}$
5. $4x^2 - 20x = 11$
6. $\frac{2x+3}{3x+2} = \frac{2x-7}{x+2}$

Part III: For nos. 7-8, formulate a quadratic equation for each word problem and solve for its solution using completing the square (3 points).

7. If a number is added to its square, the result is 42. Find the number.
8. The sum of the squares of two consecutive integers is 85. What are the numbers?

Case Studies: Pre-Service Mathematics Teachers' Integration of Technology into Instructional Activities Using a Cognitive Demand Perspective

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Abstract: This study aims to investigate pre-service teachers' integration of technology and how the integration of technology influences the level of cognitive demands of the mathematics tasks in their mathematics technology activities. The purpose of this study was to investigate the various levels of cognitive demands of mathematical tasks created or modified by pre-service mathematics teachers for technology activities. This study presents case studies of pre-service teachers chosen from a group of participants, and these PSTs come from a variety of backgrounds. Showcase Portfolios and lesson plans were gathered in order to comprehend the selection/creation of mathematical tasks and how they were intended to be implemented in the classroom by PSTs. This study offers suggestions for teacher preparation to integrate technology into mathematics instruction in ways that support students' learning through a review of results, the cognitive demands of mathematical tasks in PSTs' technology activities.

Keywords: cognitive demands of the mathematics tasks, case studies, technology integration, pre-service mathematics teachers

INTRODUCTION

We live in a technological and mathematically advanced era. New technologies are developed on the basis of mathematical knowledge (Kilpatrick, Swafford, Findell, & National Research Council, 2001); thus, people must be able to understand and do mathematics in order to effectively participate in opportunities to shape the future (National Council of Teachers of Mathematics [NCTM], 2000, 2014). Today's students cannot survive economically in the twenty-first century without technology-supported learning opportunities, and traditional education cannot provide these opportunities for students (International Society for Technology in Education, [ISTE], 2000). One of the six central principles for school mathematics addressed in the NCTM Principles to Actions: Ensuring Mathematical Success for All (2014) is technology. According to the NCTM technology principle, "technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning" (p. 24).

Over the last 40 years, teaching strategies and school curricula have evolved significantly (Heddens & Speer, 2006). Today's students are frequently bored during direct instruction lessons in mathematics and other subjects because they only need to listen to the lecture and sit in their chair (Schrum & Levin, 2009). Because today's students have grown up in a technological era, our educational system should take this into account when developing curriculum and instructional strategies. "Incorporate and support the effective use of appropriate tools and technology in mathematics curriculum standards across all grade levels" (NCTM, 2014, pp. 111-112). Incorporating technology into the educational process provides students with an interactive learning experience that allows them to remain engaged in the subject matter (Haleem et al., 2022). Furthermore, using technology to teach mathematics increases students' involvement in the learning process (Getenet, 2020).

Students in the twenty-first century differ from previous generations. Students must be taught differently because they learn and think differently than adults (e.g., teachers and school leaders), who are referred to as digital immigrants (Prensky, 2001; Schrum & Levin, 2009). The primary distinctions between digital natives and digital immigrants are their levels of comfort with technology, as well as their approaches to information processing and learning through technology (Cunningham, 2012; Zur & Zur, 2011). Technology is simple to use and understand for digital natives (Schrum & Levin, 2009). When compared to digital immigrants, digital natives use technology to access information quickly and effectively communicate with their peers (Cunningham, 2012). The term "digital natives" refers to two fundamental expectations. To begin, educational methods should be tailored to promote rapid, relevant, and pragmatic learning. Second, there is a need for educators to develop a common technological vernacular that speaks directly to this demographic (Zuluaga Trujillo & Gómez Montero, 2019).

Technology has the potential to improve the quality of mathematical investigations, present meaningful mathematical ideas to students and teachers from various perspectives, and change traditional ways of doing mathematics (NCTM, 2000). The use of technology in the classroom may increase children's involvement in the learning process (Hallem et al., 2022). According to The NCTM Principles to Actions: Ensuring Mathematical Success for All (2014), technology not only improves students' understanding and learning of mathematics, but it also assists teachers in making instruction more effective and meaningful for students. Effective mathematics instruction is required for all students in all classrooms to improve their mathematical understanding. Teachers use various teaching styles and strategies to teach specific mathematical concepts, and there is no one way to teach. Teachers' mathematical knowledge and understanding are important factors in influencing decisions and actions in their mathematics classrooms to improve students' learning (Anthony & Walshaw, 2009; Ball, Thames, & Phelps, 2008).

One of mathematics teachers' responsibilities is to provide various opportunities for their students to develop mathematical thinking. Teachers require resources to expand their knowledge and refresh their strategies for effective mathematics teaching and learning (NCTM, 2000). Teachers plan the mathematical tasks that will be used in mathematics lessons and design how these tasks will be implemented in class to improve students' thinking. A mathematical task is defined as "a classroom activity designed to direct students' attention toward a specific mathematical concept, idea, or skill" (Henningsen & Stein, 1997, p. 528). Tasks are identifiable and meaningful elements that are used to evaluate and create curriculum, teaching methods, and assessment strategies (Tekkumru-Kisa & Stein, 2015). According to research, mathematics tasks are critical for students' learning and for improving their reasoning skills (Boaler & Staples, 2008; Stein & Lane, 1996). Other essential roles of teachers include focusing on the relationship between tasks and student thinking, selecting high-level tasks (e.g., tasks that promote higher order thinking) for mathematics instruction, and implementing these tasks in ways that maintain high-level cognitive demands (Boston & Smith, 2009; Stein, Grover, & Henningsen, 1996). The use of various, meaningful, and valuable mathematical tasks is associated with effective teaching and improving students' mathematical skills (Glasnovis Gracin, 2018).

The selection of mathematical tasks has significant implications for students' understanding of mathematics as well as the quality of their mathematical thinking and learning. As a result, it is critical to comprehend the role that technology may play in the tasks that teachers choose to assign to their students. Mathematics teachers are increasingly likely to use technology-enhanced teaching methods and integrate technology into their classroom practices (Joubert et al., 2020). Mathematics teachers' roles in the classroom are critical for the effective use of technology in ways that support students' mathematical understanding (NCTM, 2000, 2014). Mathematics teachers are not replaced by technology, but rather make decisions about how and when to use technology as a supplement in the teaching and learning environment (NCTM, 2000, 2014). Sherman (2014) emphasized the significance of using technology to assist students in improving their high-level mathematical thinking. Using technology to support student learning entails using technological tools to provide and sustain student engagement in high-level tasks and thinking.

Attending to the cognitive demand of technology tasks used in mathematics teaching and learning serves as a productive focus for effectively using technology; however, research shows the complexity of teaching mathematics using cognitively challenging tasks (Boston & Smith, 2009; Henningsen & Stein, 1996). Sherman (2014) observes that teachers struggle to maintain high-level demands during implementation (i.e., throughout a lesson) when using technology, despite having selected and set up high-level tasks at the start of the lesson. Professional development can help teachers carry out high-level tasks (Boston & Smith, 2009, 2011). As a result, teachers require training to influence the use of technology in education, which should begin in teacher preparation programs.

Teacher Preparation

Training in how to use technology effectively to support students' mathematical learning should begin in teacher preparation programs. In teacher preparation programs, future educators are prepared to gain pedagogical and subject matter knowledge as well as early teaching practice (Feuer, Floden, Chudowsky, & Ahn, 2013). Furthermore, teacher preparation programs should provide prospective teachers with the tools they will need in the classroom, such as educational technology (Edutopia, 2008). In her article, Niess (2008) states that "with the addition of an integration of new and emerging twenty-first century technologies as tools for learning, the preparation of teachers must evolve toward preparing preservice teachers to teach in ways that help them to guide their students in learning with appropriate technologies" (p.224).

Hence, it is critical that training in the effective use of technology in pedagogy (processes, practices, and methods of teaching and learning) and content (mathematics subjects such as number and quantity, algebra, functions, geometry, statistics, probability, and calculus) begin in mathematics teacher preparation programs. ISTE (2000) created technology standards to help pre-service teachers integrate technology into their classrooms. Teaching with technological tools should be emphasized in teacher preparation programs (Mishra & Koehler, 2006), as many pre-service teachers, and even in-service teachers, are unaware of the technologies availability for use in the classroom (Lin, 2008). The significance of teacher preparation programs cannot be overstated, because such programs can provide positive experiences with technology for PSTs in mathematics teaching and learning (Browing & Klespis, 2000). According to Garofalo et al. (2000), "PSTs need to develop technology skills, enhance and extend their knowledge of mathematics with technological tools, and become critical developers and users of technology-enabled pedagogy" (p. 86).

After graduating, pre-service teachers (elementary, middle, and secondary level) are expected to teach mathematics lessons and, ideally, integrate technology into their instruction, but many of them have not had enough opportunities during coursework to learn how to integrate technology effectively into their lesson activities. There are numerous technology tools available for pre-service teachers, as Johnston (2009) stated that "little is known about how pre-service elementary teachers evaluate technology tools as they plan for instruction" (p.1). According to the literature, PSTs require opportunities during their preparation program to plan and implement technology-enhanced lessons. Understanding how PSTs can be supported to plan lessons that integrate technology to effectively support students' mathematics learning will make a significant contribution to the field's knowledgebase.

In conclusion, technology is critical in education, particularly in the teaching and learning of mathematics. There are numerous technological tools available to teachers, and teachers must choose and implement technology in ways that support students' mathematical learning. Teachers require training to effectively use and integrate technology in their lesson activities, and this training should begin in teacher preparation programs.

Significance of Study

When considering the role of technology in addressing student learning, there are two important and distinct approaches to consider: (a) the quality of instruction and (b) the impact on student learning. These two approaches are linked to and influence student learning in education, particularly mathematics education. Many studies investigated how technology has affected students' learning and understanding of mathematics (such as Shin, Sutherland, Norris, & Soloway, 2012), and a few key studies have looked at how using instructional technology affects teachers' task implementation and students' complex thinking in the classroom (e.g., Sherman, 2014). According to Hollebrands, Conner, and Smith (2010), the majority of studies have focused on the use of technology and how it affected the learning of the NCTM Content Standards (number and quantity, algebra, functions, geometry, statistics, probability, and calculus), but fewer studies have focused on how technology supports learning of the NCTM Process Standards (problem solving, reasoning and proof, communication, connection, and representations).

According to Rice, Johnson, Ezell, and Pierczynski-Ward (2008), addressing learners' needs, using best teaching strategies, and teaching the standards are insufficient without the integration of technology for the process of effective planning. Few studies have been conducted on how PSTs use and integrate instructional technology for instruction. The following research questions will be addressed in this study:

- What is the level of the cognitive demands of mathematical tasks created or modified by pre-service mathematics teachers for technology activities?
- How does the integration of technology change the level of cognitive demands of mathematics tasks in mathematics technology activities?

METHOD

The purpose of this study was to investigate the various levels of cognitive demands of mathematical tasks created or modified by pre-service mathematics teachers for technology activities. This study presents case studies of pre-service teachers chosen from a group of participants, and these PSTs come from a variety of backgrounds. This section's comments and descriptions are based on lesson activities and/or samples of student work (for Showcase Portfolios only). Showcase Portfolios and lesson plans were gathered in order to comprehend the selection/creation of mathematical tasks and how they were intended to be implemented in the classroom by PSTs.

Participants

This study chose five PSTs with diverse academic backgrounds, including two elementary level PSTs (Zack and Emily), two secondary level PSTs (Dora and Carrie), and a group of middle level PSTs (three PSTs collaborated to create this activity). A pseudonym was used to identify all data collected from participants, and the pseudonym is used throughout this paper (Table 1).

Participant (Pseudonym)	Grade Level	Technology	Task sources
Zack	Elementary Level	Smartboard	NCTM/Illumination
Emily	Elementary Level	iPad or a computer	NCTM/Illumination
Dora	Secondary Level	Smartboard	NCTM/Illuminations
Carrie	Secondary Level	Graphic Calculator	Created herself
Group of PSTs	Middle Level	Smartboard	Exchange Smarttech website

Table 1: Participant Information

Research Design

The researcher presented case studies to further investigate the understanding of how pre-service mathematics teachers selected mathematical tasks and aimed to implement them in the classroom environment. Case studies, according to Yin (2003), are "an empirical inquiry that investigates a contemporary phenomenon within its real-life context" (p.13-14). The case studies can be single or multiple case studies, and in this study, a multiple case study approach was chosen because the grade level of technology activities created by PSTs differed in each case.

Data Collection

This data source was examined to determine whether the level of cognitive demands was maintained, decreased, or increased during implementation, as well as student response. This study examined the types of mathematical tasks chosen by PSTs, how PSTs implemented these tasks, how students were expected to answer these tasks, and their relationships. The technology activities from the mathematics pedagogy course and the student teaching showcase portfolio (e.g., copies of technology activities used while student teaching and samples of student work from those activities) were collected for this purpose.

Data Analysis

The Instructional Quality Assessment (IQA) Mathematics rubrics were used to analyze technology activities (Boston, 2012). The IQA Mathematics Rubrics were used to determine the cognitive demand of the instructional task as well as the level of cognitive process engaged in by students while working on the task. "The IQA Toolkit was created to provide statistical and descriptive data about the nature of instruction and students' learning opportunities" (Boston, 2012, p. 5). The Levels of Cognitive Demand and the Mathematical Tasks Framework were used to create IQA rubrics.

	Task Type	
Low Level Cognitive Demands	Memorization	Involve either reproducing previously learned facts, rules, formulas, or definitions or committing facts, rules, formulas or definitions to memory. Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure. Are not ambiguous. Such tasks involve the exact reproduction of previously seen material, and what is to be reproduced is clearly and directly stated.

		Have no connection to the concepts or meaning that underlie the facts, rules, formulas, or definitions being learned or reproduced.
	Procedures without connections	Are algorithmic. Use of the procedure either is specifically called for or is evident from prior instruction, experience, or placement of the task. Require limited cognitive demand for successful completion. Little ambiguity exists about what needs to be done and how to do it. Have no connection to the concepts or meaning that underlie the procedure being used. Are focused on producing correct answers instead of on developing mathematical understanding. Require no explanations or explanations that focus solely on describing the procedure that was used.
High Level Cognitive Demands	Procedures with connections	Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas. Suggest explicitly or implicitly pathways to follow that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts. Usually are represented in multiple ways, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations helps develop meaning. Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with conceptual ideas that underlie the procedures to complete the task successfully and that develop understanding.
	Doing Mathematics	Require complex and nonalgorithmic thinking—a predictable, well-rehearsed approach or pathway is not explicitly suggested by the task, task instructions, or a worked-out example. Require students to explore and understand the nature of mathematical concepts, processes, or relationships. Demand self-monitoring or self-regulation of one's own cognitive processes.

	Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task. Require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions. Require considerable cognitive effort and may involve some level of anxiety for the student because of the unpredictable nature of the solution process required.

Table 2: Four Types of Mathematical Tasks (Stein, Smith, Henningsen, & Silver, 2009)

The IQA Mathematics rubrics were used to assess the instructional quality of technology-based instructional activities using three indicators: the written instructional task, task implementation, and expected student responses. Data from PST technology activities will be graded using the Instructional Quality Assessment (IQA) Academic Rigor (AR) in Mathematics rubrics for Task Potential, Described Implementation, and Expected Student Responses.

Potential of the Task. The cognitive demand of the mathematical task as it appears (i.e., as written or on screen) in the technology activity is coded as The Potential of the Task. The original IQA Academic Rigor 1 (AR1) rubric will be used to code each task. The researcher coded Potential of the Task as "did the task have potential to engage students in rigorous thinking about challenging content?"

Described Implementation/Implementation. Task implementation is described as the level at which the teacher supports students to engage with the task throughout the lesson, or how tasks are enacted during instruction. For data from PSTs' student teaching (e.g., instructional activities and student work), the cognitive process evidence in students' written work will be scored for Task Implementation using the IQA Mathematics Assignments-Academic Rigor rubric for Implementation (AR2). For data from the methods courses, PSTs' technology activities will be coded for "Described Implementation" based on the description of how the PST aims to use the technology tasks in the instructional activity. The rubric for Described Implementation was modified from the original IQA Mathematics Academic Rigor-Implementation rubric (AR2) and was tested during the pilot study and another study of cyber-based curriculum.

Expected Student Responses/Student Responses. Expected student response is the extent to which students show their work and explain their thinking about the important mathematical content. The Expected Student Response rubric was modified from the original Academic Rigor 3 (AR3) Elaborates of Student Responses rubrics in the IQA Mathematics Assignments rubrics and tested in the pilot study. The modified rubric will be used to score "expected students' responses" in PSTs' technology activities from the methods courses. The original "Elaborates of Student Responses" rubric will be used to score samples of students' work from PSTs' student teaching lesson activities.

CASES

The cases presented in this section demonstrate: 1) how PSTs used the same task in different ways; 2) how PSTs maintained high level cognitive demands during implementation and student responses; 3) how PSTs reduced high-level cognitive demands during implementation and student responses; and 4) how PSTs increased high-level cognitive demands during implementation and student responses. These cases were chosen because they demonstrated how PSTs attempted to implement various tasks in various ways.

Integration of same task differently

The first case illustrates how the same technology task is described and used differently in different lesson activities. This task is retrieved from http://illuminations.nctm.org/Activity.aspx?id=3540 and is illustrated in Figure 1.

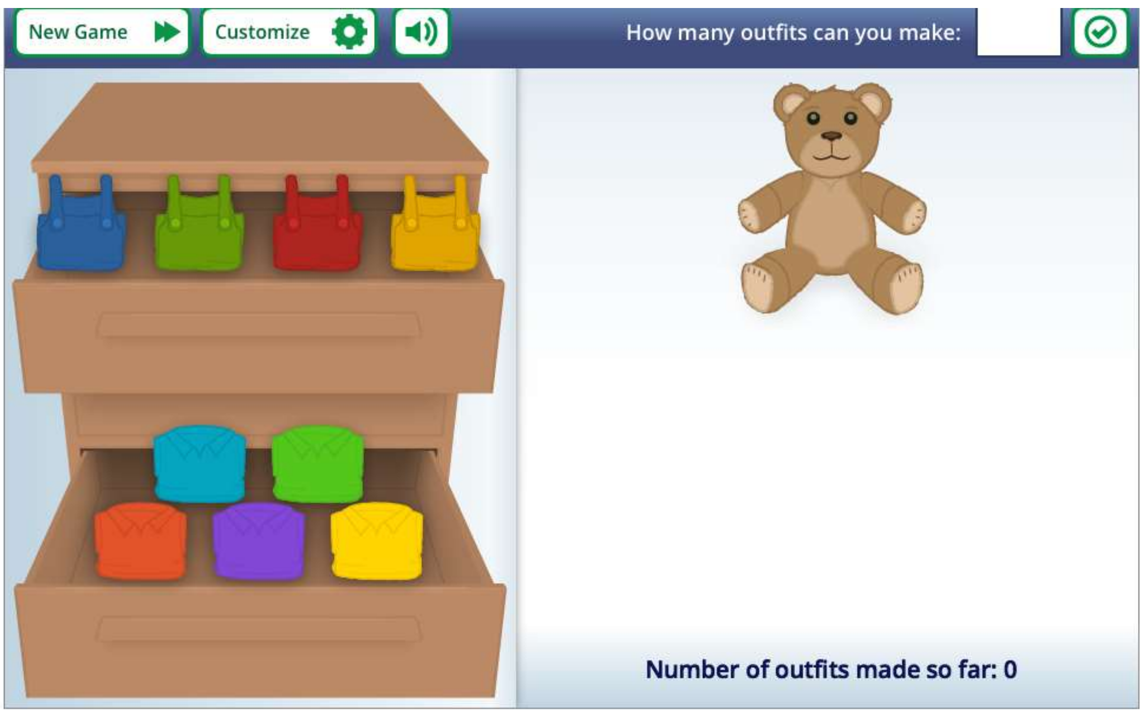


Figure 1: Screenshot of Bobbie Bear activity

This "Bobbie Bear" activity can be used in a Pre-K to Fifth Grade classroom to help students learn about using counting strategies to see how many different outfit combinations they can make for Bobbie Bear. By putting together different outfit combinations, students can learn about combinations, addition, and multiplication. The customized settings differ by grade; the only thing that changes is the number of shirts and pants. The teacher can specify how many different pairs can be created and which levels of difficulty the students use. The activity's instructions are as follows: "Bobbie Bear is planning a vacation and wants to know how many outfits can be made using different colored shirts and pants." "How many outfits can you put together?"

This activity also includes five questions for the teacher to use as a source of exploration with the students in order to broaden their knowledge (however, there is no lesson plan, suggested activity, or handouts that correspond with the applet).

- How many outfits do you think can be made?
- How do you know when you have made all the outfits?
- If you are missing an outfit, how do you find out which one it is?
- How can you organize your work to make answering these questions easier?
- Try your strategy for more shirts and pants using the Customize button.

The Task's Potential receives a 3 because students are asked questions that allow them to identify the combination of different colored shirts and pants. The task has the potential to engage students in the process of making sense of mathematical concepts and procedures. Because the task does not necessitate an explanation or evidence of students' reasoning and understanding (e.g., generalizing a shortcut or explaining why repeated addition, multiplication, a tree diagram, or the Fundamental Counting Principle is an appropriate strategy), it does not receive a 4, and thus does not receive a 4.

The Described Implementation and Expected Student Response scores can differ depending on how PSTs describe the task or technology's implementation within the instructional activity. The researcher provides examples from two cases below of how PSTs implement the same task in different ways and expect different student responses.

Zach is the first PST, and he was enrolled in the PK-4 Numeracy Pedagogy course (e.g., elementary mathematics methods). Zach used the SmartBoard to demonstrate the "Bobbie Bear" activity by incorporating the National Council of Teachers of Mathematics (NCTM) Illumination website. The reason for choosing this website was to address important mathematical content, as the activity he chose includes a variety of activities with addition. Zach emphasized the importance of teaching children addition concepts at a young age because "it is the foundation of a lot of different mathematical concepts they will encounter later in life."

Students learn about combinations and what they mean by adding up the various outfit combinations in this activity. Zach described task implementation as the teacher beginning the lesson by explaining different combinations and providing examples of different combinations. The implementation is then explained by Zach as follows:

> When the class has a solid foundation, the teacher can poll the students to see how many different outfits they can make for Bobbie Bear. After recording the class estimate, the teacher can direct students to come up to the board and drag the two pieces of clothing onto the bear. The teacher will then repeat this process until the class agrees that no more combinations are possible. The teacher will then be able to compare the class's estimate to

the number of outfits they were able to produce. The teacher can then check the students' answers, and the program will tell the class whether or not they were correct.

He described the procedure-level implementation, and his Described Implementation score is a 2. Students must focus on correctly executing a procedure to obtain a correct answer, rather than exploring, building meaning, explaining, or supporting their ideas. In fact, the described implementation makes no references to addition. Student Response receives a score of one because students are only asked to provide a brief numerical answer and find the correct number of combinations by typing numbers into the box. Zach reduced the cognitive demands for Described Implementation and Expected Student Response from high to low.

Emily is the second PST, and she is enrolled in the PK-4 Numeracy Pedagogy course. In her activity, the children could do the "Bobbie Bear" activity together on an iPad or a computer. She chose this project due to the fact that "this would be a fun interactive way for the students to apply their probability and computing possibilities knowledge in a fun and exciting way using technology" . During the implementation, she wanted the students to share their various problem-solving strategies. Furthermore, she stated that this activity could be used as an informal assessment of the children's knowledge: "While the students were playing this game, I could formatively assess them by walking around the room and seeing different strategies the students are using within their problem solving."

Emily described implementation at the "procedures with connections" level, and the Described Implementation score is 3, because students create meaning for mathematical procedures and concepts but are not explicitly required to produce explanations (e.g., to explain why 3 shirts and 4 pants result in 4 x 3 or 12 outfits), so it does not score a 4. Expected Student Response receives a 3 as well, because students must provide evidence of mathematical thinking and reasoning, such as multiple strategies, but no explanation is required. Emily keeps the cognitive demands for Described Implementation and Expected Student Response at the same level.

Lesson plans are part of the intended curricula, as described in Chapter 2, and the teacher's thinking about how lessons should be taught can be reflected in lesson plans (Remillard, 1999; Stein, Remillard, & Smith, 2007). Both PSTs chose the same task using the same technology activity (intended curriculum), but they aimed to enact the activity in different ways (enacted curriculum).

Maintenance of High-Level Cognitive Demands

This case demonstrates how PSTs maintained the cognitive demands of mathematical tasks while implementing them and anticipating student responses. Dora, the PST, was enrolled in Teaching Secondary Mathematics (e.g., secondary mathematics methods course). In a high school level (Grades 9-12) Algebra class, she creates an activity that involves the use of virtual Algebra tiles. Dora chose Algebra tiles, which are mathematical manipulatives designed to help students visualize symbolic representations through concrete models. Algebra tiles provide students with an alternative method of solving algebraic problems other than abstract manipulation. Algebra tiles can be used to practice a wide range of mathematical concepts, such as adding and subtracting integers, multiplying polynomials, factoring, and completing the square.

The task is to solve linear equations using Algebra tiles, and she chose an applet from the NCTM illuminations website (http://illuminations.nctm.org/activity.aspx?id=3482), as shown in Figure 2. Dora believes that this Internet applet is beneficial to students because it allows them to use technology to solve a mathematical concept rather than pencil and paper, and it allows students to visually see what they are doing to solve an equation.

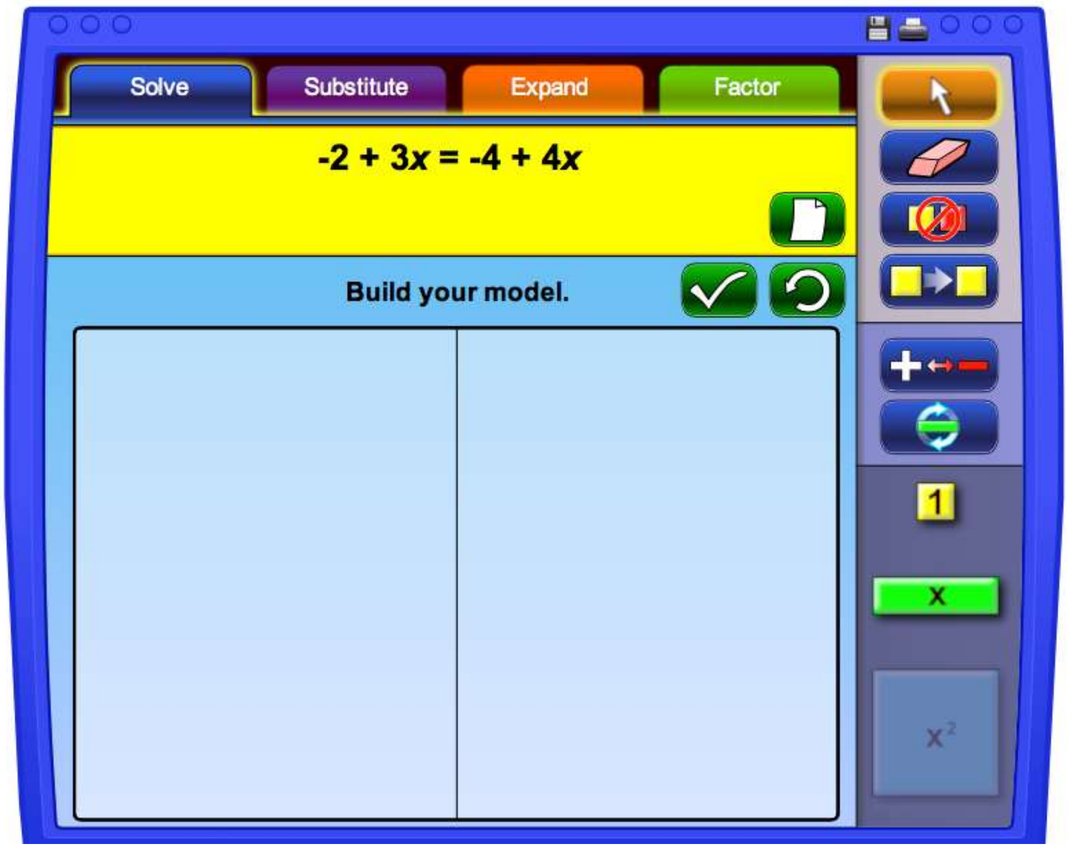
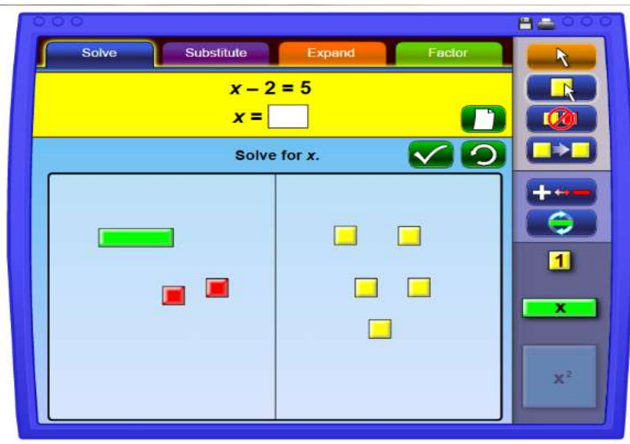


Figure 2: Screenshot of NCTM illuminations website

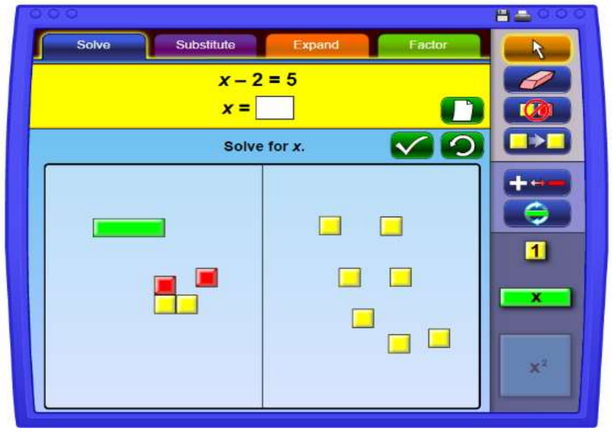
The task requires you to build a model and solve an equation. The website also includes a list of activities that students can do with applets: "Learn how to represent and solve algebra problems by using tiles to represent variables and constants." Solve equations, use variable expressions as substitutes, and expand and factor. "Flip tiles, remove zero pairs, copy and arrange your way to a better understanding of algebra." Because students are asked to build their own model, the Task's Potential is a 3. The task has the potential to engage students in the process of making sense of mathematical concepts and procedures. Dora outlined the process in detail:

1. Start with an equation

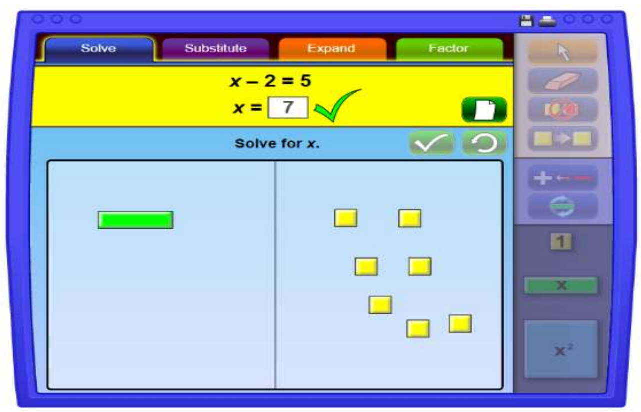
2. Use the pointer tool and place the correct pieces in the workspace. After you build the model of the given problem, check your answer to move on to the next step. Only tile type, tile quantity, and workspace area are checked, not the way in which tiles are arranged.



3. Try eliminating the necessary tiles to create zero pairs. Remember, what you do to one side, you must do to the other side!



4. After you solve the problem, check your answer.



5. Practice: Solve the following equations using the Algebra tiles:

a) $4x - 1 = 2x + 3$ c) $4x - 3 = 5$

b) $2x + 2 = 4$ d) $5x - 5 = 4x + 2$

Figure 3: Screenshot of Dora lesson plan

Dora planned to ask three questions after the students practiced above problems. These questions are:

1) How do the Algebra tiles allow you to better visualize the concept of zero pairs?

2) Explain the phrase "whatever you do to one side, you must do the exact same thing to the other side"?

Dora described procedures with connection level implementation as requiring complex thinking. The Described Implementation score is 4 because students must explain and comprehend the nature of mathematical concepts and procedures. Expected Student Response also receives a 4 because students must provide evidence of mathematical thinking and reasoning, such as multiple strategies, as well as explanation. Dora's described implementation maintained the high level cognitive demands of the original task and increased the score level from 3 to 4 for Described Implementation and Expected Student Response.

This case illustrates the preservation of high-level task demands for described implementation and anticipated student response. This case is significant because task implementation resulted in higher student achievement by maintaining the cognitive demand of instructional tasks. Figure 4 depicts the patterns of set up, implementation, and student learning described by Stein and Lane (1996). High-level cognitive demands during task setup and maintenance lead to high-level student learning.

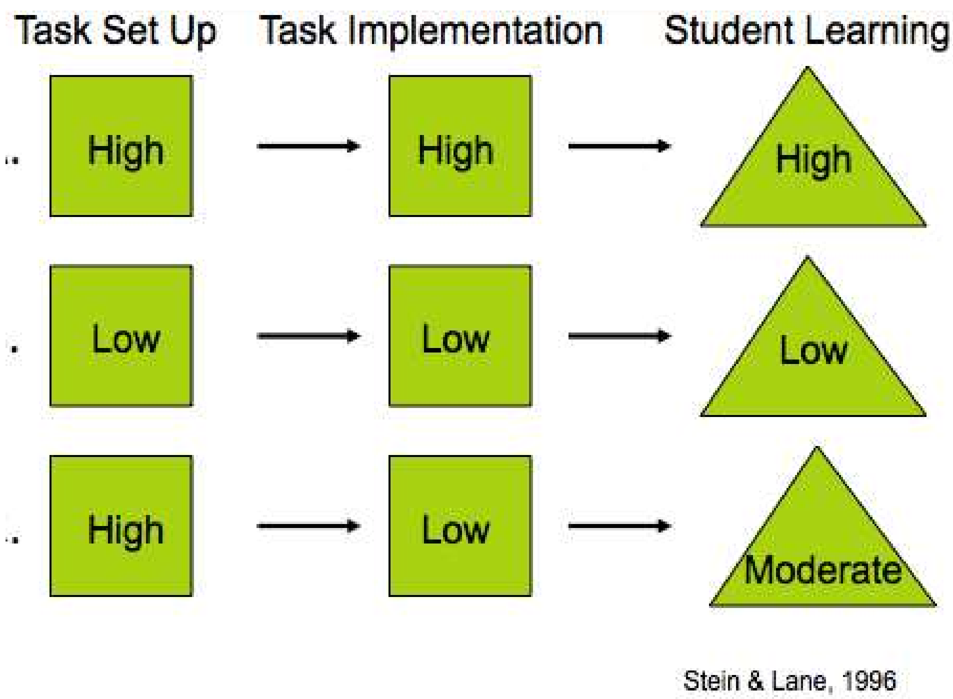


Figure 4: Patterns of Set up, Implementation, and Student Learning

Increasing Low-Level Cognitive Demands

In this case, it is discussed how PSTs increased the cognitive demands of mathematical tasks during implementation and the expected student response. PSTs (working in groups of three) from the Teaching Middle Level Mathematics (middle level mathematics methods) course created this activity. This activity demonstrates how PSTs increased low-level mathematics tasks with high-level cognitive demands during Described Implementation and Expected Student Response. It is a SmartBoard activity called "Perimeter Patch" that deals with the concepts of area and perimeter. Figure 5 shows the activity that was chosen from http://exchange.smarttech.com/details.html?id=a06612eb-f7ec-43b4-9ae8-3e5b9784a7f1.

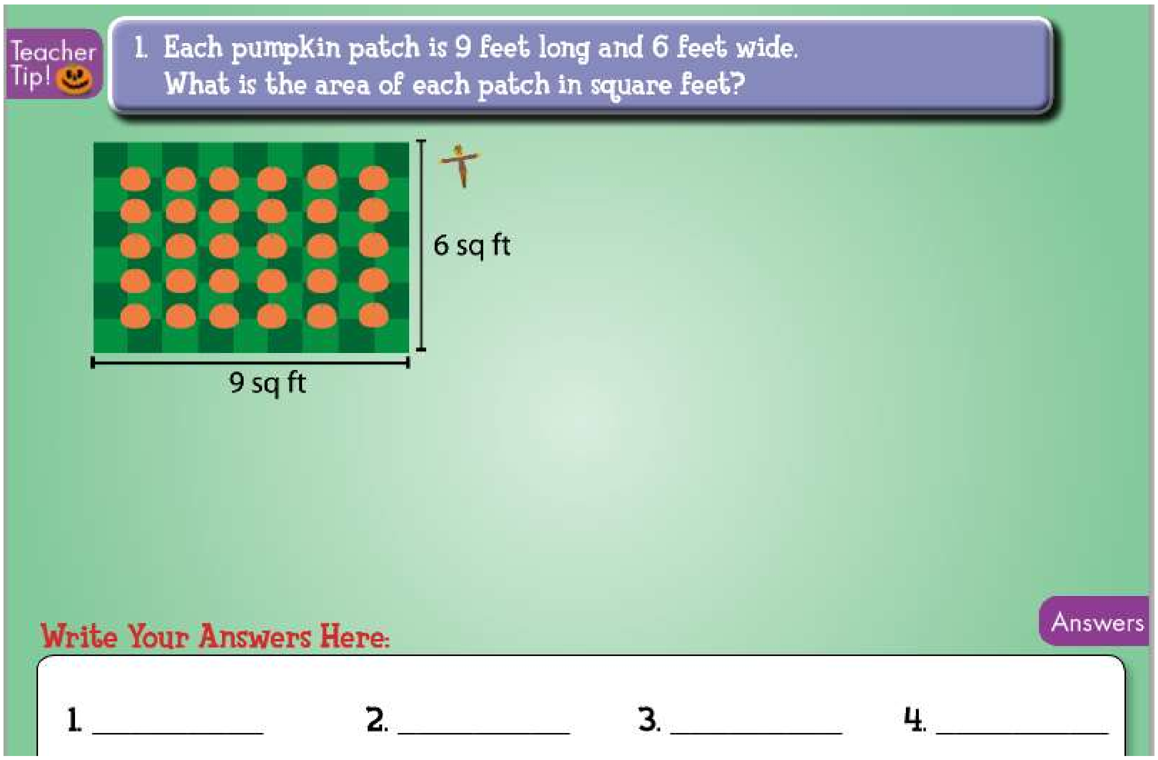


Figure 5: Screenshot of Middle Level PSTs' Lesson Activity

SmartBoard was chosen as a technological tool by this group because "it allows students to complete activities while having a visual representation, and they are also able to interact with the SmartBoard throughout the lesson as they work to grasp the concepts of area and perimeter." This SmartBoard lesson is a colorful and engaging way for students to learn about area and perimeter while interacting with technology. This activity can also help students apply the concepts of area and perimeter to real-life situations and understand why they are important."

"Each pumpkin patch is (9) feet long and (6) feet wide," says the task. "How big is each patch in square feet?" The task asks 8 similar-format questions (with different numbers for length and width) and allows students to "Write your answers here." The Task's Potential receives a 2 because it does not require students to make connections to concepts or meaning of content (e.g., students could produce the answers procedurally or from memory without making any connections to area, length, width, or square feet) and the task's focus is writing the correct answer.

The PSTs described implementation as follows:

> The following SmartBoard slides in this activity will look at pumpkin patches with different sizes but the same perimeter. Then we'll look at pumpkin patches that are the same size but have a different perimeter. Students will understand that area and perimeter are not always related, and that just because two objects have the same perimeter, they do not have to have the same area, and vice versa.
>
> The following section of the SmartBoard lesson will consist of a problem for students to solve. I'm going to demonstrate an empty pumpkin patch and ask: "If each block of the pumpkin patch counts for one square yard, and 4 pumpkins can fit in each square yard, then how many pumpkins can fit in the patch if the area of the pumpkin patch is 25 blocks?" The blank pumpkin patch will be displayed on the SmartBoard, and students will be able to come up to it and drag and drop pumpkins into each of the squares as they work to solve the problem. Students can also use the manipulatives provided at each table to assist them in solving the problem. Some students may be able to create a formula and solve the problem using worksheets. We will solve the problem on the Smartboard after students have solved it on their own and demonstrate the various ways to find the answer.

The PST group described implementation at the procedure with connections level, and the Described Implementation score is 3. The PSTs want their students to use a variety of strategies and manipulatives to complete the task. The perimeter and area questions require students to engage with and comprehend mathematical concepts. Expected Student responses receive a 3 as well because they were asked to create a formula or use multiple strategies or diagrams to find the correct answer and demonstrate their understanding of perimeter and area.

This case demonstrates how teachers and PSTs can increase the task's cognitive demand during instruction. This case is significant because enacting this task with high level cognitive demands results in various types of student thinking and opens up opportunities for higher order thinking. Furthermore, assigning tasks with a higher cognitive demand to students during instruction can result in higher achievement and conceptual understanding.

Decline of High-Level Cognitive Demands

This case shows how PSTs reduced the cognitive demands of mathematical tasks during implementation, lowering the expected student response from high to low. Carrie is the PST, and she has finished her student teaching. This case is based on a task, implementation reflection, and student work samples submitted as part of her student teaching Showcase Portfolio.

This activity is designed for a 9th and 10th grade Honors Algebra 2 class to create an equation for a quadratic relationship. The goal of this task is to learn how to use a graphing calculator to calculate a quadratic equation that passes through three given points. Using a graphing calculator assists students not only in creating the equation but also in understanding how they will be able to apply this knowledge when working with polynomials.

Figure 6a depicts the task, while Figure 6b depicts the calculator instructions. A mathematical task, as previously stated, can be a single problem or exercise (simple or complex and multi-step) or a collection of related problems or exercises that focus students' attention on a specific mathematical idea (Stein, Smith, Henningsen, & Silver, 2009). This group of related problems is graded as a single task. The Task's Potential receives a 4 because it requires students to engage in complex mathematical thinking and provide an explanation. The final question is "Calculate the revenue if the t-shirts were sold for $4 each, explain what this would mean," and this question earns the task a 4 on the scale.

The school store at Norwin sells T-shirts among other items. The table shows data from the last four years for the price charged for a T-shirt, x, and the total revenue earned from selling them, y.

X	8	10	12	14
Y	1180	1450	1675	1550

1st Observe the table and predict what price should be used to maximize revenue.

2nd Use a graphing calculator to find the best-fitting quadratic model for the data in standard form. (see front board for instructions)

3rd Plot the scatter plot on the calculator

4th Graph the best-fitting quadratic on the calculator

5th Calculate the price of the t-shirts that would maximize the revenue

6th Calculate the total number of t-shirts sold when maximizing the revenue

7th Calculate the revenue if the t-shirts were to be sold for $4 each, explain what this would mean.

Figure 6a: Screenshot of Lesson Activity

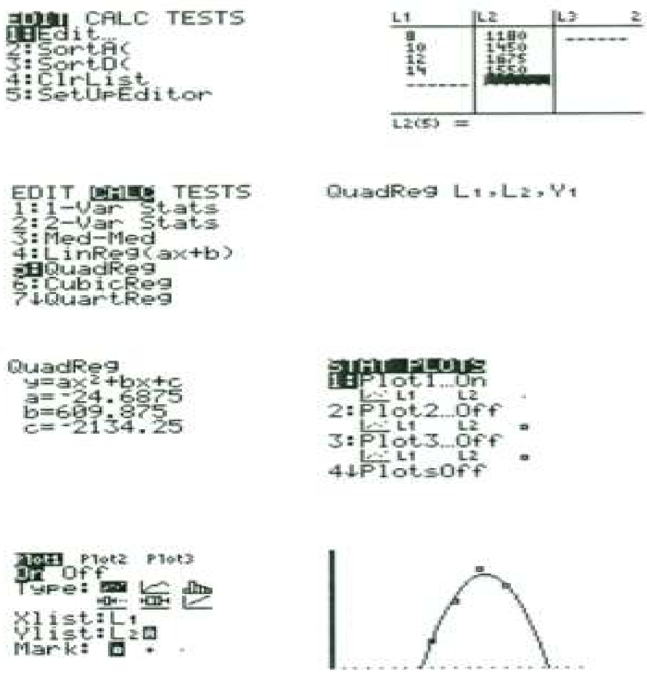


Figure 6b: Screenshot of Calculator Page

Graphing Calculator Instructions for Quadratic Functions

1. STAT Button
   a. 1.Edit
   b. Put x values in
   c. Put y values in
2. 2nd Y= (Stat Plot)
   a. 1. Plot 1
   b. Enter
3. Zoom Button
   a. 9. ZoomStat
4. STAT Button
   a. Right to CALC
   b. 5. QuadReg
5. 2nd STAT (List)
   a. 1. L1
   b. Enter
   c. ,
6. 2nd Stat (List)
   a. 2. L2
   b. Enter
   c. ,
7. VARS Button
   a. Right to Y-VARS
   b. 1. Function
   c. 1. Y1
   d. Enter
8. You should then get your quadratic equation on your screen that passes through those points.

Figure 6c: Screenshot of Calculator Page

Carrie began the class by distributing the day's warm-up problem and handing out the worksheet with the graphic calculator instructions and screen shots on it. This is useful for students to refer to as they work through the problem. Carrie walked around the classroom, assisting students as they worked through this worksheet. She intended to provide opportunities for students to apply critical thinking and problem-solving skills.

However, student response is a 2 because students only provide one-word descriptions or simply solve the task without providing an explanation. Students were expected to plot the scatter plot on the calculator, but there was no evidence of student work with the graphing calculators on their worksheets. Carrie received a 2 for implementation because she intended to use a graphing calculator to teach a quadratic equation, but she describes how limited access to technology in her class made incorporating the graphing calculator into the lesson difficult. Carrie was able to secure laptops for students who did not have their own graphing calculator, and if necessary, they used a website with a graphing calculator at home. Because some students were unable to use graphing calculators in class, they were unable to engage in high-level thinking and reasoning during the lesson.

In this case, technology would have served as a reorganizer, but limited access to the technology prevented students from making connections with representations (High Level Cognitive Demands) that the technology would have illustrated. Furthermore, as stated by Stein and Lane (1996), the decrease in cognitive demands during implementation resulted in moderate student learning.

CONCLUSIONS

Five different case studies were described in this study. These cases discuss how PSTs aimed to implement the same task differently, as well as to maintain, decrease, and increase the level of cognitive demands during the described implementation. These cases can assist teacher educators and PSTs in understanding how to design and implement instructional activities and technology within the context of mathematics for students' higher mathematics learning and success. These examples are significant because the selection and implementation of instructional tasks has an impact on students' mathematical understanding. Maintaining the cognitive demand of instructional tasks through task implementation, as defined by Stein and Lane (1996), resulted in higher student achievement.

The study's recommendations can be used as a guide in mathematics teacher preparation programs. The findings of this study can help mathematics teacher educators prepare PSTs to use technology to support students' high level mathematical thinking by providing a framework (e.g., attention to cognitive demands) and examples. While the focus of this study was on PSTs, considering cognitive demands when planning instructional activities is also a useful framework for classroom teachers. Similarly, while this study focused on mathematics content, the findings have the potential to guide mathematics or instructional technology courses in universities that are preparing PSTs to incorporate technology into instruction in ways that support students' learning.

It is critical to train PSTs to use technological tools in ways that support students' high-level thinking in any subject. Teacher educators should be aware of the importance of attending to cognitive demands when using technology as a teaching and learning tool, and future teachers should be prepared with this goal in mind. The study could provide resources and/or materials for mathematics teacher educators to consider various levels of cognitive demands of tasks in and outside of mathematics.

Several studies found that the use of technology had a positive influence on student achievement (Bebell & Kay, 2010; Bebell & O'Dwyer, 2010; Higgins, Huscroft-D'Angelo, & Crawford, 2019; Ran, Kim, & Secada, 2022; Shapley, Sheehan, Maloney, & Caranikas-Walker, 2010; Suhr, Hernandez, Grimes, & Warschauer, 2010). Because they grew up in a technologically advanced world, students in the classroom are digital natives (Prensky, 2001). Today's PSTs can integrate technology into their lesson plans and indicate that they are open to the idea, but they need guidance to do so effectively. This guidance is a blend of technological, pedagogical, and content knowledge. Not only must teacher education programs address pedagogical and content knowledge, but also the use of technology within specific pedagogy (e.g., learner-centered classrooms) and content (e.g., mathematics). Method courses provide future teachers with pedagogical content knowledge while also providing opportunities for PSTs to increase their technology knowledge within the context of pedagogical and content-related goals. Mathematics teacher educators should help PSTs effectively design technology-based instruction, help PSTs integrate technology into lesson plans, and provide opportunities for PSTs to use technology in field experience or student teaching classrooms. According to the findings of this study, one productive path would be to provide guidance to PSTs on how to maintain or increase the level of cognitive demands.

Classrooms, especially for PSTs, are complex environments. Prospective teachers need more opportunities to design and implement technology-based instructional activities that support students' learning. According to Haryani and Hamidah (2022), technology-integrated worksheets allow students to explore content rather than simply answering questions, increasing student engagement and understanding in discussions. PSTs, in particular, require opportunities to teach these activities as part of teacher preparation programs, field experiences, and student teaching in order to be prepared and comfortable incorporating technology into their future classrooms. The use of technology by PSTs is unlikely to be successful unless it is practiced prior to and during student teaching. PSTs, for example, can practice by developing and delivering technology-based instructional activities that combine technology, pedagogy, and content knowledge. As a result, teacher education programs should provide opportunities for PSTs to incorporate technology into methods courses and student teaching placements. The ultimate goal is for PSTs to apply what they've learned in their future classroom settings. The level of cognitive demand for mathematical tasks is implemented at a lower level than predicted during implementation (Boston, Candela, & Dixon, 2019; Henningsen & Stein, 1997). In practice, high-level tasks in textbooks or lesson plans should be reduced (Dede, Ünal, &Ylmaz, 2023). Because most mathematical tasks in activities and implementations have a low cognitive demand (Dede, Ünal, &Ylmaz, 2023; Reçber & Sezer, 2018), this study will shed light on how mathematical tasks are designed and integrated in practice.

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Appendix A

Academic Rigor 1: Potential of the Task

Instructional Quality Assessment (IQA) in Mathematics Rubrics (Boston, 2012)		
AR1: ***Potential of the Task***		
High-Level Cognitive Demands		The task <u>has the potential to engage students in exploring and understanding the nature of mathematical concepts, procedures, and/or relationships</u>, such as (from Stein, et al., 2009): • Doing mathematics: using complex and non-algorithmic thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example); or • Procedures with connections: applying a broad general procedure that remains closely connected to mathematical concepts. The task <u>must explicitly prompt</u> for evidence of students' reasoning and understanding. For example, the task MAY require students to: • solve a genuine, challenging problem for which students' reasoning is evident in their work on the task; • develop an explanation for why formulas or procedures work; • identify patterns;…justify generalizations based on these patterns;…
		The task <u>has the potential to engage students in complex thinking or in creating meaning for mathematical concepts, procedures, and/or relationships.</u> However, the task does not warrant a "4" because: • the task does not explicitly prompt for evidence of students' reasoning and understanding. • students may need to identify patterns but are not pressed to form or justify generalizations; • students may be asked to use multiple strategies or representations but the task does not explicitly prompt students to develop connections between them;…
Low –Level Cognitive Demands		The potential of the task is limited to engaging students in using a procedure that is either specifically called for or its use is evident based on prior instruction, experience, or placement of the task…. The task does not require students to make connections to the concepts or meaning underlying the procedure being used… (e.g., practicing a computational algorithm).
		The potential of the task is limited to engaging students in memorizing or reproducing facts, rules, formulae, or definitions…

Appendix B

Academic Rigor 2: Implementation of the Task

	Implementation of the Task (Boston, 2012)
	Students engage in using complex and non-algorithmic thinking or by exploring and understanding the nature of mathematical concepts, procedures, and/or relationships.
	Students engage in complex thinking or in creating meaning for mathematical proceduresand concepts BUT the problems, concepts, or procedures do not require the extent of complex thinking as a "4"; OR The "potential of the task" was rated as a 4 but students only moderately engage with thehigh-level demands of the task .
	Students engage with the task at a procedural level. Students apply a demonstrated orprescribed procedure. Students may be required to show or state the steps of their procedure, but are not required to explain or support their ideas. Students focus on correctly executing a procedure to obtain a correct answer.
	Students engage with the task at a memorization level. Students are required to recallfacts, formulas, or rules (e.g., students provide answers only). OR The task requires no mathematical activity.
N/A	Reason:

Appendix C

Academic Rigor 3: Expected Student Response

Expected Student Response (Boston, 2012)	
	The expected student response provides evidence of students' mathematical thinking and reasoning (such as multiple representations or strategies, diagrams, etc.)AND an explanation is explicitly required.
	The expected student response provides evidence of students' mathematical thinking and reasoning (such as multiple representations or strategies, diagrams, etc.) BUTno explanation is required.
	The expected student response is a computation or procedure,…or procedural explanationsuch as "Show your work." Students are not required to demonstrate connections to mathematical concepts in theirresponse to the task, even if task itself provided opportunities for connections.
	Students *are asked to* provide brief numerical or one-word answers (e.g., fill in blanks,provide only the result or answer).
N/A	Reason:

Examining Pre-service Mathematics Teachers' Pedagogical Content Knowledge (PCK) during a Professional Development Course: A Case Study

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Abstract: Peer teaching is a valuable practice that helps pre-service teachers to acquire the knowledge and the expertise required for teaching as well as improving their pedagogical content knowledge. For that reason, this study focused on the pre-service mathematics teachers' peer teaching, with the intention of exploring their pedagogical content knowledge as mirrored in peer teaching. The participants were four pre-service secondary mathematics teachers. Each participant designed a lesson plan and conducted peer teaching on a mathematics concept of her choice. The lesson plans and videotapes of the participants' peer teaching were the data collection instruments. The components of pedagogical content knowledge were used to analyze the data. The findings revealed that the pre-service mathematics teachers were knowledgeable about the subject content matter, but their knowledge of learners, understanding of instructional strategies and familiarity of context was inadequate about the topic of their peer teaching. The study recommends an amalgamated teaching of mathematics content courses and pedagogy courses.

Keywords: *pedagogical content knowledge, pre-service mathematics teacher, peer teaching*

INTRODUCTION

One of the crucial goals of mathematics teacher preparation curricula is to assist trainee teachers in improving their mathematics content knowledge and expertise for good teaching using their assignments, tests and practice. Mathematical content and general pedagogy courses are taught in mathematics teacher training institutions to enhance the development of trainee teachers' mathematical knowledge and pedagogical knowledge. Nevertheless, teachers not only require to be knowledgeable about mathematics content and pedagogy but in addition, knowledge of learners, instructional resources, curriculum and assessment as well as the ability to interlink them efficiently (Shulman, 1986). Peer teaching is considered to be one of the teacher training components that help trainee teachers in developing their skills of interlinking various forms of knowledge for realistic teaching. During peer teaching trainee teachers focus on how difficult or easy is a specific topic to the learners, learning objectives specified for a particular concept in the mathematics syllabus, teaching approaches and resources that enable learners' learning as well as assessing learners' intellectual capacity (Adu-Yeboah & Yaw Kwaah, 2018; Kilic, 2010).

For the purposes of providing trainee teachers with practical skills of teaching before they are deployed for practicum, trainees take part in micro-teaching and peer teaching activities. Peer teaching is carried out in artificial environments to prepare the trainees for the real classroom setup. Peer teaching provides the trainees an opportunity to use the mathematical subject knowledge in addition to instructional knowledge they have learned during training (Kartal, Ozturk, & Ekici, 2012). Hence, trainee teachers ought to nurture and develop pedagogical content knowledge (PCK) (Shulman, 1986). PCK skills and ability can be developed through reflection and practice (Mason, 1999; Ayhan, 2012). According to Kind (2010) the possession of good content matter, classroom familiarity as well as having emotional qualities such as individual self-confidence as well as providing a supportive working environment where collaboration is encouraged for the development of PCK in trainee teachers.

The PCK domains are integrated and connected hence cannot be regarded as distinct knowledge bases (Cochran, De Ruiter, & King, 1993). Although the domains of PCK are interrelated, scholars have concentrated more on how several opportunities provided in pre-service teacher training courses support the growth of PCK components separately (Kilic, 2010; Rianasari, 2017). Lee, Brown, Luft and Roehrig (2007) observed that pre-service teachers have fragmented pedagogical content knowledge, hence, it is vital to support the integration of PCK domains (Aydin & Boz, 2013). For that reason, further research is necessary concerning the nature and growth of the interface amongst pre-service teachers' pedagogical content knowledge domains, also with regard to how several situations inspire the interface of PCK domains (Aydin, Demirdogen, Nur Akin, Uzuntiryaki-Kondakci, & Tarkin, 2015). In view of that, the current study intended to explore pre-service teachers' growth of the integrated PCK domains during peer teaching in a mathematics pedagogy course.

PEDAGOGICAL CONTENT KNOWLEDGE

Pedagogical content knowledge forms the basis for several teachers' actions during the teaching and learning process. According to Shulman (1986), pedagogical content knowledge is the technique of demonstrating and communicating the content so that it is understandable to the learners. For Shulman (1987) PCK is the amalgamation of subject matter and instruction in comprehending specific concepts and difficulties. His PCK takes account of knowledge possessed by learners, knowledge of instructional resources as well as educational context knowledge. Cochran, De Ruiter, and King (1993) retitled Shulman's PCK as pedagogical content knowing (PCKg) which is more aligned to the constructivist view of teaching and learning. As stated by them, PCK is dynamic. To put more emphasis on the dynamic nature of PCK, the term "pedagogical content knowing (PCKg)" was used, which according to them is the teacher's combined comprehension of the characteristics of the learners, pedagogy, the learning environment and the subject matter content. According to them, pedagogical and content knowledge ought to be developed in the environment in which the teachers understand the learners as well as the environmental context of learning. PCKg is an amalgamation of the teachers' knowledge of pedagogy, subject matter, environmental contexts and learners.

SUBJECT MATTER KNOWLEDGE

Subject matter knowledge is the content of the mathematics concepts that are taught to the learners (Koehler, 2011; Shulman, 1986). The subject matter knowledge consists of specifics of the guidelines of proof and evidence, facts, major ideas, concepts, propositions, interactions, theories, suitable illustrations that make the content understandable, principles that are taught and learned, processes of a specific concept and information of descriptive contexts that consolidate and link notions. Subject matter knowledge is important in teaching any topic and it influences how teachers think and implement the curriculum (Shulman, 1986; Cochran, et al., 1993). Teachers must not merely possess the content of teaching the concept but should be able to adjust this knowledge to make it easy to learn by learners (Marcon, Graça, Nascimento, Milistetd, & Ramos, 2015). Teachers' subject knowledge enables them to comprehend school and classroom organization in addition to administration, the societies in which the learners come from as well as the institutional, constitutional, and political extents that affect the educational organization (Marcon, et al, 2015).

KNOWLEDGE OF PEDAGOGY

According to Shulman (1986), knowledge of pedagogy comprised teaching approaches, learning processes and classroom assessment. A deep understanding of pedagogical knowledge includes an understanding of learners' construction of knowledge; comprehend cognitive, societal, and understand the development of theories of learning as well as how those theories relate to the learners in the classroom; and being cognizant of what would be happening in all sections of the classroom and being able to handle a number of classroom events. In a nutshell, pedagogical knowledge consists of all issues linked to how learners learn, how they are assessed, how the classes are managed, knowledge of learners' characteristics, and the development and implementation of the lesson plan (Shulman, 1986). Pedagogical understanding consists of curriculum knowledge and knowledge of educational objectives and purposes as well as an understanding of what constitutes good teaching, taking into account the best teaching methods in a given situation, which are guided by suitable learning theories (Cochran, et al., 1993). Knowledge of pedagogy helps in developing teaching concepts and instructive principles, identifying pedagogical approaches, and planning, organizing as well as creating teaching and learning activities (Grossman, 2008; Marcon, et al, 2015).

KNOWLEDGE OF CONTEXTS

Knowledge of context includes teachers' understanding of the environmental contexts of learning, which comprise resources, learners' socio-economic background, curriculum, cultural, political, social, classroom conditions, availability of time for teaching and learning and physical environmental situations that affect teaching and learning.

KNOWLEDGE OF STUDENTS

This involves teachers' understanding of learners, such as skills, learning approaches, ages, developmental stages, attitudes, inspirations, and previous conceptions of a subject (Cochran, et al., 1993). Teachers need to be aware of the learners' misconceptions in a particular topic for them to understand learners' actions and ideas. The teachers need to know learners' previous knowledge before introducing new concepts. The student knowledge domain is important because teachers have to understand and consider their learners' needs including differences in learners' thinking, views, experiences, and knowledge of the environmental context (Marcon, et al, 2015).

THE INTERPLAY OF PCK COMPONENTS

Cochran, et al. (1993) stated that the amalgamation of the four domains consists of PCKg. The authors reported that teacher education must promote the acquisition of PCKg through offering concurrent experience of the four domains. According to Cochran, et al. (1993) PCK grows with the concurrent comprehension of the four domains of content matter, pedagogy, environmental context and students. The four domains must not be attained separately but must be attained simultaneously during training (Cochran, et al., 1993). According to Cochran, et al. (1993) the four domains might develop in an amalgamated way if the trainees experience the four domains simultaneously. PCKg growth is frequently complemented by educational change and notion amalgamation that comes from several hours of training, observing as well as imitating by trainees individually in addition to peers' teaching (Cochran, et al., 1993). According to Smith and Neale (1989) and Chien, Rohaida Mohd and Siow (2015), the amalgamation of PCK domains is crucial to the active teaching of mathematics and the more the domains are amalgamated the stronger they become and they result in a more developed PCK. Teaching is therefore an act of integrating all the PCK domains (Ayhan, 2012). Pre-service teachers usually acquire their pedagogical and subject matter knowledge from different academic departments (Ayhan, 2012). Cochran, et al (1993) buttressed a more all-inclusive approach to teacher education by disapproving the isolated acquisition of pedagogical and subject matter knowledge. Marcon, et al. (2015) also reported on the lack of evidence that shows the progress of PCK`s acquirement throughout the teacher training program. Pre-service teachers' PCK is elementary and inadequate (Cochran, et al., 1993). Therefore, the focus of the current study is on the pre-service teachers' development of PCK during peer teaching.

PEER TEACHING

PCK is making use of the comprehending of content knowledge, learning procedures, and approaches for teaching a particular mathematics concept in a manner that allows effective knowledge construction in a particular context by learners (Cochran, King, & De Ruiter, 1991). Peer teaching is an essential technique in teacher training institutions that significantly contribute to pre-service teachers' PCK development (Baştürk, 2016). Through peer teaching, trainee teachers have a chance of putting into practice their content knowledge and pedagogical content knowledge (PCK) (Baştürk, 2016). Pre-service teachers have a chance of identifying as well as improving their weaknesses of teaching skills, for instance, lesson planning and development, classroom management, organization of group work and many others. Peer teaching is an essential component for trainee teacher preparation programs that are more realistic as compared to traditional teaching (Baştürk, 2016). Peer teaching experiences help trainee teachers to be familiar with the actualities of teaching, to have a chance to have an understanding of their duties as teachers, to recognize the significance of making preparations for teaching, making decisions, and putting into practice instructional techniques, to cultivate and develop their teaching expertise, as well as building their sureness for teaching (Subramaniam, 2006; Baştürk, 2016).

Pre-service teachers are also required to master a number of basic skills of opening and closing lessons, asking questions as well as providing reinforcement, clarifying and giving varying incitement (Rianasari, 2017). According to Magnusson, Krajcik, & Borko (1999), teaching experience is an essential aspect for the growth of PCK. Consequently, pedagogical content knowledge comprises of experimental knowledge and skills acquired from teaching practice and are also an amalgamated organization of knowledge, ideas, theories, philosophies and principles developed by teachers during teaching practices (Magnusson, Krajcik, & Borko, 1999; Kartal, Ozturk, & Ekici, 2012).

Pedagogical content knowledge develops in conjunction with teaching experience (Rianasari, 2017). According to Lee, Brown, Luft and Roehrig (2007), courses that are taken by pre-service teachers during their training help in the growth of PCK. One of such courses is the mathematics methodology course in which peer teaching practices are carried out. Peer teaching is a hands-on method that provides teaching expertise as an output where there is an opportunity for analysis (Kartal, Ozturk, & Ekici, 2012). Thus, peer teaching contributes greatly to pre-service teachers' development of PCK (Kartal, Ozturk, & Ekici, 2012). Peer teaching involves a number of practices and ideas regarding how a topic might be taught better using diverse approaches, techniques and methods, how the topic might be made more comprehensible for learners (Kartal, Ozturk, & Ekici, 2012). Pedagogical content knowledge classifies the distinct forms of knowledge that are ideal for teaching (Shulman, 1987). Trainee teachers' growth of Pedagogical Content Knowledge skills can be measured from two facets, specifically, being able to develop a lesson plan and to implement the lesson plan in the class (Karim & Danaryanti, 2020). Hence, peer teaching contributes to the trainee teachers' development of their knowledge of teaching a specific subject-matter, pedagogy, and learners. Given the explanations provided above, this study focuses on the pre-service mathematics teachers' growth of pedagogical content knowledge during peer teaching. The following research question guides the current study: How does a mathematics methodology course influence pre-service teachers' development of pedagogical content knowledge?

METHODOLOGY

The current study used descriptive case study to describe in detail the pre-service teachers' growth of PCK in a mathematics methodology course. Saunders, Lewis and Thornhill (2009) pointed out that descriptive research depicts a precise of events, situations or individuals. This design helped the researchers to describe in detail the relevant aspects of the pre-service teachers' PCK development.

PARTICIPANTS

The population of this study comprised four second-year undergraduate female pre-service mathematics teachers taking a mathematics methodology course at a university in Zimbabwe. After completing the second year the pre-service teachers are deployed into schools for a year for their practicum. The teachers were diverse in terms of ethnicity. The four female pre-service teachers enrolled in the methodology course were invited to take part in the study at the beginning of the semester and volunteered to participate in the study by signing their consent forms.

CONTEXT OF STUDY

Pre-service teachers' PCK needs to be developed for effective mathematics teaching. For that reason, courses focusing on teaching about the connection of subject matter content knowledge and pedagogy integration have been of great importance particularly for the teacher training department at the university under study. One such course is the mathematics methodology course offered to pre-service teachers before they are deployed for practicum. The course aimed to equip the trainee teachers with the expertise required for the actual teaching during practicum and after graduating. In addition, to the provision of practical experience on how to integrate theory and practice through the amalgamation of content and pedagogy, the course could be ascribed as focusing on the integration of subject matter content and pedagogy to become Pedagogical Content Knowledge (PCK). The implementation of the methodology course was designed based on the notion that the course would support trainee mathematics teachers' growth of Pedagogical Content Knowledge (PCK). The implementation of the course took 12 weeks, which is 48 hours per semester excluding examinations. The course comprised of topics such as aims of teaching mathematics and the mathematics curriculum, interpretation and development of a school syllabus, scheming and lesson preparation, teaching approaches, assessment and evaluation, question techniques, teaching/learning challenges in mathematics, peer teaching as well as a focus on content courses such as set theory, vectors, matrices, functions and their graphs, linear programming, simultaneous equations. Trainee teachers were required to develop a lesson plan based on the knowledge they gained during theoretical and practical lectures, in which they had to provide a description of what they were going to teach and how they were to teach it. In an attempt to meet the demands of such a lesson plan, pre-service teachers had to make use of their mathematics content knowledge, curriculum knowledge, pedagogy knowledge, knowledge of contexts and knowledge of learners and knowledge of pedagogy thus pedagogical content knowledge. They were also required to present peer teaching sessions based on the lesson plan they would have developed. Pre-service teachers assess and learn from their teaching practices and experiences during and after the implementation of the lesson. Teachers tend to rely on their experiences when teaching (Kilic, 2010), hence, peer teaching experiences might help in developing their knowledge domains. The peer teaching sessions were implemented during the eleventh and twelfth weeks.

PEER TEACHING PROCEDURES

In this study, peer teaching is whereby the pre-service teacher teaches her peers. Each pre-service teacher was asked to plan and present her lesson during the eleventh and twelfth weeks of the methodology course. Each peer teaching lesson was to be presented within 45 minutes followed by a brief discussion of the lesson. Additionally, the researchers and the peer teachers provided written feedback to the peer teacher. In the current study, peer teaching was used to assess trainee teachers' pedagogical content knowledge. The context of peer teaching was considered to be safe from the known constrictions that possibly obstruct the teaching of mathematics, for instance, classroom management, institutional constraints and the pressure to cover mathematics content.

Peer teaching provided the pre-service teachers with an opportunity of planning mathematics lessons and implementing their lesson plans, as well as observing their peers' teaching in the context of numerous mathematics content and teaching methods. Essentially, it must be noted that peer teaching cannot reflect the actual classroom context. There are chances that teachers might not implement the same teaching presented in peer teaching in their actual classroom situation. This is one of the limitations of the current study that must be well-thought-out in interpreting the findings.

DATA COLLECTION INSTRUMENTS

Two types of data collection instruments were used, written documents and classroom observations. The written documents comprised pre-service teachers' lesson plans. The lesson plans were prepared in such a manner to meet the requirements of the secondary school mathematics curriculum. The participating pre-service teachers were observed during peer teaching using an observation instrument. The lesson presentation was video recorded. A total of four pre-service lesson plans and videotaped lessons were gathered.

DATA ANALYSIS

The researchers read lesson plans and watched each of the videotaped lessons. Each peer teaching (videotapes and lesson plans) was coded and then categorized into themes as described by Patton (2002). The data were categorized into four themes: subject matter knowledge, Knowledge of teaching approaches, knowledge of learners, and knowledge of context. Specifically, each peer teaching was analyzed as per the four themes, using the following procedures: (0) if the pre-service teacher did not state the teaching approach in the lesson plan, (1) if the pre-service teacher stated the teaching approach. The four pre-service teachers were coded L, M, N and S for confidentiality purposes.

FINDINGS

The findings of this study were presented in two categories as an assessment of the four teachers' PCK through lesson planning and assessment of the pre-service teachers' PCK through observation.

ASSESSMENT OF THE FOUR TEACHERS' PCK THROUGH LESSON PLANNING

The knowledge that is essential for planning a good mathematics lesson is an aspect of the domains of pedagogical content knowledge (Prescott, Bausch, & Bruder, 2013). This section discusses the pre-service teachers' PCK through lesson preparation. Table 1 shows each pre-service teacher's evaluation regarding their knowledge of the subject matter.

Subject matter knowledge	Pre-service teachers			
	L	M	N	S
Major concepts to be taught were indicated in the lesson plan	1	1	1	1
Possible mathematics processes to be taught to the learners were indicated in the lesson plan	1	0	1	1
The lesson plan mirrors correct concepts related to the topic to be taught	1	1	1	1
Methods of summarizing the lesson through learners' involvement were indicated in the lesson	1	0	1	1

Table 1: Pre-service teachers' subject matter knowledge

An examination of the four pre-service teachers' lesson plans revealed that the lesson plans contained correct information regarding the concepts to be taught implying that they have the required knowledge about the content of the concepts to be taught. Teacher M did not indicate possible mathematics procedures associated with the concepts that she intended to teach. All four lesson plans comprised correct information about the topics they were going to teach. Teacher M also did not indicate ways of reflecting or making a summary by involving the learners.

Table 2 shows pre-service teachers' knowledge of teaching approaches.

Knowledge of teaching approaches	Pre-service teachers			
	L	M	N	S
Suitable teaching approaches were stated in the lesson plan	1	1	1	1
Inclusion of different instructional approaches in the lesson plan	1	0	0	0
Examples including real-life examples and analogues to be used during teaching indicated in the lesson plan	1	0	1	1

Table 2: Pre-service teachers' knowledge of teaching approaches

The four pre-service teachers (L, M, N, and S) stated the teaching strategy to be used during. Only the pre-service teacher (L) stated alternative teaching strategies such as group work activities to be used during the lesson. The examination of the lesson plans of the three pre-service teachers (M, N and S) showed that they have insufficient knowledge of teaching approaches to teach mathematics concepts as they heavily relied on the demonstration method which makes them the main imparters of mathematics knowledge.

Table 3 shows the pre-service teachers' knowledge of learners.

Knowledge of learners	Pre-service teachers			
	L	M	N	S
Inclusion of learners' prior knowledge in the lesson plan	1	1	1	1
Inclusion of learners' possible difficulties in the lesson plan	1	1	1	1

Table 3: Pre-service teachers' knowledge of learners

From the lesson plan analysis, it was noted that all four pre-service teachers indicated learners' prior knowledge and their possible difficulties in their lesson plans.

Pre-service teachers' knowledge of contexts is indicated in Table 4 below.

Knowledge of contexts	Pre-service teachers			
	L	M	N	S
Stating resources/media to be used during the lesson	1	1	1	1
Involvement of learners in the utilization of stated resources/media	1	0	1	1
Relevance or appropriateness of stated resources/media to be used to the concepts to be taught	1	0	1	1

Table 4: Pre-service teachers' knowledge of contexts

All the pre-service teachers stated the media/ resources in their lesson plan. Pre-service teachers L, M and S showed how the resources were to be used by students and the resources were relevant to the topics to be taught. Although, stated the resources to be used during the lesson plan did not show how the students were to utilise those resources. And an analysis of the stated resources revealed that those resources were not relevant to the topic that was to be taught.

ASSESSMENT OF THE PRE-SERVICE TEACHERS' PCK THROUGH OBSERVATION

The lesson observation focused on examining the pre-service teachers' subject matter knowledge, knowledge of teaching approaches, knowledge of learners and knowledge of context. The topics in which lessons were observed were number patterns, simultaneous equations, matrices and sets. Table 5 presents the evaluation of the pre-service teachers' peer teaching in terms of the participants' knowledge of the subject matter.

Knowledge of the subject matter	Pre-service teachers			
	L	M	N	S
Demonstrating mastery of concepts being taught.	1	1	1	1
Identification of crucial mathematical domains in the concept of mathematics that are essential for comprehending as well as applying that concept.	1	1	1	1
Explaining learning objectives related to the concept being taught	1	1	1	1
Performing procedures for solving mathematical problems	1	1	1	1
Makes connections between concepts and topics, including interdependence of concepts	0	0	0	1

Table 5: Pre-service teachers' subject matter knowledge

Generally, the pre-service teachers revealed that they had the required knowledge of the subject matter as they were able to present correct mathematical ideas about the concepts that they choose to teach. This could be due to the fact that they choose the topics that they were good at. However, three teachers could not make connections between concepts and topics, including the interdependence of concepts, excerpt for pre-service teacher S.

Pre-service teachers' knowledge of teaching approaches is shown in Table 6 below.

knowledge of teaching approaches	Pre-service teachers			
	L	M	N	S
Using proper strategies or approaches to teach concepts	1	1	1	1
Using real-life examples and analogies when teaching	0	0	0	1
Utilizing different instructional strategies when teaching	0	0	0	0
Using various illustrations when teaching such as formulae and graphics	0	0	0	1
Actively engage the learners	1	0	0	1
Checking for learner understanding during interactive teaching.	0	0	0	1

Table 6: Pre-service teachers' knowledge of teaching approaches

Generally, the pre-service teachers lacked the knowledge of teaching approaches for the mathematics concepts that they had chosen for their peer teaching. All the teachers demonstrated insufficient knowledge to teach the topics that they selected. Three teachers did not use real-life examples and analogies when teaching. All the teachers used the demonstration method; they did not vary their teaching approaches to meet the diversity of the learners. Although pre-service teachers N and S engaged the learners, they asked recall questions that were purely low order level. These types of questions did not engage learners in mathematical thinking. Only pre-service teacher S checked for learners' understanding during interactive teaching.

Table 7 shows an evaluation of the teachers' peer teaching concerning the learners' knowledge.

Knowledge of learners	Pre-service teachers			
	L	M	N	S
Identifying learners' particular ways of thinking about a concepts	0	0	0	0
Identifying certain learners who have misunderstandings about the concept, then giving an explanation	0	0	0	0
Identifying tasks that students feel are difficult to do	0	0	0	0
Linking concepts to the knowledge possessed by learners	0	0	0	1

Table 7: Pre-service teachers' knowledge of learners

The study showed that the pre-service teachers had insufficient knowledge of learners in the mathematics concepts that they had chosen to teach during peer teaching. However, pre-service teacher S made an effort to make links to students' prior knowledge to the topic she was teaching, whilst the three teachers could not make the connections. The pre-service teachers seemed not to have adequate knowledge to identify tasks that were challenging for the students in the topics that they selected to teach.

Pre-service teachers' knowledge of contexts is shown in Table 8.

Knowledge of context	Pre-service teachers			
	L	M	N	S
Linking concepts learned with everyday life	0	0	0	1
Using proper learning resources (locally, contextualized)	0	0	0	1
Linking concepts learned with other mathematics concepts related topic concepts	0	0	0	1
Involving learners in the utilization of resources/media	0	0	0	1

Table 8: Pre-service teachers' knowledge of contexts

Generally, the pre-service teachers demonstrated inadequate knowledge about knowledge of context. The three teachers were not able to link concepts learned with everyday life, use proper learning resources, linking concepts learned with other mathematics topics as well as involving learners in the utilization of resources/media. It appeared that pre-service teacher S was the only one who demonstrated adequate knowledge about the knowledge of context.

DISCUSSION AND CONCLUSION

PCK is essential in pre-service mathematics teacher preparation. Teaching experiences such as peer teaching influence pre-service teachers' PCK which concurs with Rianasari (2017) that PCK is generally influenced by pre-service teachers' teaching experiences. Mastering the content only is not sufficient for teachers to teach, they must have the knowledge of pedagogy that would enable them to transfer the content in a manner that the students would understand. Lim (2007) pointed out that successful teaching of a particular mathematics concept hinges on the deepness and comprehensiveness of the teachers' pedagogical content knowledge for the reason that, before teaching any lesson, it essential for the teacher to plan the lesson, select a teaching approach as well as choosing the content that is appropriate for the student's level of understanding. Teachers must have both the content and pedagogy knowledge which includes knowledge of learners, contexts and instructional strategies.

PRE-SERVICE TEACHERS' SUBJECT MATTER KNOWLEDGE

An analysis of both the lesson plan and the classroom observation demonstrated that the pre-service teachers had the knowledge of the subject matter required to teach the topics that they choose. The pre-service teachers' adequate knowledge of the subject matter could be due to the content courses that they had taken during training (Lee, Brown, Luft, & Roehrig, 2007). In addition, their knowledge of the subject matter was influenced by the fact that they were allowed to choose the topics that they were more knowledgeable about. Their knowledge of the subject matter could also have been influenced by the content which they have learned from high school and university as reported by Chien, Rohaida and Siow (2015).

PRE-SERVICE TEACHERS' KNOWLEDGE OF TEACHING STRATEGIES

The pre-service teachers' knowledge of teaching strategies in both the lesson and classroom observation was inadequate. All the teachers used only one method of teaching that is the demonstration method. The findings of this study concur with Kilic (2010) who also noted that pre-service teachers had inadequate knowledge of teaching approaches and representations as they view teaching as telling and demonstrating the procedures to the students so that they practice them. Although, pre-service teacher L indicated that she would use group work activities in her lesson plan, this was never used during her lesson delivery. Teachers' PCK encompasses their ability to use various teaching approaches that are appropriate for various learners with varied interests, abilities and learning styles.

PRE-SERVICE TEACHERS' KNOWLEDGE OF LEARNERS

Although, all the pre-service teachers indicated in their lesson plans the possible students' difficulties and their prior knowledge, their classroom observation did not reflect so. The classroom observation revealed that the pre-service teachers did not pay attention to the students' prior knowledge, misconceptions and difficulties; therefore, they lacked the knowledge of students. Since the pre-service teachers did not vary their teaching strategies, this also implies that they were not able to cater for the students' learning differences. They used the demonstration method only that disregarded students' differences and needs. The findings of this study are in line with (Kula Ünver, Özgür, & Bukova Güzel, 2020) who pointed out that pre-service teachers' knowledge of learners was very poor.

PRE-SERVICE TEACHERS' KNOWLEDGE OF THE CONTEXT

Even though the pre-service teachers demonstrated their knowledge of the context in lesson plans, they, however, demonstrated inadequate knowledge of context during a classroom observation. This could have emanated from the idea that the pre-service teachers had acquired the knowledge of context from their courses in the teacher education program that they applied in lesson planning but had a limited teaching experience which hinders their ability to consider the knowledge of context during peer teaching.

Overall, the pre-service teachers had inadequate PCK. This finding is in line with Chien, Rohaida and Siow (2015) who reported insufficient pre-service teachers' PCK. Personal learning history, teacher education and teaching practice experience influence teachers' PCK (Cochran, et al., 1993; Chien, Rohaida & Siow, 2015). In this study, teacher education could be the major factor contributing to the pre-service teachers' inadequacy in PCK, because the mathematics content courses are taught in isolation from the methodology courses. The mathematics content courses are learned for three semesters (one and a half years) before introducing the methodology course in the fourth semester (after one and half years) and the courses are offered from the different departments. This was also reported by Ayhan (2012) who reported that pre-service teachers generally acquire their pedagogical and subject matter knowledge from different academic departments. Cochran, et al (1993) was against the isolated acquirement of pedagogical and subject matter knowledge. The study recommends amalgamated teaching of both mathematics pedagogy and content courses. Mathematics teachers must be educated in both mathematics courses and mathematics pedagogy courses as combined courses and not as separate courses.
In this research pre-service teachers' growth of the integrated PCK domains during peer teaching in a mathematics pedagogy course was investigated. In order to get detailed information, more studies could be done using other data collection methods such as interviews that solicit pre-service teachers' views on PCK.

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The Problem Corner

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The Purpose of **The Problem Corner** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem and its solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Greetings, fellow problem solvers!

As the editor of **The Problem Corner**, I'm thrilled to announce that I've successfully received answers for both Problem 22 and Problem 23. I'm proud to report that all solutions were not only accurate but also demonstrated the effective application of strategies. My primary goal is to present what I believe are the best solutions to contribute to enhancing and elevating mathematical knowledge within our global community.

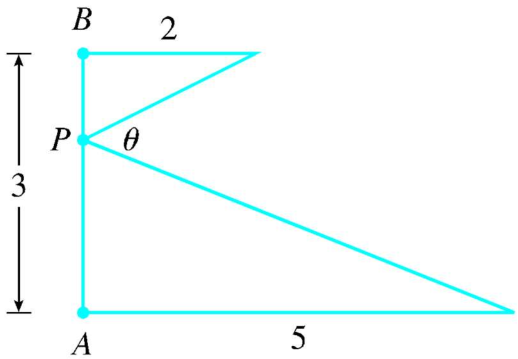
Solutions to **Problems** from the Previous Issue.

Interesting "optimization" problem.

Problem 22

Proposed by Ivan Retamoso, BMCC, USA.

In the illustration below, at what distance from B should point P be positioned to maximize the angle θ?

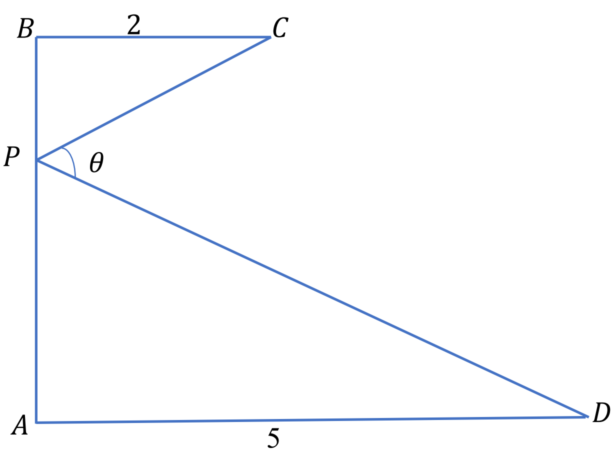


First solution to problem 22

By Dr. Abdullah Kurudirek, Mathematics Ed. Department, Tishk International University.

Our initial solution cleverly utilizes trigonometry, specifically employing the tangent function along with its inverse and derivative, to maximize the value of the angle θ.

Solution.
$CB \perp AB$ and $DA \perp AB$.
$AB = 3$.



First of all we should remember that this is an optimization question!
According to the given information, to make our operations understandable, we can use some labels as follows.
$\angle BPC = \beta$, $\angle APD = \alpha$, and for the segments $AP = x$ so $BP = 3 - x$ where $AB = 3$.
Since α, β and θ are on the same straight line the sum of them must be π. So, $\alpha + \beta + \theta = \pi$ and now let us solve this for θ

$$\theta = \pi - (\alpha + \beta) \qquad (1)$$

By using trigonometry in these right angled triangles we get $tan\alpha = \frac{5}{x}$ then (by using inverse trigonometric function) $\alpha = arctan\left(\frac{5}{x}\right)$ and in a similar manner $tan\beta = \frac{2}{3-x}$ then $\beta = arctan\left(\frac{2}{3-x}\right)$. Let substitute these in the equation (1) and get

$\theta = \pi - arctan\left(\frac{5}{x}\right) - arctan\left(\frac{2}{3-x}\right)$ now differentiate both sides w.r.t x and get

$\frac{d}{dx}[\theta] = \frac{d}{dx}\left[\pi - arctan\left(\frac{5}{x}\right) - arctan\left(\frac{2}{3-x}\right)\right]$

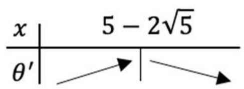
$$0 = 0 + \frac{\frac{5}{x^2}}{1+\left(\frac{5}{x}\right)^2} - \frac{\frac{2}{(3-x)^2}}{1+\left(\frac{2}{3-x}\right)^2}$$

After necessary algebraic operations, we get

$$\frac{5}{x^2+25} - \frac{2}{(3-x)^2+4} = 0$$

and when we solve this equation

$x^2 - 10x + 5 = 0$ then $x_{1,2} = 5 \pm 2\sqrt{5}$ but $x = 5 + 2\sqrt{5} > 3$ so there is left just $x = 5 - 2\sqrt{5}$ and when we check this on the behaviour table we see that θ reaches its maximum.



Hence, $AP = x = 5 - 2\sqrt{5}$ and $BP = 3 - x = 2\sqrt{5} - 2$

Obtuse isosceles triangle inside a circle problem.

Problem 23

Proposed by Ivan Retamoso, BMCC, USA.

Calculate the radius of the circle in which an isosceles triangle, with a base of 24 inches and legs each measuring 15 inches, is inscribed.

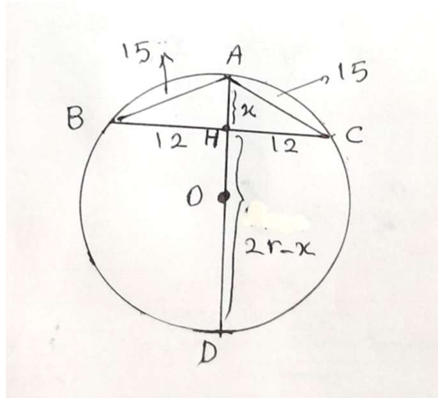
First solution to problem 23

By Dr. Hosseinali Gholami, University Putra Malaysia, Serdang, Malaysia.

Remarkably, our solver provides two simple and elegant solutions, complemented by graphs illustrating the obtuse isosceles triangle with its circumcenter located outside the triangle.

Solution 1:
As respect to the following shape, we obtain:

$$AH^2 = AC^2 - HC^2 \Rightarrow AH^2 = 15^2 - 12^2 = 81 \Rightarrow AH = 9.$$



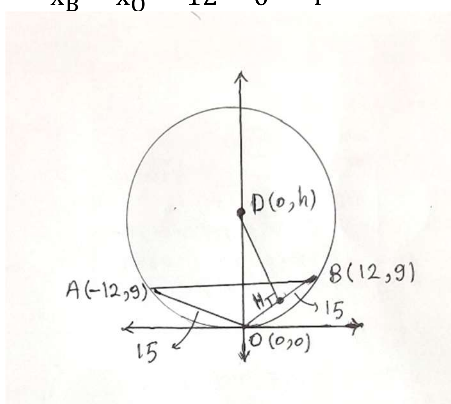
According to the Intersecting Chords Theorem we have:

$$AH \times HD = BH \times HC \Rightarrow x(2r - x) = 12^2 \Rightarrow 9(2r - 9) = 144 \Rightarrow r = 12.5.$$

Solution 2:

Based on the figure below, the point $H(6, 4.5)$ is the middle point of segment OB.

$$a_{OB} = \frac{y_B - y_O}{x_B - x_O} = \frac{9 - 0}{12 - 0} = \frac{3}{4} \Rightarrow a_{DH} = -\frac{4}{3}.$$



The equation of DH is $y = -\frac{4}{3}x + b$.

The coordinates of H satisfy in the above equation.

$$4.5 = \frac{-4}{3} \times 6 + b \Rightarrow b = 12.5.$$

The point D is located on the segment DH thus we have:

$$h = \frac{-4}{3}(0) + 12.5 \Rightarrow h = 12.5.$$

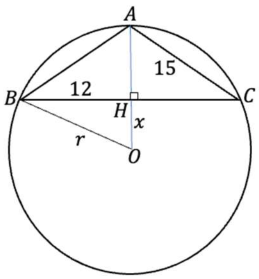
Therefore, it is clear that $r = OD = 12.5$.

Second solution to problem 23

By Dr. Abdullah Kurudirek, Mathematics Ed. Department, Tishk International University.

This alternative solution relies solely on the Pythagorean theorem and symmetry, as clearly demonstrated in the figure provided by our solver.

Solution.
We are given that the base of an isosceles triangle is 24 inches its legs are 15 inches and it is inscribed in a circle.
Let us draw a perpendicular that passes through the center of the circle to the base of ΔABC as follows.



The illustration shows that OA is the radius of the circumcircle. Since H is the middle of the base BC (where ABC is an isosceles triangle), therefore
$BH = HC = \frac{24}{2} = 12$ inches and knowing that $AB = AC = 15$ inches.
Now, using the Pythagoras theorem in ΔABH or ΔACH we get $15^2 = 12^2 + AH^2$ then $AH = 9$ inches.
Knowing that $AO = AH + HO$ where $AO = r$ and then get $OH = x$ so, $r = 9 + x$ and apply the Pythagoras theorem one more times in right triangle OHB and get $r^2 = 12^2 + x^2$. Let's reduce the unknown number to one by writing $r - 9$ instead of x. So, after all necessary algebraic operations we get $r = \frac{25}{2}$ inches.

Third solution to problem 23

By Irfan Rahman, Borough of Manhattan Community College, Bangladesh.

This alternative approach takes a distinct path by first employing the Pythagorean theorem. It then incorporates the triangle's area and links it with the radius of the circle through a specific formula.

Solution.
We're dealing with an isosceles triangle where:

- The base b is 24 inches.

- The two equal legs l are 15 inches each

Now we need to find the Height of the Triangle:

To find the height h, we'll split the triangle into two right triangles by drawing a line from a vertex at the base to the opposite vertex. This splits the base into two equal parts, each measuring $\frac{b}{2} = \frac{24}{2} = 12$ inches.

using the Pythagorean theorem $a^2 + b^2 = c^2$ where c is the hypotenuse we have:

$h^2 + 12^2 = 15^2$

Solving for h now, $h = 9$ inches

Now, we need to calculate the area of triangle. The area A of the triangle can be found using the formula:

A = ½ * base * height
A = ½ * 24 * 9 = 108 square inches
Finally, we need to find the Radius of the Circumscribed Circle. For a triangle inscribed in a circle, the radius R of the circumscribed circle can be found using the formula:
$A = \frac{abc}{4R}$
where a, b, and c are the sides of the triangle. Here a = 15 inches and b = 15 inches (because they both are the legs) and c = 24 inches (the base)

$R = \frac{abc}{4A}$ = (15 * 15 * 24) / (4*108) = 12.5 inches

The radius of the circle in which this isosceles triangle is inscribed is 12.5 inches.

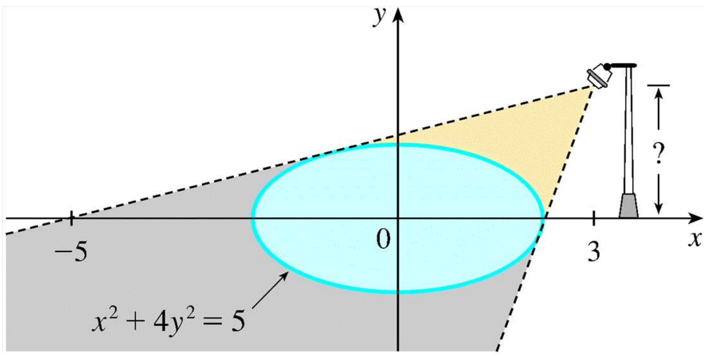
Dear fellow problem solvers,

I'm glad you found solving problems 22 and 23 enjoyable and that you've picked up some new strategies for your mathematical toolkit. Let's now transition to our next pair of problems to continue enhancing your skills.

Problem 24

Proposed by Ivan Retamoso, BMCC, USA.

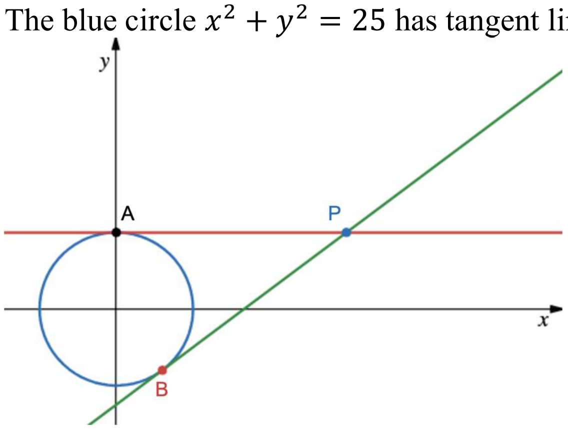
The diagram illustrates a lamp positioned three units to the right of the y-axis and casting a shadow due to the elliptical region defined by the equation $x^2 + 4y^2 \leq 5$. Given that the point $(-5, 0)$ lies on the shadow's edge, how far above the x-axis is the lamp located?



Problem 25

Proposed by Ivan Retamoso, BMCC, USA.

The blue circle $x^2 + y^2 = 25$ has tangent lines at the points A and B.



The point B has x-coordinate 3.
The tangent lines meet at the point P.
Find the coordinates of the point P.