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by Ivan Retamoso, the Editor of MTRJ's *The Problem Corner*

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I am delighted to announce that our MTRJ e-journal is gaining global significance as time goes by. As evidence of this, the current issue features ten papers from diverse countries: Thailand, Philippines, India, Turkey, Colombia, Malaysia, Mexico, Nepal, South Africa, and Rwanda. It brings me great satisfaction to be part of a mission that aims to make mathematics accessible to all. I firmly believe that our world would greatly benefit from a deeper understanding of mathematics, often referred to as "the language of the universe." Regarding the content of this spring issue of MTRJ, it begins with a focus on Teaching Practice in the Classroom context, exploring the Teaching Process, Learning Process, and Thinking Process. This particular study was conducted in Thailand. Next, we present a research paper from the Philippines that delves into online collaborative learning and its applicability in various scenarios of traditional learning, both quantitatively and qualitatively.

Continuing on, we showcase a paper written by a young contributor from India, Sameer Sharma. His research centers on the study of loops and spaces, presenting new findings and formulas. It is worth noting that his work was inspired by Professor James Tanton from Princeton University, specifically his intriguing project titled "A Math Mystery for Young Mathematicians."

Moving forward, we present a study conducted in Turkey that aims to examine the effects of a course module designed to enhance the proof schemes of pre-service mathematics teachers. Additionally, we feature research evaluating the effectiveness of the online learning platform Brilliant.org in improving the academic performance of students from four public schools in Barranquilla, Colombia.

Continuing the journey through this issue, we delve into a qualitative study from Mexico that analyzes how fourth-grade elementary school students (ages 9 to 10) solve and interpret non-routine problems, focusing specifically on division measurement and division-partition with remainder.

Furthermore, we present a study on the effectiveness of teaching and learning mathematics online, focusing on teachers' perspectives and the real-life situations they encounter in Nepal.

As we progress, we encounter a study that aims to examine teachers' understanding of the importance of instructional time as a valuable teaching resource for developing learners' relational understanding of mathematics in South Africa.

Lastly, we feature a study from Rwanda that explores the enhancement of learning limits of functions through the utilization of multiple representations.

As customary, this issue concludes with *The Problem Corner* section, presenting new collaborations from problem solvers as well as new challenges. I am confident that these engaging mathematical problem-solving activities will capture your attention and ignite your passion for mathematics.

Contents

Mathematics Student-Teachers' Views of Teaching Practice in Classroom Context

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Abstract: This study aimed to analyze views of mathematics student-teachers on teaching practice in classroom contexts by subsuming following the new didactic triangle, which was used as a conceptual framework: 1) Teaching Process, 2) Learning Process, and 3) Thinking Process. A participative research design was employed for the research methodology. The participants included seven fourth-year student-teachers, a purposive group of a case study that had been teaching practice in schools of the mathematics teacher education program for two weeks. They voluntarily participated in schools using Lesson Study and Open Approach and collaboratively designed lesson plans coached by researchers. After two-week teaching practices in school contexts, they were asked to reflect on their teaching practice covering 1) whether their teaching practices are accomplished, show some evidence of students' ideas, 2) identify problems found in their teaching practice, and 3) identify improving aspects of teaching practice for subsequent lessons' improvement. Results of the study showed that the mathematics student-teachers' views are reflected on their teaching practice in the actual classroom context following the aspects of the new didactic triangle and correlate with teaching mathematics through problem-solving, which is a mainstream and widespread pedagogical approach.

INTRODUCTION

Teacher preparation is one of the crucial aspects of the educational reform movement in many countries. In Thailand, especially in the mathematics teacher education program, most teachers' colleges are attempting to improve the teaching practice of student-teachers by improving courses and supporting teaching practice (Ball & Cohen, 1999; Inprasitha, 2015; Hancherngchai, Inprasitha, & Thinwiangthong, 2017).

In the case of the mathematics education program of Lampang Rajabhat University, Thailand, there are some endeavors to improve its course and support its teaching practice. In addition, in the case of encouraging the teaching practice for its student-teachers, the Lesson Study and Open Approach are involved in this developmental process.

In this research context, the teaching practice was driven by Lesson Study and Open Approach, which aimed to improve teacher education programs and professional development in Thailand since 2002 (Inprasitha, 2011; 2015; 2022). Two innovations, Lesson Study and Open Approach, have been proposed to change the paradigm of teaching practices from a product-oriented approach to a product-process-oriented approach, and improve the teaching practices consecutively (Inprasitha, 2004).

This paper focused on the views of the student-teachers after their teaching practice in a school context of Lesson Study and Open Approach in relation to components of a didactic triangle in mathematics to demonstrate how the program encouraged the student-teacher's teaching practice. This process of research will be brought to improve a forthcoming (mathematics) teacher education program.

RESEARCH OBJECTIVE

This study aimed to subsume the reflections of the student-teachers relating their views of teaching practice derived from the new perspectives of a didactic triangle; the teaching processes, learning processes, and thinking processes. This teaching practice was in the classroom using an Open Approach as a teaching approach and Lesson Study as a way to improve the teaching approach of the student-teachers of the mathematics teacher education program, Faculty of Science, Lampang Rajabhat University, Thailand. For achieving this research objective, the after-teaching-practice reflections of the student-teachers and the new didactic triangle (Inprasitha, 2014) had been adapted to consider views of their teaching practice.

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

A literature review was accomplished concerning the teaching practice, which is the foundation for the teacher education program (Jarrah, 2020). For this reason, the teaching practice could be reflected in the views of student-teachers, which are related to the didactics or teaching in their classrooms.

Teacher Education Programs and Teaching Practice

Teaching practice is a fundamental element of teacher education programs. It is not only because it provides student-teachers with experience in a real classroom context, but it is also an opportunity for student-teachers to implement pieces of stuff they have learned about their subject matters (Jarrah, 2020). Therefore, the teaching practice is also crucial for professional and personal development as the growth of student-teachers (Azhar & Kayani, 2017). In addition, effective teachers are expected to possess a richness of content knowledge, suitable theoretical foundation, and competence in pedagogical and instructional strategies (Cochran-Smith & Lytle, 1999).

Across the twentieth century, there is consistently a requirement for teacher education which has been severed by a persistent split between subject matter and pedagogy knowledge. In other words, fragmenting teaching is a gap in teacher education (Ball & Bass, 2000).

Consequently, to do the teaching practice successfully, teacher education programs need to provide knowledgeable teachers who are willing to learn about their students as learners. Moreover, the student-teachers must acquire the skills, disposition, and instructional strategies required to teach such subject matters. In addition, teacher education program needs to provide opportunities for student-teachers to connect different sorts of knowledge (Kilpatrick, Swafford, & Findell, 2001).

Didactics of Mathematics

The ZDM-The International Journal on Mathematics Education in 2012 has a theme of "New Perspectives on the Didactic Triangle: Teacher-Student-Content" which called for various perspectives of mathematics education researchers about fundamental relationships within the didactic triangle or instructional triangle. This volume provides integration of technology roles in teaching mathematics, the researcher in mathematics teaching developmental research, and mediating complexes in the student-teacher-content vertices (Goodchild & Sriraman, 2012).

There are also researchers in the mathematics education community who have been interested in the research of the didactic triangle or didactic theory over decades ago and are still relevant today, for example, Brousseau (1997), Kilpatrick, Swafford, & Findell (2001), Cohen, Raudenbush, & Ball (2003), Straesser (2007), and Ruthven (2012). These studies are almost related to how teachers might be empowered to become aware of and work on relationships between themselves (the teacher), their students, and mathematics (Goodchild & Sriraman, 2012). Likewise, Brousseau (1997) raised a domain of the didactic theory regarding the theory of didactical situations in which the teacher creates a milieu in which students engage with mathematics. Regardless, there are some attempts to extend more vertex of the didactic triangle. For example, Ruthven (2012) and Rezat and Straßer (2012) stated that technology or artifacts were fundamental constituents of the didactic triangle, resulting in the didactic tetrahedron or socio-didactic tetrahedron.

Teaching through Problem-Solving

Problem-solving is central to mathematics knowledge construction (Pehkonen, 2019). In a learning and teaching atmosphere, in other words, mathematical problem-solving is significant and challenging as it is the heart of mathematics (Jarrett, 2000). Furthermore, teaching and learning mathematics through problem-solving support the development of learners with a deep understanding (Inoue et al., 2019).

In teaching through problem-solving, learning emerges during the problem-solving process. When the students solve the problems, they might use any procedures, recall any pieces of knowledge they have learned, and convincingly identify their ideas. In the learning atmosphere, the teacher should provide opportunities for the students to express various solutions to their class and learn mathematics through meaning negotiation and grasping a shared understanding. These activities support the students in clarifying their ideas and acquiring different perspectives from their peers (Cai et al., 2003).

Moreover, Takahashi (2021) stated that teaching through problem-solving (TTP) is how Japanese teachers teach new mathematical notions by providing students with compelling mathematical challenges to solve on their own and discussing with the students to find shared conclusions, as in the TIMSS video. It was called Structured Problem-solving (Stigler & Hiebert, 1999). Additionally, teaching mathematics through problem-solving bestows educators an instrument for restructuring their lesson and curriculum design to create creative and adaptive problem-solving simultaneously (Takahashi, 2021).

Lesson Study and Open Approach

The traditional teaching approach in Thailand emphasizes on product-oriented approach as the teacher needs only an answer from the students. Consequently, there is an endeavor to shift a paradigm of teaching practices from a product-oriented approach to a product-process-oriented approach by introducing innovations since 2002; Lesson Study and Open Approach. In other words, the teacher needs to go beyond the answer. The process of solutions and the reason behind these solutions should also be emphasized. This endeavor has been consecutively improving teaching practices (Inprasitha, 2014).

In addition, the Open Approach is composed of four phases; 1) Posing Open-ended Problems, 2) Students' Self-learning, 3) Whole Class Discussion and Comparison, and 4) Summarizing through Connecting Students' Mathematical Ideas that emerged in Classroom. This teaching approach emphasizes individual differences, incredibly individual differences in students' thinking. Teachers try to collect their students' ideas to conduct a summarization in accordance with the student's ideas. The Lesson Study, moreover, focuses on improving the collaborative working of teachers for improving the teaching approach, composed of three steps; 1) Collaboratively Design Research Lesson (Co-Plan), 2) Collaboratively Observe Research Lesson (Co-Do), and 3) Collaboratively Reflect on Teaching Practice (Co-See), and done this collaborative work in a week or weekly cycle (Inprasitha, 2011; 2015; 2022). These two innovations are incorporated in the second step of the Lesson Study or Collaboratively Do, as in figure 1, called a Thailand Lesson Study incorporated Open Approach (TLSOA) (Inprasitha, 2022).

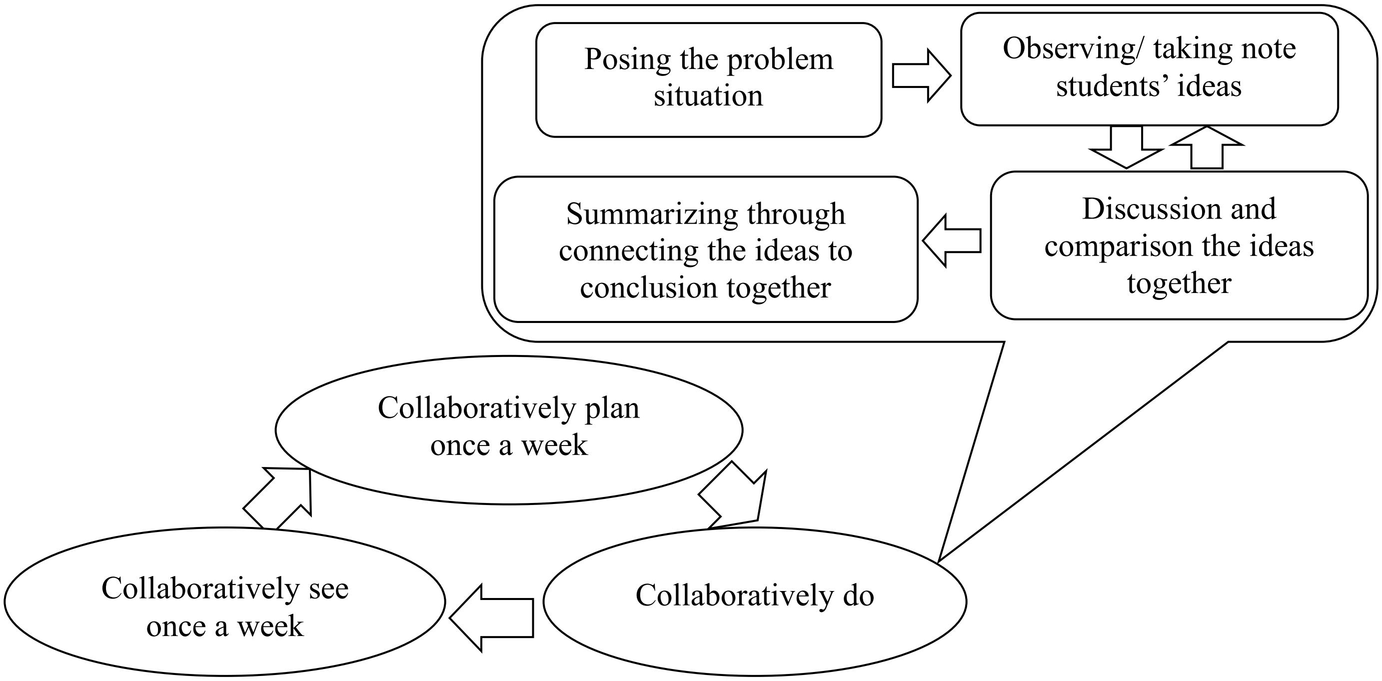

Figure 1: The cycle of Lesson Study and Open Approach
(Adapted from Inprasitha, 2011; 2015; 2022)

Additionally, the Lesson Study encourages the teachers to have a Community of Practice (CoP), described as the community as a way of talking about the social form in which members of the community, defined as participation in the community members, is recognized as competence (Wenger, 1998).

In the new context of teaching and learning mathematics, Inprasitha (2014) has proposed three classroom components as a new didactic triangle: the Teaching Process, Learning Process, and Thinking Process. These components are related to the student's ideas used for accessing the student's learning or thinking processes.

There is, consequently, an extended perspective on the three components of the didactic triangle: teacher, student, and content concerning the teaching approach that emphasizes teaching through problem-solving and utilizes the students' ideas in steps of reflecting, planning, and observing as the Lesson Study step: Teaching Process, Learning Process, and Thinking Process, as in figure 2, respectively. The new didactic triangle (Inprasitha, 2014) was employed as a theoretical framework to subsume views associated with teaching practice in the classroom of the student-teachers in the mathematics teacher education program. Details of three constituents in the new didactic triangle are as follows.

1) Teaching Process is related to the teacher's capabilities to engage the students to have their problems from problem situations, preferring to use semi-concrete aids to extend the students' ideas that occurred in the classroom and encouraging the students with questions.
2) Learning Process is related to how each student will master their experience with the condition and context of a problem situation. The students afterward could solve the problem situation in various ways. In addition, the students could share their ideas with their friends in the classroom.
3) Thinking Process is related to the student's engagement with the problem situation and encountering a problematic condition. The students will proceed with ways of solving the problem. The students then formed their ideas using 'how to' from previous lessons to think about and solve such problematic conditions to generate mathematical ideas.

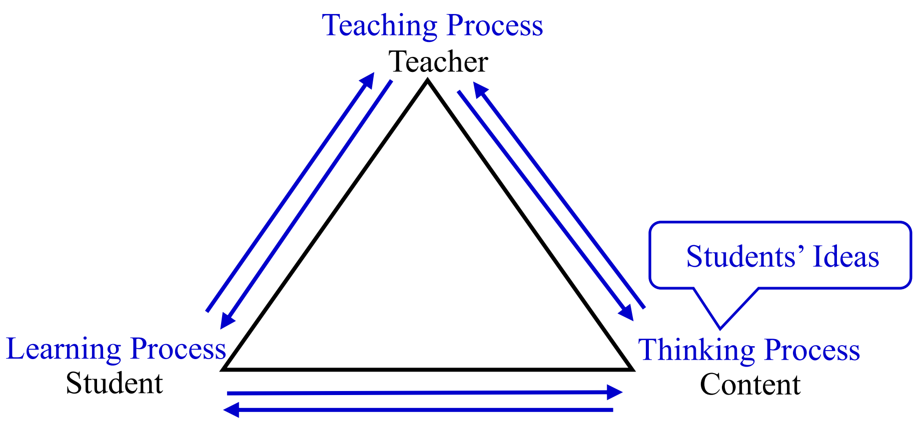

Figure 2: New didactic triangle (Adapted from Inprasitha, 2014)

RESEARCH METHODOLOGY

Research Design

This study was a case study of student-teachers in a teacher education program who had two-week teaching practice in a school context. The study employed a qualitative research methodology as a participative research design in which the researchers participated in a collaborative design (Collaboratively plan) and a collaborative reflection (Collaboratively see) of the Lesson Study processes with the mathematics student-teachers who served as a purposive group of this study.

Participants

In this study, the researchers surveyed all 52 student-teachers in the fourth year of the five-year teacher education program in the 2020 academic year. This study was done in the course of mathematics education called 'Teaching Experiment'. This course provided an experience for student-teachers in a school context as they will have their mathematics classroom at the primary or lower secondary school level for 8-12 hours a week (Mathematics Program, Faculty of

Science, Lampang Rajabhat University, 2014). Seven fourth-year student-teachers voluntarily participated (McLain & Kim, 2018) in the schools that use the Lesson Study and Open Approach. They were a purposive group that had been doing the Lesson Study and Open Approach for two weeks in a school context. To protect the identities of the purposive group, the researchers abbreviated each participant by applying a code student-teacher added by a capital letter, i.e., student-teacher A-G.

In addition, the mathematics student-teachers of Lampang Rajabhat University learn about subject matters in courses of a five-year mathematics teacher education program. They had opportunities to attend extra-curricular activities during four years of the teacher education program. The extra-curricular activities, such as National Open Class, were part of school mathematics (Klein, 1982). These activities applied the Lesson Study and Open Approach processes; co-plan, co-see, and co-reflect, to cooperate with in-service and pre-service teachers of the mathematics teacher education program working in the school context.

Data Collection

These mathematics student-teachers used Japanese textbooks translated into Thai versions (Inprasitha & Isoda, 2019) to collaboratively design lesson plans coached by the researchers on their lesson study team at the university. They then went to the schools with the lesson plans and taught in the classroom context using the Open Approach as a teaching approach. Furthermore, they and in-service teachers used a weekly cycle of the Lesson Study to collaboratively reflect on getting aspects used to improve the students' ideas via collaboratively designing activities used in their lesson plans.

Figure 3: Japanese textbooks translated into Thai versions since 2010
(Inprasitha & Isoda, 2019)

After the two-week teaching practice in the classroom, the student-teachers of the mathematics teacher education program reflected on their teaching practice according to the following aspects.
1) whether their teaching practices are accomplished, show some evidence of students' ideas,
2) identify problems found in their teaching practice,
3) identify improving aspects of teaching practice for the following lesson's improvement,
with evidence such as two consecutive lesson plans, photos of students' ideas or worksheets, bansho (blackboard used in the classrooms), and students' solving problems.

Data Analysis

The data from reflections about teaching practices for two weeks in a school context by the student-teachers and their pieces of evidence accompanied the data from the collaborative planning phase. The research lessons of the student-teachers and the researchers, and the phase of collaborative teaching and observing the research lessons of each student-teachers with their partner in the school context were analyzed using content analysis (Bengtsson, 2016). Then, they were classified into the following components of the new didactic triangle (Inprasitha, 2014).

RESEARCH RESULTS

After the two-week teaching practice in the actual classroom setting of school contexts, the seven student-teachers were asked to reflect on the abovementioned aspects of their teaching practice in mathematical classrooms.

1) Reflections of student-teachers related to the teaching process

(1) engaging the students to have their problems from problem situations

"When posing an open-ended problem, students couldn't access a problem situation. This situation makes them confused with the teacher's direction, and students couldn't see what their problems are."

Student-Teacher A (January 13, 2021)

"To improve in teaching is how to transform problem situations into the students' problems because the students don't realize that is the problem to solve."

Student-Teacher C (January 13, 2021)

The student-teachers reflected on an awareness of the problematic conditions by identifying a state in each student that could not solve the problems or obstacles that the students confronted with the problem situation students had been solving. Moreover, the student-teachers realized how to improve their teaching practice, emphasizing teaching through problem-solving, conveying the problem situations as their students' problematic conditions or problematization.

(2) preferring to use semi-concrete aids to extend the students' ideas that occurred in the classroom and how to connect them

"The students couldn't move erasers in a worksheet, so they can't find the total number of those erasers. A teacher should have additional material (blocks) for helping students in counting the erasers."

Student-Teacher A (January 13, 2021)

"The students couldn't realize that 9 is missed 1 to become 10 because the teacher had only blocks or missed a ten-bar. So, the teacher should prepare the tens-bar for review this lesson before starting the next lesson."

Student-Teacher D (January 13, 2021)

The student-teacher reflected on supporting materials that the student-teachers need to use to broaden the students' ideas out to other students in the classroom to learn together, e.g., blocks, tens-bar, and figures that can be moved on the blackboard. Additionally, the student-teachers realized that they had prepared to get enough of these supporting materials to improve the students' ideas by ultimately connecting them to mathematical ideas.

(3) teachers' role in encouraging students with questions

"The teacher asked students the most straightforward way how to find the answer of 9+4. The students could realize that they should make 10 first and the answer a product of the addition of 10 and 3 to become 13."

Student-Teacher D (January 13, 2021)

The student-teacher reflected on how to grasp the lesson's purposes by asking critical questions. These questions should be prepared as a sequence of the lessons and considered as the specific students' ideas occurring in the third phase of discussion and comparison of the student's ideas. Finally, the student-teachers realized that these critical questions would support the students to think about "how to learn" on their own for the upcoming lessons.

2) Reflections of student-teachers related to the Learning Process

(1) Students can solve the problem situation in various ways

A grade-1 problem situation of how many stamps there are related to numbers more than 10 and less than 100.

"The students could be able to count by 1, 2, 3, 7, or 10."

Student-Teacher A (January 13, 2021)

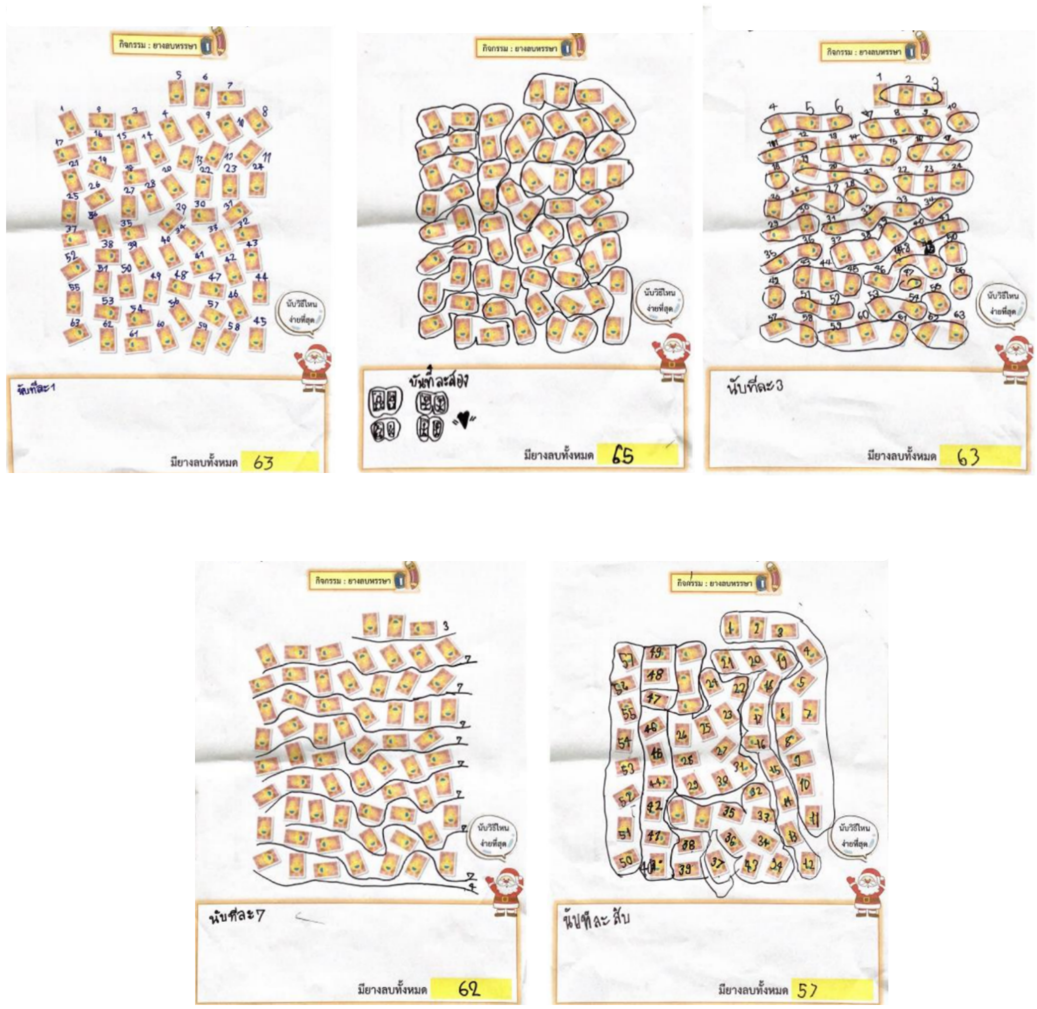

Figure 4: Students' worksheets representing ideas of numbers more than 10 and less than 100
(a) count by one (b) count by two (c) count by three (d) count by seven (e) count by ten

From a grade-2 problem situation of length, the students will learn how to compare the length of pieces of stuff. The students will be forced to use things around them as measuring tools.

"The students used a palm, a pencil, and an eraser to be measure tools. Furthermore, one of the students used his body as a measuring tool."

Student-Teacher E (January 13, 2021)

The student-teachers reflected on the students' various tools to solve the problematic conditions. Eventually, the student-teachers realized that the opportunity they provided for their students was in the phase of students' self-learning by solving the problem with themselves or their peers. In addition, the student-teachers only observed or took note of the student's ideas, whether those ideas were as had been anticipated thoroughly.

(2) Students share their ideas with their friends

For a grade-4 problem situation of pouring water by estimating 1 liter into a kettle without any measuring devices, the student-teacher let their students solve the problem by themselves and share their ideas in groups. In the third classroom phase, student-teachers asked their students to share their ideas on representing the amount of water in front of the class.

Figure 4: Atmosphere of classroom
taught by Student-Teacher F (January 13, 2021)

The student-teachers reflected on the opportunities they provide for their students to share the ways used to solve the problem situations with all peers. Accordingly, the student-teachers could record the students' ideas on a blackboard, and the students could visualize back and forth along the lessons. The student-teachers realized that these opportunities would encourage the students to be conscious of their ideas and learn with peers' ideas to examine whether they are reasonable.

3) Reflections of student-teachers related to the Thinking Process

From a grade-4 problem situation of an unknown unit (deciliter), the students used their prior knowledge to access the meaning of deciliter.

"The students adapted 'how to learn' from the previous lesson (group of ten) to be a tool to access an unknown water level by separating the beaker into 10 parts equally, to deliver 10 dl as 1 L."

Student-Teacher F (January 13, 2021)

From a grade-1 problem situation of addition (2) in which two numbers are less than 10, and the addition result is more than ten, student-teacher G reflected on the necessity of reviewing using a diagram of adding two numbers that are less than 10. Moreover, the addition result is also less than ten.

> *"The students' difficulty was incorrectly using diagrams such as composing and decomposing. The student-teacher realized that there should be reviewing of the diagram in the introduction part of the lesson."*
>
> *Student-Teacher G (January 13, 2021)*

The student-teachers reflected on the students' ways of solving the problem situation after the students engaged with the problematic conditions by preferring "how to learn" from previous lessons. Consequently, the student-teachers realized that the students would solve the problem properly when they are conscious of their own "how to learn" and eventually use them to solve the problem.

DISCUSSION AND CONCLUSION

Based on research results, a discussion could be divided into three parts as the research results followed the theoretical framework of a new didactic triangle: the view of student-teachers related to the teaching process, learning process, and thinking process.

1) View of student-teachers related to the Teaching Process: awareness of problematic conditions that the student-teachers should consider when they design problem situations. In the teaching phases, the student-teachers should support their students to express ideas and encourage them to deliberate their ideas for solving problems.

The student-teachers reflected on the teachers' roles in engaging the students to have their problems from problem situations or problematic mentioned by Isoda (2010) that there is a local theory of problem-solving approach, for instance, the difference between problem situation (task) and problematic (problem). Moreover, these corresponded to Isoda and Katagiri (2012) as "problematic" is an essential element of the problem-solving approach because it is necessary for children to learn by/for themselves, and it is also related to the objective of the lesson. Consequently, the teachers will collect answers only good answers if there are without the problematic condition. However, with various kinds of answers, children can discuss which answers are appropriate by themselves.

The connection between real-world experience and formal mathematics concepts or the mathematical world is enormous and complex (Bransford, Brown, & Cocking, 1999). Therefore, the student-teachers reflected on using semi-concrete aids to clarify and extend the students' ideas to consecutively establish mathematical ideas starting from the students' real-world experiences. They also used them to create the problem situations, using semi-concrete aids, and forming the students' mathematical ideas by comparing and discussing the students' ideas or "flow of lesson" (Inprasitha & Isoda, 2019; Intaros & Inprasitha, 2019).

The student-teachers reflected on questioning to encourage the students' thinking about the problem situations to recall how to solve problems by utilizing what they have learned. In collaboratively planning the lessons, the student-teachers anticipated the students' ideas or ways of solving problems and prepared questions that would be used when the student's responses followed the students' anticipated ideas. These student-teacher roles align with Ulep (2015) that the teachers accommodate students to consider how they knew if their answers were correct. They let students evaluate which correct solutions they preferred and share their reasons with other students using questions. In addition, these ideas are consistent with Takahashi (2021), who states that during the lesson study processes are ongoing, the teachers conduct an in-depth study of the mathematics and curricular material related to the lesson's objective. The teacher must, moreover, consider the student's prior knowledge to choose an appropriately challenging problem that will accomplish the lesson's objective and anticipate the students' possible response to the problem to plan how to engage those students' ideas. This teaching approach is teaching through problem-solving, and the role of teachers in guiding mathematical discourse is a remarkably complex activity (Cai et al., 2003). Furthermore, by devoting a suitable time to discuss ways of solving the problem, teachers should decide what aspects of a task would be highlighted and how to orchestrate the students' ideas. Teachers should also decide what questions are used to challenge those with different expertise of the students and how to support students without taking over the thinking process for them and therefore eliminating the challenge of the problem (National Council of Teachers of Mathematics, 2000).

2) View of student-teachers related to the Learning Process: awareness of providing the opportunity for their students to solve the problem in various ways and share these ideas with the other students to grasp the purpose of the lessons.

The student-teachers reflected on students' ways of solving problems in various ways by using the students' prior knowledge, experiences, and what they have got, such as a palm and a pencil, to solve the problems. These scenarios would happen when the teachers provide the opportunity for the students to solve by themselves and prepare problem situations related to the student's experience. This is consistent with Gueudet et al. (2017), which mentioned transitions in mathematics education, e.g., a transition between in- and out-of-school mathematics, which is connected by including cultural contexts or sociocultural perspectives of students' contexts in school mathematics.

The student-teachers who used the Open Approach as their teaching approach will provide opportunities for students to share ideas after the students solve the problems on their own or by themselves. This step of the Open Approach is similar to the Japanese teaching approach called 'Nariage' (Shimizu, 1999), which is dynamic and collaborative during the class discussion by looking back on students' ways of solving problems. This teacher's role is worth it as they help the students derive the essential facts, concepts, and procedures. Therefore, it differs from the teachers' role in a traditional teaching approach in which the students do not explore any new mathematical concepts.

3) View of student-teachers related to the Thinking Process: awareness of 'how to learn' that the students have to develop through their ways of solving problems and become aware of these ideas used to solve the problems.

The student-teachers reflected on "how to" what is in accordance with Isoda and Katagiri (2012), who stated that teaching children how to develop mathematics by nurturing children who think and learn mathematics by and for themselves. Well-nurtured children are given problem situations to consider the next step by themselves and imagine the next step for themselves. Likewise, these are affected by the translated Japanese textbooks, which are based on a 'think about how to' characteristic of problem situations. When the students learn mathematics as a sequence of textbooks, it could be called 'today's learning is preparing for the next day's learning (Inprasitha & Isoda, 2019).

The study could be concluded that the views of student-teachers reflected from their teaching practice in the actual classroom context follow the aspects of the new didactic triangle, which is composed of the teaching process, learning process, and thinking process. The awareness of the student-teachers in each aspect of the didactic triangle is taken from the teaching practice using the Open Approach. These aspects of the new didactic triangle are correlated with teaching mathematics through problem-solving, which is a mainstream mathematics teaching approach and a widespread pedagogical approach (Takahashi, 2021).

IMPLICATION AND FURTHER RESEARCH

The results of this research shed some light on teaching mathematics through problem-solving as a new mathematics teaching and learning approach for school context and mathematics teacher education programs. Nevertheless, the research is a case study that has a limited number of participants because it focused on only seven student-teachers in the mathematics teacher education program. Further research should level the number of participants up to other mathematics teacher education programs' students who are encouraged to apply the Lesson Study and Open Approach as a way of teaching practices in school contexts.

The two-week teaching practice is the experience in an actual classroom context that the student-teachers could learn before being a pre-service teacher for a year in the next academic year. There should be more than two weeks, i.e., four weeks, for teaching practice in a school context. This will enhance the teaching practice of student-teachers by using the Lesson Study and Open Approach, which emphasizes students' ideas to collaboratively plan, teach and observe, and reflect to improve the problem-solving lessons. The at-least-four weekly cycle of the Lesson Study will be an effective flow for the student-teacher to perceive a suitable tendency to use students' ideas to improve their lesson plans and their teaching practice in the school context.

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Online Collaborative Learning: Applicability in Comparison with Individual Learning and Face-to-face Collaborative Learning

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Abstract: This classroom-based action research aimed to determine the applicability of implementing Collaborative Learning in online setting to help students develop their interaction with one another and improve their academic performance. Participants of the study consisted of Grade 8 intact pilot section of a public school in an urban community which yielded only 20 valid responses. Student participation during online collaborative learning made use of the first three stages of Siemen's Connectivism taxonomy. Quantitative data were taken from Student Attitude Survey, individual and group learning activity sheets, and achievement test, while qualitative data were taken from focus group discussions. Transcripts of the recorded online collaborative learning were analyzed using content analysis following Garrison's Practical Inquiry Model. Results showed that there is a significant increase in students' scores in the activities done collaboratively than those done individually. However, only 12 out of the 20 students successfully passed the achievement test. Additionally, results of the content analysis of the video recording transcripts show that among the four categories of Garrison's Practical Inquiry Model, students made more extensive use of integration and exploration during online collaborative learning. In terms of students' attitudes toward the intervention, the results indicated a positive response with regard to their experience with online collaborative learning. Analysis of the focus group discussions using interpretative phenomenological analysis revealed similarities and differences in collaborative learning between an online and face-to face learning.

INTRODUCTION

When the World Health Organization declared COVID-19 as a pandemic due to the widespread of the Novel Coronavirus disease, lockdowns and social distancing protocols disrupted everyone's normal everyday life which led to the closure of large gathering venues such as movie theaters, museums, churches, and even educational institutions (Sahu, 2020).

In compliance to the directive of the secretary of Department of Education concerning the preventive measures against COVID-19, all schools in the Philippines implemented distance learning as delivery mode (Department of Education, 2020). In line with the different learning modalities during distance learning, some public schools offer a variety of learning delivery modes to students, whichever was suitable to their situation. Students who cannot afford to have an access online may enroll in printed modular learning. While those who might have access to internet although possibly unstable, may enroll in blended learning – a mix of online and modular learning – as their mode of instruction during the pandemic. Lastly, those who have stable internet connections but cannot to go to school to retrieve and/or submit modules in compliance to the preventive measures against COVID-19 may enroll in online learning.

Throughout the school year 2020-2021, the students were learning in isolation at home either through following a printed learning module or through synchronous and asynchronous online learning. This raised doubts on the quality of learning as there is lack of social interaction and communication. Studies have shown learners in online education need various support not only from instructors, but also from peers since peers are essential in forming learning environment where they may collaborate and be engaged in learning (Wang et al., 2019; Tay et al., 2021). Furthermore, teachers could adopt collaborative learning in their online classes to develop learners' practical skills and knowledge, collaborative problem-solving skills, and teamwork skills, especially in mathematics (Tsai, 2010). For this reason, this study examines the implementation of collaborative learning in online distance learning during the pandemic.

LITERATURE REVIEW

Online collaboration is the online version of the traditional in-class collaborative learning (Ku et al., 2013) with the exception that group meetings in online setting are held synchronously or asynchronously via the internet. In this approach, it is possible for students to interact with one another despite limitations in time and locations (Nurdiyanto et al., 2017). Furthermore, collaborative learning in online environment develops interaction among learners and a sense of social presence, which promotes students' improvement of learning and their capability to adapt to various teaching techniques, as well as their motivation and satisfaction (Magen-Nagar & Shonfeld, 2018).

The implementation of collaborative learning in mathematics online class was anchored on the first three stages of connectivism together with Garrison's Practical Inquiry Model (PIM). Connectivism emphasizes that in an online learning environment, the most common means of finding and producing knowledge is through interaction and dialogue (Siemens, 2005). This framework is helpful to understand collaborative learning in an online setting. Learning in the digital age, according to Siemens, is no longer based on individual knowledge acquisition, storage, and retrieval; instead, it is based on connected learning that takes place through interaction with difference sources of information as well as involvement in communities of common interest, social networks, and group tasks. Moreover, one of Connectivism's most notable aspects is interaction and connections. It refers to the connections between nodes in a network that allow information and knowledge to flow (Banihashem & Aliabadi, 2017). With students viewed as nodes in a network of learning system, this theory supposes that learning rests on diversity of opinions among students forming connections and collaboration with other nodes (AlDahdouh, 2018).

Garrison's PIM addresses cognitive presence in order to provide a tool for evaluating critical discourse in a discussion (Garrison et al., 2009). It was used in other studies as a coding framework to investigate the students' online discussion (Liu & Yang, 2012). In order to explain and comprehend cognitive presence in an educational setting, the practical inquiry model suggests four phases – triggering event, exploration, integration, and resolution. The stages of the practical inquiry model are an idealized logical structure of the critical inquiry process and should not be regarded as constant but changes over time. The PIM is more involved with thinking processes than with individual learning results. As a result, it may be utilized in online conversations to measure critical discourse and higher-order thinking (Garrison et al., 2009). In addition, the PIM was determined to be the best applicable to analyze cognitive dimension and provides a clear description of the knowledge-building processes that take place in an online discussion (Schrire, 2004).

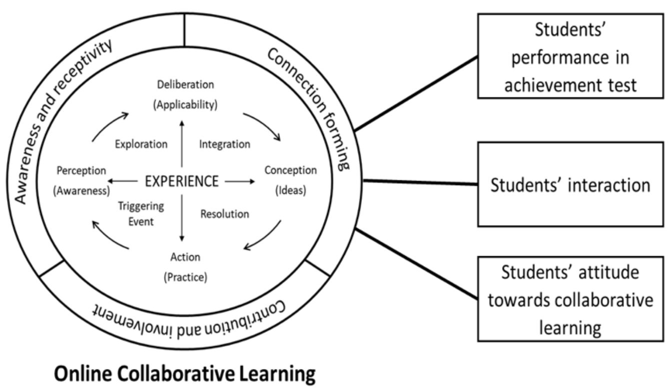

Figure 1: Framework of the Study

As presented in Figure 1, online collaborative learning (OCL) is anchored in connectivism, which has three major components namely (1) awareness and receptivity, where learners acquire basic knowledge for handling information abundance and have access to resources and tools; (2) connection forming, where learners use the tools and understand the acquired knowledge from the first stage to be able to form connections with their network and (3) contribution and involvement, where the learner begins to actively participate in the network, and allows it to recognize the learner's resources, contributions, and ideas, resulting in reciprocal understandings and relationships (Sitti et al., 2013).

Concurrently, online collaborative learning is anchored on the four phases driven from Garrisons' PIM namely which are the four major phases of cognitive presence that can be seen in students' online discussions (Garrison et al., 2009). The first one is triggering event which is the process of becoming aware of an issue by starting an investigation. The second phase is called exploring, which is where students are looking for relevant information, reflecting, and exchanging explanation to solve a problem. The third phase is integration, a process of making meaning from a variety of sources and suggesting solutions to the given problem. Lastly, resolution is a phase in which implementing or defending potential solutions with a new idea occurs. Students may return to a prior phase, hence these four phases do not necessarily have to be in sequential order (Swan et al., 2009).

PROBLEM STATEMENT

This study intends to determine the applicability and viability of collaborative learning in synchronous online class setting. Specifically, it seeks to answer the following:

1. How are the learning activities developed?
2. What are the students' performances in their individual learning tasks, collaborative learning tasks, and achievement test in mathematics?
3. How interactive are the students during collaborative learning in an online setting based on the four categories of Garrison's Practical Inquiry Model – Triggering Event, Exploration, Integration, and Resolution?
4. What are students' attitudes towards online collaborative learning?
5. How do students perceive online collaborative learning as compared to face-to-face collaborative learning?
6. How do the students interact based on the first three stages of Connectivism – awareness and receptivity, connection-forming, and contribution and involvement?

METHODOLOGY

The study is an action research that utilized mixed methods approach. The qualitative data consists of students' perceptions on online collaborative learning compared to face-to-face collaborative learning, and transcripts of students' interactions during collaborative learning. Quantitative data consist of students' self-report attitudes towards the implemented collaborative learning, students' academic performance using classroom assessment tools such as activity sheets and achievement test.

Since very few students in the public school where the study was conducted can afford online learning, only an intact class of 40 students who have a gadget and an access to internet connection was requested to participate in the study. However, in the observance of research ethics, only 28 students gave their consent. And out of these, the study yielded 20 valid responses. From the 20 valid responses, there were 9 participants who missed at least one online group work, leaving only 11 who were able to attend all the meetings of online collaborative learning. The class was handled by the first researcher and was divided into small groups with 3-5 members each group for the implementation of online collaborative learning (Moreno et al., 2021). Each group was heterogenous (Murphy et al., 2017) made up of low-, average- and high-ability students.

The research instruments used were online learning activity sheets, a validation guide, an achievement test, an attitude survey questionnaire, and a focus group discussion (FGD). The activity sheets were face and content validated by 3 experts with the use of an adapted validation questionnaire (Yildirim & Orsdemir, 2013). All items in the validation questionnaire used a 5-point Likert scale. The achievement test consisted of a 50-item multiple choice summative assessment that involved all the said topics in mathematics. The Garrison's PIM was used to analyze and determine students' cognitive presence during online discussion transcripts. The student attitude survey questionnaire on online collaborative survey questionnaire was adopted from Korkmaz (2012) and the focus group discussions was used to determine student views on online collaborative learning.

Prior to each online collaborative learning session, students answered the learning activity sheets individually. These were not graded. Students underwent online collaborative learning where they answered the same activity this time as a group. Student mean scores on the activities done individually and by group were compared using paired samples t-test. This is to determine if there is a significant change of scores in the activity when independently and collaboratively. After having gone through all the class activities, an achievement test was administered to the students.

During the OCL, each group activity was recorded. Transcripts of the video recordings were analyzed using content analysis with the use of Garrison's Practical Inquiry Model. Subsequent to the online collaborative sessions, the students answered the student attitude survey and a focus group discussion (FGD) took place. The participants' responses were also recorded and transcribed. The interpretative phenomenological analysis was utilized to analyze students' responses in FGD.

RESULTS

The following results are presented following the order of the corresponding research questions it seeks to answer.

Development of Learning Activities

The learning activity sheets are researcher-made formative assessments items of which were selected based on the Department of Education's most essential learning competencies. The tasks contained probing and guide questions eliciting students' prior knowledge or what they know towards discovering what they do not know in accomplishing the tasks. These are also concept building activities prior to class discussion where students construct knowledge and skills predefined in the lesson. Furthermore, concept building using real-life problems on the following topics corresponding to the online collaborative learning were utilized: linear inequalities in two variables; systems of linear inequalities in two variables; and relations and functions, and dependent and independent variables.

The activity sheets underwent two cycles of revisions and feedback between the two researchers. For the first two activities, only minor revisions took place. For example, in the first activity (problems involving linear inequalities in two variables), the graph of an inequality, which is a half plane, was introduced to the students. After being able to describe the graph based on the points (solutions) on the coordinate plane, they will see that some problems are best represented by an inequality rather than an equation. Subsequently, another problem of the same type was given in the activity. At this point, as they were able to determine that the problem was best represented by an inequality, they were asked in the activity how the graph of an inequality could be determined in terms of the boundary line and the shaded half-plane.

For the third activity, validators commented the questions made were very superficial, thus requires major revision. It was brought up during the cycle of revisions that the concepts should be presented based on real-life. Consequently, the major changes consist mainly in the use of real-life situations for relevance in students' experience and for eliciting students' prior knowledge.

Table 1 shows validators' ratings of the activity sheets. All item responses indicate a positive strong agreement with regard to the validity of the activity sheets. There were few comments brought up by the validators most of which were minor changes in the rubric to be used for the activity. This is consistent with their ratings as indicated in item 5, "The activity will be scored using an appropriate rubric" obtained the lowest mean score among all items in the validators' questionnaire.

Indicators	Activity 1	Activity 2	Activity 3	Overall Mean
1. The questions are aligned to the learning competency intended to be achieved.	5	5	5	5
2. The activity which I assign requires the students to use cognitive skills such as critical thinking, problem solving, and comprehending.	4.67	4.67	5	4.78
3. The activity which I assign are suitable to the students' level.	4.67	5	5	4.89
4. The activity which I assign can be applied in real-life situations.	5	5	5	5
5. The activity will be scored using an appropriate rubric.	4.33	5	4.67	4.67
Overall Mean	4.73	4.93	4.93	4.86

Table 1: Validators' Mean Scores of the Learning Activities

Student Performance

The following shows students' performances on the individual and online collaborative learning activities, and their achievement test as indicated by their individual scores in Figures 3 to 9.

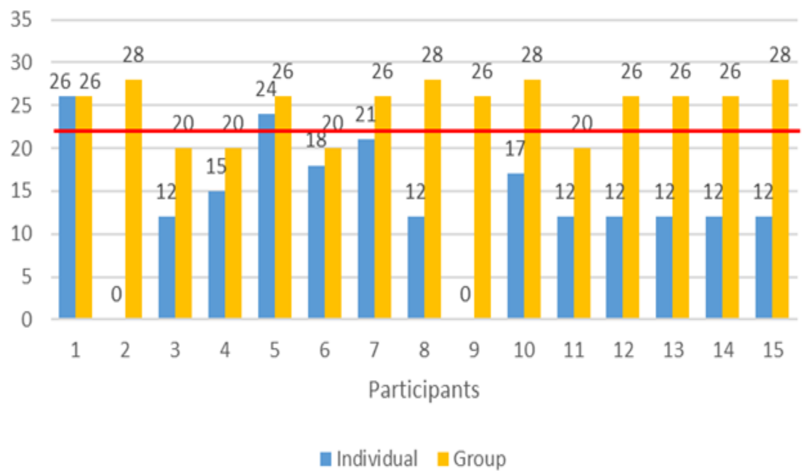

Figure 2: Individual and Group Activity 1 Scores of each Student

As can be gleaned from Figure 2, there were 15 students who participated in the first activity. The lowest among the scores in the individual activity 1 was 0, which were the submitted activities with no answer, while the highest was 26 out of 36. The mean score in this individual activity is 13.67 (*SD* = 7.26). The passing score is 22, and Figure 2 shows that only two students passed the individual activity 1.

With the same activity sheet done collaboratively, Group 2 (Participants 2, 8, 10, and 15) obtained the highest score of 28, both Group 1 (Participants 1, 5, and 13) and Group 3 (Participants 7, 9, 12, and 14) scored 26, and the lowest, Group 4 (Participants 3, 4, 6, 11), scored 20. This reveals that there were 4 students who also obtained the highest score amongst the 15 participants, and 4 students who not only scored the lowest, but also did not meet the passing score, which is 22. The overall mean of scores of each student in this group activity was 24.93 (*SD* = 3.20).

It can also be seen in the figure that only Participant 1 did not have any changes with his/her score. This is because among the members of his/her group, participant 1 has the highest score in the individual activity. Thus, participant 1 was the one who helped his/her groupmates, which resulted to no improvement to his/her score in the group activity. For the rest of the participants, there was an increase of scores between their individual and their group work.

Figure 3 presents that there were 18 students who participated in this activity. Two scored 33 out of 36 were the highest among the participants, while 10 students who obtained the score of 12 were the lowest in the second individual activity. The mean of the scores in this individual activity is 17.33 (*SD* = 7.72). The passing score is 22, and Figure 3 shows that only 6 students passed the individual activity 2.

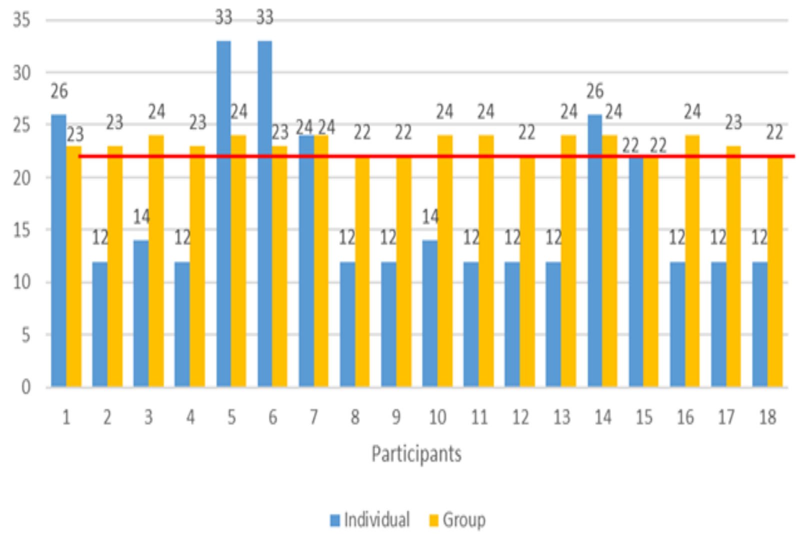

Figure 3: Individual and Group Activity 2 Scores of each Student

In the group work of the same activity sheet, the highest score was 24, which was obtained by both Group 3 (Participants 10, 11, 14, and 16) and Group 4 (Participants 3, 5, 7, 13), Group 2 (Participants 1, 2, 4, 6, and 17) scored 23, while Group 1 (Participants 8, 9, 12, 15, and 18) had the lowest score of 22. This implies that 8 students obtained the highest score of 24 in this group activity. Since the passing score is 22, Figure 3 also shows that all of the groups met the passing score in the given learning task. The overall mean score in this group activity was 23.17 (*SD* = 0.86).

It can be noticed in Figure 3 that Participants 1, 5, 6, and 14 had a decrease in their score during their collaborative work, while Participants 7 and 15 did not have any changes with their scores. These participants were actually the ones who had higher scores in their individual activity amongst their groupmates. Moreover, during online collaborative learning, they were the one who facilitated and helped their groupmates. As a result, instead of having improvements in their scores in the group activity, their scores either were pulled down or had no improvement at all because they were the ones assisting their groupmates and no one was actually helping them. The rest of the participants, however, had an increase in their scores in the second group activity.

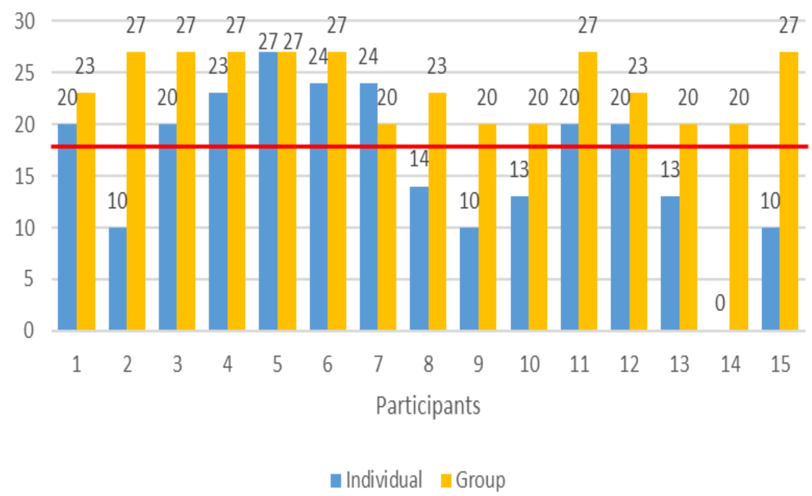

Figure 4: Individual and Group Activity 3 Scores of Each Student

Figure 4 illustrates there were 15 students who participated in this activity. Amongst all the participants in the individual activity 3, the highest score obtained was 27 out of 30, while the lowest was 0. The mean of the scores is 16.53 (*SD* = 7.29). Since the passing score is 18, and Figure 4 shows that there were only 8 students who passed during the individual activity.

During the collaborative work of the same activity sheet, the highest score obtained was 27, which was from both Group 2 (Participants 2, 3, 5, and 11) and Group 4 (Participants 4, 6, and 15). Group 3 (Participants 1, 8, and 12) scored 23, while Group 1 (Participants 7, 9, 10, 13, and 14) obtained a score of 20, which is the lowest. This indicates that a total of 7 participants got the highest score in this activity, and 5 were the lowest. Figure 4 also shows that all of the groups met the passing score in the given learning task, which is 18. The overall mean of scores of each student in this group activity was 23.87 (*SD* = 3.23).

It can be seen in Figure 4 that Participant 5 did not have any changes in his/her score during the intervention, while participant 7 had a decrease in his/her score during online collaborative learning, and the rest of the participants' scores increased on their online collaborative work. Participants 5 and 7 were the highest in the individual activity among the group. Just like in the previous activities, they were the ones assisting their groupmates and facilitating their online group work. Since no one was actually helping them, their scores in the online group work either were pulled down or had no improvement at all.

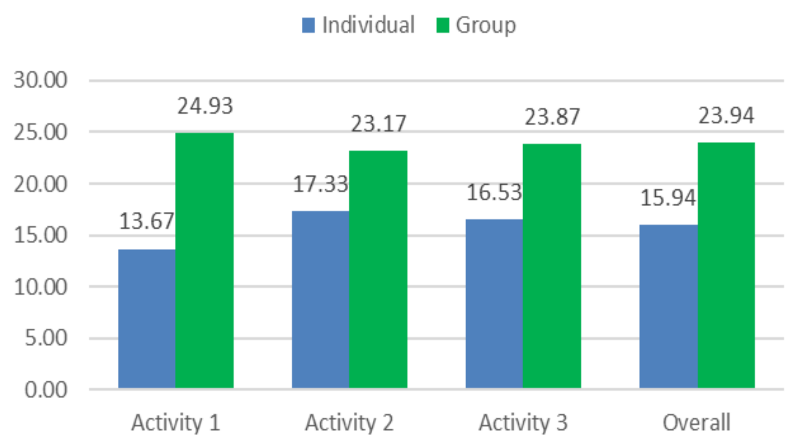

Figure 5: Mean Scores of Students in Both Individual and Group Activities

A paired samples *t*-test was used to determine if there is a significant increase in the scores of the activities when done by group. Results revealed that the mean score in the individual and group activities 1, 2, and 3 have significantly increased at 0.05 level, $t = -5.261$ with *p*-value of 0.00006, $t = -3.265$ with *p*-value of 0.00227, and $t = -4.3122$ with *p*-value of 0.00036 respectively. Consistently, students' overall mean score in the individual activities 15.94 ($SD = 2.64$) has significantly increased to 23.94 ($SD = 2.64$), the overall mean score of the group activities at 0.05 level, $t = -7.183$ with *p*-value of 2.1544×10^{-9}.

The achievement test was administered after the 3-week implementation of online collaborative learning to determine the improvement of students' academic performance after the intervention.

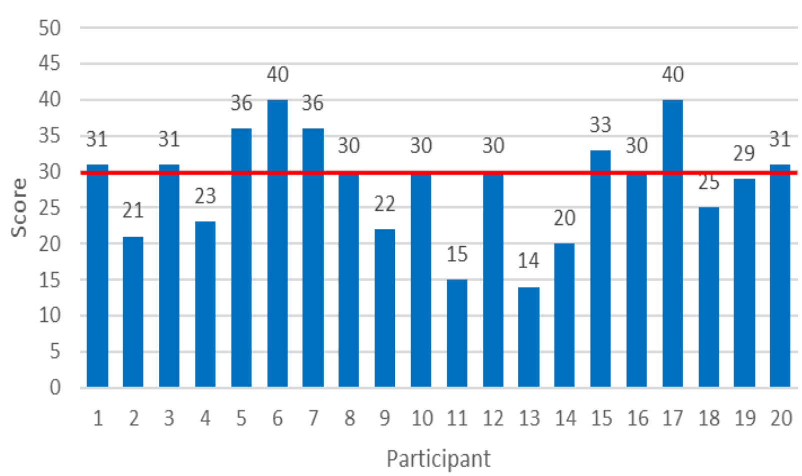

Figure 6: Achievement Test Scores

As seen in Figure 6, two participants who scored 40 out of 50 are the highest among the group, while the participant who obtained the score of 14 is the lowest. The passing score is 60% of the total items, which is 30 points. It can also be seen in Figure 6 that out of 20 participants, there were 12 who passed the achievement test.

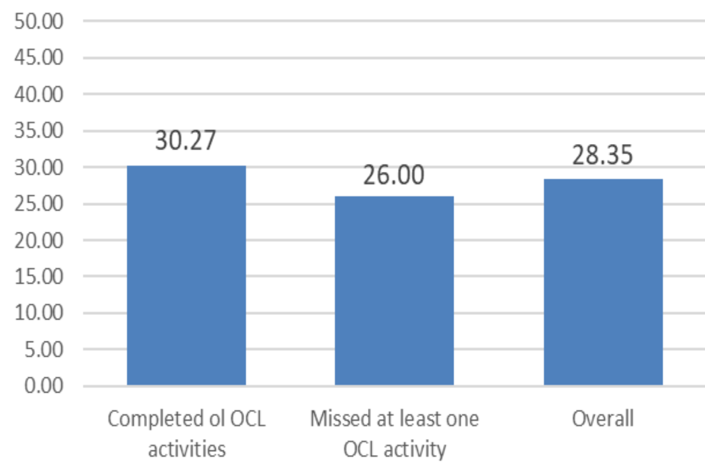

Figure 7: Mean of Students' Scores in Achievement Test

Figure 7 shows the mean scores and standard deviations of the summative test scores of those who were able to complete all the online collaborative learning, those who missed at least once, and the overall. A t-test of two samples showed that there is no significant difference between those who have completed online collaborative learning sessions and those who have missed at least once at 0.05 level, $t = 1.761$ with p-value of 0.1129. Although there is no significant difference between the mean scores of those who completed the online collaborative activities and those missed at least once, those who completed have a mean score of 30.27, which is numerically higher than those who missed a group activity with mean score of 26.00.

The percentage of each score was computed and grade percentages scale description provided by the Department of Education were used to describe the percentage scores in achievement test (Department of Education, 2015).

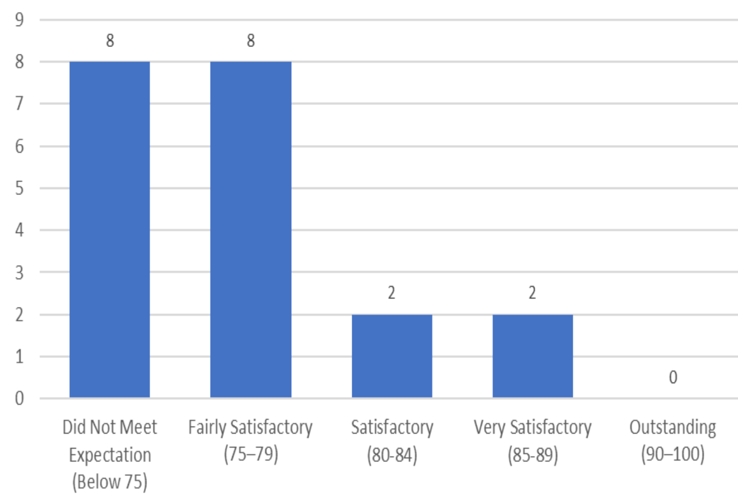

Figure 8: Description of Percentage Scores of the Participants

Figure 8 shows 8 students did not meet the expectation, 8 students were fairly satisfactory, two were satisfactory, and two were very satisfactory. None of the participants was able to obtain a score that is described as outstanding. However, the majority still passed the achievement test.

Student Interaction based on Garrison's Practical Inquiry Model

During online collaborative learning, the students had to meet using the Zoom Software and answer the activities by group. Figure 9 shows a screenshot of a video recording, which illustrates students' group work during online collaborative learning.

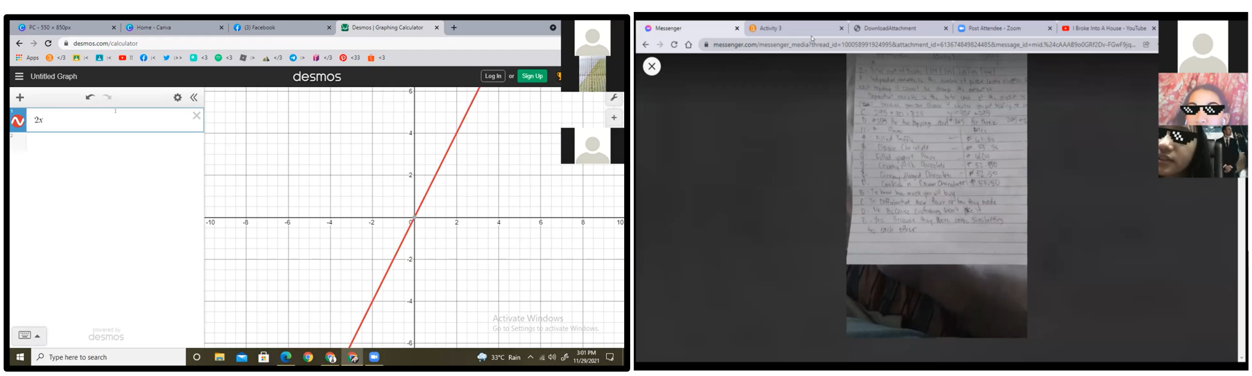

Figure 9: Screenshots of video recordings of online collaborative learning

With the use of content analysis with respect to Garrison's Practical Inquiry Model, the number of messages in each group during online collaborative learning were obtained and were grouped into four categories namely triggering event, exploration, integration, and resolution, following the indicators in Table II (Garrison et al., 2009; Rodriguez, 2014).

Category		Indicators	OCL 1 (%)	OCL 2 (%)	OCL 3 (%)
1. Triggering	a.	Recognizes or identifies problems, concepts, or issue.	1.23	0.84	0.00
	b.	Describes only the assigned problem.	0.61	0.00	0.00
		Sub-total	1.84	0.84	0.00
2. Exploration	a.	Adds to established points but does not systematically defend/justify/ develop.	19.63	28.57	25.93
	b.	Presents relevant background information related to discussion topic.	0.61	1.68	1.85
	c.	Adds suggestions about discussion topic.	1.23	0.00	0.00
	d.	Asks questions seeking specialized information.	9.20	0.84	0.93
	e.	Offers opinions	3.07	0.00	0.93
		Sub-total	33.74	31.09	29.64
3. Integration	a.	Explores potential solutions, applications, or conclusions.	25.15	19.33	36.11
	b.	Draws conclusions or summarizes discussion.	14.11	18.49	12.96
	c.	Reference to previous message followed by substantiated agreement, for example, "I agree because . . ."	1.84	0.84	0.00
	d.	Substantiated building on, adding to others' ideas.	0.00	0.84	0.00
	e.	Synthesis: Connecting ideas. Integrating information from various sources.	6.13	5.04	0.93
	f.	Providing rational, justifications.	0.61	0.00	0.00
		Sub-total	47.84	44.54	50.00
4. Resolution	a.	Applying, testing, defending, or critiquing solutions or conclusions.	11.04	17.65	16.67
	b.	Suggests applications or action to take.	4.91	5.88	2.78
	c.	Commits to solutions or conclusions	0.61	0.00	0.93
		Sub-total	16.56	23.53	20.38
		TOTAL	100	100	100

Table 2: Students' Interaction based on Garrison's Practical Inquiry Model

A total of 390 dialogues and messages, both oral and written, were found in the three online collaborative learning activities. OCL 1 included a total of 163 messages, OCL 2 contained 119 messages, and OCL 3 had 108 messages. For all the three sessions, the average percentage of the messages resulted in 0.89% triggering events, 31.49% exploration, 47.46% integration, and

20.16% resolution. Among the four categories, the lowest percentage was in triggering events (0.89%) and resolution (20.16%). The messages classified as triggering events mainly comprised of explanations of the problems in the activities, and the messages classified as resolution mainly comprised of giving justifications or explanations regarding their answers in the activity (Rodriguez, 2014). Messages accounted for the highest frequencies were integration (47.46%) and exploration (31.49%).3

As mentioned, the lowest category is triggering events which only took place when any of the participants cannot understand a question on the activity, and their groupmates had to explain it. Here are some statement samples that were described as triggering events:

> "*Values listed on the previous item will be plotted on the Cartesian Plane*."

> "*It says in the problem, 'how many washable and disposable masks does Yasmin need to sell to earn at least 100?' in the example given, if she sells 1 washable and 4 disposable masks, she will earn P105, which is greater than P100*."

The category exploration can be seen during online collaborative learning. Below is an excerpt of the dialogue between two students, which illustrates the category exploration from Garrison's Practical Inquiry Model, while they were answering the given question:

> Yasmin is selling a washable face mask (x) for Php 25 each and a disposable face mask (y) for Php 20. How many of both masks does she need to sell to make at least Php 100? Question: Do you think an equation represents the solution set of the problem? If not, is an inequality? Explain.

> *Student A*: What is an inequality?

> *Student B*: I think that's when the symbol is a greater than, less than, greater than or equal to, or less than or equal to. Which in an equation, the symbol used is an equal sign.

The category integration can also be seen from students from the other group while answering the same question. Here is a sample of dialogue of students that illustrates the category integration from Garrison's Practical Inquiry Model wherein the student is trying to verify their ideas from another source of information.

> *Student C*: What is an equation? What is the difference between an equation and an inequality?

> (*Student C searches in Google*)

> *Student C*: Equation – a mathematical statement that shows equal value of two expressions. Inequality – a mathematical statement that is less than or more than the other.
>
> *Student D*: In an equation, the symbol is an equal sign, while in an inequality, the symbol used is greater than, less than, greater than or equal to, or less than or equal to.
>
> (*Student C continues to search in Google for more input*)
> *Student C*: an equation uses factors like x and y, while an inequality uses symbols such as less than or greater than.
>
> *Student D*: yes, yes. So, the answer in this question is: it is an inequality because it has an inequality symbol.

Lastly, the category resolution from Garrison's Practical Inquiry Model was seen as the students answered a question from activity 3. Below is a sample statement of a student that reveals the category resolution where the student applied what s/he learned from the lesson in another problem.

Question: Patricia is going out for a pizza. The pizza costs P295 plus P30 for each extra topping, which is represented by x. The total cost of the pizza is represented by t. What is the independent variable? What is the dependent variable? Explain your answer.

> *Student E*: I think the independent variable is the number of extra toppings because this varies and is represented by the variable x as indicated in the problem and the total cost of pizza is the dependent variable because it's affected by the cost of the number of extra toppings.

Student Attitude towards OCL

Table 3 presents the means and standard deviations of student attitude ratings on online collaborative learning.

Survey Items	Mean	*SD*
1. I enjoy solving problems regarding the group project using Online Collaborative Learning my group members.	3.95	0.76
2. Being interactive with the other group members using Online Collaborative Learning increases my motivation for learning.	4.00	0.79
3. I enjoy experiencing cooperative learning using Online Collaborative Learning with my group members.	4.15	0.67
4. Online Collaborative Learning improves my social skills.	3.95	0.60
5. I enjoy helping others in Online Collaborative Learning.	4.25	0.55

6. Online Collaborative Learning is very entertaining for me.	4.10	0.79
7. Online Collaborative Learning helps me feel better psychologically.	3.75	0.97
8. More ideas come up as a result of Online Collaborative Learning.	4.50	0.61
9. I think that I have had/will have more successful results since I work with a group in Online Collaborative Learning.	4.25	0.72
Overall	**4.10**	**0.74**

Table 3: Mean Scores and Standard Deviations on Student Attitude Survey (n=20)

As per the mean scores and standard deviations on Student Attitude Survey responses (see Table III), the highest mean among the questions is item 8, "More ideas come up as a result of online collaborative learning." These participants strongly agree to this statement as this item garnered a mean score of 4.50 with standard deviation of 0.61. On the other hand, the item that has the lowest mean score is item number 7, which states, "Online Collaborative Learning helps me feel better psychologically." It may be the item lowest among the others, albeit students still agree to this statement as it garnered a mean score of 3.75 with standard deviation of 0.97.

Student Perception towards OCL Compared to Face-to-Face Collaborative Learning

The students' responses during the focus group discussion were transcribed and these went through the four stages of interpretative phenomenological analysis (IPA) namely (1) multiple reading and making notes, (2) transforming notes into emergent themes, (3) seeking relationships and clustering themes, and (4) writing an IPA study (Pietkiewicz & Smith, 2012).

The first step entails reading the transcript several times carefully. Since the video recordings were available, they were also watched multiple times. Here, notes were added based on observations and views about the interview experience, as well as any other relevant ideas and remarks. They were concentrated on content (what is really being discussed), language used (e.g., metaphors, symbols, repetitions, and pauses), context, or early interpretive remarks. Some personal related comments were also created (Pietkiewicz & Smith, 2012).

Afterwards, the focus shifts from the transcripts to the notes. Nonetheless, a thorough and complete reports prepared in the process indicate accurate reflections of the original material in order to turn notes into emergent themes. The researcher tried to come up with a concise statement at a little higher degree of abstraction that might correspond to a psychological conceptualization. Nonetheless, this is based on the specific details of the participant's experience (Pietkiewicz & Smith, 2012).

Next, links between emergent themes were established by grouping them together based on conceptual similarities, and providing a descriptive title to each cluster. This entailed collecting a list of themes for the entire transcript before searching for linkages and clusters. Some of the themes were discarded because they did not fit well with the emergent framework or their evidence foundation was insufficient. Numerous superordinate themes and subthemes were included in the final list (Pietkiewicz & Smith, 2012). Table IV shows samples of the researchers' comments, emergent themes and their theme clusters.

Exploratory Comments	Emerging themes	Theme Clusters
The student liked working collaboratively as a group in an online environment since during groupwork, the student can talk to his/her classmates and be friends with them. However, the same thing can also happen if they work collaboratively in face-to-face setting. They would still be able to communicate with one another and be friends.	Student Interaction	Similarities between Online Collaborative Learning and Face-to-Face Collaborative Learning
Working in groups is more enjoyable and a motivation not to get lazy because the student knows that contribution is needed during group work. A student gets encouragement when he/she gets help from his/her groupmates.	Encouragement/ Motivation among learners	
The student would not have learned in class if he/she had to do the activities alone because in collaborative learning, the students would be able to compare their insights with their groupmates, and in case of any error, the groupmates will be able to correct him/her and point out where she went wrong. With this, the student would be able to learn more from the group.	Student Assistance	
Unlike in face-to-face collaborative learning, students need internet connection and gadgets.	Resources	Differences between Online Collaborative Learning and Face-to-Face Collaborative Learning
This student is not really fond of the online collaborative learning as some of his/her groupmates would either have their microphone off the whole time, or they would just leave the Zoom room. In face-to-face, students could just be passive and appear disinterested.	Student Participation	
Students encountered difficulties in online collaborative learning: unstable internet connection and few technical difficulties, which do not happen during face-to-face collaborative learning.	Technical difficulties and Glitches	

TABLE 4: Emergent Themes from Theme Clusters and Formulated Meanings

A narrative summary of the study is then written. Each theme was discussed and illustrated with the researchers' remarks together with the excerpts from focus group discussions. The table of themes is then transformed into a convincing narrative that communicates to the reader the key experience things discovered throughout the analytical process (Pietkiewicz & Smith, 2012).

In this step, similarities and differences between online collaborative learning and face-to-face collaborative learning were identified. During the focus group discussion, students find collaborative learning in both settings promote student interaction, generate more ideas compared to individual learning, provide peer assistance and encouragement among learners, develop soft skills such as time management and communication skills, and produce better outputs compared to the individual output. Likewise, students mentioned differing experiences between face-to-face collaborative learning and online collaborative learning such as more resources are required when conducting online than when it is in face-to-face, technical difficulties and glitches can happen during online, and differences in students' participation.

Student Interaction in OCL

During the process of online collaborative learning, the first three stages of the taxonomy of Siemen's Connectivism, namely awareness and receptivity, connection-forming, and contribution and involvement, were observed.

Awareness and Receptivity

This is the first stage of Siemen's Connectivism taxonomy (Sitti et al., 2013). In here, learners acquire basic knowledge for handling abundant information through access to resources and tools. In this study, before the participants do the activities collaboratively, they had to accomplish the same activities alone. Here, they were given the opportunity to look for sources of information that helped them not only answer their individual activities, but also share the information they obtained to their group. In this study, students made use of video recordings of lessons, Desmos, self-learning modules, and other accessible resources online as their sources of information and shared it with their groupmates.

Connection Forming

In this stage, learners use the tools and understand the obtained knowledge from the first stage to be able to form connections with their networks (Sitti et al., 2013). Here, they are engaged as they share new resources and technologies in their learning environments. During this study, as the students were doing their activities through online collaborative learning, they shared the information they acquired from various resources such as online learning materials, online math tutorials, and online graphing tools, that they accessed in the first stage. They started imparting their knowledge obtained to their group, as they answered the activities that were given to them. This was done by showing their written notes through their cameras, sharing their screens during online collaborative learning, or writing on the screen with the use of the Zoom Applications' annotation tool.

Contribution and Involvement

In the third stage, the learner begins to actively participate more in the group's activities (Sitti et al., 2013). This active participation allows other members of the group to recognize the learner's resources, contributions, and ideas, resulting in reciprocal understandings and relationships. During online collaborative learning, the students acknowledged their groupmate's shared information, and the group discussed about it further. The learners assessed and verified each knowledge shared among the group for students' enlightenment and better understanding of the lesson. This led to clarification, more ideas, and the development of relationships among the learners during the activity.

DISCUSSION

Developing learning activities for online collaborative learning requires conscientious, purposeful and thoughtful efforts making sure that the activities are relevant to students and could elicit their prior knowledge and experience. Allowing colleagues teaching the same discipline especially the experts and seasoned teachers to provide constructive comments could greatly improve the learning activities. The use of a validation guide also proved useful to elicit suggestions.

The mean scores in the activity sheets revealed that students performed better when doing tasks by group rather than doing it individually. This served as a good indicator that the intervention was effective for the students' learning. However, the results of the achievement test do not support the results of the activity sheets for it has revealed very low raw percentage scores despite the intervention. There may be factors underlying this incident, such as some learners may be free-riders during the online group activity. Free-riding can be very unhelpful to the learning outcomes intended during group work (Hall & Buzwell, 2012). Likewise, the study's results show strong reliance from the more capable students to the point that the latter did not get anything from the former. This can be seen in the group score of the more capable students which were not improved from their individual scores. Another factor that may explain this incident is that some students were shy to participate in the online collaborative learning. The lack of teamwork and interpersonal skills may not only hinder group interaction but may also restrain individual and collaborative learning. Student's lack of confidence might affect his/her learning during online collaborative learning since he/she might not be able to fully contribute to the assigned task (Le et al., 2016).

Students' interaction during collaborative learning in an online setting were analyzed based on the four categories of Garrison's Practical Inquiry Model – Triggering Event, Exploration, Integration, and Resolution using content analysis with respect to Garrison's Practical Inquiry. Results show that messages accounted for the highest frequencies were integration and exploration. Similar to other studies that employed the Garrison's Practical Inquiry Model and content analysis to evaluate learning in discussions, high frequency in integration and exploration was also found (Garrison et al., 2009; Schrire, 2004; Rodriguez, 2014). In this study, integration garnered the highest percentage as they integrated other sources, such as Desmos, a graphing software, when answering the activities. Triggering events obtained the lowest score because as the groups were doing their group activities, they did not try to analyze, explain and solve the problems anymore. The participants would just ask each other their answers to the given items, and if their answers were the same, they would not have any discussion anymore and just moved on to the next problem. This scenario would mean that somehow, the purpose of having the activity done individually first so they could be able to contribute to the group and be accountable with each another's learning is defeated. In this study, the high-performing students failed to dig deeper and assess if their low-performing groupmates really understood the lesson. This may be improved by encouraging the students to always further explain their answers to the group whenever they present their thoughts and ideas. Likewise, the more abled and knowledgeable student may also be encouraged to facilitate and stir up their group's discussion not only by being more directive but motivate their groupmates to share their thoughts on the ideas brought up by their peers. Nonetheless, students may be empowered to look for answers if teachers do not transmit all information and just provide prompt and probing questions as this information are readily available in the internet within students' reach during online collaborative activities.

Nonetheless, the overall mean of the students' responses in the Student Attitude Survey generally indicates a positive agreement with regards to their attitudes with online collaborative learning. Students' top answer being on the aspect of coming up with more ideas means that when working in groups, students strongly agree that they are not only limited to their own thoughts and ideas. They are given an opportunity to realize and explore more ideas that are actually beyond what they can think of. Students find "online collaborative learning help them feel better psychologically" the least in agreement. It may be the item lowest among the others, albeit students still agree to this statement as it garnered a mean score of 3.75 with standard deviation of 0.97. This is because students feel more confident in their ability to understand skills learned when students can practice, investigate, and explore abilities with a groupmate rather than in isolation. Collaboration is enjoyable for students, as we discovered, because they enjoy the social, cognitive, and emotional benefits of working together (Backer et al., 2018). Moreover, students' attitude towards online collaborative problem solving positively affects the relationship and impact during with fellow learners online, which results to positive knowledge contribution during collaborative work (Panigrahi et al., 2018).

It may be inferred from the results that some of the collaborative learning opportunities and experiences in face-to-face may also be found online: student interaction with each other, more ideas are created, student assistance, encouragement among learners, development of skills, timeefficiency in finishing a task, and better output produced. Nonetheless, there are also differences identified by the students between collaborative learning done face-to-face and online. These are in terms of the resource requirements, modalities of student participation, and the various advantages and disadvantages one have over the other – e.g., interrupted interaction due to technical glitches and difficulties in OCL is viewed as a disadvantage and the need to be physically present is a disadvantage in the face-to-face collaborative learning. Moreover, students expressed online collaborative learning served as an avenue for them to exhibit cooperation and participation even in an online learning set up. This confirms the notion that in the process of online collaborative learning, the first three stages of the taxonomy of Siemen's Connectivism, namely awareness and receptivity, connection-forming, and contribution and involvement are evident. The analysis was further made possible by the interpretative phenomenological analysis.

CONCLUSION AND RECOMMENDATIONS

On the basis of the results, the researchers deemed collaborative learning in online setting is applicable. This affirms that collaborative learning in online environment develops interaction among learners and a sense of social presence, which promotes students' improvement of learning and their capability to adapt to various teaching techniques, as well as their motivation and satisfaction (Magen-Nagar & Shonfeld, 2018). However, its implementation is not without fault nor challenges.

We conclude that one prominent challenge in this study is the needed infrastructures for synchronous online collaborative learning, which resulted in a low number of respondents. It is suggested that for future research studies, a similar study may be implemented in a much larger group of participants for a larger extent of generalization.

Aside from the obvious challenge of the required resources for synchronous online collaborative learning, there were emerging challenges in the need for clear guidelines and mechanisms such as the use of self-and-peer assessment in OCL (Caspari-Sadeghi et al., 2022) to avoid free-riding, to maximize learning for retention beyond the group activity for individual learning reflected in achievement test, and to benefit the more knowledgeable and capable students in mathematics as well as the other peers.

Thirdly, doing the activity individually allowed students to reflect on their own and see how much they could contribute to the online collaborative learning which consequently saves some interaction time but this offsets some opportunities to brainstorm in OCL. In this study, it was envisioned that students could bring with them ideas in OCL for discussion from their individual task into the group by taking more time to try to understand, justify, and defend one another's solutions and come to a resolution for a more meaningful exchange of ideas and productive dynamics of collaboration, instead of sheer comparison of answers (Retamoso, 2022).

Furthermore, aside from the use of Garrison's PIM, a closer look at the student exchange of ideas using Sfard's (2007, as cited in Gavilán-Izquierdo et al., 2022) sociocultural theory of commognition may reveal more of students' mathematical thinking through discourse during OCL.

The study conducted a synchronous collaborative learning but future studies may also look into the applicability of online asynchronous collaborative learning. As a result, teachers must always be reminded that when implementing collaborative learning in an online setting, different approaches may be applied as compared to face-to-face collaborative learning.

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The Theory on Loops and Spaces – Part 2

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Abstract: The study of loops and spaces in mathematics has been the subject of much interest among researchers. In Part 1 of "The Theory on Loops and Spaces," published in the "Mathematics Teaching Research Journal," introduced the concept and the basic underlying idea of this theory. This article continues the exploration of this topic and aims to advance the understanding of the theory through observation and analysis of patterns. A systematic examination of intersection points and their numerical Sum is conducted, and the effect of the order of numbering on the analysis is analyzed. Furthermore, the physical implications of the theory are discussed, and the validity of the theory in the third dimension is confirmed through analysis. This article provides a solid foundation for the understanding of the elementary principles of Graph Theory and paves the way for the development of more advanced theorems in the field. Additionally, the article demonstrates how patterns in nature can be analyzed and expressed mathematically, offering a unique perspective on the interplay between mathematics and the natural world. The work is inspired by the video posted by mathematician Dr. James Tanton on his YouTube channel on September 27th, 2021.

INTRODUCTION

The world of loops is a captivating one, from our childhood days of doodling simple shapes to the complex celestial orbits that surround us. In mathematics, space is defined as the region surrounded by a loop, with basic loops such as circles enclosing only one space within their boundaries. But what happens when a loop intersects with itself, creating an intersection point? This simple change in the structure of the loop now creates two distinct spaces, opening up a whole new realm of exploration.

In Part 1 of our investigation, we ventured into exploring the properties of loops, including the remarkable ability to trace the entire loop without lifting the pencil and the possibility of coloring the loop with two colors such that no two same-colored spaces share a common boundary. In this article, we continue our journey by exploring the Number of intersection points in a loop and the significance of their numerical Sum. We also delve into the application of these findings in the realm of three dimensions and consider the potential physical implications of our results.

DEFINITIONS AND NOTATIONS

Before we begin, we will discuss a few definitions and notations which will be used in this article.

Definitions:

- Loop: A closed curve in a space whose starting and ending point is the same.
- Directed Graph: The kind of loops we study in this article are formally known as a directed graph, also called a digraph, in graph theory. A digraph is a graph in which the edges have a direction.
- Space: The region enclosed within the loop, excluding the boundary of the loop.
- Ray: A segment of the loop that either begins or ends with a fixed intersection point P and contains no intersection point other than P.
- Intersection Point: A point created when the loop cuts itself. An Intersection point produces a minimum of 4 rays either going in or out of the point.
- Single intersection point: A point through which the loop passes only once. It contains only 4 rays either going into the point or emerging out of the point.
- Multiple intersection point: A point through which the loop passes more than once. It contains more than 4 rays either going in or emerging out of the point.
- Value of a Point, $V(P)$: If we consider a point as a source of two or more rays emanating from it in opposite directions. For instance, if we take a point on a line, then we have two rays emanating from that point. Similarly, a point with two intersecting lines will have four rays. Now the Value of a Point, P, will be evaluated as:

$$V(P) = 1 + \frac{n-4}{2} = \frac{n-2}{2} \quad (1)$$

  Where 'n' is the Number of rays through that intersection point.
- Numerical Value ($\mathcal{N}$): It is the Number assigned to each intersection point created while tracing the loop as 1, 2, 3, and so on in a definite order.
- I_P: Set of all intersection points passed when loop L is traced starting from a fixed point P.
- Number of Crossings: Number of times an intersection point is crossed while tracing the loop. It is equal to half the Number of rays that emerge from an intersection point.

Notation:

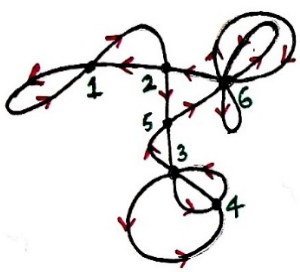
- If we follow the direction of the loop, and we choose two consecutive points, P and Q, on the loop. If we pass through P before passing through Q, then we write $P \rightarrow Q$.

ANALYSING INTERSECTION POINTS

Suppose we trace a loop starting at a fixed point. Let P be an intersection point. In this section, we will obtain a formula for the Number of intersection points passed before reaching P.
Before we begin, we will motivate our result using the following two examples:

Example 1:

An important point to note is that we are tracing along the arrows starting from $1 \rightarrow 2 \rightarrow 3\ldots$ and so on.



Here in Figure 1, we see that we have 2 multiple points, the point labeled 3 with 6 rays and the point labeled 6 with 8 rays, and 4 single intersection points labeled 1, 2, 4, and 5.

Let us observe either of the single intersection points labeled 1, 2, 4, and 5. For instance, let us take point labeled 1, then we will pass through 13 intersection points in the order $2 \rightarrow 5 \rightarrow 3 \rightarrow 4 \rightarrow 3 \rightarrow 4 \rightarrow 3 \rightarrow 5 \rightarrow 6 \rightarrow 6 \rightarrow 6 \rightarrow 6 \rightarrow 2$.

However, if we observe 3, we will pass through a total of 12 intersection points in the order $4 \rightarrow 4$ and $5 \rightarrow 6 \rightarrow 6 \rightarrow 6 \rightarrow 6 \rightarrow 2 \rightarrow 1 \rightarrow 1 \rightarrow 2 \rightarrow 5$, which gives a total of 1+1+10 = 12, or if we observe 6, then we will pass through a total of 11 intersection points in order $2 \rightarrow 1 \rightarrow 1 \rightarrow 2 \rightarrow 5 \rightarrow 3 \rightarrow 4 \rightarrow 3 \rightarrow 4 \rightarrow 3 \rightarrow 5$.

Let us analyze it further in terms of V_P, Number of rays, and Number of crossings

Point Label	Value	Rays	Crossings (Points Passed)
1	1	4	2
2	1	4	2
3	2	6	3
4	1	4	2
5	1	4	2
6	3	8	4

Table 1: The table shows the Value of each point labeled in Figure 1, along with the Number of rays and crossings at each point.

From table 1 above, the Number of crossings is exactly half of the Number of rays. Now, we will use this observation for further results.

Example 2:

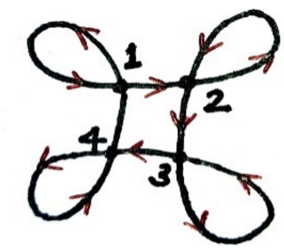


Figure 2: Example 2

In figure 2, we see that we have 4 intersection points labeled 1, 2, 3, and 4, all of which are single intersection points. If we observe either of these points, for instance, let us take point labeled 2, then we will pass through 6 intersection points before reaching 2 again in the order: 3 → 3 → 4 → 4 → 1 → 1.

In fact, we can show that the Number of intersection points passed on tracing the whole loop containing single intersection points will be even.

Theorem 1: If there are only single intersection points in the whole loop, then on tracing the loop starting from a fixed point P, the Number of intersection points passed before reaching P will always be even.

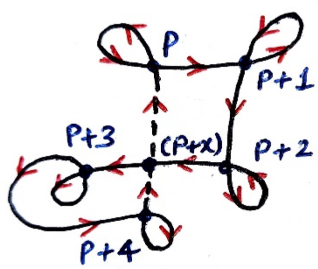


Figure 3: Theorem 1

Proof: Let us take a loop consisting of only single intersection points and fix a point P, then $P + x$ is the x^{th} point from P and the dashed line indicates that there can be some extended loop that is not shown for simplicity.

If we trace the loop starting from P in the order: $P \rightarrow (P + 1) \rightarrow (P + 1) \rightarrow (P + 2) \rightarrow (P + 2) \rightarrow (P + x) \rightarrow (P + 3) \rightarrow (P + 3) \rightarrow (P + 4) \rightarrow (P + 4) \rightarrow (P + x) \rightarrow P$.

From above, we observe that we have crossed every point twice. So, the total Number of intersection points passed by tracing the entire loop once is 2 N, where N is the total Number of Intersection points.

Hence, the Number of Intersection points passed before reaching P will be 2N – 2 = 2 (N – 1). Thus, we have the result when there are only single intersection points.

Thus, by the above examples, we make the following observations:

Observations

1. After passing through an arbitrary intersection point P, the total Number of intersection points passed before reaching P is solely governed by the Value of intersection points.
2. If we trace the loop and pass through an intersection P, then the Number of Intersection points passed before reaching P depends on the 'Total Number of Intersection Points" present in the loop.

The above observations give us the following results:

Lemma 1: In any given loop L, every multiple intersection point (if it exists) must satisfy the condition of a single intersection point; that is, it must have a minimum of 4 rays.

Proof: We know that by the definition of intersection points, every intersection point must have a minimum of 4 rays. Moreover, a single intersection point contains exactly 4 rays either going into the point or emerging out of it. Hence by the definition of multiple intersection point, every multiple intersection point (if it exists) must satisfy the condition of the single intersection point.

Theorem 2: If we have a loop L having N intersection points. Suppose there are S single intersection points and N – S multiple intersection points. Let P be the point on the loop L. Then, starting from P, if we trace the loop, the Number of intersection points passed before returning to P is given by:

$$2 \times (N-1) + \sum_{\substack{Q \in I_P \\ Q \neq P}} (V_Q - 1) \tag{2}$$

Where, I_P: Set of all intersection points passed when loop L is traced starting from a fixed point P, and V_Q is the Value of point Q.

Proof:

We will consider two cases:

<u>Case 1</u>**:** Considering the loop, L consists of only single intersection points. N = S:

Proof, in this case, follows from Theorem 1, and we have: starting from fixed point P, on tracing the loop, the Number of intersection points passed before reaching P is given by $2 \times (N-1)$.

<u>Case 2</u>**:** Considering the loop, L consists of both single and multiple intersection points N ≠ S:

Consider a loop L with N intersection points labeled as P, $P+1$, $P+2$, … $P+m$ in such a way that P has r_0 rays, $P+1$ has r_1 rays, $P+2$ has r_2 rays,… $P+m$ has r_m rays, and $r_i \geq 4 \; \forall \; i \in \{0, 1, 2, \ldots m\}$. Now, assuming that we are tracing the whole loop starting from P, then,

By lemma 1, from each intersection point, 4 rays are used in making single intersection points, giving us $2 \times (N-1)$ intersection points (case 1).
The remaining ($r_i - 4$, $\forall$ i ∈ $\{1, 2, \ldots m\}$) rays will account for the multiple intersection points.
Now, the Number of crossings given by each ($r_i - 4$) rays is equal to $\frac{r_i - 4}{2}$ (3)

This can be rewritten as:

Number of crossings = $\frac{r_i-2-2}{2} = \left(\frac{r_i-2}{2} - \frac{2}{2}\right) = V_{P+i} - 1, \ \forall \text{ i} \in \{1, 2, \dots m\})$ (4)

Summing over the whole loop (excluding P), we get: $\sum_{\substack{Q \in I_P \\ Q \neq P}} (V_Q - 1)$, where $V_Q = V_{P+i}$

Hence, combining both the results,

Total Number of Intersection points that belong to I_P will be given by:

$$2 \times (N-1) + \sum_{\substack{Q \in I_P \\ Q \neq P}} (V_Q - 1) \quad (5)$$

Remark:

In case 1 of single intersection points, the Value V_Q of all the points is 1. Hence, $\sum_{\substack{Q \in I_P \\ Q \neq P}} (V_Q - 1)$ will be equal to 0, and we get the Total Number of Intersection points that belongs to I_P as $2 \times (N-1)$.

SUM OF INTERSECTION POINTS

This section will find the relation to calculate the Sum of the 'Numerical Value' of intersection points as we trace through the loop once.

Before we begin, let us assume that while tracing the loop, the total Sum of Intersection points passed with respect to a fixed-point P is defined based on the Numerical Value given to the intersection points (in a definite order). That means that if we change the order of numerical values of the points, the total Sum of the intersection point will change, and further analysis will be affected.

Now, let us consider the following example:

Example 3:

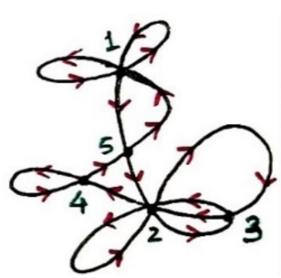


Figure 4: Example 3

In Figure 4, we will find the sequence of intersection points as we trace the loop starting from 1. The sequence is unique to every loop and depends on the numbering of intersection points.

So, the sequence for the Loop in Figure 4 will be:

$$1 \to 1 \to 1 \to 5 \to 2 \to 3 \to 2 \to 2 \to 3 \to 2 \to 4 \to 4 \to 5$$

By observing the sequence, we can have the following lemma:

Lemma 2: While tracing the loop, we will pass through an intersection point one more than the Value of that intersection point.

Proof: By the definition of the Value $V(P)$ of the intersection point, we have,
For an intersection point with 'n' rays, the Value of the intersection point is given by:

$$V(P) = 1 + \frac{n-4}{2} = \frac{n-2}{2} \quad (6)$$

Simplifying equation 6, we get,

$$n = 2 \times V(P) + 2 \quad (7)$$

Now, by definition, the Number of crossings is always equal to half the Number of rays hence,
The Number of crossings will be equal to $\frac{n}{2} = \frac{1}{2}(2 \times V(P) + 2)$ (8)

$$\text{Or,} \quad \text{Number of crossings} = V(P) + 1 \quad (9)$$

Hence, while tracing the loop, we will pass through an intersection point one more than the Value of that intersection point.

Theorem 3: While tracing the loop and defining the sequence with respect to a fixed-point P, the Sum (S) of all the numbers appearing in the sequence is given by:

$$S = \sum_{Q \in I_P} (1 + V(Q)) \times \mathcal{N}_Q \quad (10)$$

Proof: From lemma 2, we know that the Number of crossings for any fixed point Q is given by:
Number of crossings = $V(Q) + 1$
So, the Number of times point P appears in the sequence is equal to $V(Q) + 1$, which gives the Sum of the Number of times point P appears in the sequence as: $(V(Q) + 1) \times \mathcal{N}_Q$
Hence, summing over the whole sequence, we have,

$$S = \sum_{Q \in I_P} (1 + V(Q)) \times \mathcal{N}_Q \quad (11)$$

Now, returning to Example 3,
So, the Sum of the sequence corresponding to Figure 4 will be:

$$S = (1 + 2) \times 1 + (1 + 3) \times 2 + (1 + 1) \times 3 + (1 + 1) \times 4 + (1 + 1) \times 5 = 35$$

Let us now analyze that; if we trace the loop and pass through an intersection P then the Sum of the numerical Value of Intersection points passed before reaching P, For that, we will refer to the table below made by referring to Figure 4:

Point	Intersection points passed	Sum
1	5 → 2 → 3 → 2 → 2 → 3 → 2 → 4 → 4 → 5	32
2	4 → 4 → 5 → 1 → 1 → 1 → 5; 3 and 3	27
3	2 → 2 and 2 → 4 → 4 → 5 → 1 → 1 → 1 → 5 → 2	29
4	5 → 1 → 1 → 1 → 5 → 2 → 3 → 2 → 2 → 3 → 2	27
5	1 → 1 → 1 and 2 → 3 → 2 → 2 → 3 → 2 → 4 → 4	25

Table 2: Sum of Intersection Points passed

From the above table, we can relate the total Sum of the sequence, and the Sum of numerical values of intersection points passed before reaching P.

Corollary 1: Given a loop 'L,' if we trace the loop and pass through an intersection P, then the Sum of the numerical Value of Intersection points passed before reaching P will be given by:

$$S_P = S - (1 + V(P)) \times \mathcal{N}_P \quad (12)$$

Where 'S' is the total Sum of intersection points given by equation 10.

Proof: From theorem 3, we have the total Sum of intersection points given as:

$$S = \sum_{Q \in I_P} (1 + V(Q)) \times \mathcal{N}_Q \quad (13)$$

Now, since we are excluding P in our analysis, so the Sum corresponding to point P will be subtracted from the total Sum and will be given as:

$$\sum_{Q \in I_P} (1 + V(Q)) \times \mathcal{N}_Q - \{(1 + V(P) \times \mathcal{N}_P\} \quad (14)$$

Which, on simplification, gives:

$$S_P = S - (1 + V(P)) \times \mathcal{N}_P \quad (15)$$

Hence, the Sum of the numerical Value of Intersection points passed before reaching P will be given as $S - \{(1 + V(P)) \times \mathcal{N}_P\}$

Now, again returning to Example 3,

If we apply the above corollary 1 for point 3 (Figure 4), then the Sum of the numerical Value of Intersection points passed before reaching 3 will be given as $35 - \{(1 + 1) \times 3\} = 29$

Remark: Special case of a simple loop:

If we have a loop with all the intersection points of 'value 1', starting from any arbitrary point P, the Sum of the numerical Value of intersection points passed as we trace the loop before returning P will decrease by 2 for each consecutive point. So, if we have the total Sum of numbers in the sequence of intersection points equal to 'S,' then starting from point' 1,' the Sum of the numerical Value of intersection points passed while tracing the loop before returning to '1' will be 'S – 2', for '2' will be '(S – 2) – 2', for '3' will be '{(S – 2) – 2} – 2' and so on. This happens so because, while we are starting from '1', we are subtracting (2 × 1) from S, while we are starting from '2', we are adding (2 × 1) but subtracting (2 × 2), similarly when we start from '3', we are adding (2 × 2) but subtracting (2 × 3), so overall we are subtracting '2' in every consecutive step.

Also, in this case, if we have total 'N' intersection points, each having the value 1, then the Sum of numbers of the sequence generated while tracing the loop is given by:

$$S = 2 \times \frac{N \times (N+1)}{2} = N \times (N + 1) \quad (16)$$

This happens because, in this case, the Value of each intersection point is '1' implies that every intersection point occurs twice (1+1) while tracing the loop. So, if we have 'N' intersection points, then the Sum of numbers of the sequence generated while tracing the loop is two times the Sum of natural numbers from 1 to N. then, from arithmetic progression, we have the Sum of natural numbers from 1 to N is given by {N × (N+1)} / 2. So, we get the Sum of the sequence of the numerical Value of intersection points as N × (N+1)

PHYSICAL SIGNIFICANCE OF LOOP THEORY – MOTIVATION FOR HIGHER DIMENSION

The study of loops and spaces has important implications in the field of astronomy, particularly in the analysis of orbits. The theory developed in "The Theory on Loops and Spaces" provides valuable insights when dealing with the intersection points of multiple orbits or when celestial bodies undergo periodic changes in their orbits (Figure 5).

This theory is especially useful in predicting the movement of celestial bodies and creating mathematical models of their behavior. For example, it can be applied to the study of space debris, a complex network of abandoned satellites and other objects, to determine the probability of collisions between these objects. This information can be used to quickly estimate the likelihood of collisions between celestial bodies (Figure 6).

The theory developed in "The Theory on Loops and Spaces" represents a valuable contribution to the field of astronomy and has the potential to revolutionize our understanding of the movements and interactions of celestial objects in the cosmos.

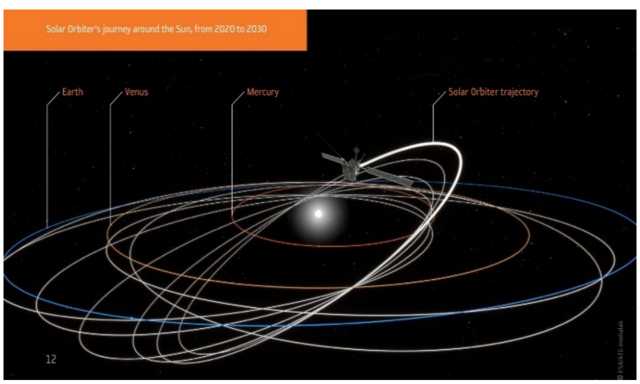


Figure 5: Solar Orbiter Trajectory intersecting with Planetary Orbits
Image Source: ESA Operations on Twitter

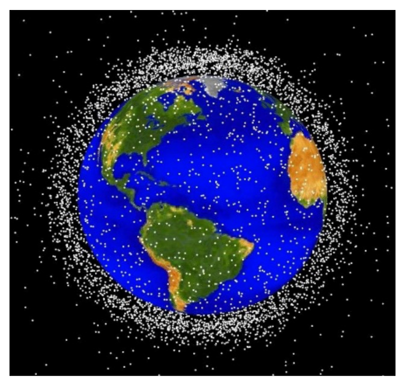


Figure 6: Space Debris Network Around Earth
Image Source: Space Debris | NASA

ILLUSTRATION IN THREE DIMENSIONS

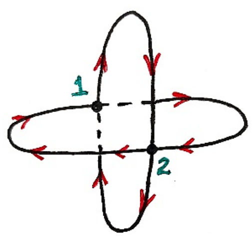


Figure 7: Example 4

The results that we have obtained for the two-dimensional space could be extended for the three-dimensional space in some cases. To verify that, let us consider the following example.

Example 4

On a Blank Piece of paper, we slowly start drawing a self-intersecting loop in 3 – D, as shown in Figure 7. Here dotted lines represent the loop behind the planes.

If we first draw the vertical loop, after completing one vertical loop, we draw a horizontal loop that cuts the vertical loop at point 2; then, we count the Number of spaces created in the following table given below:

Piece	Intersection Point	Region/Space	The Number assigned to that Intersection Point	Value
-	+1	+1	1	0
-	+1	+1	2	+1
+1	-	+1	1	+1

Table 3: Tally of Number of Regions, Intersection Points, Loops and Value of Intersection Points in 3 – D.

Here, when we complete one Vertical Loop and return to point 1, the Value of that point (1) does not increase to +1 because it formed one closed curve and did not form any intersection point, so the Value depends on the intersection point. After completing one closed vertical loop, we have formed one vertical space. Then, we draw the horizontal loop, which intersects the vertical loop at point 2; another space is created as the right part of the horizontal loop, followed by line joining points (1) and (2). When we complete the remaining horizontal loop, 1 Piece of combination loops is completed, and one more extra space is created by the left part of the horizontal loop, followed by line joining points (1) and (2). So, we have a total of 3 Spaces created with 2 intersection points and 1 Piece, and the Value of each intersection point is also preserved. Hence, this shows that our theory could also be extended to 3 – dimensional space.

As a future direction in this area, it would be beneficial to extend this work to include proof of the above results in a general 3 – dimensional space or even a general n – dimensional space.

ACKNOWLEDGEMENT

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I am truly thankful for their support and guidance, and I hope that this article will inspire others to delve further into the fascinating world of loops and spaces.

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An Intervention Study for Improving Pre-service Mathematics Teachers' Proof Schemes

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Abstract: This study aims to examine the effects of a course module aimed at improving pre-service mathematics teachers' (PMTs) proof schemes. This study used designed-based research. Participants of the course were 22 PMTs in a teacher preparation program. We obtained data from two surveys, one of which consists of proof questions. The other requires identifying the proof schemes of ten 9th grade students in a scenario including an excerpt of a hypothetical discussion with their teacher. We implemented both surveys before and after the intervention. In addition, semi-structured interviews were conducted with four PMTs. Findings indicated that the module had a significant and large effect on PMTs' proof schemes in favor of post-intervention. Most of the PMTs transformed their external and empirical proof schemes to the analytical ones. Findings also indicated that PMTs had difficulties identifying students' symbolic, example-based, transformational, and axiomatic proof schemes but overcame these difficulties after the intervention.

Keywords: Designed-based research, intervention study, proof schemes, proof teaching, teacher preparation

INTRODUCTION

Helping students understand proof and developing their proving techniques is a challenging field in mathematics education research (Marrades & Gutiérrez, 2000). Undergraduate courses often focus on how to write proofs rather than how to best use proofs in a high school classroom (Dickerson & Doerr, 2014). Likewise, PMTs reported little opportunity to deal with proof in their university experience and guidance to teach proof (Sears, Mueller-Hill, & Karadeniz, 2013). However, teachers' arguments should be strong, and at the same time, they should raise students with strong arguments. Therefore, mathematics teacher educators should look for different ideas to teach proof in teacher preparation courses (Stylianides & Stylianides, 2017). Lack of such courses and instructional materials in pre-service teacher education programs calls for well-designed intervention studies that foster PMTs' knowledge of proof. In response to this call, this study aims to report the findings of an intervention in the context of a course module that aims to develop PMTs' proof schemes.

Proof Schemes

Proof schemes are cognitive characteristics of the proving process (Harel & Sowder, 1998) and "consists of what constitutes ascertaining and persuading for that person (or community)" (Harel & Sowder, 2007, p. 809). The types of justification people use to prove a proposition determine their proof schemes (Harel & Sowder, 1998, 2007). These arguments may be weak exogenous justifications (Sowder & Harel, 1998), justifications based on experiences more advanced than exogenous arguments (Harel, 2007; Harel & Sowder, 1998). They may also be strong justifications such as generality, operational thought, and logical inferences (Harel, 2007; Sowder & Harel, 1998).

Harel and Sowder (1998) discovered undergraduate students' categories of proof schemes each of which "represents a cognitive stage and intellectual ability in students' mathematical development" (p. 244). Harel and Sowder (1998) offered three main categories of proof schemes and their sub-categories, each characterized by one's methods of justification. In the first category, the *external proof schemes*, an external source convinces the student. This source could be an authority (e.g., a teacher or a textbook). In this case, it is called the *authoritarian proof scheme*. The external source might also be the form or appearance of arguments, e.g., proofs in geometry must be in two columns (Harel & Sowder, 2007). In this case, it is called the *ritual proof scheme*. The last sub-category of an external proof scheme is the *symbolic proof scheme* which includes meaningless manipulations of symbols e.g. $\frac{x+y}{y+z} = \frac{x}{z}$ (Harel, 2007).

The second category is the *empirical proof schemes*. For this scheme, "conjectures are validated, impugned, or subverted by appeals to physical facts or sensory experiences" (Harel & Sowder, 1998, p. 252). It has two sub-categories: (a) the *example-based proof scheme* that relies on evidence from one or more examples, and (b) the *perceptual proof scheme* that relies on intuition or perception to convince or to be convinced (Harel, 2007).

At the highest level of justification, the third category is the *analytical proof schemes* in which conjectures are validated using logical deductions (Sowder & Harel, 1998). It has two sub-categories: transformational and axiomatic. The *transformational proof scheme* relies on generality, operational thought, and logical inference (Harel, 2007; Sowder & Harel, 1998). Generality is concerned with justifying "for all" and operational thought occurs when a student "forms goals and subgoals and attempts to anticipate his/her outcomes during the proving process" (Harel, 2007, p. 67). Finally, mathematical justification should be based on the rules of logical inference (Harel, 2007; Harel & Sowder, 1998). In addition to these three characteristics, in the *axiomatic proof scheme*, proving processes are built upon an axiomatic system; therefore, they must start from accepted principles (Harel, 2007). Unlike the first two categories, analytical proof schemes require formal proofs and those who have this scheme use appropriate proof methods. Both teachers' and students' proof schemes shape the teaching of proof in the classroom because proof schemes are characterized by one's methods of justification to convince himself or others about the truth or falsity of a proposition (Harel & Sowder, 1998; Sowder & Harel, 1998). Teachers might often use external and empirical arguments in their teaching. However, the use of this schemes causes difficulties in proof teaching. Because students have difficulties bridging the gap between informal and formal argumentation (Heinze & Reiss, 2003). For this reason, for teaching proof effectively, firstly, teachers should have analytical proof schemes when making proofs because they are at the top level of justification types and accept formal proof (Sowder & Harel, 1998). Nevertheless, research studies in the mathematics education literature indicated that pre-service and in-service mathematics teachers' proof schemes were not at the desired level (Manero & Arnal-Bailera, 2021; Sears, 2019). Therefore, this study aims to develop pre-service mathematics teachers' analytical proof schemes.

As future teachers, PMTs should determine the types of justifications used by their students and help them enhance their justification types to reach the analytical level. PMTs should identify students' valid and invalid arguments (Lesseig, Hine, & Boardman, 2018). However, research shows that they have difficulties in evaluating proofs (İmamoğlu & Yontar-Toğrol, 2015; Sears, 2019). Therefore, another aim of this study is to improve pre-service mathematics teachers' knowledge of identifying proof schemes used by students.

In line with the aims of the research, we designed a course module and teaching materials to address the deficiencies about proof schemes reported in the literature. The course module was integrated into a teacher preparation program. In doing that we aim to guide mathematics education researchers and teacher educators in designing similar modules on proof and proof schemes.

METHODOLOGY

This study is part of a wider study which used designed-based research (DBR). DBR "is an emerging paradigm for the study of learning in context through the systematic design and study of instructional strategies and tools" (Design-Based Research Collective, 2003, p.5). A proof course was designed and implemented in a teacher preparation program in a state university in Istanbul, Turkey. The current study focuses on a module on proof schemes. Within this module, DBR is used to develop and examine PMTs' proof schemes. The current study reports the first cycle of these processes and makes suggestions for a second cycle.

Participants and Context of the Study

Participants were 22 PMTs (fourteen females and eight males) who enrolled in an elective course called "Proof in Mathematics Teaching." Before this course, they were familiar with mathematical proofs. All participants signed a consent form that explains the aim of the study and ethical issues.

We selected four PMTs using the theoretical sampling technique (Mason, 2002) based on their answers to the survey questions for a further qualitative investigation. PMT1 often used the analytical scheme, PMT2 the empirical scheme, PMT3 the external scheme, and PMT4 was the one who left the survey questions unanswered the most and could not complete the proofs (i.e., considered as not having a particular proof scheme).

Design of the Module

Therefore, we first specified the objectives of the module considering the related literature on proof schemes as reported above. We specified two learning objectives (Cihan, 2019; Cihan & Akkoç, 2019):

- PMTs will be able to use analytical proof schemes in their proofs.
- PMTs will be able to identify students' proof schemes.

In the wider study (Cihan, 2019), we designed a 15-week course. It had various modules (i.e., proof components, methods, identifying proof schemes, student difficulties with proving, reasons behind student difficulties, and teaching strategies to overcome student difficulties). Within the scope of this current study, we prepared an eight-week module to improve PMTs' proof schemes. The module's content consists of proof methods, proof classifications, and proof schemes. As a result of expert opinions, we decided to prepare worksheets and scenarios to improve proof schemes of PMTs.

Implementation of the Module

The first author was the tutor of the course module, while the second author had the role of a coordinator and a non-participant observer. The teaching methods included lecturing, questioning, discussion, problem-solving, case study, and scenario-based teaching. The course addressed various proof methods (proof by induction, direct proof, proof by cases, proof by contradiction, proof by contrapositive, proof by counterexample, proof by exhaustion) using hands-on activities as PMTs worked in groups of five or six. Later, the tutor explained Sowder and Harel's (1998) three main proof schemes and seven sub-proof schemes.

The tutor asked PMTs to prove theorems at high school and undergraduate levels using worksheets. We prepared worksheets following the axiomatic structure specific to each proof method. Each worksheet starts with a sentence stating the assumptions and ends with a closing sentence stating that the proof has been completed. The tutor asked PMTs to fill in the gaps between these sentences according to the proof methods. These gaps were specific to the proof methods. For example, for a theorem to be proved by the weak inductive proof method, worksheets had gaps for the basic step (or the initial step), the persuasion step, the inductive hypothesis (or the inductive assumption), and the inductive step (See Appendix 1). To exemplify the method of proof by cases, the worksheets had gaps to be filled by PMTs for proving all the cases. Worksheets for the proof by contrapositive had gaps to write the expressions $p \Rightarrow q$ and $\neg q \Rightarrow \neg p$, and to prove $\neg q \Rightarrow \neg p$.

Using these gaps in the worksheets, we aim to help them to decide about the steps in different proof methods. We devoted one week for each proof method. PMTs worked individually and as a group. Later, the tutor completed the proofs at the axiomatic level. The class discussed the gaps in the argument chain of different proofs used by PMTs.

We presented Sowder and Harel's (1998) classifications of proof schemes to the PMTs using two scenarios. PMTs worked first in groups and then individually to identify the students' proof schemes in these scenarios. Finally, the class discussed proof schemes of the students in the scenarios. PMTs took on the role of a teacher when determining students' schemes and were asked how they would respond to the students. After the intervention, we evaluated the learning objective related to proof schemes using a different scenario.

Evaluation of the Module

For this study, we evaluated the effect of the module on achieving the learning objectives. We considered pre-test, post-test, and interviews to evaluate them. As a result of these evaluations, which will be presented as the findings of this study, we revised the course module to be implemented in the next cycle to improve the intervention because DBR goes beyond designing and testing specific interventions and contributes to learning and teaching theories (Design-Based Research Collective, 2003). We will attend to these issues in the discussion section. Based on our experience in conducting the module and findings of our research, we have prepared a practical guide for practitioners who might want to use the module (See Appendix 2).

Data Collection Tools

We collected data using the proof survey, a scenario-based survey, and semi-structured interviews. We conducted the surveys and interviews twice, before and after the intervention (fifteen weeks later).

We designed a proof survey to explore the proof schemes used by PMTs (See Appendix 3). Proof survey consist of seven types of proving questions on mathematical topics such as Fibonacci sequence, matrices, inequalities, functions, divisibility, trigonometry, and numbers. Six of these questions require proofs of the theorems and the other a refutation of a false proposition. We took expert opinion about whether survey questions can be proved with seven different proof methods (proof by induction, direct proof, proof by cases, proof by contradiction, proof by contrapositive, proof by counterexample, proof by exhaustion), and whether the questions reveal PMTs' proof schemes. Experts gave their opinions during a workshop that introduced the purpose of the study, research questions, methodology, data collection tools, and data analysis techniques. We delivered expert opinion forms, and experts discussed their inputs. For the reliability of the survey, we conducted a pilot study with ten PMTs who were studying in the second year of a teacher preparation program in another university. After taking expert opinions and conducting the pilot study, we revised the survey questions so that PMTs prefer each of the proof methods above. Thus, the survey was revised to include seven questions for which seven proof methods are the most appropriate. We also wanted to observe whether PMTs could use each proof method using the analytical schemes.

The first author interviewed four PMTs individually using a semi-structured interview form. We asked PMTs follow-up questions during the interviews to obtain in-depth information for their answers to each question in the survey. Questions in the form explored whether PMTs had a proof scheme, and if so, which scheme they had. The interviewer asked PMTs whether the proof was sufficient to convince themselves or others, whether there were missing points in their proofs, whether a different proof was possible, and whether they had any difficulties. He also asked them to make sense of their steps in the proof and follow-up questions based on their proof schemes.

We also designed a scenario-based survey to explore how PMTs identified students' proof schemes (See Appendix 4). Scenarios include "short stories about hypothetical characters in specified circumstances" to whose situation the respondent is invited to respond (Finch, 1987, p. 105). In this study, the scenario includes an excerpt of a hypothetical discussion among a mathematics teacher and ten 9th-grade students (age of fifteen). The topic of the scenario is set theory, which is a typical topic in the 9th-grade curriculum in Turkey, the basis of many mathematical subjects (e.g., functions, derivatives) and has an essential place in the use of mathematical language. In addition, set theory is suitable for the use of different proof schemes (See Cadwallader-Olsker, 2011). The class discusses the truth of a proposition. PMTs are familiar with set theory from undergraduate courses. In the scenario, the teacher presents the following proposition and asks students whether this proposition is true or false, and to justify their answers:

"Let X, Y and Z be sets. If $X \subset Z$ and $Y \subset Z$ then $X \subset Z$".

Students' excerpts illustrate Sowder and Harel's (1998) proof schemes. The scenario in the survey was different from the one used in the module for validity concerns. The topic was also different. PMTs filled the scenario-based survey before and after the intervention (fifteen weeks later). They worked on handouts that included the scenario and the course objective. The survey included the question: "*Determine the arguments (justifications) used by the students in this scenario to prove the truth or falsity of the statement*."

We ensured the validity and reliability of the scenario-based survey in a similar way as for the proof survey. For validity concerns, researchers conducted a workshop with ten experts in mathematics and mathematics education. Considering the expert opinions, we specified the number of students as ten which is bigger than seven (the number of sub-proof schemes), to prevent PMTs from matching students' work to the proof schemes. Based on expert opinion, we prepared a coding key for proof schemes (See Appendix 5).

We used the same scenario and asked the following questions to the four PMTs during the semi-structured interviews: (a) identify students' argument (justification type), (b) whether students' answers convinced you or would convince others (c) how would you intervene with students' answers if you were the teacher.

Data Analysis

We used Sowder and Harel's (1998) categories for proof schemes to analyze data to explore the module's effectiveness. We used the Wilcoxon Signed Rank Test (Wilcoxon, 1945) to investigate whether the module significantly affected PMTs' proof schemes. Probability value $p < 0.05$ was decided to be enough to be a statistically significant difference with a confidence level of 95%. The effect (r) size was calculated using the formula $r = z/\sqrt{n}$ (Pallant, 2007). According to Cohen (1988), the effect size of $r = 0.1$ is considered small, $r = 0.3$ medium, and $r = 0.5$ big.

We analyzed the answers to the proof survey by descriptive analysis. One hundred fifty-four responses given by 22 PMTs to seven questions in the survey were coded as "external," "empirical," "analytical" and their sub-schemes. We coded no responses and incomplete proofs as "without a scheme." We considered answers that the prover is not convinced by himself also in this category because, for a person to have a proof scheme, he must first be convinced of the proof he made (Harel & Sowder, 1998; Sowder & Harel, 1998). We scored the analytical schemes as three points, empirical schemes as two points, external schemes as one point, and "without a scheme" answers as none. External scheme is ranked higher than "without a scheme" because one having an external scheme provides a justification for the truth or falsity of a proposition.

We analyzed the answers to the scenario-based survey also by descriptive analysis. Each participant identified 10 students' proof schemes. Therefore, there are a total of 220 answers (22 PMTs times 10 schemes). We coded quantitative data obtained from the survey as "correct" (proof scheme was identified correctly), "incorrect" (proof scheme was identified incorrectly), or "no response." We scored the correct answers as one point, others as none.

For the qualitative analysis, we used descriptive content analysis of interview data. We determined the thematic framework as analytical, empirical, external, and without a scheme to explore proof scheme using the interview data. If a PMT has doubts that the proof he made is not convincing for himself and others, or if he could not complete the proof, we coded it as "without a scheme." We considered PMTs who did not have any problem with persuasiveness, answers of those who resorted to authority for proof in the interviews, and who thought that this was valid for proof and gave answers in this direction to be having an "external (authoritarian) proof scheme." We coded the answers of the PMTs who used symbols without a meaning thinking that they were making valid proofs as "external (symbolic) proof scheme." The PMTs who tried to imitate the proofs they made in their past learning experiences focused only on the proof's form or appearance and not its content or logic. We coded them as an "external (ritual) proof scheme." We coded the answers of the PMTs who thought that one or more examples are sufficient for proof as an "empirical (example-based) proof scheme." We considered proving by drawing figures to indicate an "empirical (perceptual) proof scheme." We coded the answers reaching a generalization with logical inference rules and operational thinking and making sense of the steps taken as “analytical (transformational) proof scheme,” the answers reaching generalization in an axiomatic structure and making sense of this as “analytical (axiomatic) proof scheme.” We also used descriptive content analysis of the interview transcripts of PMTs’ responses to the scenario-based survey to elaborate further on how they identified the students’ arguments (justifications) in the scenario using Sowder and Harel’s (1998) classification and the teacher’s response guiding the student.

RESULTS

This section first presents the findings related to the effects of the intervention on PMTs’ proof schemes. We will compare the proof schemes of 22 PMTs before and after the module based on the analysis of their responses to the proof survey (See Appendix 3). We will present the excerpts from the interviews to exemplify how PMTs’ proof schemes evolved from the external and empirical schemes to the analytical schemes. The second section will present the findings related to the effect of the intervention on the way 22 PMTs determined the proof schemes of the students in a scenario.

The Effects of the Intervention on PMTs’ Proof Schemes

We evaluated PMTs’ written proofs and Table 1 presents the frequencies of the proof schemes used by 22 PMTs before and after the intervention.

		External			Empirical		Analytical			
Post-Pre	f and f_{sum}	Authoritarian	Symbolic	Ritual	Example-based	Perceptual	Transformational	Axiomatic	Without a scheme	Total
Pre	f	26 (30)*	35	5	27	5	25	4	27	154
	f_s		66		32		29		27	
Post	f	8	20	4	2	0	8	110	2	154
	f_s		32		2		118		2	

Table 1: Frequencies for PMTs’ Proof Schemes (*Note.* *Before the intervention, more than one proof scheme was used in four questions. We considered those at the higher level as the scores.)

If PMTs had no response, could not complete their answers, or were not convinced, then we considered their answers as “without a scheme.” The frequency of this category decreased from

27 to two throughout the intervention. PMTs used two different main proof schemes (external and empirical) in four of the responses given before the intervention. We coded these responses as the empirical scheme. After the intervention, the frequencies of the external and empirical proof schemes decreased while the frequency of the analytical schemes increased dramatically (from 29 to 118).

We explored the turn from external schemes and empirical schemes to analytical schemes in-depth during the interviews. For example, PMT3 mostly used external proof schemes before the intervention. In response to the question, "*Prove that* $x + \frac{1}{x} > 1$ *for* $\forall x \in R_{>0}$", he just manipulated the symbols. His approach was like problem-solving without using a proof method and displayed the characteristics of the external (symbolic) proof scheme. After the intervention, he chose the method of proof by cases and showed that the inequality was satisfied for the two cases ($0 < x < 1$ and $x \geq 1$) using the axiomatic structure. After the intervention, his proof scheme transformed into an analytical (axiomatic) scheme. Another question in the proof survey was "*Prove that* $\sqrt{5}$ *is not a rational number*". Before the intervention, he developed an argument based on a rule by stating that "*Root numbers are generally irrational numbers. Since* 5 *is not a perfect square, it is not rational*," and he was convinced about the correctness of the proposition in this way. In the interview, he displayed the characteristics of the external (authoritarian) scheme. After the intervention, he used the method of proof by contradiction; he assumed that $\sqrt{5}$ is rational, that is, $\sqrt{5} = \frac{a}{b}$, $a, b \in Z$ and $(a, b) = 1$. He then reached a contradiction that $(a, b) \neq 1$. During the interview, he made sense of the proof method and all the steps he took. With his answers, he showed all the characteristics of the axiomatic scheme.

PMT2's proof scheme was mostly empirical before the intervention. She did not use any specific proof method to prove that "$\forall m \in Z^+$, $m^3 - m$ *is divisible by* 6". She mentioned that $m^3 - m$ is a multiple of 6 for $m = 1, m = 2, m = 3$, and $m = 4$ until she convinced herself. After the intervention, she preferred the method of proof by exhaustion, considering that $m \in \{0,1,2,3,4,5\}$. In short, she moved from the example-based scheme to the axiomatic scheme. When responding to another question in the proof survey, she used the perceptual proof scheme (empirical) before the intervention, as shown in Figure 1.

Let $K, L,$ *and* M *be non-empty sets and* $g: K \to L$ *and* $h: L \to M$ *be two functions. Prove that if* hog *is a one-to-one function then* g *is also a one-to-one function.*

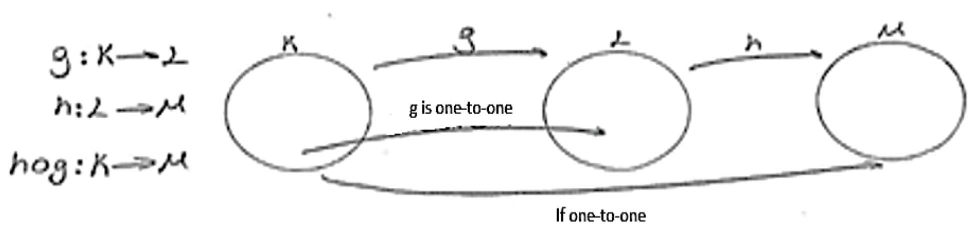

Figure 1: A drawing implying the perceptual proof scheme (PMT2, Pre-Intervention)

During the interview, PMT2 stated that the figure above was convincing for her and enough to convince others:

I: Which proof method did you use for this question?

PMT2: I drew a figure

I: How did you draw it?

PMT2: I checked the definitions of composite functions and one-to-one functions.
I: Any method?

PMT2: Actually, I did not use any method.

I: Is it possible to prove without a proof method?

PMT2: It was enough for me to do it, as in the diagram, intuitively.

I: Is this proof enough to convince you or others?

PMT2: Intuitively but if any other person did it then I would be convinced. I would also convince others.

The excerpt above is an indication of the perceptual scheme since PMT2 did not choose any method of proof and relied on a diagram. The following excerpt underlines her perceptual scheme further:

I: Are there any deficiencies in your proof?

PMT2: Even so, it is a convincing proof. Intuitively.

I: Is there a different proof other than yours?

PMT2: There are some others using mathematical expressions. But it explains this figure.

I: Is this enough to draw a figure in this question?

PMT2: Yes, if it is explanatory and meaningful.

After the intervention, she transformed into the analytical (axiomatic) scheme. She wrote the following proof:

Let g be a function that is not one-to-one. Let's prove that the function hog is not one-to-one.

If g is not one-to-one, then there exists $x, y \in K$ such that $g(x) = g(y)$. (1)

Then $(h \circ g)(x) = h\big(g(x)\big) = h(g(y)) = (h \circ g)(y)$ (2)

If $(h \circ g)(x) = (h \circ g)(y)$, then $h \circ g$ is not one-to-one. (3)

The proof ends here.

As her proof shows, accepting that the function g is not one-to-one, she showed that the composite function is not one-to-one in the axiomatic structure using the definition of one-to-one-ness. She applied the proof method correctly. The excerpt below indicates that she was aware of the proof method she chose:

I: Which proof method did you use for this question?

PMT2: Proof by contraction.

I: Why did you choose this method?

PMT2: Direct proof was also possible, but I think proof by contraction was the easiest for this question instead of looking for one-to-one-ness in composite functions.

I: Can you explain the method you chose?

PMT2: When the truth of the proposition $p \Rightarrow q$ is asked, we prove that $\neg q \Rightarrow \neg p$.

During the interview, she explained why her proof was convincing for her and others:

I: Is the proof method you choose suitable for this question?

PMT2: Yes.

I: Is this proof enough to convince you or others?

PMT2: It was very convincing for me.

I: Does it convince others?

PMT2: It does. If people know what a composite and one-to-one function is.

I: Can you make sense of each step in your proof?

PMT2: I can make sense. Suppose g is not one-to-one. From here, using the definitions of composite and one-to-one functions, I proved that the composite function is not one-to-one.

I: Did you have any difficulties when proving?

PMT2: No …at least I know the definitions and rules.

As the excerpt above shows, she could make sense of the steps she took in the proof. Based on the definitions, she reached a generalization using operational thought and logical inference which are indications of the axiomatic scheme.

PMTs who did not embrace any specific proof scheme tended to demonstrate the characteristics of the analytical proof schemes after the intervention. PMT4 left most of the questions in the proof survey unanswered before the intervention. In the interview, she reported that this situation was related to a lack of knowledge of proof methods and the content. For example, for the question, "*Let X be a matrix. $\forall X$ prove that XX^T is a symmetric matrix*", she stated that she could not prove it because she did not know the correctness of the proposition herself. Hence, she said that she did not have a convincing argument. PMT4 could not give any response which belonged to any scheme for this question. After the intervention, she chose the direct proof method and performed the following operation: $(XX^T)^T = (X^T)^T X^T = XX^T$, starting with the definition of a symmetric matrix. In other words, she used the axiomatic scheme.

PMTs failed to use the analytical scheme before the intervention. For example, when PMT1 was proving the statement "*Let f_m be an element of the Fibonacci sequence. If $m \geq 2$ then $(f_m)^2 - f_{m+1} \cdot f_{m-1} = (-1)^{m+1}$*" (Bloch, 2011) before and after the intervention, she proved the base case ($m = 2$) and assumed that the statement holds for $m = k$ for any k. However, when proving that it holds for $m = k + 1$, she did not use the axiomatic structure before the intervention. Instead of reaching the induction step from the induction hypothesis, she performed the necessary transformations of the induction step (logical inferences, procedural thinking, and generalizations) and completed the proof. After the intervention, she managed to do all these in an axiomatic structure by starting with the induction hypothesis, constructing a chain of arguments, and reaching the induction step. In other words, she moved from the transformational scheme to the axiomatic scheme throughout the intervention.

We can conclude from the findings presented in Table 1 and as emerged from the interviews that the module effectively reached the learning objective "PMTs will be able to use analytical proof schemes in their proofs. We used the Wilcoxon Signed-Rank Test to decide whether this effect was significant (See Table 2).

Post-intervention—Pre-intervention	N	**Mean Rank**	**Sum of Ranks**	Z	p	r
Negative Ranks	0^a	0.00	0.00	-4.113^*	<0.001	-0.88
Positive Ranks	22^b	11.50	253.00			
Ties	0^c					

Table 2: Wilcoxon Signed-Rank Test Outputs Regarding PMTs' Proof Schemes *(Note.* * Based on negative ranks, [a] Post-intervention < Pre-intervention, [b] Post-intervention > Pre-intervention, [c] Post-intervention = Pre-intervention.)

Table 2 shows a significant difference ($Z = -4.113$, $p < 0.001$) between scores obtained before and after the intervention. Since the absolute value of effect size (r) is 0.88, greater than 0.50, we can say that the module had a large effect size on the scores of proof schemes in favor of post-intervention.

The Effects of the Intervention on PMTs' Identification of Proof Schemes

In addition to PMTs' proof schemes, we also examined the effect of the intervention on the way PMTs determined the proof schemes of the students in the scenario and found that they better identified students' proof schemes after the module. Table 3 presents the frequencies of the PMTs' correct and incorrect answers for identifying the proof schemes of ten students in the scenario before and after the intervention. The scenario included 10 students and the table shows the answers from 22 PMTs for identifying each student's proof schemes.

Student number	Type of Proof Scheme	Correct (Pre-/Post-)	Incorrect or Empty (Pre-/Post-)
Student 1	Authoritarian	19/18	3/4
Student 2	Authoritarian	19/22	3/0
Student 3	Perceptual	17/22	5/0
Student 4	Ritual	18/22	4/0
Student 5	Symbolic	0/20	22/2
Student 6	Example-based (a single example of finite sets)	21/22	1/0
Student 7	Example-based (a single example of infinite sets)	13/21	9/1
Student 8	Example-based (multiple)	14/21	8/1
Student 9	Transformational	2/22	20/0
Student 10	Axiomatic	12/22	10/0
Total		135/212	85/8

Table 3: The frequencies of PMTs' correct and incorrect answers for identifying proof schemes

As can be seen in Table 3, findings indicate an improvement of the way PMTs identified students' proof schemes. The number of correct answers increased by 77. As Table 3 shows, PMTs had difficulties identifying symbolic, example-based, transformational, and axiomatic proof schemes before the intervention. They had overcome most of these difficulties after the intervention. Below, we illustrate these improvements with specific examples.

Student 5 in the scenario has the *symbolic proof scheme*. He justifies his answer considering the number of elements: "If $X \subset Y$ then $S(X) < S(Y)$ and if $Y \subset Z$ then $S(Y) < S(Z)$. Therefore $S(X) < S(Z)$ that is $X \subset Z$". Using this statement which is wrong, Student 5 uses shallow symbolic manipulation. None of the PMTs correctly identified that this student has a symbolic proof scheme because they also assumed it was a valid proof. PMTs thought that Student 5 in the scenario "*puts forth other cases based on known facts or data*" or "*justified it using logical inference*" and did not notice that Student 5 used symbolic manipulations. After the intervention, 20 out of 22 PMTs identified Student 5's proof scheme correctly. They used the terminology of the proof scheme framework. PMT1, before the intervention, mentioned that Student 5 completed the proof using the mathematical expression and did not notice the errors in the procedures, and she was convinced. After the intervention, she mentioned that Student 5 had the external (symbolic) proof scheme since the student manipulated symbols incorrectly, and she described the scheme's characteristics. Furthermore, she said that she was not convinced by Student 5's justifications and would tell the student his wrong symbolic manipulations.

In the scenario, we prepared three different cases of the *example-based proof scheme* using (a) a single example of finite sets (Student 6: Let $X \subset Y$ and $Y \subset Z$. Let $X = \{1,2\}$, $Y = \{1,2,3\}$ and $Z = \{1,2,3,4\}$, Since $\{1,2\} \subset \{1,2,3\} \subset \{1,2,3,4\}$ then $X \subset Z$), (b) a single example of infinite sets (Student 7: $N \subset Z$ and $Z \subset R$. Therefore $N \subset R$), and (c) multiple examples (Student 8: If each one of us in the class finds an example to show the truth of the proposition, then we can reach a generalization). Before the intervention, 21 out of 22 PMTs noticed that Student 6 relied on only one example. After the intervention, all PMTs identified the proof scheme of Student 6 correctly. For the cases of (b) and (c), frequencies of correct answers increased considerably after the intervention. For (b), after the teacher called out for a more general example, Student 7 justified his answer using the sets $N, Z,$ and R, which are infinite. We consider this as an "example-based proof scheme using a single example," as in the case of (a). However, before the intervention, nine PMTs could not identify the proof scheme in the case of (b) because they thought this was a generalization. Since they considered the student's proof scheme a generalization rather than the example-based scheme, we coded their responses as incorrect. However, after the intervention, they improved in identifying this scheme (21 out of 22 PMTs answered correctly). Student 8 suggested that if each student find one example, then there would be many examples to justify the truth of the proposition. 8 out of 22 PMTs could not identify Student 8's scheme as "*example-based*" before the intervention thinking that multiple examples were convincing for a generalization. After the intervention, 21 out of 22 PMTs correctly identified Student 8's justification as an "*empirical (example-based) proof scheme*." As an example, we present the responses from PMT3.

I: What is this student's argument (justification)? (Pre-intervention)

PMT3: If each student in the class gave an example, different examples would be sufficient for a generalization.

I: Did this student's justification convince you?

PMT3: So, what the student did, led to a proof. He convinced me.

Before the intervention, he thought that students' different examples were convincing to reach a generalization. He failed to refer to the properties of the example-based scheme prior to the intervention. However, he did not find them convincing and pointed out the need for a generalization after the intervention:

I: What is this student's argument (justification)? (Post-intervention)

PMT3: He gave a lot of examples.

I: Did this student's justification convince you?

PMT3: It has to be a generalization to persuade, but it is not. So he didn't.

I: What kind of approach would you take if you were the teacher in the scenario?

PMT3: I would say what he did was giving a lot of examples. I would say this is not a proof.

I: What is the main proof scheme this student has?

PMT3: Empirical proof scheme.

I: What is the sub-proof scheme that this student has?

PMT3: Example-based proof scheme.

I: What are the features of the scheme that this student has?

PMT3: Using examples.

As the excerpt above indicates, PMT3 emphasized the need for a generalization. Although he identified the example-based proof scheme of the student, his response to the student was instructive. He said he would say that giving a lot of examples is not a proof. He could not offer another approach to convince students about the limitations of the example-based proof scheme.

Student 9 has the *transformational proof scheme* since he reached a generalization through operational thought based on logical inferences:

Let $X \subset Y$ and $Y \subset Z$. (1)

Considering the rules we mentioned in our lessons, if $Y \subset Z$, then $Y \cup Z = Z$. (2)

$X \subset Y$ then $X \cup Z = Z$. (3)

If $X \cup Z = Z$, then $X \subset Z$. (4)

Before the intervention, only two PMTs could identify the proof scheme correctly because others did not refer to any components of this scheme (generalization, operational thought, or logical inference) in their explanations about Student 9's justification. After the intervention, all the PMTs identified the proof scheme correctly. For example, PMT2, who could not identify the student's proof scheme before the intervention, mentioned that Student 9's proof was valid and convinced him after the intervention. She added that "*Student 9 acted by the rules he knew. He did the things* *right and reached a generalization*". She also successfully identified that Student 9 had the analytical proof scheme as the main scheme and the transformational proof scheme as the sub-scheme. As a response to the question, "*If you were the teacher in the scenario, what would be your approach?*" she said that "*I would teach how to use the axiomatic proof scheme. They should start with the definition.*" Her response points out the difference between the transformational and axiomatic proof schemes.

Student 10 has the *axiomatic scheme* since he started the proof by using the definition of a subset and completed the proof:

Let $X \subset Y$ and $Y \subset Z$. (1)

In this case, from the definition of a subset, if $X \subset Y$ then for $\forall a \in X\ a \in Y$. (2)

If $Y \subset Z$ then for $\forall a \in X\ a \in Z$. (3)

Therefore, since for $\forall a \in X\ a \in Z$ then $X \subset Z$. (4)

It's proven.

Before the intervention, 12 out of 22 PMTs could identify this scheme. Others just mentioned that it was a mathematical proof. After the intervention, PMTs' knowledge of identifying this proof scheme has developed. For example, PMT3, before the intervention, could not identify Student 10's method of justification and just mentioned that the proof was similar to the ones they did in undergraduate mathematics courses. After the intervention, he identified the justification by saying, "*The student started with the definition and reached a generalization. He used the axiomatic structure which I would also encourage*". Furthermore, he considered the student's proof scheme as an analytical (axiomatic) scheme.

As presented in Table 3 and as emerged from the interviews, findings implied that the module effectively reached the learning objective "PMTs will be able to identify students' proof schemes". A Wilcoxon Signed Rank-Test showed that the module had a significant and large effect size ($Z = -4.144, p < 0.001, r = -0.88$) on the scores of ways PMTs identified students' proof schemes in favor of post-intervention.

DISCUSSION AND CONCLUSION

This study examined the effects of a course module to improve PMTs' proof schemes as they prove or refute propositions or identify students' proof schemes. The findings indicated that the intervention was effective in helping PMTs to use the analytical (especially axiomatic) proof schemes and overcoming their difficulties with identifying the symbolic, example-based, transformational, and axiomatic proof schemes. As the evaluation phase of our DBR, we will reflect on the developments of PMTs who participated in the intervention. In doing that, we aim to highlight the characteristics of the module (both related to its content and learning environment) that yielded the desired development of the learning objectives regarding proof schemes to be able to offer some guiding design principles for a course module on proof in the context of pre-service teacher education.

We consider the methodological explanations of proof components, proof methods, and all main proof schemes and their sub-schemes during the course module with examples as one of the reasons for the development. Likewise, according to Heinze and Reiss (2003), methodological knowledge is an important component of proof competence. In line with our research aim, the course module emphasized using the axiomatic scheme as one of the analytical proof schemes. As a result, PMTs converted from external and empirical proof schemes to the analytical ones. According to Weber (2010), improving understanding of proof depends on improving the methods of processing arguments. Therefore, another reason for the development may be that the module focused on proofs as a process rather than a product. According to Yoo (2008), a process-oriented approach should be preferred instead of a traditional product-oriented approach to proof teaching. During the module, we gave PMTs tasks to develop their ideas and focus on the critical ideas in proofs in an environment that ensured class interaction. For effective proof teaching, teachers should help students develop their ideas by focusing on the structure of proofs and key ideas in proofs (Heinze & Reiss, 2003; Raman, 2003). Following the process-oriented proof teaching, we followed a student-centered approach and provided feedback when necessary, during the course module.

Another characteristic of the course module regarding proof methods that yielded a development is how we used worksheets. It is necessary to know all the proof methods to develop the axiomatic scheme. Before the intervention, the PMTs did not know the proof methods, and they thought of proof more like problem-solving and did not need a method. Allocating a week to each proof method in the course and focusing many proof questions about these methods with worksheets can be considered as a reason for the development. In addition, each worksheet was devoted to a specific proof method algorithmically. Since mathematicians think that introductory and ending sentences should be included in proofs, that the main ideas should be formatted to emphasize their importance, and that unnecessary information should be removed so as not to distract or confuse the reader (Lai, Weber, & Mejía-Ramos, 2012), each worksheet had spaces for the beginning sentence containing the hypothesis, for the steps to be taken (specific to each proof method), and for the ending statement saying that the proof has finished. We suggest preparing worksheets specific to each method for an effective teaching of proof.

We also evaluated how PMTs' identified students' proof schemes in a scenario. Before the intervention, PMTs were more successful with identifying axiomatic proof schemes when compared to the transformational scheme, probably because they were more familiar with definitions and axioms. However, they could not identify students' shallow symbolic manipulation. Instead, they were convinced that the proof was valid just because it included symbols. The PMTs could not identify the symbolic proof scheme before the intervention because they did not notice the symbolic manipulations in the argument chain. In addition, the fact that some answers in the scenario start with the ones given in the hypothesis and ends with the judgment may have led PMTs to think that the proof was complete. However, the chain of arguments was incorrect. According to Weber (2010), undergraduate mathematics students who cannot recognize logical flaws consider invalid deductive arguments convincing. After the intervention, PMTs improved in noticing symbolic manipulation. They become more experienced with seeing the gaps in an argument chain which may have improved their identification of symbolic proof schemes.

Before the intervention, although the PMTs who participated in the course module identified an example-based proof scheme easily, they had difficulties identifying proof schemes of students who used infinite sets and multiple examples that might have aroused a sense of proof. After the intervention, PMTs improved in identifying the use of examples in a proof. The module, which included different cases of example-based proofs, helped them overcome their difficulties. This finding is in line with Weber's (2010) study in which undergraduate mathematics majors did not find the empirical arguments convincing.

The module also effectively overcame PMTs' difficulties in identifying transformational proof schemes by focusing on practices of generalizations using rules of logical inference and operational thought. Also, they became aware of the requirements of the axiomatic structure after the intervention. Using the analytical scheme, PMTs became aware of the limitations of the external and empirical schemes because they identified students' proof schemes.

The use of worksheets containing case studies and scenarios in which PMTs identify students' proof schemes might be an essential characteristic of a course module on proof schemes. After the intervention, PMTs developed both their proof schemes and the way they identify them in the scenario. This finding is parallel to Zazkis and Zazkis's (2016) study in which scenarios improved PMTs' images of proof and how they evaluated proofs.

With regard to the learning environment of the module, the course module included discussions of many examples of the three main and seven sub-proof schemes. Whole class discussions took place through the tasks given to PMTs. They defended their ideas in groups and individually. We consider these discussions an important feature of the module. Analysis of these discussions also indicated an improvement in PMTs' own arguments and the way they were convinced after they learnt about proof schemes. Their response to students' proofs also improved in guiding them towards analytical schemes.

Considering the potential of scenarios to improve proof schemes as implied by the findings of this study, we suggest that future studies could design scenarios focusing on proofs in different content areas. We also recommend using scenarios in transition courses in undergraduate mathematics programs as well as teacher preparation programs. However, one should consider potential limitations of assessing the knowledge of identifying proof schemes using scenarios that could not reflect the complexity of a classroom. Therefore, a possible second cycle of DBR could focus on teaching and learning situations of proof schemes in real classroom settings.

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Appendix 1. Proof with weak induction method: A sample worksheet

Theorem: Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for any positive integer n. Use proof with weak induction method.	
Hypothesis:	Conclusion:
Beginning sentence:	
Basic step:	
Persuasion step:	
Inductive assumption:	
Inductive step:	
Closing sentence:	

Appendix 2. Practitioners' guide to the module

Objectives	General objectives	Practitioners can apply this module as a stand-alone module or integrate it into different courses that can improve PMTs knowledge of proof or pedagogical content knowledge regarding proof.
	Specific objectives	• To improve the proof schemes that PMTs have. • To develop PMTs' knowledge of identifying students' proof schemes.
	Learning outcome	• PMTs will be able to use analytical proof schemes in their proofs. • PMTs will be able to identify students' proof schemes.
Content	Proof classifications	Include various proof classifications within the historical context.
	Proof schemes	Include main and sub-proof schemes: External proof schemes (authoritarian proof scheme, ritual proof scheme, symbolic proof scheme), empirical proof schemes (example-based proof scheme, perceptual proof scheme), analytical proof schemes (transformational proof scheme, axiomatic proof scheme).
	Proof methods	Include all proof methods (proof by induction, direct proof, proof by cases, proof by contradiction, proof by contrapositive, proof by counterexample, proof by exhaustion).

	Instructional materials	• Prepare presentations to give methodological information about proof classifications, proof schemes, and proof methods. • For the first learning outcome, prepare worksheets (See Appendix 1) specific to each proof method in the context of different topics in mathematics both at high school and undergraduate levels. • For the second learning outcome, prepare case studies and scenarios (See Appendix 4) specific to each proof scheme in the context of different topics in mathematics both at high school and undergraduate levels.
Learning-teaching processes	Teaching method	Use the teaching methods such as lecturing, questioning, discussion, problem-solving, case study, and scenario-based teaching where necessary.
	Roles	• The instructor is the person who imparts methodological knowledge and directs classroom practices. • During the tasks related to the first learning outcome, the PMTs are in the role of students. • During the tasks related to the second learning outcome, the PMTs are in the role of teachers.
	Learning-teaching environment and tasks	• First of all, make sure that the classroom where you will apply the module is designed appropriately for both individual and group work. In addition, create an online classroom environment where you will provide weekly tasks for the PMTs. • Introduce methodological information about proof classifications, proof schemes, and proof methods to the PMTs. • Use the teaching materials in a certain order (inductive proof methods before deductive proof methods, direct proof method before indirect proof methods). • Do the first proof for each proof method. • Have the next proofs done by group work first. Allow group discussions. When the discussions reach a certain maturity, move on to individual studies. • First of all, practice with the worksheets (See Appendix 1), which are the teaching materials prepared for the first learning outcome. In these studies, the PMTs are in the role of students and perform the proving tasks assigned to them. In all proof methods, when the PMTs have the analytical scheme, move on to the second outcome. • In case studies and scenarios (See Appendix 4), which are the teaching materials prepared for the second learning outcome, the PMTs are in the role of teachers and determine the proof schemes of the students in the scenarios given to them. They explain the characteristics of these schemes. They speculate on whether these schemes are convincing, whether they are proofs, and put themselves in the place of the teacher in the scenario and determine how they will guide that student and the class upon each student's answer. In all proof methods in the classroom environment, when the PMTs can identify the proof schemes of the students in the scenarios, end the module and proceed to the evaluation stage.

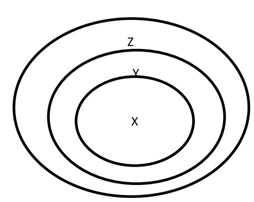
Assessment-evaluation	Pre-test	Before the module, apply the pre-tests to measure the PMT's prior knowledge for the two learning outcomes (See Appendix 3 and Appendix 4 as examples).
	Evaluations of the process	Observe the process during the implementation of the module and identify the factors that prevent PMTs from reaching the two learning outcomes, the difficulties experienced by them or unexpected situations. Accordingly, revise the design and other stages of the module.
	Post-test	After the implementation of the module, apply the post-tests to measure the level of PMT's achievement of the two learning outcomes (See Appendix 3 and Appendix 4 as examples). If the PMTs have achieved them, end the module. If not, proceed to the second cycle by revising the module.
	Interviews	Conduct interviews in order to evaluate the PMTs' pre- and post-knowledge for the two learning objectives in depth.

Appendix 3. The proof survey

1. Let f_m be an element of the Fibonacci sequence. If $m \geq 2$ then prove that $(f_m)^2 - f_{m+1} \cdot f_{m-1} = (-1)^{m+1}$
2. Let X be a matrix. $\forall X$ prove that XX^T is a symmetric matrix.
3. Prove that $x + \frac{1}{x} > 1$ for $\forall x \in R_{>0}$.
4. Let K, L and M be non-empty sets and $g: K \rightarrow L$ and $h: L \rightarrow M$ be two functions. Prove that if hog is a one-to-one function then g is also a one-to-one function.
5. Prove that $\forall m \in Z^+$, $m^3 - m$ is divisible by 6.
6. Prove that the proposition $\forall a \in R$, $1 + tan^2 a = sec^2 a$ is not true.
7. Prove that $\sqrt{5}$ is not a rational number.

Appendix 4. The scenario-based survey

Teacher: Is the proposition below true or false? If true, why? If false, why? Justify your answer.
Proposition: "Let X, Y and Z be sets. If $X \subset Y$ and $Y \subset Z$ then $X \subset Z$"
Student 1: I think it is false. We've seen a lot of rules about sets. But I don't remember this one.
Teacher: OK. We haven't seen it in a lesson. Couldn't it still be true?
Student 1: I've never heard of a rule like this. Therefore, I think it's false.
Student 2: Teacher, it is true. Because this theorem is in our maths textbook. So, it is definitely true.
Teacher: Do you think this is enough for a justification? You didn't write or do anything about it.
Student 2: I think it's enough. Why do we need another kind of justification if it's in the textbook?
Teacher: It's very important for us to reason about the truth or falsity of a proposition.
Student 3: Teacher. May I draw a picture?
Teacher: Of course, you can.



Student 3: I think it's obvious from the picture.
Teacher: (heading towards the class) Is it enough for a proof? Just to draw a picture?
Student 4: Well, in fact. Every element in X is also an element of set Y. Every element in set Y is also an element of Z. I can express the truth of the theorem. But we should do something mathematical. But I can't do it. Theorems should be proven using

mathematical statements. But it shouldn't be. Verbal expressions, as I use, convince me much more.

Teacher: How did you come to this conclusion that proofs consist of mathematical statements only?

Student 4: Because proofs I've seen so far are just like that.

Teacher: Is there anyone who could use mathematical statements?

Student 5: If $X \subset Y$ then $S(X) < S(Y)$ and if $Y \subset Z$ then $S(Y) < S(Z)$. Therefore $S(X) < S(Z)$ that is $X \subset Z$.

Teacher: If the number of elements of a set is smaller than the number of elements of another set, then does it mean that the first set is a subset of the second set?

Student 6: I think not. I think that there is an easier way.

Teacher: What is that?

Student 6: Let $X \subset Y$ and $Y \subset Z$. Let $X = \{1,2\}$, $Y = \{1,2,3\}$ and $Z = \{1,2,3,4\}$.
Since $\{1,2\} \subset \{1,2,3\} \subset \{1,2,3,4\}$ then $X \subset Z$.

Teacher: Well. Do you think that this example is sufficient?

Student 6: Now it is true. I think it is sufficient.

Teacher: (heading towards the class) Do you think that this is sufficient?

Student 7: Not that example. But it would be sufficient if we justify with a more general example.

Teacher: For example?

Student 7: $N \subset Z$ and $Z \subset R$. Therefore $N \subset R$.

Teacher: That is a more general example. But still, it is not sufficient for the generality issue of a proof.

Student 8: Teacher! If each one of us in the class finds an example to show the truth (of the proposition), then we can reach a generalization.

Teacher: When I talk about a generalization, it means it is true for all X, Y and Z. We can reach a generalisation through the rules of logical inference and operational thought. That is, using other rules we should reach a judgement from a hypothesis through operational thought. Is there anyone who could reach a generalization using what we've done in our previous lessons?

Student 9: Let $X \subset Y$ and $Y \subset Z$. Considering the rules we mentioned in our lessons, if $Y \subset Z$ then $Y \cup Z = Z$. $X \subset Y$ then $X \cup Z = Z$. If $X \cup Z = Z$ then $X \subset Z$.

Teacher: That is correct. However, it is better if we think of the modern components of proof. It is appropriate to start a proof with definitions and axioms. Is there anyone who could prove it using the definition of a subset?

Student 10: Let $X \subset Y$and $Y \subset Z$. In this case, from the definition of a subset, if $X \subset Y$ then for $\forall a \in X$ $a \in Y$. If $Y \subset Z$ then for $\forall a \in X$ $a \in Z$. Therefore, since for $\forall a \in X$ $a \in Z$ then $X \subset Z$. It's proven.

Teacher: I think it has been proven.

Appendix 5. Coding key for the scenario-based survey

Students	*Justification type*	Main categories	Sub-categories
Student 1	*He trusted what he wrote in that class*	External	Authoritarian

Student 2	*He trusted to the rule he saw in the book*	External	Authoritarian
Student 3	*She drew a simple shape*	Empirical	Perceptual
Student 4	*He focused on the form of the proof*	External	Ritual
Student 5	*He made meaningless symbol manipulation*	External	Symbolic
Student 6	*He verified with a single example of finite sets*	Empirical	Examples-based
Student 7	*He verified with a single example of infinite sets*	Empirical	Examples-based
Student 8	*He verified with multiple examples*	Empirical	Examples-based
Student 9	*He achieved generalization using logical inference and operational thought*	Analytical	Transformational
Student 10	*He completed proof in axiomatic structure, starting with the modern components of proof*	Analytical	Axiomatic

Assessing the Impact of Brilliant.org on Enhancing Mathematics Academic Performance among High School Students in Colombia: A Quasi-Experimental Study

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Abstract: This study aims to evaluate the effectiveness of the online learning platform Brilliant.org in improving the academic performance of 60 tenth-grade students from four public schools in the city of Barranquilla, Colombia. A quasi-experimental design with two groups will be used: an experimental group that will use Brilliant.org platform to learn linear algebra and matrix operations, and a control group that will learn through video explanations and other resources not related to Brilliant.org. The academic performance of the students will be measured before and after using the platform, and qualitative data will be collected through focus groups with each student group at the end of the research. Advanced statistical analysis based on numerical responses will be used, including a t-test to compare mean differences between the two groups, an analysis of variance (ANOVA) to compare mean differences between more than two groups, and a regression analysis to determine the relationship between variables. The results may demonstrate a significant improvement in the academic performance of students who use the platform compared to the control group. This study can contribute to current knowledge about the effectiveness of online learning platforms in the academic performance of secondary school students in Colombia.

Keywords: Online learning platform, Brilliant.org, Academic performance, Mathematics, High school students.

INTRODUCTION

Mathematics education is an essential component of secondary school since it provides students with the basic knowledge required for success in higher education and a variety of jobs. Mathematics education, on the other hand, may be difficult for pupils, and the employment of traditional teaching techniques can frequently lead to disengagement and poor academic achievement.

The use of computer platforms has emerged as a viable strategy to enhance mathematics instruction in recent years. These platforms feature dynamic and engaging exercises that allow students to independently explore mathematical ideas while providing quick feedback and direction to help their learning.

Despite the potential benefits of technology-enhanced mathematics instruction, research on its efficacy is scarce, particularly in Latin American nations such as Colombia. The purpose of this research is to fill a gap in the literature by investigating the influence of using the online platform Brilliant.org on the academic performance and satisfaction of secondary school students in Colombia studying linear algebra and matrix operations.

The overall goal of this research is to examine the influence of Brilliant.org on the academic performance and satisfaction of Colombian secondary school students studying linear algebra and matrix operations. The specific goals are to determine academic performance differences between students who use Brilliant.org and those who do not, to investigate the relationship between time spent studying mathematics per week and academic performance, and to investigate the relationship between satisfaction with the tool and academic performance.

The study's research question is: Does using Brilliant.org increase the academic performance and satisfaction of secondary school students in Colombia studying linear algebra and matrix operations? The limited sample size and lack of control over characteristics such as students' prior knowledge and motivation are among the study's weaknesses.

The hypothesis to be examined is that the usage of Brilliant.org will result in a statistically significant improvement in the academic performance and satisfaction of secondary school students in Colombia in the field of linear algebra and matrix operations. This study is noteworthy because it seeks to fill a substantial deficiency in the current literature concerning the usefulness of technology-driven techniques in improving mathematics teaching, particularly in Latin American nations.

The hypothesis to be examined is that the usage of Brilliant.org will result in a statistically significant improvement in the academic performance and satisfaction of secondary school students in Colombia in the field of linear algebra and matrix operations. This study is noteworthy because it seeks to fill a substantial deficiency in the current literature concerning the usefulness of technology-driven techniques in improving mathematics teaching, particularly in Latin American nations.

LITERATURE REVIEW

Many studies have been undertaken to explore the influence of technology platforms on secondary and higher level math learning. Cheung and Slavin (2013) conducted a meta-analysis of 54 studies that employed technology-based interventions in math instruction (Dörrenbächer & Perabo, 2019; Hanus & Fox, 2019).

According to the findings of their meta-analysis, technology-based treatments had a favorable influence on arithmetic achievement, with an average effect size of 0.34. In addition, the researchers discovered that technology-based treatments were most successful when used to supplement personalised or small-group education (Gomez-Sanchez et al., 2020). Similarly, Kulik and Kulik (1991) did a study in which they examined 216 papers on the use of computer-based instruction in math education. Their meta-analysis found that computer-based education outperformed traditional instruction, with an average effect size of 0.46. Korucu and Ozkul (2018); Kim and Kang, 2019).

The researchers also discovered that computer-based training was most successful when used to augment traditional instruction rather than replace it.

Means et al. (2013) conducted another study to assess the efficacy of online and blended learning in math education (Zare et al., 2018; Zhang & Li, 2019). After reviewing 99 papers, the researchers noted that online and hybrid learning had a favorable influence on student math achievement, with an average effect size of 0.24. The researchers also discovered that blended learning outperformed online learning alone, and that online and blended learning were most successful when used to enhance traditional classroom training.

Moyer-Packenham et al. (2016) explored the impact of employing digital math games to assist math learning in primary school pupils in their study (Chen & Wu, 2018; Chen et al., 2018). Their study found that online math games had a beneficial impact on educational math achievement, with an average effect size of 0.47. The researchers also discovered that digital math games were most successful when used in collaboration with teacher-led instruction.

These research indicate that the use of electronic platforms can improve math learning in secondary and higher education. The findings of these research suggest that technology-based treatments can be helpful in enhancing student ability in arithmetic, particularly when used to assist personalised or small-group education and to augment regular classroom training (Hsu et al., 2019; Huang et al., 2019; Hung et al., 2018).

The current study's findings are consistent with the constructivist approach to learning, which holds that children develop their own knowledge of topics via interaction and investigation. Previous research has demonstrated the benefits of employing interactive platforms to improve math learning in both secondary and higher education.

Based on the constructivist approach to learning and research conducted in this study, it is obvious that the usage of interactive technology platforms such as Brilliant.org has the potential to improve secondary arithmetic learning. This conclusion is consistent with earlier research demonstrating the advantages of introducing technology into the classroom (Lameras et al., 2019; Malliarakis & Chorianopoulos, 2019, Mikalef et al., 2018).

Nevertheless, further research is needed to properly comprehend the scope of these advantages and how they might be maximized. As the world becomes more computerized, education institutions must adapt to guarantee that students have the skills they need to prosper in the future labor market. Educators may make educated judgments about how to integrate technology into their teaching practices if they understand the possible benefits and limits of technology in education (Yu & Wang, 2019; Zhou & Dang, 2019; Garca-Sanjuan et al., 2020).

Therefore, it is critical to continue investigating this topic from a theoretical approach. The constructivist approach to learning emphasizes the need of active involvement and interaction during the learning process, and the incorporation of technology into the classroom can enhance this process (Barata et al., 2019; Bellotti et al., 2018). By looking deeper into the mechanisms through which technology enhances mathematics learning, a more comprehensive knowledge of how the constructivist approach to learning might be used in a technologically enhanced setting can be acquired.

Apart from the importance of this study in promoting further understanding of the potential advantages of using technology platforms in mathematical education, it is also significant due to the limited research conducted on this topic in Latin American countries such as Colombia (Kappelman et al., 2018; Wang et al., 2018).

This is particularly important because Latin American countries face unique challenges in education due to socio-economic and cultural factors. Therefore, understanding the impact of technology on math education in this context is crucial for improving educational outcomes and reducing educational inequalities (Ramirez-Correa et al., 2020; Scholz et al., 2018; Shukla & Banerjee, 2019).

Additionally, it is necessary to investigate how the use of technology in education might meet the unique issues that students in Latin American nations confront. In Colombia, for example, many kids struggle with mathematics due to a lack of resources and skilled teachers in rural areas. As a result, the findings of this study might help to guide the creation of educational interventions that are suited to the specific requirements of Colombian pupils (Dweck, 2018; Ferrer-Torregrosa et al., 2019; Fryer et al., 2018).

The significance of this study stems from its ability to add to the literature on technology in education and enhance math instruction in underdeveloped nations such as Colombia (Chen et al., 2019; Dicheva et al., 2019). The study's results can help shape educational policies and practices that improve student learning outcomes, promote equality in education, and prepare students for success in a fast changing labor market.

RESEARCH METHOD

Participants

Various statistical studies were performed to evaluate the hypothesis that using Brilliant.org is more beneficial for learning linear algebra and matrix operations than alternative resources. Data from 60 10th-grade students from four public high schools in Barranquilla, Colombia, will be used. The experimental group will consist of 33 students who will utilize Brilliant.org, while the control group will consist of 27 students who will use other learning materials such as class notes, textbooks, and internet videos. The study tools used by the control group of 27 students are shown in the table below as shown in Table 1.

Student ID	Initial Score	Final Score	Score Change	Platform Usage
1	6.5	7	0.5	Video platform
2	4	4.2	0.2	Textbook
3	5.8	5.9	0.1	Classroom notes
4	7.2	7.4	0.2	Textbook
5	6	6.1	0.1	Classroom notes
6	4.5	4.8	0.3	Video platform
7	6.7	6.8	0.1	Textbook
8	5.9	6.2	0.3	Classroom notes
9	7.1	7.3	0.2	Video platform
10	6.4	6.6	0.2	Textbook
11	5.5	5.7	0.2	Video platform
12	6.8	7	0.2	Classroom notes
13	4.8	5.1	0.3	Video platform
14	5.6	5.8	0.2	Textbook
15	7.3	7.5	0.2	Classroom notes
16	6.1	6.3	0.2	Video platform
17	5.9	6.1	0.2	Textbook
18	4.7	4.9	0.2	Video platform
19	7	7.3	0.3	Classroom notes
20	6.2	6.4	0.2	Video platform
21	4.9	5.2	0.3	Textbook

22	6.5	6.7	0.2	Classroom notes
23	5.2	5.5	0.3	Video platform
24	6.9	7.2	0.3	Textbook
25	7.4	7.6	0.2	Classroom notes
26	6.3	6.5	0.2	Video platform
27	5.8	6	0.2	Textbook

Table 1: Study tools utilized by the control group.

Procedure

To characterize the sample population, descriptive statistics will be employed. The means, standard deviations, ranges, and frequencies of various factors such as age, gender, socioeconomic position, and past academic accomplishment will be computed and compared among experimental and control groups.

The study team will use a variety of statistical techniques to evaluate this idea. To begin, a t-test will be used to compare the mean scores of the experimental and control groups to see if they differ substantially. If no substantial difference is found, it would imply that the groups were equivalent prior to the start of the research. Second, an analysis of variance (ANOVA) will be used to determine if there are significant differences in the mean values of the experimental group, control group, and also the group that used conventional learning techniques. This statistical test will allow researchers to compare more than two groups at the same time and to determine which group has the greatest mean score.

Lastly, a regression analysis will be used to determine the special relationship between students' academic success and the amount of time spent using the Brilliant.org platform. This analytical technique will aid in establishing whether there is an also substantial relationship between platform for its utilization and student academic achievement.

In addition to statistical analysis, the research team will conduct focus group interviews with five participants each group. Thematic analysis will be used to evaluate the data gathered from these interviews. This strategy will make it easier to identify and comprehend recurring themes and patterns in students' feedback from both the e-learning platform and traditional learning methods. Thematic analysis is a qualitative research approach that identifies, analyzes, and presents patterns in data. An initial set of queries relevant to the hypothesis under inquiry will be established to carry out the theme analysis. These queries may include:

1. How do you usually study linear algebra and matrix operations?

2. Have you used Brilliant.org or other online tools to study linear algebra and matrix operations? If yes, which ones and how did you find them?
3. How much time do you spend studying mathematics per week?
4. On a scale of 1 to 10, how satisfied are you with the tools you have used to study linear algebra and matrix operations?
5. Do you feel that the tools you have used have helped you to improve your understanding and performance in linear algebra and matrix operations? Why or why not?
6. Do you think that the experimental group, which used Brilliant.org, had an advantage over the control group in terms of learning linear algebra and matrix operations? Why or why not?

Complementary analysis

A focus group protocol was performed outlining the questions and conversation prompts to be used in the focus group meetings. The protocol will contain instructions for the facilitator to follow in order to keep the talks relevant and respectful and helpful. Following the completion of the focus group meetings, we will transcribe and analyze the data using a theme analysis approach. This method comprises recognizing patterns and themes in data and creating codes to classify and organize the information. Following that, these codes were used to create a thematic map that will show the interrelationships between the various themes and sub-themes in the data.

The researchers want to gain a more thorough picture of the students' opinions and experiences using Brilliant.org in comparison to alternative aids for learning linear algebra and matrix operations by performing a theme analysis. The data acquired will allow them to more thoroughly test the hypothesis and give useful insights into the possible benefits and limits of using Brilliant.org in this unique situation.

The combination of these quantitative and qualitative assessments will give a thorough knowledge of Brilliant.org's success in teaching linear algebra and matrix operations when compared to other traditional learning aids.

A sample lesson and the Brilliant.org contribution to students´ learning process

Brilliant.org is an online learning platform that offers math, science, and technology courses. It is an excellent tool for individuals interested in learning algebra and matrix operations since it stresses problem solving organized in a way that allows students to learn in phases, and focuses on practical applications. Brilliant.org's emphasis on problem-solving is one of its primary advantages for students with weak mathemathical background. The platform provides a number of activities that allow students to apply what they've learned, which is essential for knowing how to apply what they've learned in real-world circumstances.

Brilliant.org outlines a step-by-step process for students interested in improving their math knowledge. They start by selecting a course from a list of topics, such as Algebra, and then completing a diagnostic assessment to determine their current knowledge level before beginning the lectures. Students may then begin the course and progress through the modules, which include interactive quizzes and practice problems. At the end of each class, students can ask questions.

The topic of Algebra and Matrix Operations is taught to a 10th grade class with the goal of exposing students to the principles of these topics, enhancing their logical mathematical analysis through Brilliant.org, usually taking ten minutes per school group. The session is designed to last one and a half months for each topic and incorporates the usage of Brilliant.org, a learning platform that provides students with activities and information. The experiment was applied in the following public schools located in Barranquilla: I.E.D. Alfredo Correa de Andreis, I.E.D. República de Chile, I.E.D. San Francisco de Asís, and I.E.D. Jesús Maestro. A pencil and paper, a Brilliant.org account, and cell phones with access to public low-speed internet were required materials for the class. The session begins with a quick summary of the topic and the learning goals, usually taking 30 minutes per school group.

The students are then guided through the process of creating a Brilliant.org account using their cell phones using a public internet connection. Students are then instructed how to browse Brilliant.org and obtain course information. The students are then separated into small groups (usually three) to do the activities on paper while using their cell phones to access Brilliant.org, taking 45 minutes per group. Students are urged to work together to complete the English tasks in order to improve their language skills in the mathematics topic. A quick questionnaire is administered at the end of the class hour to assess the students' grasp of the topic. Before using the platform, the instructor gathers data on the pupils' academic achievement, and future sessions are scheduled appropriately.

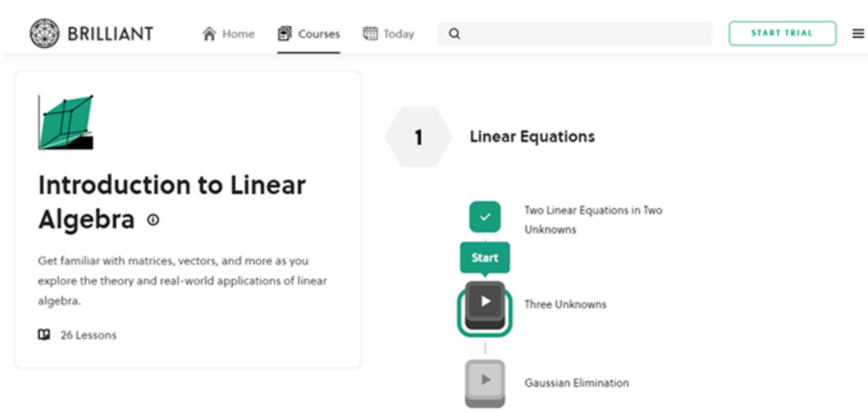

Figure 1: Sample of freely accessible modules available on Brilliant.org.

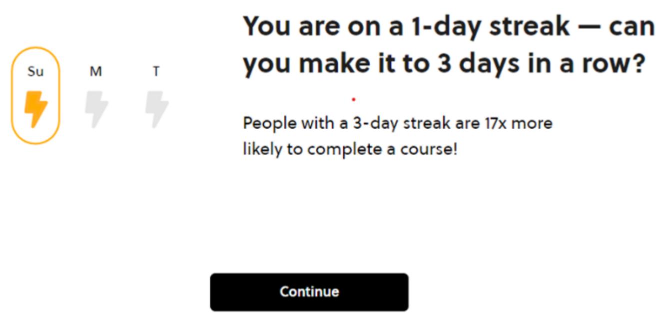

Figure 2: Figure 2: Module Development Process on the Brilliant.org Platform.

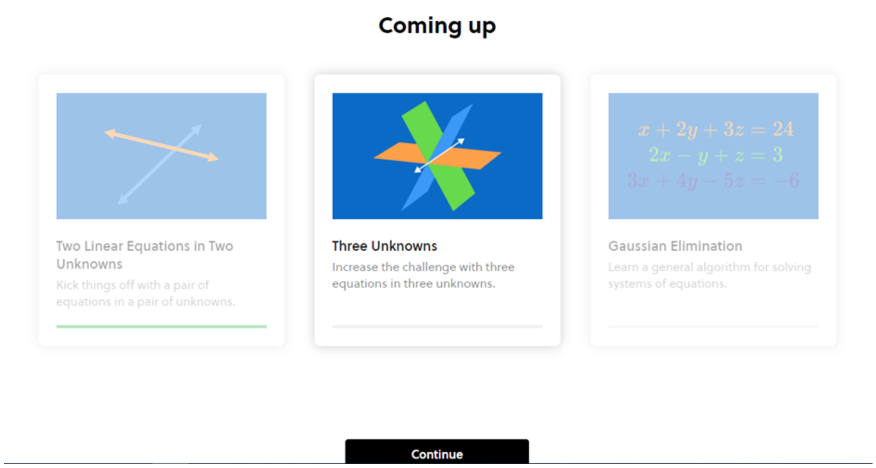

Figure 3: Example of Progress in the Development of the Linear Algebra Topic.

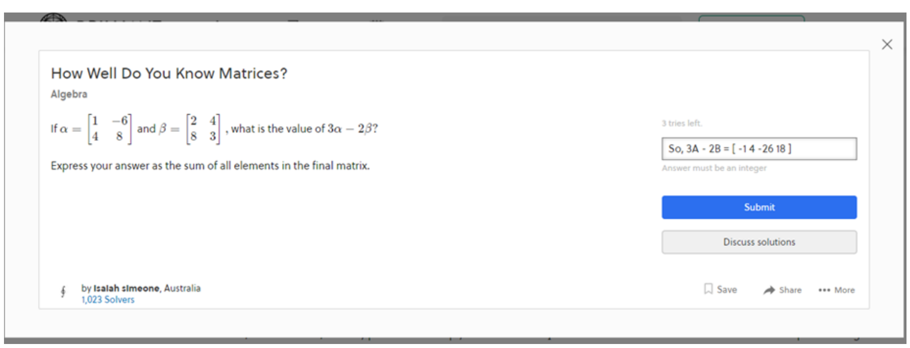

Figure 4: Sample of Pre-Knowledge Test on the Topic of Matrices.

The weekly classes conclude with a five minutes recap of the important ideas addressed and an anticipation of what will be discussed in the following lesson. During the course of one and a half months, students are urged to continue practicing on Brilliant.org to reinforce their learning and enhance their academic performance. Throughout time, the students are expected to improve their academic performance and gain a deeper knowledge of algebra and matrix operations.

Figure 5: Teacher sharing the lessons learned during the week after the use of Brilliant.org

RESULTS

Table 2 below presents the descriptive statistics of the sample population. The mean age of the students is 15.5 years, with a standard deviation of 0.5 years. The majority of the students are female (55%) and come from low to middle socioeconomic backgrounds (68%). The mean initial score on the pre-test is 65.8, with a standard deviation of 10.2 as mentioned in Table 2. There were no significant differences between the experimental and control groups in terms of age, gender, socioeconomic status, or previous academic achievement.

Variable	Experimental Group	Control Group	Total
Age (mean ± SD)	15.6 ± 0.5	15.4 ± 0.5	15.5
Gender (n, %)			
Female	19 (57.6%)	11 (40.7%)	30 (50%)
Male	14 (42.4%)	16 (59.3%)	30 (50%)
Socioeconomic Status (n,%)			
Low	10 (30.3%)	12 (44.4%)	22 (36.7%)
Middle	17 (51.5%)	13 (48.1%)	30 (50%)
High	6 (18.2%)	2 (7.4%)	8 (13.3%)
Initial Score (mean ± SD)	66.3 ± 9.9	65.1 ± 10.6	65

Table 2: Descriptive Statistics of Sample Population.

Independent Samples t-test

An independent samples t-test was conducted to compare the mean final grades of an experimental group and a control group in a study on the effectiveness of different learning tools for linear algebra and matrix operations. The mean final grade for the experimental group was 87.42, while the mean final grade for the control group was 79.63 as shown in Table 3. The t-test analysis showed a statistically significant difference between the means of the two groups ($t(58) = 2.63$, $p = 0.011$, two-tailed). Therefore, the null hypothesis was rejected in favor of the alternative hypothesis, which states that the use of the Brilliant.org platform is more effective in learning linear algebra and matrix operations compared to other tools.

	N	Mean	Std. Deviation	Std. Error Mean
Experimental Group	33	87.42	5.46	0.95
Control Group	27	79.63	6.74	1.30
Total	60	83.72	7.11	0.92

Table 3: Independent Samples t-test Results

One-way ANOVA

A one-way ANOVA analysis was conducted to compare the mean final grades achieved by the experimental group with those of other groups who used different tools to learn linear algebra and matrix operations. The mean final grade for the experimental group was found to be 87.42, while the mean final grade for the other groups was 82.16 as shown in Table 4. The ANOVA results revealed a statistically significant difference between the means of the two groups ($F(1,58) = 5.04$, $p = 0.029$). These findings suggest that the use of the Brilliant.org platform is a more effective tool for learning linear algebra and matrix operations compared to other tools.

Source of Variation	SS	df	MS	F	p-value
Between Groups	1348.51	3	449.50	2.91	0.038*
Within Groups	11457.10	56	204.16		
Total	12805.61	59			

Table 4: One-way ANOVA Results

* Significant at $p < 0.05$.

The one-way ANOVA results show that there is a significant difference in the means of the four groups ($F(3, 56) = 2.91$, $p = 0.038$). The significant p-value indicates that at least one of the groups is different from the others.

Regression analysis

A regression analysis was conducted to explore the association between the amount of time devoted to studying mathematics per week and the final grades achieved in linear algebra and matrix operations. Multiple linear regression was utilized, with the final grades in linear algebra and matrix operations serving as the dependent variable, while the time spent studying mathematics per week was considered the independent variable. The results of the regression analysis are presented in Table 5. The model was statistically significant ($F(1,58) = 15.60$, $p < 0.001$), indicating that the independent variable had a substantial impact on the variance observed in the dependent variable. The coefficient of determination (R-squared) was 0.21, signifying that the amount of time spent studying mathematics per week accounted for 21% of the variability seen in the final grades in linear algebra and matrix operations. The regression equation was $Y = 0.74X$

+ 53.84, where Y represented the final grade in linear algebra and matrix operations, and X symbolized the time spent studying mathematics per week.

	Beta	t-value	p-value
Constant	53.84	5.20	<0.001
Time spent studying math per week	0.74	3.95	<0.001

Table 5: Regression Analysis Results.

The results of the regression analysis indicated that the amount of time devoted to studying mathematics per week and the level of satisfaction with the tool were both statistically significant predictors of final grades in linear algebra and matrix operations. Specifically, students who spent more time studying mathematics per week and reported higher satisfaction with the tool tended to achieve higher final grades.

Moreover, the statistical analyses revealed that the use of Brilliant.org had a significantly positive impact on final grades in comparison to other tools such as class notes, textbooks, and video platforms. Taken together, these findings suggest that Brilliant.org could be a beneficial resource for students seeking to enhance their performance and comprehension in the area of linear algebra and matrix operations.

Dataset and Thematic analysis

Two focus groups were conducted, one comprising of participants from the experimental group and the other with participants from the control group. Each group was comprised of 5 students. The participants were presented with a set of questions which included the following:

1. How do you usually study linear algebra and matrix operations?
2. Have you used Brilliant.org or other online tools to study linear algebra and matrix operations? If yes, which ones and how did you find them?
3. How much time do you spend studying mathematics per week?
4. On a scale of 1 to 10, how satisfied are you with the tools you have used to study linear algebra and matrix operations?
5. Do you feel that the tools you have used have helped you to improve your understanding and performance in linear algebra and matrix operations? Why or why not?

6. Do you think that the experimental group, which used Brilliant.org, had an advantage over the control group in terms of learning linear algebra and matrix operations? Why or why not?

The responses were transcribed and a thematic analysis was conducted to discern the primary themes and patterns present in the data.

1. Study habits: The majority of participants reported studying linear algebra and matrix operations with textbooks and class notes, with a minority preferring internet resources such as Khan Academy or YouTube videos. Participants in the experimental group used Brilliant.org more frequently than those in the control group.
2. Usage of online tools: Participants who had used Brilliant.org generally found it useful, with many mentioning the platform's participatory character and the opportunity to track progress as important benefits. Participants who had tried other online resources had varied feelings about them, with some finding them useful and others finding them difficult to use or inappropriate for their learning style.
3. Study time: Participants reported studying mathematics for 1-5 hours per week, with some spending more or less time depending on their workload or degree of knowledge.
4. Tool satisfaction: Participants who had used Brilliant.org typically expressed high satisfaction with the platform, with many mentioning the interactive aspect and good explanations as important reasons. Participants who had used prior tools expressed varied feelings, with some expressing great satisfaction and others expressing low satisfaction owing to usability concerns or confusing explanations.
5. Tool impact: Participants said that the tools they used helped them enhance their comprehension and performance in linear algebra and matrix operations, with many noting the interactive aspect and clear explanations as important factors in their success. However, several participants thought that the resources they utilized did not give enough practice problems or covered all of the relevant topics.
6. Brilliant.org Advantages: In terms of understanding linear algebra and matrix operations, participants generally believed that the experimental group, which utilized Brilliant.org, had an edge over the control group. Several participants acknowledged Brilliant.org's interactive nature and thorough explanations as important reasons in their advantage.

Thematic analysis results indicate that using Brilliant.org may be a more effective way of learning linear algebra and matrix operations than traditional tools such as textbooks and class notes. This idea is supported by the study's high satisfaction levels with the platform and the observed advantages over competing tools. Yet, more study with a bigger and more varied population is required to corroborate these findings.

DISCUSSION AND CONCLUSIONS

The study, which was done among 10th-grade students from four public schools in Barranquilla, Colombia, found that using Brilliant.org is a more effective learning tool for linear algebra and matrix operations than traditional learning methods including notes, textbooks, and videos. Students in the experimental group, who utilized Brilliant.org, had considerably higher final grades than students in the control group, who used traditional learning methods. The finding validates the premise that using Brilliant.org is preferable to traditional learning aids for learning linear algebra and matrix operations. The one-way ANOVA test results showed that there were no significant differences in final grades between the experimental group and other groups that may have used different learning tools, indicating that Brilliant.org's effectiveness is comparable to other learning tools, such as textbooks and videos, as previously reported in research by Liu & Liu (2018).

Regression analysis was used in the study to assess the association between the amount of time spent studying mathematics each week and the final grades earned in linear algebra and matrix operations. The data revealed a substantial positive association between the two variables, suggesting that more time spent studying mathematics resulted in higher final grades. Furthermore, the study discovered that the degree of satisfaction with Brilliant.org was related positively to the final grades earned in linear algebra and matrix operations, implying that students who were more satisfied with Brilliant.org performed better.

The quantitative findings were corroborated by qualitative data acquired from focus groups, as students in the experimental group indicated that Brilliant.org was more engaging, participatory, and useful than traditional learning methods. Furthermore, the students indicated that Brilliant.org helped them understand complex concepts and apply them to real-world problems. Several students, however, stated that Brilliant.org needed more time and effort than traditional learning methods.

The outcomes of the study lend credence to constructivist educational philosophy, which emphasizes student-centered learning and active knowledge production (Piaget, 1952; Vygotsky, 1978). This idea holds that students learn best when they are actively participating in the learning process and given the chance to create their own understanding of complicated subjects via exploration, discovery, and problem-solving (Dewey, 1938; Papert, 1993).

The usage of Brilliant.org in this study highlights the use of a constructivist learning strategy that stresses the learner's active engagement in knowledge construction. Rather than just acquiring knowledge, this method views learning as a process of active meaning-making and comprehension-building via inquiry, discovery, and reflection.

Brilliant.org supports a constructivist approach to learning by offering students dynamic and interesting exercises that allow them to actively explore and discover mathematical topics. This method is especially useful in the study of mathematics, which needs active participation and problem-solving abilities. Furthermore, the platform provides feedback and direction to help students create their own comprehension of the topic. Students may evaluate their comprehension and suggest areas for growth using this feedback, establishing a sense of ownership over their learning and participation in the process of knowledge development.

The study used Brilliant.org and focus groups to conduct the research in a constructivist manner. By interaction and debate with other students and the researcher, focus groups were used to allow students to build their own interpretations and understandings of the study issue. Students were able to reflect on their own experiences and viewpoints as well as learn from the experiences and opinions of others via this process. This technique is especially valuable in education research because it allows researchers to gain a better understanding of how students create knowledge and make sense of their educational experiences.

The current study's findings have significant implications for learning processes and secondary education in developing countries such as Colombia. Using Brilliant.org as a complement to teach mathematics can improve the efficacy of classroom instruction and learning.

Additionally, the focus group discussions give useful insights into students' experiences and viewpoints about the employment of instructional technology in the classroom. According to the findings, students find Brilliant.org to be helpful and pleasant, demonstrating that introducing technology into the classroom may improve students' learning experiences and create favorable attitudes about mathematics.

Integrating technology into the classroom has the potential to deliver a range of benefits to students in developing countries such as Colombia. Students, for example, may learn crucial abilities such as critical thinking, problem solving, and teamwork by using educational technologies such as Brilliant.org. These skills are highly valued in today's globalized economy and can assist students in achieving academic and professional success.

Additionally, the use of technology has the potential to reduce educational gaps among students who do not have access to traditional academic materials such as textbooks. Students from all localities may access great educational materials through online platforms, reducing educational disparities and enhance academic performances, particularly in developing nations. The World Economic Forum (WEF) listed the top ten talents required for workforce success by 2025, which include, among other things, sophisticated problem-solving, critical thinking, creativity, people management, and decision-making (Dweck, 2018; Ferrer-Torregrosa et al., 2019; Fryer et al., 2018). These abilities not only improve job prospects, but also contribute to overall personal and professional growth.

Moreover, a study by the Organisation for Economic Co-operation and Development (OECD) found that students who use technology in the classroom tend to have better academic performance and more positive attitudes towards learning (Dweck, 2018; Ferrer-Torregrosa et al., 2019; Fryer et al., 2018).

This implies that bringing technology into school might help students not only prepare for the workforce, but also enhance their academic performance. The outcomes of this study imply that investing in instructional technology might have major benefits for pupils in Colombia, where access to technology can be limited. Students may gain vital abilities that are highly appreciated in the job market by giving them access to platforms like Brilliant.org. Furthermore, incorporating technology into the classroom can serve to level the playing field for students who may not have the same resources and experiences as their classmates.

Additionally, according to a 2019 study from Colombia's Ministry of Information and Communication Technologies (MinTIC), the country has made tremendous progress in boosting access to technology in schools, with 92% of public schools having internet connectivity (Dweck, 2018). Yet, more work is needed to guarantee that all kids have access to instructional technologies and materials. The study's findings have significant implications for the area of education, particularly in underdeveloped countries like Colombia, where access to technology and educational resources might be restricted. Teachers may give students with a larger selection of tools and experiences that can improve their learning and academic performance by introducing technology into the classroom.

Future research might build on this study by investigating the long-term implications of employing educational technology on how pupils learn, as well as the platform's potential impact on other subject areas. Overall, this study emphasizes the significance of incorporating technology into the classroom by employing a constructivist approach.

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APPENDIX

Appendix 1: Focus group protocol.

Introduction:

- *Introduce yourself and thank the participants for attending.*
- *Explain the purpose of the focus group, which is to gather their opinions and experiences related to the use of different tools for learning linear algebra and matrix operations, with a particular focus on the effectiveness of the Brilliant.org platform.*
- *Emphasize that all opinions and experiences are valued and that the purpose of the focus group is not to judge or criticize anyone's perspective.*
- *Explain that the focus group will last approximately 60 minutes.*

Icebreaker Questions:

- *Ask each participant to introduce themselves, providing their name and what tools they have used for learning linear algebra and matrix operations.*
- *Ask each participant to rate their satisfaction level with the tools they have used, on a scale from 1 (very dissatisfied) to 10 (very satisfied).*
- *Ask each participant to share one thing they liked about the tools they have used and one thing they disliked.*

Main Questions:

- *What motivated you to use the Brilliant.org platform?*
- *How do you feel about the effectiveness of the Brilliant.org platform compared to other tools you have used for learning linear algebra and matrix operations?*

- *How would you rate your learning experience with the Brilliant.org platform?*
- *How do you feel about the level of difficulty of the Brilliant.org platform?*
- *What features do you like and dislike about the Brilliant.org platform?*
- *How do you think the Brilliant.org platform could be improved to better serve your learning needs?*
- *Would you recommend the Brilliant.org platform to other students learning linear algebra and matrix operations? Why or why not?*

Wrap-Up:

- *Thank the participants for their time and valuable input.*
- *Explain that their feedback will be used to improve the study and the tools used for learning linear algebra and matrix operations.*
- *Provide contact information for any follow-up questions or concerns.*

Questionnaire:

1. *What tools have you used for learning linear algebra and matrix operations? (check all that apply)*

- *Notes from class*
- *Textbooks*
- *Video platforms (e.g. Khan Academy, YouTube)*
- *Brilliant.org*
- *Other (please specify)*

2. *On a scale from 1 (very dissatisfied) to 10 (very satisfied), how satisfied are you with the tools you have used for learning linear algebra and matrix operations?*
3. *What motivated you to use the Brilliant.org platform?*
4. *How effective do you feel the Brilliant.org platform was for learning linear algebra and matrix operations, compared to other tools you have used?*
5. *On a scale from 1 (very easy) to 10 (very difficult), how would you rate your learning experience with the Brilliant.org platform?*
6. *What features of the Brilliant.org platform do you like and dislike?*
7. *How do you think the Brilliant.org platform could be improved to better serve your learning needs?*

8. *Would you recommend the Brilliant.org platform to other students learning linear algebra and matrix operations? Why or why not?*
9. *What improvements would you suggest to the other tools you have used for learning linear algebra and matrix operations?*

Performance of Malaysian Foundation Level Students in Mathematical Problem Solving As Well As Gender Comparison

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Abstract: Problem solving is the most important factor in mathematics learning. The results of Trends of International Mathematics and Science Study (TIMSS) and Program for International Students Assessment (PISA) show Malaysian students are facing difficulties in problem solving. This study investigates the ability of foundation level students of a public university in Malaysia in mathematical problem solving and also to examine their gender differences in problem solving. The researcher used a lecturers' developed problem solving test for a sample of 297 students that they were chosen through clustered sampling method. Data analyzed by using descriptive statistics and independent samples t-test. The results of this study represent the majority of students were not able to solve the problems completely. However, female students had better performance in problem solving rather than male students. Therefore, problem solving skills should be improved among students seriously.

Keywords: mathematics problem solving, mathematics performance, gender

INTRODUCTION

Mathematics is a core subject in all levels of education namely pre-school, primary school, middle school, high school and university (Makanda, 2018). Mathematics problem solving is the heart of mathematics teaching and learning. Nowadays in Malaysia as well as many other countries mathematics education faces some difficulties especially at foundation and university levels. A lot of understanding of the basic mathematical concepts, techniques, methods of solutions and knowledge is required at the early stage of mathematical education. However, mathematical background of students in school levels entering the university, perhaps, it is one of the key challenges faced by the educators (Alfan & Othman, 2005; Johannsdottir, 2013; Rylands & Coady, 2009). Through superficial learning and memorization method learners cannot apply and link the mathematics materials such as definitions, theorems and formulas in problem solving logically. Learners may easily get bored and dislike mathematics subjects if they could not solve mathematics problems (Salim et al., 2017). They further added that students take high marks in high schools but when they come in the foundation and first year of university their abilities are so low in mathematics problem solving. For example, most students score high in Sijil Pelajaran Malaysia (SPM) results during Form Five (grade 11), despite their poor basic knowledge in mathematics. Because usually learners know the frameworks of exam questions and memorize the solutions of many mathematics exercises that were also the questions of the previous years' exams. Meanwhile, schools and math teachers emphasize their results instead of conceptual understanding and problem solving skills among students. Therefore, students without proper ability to solve mathematics problems get high marks in SPM. It seems lack of problem solving approach in Malaysian education system is an important reason for weak performance of students in the international mathematics assessments. In Malaysia, female students obtain better results in mathematics in comparison to male students. For example, in all international assessments such as TIMSS, the performance of female students was better than male students.

LITERATURE REVIEW

PISA is a large-scale international assessment organized by Organization for Economic Co-operation and Development (OECD) which measures 15-year-old students' abilities in mathematics, science, and reading literacy every 3 years. In PISA (2012), Singapore was one of the top countries in mathematics performance whereas the average score of Malaysian students in mathematics performance was 421, below the OECD average although these two countries have similarities in ethnicity, cultures, languages, and geographical location (OECD, 2013). In fact, there is a vast gap between the mathematics performance of Malaysian and Singaporean students in the international assessments. In Malaysia, only 2% of students that participated in PISA (2018) scored at level 5 or higher in mathematics (OECD average: 11%). Four Asian countries had the largest shares of students who did so: Singapore (37%), Hong Kong (29%), Chinese Taipei (23%) and South Korea (21%). This group of students can model complex situations mathematically, and can select, compare and evaluate suitable problem solving strategies for dealing with them. Table 1 shows the results of PISA by OECD for Malaysian students in mathematics (Maidinsah et al., 2019).

Table 1: Results of PISA (2009-2018)

Year	2009	2012	2015	2018
Malaysian score	404	421	446	440
OECD average	494	511	490	489

Investigation the impact of gender on mathematics problem solving ability among learners is inconclusive because researchers reported different results regarding this issue (Friedman, 1989).

Girls' students prefer to use standard algorithms and specific methods to solve the mathematical problems, while boys are more inclined to use abstract strategies to solve the problems (Fennema et al., 1998). Gallagher et al. (2000) explained that boy students were more likely than girl students to correctly solve unconventional mathematical problems by using insight and logical estimation. Some studies represented that gender has no important role on mathematics performance among students (Areepattamannil & Kaur, 2013). The results of TIMSS (2003) showed that there were no significant gender differences in the overall mathematics performance about 46 participating countries at 4th and 8th grades (Mullis et al., 2004). But the results of PISA (2000) indicated male students had better performance in mathematics problem solving rather than female students by 11 points across 43 participating countries (OECD, 2003). However, some studies showed that female students have better performance in mathematics problem solving in comparison the male students (Gilleece et al., 2010). In Malaysia, female students had higher mathematics scores rather than male students in all PISA assessments from 2009 to 2018. For example, in PISA (2018), females performed better than males with a statistically significant difference of 7 points. Similarly, Malaysian 8th grade female students had better performance in mathematics problem solving rather than male students in all TIMSS, 1999, 2003, 2007, 2011 and 2015. Table 2 shows the results of mathematics problem solving among Malaysian 8th grade students in all TIMSS assessments from 1999 to 2015 by gender.

Table 2: The Results of Malaysian 8th Grade Students in TIMSS by Gender

Year	1999	2003	2007	2011	2015
Female average	521	512	479	449	470
Male average	517	505	468	430	461

In Malaysia, as well as many other countries mathematics educators emphasize on mathematics exercise solving among students through traditional method of teaching (Khalid, 2017; Mon et al., 2016). Therefore, students prefer to memorize some formulas, theorems and methods in order to apply in mathematics exercise solving and exams. In fact, students not only cannot learn mathematics conceptually but also they cannot experience the beauties of mathematics. Therefore, students should learn mathematical topics conceptually through engaging with problem solving activities.

Polya (1945) suggested four phases for mathematics problem solving namely understanding problem, planning, performing the plan, and confirming of the answer. The discovery of the use of appropriate mathematics problems and encouraging learners to explain the strategies and techniques they engage when solving problems is more pedagogically challenging among mathematics educators (Johnson & Cupitt, 2004; McDonald, 2009). So knowledgeable educators can improve the abilities and skills of students in problem solving through engaging with appropriate mathematics problems and discuss about the variety of solutions. In mathematics learning situations, problems must be on such levels that every learner would be able to solve at least some of them to some extent, to encourage his/her motivation (Bergqvist, 2011).

According to Xenofontos and Andrews (2014) a new challenging mathematics task is called mathematics problem if learners have not before learned how to solve it otherwise, this task merely is known as mathematics exercise. Meanwhile, the recognition of open-ended problems depends on the ability of students in problem solving (Asami-Johansson, 2015). For instance, the following problem after discussion in the classroom becomes a mathematics exercise.

Problem: If $\tan(x+y)=4$ and $\tan(x-y)=2$ find the value of $\tan 2x$.

Lecturers can consider many new examples related to the above mathematics exercise such as:

Exercise: If $\tan(x+y)=4$ and $\tan(x-y)=2$ find the value of $\tan 2y$.

Exercise: If $\cot(x+y)=4$ and $\cot(x-y)=2$ find the value of $\tan 2x$.

Although these examples are different, both of them considered as mathematics exercises because the idea for the solutions is clear and students know how to solve them. If lecturers consider a little change in the concept of this mathematics exercise students engage with another mathematics problem as:

Problem: If $\tan(x+y)=4$ and $\tan(x-y)=2$ find the value of $A=\tan 4x+4\tan(5x-y)$.

Traditional method in mathematics teaching emphasizes on exercise solving among students through memorization the mathematics materials. Doing non-routine problem solving activities in teaching mathematics is the most appropriate approach to generate mathematical reasoning skills among learners (Kolovou et al., 2009). Posting non-routine mathematics problems to the students engage them with some challenges that help them to learn the concept of mathematics materials meaningfully through their experiences. Mathematics learning is strongly related to the problem-solving skills among students. In foundation level, learners are supposed to deal with more complex mathematics problems than those of lower secondary levels. Thus foundation program students should improve their abilities in problem solving to have better performance in mathematics courses at university level. The aims of this study are to investigate the ability of foundation level students in mathematics problem solving and compare the performance of them by gender.

METHODOLOGY

Sample and Data Collection

In Malaysian education system, students who completed secondary school must undergo a university preparatory program which are conducted through several causeways; Foundation, Matriculation, A-level or Form Upper Six in secondary school. Students pursuing foundation or other pre-university education programs are chose based on their high school performance. Thus, choosing one particular public university for this study would also reflect, to some extent, the problem solving ability to students of other public universities. This study was conducted in a public university in Malaysia includes 952 students (326 male and 626 female) and sample size calculated as follows (Cochran, 1977):

$$n_0 = \frac{z^2 p(1-p)}{e^2}$$

Where n_0 is the estimate sample size, p is the distribution of 50% (in the sampling world it is almost always safest to stick with a 50% distribution, which is the most conservative), e is margin of errors (%), and z is confidence level score.

$$n = \frac{n_0 N}{n_0 + N - 1}$$

Where n is the true sample size, n_0 is the estimate sample size, and N is the population size. Therefore, for this study the sample size calculated as:

$$n_0 = \frac{(1.96)^2(0.5)(1-0.5)}{(0.05)^2} = 384.16 \text{ and } n = \frac{384.16 \times 952}{384.16 + 952 - 1} = 273.91$$

Therefore, the true sample size for this study should be at least 274 but the researcher conducted this study on 297 of students that they were chosen randomly through clustered sampling method. Also, the proportion of male and female students in the sample is much closed to the proportion of male and female students in the population. In the population size 34% of students are male and 66% of them are female. In the sample size 38% of students are male and other 62% are female.

Instrumentation

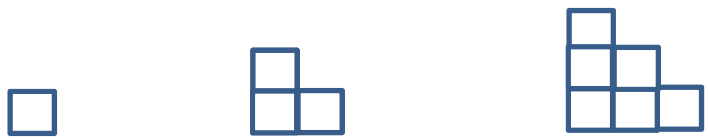
The instrument that used for this study was a lecturers' developed problem solving test contains 4 open-ended mathematics problems related to the mathematics function with half an hour time for students to answer. The researcher conducted this test for students one week after lecturers taught this topic. Table 3 shows the questions of this test.

Table 3: The Problems of the Test

Number	Item
1	Let $f = \{(1,2), (3,5), (1, m^2 - n^2), (4,7), (3, 2m - 1)\}$ is a function. Find the value of $2m + n^2$.
2	Find the domain and range of the function $g(x) = 4x^2 - 4x + 15$.

3 Is the relation $|x+y|=4$ a function? Why?

4 Look at this step pattern. In the first figure, which has one step, each side of the block is 1 cm long.
  a. Find the perimeter of a figure with 47 steps.
  b. Is there a figure with a perimeter 74 cm? If so, how many steps does it have? If not, why not?



The researcher during two sessions discussed with the Head of Mathematics Unit and a mathematics lecturer to improve the quality of questions according to the students' abilities in problem solving. So, they designed a mathematics problem solving test contains eight questions. But later the number of questions reduced from eight to four because in this educational center it was difficult to consider long time for conducting the test by lecturers. The validity of this test was confirmed by four mathematics experts from a public university in Malaysia. Also, for reliability of this test the researcher used Equivalent Forms Method and the Pearson correlation significant for this test with 35 participants outside of this research was 0.76. Finally, the questions in this test were confirmed by some experts in the Research Management Center (RMC) at the same university. In this foundation center, there were 20 classes with 45 to 50 students in each. But for English course there were 40 groups with 22 to 25 students (for English subject each class divided in two groups). The researcher conducted this test during English classes in order to have better sample size.

Also, item analysis was enforced in order to determine item difficulty. The researcher used the following formula that is more appropriate for open-ended questions.

$$p=\frac{a}{b-c}$$

In this formula, p is difficulty index and a is the total average for all scores in the item. The variables b and c are the maximum possible score and the minimum possible score for the item respectively. The item difficulty for questions 1, 2, 3 and 4 are 0.38, 0.31, 0.30 and 0.57 respectively. The item difficulty indices range from 0.30 (the most difficult item, item 3) to 0.57 (the easiest item, item 4). The difficulty indices from 0.20 to 0.80 can be used to retain the items in a standard test (Purnakanishtha et al., 2014). All the questions were in the range of standard difficulty index.

For scoring, if student doesn't understand the problem (illogical and incorrect answer) or non-answer the problem scored 0, if some steps in the solution show student understand the problem scored 1 (first step of Polya's model), if student understand and design a method for solution include some errors scored 2 (first and second steps of Polya's model) and finally completely correct answer scored 3 (all steps of Polya's model). So, the minimum and maximum scores for this test with four items were 0 and 12 respectively. Each student's exam paper was scored by two correctors. If there were no differences between their marks the researcher recorded the marks otherwise, the final mark for each student calculated according to the following rule. Assume that first and second lecturers considered two marks a and b for a student respectively the final mark (m) for this student was, $m = \left[\frac{a+b+1}{2}\right]$, where [ ] is the symbol of integral part. For instance, for two scores 9 and 10 the final mark calculated as $\left[\frac{9+10+1}{2}\right] = [10] = 10$.

In this study, the researcher first submitted the permission letter to the director of the foundation center and then coordinated with some lecturers to conduct this exam in their classes. Meanwhile, in each classroom, the researcher explained that students can participate in this exam voluntary.

Analyzing of Data

The descriptive statistics was used to find the situation of mathematics problem solving among foundation level students. The researcher analyzed the percentage of all scores of students in each mathematics problem and in overall score for this test. Also, the performances of male and female students in problem solving were compared by using independent samples t-test.

RESULTS

The results of this study discussed in two parts namely the ability of foundation level students in problem solving and the comparison of students' performance in problem solving by gender.

The Situation of Mathematics Problem Solving

In order to have better understanding about the students' performance in mathematics problem solving, Table 4 shows the mean and standard deviation of all questions and test scores.

Table 4: Mean and Standard Deviation of all Questions and Test Scores

Score	Number	Minimum	Maximum	Mean	Standard Deviation
Question 1	297	0.00	3.00	1.40	1.16
Question 2	297	0.00	3.00	1.42	0.90
Question 3	297	0.00	3.00	1.31	0.81
Question 4	297	0.00	3.00	1.68	1.13
Test	297	0.00	12.00	5.82	2.17

As respect to the Table 4, the mean of all variables are low and it seems the performance of students were poor in problem solving. The average of all mean scores for questions 1 to 4 is 1.45, it represents that the skills of students didn't allow them to solve the problems completely. Table 5 shows the students' performance for each question.

Table 5: Students' Performance for each Problem

Problem \ Mark	0 Frequency (%)	1 Frequency (%)	2 Frequency (%)	3 Frequency (%)
1	80 (26.93%)	101 (34.00%)	32 (10.77%)	84 (28.28%)
2	38 (12.79%)	141 (47.47%)	71 (23.90%)	47 (15.82%)
3	37 (12.45%)	157 (52.86%)	75 (25.25%)	28 (9.42%)
4	60 (20.20%)	73 (24.57%)	66 (22.22%)	98 (32.99%)

Table 5 shows that students usually have some difficulties in mathematics problem solving and usually they have some errors in their answers. Only small percentages of students answered completely to the questions. The percentage of complete answers for questions 1, 2, 3 and 4 were 28%, 16%, 9% and 33% respectively. It means the majority of students were not able to solve the problems completely based on the levels of Polya's model. The scores of students categorized in three groups namely, low (scores from 0 to 4), moderate (scores from 5 to 8) and high (scores from 9 to 12). Table 6 shows the frequency and percentage of different groups among students in mathematics test.

Table 6: The Frequency and Percentage of Different Groups of Students

Group	Frequency	Percentage	Mean	Standard Deviation
Low	88	29	3.31	0.98
Moderate	177	60	6.36	1.04
High	32	11	9.75	0.95
Total	297	100	5.82	2.17

The mean and percentage of each group show by Figure 1.

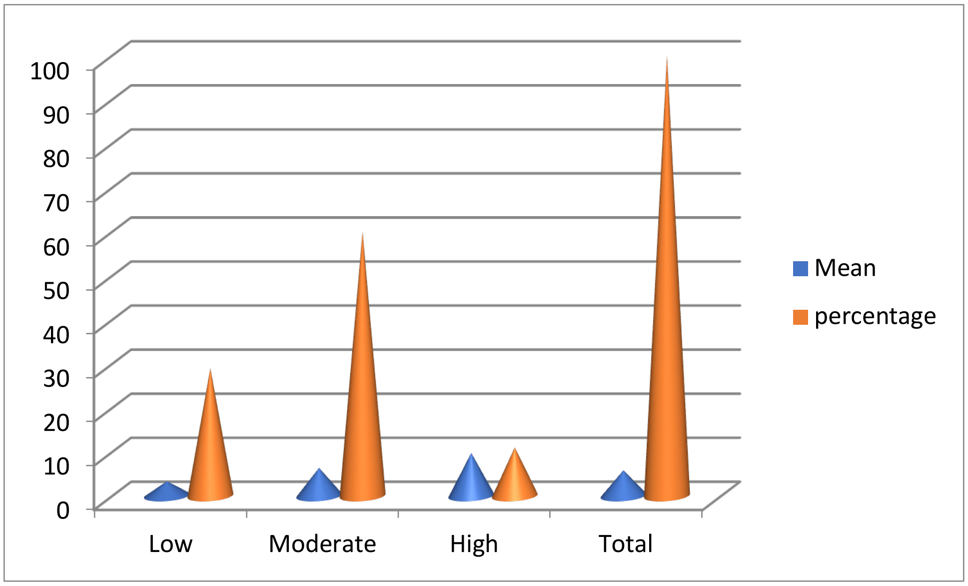


Figure 1: The Mean and Percentage of each Group

Table 6 and Figure 1 illustrate that only 11% of students had good performance in mathematics problem solving. According to the results of students in this test their abilities in mathematics problem solving should be improved.

Mathematics Problem Solving by Gender

Kim (2013) and Mishra et al. (2019) explained that by using skewness and kurtosis a Z-score could be obtained by dividing the skew values or excess kurtosis by their standard errors as:

$$Z=\frac{Skew\ value}{Standard\ error} \text{ or } Z=\frac{Excess\ kurtosis}{Standard\ error}$$

Then the normality of data based on the sample size determine as follows:

a. If the sample size is less or equal 50 ($n \leq 50$) and $|Z - score| < 1.96$ then data normally distributed.
b. If the sample size is between 50 and 300 ($50 < n \leq 300$) and $|Z - score| < 3.29$ then data normally distributed.
c. If the sample size is more than 300 ($n > 300$) and the values of skewness and kurtosis without considering the Z-scores are between -2 and 2 then data normally distributed.

Since the absolute values of Z-scores for all groups in Table 7 are less than 3.29 so the scores doesn't differ from normal distribution.

Table 7: The Normality of Scores

Group	No.	Skewness	Standard Error (Skewness)	Z-score (Skewness)	Kurtosis	Standard Error (Kurtosis)	Z-score (kurtosis)
Male	114	-0.030	0.226	-0.132	-0.356	0.449	-0.792
Female	183	0.339	0.180	1.883	0.031	0.357	0.086
All	297	0.169	0.141	1.198	-0.032	0.282	-0.113

Table 8 shows the mean and standard deviation of all mathematics problems and test scores for male students.

Table 8: Mean and Standard Deviation of problems and Test Scores for Male Students

Scores	Number	Minimum	Maximum	Mean	Standard Deviation
Problem 1	114	0.00	3.00	1.25	1.19
Problem 2	114	0.00	3.00	1.38	0.82
Problem 3	114	0.00	3.00	1.24	0.83
Problem 4	114	0.00	3.00	1.52	1.15
Test	114	0.00	10.00	5.41	2.19

Table 9 shows the mean and standard deviation of all mathematics problems and test scores for female students.

Table 9: Mean and Standard Deviation of Problems and Test Scores for Female Students

Scores	Number	Minimum	Maximum	Mean	Standard Deviation
Problem 1	183	0.00	3.00	1.49	1.13
Problem 2	183	0.00	3.00	1.45	0.95
Problem 3	183	0.00	3.00	1.36	0.79
Problem 4	183	0.00	3.00	1.77	1.11
Test	183	1.00	12.00	6.08	2.13

According to the Tables 8 and 9 the performance of female students was better than male students in all mathematics problems. Figure 2 compares the performance of students in all mathematics problems and test by gender.

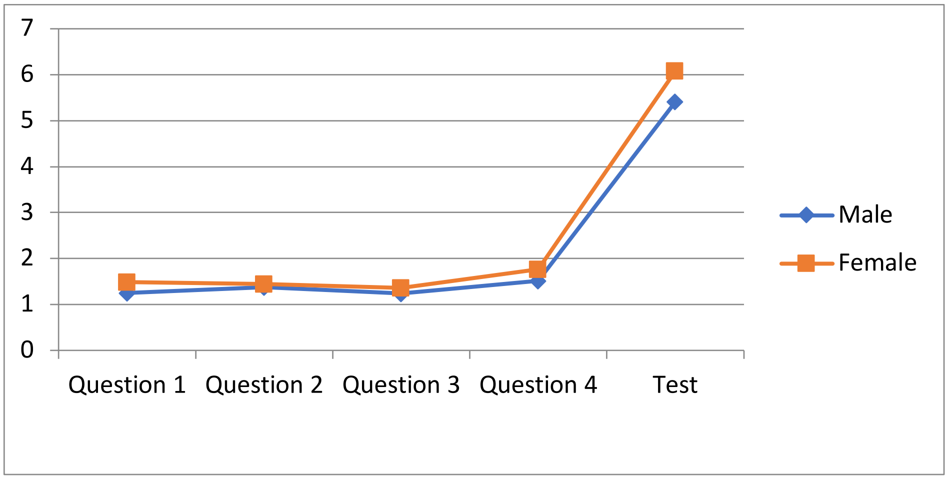


Figure 2: Performance of Students in Problem Solving by Gender

The researcher used independent samples t-test to compare the performance of students in problem solving by gender statistically. Table 10 shows the equality of variances for both groups (male and female students) statistically $F(1, 295) = 0.516$, $P > 0.05$.

Table 10: Levene F Test (consistency of error variances)

F	df1	df2	Sig
0.516	1	295	0.473

Table 11 shows the results of independent samples t-test to compare the performance of students in mathematics problem solving by gender.

Table 11: Results of Independent Samples t-test for Problem Solving by Gender

Group	Number	Mean	Standard Deviation	t	df	Sig
Male	114	5.41	2.19	-2.624	295	0.009
Female	183	6.08	2.13			

The result of Table 11 shows that there is a significant mean difference between the performance of male ($M = 5.41$, $SD = 2.19$) and female ($M = 6.08$, $SD = 2.13$) students in mathematics problem solving $t(295) = -2.624$, $p < 0.05$. In other words, female students had better performance rather than male students in mathematics problem solving.

DISCUSSION

This current study investigated the ability of students in problem solving about mathematics functions that considered as a problematic topic for students to learn. The concept of mathematics function is a central and practical but difficult topic in secondary school curricula (Akkus et al., 2008; Ponce, 2007). For example, "the topics inverse function and composite function is more conceptual and challenging among educators to transfer to students" (Oehrtman et al., 2008, p. 39). Michelsen (2006) explained that modeling the real-world problems is one of the most common applications of the mathematics functions in different areas of studies. Since mathematics function is one of important topics that used in all mathematics courses at the university level, students in foundation level need to have suitable knowledge about it. But this is an important question "how students can learn the functions conceptually?". Problem solving approach should be common in foundation centres because this level of education is the border between high school and university level. So, mathematics lecturers during one year (two semesters) of foundation program should be able to improve the ability of students in problem solving.

Mathematical problem solving is a big challenge among lecturers and usually students have some difficulties against mathematics problem solving (Gholami et al., 2021; Khalid, 2017). Many researchers have explained that students usually receive mathematics contents which emphasize on the solving of routine exercises and they follow the steps that mathematics educators explained to them (Intaros et al., 2013; McDonald, 2009; Mon et al., 2016; Tambychik & Meerah, 2010). Gholami et al. (2019) explained that "students have the impression that they only need to memorize the formulas, theorems, shortcuts and methods to apply in exercise solving and in preparing for examinations" (p. 307). In fact, students are seldom engaged with open-ended problem solving during their mathematics courses. So, usually they cannot solve the new mathematics exercise if lecturers change it slightly. It seems students have poor basic knowledge in mathematics from previous years although their grades in mathematics are usually excellent. In this situation, students cannot learn the mathematics conceptually and experience the beauties of it. Another reason for low ability in problem solving among students of this foundation centre related to the superficial teaching because lecturers need to cover a lot of topics during each semester. Lecturers prefer to teach exactly the same textbook materials. Therefore, this method of teaching encourages students to use the memorization method in learning mathematics. The results of this study confirmed that in this foundation centre, mathematics problem solving activities should be improved among students.

Also, the results of this study illustrated that the performance of female students ($M = 6.08$, $SD = 2.13$) in mathematical problem solving were better than male students ($M = 5.41$, $SD = 2.19$). This finding is in line with the results of Malaysian students in all TIMSS assessments from 1999 to 2015 (Mullis et al., 2016). More studies need to conduct in different educational levels and contexts of Malaysia to find why this issue happens in this country whereas in many of countries male students have better performance in mathematics problem solving rather than female students. It seems the situation of mathematics teaching and learning in Malaysia such as memorization method prepares better opportunity for female students to have higher marks compare male students. Because usually female students are harder working and sensitive about their results therefore female students are more likely to have better results in mathematics exams through learning some standard algorithms and specific methods.

CONCLUSION

Malaysian Ministry of Education aims to be one of top countries in the international assessments such as TIMSS and PISA in 15 next years, thus, the government is very serious to enhance the quality of mathematics teaching and learning (Ministry of Education, 2014). The low quality of mathematics problem solving in Malaysian education system is a serious alarm for Ministry of Higher Education, Ministry of Education, policy makers, administrators and mathematics educators to have better plan and strategy to improve the abilities of students in mathematics problem solving. Malaysian Ministry of Education needs to know "why female students have better performance in mathematics problem solving?". Malaysian government should enhance the ability of students in problem solving in order to obtain better results in the international assessments. In summary, likely poor basic knowledge in mathematics among students, traditional teaching methods, a lot of topics and emphasize on the grades of students instead of the learning quality are some of important reasons for the low ability of students in mathematics problem solving. Based on the results of international assessments such as TIMSS and PISA, some factors such as students' level of self-confidence in mathematics problem solving ability, the value of mathematics in the real life and their future careers, and the amount of time students spend doing mathematics homework were considered as possible factors affecting this differences. In Malaysia, most mathematics educators teach mathematical material through the traditional method. In this method, students usually follow the mathematical procedures and rules taught by the educator to perform similar exercises. In other words, students prefer to use the memorization method in learning mathematics. It seems that Malaysian female students show more responsibility in doing assignments and participating in class activities, which not only increases their self-confidence, but also makes them superior to males in the mathematical exams and international assessments. Successful countries in international assessments such as Singapore and Japan emphasize the mathematical problem solving approach among educators in order to improve the ability of students in solving non-routine problems. Mathematics educators as the responsible group for delivering the curriculum should be more competent in problem solving in the taught lessons in order to improve the ability of students in problem solving (Pineiro et al., 2021). Therefore, Malaysian Ministry of Education needs to improve the ability of mathematics educators through professional development programs in problem solving, improve the ability of students in problem solving and critical thinking by engaging them with suitable problem solving activities as well as enhancing textbook materials based on problem solving approach to see better outcome among Malaysian students in the future international assessments.

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Division problems with remainder: A study on strategies and interpretations with fourth grade Mexican students

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Abstract: The present research focuses on analyzing how fourth-grade elementary school students (ages 9 to 10) solve and interpret the result of non-routine problems, precisely division measurement and division-partition with remainder. The methodology is qualitative, with a descriptive and interpretative approach. The information was collected using a questionnaire consisting of three problems (two of quotitive division and one of partitive division) and a clinical interview. The results showed the importance of using the division, multiplication, and addition algorithms to give a realistic answer to the problems. In the same way, it was possible to demonstrate the graphic strategy combined with counting to give a realistic answer to the problem. However, students were found to use division correctly but without an interpretation of the remainder or quotient. Likewise, they struggled to choose the correct procedure to solve the problem. These data suggest including realistic problems in mathematics classrooms to make sense of mathematical concepts in real life or the student's context. Likewise, this study provides implications on the mathematical problems that the teacher proposes in the classroom, where not only the division algorithm should be taught mechanically, nor focus on posing routine problems that lead the student to use a single heuristic resolution strategy. Essentially, the teacher is required to include real-world problems, where the student can awaken creativity to represent in different ways the understanding of a problem and, therefore, different strategies to solve it. In addition, that the student has the ability to check the result of the problem, with the conditions, situations or circumstances imposed by reality or everyday life.

Keywords: *Division problems with remainder, division-quotitive, division-partitive, resolution strategies, primary education.*

INTRODUCTION

To talk about problem-solving is to consider the importance of schools as a suitable setting for learning. It is here where the student manages to develop mathematical strategies to face challenges where the solutions to these do not require an obvious answer. However, different researchers interested in the subject show that the problems brought to the mathematics classroom lack authenticity. That is, they leave aside the students' context and reality. These types of problems do not help students learn to apply mathematical procedures in situations in their daily lives (Chamoso et al., 2014; Jiménez & Ramos, 2011).

Frequently, problem-solving is only a processing of the numerical data that appear in the statement without understanding what is being sought (Eslava et al., 2021). For Vicente et al. (2008), the most significant difficulties in students occur when working with real-world problems. In this sense, one of the most studied non-routine realistic problems, and where difficulties arise, are division problems with remainder (DWR). That is, problems in which the dividend and the divisor are integers, and the division gives rise to a non-integer result. In this type of problem, the result must be interpreted by the real-world constraints that give meaning to the problem (Verschaffel et al., 2009). According to Jiménez (2008), "realistic problems are verbal problems that describe real-life situations and that the application of an arithmetic operation does not simply lead to the solution of the problem" (p. 38). For Dewolf et al. (2014), verbal problems are an important way to bring the real world into the mathematics classroom and to teach mathematical modeling and applied problem-solving. A specific feature of realistic verbal problems is that they often do not contain all the information required to obtain a correct solution (Krawitz et al., 2018). For Payadnya (2021), realistic problems play an important role when you want to learn mathematics. That is, on the one hand students are required not only to understand the concept but also to apply the concept to solve everyday problems. Therefore, in verbal problems, the student must use real-world knowledge to give meaning and coherence to his or her answers. In that sense, the real world is the starting point where the development of mathematical concepts takes on meaning and relevance (Agustina et al., 2021).

An example of a realistic problem found in the literature was: "An army bus holds 36 soldiers. If 1128 soldiers are being transported by bus to their training site. How many buses are needed?" (Carpenter et al.,1983, as cited in Verschaffel et al., 2009). Two situations may arise in this problem. The first is related to not having difficulty in identifying division as the correct solving operation, and the second is linked to the tendency to give an incorrect answer 31.3 buses because they do not emphasize the non-integer quotient, taking into account the situation of the problem (Lago et al., 2008).

The structure of the division problems with remainder allows us to see which element of the division (remainder or the non-integer quotient) the analysis is focused on. According to Fischbein et al. (1985). They propose two intuitive problem models of division: the quotitive and the partitive. The first could also be called division by measure, which seeks to determine how many times a given quantity is contained in a larger quantity. For this type of division it is required that the dividend must be greater than the divisor. That is, it is established how many times a given quantity is contained in a larger one (Lago et al., 2008). An example of this model is shown by

Zorrilla et al. (2021) "9 fans want to travel to the stadium in another city. Each cab can carry four fans. How many cabs do they need?" (p. 1322).

In the case of partitive division model we rely on what is established by Fischbein et al. (1985): “an object or collection of objects is divided into a number of equal fragments or subcollections. The dividend must be larger than the divisor; the divisor (operator) must be a whole number; the quotient must be smaller than the dividend (p. 7). Likewise, this refers to the approximation of the partitioning. That is, a given number of equivalent groups is formed to define the number of each group (Lago et al., 2008). For Buforn (2017), split-partite problems refer to problems where the number of objects per group is unknown. An example of this model is also shown by Zorrilla et al. (2021) "A dance academy has distributed in a class eight tickets for a musical. The dancers in the class were three, and all received the same number of tickets. How many tickets did each dancer receive?" (p. 1322).

It should be emphasized that these two types of division models with remainder are the ones we intend to address in the present research. The analysis we want to develop also focuses on how students interpret both the remainder and the non-integer quotient.

In that sense, interpretation plays a vital role in these division problems with remainder. That is, not only is it required that the student uses the mathematical algorithm correctly, but also that the answer makes sense with the real situation of the problem. This implies two situations: in the first one, it must be kept in mind that the existence of the remainder forces the student to recognize the value of the quotient plus one unit as a result. The remainder is not contemplated in the second, but the non-decimal quotient is based on the partition's context (Buforn, 2017; Lago et al., 2008; Parra & Rojas, 2010; Verschaffel et al., 2009).

On the other hand, further research has focused on studying the use of strategies in division problems with remainder in elementary school students (Downton, 2009; Ivars & Fernández, 2016; Sanjuán, 2021; Zorrilla et al., 2021). In that sense, Downton (2009), in his study with third-grade students (ages 8 to 9 years), evidenced the use of modeling, multiplicative thinking, repeated addition/subtraction, and counting that were employed as strategies in both division models.

For their part, Ivars and Fernández (2016) in their research with students from 6 to 12 years old, evidenced modeling and counting strategies in both division models and number fact strategies and multiplication of equal addends in division-measurement problems. In Sanjuán's study (2021) with students aged 6 to 12, strategies such as direct modeling, repeated addition, grouping, combination, and the multiplicative strategy were found in division-measure problems. As for division-partition problems, non-anticipative, additive, and multiplicative coordination strategies were used. Finally, Zorrilla et al. (2021), in their study, coincide with the strategies observed by Ivars and Fernandez (2016), highlighting a strategy modeling with counting (graphical strategy), additive and subtractive strategies (use of successive addition-subtraction), and known number facts and those derived from addition and multiplication.

In this research, we analyzed how fourth-grade elementary school students solve division problems with quotient and remainder considering the division-quotitive and division-partitive models. With this objective, we seek to answer the following questions:

How do students interpret the quotient and the remainder in non-integer division problems? What strategies do students employ in solving division problems with remainder?

Method

The present research is qualitative, descriptive, and interpretative, according to Hernández (2014), since it attempts to make sense of the phenomena in terms of the meanings people give them. A questionnaire validated by a group of researchers was first applied. The participants were 50 fourth-grade elementary school students from a private school in Puebla, Mexico, whose ages ranged between 9 and 10 years old, selected by convenience. This questionnaire is made up of three non-routine problems of multiplicative structure, two division-measurement problems, and one division-partite problem (see Table 1). Verschaffel et al. (2009) propose three situations for these types of problems. The first requires the quotient to be rounded up, the second consists of rounding down, and the last suggests keeping the result of a division with the remainder as a decimal.

In this sense, Zorrilla et al. (2021) classified these situations into three types: the first implies that the presence of the remainder forces to recognize as a solution the value of the quotient plus one unit (type 1). The remainder is not considered in the second but the non-decimal quotient (type 2), and the third is the quotient plus the fractional part of the remainder (type 3). It should be noted that we will focus on type 1 and type 2 situations (see Table 1).

Questionnaire problems	Problem Characteristics
A candle vendor in the Emiliano Zapata market in Puebla, Mexico, packs 30 candles in a box. How many boxes will the vendor need if he has to pack 122 candles?	Quotitive Division. The remainder forces to recognize, as a result, the value of the quotient plus one unit Zorrilla et al. (2021) (Type 1).
A museum has given away 75 tickets to an art exhibition to 14 schools. The schools have received the same number of tickets to be distributed to their best students. How many tickets does each school receive?	Partitive Division. The remainder is not considered. The solution is the non-decimal quotient Zorrilla et al. (2021) (Type 2).
Twenty-two players of the Puebla soccer team want to travel by cab to the training sports venue. Each vehicle can carry four players. How many cabs do they need?	Quotitive Division. The remainder forces to recognize, as a result, the value of the quotient plus one-unit Zorrilla et al. (2021) (Type 1).

Table 1. Measurement and partitive division problems

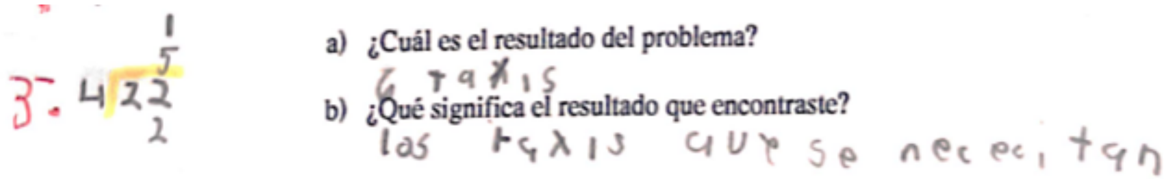
Secondly, an interview was conducted with the seven selected students to understand how children construct their worlds, think, and work cognitive processes (Ginsburg, 2009). That is, to know how they solve problems and interpret the quotient and the remainder in the division-measure and division-partite problems. It should be noted that the interview was applied to seven students, which were audio-recorded and immediately transcribed for subsequent analysis and triangulation of the information. In addition, the students were assigned codes S1, S2, S3, S4, S5, S6 and S7 for the interviews' excerpts. The letter R stands for the researcher.

Results and Analysis

The results presented in this section are organized into two sections. The first section presents the analysis of the responses to the division problems with residue, distinguishing their realistic and unrealistic character. The responses were classified into realistic responses with the application of division, realistic responses without the application of division, unrealistic responses applying division, and other responses. The second section presents the strategies used by the students when solving problems 1 and 3 of type 1 of the quotitive division model. The strategies evidenced are realistic responses.

Realistic responses with the application of division.

In these answers, the individual uses division appropriately and considers the realistic aspects of the problem. The solver interprets the remainder or the quotient to give a realistic answer. This response is evidenced in problem 3 of type 1 of the quotitive division model, as shown in Figure 1.



a) What is the result of the problem?

"6 taxis"

a) What does the result you found mean?

" The taxis you need "

Figure 1. Realistic response to the division operation.
Source: student's response (4th grade)

In the interview, the student was asked the following questions:

R: What is the exercise asking you?

S1: How many cabs do the soccer team players need to go to the sports venue?

R: What procedure or operation did you perform there?

S1: A division.

R: You tell me that they are...

S1: 6

R: Why do you arrive at six cabs? Could you explain it to me?

S1: Because I divided: 22 by 4, it gives me 5, 2 are left, and the other two players leave in another cab. That would be six cabs. Five full cabs and another one with two players (4th-grade student, Interview excerpt, May 13, 2022).

Here the student uses division as an adequate procedure, expressing that to reach the problem result, he had to divide 22 by 4. In addition, he considers the real facts of the problem when he emphasizes that six cabs are needed to transport the 22 players to the sports venue. This is because the student understood the problem and interpreted the remainder of the division (2 players), assigning to this remainder an extra cab. He emphasizes that the division gives five as the quotient, and two are left over, expressing that there are five cabs full of players and another with two players for a total of six cabs.

Another realistic response result applying the division algorithm was evidenced in problem 2 of type 2 of the partitive division model, as shown in Figure 2.

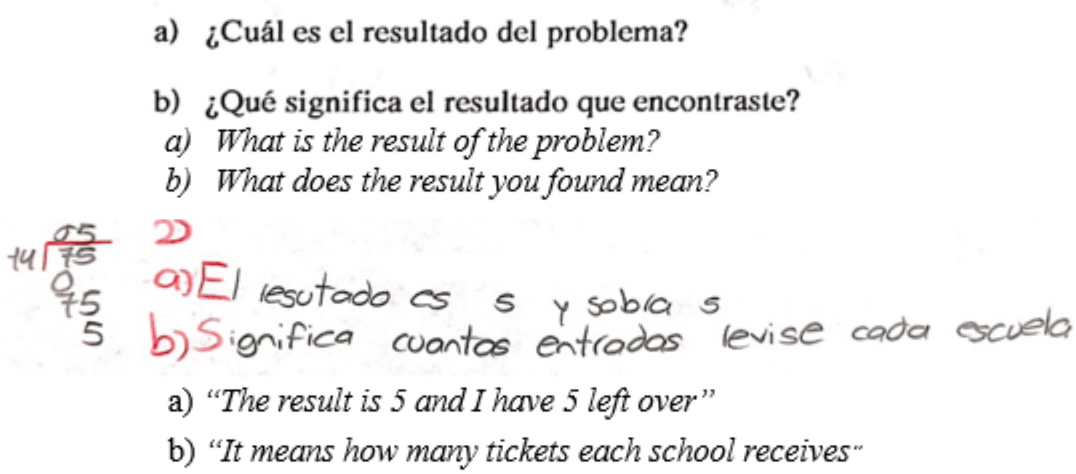


Figure 2. Realistic response to the division operation.
Source: student's response (4th grade)

In the interview, the student was asked the following questions:

R: *What is the problem asking of us?*

S2: *A museum has given away 75 tickets to 14 schools. Each one is going to send its best students. So I divided to find out how many tickets each school gets.*

R: *Do you think you can do another procedure here?*

S2: *A sum could be*

R: *Can you explain how you would use addition in this problem?*

S2: *Adding until I get a result, but not more than the number of tickets we were given, and putting the remainder as a remainder.*

R: *Ok, so how many tickets does each school get?*

S2: *Five tickets*

R: *Ok, what does this residue mean?*

S2: *The leftover tickets*

R: *Would it be convenient to distribute those five tickets among the fourteen schools?*

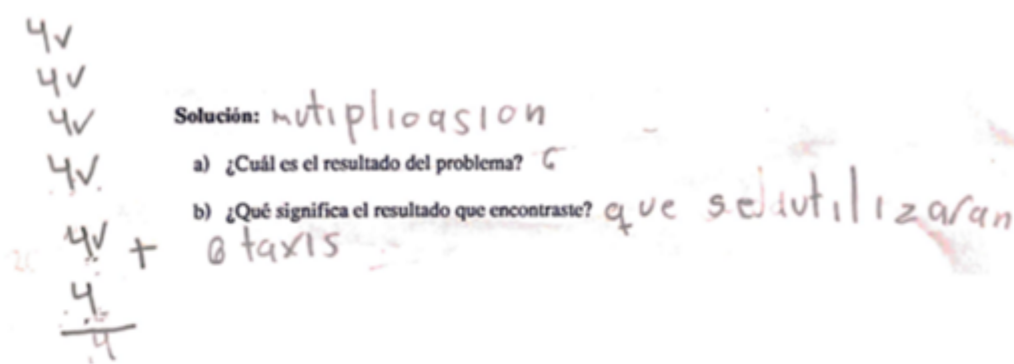
S2: No because it would be unfair

R: What would be the most convenient thing to do with those five tickets?

S2: Give them as a gift to someone else. (4th-grade student, Interview excerpt, May 13, 2022).

Here the student applies the division algorithm as a suitable procedure to answer the problem. In addition, he expresses that the repetitive addition would be another procedure that could be applied as long as it did not exceed the total number of entries and that the leftover entries would be the remainder. She also states that she interpreted the remainder as the five entries that were left over and the quotient as the number of entries that should be distributed to the fourteen schools equally (five entries for each school). In other words, the student interpreted the numerical answers correctly (context of the distribution) and successfully solved the problem.

Realistic responses without the application of division. These are responses where the subject uses arithmetic operations other than division and manages to interpret the real situation of the problem. That is, the solver uses addition and multiplication as an adequate procedure. Likewise, they interpret the reality or situation described in the problem to solve it. This situation was evidenced in problem 3 of type 1 of the quotitive division model, as shown in Figure 3.



Solution: "Multiplication"

a) What is the result of the problem? "6"

b) What does the result you found mean? "That 6 taxis will be used"

Figure 3. Realistic responses without the application of division.
Source: student's response (4th grade)

In the interview, the student was asked the following questions:

R: What is this problem asking you?

S3: I am being asked if 22 players of the soccer team want to travel in cabs to the training sports venue. Each vehicle can only carry four players. How many cabs are needed? I did multiplication.

R: Using multiplication, how did you arrive at the answer of 6 cabs?

S3: If each cab can carry 4, because 4 times... no I had to divide 22 by 4 and I got six.

R: Explain to me why you got the answer six cabs?

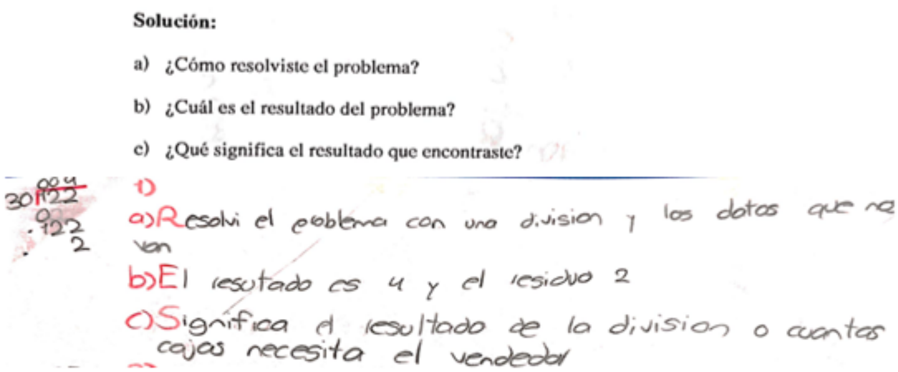
S3: The four players that fit in the cab I added them until I got 22, I also multiplied 4 x 4=16, 4 x 5=20, 4 x 6=24, now 4 x 5=20, the players are complete in the cab, and then I have two players left over.

R: What happens to those two players left over?

S3: They can go in a sixth cab, then they are 4 x 6= 24, but it happens, I know I have only 22 players, but the two missing players can go in a cab that fits 4, and nothing happens (4th-grade student, Interview excerpt, May 13, 2022).

Here the student used addition and repetitive multiplication as the appropriate procedure to answer the problem and gave it the correct interpretation. However, at one point in the questioning, he manages to reflect by stating that he has divided. However, when he begins to explain what he has done, he seems to have used addition and multiplication. That is, the student expresses that to arrive at the answer of 6 cabs, he added four by four until he got 22 players. In explaining how he performed the multiplication, he emphasizes that the product of $4\times5=20$ represents the players that can travel in the five cabs with four people. He also considered that the two surplus players could travel in a sixth cab, considering the multiplication of $4\times6=24$, although it exceeds the total of 22 players that must be transported.

Unrealistic responses applying division. Although using division as an appropriate arithmetic operation, the answers given by the subject did not take into account the real part of the problem. The solver does not interpret the division's elements (the quotient and the remainder). An example of this unrealistic response to problem 1 of type 1 is shown in Figure 4.



Solución:

a) ¿Cómo resolviste el problema?

b) ¿Cuál es el resultado del problema?

c) ¿Qué significa el resultado que encontraste?

1)
a) Resolvi el ooblema con una division y los datos que me dan
b) El resutado es 4 y el residuo 2
c) Significa el resultado de la division o cuantas cajas necesita el vendedor

a) How did you solve the problem?

"I solved the problem with a division and the data they gave"

a) What is the result of the problem?

" The result is 4 and the residue 2"

a) What does the result you found mean?

"It means the result of the division or how many boxes the seller needs

Figure 4. Unrealistic responses applying division
Source: student's response (4th grade)

Here the student divides 122 candles among the four boxes, obtaining; as a result, four boxes in the quotient and two candles as the remainder. Taking into account the above in the interview, he was asked the following questions:

R: *In the first problem: What was I asking you to do?*

S4: *Divide how many boxes the salesman will need to put the candles in, and I already had to divide and get the result for the questions you were asking me.*

R: *So for this problem, you divided, right?*

S4: *Yes*

R: *What's the answer to this question? How many boxes will the salesperson need?*

S4: *Five boxes, right?*

R: *Five boxes? Why?*

S4: *Ah no. Four boxes.*

R: *And those two that are loose, what happens to them?*

S4: They can't go because the boxes are already full. Yes, they could fit, but they would not be well arranged and out of the box.

R: What other option would you give?

S4: Buy another box

R: Apart from the division, do you think you could apply another procedure?

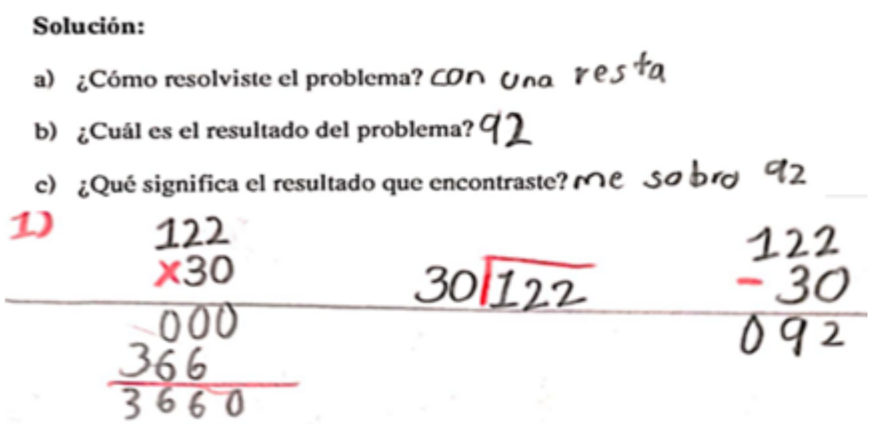
S4: No.

R: Then how many boxes would you need to ship the candles in total?

S4: 5 (4th-grade student, Interview excerpt, May 13, 2022).

In this interview excerpt, the student expresses that she used division to solve the problem. Furthermore, although she correctly applied the division procedure, she could not interpret the remainder (the two leftover candles). That is, the student did not consider the problem's real part. When answering, the student forgets the problem's text and takes the one found with the division algorithm as the correct answer, thus generating an incorrect result. It is worth noting that at one point in the interview, the student reflected that in order to pack the two leftover candles, an additional box was needed. That is, concluding that five boxes were needed.

Other responses. The solver's responses in this situation are influenced by an inadequate procedure or calculation of the arithmetic operation. In addition, there is no interpretation of the result. An example of these responses was evidenced in problem 1 of type 1 in the quotitive division model, as shown in Figure 5.



a) How did you solve the problem? "With a subtraction"
b) ¿What is the result of the problem? "92"
c) What does the result you found mean? "I have 92 left over"

Figure 5. Other responses
Source: student's response (4th grade)

Here the student applies different arithmetic operations without arriving at a reasonable answer to the problem in question. The student performed arithmetic operations: from the application of subtraction and multiplication to division. However, there was no success in solving the problem. In the interview, the student was asked the following questions:

R: In the first problem, what is he telling you, or what data is he giving you?

S5: In the Emiliano Zapata market in Puebla, Mexico, a candle vendor packs 30 candles in a box. How many boxes will the vendor need if he has to pack 122 candles?

R: What is the problem asking you?

S5: How many boxes will the seller need?

R: What procedure do you think should be done for this problem?

S5: A...

R: Here you said a subtraction. What should be subtracted?

S5: One hundred twenty-two minus thirty

R: Minus 30, which is what you did here.

S5: Yes.

R: But here I see you were going to divide. Why didn't you do that?

S5: I couldn't, so I tried subtraction, giving me a result I thought it would be.

R: Why did you do multiplication?

S5: Because I was starting from multiplication to subtraction, then division, subtraction, multiplication, and addition [A confusing sentence, and the investigator did not go deeper].

R: And why did you stick with the subtraction result?

S5: Because I feel that they are the boxes that are going to accommodate 122 candles.

R: Ok, well, do you think that doing multiplication, division, or addition can also give me the correct result?

S5: No, because multiplication will be a more significant number, the division will be a bigger number, and the addition will be a bigger number.

R: Ok, then it's the subtraction. (4th-grade student, Interview excerpt, May 13, 2022).

In this interview excerpt, the student is convinced that the correct procedure is subtraction. In addition, he justifies that with this answer, the 122 candles can be accommodated, and that multiplication, division, or addition would be a more significant number. In this problem, there is the belief that multiplication, addition, and division imply a more significant number and subtraction a minor number without considering what the problem is posing. In this case, the student shows an incorrect understanding of the problem and a lack of clarity when choosing the correct operation. The above could be why students tend to focus on superficial aspects of the problem statement and select inappropriate solution procedures.

Strategies used in realistic responses

Graphic strategy. According to Zorrilla et al. (2021), this type of strategy occurs when the subjects offer a solution in which they use a drawing or diagram to solve the problem. For Matalliotaki (2012), drawings are one of how children express a complex phenomenon by facilitating the expression of the spatial relationships of objects. In that sense, the representation created by the students is helpful in interpreting the result, keeping in mind the real part of the problem. An example of this strategy is shown in Figure 6. The student drew the cabs needed to take the players, evidencing that the student is aware of the meaning of the rest.

Figure 6. Example of graphical strategy-counting in realistic responses.
Source: student's response (4th grade)

In the interview, the student was asked the following questions:

R: What does the picture you drew in this problem mean?

S6: The cab that takes the players

R: Well, explain to me what does this drawing consist of? Why do you place six cabs?

S6: Because there are 22 players

R: Ok, how many players are in each cab?

S6: Four players

R: Why do you put two players in the last cab?

S6: Because they are 1,2,3,4,5,6,7,7,8,910,11,12,13,14,15,16,17,18,18,19,20,21 and 22.

R: ok

S6: And here, fit two because they are all complete of 4.

R: Ok, they are complete of what?

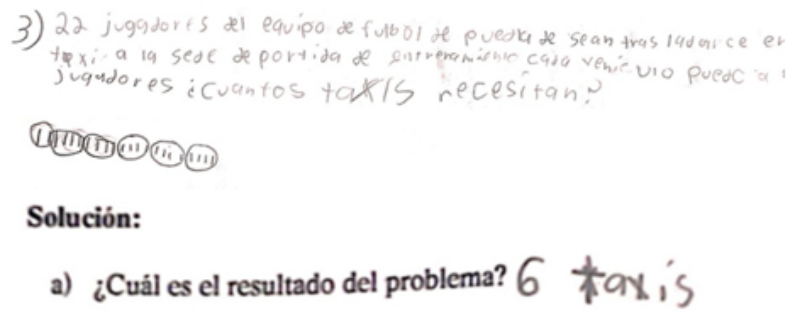
S6: Of four

R: *Does it matter that the last cab is of two players?*

S6: *It doesn't matter (4th-grade student, Interview excerpt, May 13, 2022).*

Here the student employs the graphing and counting strategy to represent the solution to the problem. In the first instance, the graph made by the student illustrates the number of players with the total number of cabs that can be used to transport the 22 players. For this strategy, the student keeps in mind that the total number of players who can go in a cab is 4. The counting strategy is evidenced when he started counting from 1 to 22, concluding that a sixth cab was used. In addition, he considered that they completed five cabs with 20 players, and the remaining two could travel in a different cab.

Another realistic response where the graphing and counting strategy was used, as evidenced in problem 3 of type 1 of the quotitive division model, as shown in Figure 7.



a) ¿What is the result of the problem?

"6 taxis"

a) What does the result you found mean?

Figure 7. Example of graphical strategy-counting in realistic responses.
Source: student's response (4th grade)

In the interview, the student was asked the following questions:

R: *In the third problem, what is the problem asking us, what data does it give us?*

S7: *22 players of the Puebla soccer team want to travel by cab to the training sports venue. Each vehicle can carry 4 players. How many cabs do they need?*

R: *What is the problem asking you?*

S7: *How many cabs do you need to go to the stadium?*

R: How many cabs do you need?

S7: Six cabs

R: How did you arrive at that answer?

S7: By drawing

R: What is the meaning of this drawing you made here?

S7: I put twenty-two lines and enclosed in fours.

R: Ok, you enclosed four by four, and in this last one how many were left?

S7: There are two

R: How many players are in this last cab?

S7: Two players

R: So how many cabs are needed?

S7: Six cabs (4th-grade student, Interview excerpt, May 13, 2022).

In this problem the E7 answers the question How did you solve the problem? giving as answer "6 taxis" and the second question What does the result you found mean? The student did not submit any response. First, the student used a graphic strategy or drawing to interpret and solve the problem. In addition, during the questioning, it was possible to identify that in addition to using the drawing, the student made a count by enclosing the lines representing the 22 players, four by four. In this problem, the student formed six groups representing the six cabs where the players should be transported, specifying that there are five groups with four players and one group with two players for a total of 22.

In summary, the results presented are a sample of how students interpret the remainder and the quotient in realistic problems. In that sense, in problems 1 and 2 (See Figures 1 and 2), students show a realistic response to having the division algorithm as a procedure. In these division models, students correctly interpret the result based on the reality of the problem. In the first instance, E1 interprets the remainder as adding one unit to the quotient. On the other hand, S2 interprets the context of division as a non-decimal result without emphasizing the remainder. These results disagree with those obtained by Zorrilla et al. (2021), who state that the algorithm's application does not result in a realistic response.

On the other hand, in some cases, when the student faces a realistic problem, it seems that he/she uses the arithmetic operations he/she handles correctly and uses it to solve the situation. In our study, the written production of S3 (See Figure 3), the procedure did not focus on the division algorithm but on multiplication and addition, thus generating a realistic response.

On the other hand, using division as an adequate procedure does not guarantee the student a correct interpretation of the result (See Figure 4). Rodríguez et al. (2009) pointed out that the choice of division as a solution procedure did not rule out inadequate numerical results. The division algorithm used in problem one by S4 was performed correctly; however, the difficulty was evidenced when associating the answer with the context of the problem. Galvão & Labres (2006) mention that children do not consider the remainder as a component of the division related to the other components. Another difficulty evidenced was considering the result of subtraction as appropriate without first interpreting problem 1 (quotient plus the unit) simply by considering that the result of multiplication, division, and addition is associated with a more significant number (See Figure 4). Jiménez et al. (2011) state that it is striking that, in order to solve problems with an apparently additive structure, many of the errors were produced by the application of subtraction, multiplication, division, or the combination of two arithmetic operations. That is to say, when solving a realistic problem; children tend to have difficulty selecting the arithmetic operation with which they intend to operate and making sense of the answer.

On the other hand, this study is a sample of the different resolution strategies that students use to give meaning and interpretation to the result of the problem. An example is a graphic and counting strategy, where S6 and S7 made a drawing to represent the situation of the problem. In problem three, the lines and dolls represent the players, and the circles and cars are the vehicles that will be used to move. In this representation, the student performs a count in rounds or consecutively. These strategies were also evidenced in research (e.g., Downton, 2009; Ivars & Fernández, 2016; Sanjuán, 2021; Zorrilla et al., 2021).

Discussion

In the answers given by the students in problem 3 of type 1 as shown in Figures 1, 3, 6 and 7, the use of the algorithm of division, multiplication, and the fact of including drawings as a graphic strategy and repetitive counting are evident. These heuristic problem-solving strategies allowed students S1, S2, S6, and S7 to arrive at the correct solution and interpret the remainder or residue in terms of the actual situation or circumstance of the problem. In the case of using the division algorithm properly, Lago et al. (2008), pose that students' answers when applying correct resolution procedures are usually accompanied by correct interpretations. This is evident if we consider both the division model and the types of subtraction.

In the case where the student uses multiplication as a calculation to solve the problem, Downton (2008) asserts that young children can solve complex division problems when provided with a problem-solving learning environment that encourages them to draw on their intuitive thinking strategies and knowledge of multiplication. On the other hand, the fact of including graphic strategies such as drawing and performing successive counts, allowed the students to solve the problems. In this sense, Ivars y Fernández (2016), state that the student graphically represents the sharing process and counts the elements a group has. Likewise, using graphical strategies to represent the problem facilitates students to identify the structure underlying the problem (Santos, 1997, cited in Zorrilla et al., 2021).

In problem 2 of type 2 (See Figure 2) S2 interpreted the numerical answers correctly (context of the distribution) successfully achieving the solution of the problem. This result agrees with that obtained in Lago et al. (2008), where in this type of problem (non-decimal quotient), students always interpret the numerical answer with a high percentage of correct interpretations. However, it disagrees with the findings of Zorrilla et al. (2021), wherein in non-decimal quotient problems, realistic responses decrease from 81.4% to 62.5% between fourth and fifth grade. This decrease coincides with the increase of unrealistic responses in fifth grade, in which students solve the problems by giving the decimal quotient as the solution without considering the distribution context.

In the case of problem 1 type 1 as shown in Figures 4 and 5, on the one hand S4 applies procedure or algorithm of division properly, however it does not interpret the rest in terms of the actual situation of the problem. That is, the student forgets the text of the problem and gives as a correct answer the one found with the division algorithm, thus generating an incorrect result. For Verschaffel et al. (2009), students' weak performance in DWR problem solving is because they provide many mathematically correct but situationally inappropriate answers. Likewise, students present difficulties with problem situations about division. In addition, they require the activation of realistic considerations and sense-making to give an adequate interpretation of a non-integer quotient. Furthermore, Galvão & Labres (2006) posit that, children do not realize the meaning of the remainder in solution processes when it comes to divisions. In some cases, the remainder is seen as something superficial and not part of the problem's interpretation.

İncikabıa et al. (2020) showed that most students using the division algorithm successfully applied the operation steps but had difficulty interpreting the remainder. This result is interpreted from a clause of the didactic contract called formal delegation proposed by D'Amore and Martini (1997). According to the authors, solving a school problem coincides with finding the most appropriate operations, i.e., interpreting the text arithmetically and moving from natural language to arithmetic expression. The result of this operation is interpreted as the answer to the problem. At the end of this phase, the solver forgets the text and focuses on solving the operation.

Finally, in the S5 case at the time of solving problem 1 fails to understand the problem or identify the appropriate heuristic resolution strategy to solve it. That is, the fact of not understanding the problem leads the student to perform different arithmetic operations using trial and error but without reaching the correct solution. For Inoue (2005), students execute arithmetic operations without thinking, without evaluating their actions about common sense understanding of real-life practices. In that sense, Cooper & Harries (2005), when faced with a contextualized division with remainder, students fail to identify and use the appropriate operations. Likewise, S5 considers that multiplication, addition and division imply a larger number and for subtraction a smaller number without taking into account the understanding of the problem. For Jiménez and Ramos (2011), the incorrect beliefs generated by the didactic contract in the classroom seem to be responsible for the greater or lesser difficulty in solving the problems. In that sense, Parra and Rojas (2010) found that students perform any operation, following the rules learned in the solution of school arithmetic problems, in which the important thing seems to be to identify numerical data and use an arithmetic operation to give a numerical answer. In addition, As Rodríguez et al. (2009) state, students usually misinterpret problems from the beginning, and this initial error guides the entire solution process, including the interpretation stage. Also, students' difficulties in DWR problems did not seem to stem from the lack of interpretation of the correct numerical answer but the initial misunderstanding of the problems.

Conclusion

In elementary school, division problems with remainder (DWR) have been a source of conflicts or difficulties for students in finding the appropriate algorithm and making real sense of the problem. Likewise, the lack of contextualized problems in the classroom means that students are only prepared to solve routine problems. Lagos et al. (2008) argue that mathematical concepts must be perceived as useful in real life and that mathematics classes should favor understanding and reflection. For Inoue (2005), real-life knowledge plays an important role in mathematical thinking. In this sense, it is necessary for teachers' lesson plans to include this type of division problem with remainder and encourage students to use different resolution strategies.

In the problems of division with rest raised in the present research, it would be expected that the students of 4th grade of basic primary would solve these problems taking into account the following aspects: Read the problem carefully in order to understand it. That is, here the student must keep in mind the data or information provided by the problem and therefore the question that arises and the situation or circumstances that the problem presents. In addition, the student must create or design a path for resolution of the same, including heuristic resolution strategies. For example, mental calculations, trial and error, making a representation, outline or diagram, making a table and illustrations or drawings. Once this plan is executed and an answer to this problem is available, the student must relate the situation described in the problem with his reality. That is, the fact of relating it to your daily life will allow you to understand it better.

In that sense, teachers in their classrooms could present their students with different heuristic resolution strategies, making explicit the role played by the quotient and the rest in problems of this type. It is also suggested that students discuss among themselves and with the teacher the different meanings and interpretations that arise when solving this type of problems, specifically those of division with rest. The above could be useful to make it clear to the student that the realization of the algorithm in a strict way without making use of the interpretation of the result and reality is not always correct.

However, the analyzed results show that even though the student uses the division algorithm and correctly interprets the problem, it is still evident that some students use this algorithm without interpreting reality. It should also be noted that using arithmetic operations is not the only way the student uses to solve the problem. This is due to the use of strategies, such as the graphic strategy combined with skip counting or successive counting, where the student shows an answer coherent with the real part of the problem.

A possible limitation of our investigation would be the fact that we have not considered division problems in which, both in the dividend and in the divisor the figure of the units is zero. In the present work, problems related to the division with rest (DWR) were addressed. An example of this could be the following problem: "In a stationery store the employee needs to store 130 pencils in boxes with capacity for 20. How many boxes will you need?" One of the strategies that a student could have used would be to omit the zeros that appear in the quantities and therefore consider that the quotient and the rest would be respectively 6 and 1, showing with it the attachment to the algorithm (formal delegate) (D'Amore, 2006). However, the interpretation of the rest in realistic terms would be that the remaining 10 pencils would merit the use of an additional box, and therefore, the answer to the problem would not be 6 but 7 boxes.

Finally, it is suggested to present students with these types of problems to develop different strategies according to the context and the meaning given to the problem's solution. We consider that this would help them to give meaning to the procedures with arithmetic operations. For Ivars and Fernández (2016), giving students opportunities to solve problems using their strategies allows them to propitiate the confluence of their abilities with more formal approaches.

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A Case Study on Effectiveness of Online Teaching and Learning Mathematics: Teacher's Perspective

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Abstract: *The study on the effectiveness of teaching and learning mathematics online focused on teachers' perspectives on the experiences they face in real-life situations in the Nepalese context. This study examines the effectiveness of online mathematics teaching and learning in higher education during the COVID-19 pandemic. It is based on a qualitative exploratory case study design. The results of the study are based on the holistic views of eight persistently selected university mathematics teachers. It examines the effectiveness of teaching mathematics online covering four areas; teacher preparation and technological competence, the issue concerning students, access to resources, and the nature of mathematics. Teachers' technical competence and preparation for online lessons were rated as satisfactory. Similarly, student-related issues were found to be uncooperative and dependent on teacher guidance during online learning. Likewise, students were found to benefit according to their access to resources. Urban students have an advantage in online learning over remote students. Similarly, the results suggest that online math classes were perceived as difficult due to the nature of mathematics. Thus, the study concludes that higher-level online mathematics education has several determinants that make online mathematics education effective compared to face-to-face classroom instruction.*

Keywords: Nature of mathematics, Online teaching and learning, Teacher's perspective, Technological competency

INTRODUCTION

The teaching and learning process is considered to be started at the establishment of human civilization. In the early days, the Gurukul system of teaching was popular in Nepal. Gurukul is an educational system in which one is educated under the full guidance of a guru or teacher. The main aim of the Gurukul education system was to transform ethics, discipline, cultural values, and behaviors concerning religion through the teachers' role models. Religious and spiritual knowledge and real-life practice were the main parts of the study (Paudel, 2021). Gradually, changes have been made in the overall education system concerning the need and interests of the nation and its citizen. In the present context, the process of teaching and learning has been accelerated due to the prevailing context, innovations in subject matter, and modern technology (Kunwar et al., 2022). It is a difficult job in an open inquiry environment the teacher requires a variety of proficiency skills (Milin Sipus et al., 2022).

The emergent use of information communication technology (ICT) has also transformed the whole education system. It has been changing society gradually from analogous systems to digital technology (Martin, 2020). It is also connecting globally and leading to transform the entire system, and the process of digitalization has also been inspiring to transform everything in the world (Schwab & Davis, 2018). The digital revolution has made human life easier, faster, and more independent (Bello et al., 2021), however, it has also been separating people from physical contact and characterizing them as more mechanical or instrumental (Khanal et al., 2022; Kunwar et al., 2022). This digital evolution has been compelling people to disconnect from the real world into the virtual world. Online teaching and learning refer to interactive activities for instance student-teacher interaction, student-student interaction, student-content interaction, and student-technology interaction to provide meaningful learning experiences for students through the online system (Turley & Graham, 2019). It is a pedagogical modification in knowledge transformation with the help of digital technology.

In the area of education, the advancement of technology has been seen reflected in the execution of ICT (Hernández et al., 2016), which directly affects the plans of training and instructional processes in the different stages of education (Moreno-Guerrero et al., 2020). All such innovations lead to new and better learning experiences, by means of creating new spaces in which they perform teaching practices (Li et al., 2019). Society is moving toward the utilization of technology in their daily life (Garzón et al., 2020). The use of technology in society has transformed into a new phase in the life of all human beings (Khanal et al., 2022). Various aspects of human activity for instance work, society, and education have been changed due to infrastructural development and technological support (López-Belmonte et al., 2020; Moreno-Guerrero et al., 2021).

In the case of mathematics teaching and learning, it requires sufficient interaction and feedback. Mathematics education intends to promote activities related to meaningful teaching and learning, inspiring students through advancing the regeneration of intended mathematical concepts (Milin Sipus et al., 2022). Online learning also ensures interaction by means of technological tools like human-computer interaction, different forums, PowerPoint, blogs, online discussion groups, podcasts, media, live chat, and visual communication (Beldarrain, 2006). However, such technological tools cannot be enough to make sure for the process of effective learning and interaction to teach mathematics online (MacLaren, 2014). This suggests the maximum utilization of student-teacher interaction to reduce the feeling of loneliness in online teaching and learning and assist them to exhibit the stepwise solving process of mathematics problems in a digital setting (Karal et al., 2015). In mathematics learning, there is no constant interaction, it is accounted as a crucial tool to understand the immediate learning environment and characterize the various facts, whether technical, scientific, and/or social, that takes place in today's world (Williamson, 2018).

The language of mathematics permits to clarify with detail and accuracy the phenomena that occur (Kartal & Caglayan, 2018) in the usual events of life, which makes it possible to change all these facts into logical information and knowledge (Yagci & Uluoz, 2018). It implies that mathematics can be considered an instrumental subject because it assigns the foundations for knowledge acquisition in other fields of education such as sociology or political science that are very important (Lewis et al., 2017). Likewise, mathematics allows for the development of the student's creativity, intelligence, entrepreneurship, autonomy, or the enhancement of self-esteem and extends to the social and environmental aspects (Paechter et al., 2017).

The use of digital technology in teaching is not a new event in higher education (Kopp et al., 2019). The academicians seem positive to adopt digital technology for making them competitive in the prevailing context, however, they are lethargic to adopt digital technology in practice (Flavell et al., 2019). The proper use of digital technology can enhance student motivation and self-efficacy (Zhang et al., 2021). Students' interest and awareness of digital technology can highly contribute to the effectiveness of online learning (Dhawan, 2020). The effective use of online learning also depends upon contextual factors such as the willingness of the learner, infrastructural development, access to digital devices, and the learning environment (Adhikari et al., 2022; Khanal et al., 2022). In the present context, several underdeveloped and developing countries are facing problems concerning utilizing online classes due to limited ICT-related infrastructural development, internet connection, and human resources (Adhikart et al., 2022; Pham & Nguyen, 2020). Similarly, the students are also facing problems due to limited learning space in their homes and a disturbed learning environment for online classes (Khanal et al., 2022; Zhang et al., 2021). Therefore, such managerial aspects, learning environment, and access to learning resources are creating problems in implementing effective online classes (Adarkwah, 2021; KC, 2020).

Most of the students in higher education in Nepal are still unfamiliar with online learning. In the meantime, the transformation of teaching and learning such as a change in approaches, assumptions, methodology, and overall practice to another method, system, and perspective generally hampers many students and teachers in their teaching and learning process (Devkota, 2021; KC, 2020). Teaching online at the university level requires more preparation technically and pedagogically with necessary online resources for both students and teachers (Khanal et al., 2022; Mishra et al., 2020). In teaching and learning mathematics, different software related to mathematics can help to teach and learn effectively but the training about using such software has not been given sufficiently to both teachers and students (Adhikari et al., 2022; Khanal et al., 2022). In addition, the non-availability of regular high-speed internet, electricity, a well-managed classroom, and sufficient e-learning materials that facilitate students and teachers to interact with each other during class also pose a major problem (Devkota, 2021; Khanal et al., 2022). The inaccessibility of the proper resources to the students has also been increasing the education gap (Adhikari et al., 2022; Kunwar et al., 2022). The students should be attentive to every mathematical fact to develop new mathematical ideas for constructing new knowledge. Thus, the students should be regular, attentive, interactive, and motivated, however, the unstable electricity and an internet connection also have been causing the absence of the student in the online class.

LITERATURE REVIEW

In the Nepalese context, online teaching and learning can be considered a new experience. The rapid progression of technology helps to expand and offer online learning around the world and also makes it popular. Online learning is fully technology-based learning, so the growth of online learning depends on the development of technology. The use of online teaching and learning depends on accessibility, advancement in communication technologies, and other resources (KC, 2020; Mishra et al., 2020). The teacher should be completely familiar with how to use different software and online application for effective content delivery (Adhikari et al., 2022; Kunwar et al., 2022). It is one kind of pedagogical transformation in education. Thus, it forced us to rethink and redesign the total education system. Online teaching and learning has become a global avenue to assist students learning in a time of schools closing due to the COVID-19 pandemic. In Nepal, generally, schools and colleges are using different online web applications such as Zoom, Google Meet, Microsoft Teams, Skype, Viber, Messenger, WhatsApp, etc. to deliver the subject matter at a closer time of schools and colleges (Khanal et al., 2022). The proper situation for delivering online teaching and learning has not been prepared to this date in terms of the technological system, knowledge, and infrastructural development (KC, 2020). However, the repeated onsets of the COVID-19 pandemic compel the educational authority to adopt online teaching and learning forcefully. Thus, the full effort could not have been implemented for the effective delivery of the classes due to various causes such as poor physical infrastructures, different resources, and pedagogical viewpoints (Devekota, 2021; Kunwar et al., 2020). Likewise, student background, practices, teachers' traditional teaching habits, and school location have made it more challenging to implement online teaching and learning (Devekota, 2021; KC, 2020). Consequently, Devkota (2021) further states that the students who stay in the urban region and have well access to digital assets and internet services have more advantages in comparison to their rural colleagues and this condition has been generating a matter of inequality in the subject of quality, uniformity, equality, and validity. On the other hand, both the teachers and the learners are also low motivated and anxious to active participation in teaching and learning due to the fear and terror of the current COVID-19 pandemic situation.

Thus, online teaching and learning became a forced pedagogical shift from the conventional method to the modern approach due to the emergency of COVID-19 (Kunwar et al., 2022). It also forced the educational institution to provide teacher training for its implementation and to the government for the infrastructure development, connectivity of electricity, and extension of internet facility to get access for each learner in the country. In the diversified geopolitical condition of Nepal, there is no equal access to stable connectivity (Devkota, 2021; KC, 2020). Access to resources has a greater impact on virtual teaching and learning in higher education in Nepal (Adhikari et al., 2022; Khana et al., 2022; Kunwar et al., 2020). In the same way, the unstable electricity and internet facilities hamper both the students and the teacher from their regular and active participation in learning (Adhikari et al., 2022; Kunwar et al., 2022). The facility of internet and electricity are mostly centered in rural areas (Devkota, 2021). Such disparity affects not only the implementation of online education but also impacts the widening of the education gap.

A survey report conducted in Indrawati Rural Municipality among school students and parents revealed that less than 10% of the students had access to the internet in their homes (Baral, 2022; KC, 2020). This scenario probably represents the existing condition of internet access in most of the rural areas of the country and access to digital devices as well (Dawadi et al., 2020; Laudari et al., 2021). An economic survey report of Nepal conducted in 2020 showed that 8366 schools have computers among the 29,707 public schools (MoF, 2020). Also, it is found that 12% of the public schools out of 29,707 found to be capable to offer information technology-based teaching with internet connectivity (Baral, 2022). The government of Nepal has already launched the National ICT policy 2015 focusing on developing ICT literacy through integrating ICT in classroom activities (MoCIT, 2015), and it was also emphasized in the school sector development plan 2016-2023 (MoE, 2016) to integrate and implement the current technologies into classroom practice (Joshi et al., 2021). However, the lack of infrastructural development, connectivity, and ICT-based teacher training, the proper implementation of ICT in the classroom has not been found satisfactory level (Rana et al., 2020). As reported by Adarkwah, (2021), people's feelings toward online learning, technological skills, and personal skills can hamper effective online learning. In the context of mathematics teaching, these constraints may or may not impact mathematics online classes however, student participation is a key influencing factor for their achievement in an online setting (Ayouni et al., 2021).

In Nepal, most urban private schools have been adopting the online teaching and learning approach as an excellent transitional remedy for the teaching and learning crisis produced by the pandemic (Khanal, 2020). Some of the urban public schools also have been implementing online as an alternative approach. All the universities have also provisioned an online teaching and learning system to deliver their courses (KC, 2020). Thus, most of the academic institutions in Nepal have started to teach an online mode to make it viable by replacing it as an alternative method to conventional teaching and learning approaches (Khanal et al., 2022; Paudyal, 2020). The immediate closedown of schools and colleges has enforced and also obliged the education institutions to think, manage, and organized a different process of education delivery and reforming pedagogy (Adhakari et al., 2022; Pokhrel & Chhetri, 2021). However, the sudden incident influenced each education policymaker, school principal, and teacher to rethink and look for an alternative to the face-to-face mode of instruction for assurance of children's education rights and enforced them to move fast toward the online mode of instruction (OECD, 2020). Almost educational institutions in Nepal have shifted forcefully into the online mode of teaching however it was new practice and experience for many of the teachers and students (Adhikari et al., 2022; KC, 2020; Kunwar et al., 2022; Paudel, 2021).

On this backdrop, the article investigates the effectiveness of online teaching and learning mathematics at the bachelor's level concerning teachers' perspectives they face while implementing online teaching and learning during the time of the COVID-19 crisis. The study is mainly concerned with the lived experiences of teachers teaching online and face to face, their students' responses about the effectiveness of content delivery, and the transformation of learning through the online classroom. Specifically, it explores the self-assessment of the teacher about the effectiveness of online mathematics teaching, adopting technology into the online classroom, and their students' participation, progress, and responses. Considering the above prevailing context in mind, it provides insight into the real ground of online instruction at the university level in the crisis times of COVID-19. This research intends to identify the teachers' perspectives regarding teaching mathematics through the online mode in Nepal. The study is based on the following research questions:

i) What are the problems encountered by mathematics teachers in teaching mathematics online?

ii) What are the experiences of the teacher about implementing an online mode of teaching mathematics?

METHOD

This study is based on the teacher's perspective on the effectiveness of online mathematics teaching and learning. The study is centered on the descriptive research method with a qualitative research approach. It is based on a qualitative exploratory case study design depending on the responses of the mathematics teachers teaching mathematics classes at the bachelor level.

Participants

In this study research, purposive sampling was used to describe and interpret the teacher's perspective on the effectiveness of online teaching and learning mathematics at the bachelor level during the COVID-19 pandemic. In this study, eight mathematics teachers teaching at the bachelor level in the faculty of education, and humanities and social sciences, in Mahendra Ratna Multiple Campus Ilam, Mahendra Ratna Campus, Kathmandu and Sanothimi campus, Bhaktapur were chosen as the participants. One teacher of the management faculty who teaches statistics was chosen as the participant. Similarly, two mathematics teachers from the humanities and social sciences faculty and the remaining eight mathematics teachers from the faculty of education were chosen as the participants for the study.

The participants belonging to this study were based in the urban area of Ilam Municipality, Province 1, Kathmandu Metropolitan city and Bhaktapur, Bagmati Province. Also, there is a linkage between the participant and the background of the college students with whom the participants deliver online teaching and interact with them. So the geophysical background of the student is directly linked with the feeling and perspectives of the participant. In this study, the participant interacts with the majority of students that come from the rural areas of Ilam and other neighboring districts. In recent years, a very low number of students are found to be admitted with majoring in mathematics in the overall college and universities in Nepal. This study is limited by the small sample size of mathematics teachers teaching at the bachelor level.

Participant	Teaching experience	Age	Workload	No. of Student
A	3 Yrs.	28 Yrs.	12 periods per week	11
B	16 Yrs.	37 Yrs.	24 periods per week	13
C	28 Yrs.	58 Yrs.	27 periods per week	10
D	18 Yrs.	40 Yrs.	24 periods per week	12
E	25 Yrs.	51 Yrs.	15 periods per week	14
F	22 Yrs.	48 Yrs.	12 periods per week	21
G	24 Yrs.	52 yrs.	12 periods per week	21
H	27 Yrs.	57 Yrs.	12 periods per week	24

Table 1: Participants Details

Instrument

In this study, the investigator used the in-depth interview as the instrument to explore the data about the case of the specific phenomenon. Since the study is based on a descriptive qualitative case study design, the teacher's perspective regarding online mathematics teaching and learning activities was explored by employing an in-depth interview. Thus, the data were based on the interviews of the teacher concerning their classroom experiences, students' participation, access to time and resources, and completion of class work of the students, etc. focusing on the holistic experiences of the mathematics teacher and what they face in the situation. The participants were interviewed based on the research questions regarding the effectiveness of online teaching and learning mathematics individually.

Data Analysis

The study uses the different stages of obtaining qualitative descriptive data before formulating the results as transcribing, initial coding, analyzing codes, identifying themes, and data presentation. The views of the participants regarding mathematics teaching online after transforming into consistent themes have been analyzed using some particular statements. Such self-reported views of the teacher about their classroom practice or online instruction can reflect the actual situation regarding the problems of teaching mathematics online in higher education. Thus, the actual problems of online teaching mathematics were analyzed thematically on the basis of four themes. The process of analysis was rooted in the eight different participants' experiences, feelings, observations, and opinions about the actual practices and their differences in views on meaningful learning. It was especially focused on describing teachers' observations, perceptions, and self-practices in the classroom context. The details of the problems regarding mathematics teaching online are presented in Figure 1 as the thematic framework of the study.

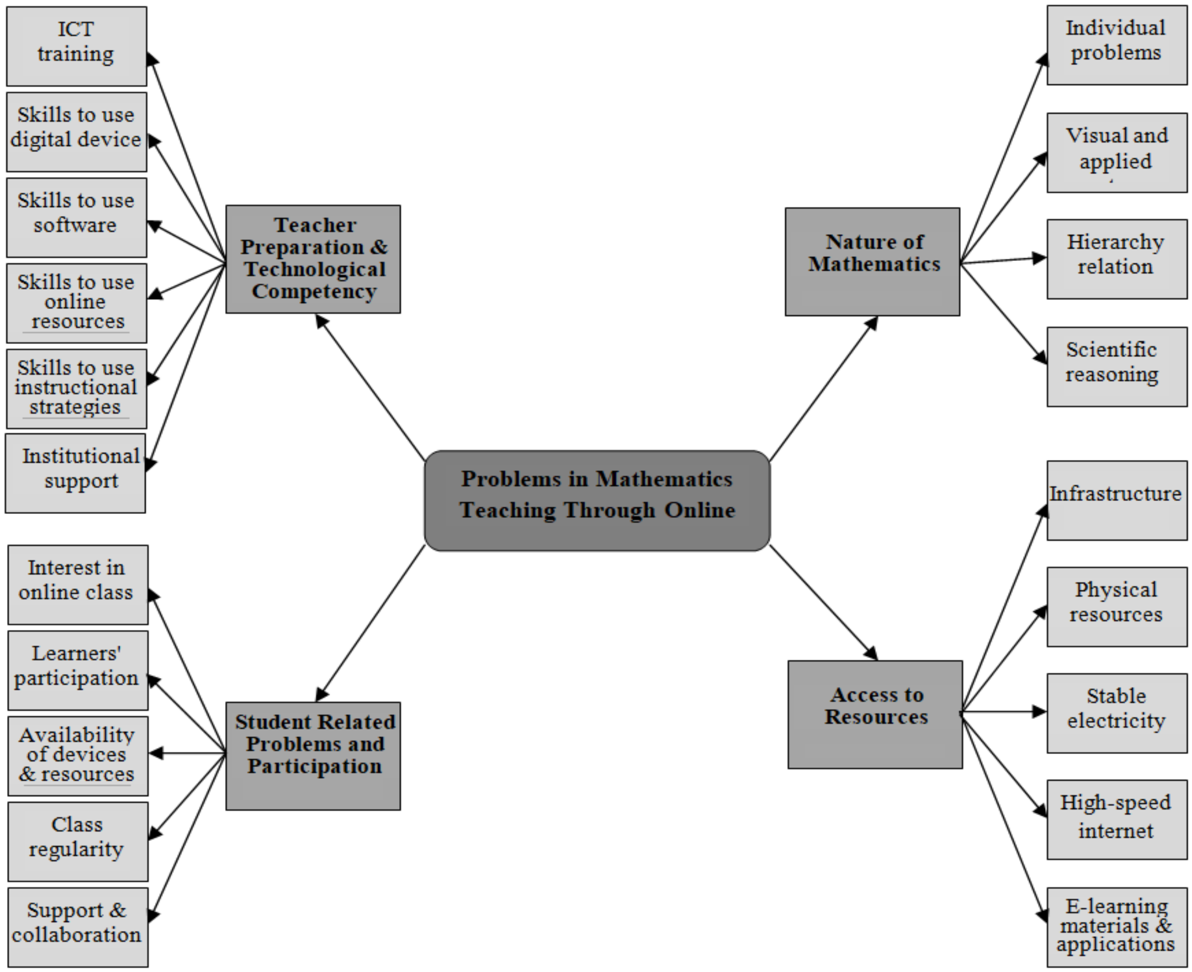

Figure 1: Thematic Framework

RESULTS

The results of the study are mainly based on the teachers' knowledge about their students' mathematics learning activities during online classes in three different faculties; humanity and social sciences, management and education such as student motivation level, collaboration and cooperation, student performance, class activity, class work, and the completed mathematical tasks. The study mainly focuses on four main thematic areas based on various sub-themes. The results of the four thematic areas are discussed in separate headings.

Theme 1: Teacher Preparation and Technological Competency

Theme one illustrates how effectively the teachers prepared the lesson and implemented it by combining technological tools. This theme mainly focuses on the effectiveness of teacher preparedness and skill in delivering lessons in both synchronous and asynchronous formats. Teacher preparation and technological competency are the most essential aspects of the teacher for the effective execution of online teaching-learning. Teacher preparation for an online class is more time-consuming and skillful work in its place in a physical classroom setting. The teacher should have sufficient knowledge and skills regarding the use of new technologies and plan for instruction in innovative ways to carry on students engaged. Only expertise in the subject matter does not mean effective teaching, it requires professional knowledge for delivering content efficiently using technology. Mathematics teachers must have extensive expertise in teaching mathematics content and language to bring mathematics ideas live in online teaching (Khanal et al., 2022). Similarly, online lessons require teachers to have technical knowledge with content knowledge and educational content knowledge (Chand et al., 2020). The teacher's creativity with good communication skills is also an essential part of online teaching that can help the student to retain information and engage them more effectively. An online class allows live virtual instruction, online assignments, project work, and virtual activities that permit them to learn new things.

In this context, three participants shared their views: *"We all teachers do have not the same level of technological competency to use appropriate mathematical software as well as a basic skill of handling online class*es *while delivering subject matter. We are facing several difficulties regarding technical knowledge due to the absence of prior knowledge and skills in this field."* But the online teachers need to be perfect to handle online tools and technology so that they can help their students when they asked. Thus, the teacher should be up-to-date in both knowledge and skills regarding the use of the latest technology.

Likewise, concerning the skill-related topic that is based on different drawing, construction, and conceptual understanding, two participants viewed: *"Some topics that required more conceptual understanding and skill related topic like construction, it takes more time to teach through online class and also very difficult to make a clear concept about the topic than the physical class*." In virtual classes, the teacher should more dynamic and expert in the subject matter and way of interaction too. In such conditions, the teacher can perform the class in exciting ways to keep students engaged with skill-related problems. Sometimes they should go beyond the subject matter to support the students individually according to their needs, interest, and background.

In addition, at the beginning stages of the virtual class, the teacher should spend more time preparing different teaching resources. They should also engage in the teaching lesson, providing support to the students, reviewing and clarifying the students' confusion about the contents. Since the use of online teaching is newly applied, thus the teachers are fully engaged to prepare their lessons.

In this regard, four participants expressed about over workload: *"The duration of 50 minutes physically single in a room and the over workload due to preparing resources and only virtual interaction sometimes makes very tedious while delivering the lesson."* An effective virtual class requires more preparation time, and dedication to build up the necessary skills for each teacher.

An online teaching journey is a more patient and time-consuming task for each educator to get success in teaching.

On the other hand, most of the students studying at this level are from diverse locations. Particularly, in remote areas, there is not enough connection to electricity and the internet which also hamper both the students and the teacher to implement virtual teaching and learning. Even in urban areas, there is no stable electricity and internet connection.

The sudden change of delivering classroom learning from physical face-to-face to online mode of teaching has created big challenges for both students and teachers due to various causes such as access to devices, the internet, appropriate physical learning space, and the learner's habit of learning. Regarding unstable network connection and electricity, three participants said: *"It is more challenging. ……sometimes only the teachers are crying alone on their computers or smartphones. Their students have already disconnected due to either caused by electricity or internet connection."*

Online teaching at the university level needs more technical as well as pedagogical preparation to use effectively different online educational resources and applications. It is also necessary to make students able to do online activities, project work, and provide more opportunities to learn beyond the class activities. In this regard, six participants shared their views regarding low habituation, resources, and infrastructure: "…..*some students could not find themselves engaged in such tasks rather than making notes from the teacher's discussion and presentation. They found less engagement in virtual class rather than the physical class."* The learning environment shifted from classroom to home learning environment creates another concern for students. Due to no access to electricity or the internet and disrupted learning environments by noise and other disturbances, students' active participation in online learning can be hampered. Also, the habituation of physical classroom learning habits can cause feeling uneasy to adopt a new learning situation.

Theme 2: Student Related Problems and their Participation

Theme two depicts students' participation in online class learning in terms of attending class regularly, timely engagement in learning activities, and participation in discussion. There were several other student-related issues, such as Internet connectivity, digital devices, and resources that also affected learner participation. Likewise, learners' readiness to learn, interest in online class participation, and support from the institution and parents are the necessary aspects of online teaching and learning. Online teaching and learning also require student collaboration such as peer collaboration, and student-teacher collaboration which makes student-centered learning (Chand et al., 2020). Concerning students' participation in online classes regularly and punctually, the participants expressed bitter experiences. Some student does not join the online class at a sharp time due to different causes like unstable electricity, the internet, or their interest. High levels of motivation of the student toward learning mathematics create more engagement opportunities in learning and hence help to improve students' skills and outcomes. Similarly, the unavailability of basic online resources and the appropriate personal learning space for the student at their home also make hamper attending timely and regular online learning activities.

In this regard, three participants shared their views: *"Some of the students mostly join late in the learning platform 'Microsoft Teams' and also irregular. These students generally do not participate actively in the learning process. Some students like to of their video and sound too, and we cannot observe their learning activity."*

The causes of interruption and irregularities can also depend upon the student's attitudes, skills related to the use of technology, and other external barriers. Students feeling toward online learning, loneliness, and limited access or no access to resources due to living in rural areas can also hinder them to participate actively in the online learning platform. In the issue related to students' active participation in the teaching-learning process in the online class, three participants expressed their views as such: *"The student who is irregular and always joins late in the learning platform has the same answer and always tries to escape from the teacher. Such students always lack discussion, class work, and doing exercise"*. Particularly, students from remote areas mostly do not present at online classes timely and regularly and the teacher cannot see physically what they are doing. Sometimes the teacher should wait for them a long time to ask and teach something related to the lesson.

In the same way, two participants expressed their views in different ways regarding the student's active participation in an online class: *"....self-directed and motivated students always take active participation in virtual class also, but the poor and lazy students try to avoid the class frequently". Even, they do not respond to the chat written by the teacher."*

The habituation of the traditional teaching style can have an impact on teaching and learning mathematics in online classes. In the physical class, students usually copy the problem solved by the teacher, and the teacher also uses the talk and chalk method. The student can feel difficulty in an online class to contact and interact with their teachers to seek help in the difficult learning concept and topics due to their habituation in the traditional physical face-to-face class. In this regard, three participants viewed: "*…the poor students often look passive or less engaged in learning mathematics. They mostly depend on their teachers or they put all of their focus on their teacher's activity and instruction. They like to listen to their teacher; they often do not like to collaborate in the learning activity."*

They often like to express their queries orally. Most of the students do not communicate actively in an online class. The cause may be poor communication skills through technology or causes of low-speed internet and other obstacles. However, such obstacles led to challenges in interacting, communicating, and engaging for effective online classes.

Theme 3: Access to Resources

The third theme, access to resources mainly comprises four different aspects. 1) Access to physical resources such as a separate well-managed online classroom, and digital resources (personal laptop, Smartphone, and writing pad). 2) Access to stable electricity. 3) Access to stable high-speed internet. 4) Access to different e-teaching and learning materials and applications 5) Access to training for operating devices and different programs. Such access to resources can make the delivery of online teaching-learning easy and effective for both students and teachers.

In this regard, seven participants viewed personal digital resources: *“Most of the students do not have their laptop for the online class, and they use their Smartphone. They can listen to every activity properly but face difficulty in seeing figures, charts, and other small lettered presentations due to small screen."*

Some of the students located in remote areas have limited digital devices and also do not have access to stable electricity and internet facility.

In this concern, six participants shared the same experiences about access to electricity and internet facility: *“Very few classes have been conducted, joining with all the students throughout the class time. Otherwise, mostly, someone else has faced the problem of electricity, internet, low-speed internet, or internet connection problem. Such incident has been happening, sometimes only students are found discussing in the online class and the teacher could not join….."*

Generally, urban students have better access to digital devices and the internet. Most of the students in mathematics class are from remote areas, and they are from varied geographical locations.

On the topic, of student class absentees due to geophysical situations and other disturbances, four participants remarked: *“The incidental absence of the students in the online class also hampers the other students because the absent student asks about the previous lesson, and it is also necessary for the teacher to clear the previous class. So it takes more time to complete the current lesson.”*

A large number of students from remote areas have been studying higher education. Some of the students still have out of reach to telephone and electricity.
In this regard, two participants argued: *“Sometimes, the internet does not work properly for 2-3 days and could not conduct the class. Particularly, in summer, the problem of unstable electricity creates more problems in online teaching and learning.”*

In mathematics teaching and learning, access to different e-teaching and learning materials and the relevant software plays a significant role. However, due to the geographical location, it is also difficult to find timely such materials.

In this regard, four participants shared their views: *"...such e-learning material has a significant role in teaching and learning mathematics for the students who stay in the remote areas and the students of urban areas as well, however, we could not able to provide them such relevant resources and applications."*

Thus, it is comparatively more difficult for students from remote areas terms of getting resources and acquiring skills for using the online application than urban students.

Theme 4: Nature of Mathematics

The specific nature of mathematics comprises diverse disciplines and deals with the logical analysis of inferences, events, truth, and observations using different mathematical procedures and models. It is used to explore natural phenomena, human behavior, and complete social systems. It is considered a much more difficult task to work alone and self-study to grasp the mathematical concept. The nature of mathematics requires a high level of symbolic and schematic representation in mathematics education (Chand et al., 2020). Also, online education requires gestures and multimodal communication. Thus, it is necessary extra skills in handling mathematics classes for the teacher to create a proper learning environment when teaching online. The nature of mathematics is hierarchical and each topic is related to another. Each individual student also has difficulties with different concepts and problems related to mathematics. So well, conceptual knowledge of the subject matter is needed before moving to another topic. Also, mathematics has a visual nature and it is described as visual art. Hence, it is difficult to create a proper learning environment and address such nature of mathematics in remote teaching. Mathematics can learn effectively through group work and collaboration with face-to-face engagement due to its logical and skill-related nature. In this study, the theme, the nature of mathematics has been discussed on the student's individual problems regarding learning mathematics, the visual and applied nature of mathematics, hierarchical relation, and scientific reasoning.

Online learning has flexible characteristics, there is no need for a fixed time and schedule for teaching (Hassan & Mirza, 2020). In this concern, two participants expressed their views regarding the nature of mathematics. "*…..due to the abstract nature of mathematics, it is very difficult to make concentrate the students for learning mathematics. The teacher cannot pay equal attention to the students which distracts the weak and passive learner, and they lag behind day by day and only the active learner gets more chances and benefits from learning mathematics*." Most students feel that learning mathematics a harder to work on individually, however, it is considered easier in group work.

On the other hand, the nature of mathematics is visual (Ní Fhloinn & Fitzmaurice, 2021). So they need to see as well as hear their learner properly to ensure when teaching and learning online. Direct one-to-one relation is necessary for effective learning of mathematics due to its abstract and applied nature. In this regard, two participants claimed that "*mathematics combines different* *models, theorems and applications to establish relations and solve different practical problems in different fields of education which necessitates one-to-one relations between the learner and the instructor. However, it is very difficult to establish one-to-one relationships with weak students in online classes. They usually do not ask questions or discuss the topic, and they only like to be a passive listener. It may be caused by the unavailability of proper learning space at their home."*

In mathematics, the development of some ideas is considered a fundamental aspect of the discipline and is regarded as very crucial. Equally, it is also difficult to create a learning environment to develop such ideas and knowledge by just talking in front of the laptop screen. Each and every topic of mathematics exist a hierarchical relation, the new topic can learn only after getting the complete knowledge and skill of the preceding one. In this regard, three participants argued: *Each topic of mathematics is so interconnected that no topic can be taught after getting the proper understanding of the previous one. Due to the irregularity of the students in the online class and the student's individual differences, it is very difficult to move to a new topic or very difficult to deal effectively with each next topic, especially for weak students. The continuation of the problem of hierarchical knowledge of mathematics has been increasing the gap between weak and smart students and de-motivating learning mathematics for weak students.*

Mathematics is a subject of scientific reasoning. Clear knowledge, skills, and ideas are needed for each mathematical term that enables the learner to think logically as well as critically. Scientific reasoning comprises different skills such as inquiry, experimentation, evaluation of the evidence, argumentation, and implication. Such skills help in the formation of concepts and theories regarding mathematics. It is also a capacity to understand and generation of scientific knowledge that is required in learning mathematics. In this concern, two participants argued that "*the study of mathematics underpins the use of inductive reasoning and axiomatic method, it begins with specific facts, symbols and particular applications following the universal laws. Individual students need logical thinking and scientific understanding that comes from interaction, logical argument, and cognitive resources while solving the problem. However, it is very difficult to manage such situations in an online class."* Thus, due to the nature of mathematics, each student should enable to understand and be familiar with each concept, statement, and procedure before solving such problems. However, most of the students have difficulties in solving such problems due to the absence of scientific reasoning skills.

DISCUSSION

The study has been carried out employing the explorative case study research design. So the evidence explored by the participants also supports the objective of the study and overall understanding of the situation. In this section, the results obtained from in-depth interviews are discussed on the basis of major themes and initial codes related to the teachers' perspectives.

The results based on the theme 'teacher preparation and technological competency' their expressions show that the unequal technological competency and basic skills of the teacher in handling the online class have been affecting effective online class delivery. As claimed by Panthi and Belbase (2017) empowering or disempowering the students in the classroom learning mainly depends on the teacher's pedagogical skill through their actions. Thus, it seems to improve the teachers' teaching skills to enhance effective online class delivery. Also, it may be the cause of the poor experience in using such technological tools in teaching mathematics (Adhikari et al., 2022; Rana, Greenwood & Fox-Turnbull, 2019). This may be the cause of creating low motivation in the students toward online teaching. However, teaching online possesses enough technological skills to access a range of technological tools and resources, teaching and learning capabilities, and limitations of the tools and technical potentiality for effective class delivery (Albrahim, 2020; Khanal et al., 2022). Similarly, the topics related to skills and conceptual understandings were found difficult to deliver effectively in class due to the absence of proper skills and time limitations (Khanal et al., 2022). Lack of such skill not only makes discomfort for the teacher but also creates low motivation for the students towards learning mathematics. This finding is consistent with the finding of Kunwar et al. (2022). So each teacher needs to be familiar with the use of software and different mathematical applications. Teachers are habitual to use a learning platform, Microsoft Teams but are not perfect to use different mathematical software to teach mathematics (Kunwar et al., 2022). Also, as concluded by Joshi et al., (2020) university-level teachers have less practice in online teaching. The majority of the participants' views correspond to over workload due to preparing materials and multiple roles and responsibilities. Conversely, the over workload of the teacher affects the delivery of quality education focusing on the needs of learners (Pacaol, 2021). This saying also shows that the teachers are also feeling bored and are not curious about their job. In such a situation, the teacher cannot entertain delivery spontaneously and cannot deliver the subject matter effectively.

The participants' responses regarding unstable network connections and electricity, it has also been creating a major problem to conduct the online class. According to Devkota (2021), different students from rural areas of Nepal are still experiencing disconnection from higher education due to not having the required physical infrastructure of electricity and internet access. Thus, the unavailability of high-speed internet and electricity has also been creating problems in the effective delivery of online classes (Adhikari et al., 2022; Dawadi et al., 2020; Baral, 2022; Khanal et al., 2022; Laudari et al., 2021). Therefore, it is necessary to focus on infrastructural development and extension to remote areas with a high-speed and stable internet connection to address the problem. The results obtained from the theme 'student participation in online class have been discussed focusing on regularity, punctuality, engagement in-class activities, and participation in discussion. Mathematics education is an interesting, relevant, and applicable subject (Milin Sipus et al., 2022). So the participation of the student in the virtual class is considered most important because it reinforces the teacher for effective teaching and also enables the students to acquire more knowledge and skills through the discussion between teacher and students, more engagement in the classroom, and sharing the idea among them. So, the students should be observed regularly, whether they are interacting with each other during the class or not. However, the result is found opposite. Student interaction in online classes was found comparatively low. This finding is similar to the finding of Khanal et al. (2022) and Mamolo (2022) that students do not take part actively in online classes while teaching abstract mathematical concepts.

In the case of the students' punctuality and regularity, it is found that some of the students do not take part actively, i.e. some join late, try to escape to participate in class discussion, and do not take part in class work an online class. Therefore, encouragement to the student is needed to create a conducive learning environment that stimulates the student to be actively involved in the classroom (Bringula et al., 2021). According to Milin Sipus et al., (2022), the teacher creates numerous creative moments by generating opportunities and challenges to make their learner able to face the situation and to support the productive exchange of mathematical ideas. Similarly, it is difficult to increase cognitive as well as behavioral participation in virtual classes in comparison to the face-to-face real classroom setting (Kunwar et al., 2022). In terms of student punctuality and regularity, similar results are observed by Yusuf and Al-Banawi (2013) and Bringula et al. (2021) that there are some obstacles in the process of online learning, such as less motivation, delayed feedback, sometimes not available of the teacher or student in class and isolation feeling due to lack of physical face-to-face class. Novljan & Pavlin (2022) argued that the student can acquire new knowledge and skills outside the physical class too. It indicates that students can be taught effectively by creating an inspiring learning environment in an online class.

The habituation of the teacher in traditional teaching and learning has found an impact on both students and teachers in online teaching. Traditionally with teacher-centered learning habits, students were also found to be passive and dependent on their teacher (Poudel, 2021). They were also found to prefer the traditional teacher-centered method to collaborative learning. It may be the postback effect of traditional learning. Similarly, teachers were also found the same (Kunwar et al., 2022). This may be caused by low levels of technical skills and a lack of collaboration among themselves both the students and teachers regarding online teaching and learning. According to Sun et al., (2017), habit is the human characteristic that acts to make easy our cognitive load while making the decision. So it will come spontaneously while performing the behavior.

The results of the theme 'accesses to resources' have been discussed in different four aspects: access to physical resources, stable electricity and stable high-speed internet, and different e-teaching learning materials and applications. The result shows that every student has no personal computer, and they use their Smartphone in online classes. So, they can listen to the teacher in the class but do not take part actively in the learning process (Kunwar et al., 2022). On the other hand, access to electricity, internet facility, geophysical situation, and other disturbances have been found to create problems of irregularity and absence for both student and teacher which cause to hamper the effectiveness of delivering the online class (Devkota, 2021; Poudel, 2021). This also confirms the result of (Adhikari et al., 2022; Khanal et al., 2022). Thus, online teaching was found more time-consuming due to frequent revising of the lesson for an absent student who could not attend the class by unstable electricity or the internet. On the other hand, it is a very tedious process due to unnecessary hurdles like unstable internet and power supply (Kunwar et al., 2022). The students in urban areas who have better access to resources are getting more benefits; conversely, the students in remote areas who have low or no access to resources are losing their opportunities. This result also conforms to the results of Devlota (2021) and Adhikari et al., (2022) that the students who stay in an urban location and have access to the internet, electricity, and resources have benefitted more than the rural students. The inaccessibility of such resources to students has been increasing their gap in education ((Kunwar et al., 2022; Poudel, 2021). Therefore, as suggested by Lin et al, (2021), more attention should be given to the learners in online learning to address their needs and enhance the value of overall learning (Lin et al., 2021). It was also found that students have been facing difficulty to study at their home due to the improper learning environment such as small spaces, noise, and distractions (Baticulon et al., 2021). Thus, it is a great consideration about selecting the best method of teaching mathematics online that can support successfully and bridge the gap between the learners.

In the theme nature of mathematics, individual problems of the students in learning mathematics, the visual and applied nature of mathematics, hierarchy, and scientific reasoning were discussed. It was found that online learning has been creating a gap between learners. Also, it was found very difficult to make the concentration on the weak learner due to the abstract nature of mathematics. This result is similar to the result of Ní Fhloinn & Fitzmaurice (2021) that teaching mathematics remotely is much harder and also difficult to address the weak student. The nature of mathematics is visual and can be taught effectively through creating one-to-one relations with the students. The result showed that it was very difficult to treat individually the student in an online class in comparison to a physical class. This result also confirms the results of Ní Fhloinn & Fitzmaurice (2021). They claimed that students do not learn mathematics effectively only through self-study, but they need individual support in each step while solving problems. Each topic of mathematics exists in a hierarchical relationship, and no one can learn the next topic without acquiring the knowledge and skills of the previous topic. In this regard, it was found that irregularity in the class due to various causes like electricity, unstable internet, and the student's motivation toward learning online class has been disturbing learning mathematics effectively and the student absenteeism in a single class has been affecting the next class or topic due to hierarchical nature of mathematics (Ní Fhloinn & Fitzmaurice, 2021). This result also aligns with the results of Khanal et al., 2022: Devkota, 2021). Also, the scientific reasoning which is applied in mathematics learning is found difficult to develop in the learners through online learning. Because, scientific reasoning can be developed through interaction, observation, and logical argument in the group which is difficult in online mode. This result also aligns with the results of Ní Fhloinn & Fitzmaurice (2021).

CONCLUSION

The pandemic situation emerged by COVID-19 has caused the closure of educational institutions and shifted the physical face-to-face teaching-learning into virtual classes. In this situation, there are two options, either close the institution or implement online teaching and learning. It was a sudden change in pedagogy or a force paradigm shift in teaching and learning (Kunwar et al., 2022). It is a more complex task to manage and conduct effectively in the current pandemic situation. This paper has attempted to find out the effectiveness of online teaching and learning mathematics in higher education. This study has explored the results through the mathematics teachers' viewpoint that what they exactly face in the situation.

The study extracts the conclusion from the four thematic areas of teacher preparation and technological competency, student-related problems and their participation, access to resources, and the nature of mathematics. Since the change in pedagogy is sudden then, the condition of teacher preparation and technological competency in terms of ICT-related skill and knowledge was not found high. Student participation in online learning was also found less engaged in learning mathematics and mostly dependent on the teachers' activity and instruction. They were found more engaged in individual tasks, but with less effort in collaborative work. Similarly, there is greater variation in terms of access to resources between the students of remote areas and urban areas. Most urban students found benefited more due to the access to electricity, the internet, and other e-teaching and learning materials and mathematical applications than the students of remote areas. According to Albrahim (2020), online teaching entails trained and skillful teachers, especially in mathematics due to its nature. However, the study found a gap in technological knowledge and skills among the teachers for an online class in higher education in Nepal. The results, thus obtained, can be specified for the initial understanding due to incorporating the small size of the sample and the single dimension of the data. Thus, the result demands a more in-depth study involving a large sample size and multi-perspective for more generalizable results. Thus, the study concludes on the basis of the insights drawn out from the mathematics teacher teaching online mathematics at a higher level of mathematics has not been executed effectively in comparison to physical face-to-face classroom teaching.

Similarly, the study recommends that teachers need to have continuous professional training to acquire the proper knowledge and skill in using different mathematical software for effective virtual class delivery. Deliberate and consistent efforts should be made by the teachers to attract the students to the virtual class and to make them more active, engaged, and keep continuing their participation and collaboration in the class and beyond. Infrastructural development and connectivity should be developed for stable electricity and the internet.

Conflict of Interest

It is declared that the authors have no conflict of interest.

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Social Time as a Pedagogical Toll for Meaningful Mathematics Teaching and Deeper Learning

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Abstract: The study reported in this paper was conducted to examine four South African grades 7 teachers' understanding of the importance of instructional time as a teaching resource to develop learners' relational understanding in mathematics. A constructivist philosophical approach, document analysis, lesson observations and interviews were used to collect data. The findings indicate that the four teachers faced, to varying degrees, challenges in using instructional time as expected. Two did not help learners understand the relationship between the concepts taught and the examples selected to scaffold learning. They prioritised drawing learners' attention to correct responses to questions posed. Only two teachers used strategies that encouraged the learners to share experience, collectively reflect on individual taken-for granted conceptions, probe and identify how they could be used to explain mathematical concepts. The conclusion is that the teachers' lack of understanding the pedagogical significance of instructional time as social time highlighted inadequate curriculum and subject content expertise and supported the general concern about the quality of mathematics teaching within the country.

Keywords: *mathematics, instructional time; teaching; relational understanding*

INTRODUCTION

In South Africa, all schools, private and public, use the National Curriculum Statement (NCS) Grades R-12, as official curriculum policy (Department of Basic Education [DBE], 2011). The NCS comprises three policy documents that guide teaching and learning, namely, the Curriculum Assessment Policy Statement (CAPS) for each approved school subject; the National policy for the promotion requirements of the NCS for each programme and the national protocol for assessment Grades R-12. These policy documents are central in indicating, respectively, what should be taught and learnt in a particular grade, how subject content is organised for the different phases and associated grades and how learners progress to the next grade within the schooling system and finally, how the taught and learnt content should be assessed. For example,

mathematics has to develop a critical awareness of how mathematical relationships are used in social, environmental, cultural, and economic relations (DBE, 2011). Problem-solving and cognitive development should be fundamental to all mathematics teaching and based on content that is grade appropriate. Teachers are expected to create environments in which learners feel free and value each other's contribution. Classrooms as judgment free zones have to facilitate learning in ways that are meaningful to cultures or concepts used routinely in the learners' lives. Learners should be treated equally irrespective of their socioeconomic status, physical conditions, and intellectual ability to ensure the attainment of the set outcomes.
It is thus reasonable to describe the teaching approach that is promoted by the NCS in mathematics and the CAPS as supporting Realistic Mathematics Education (RME) – a mathematics teaching approach that is supposed to make mathematics learning fun and a meaningful experience by introducing situational problems from contexts within the learners' experiential realm (Loc & Hao, 2016; Yet et al., 2017; Yuanita et al., 2018).

RME as a teaching approach

This approach was developed in 1971 by the Freudenthal Institute in the Netherlands (Loc & Hao, 2016; Theodora & Hidayat, 2018; Yet et al., 2017; Yuanita et al., 2018). It emphasises the need to provide learners with opportunities to construct mathematical knowledge through managing and processing real-life situational problems. For example, Yuanita et al. (2018) describe five main criteria that guide the utilisation of the RME approach, namely (i) acknowledging learners' experience in daily living, (ii) modelling reality and converting it into formal systems through the mathematical vertical process, (iii) using learners' active (learning) style, (iv) using discussion, question-and-answer teaching and learning styles and strategies to cultivate mathematical skills and knowledge, and (v) establishing relations between concepts and topics for the holistic learning of mathematics concepts. Following the criteria, RME starts with selecting problems that are relevant to learners' experience and knowledge and the teacher acting as a facilitator who guides learners as they solve contextual problems. The problems help to contextualise mathematical tasks. First, they must appeal to learners and second, they must relate the tasks or problems to their daily living and situations in ways that evoke experiences, thoughts, cultural and historical sentiments that learners can draw on to share subjectivities, perspectives, and experiences.

The teacher models a mathematical problem (task) to provide learners with opportunities or a context in which they can develop, based on experience, meaningful ways of solving issues they face in their day-to-day living and, as they do so, further develop logical thinking through vertical and horizontal mathematization processes. As a facilitator of learning, the teacher may, for example, employ question and answer strategies to provoke learners' interactions, engagement, critical thinking, creativity, and innovation in resolving the given situational problem. The interactions and engagement with each other create opportunities crucial for consensus on the processes and procedures to share experiences, perceptions and beliefs that should be used to resolve problems. The intersubjectivities that result from these activities are thus products of a willingness to engage multiple frames of reference. Through drawing on what is familiar, the learners can collectively reconstruct their mathematical knowledge as they shape, reshape, and remap their previously taken for granted knowledge to develop solutions to the given problems. As they engage and interact, they (learners) establish relationships between concepts and topics taught and develop a holistic (relational) understanding of mathematical concepts rather than seeing them as disintegrated concepts applicable to isolated topics. It is in this sense that RME may be viewed as promoting intersubjectivity as a context for meaningful learning.

Intersubjective relations as context for learning

Heidegger (1962 cited in Stroh, 2015) sees learning as a social process that depends on intersubjective relations embedded in social time. He argues that people gain their understanding of being from the communities to which they belong. Since meaning is embedded in the community into which an individual is socialised, as members of the community, individuals understand the social practices and roles within that community. However, such practices and social roles are not understood in isolation but learnt in relation to the social context in which meaning is generated; that is, in the intersubjective totality of the community. As individuals acquire knowledge by sharing their personal perspectives with others, their perspectives and those of others become the basis for taken-to-be-shared learning and understandings of social practices and roles.

As social beings, people can move within and across different social spaces, thus enabling border crossing. For example, in the classroom, interactions and engagements can enable learners to share historical contexts and embodied ways of Being and by so doing, learn from each other. For this reason, culturally responsive teachers adapt interactively (Deady, 2017), their teaching styles and strategies to suit learners' backgrounds and different levels of readiness to learn. As regards mathematics, Mogari (2014, p.3) argues that such teachers can reaffirm and restore the cultural dignity of learners. They (teachers) employ, for example, the ethno-mathematical approach in their classes, which is "an activity-oriented pedagogy that focuses on the mastery of mathematics content and induces affinity through relevant real-life activities that are familiar to learners" . It is such activities that Compton-Lilly (2015) would describe as creating social time; that is, experiential and therefore, a subjective and anthropomorphic sense of time. This is a conception of time that allows learners to make sense of themselves through anthropomorphism of the activities, experiences and relationships (Compton-Lilly, 2015, p. 4) that help them understand subject content.

Compton-Lilly (2015) asserts that when instructional time is used to create environments in which learners are free, value each other's contribution and ask questions as suggested by the CAPS, the time can be described as social because it influences how interactions and relationships are guided. Instructional time as social time provides a subjective, anthropomorphic, and experiential sense of time. The manner in which learners experience curriculum activities shape how they make sense of themselves, relationships amongst themselves and with teachers and, above all, what they are taught. It is in this sense that instructional time serves as a significant resource when devising strategies and designing activities that are aimed at facilitating learning. In Compton-Lilly's (2015, p. 4) view, "all that people have lived and understood as well as ways they make sense of themselves, their experiences, and their relationships is dependent on social time'.

In South Africa, the Low Achievement Trap report published by the Human Sciences Research Council (HSRC) in 2012 indicates a shortage of appropriately qualified teachers, especially in mathematics, science, and foundation phase.

The Trends in International Mathematics and Science Study (TIMSS) and the Southern and Eastern Africa Consortium for Monitoring Education Quality (SACMEQ) (Spaull, 2013) also indicate knowledge gaps between South African learners and those from other similar middle-income or less developed countries and also amongst learners within South Africa based on their location. While there are studies that focus on the use of instructional tools to facilitate learning, for example, digital gaming (Keeble, 2008) or digital Game-Based Learning approach (Makri, Vlachopoulos & Martina, 2021), we could not trace any that looked specifically at instructional time as a teaching rather than regulatory tool.

The study we report on in this paper sought to examine teachers' understanding of the importance of instructional time and how they translated it in practice to enhance meaningful or culturally relevant mathematics learning. We wished to answer the following research questions:

a) How do the grade 7 mathematics teachers in Vhembe district interpret the instructional time stipulated for specific topics in the CAPS? and
b) How do the teachers translate their conception of instructional time in practice?

METHODOLOGY

A constructivist philosophical approach (Creswell & Poth, 2018; Grover, 2015; Merriam & Grenier, 2019) or interpretivist paradigm (Kivunja & Kuyini, 2017; Lincoln & Guba, 1985; Ndlovu, 2021) of the qualitative approach was used for the study. In terms of the paradigm, people create their own meaning of social and psychological phenomena through interacting with the world and making sense of their experiences based on historical, cultural and societal perspectives (Grover, 2015).

Schools that had information-rich teachers who had been teaching mathematics as proposed by the CAPS were purposively sampled (Babbie & Mouton, 2001; Saunders, Lewis & Thornhill, 2009; Wagner, Kawulich & Garner, 2012). They (schools) had performed above 70% in grade 7 for four consecutive years despite their overcrowded grade 7classes of between 62-84 learners. Eight out of 19 schools in one circuit in the Vhembe district satisfied the criteria. The district is one of the five mega departments of education in the Limpopo Province of South Africa. Access to the schools was convenient. A manager issued a circular to all 19 schools informing them of the study the researchers wished to conduct. Once the departmental permission to conduct research was obtained, eight agreed to be involved. However only four teachers consented to participate in the study. The teachers' experience ranged between three and 26 years.

The CAPS for mathematics was analysed to identify the instructional time and teaching strategies it suggested for the different mathematics topics that had to be taught at grade 7. Lesson observations were conducted to capture the teachers' pedagogical practices. Interviews were used to establish their understanding of (i) the prescribed instructional time and (ii) implications of the suggested teaching strategies; that is, what activities and processes were to be prioritised and how they were to be designed to help learners understand the relations between their life experiences and the subject content within the prescribed time.

Transcription of the recorded data formed part of the data analysis. As the process unfolded it was possible to establish teachers' understanding of the significance of the officially allocated instructional time, how they made sense of it and translated it during lessons. Words and phrases used to refer to these aspects and how they were related guided data coding, categories and themes (Emerson, Fretz & Shaw, 2011). Views that were related to the time and what it implied as regards the proposed teaching approach indicated teachers' understanding of mathematics as a particular type of knowledge and provided codes that were linked to categorize their curriculum expertise as mathematics teachers. The categories were afterwards used to create themes that we used to organize data presentation. To validate the research, amongst others, the four trustworthiness components; namely, credibility, dependability, transferability and confirmability (Kivunja & Kuyini, 2017; Lincoln & Guba, 1985) were used as criteria.

RESULTS

Teachers' conception of the allocated instructional time

To make sense of the CAPS for Mathematics allocation of the 45% teaching time (DBE, 2011, p. 157), the teachers seemed to rely on the recommended teaching approaches, for example, drawing on the experiences of the learners and using application, clarity/understanding and explanation/analysis questions to facilitate conceptual understanding. Opinion/synthesis questions hardly featured. Application type questions established whether the learners were able to use procedural knowledge or routine procedures accurately. Clarity seeking questions focused on learners' justifications that linked aspects in their contexts to the development of abstract-general mathematical concepts. Explanation questions focused on subject content and relevant concepts to clarify the relationship or lack thereof of the content to the abstract-general conceptions. Opinion questions required the learners to use their prior knowledge to explain new concepts and how they could be applied in a specific context. The following are examples of lessons that illustrate how teachers understood the time allocated for teaching different concepts/topics as mainly prioritising awareness of correct responses rather than the learning process.

Teacher 2's lessons

The lesson was on graphs and allocated 60 minutes. The teacher had to teach to help learners develop "the ability to analyse and interpret global graphs (with special focus on trends and features of constant, increase or decrease), and draw global graphs from given description of a problem situation" (DBE, 2011, p. 65). However, Teacher 2 facilitated/promoted learner interactions that mainly reinforced correct responses.
The lesson started just after the lunch break. The bell rang at 12h17 and the class was mostly occupied by learners at 12h20. The teacher entered the class at 12h23 together with some learners who were late. Pseudonyms are used for the learners.

EXCERPT 1: *The teacher greeted learners, …, and started the lesson by indicating that it was on graphs. Thereafter, the learners were asked to turn to page 223 of the textbook and the teacher introduced the lesson by reading the following excerpt.*

Teacher 2: *Two motorists (A and B) are travelling at a speed of 120km/h and 100km/h respectively for 5 hours, what does the word respectively mean? … this is shown on the graph. The graph shows the speed done by each motorist and relationship between the time taken and the distance travelled by the motorists (points at the graph below).*

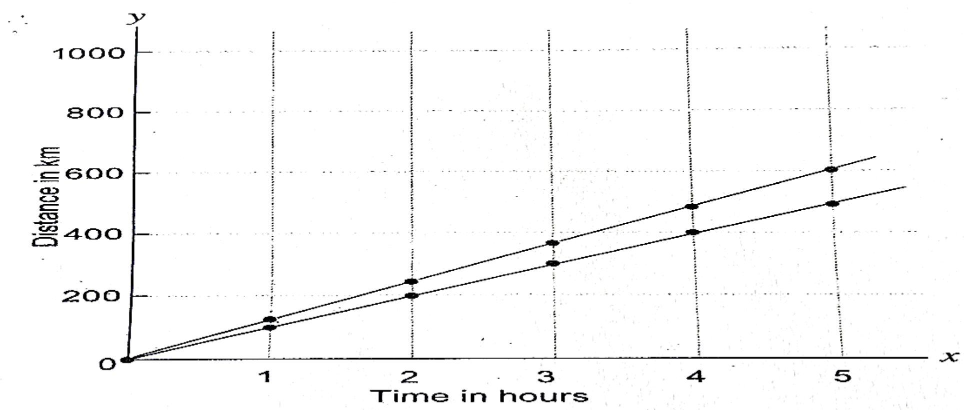

Figure 1: The graph shows time taken and distance travelled by two motorists respectively

So, you can see from the graph here, how far the motorists travelled in three hours? … how far did each motorist travel in three hours? …. remember we are comparing two motorists.

Learners: *Yes.*

Teacher 2: *The one who is travelling at 120 km/h and the other one travelling 100 km/h. How far did each travel in 3 hours …. Thapelo?*

Learner 7: *300 km and 390 km.*

Learner 8: *300 km and 600 km.*

Teacher 2: *Yes. (Pointing at another learner)*

Learner 9: *360 km and 300 km.*

Teacher 2: *360 km and 300 km. (lesson continues)*

The excerpt had to assist learners determine the correct responses by mapping and relating the provided information based on what they had been previously taught. Therefore, it was not enough to simply repeat questions until a correct response was given. The learners needed to be given opportunities to demonstrate that they understood the relevant processes and principles (mathematical representation) that applied to the subject matter knowledge (SMK) taught. To meet the requirements of the objectives of the lesson, it was important for the teacher to not only use a problem to teach the interpretation and calculation of values but also s/he needed to use the values to explain why "constant, increase or decrease" are referred to as graph features and trends. Doing so required that learners be taught how to identify the trends in the graph and, then analyse a pattern to determine whether it is constant, increasing, or decreasing (monotonically or otherwise). Pattern analysis would afford learners opportunities to use their prior knowledge of numbers, operations, and relationships (specifically counting forward, backwards, and in groups). The analysis would also enable learners to link the abstract-apart mathematical object (graph) to the abstract-general conceptions (describing rules generating the patterns that determine features of the graph). Pattern analysis would create a realistic modelling process; making it possible to imagine the lesson content in relation to their personal experiences and, thus trigger an interactive sharing of experiences, thoughts, views and understandings (Loc & Hao, 2016; Scheiner, 2016; Yet et al., 2017; Yuanita et al., 2018) to draw on and determine what is important to consider when plotting global graphs based on a given description. Similar patterns of questioning were noted in Teacher 4's lesson on common fractions.

Teacher 4's lesson

The use of teacher-posed questions by Teacher 4 was to elicit prior knowledge from short term memory of the content taught. Only one learner asked a clarity seeking question. The question sought to establish understanding whether sharing unequally would still constitute a fraction, but the teacher missed the gist of the question and left it unanswered. Again, in this instance, the teacher had missed the socio-instructional time opportunity to use engagement and interactions as scaffolding of meaningful mathematical conceptual development. The learners needed a platform for discussing the concept of a fraction, thus creating possibilities for realisation, but the teacher missed the opportunity of using the question to engage the learners and explain why unequal sharing cannot lead to a proper fraction. This could have been done by indicating that the proper procedure to share unequal wholes was to first divide equally each whole and share the parts thereof, rather than the undivided wholes and then draw on the real-life examples to explain the significance of the procedure. The missed opportunity highlighted, for example, Dhlamini et al. (2019) and Mabotja et al.'s (2018) assertions that the inadequate capacity of teachers to teach concepts in a way that facilitates meaningful understanding, compromises the recognition and actualisation rules that are important in the formation of mathematical concepts through abstraction (empirical and structural) from physical and social experiences. Perhaps it is such oversights that have persuaded Spaull (2013, 2015) to link South African learners' poor performance in mathematics to the teachers' low mathematical knowledge for teaching (MKT), which underscores lack the professional expertise to create situations and contexts in which learners can intersubjectively engage and interact, making learning anthropomorphic without compromising the essence of the subject content. In Spaull's (2013) view, teachers with limited content knowledge will not be able to simplify specific aspects of the subject content and make it relevant or interesting when relating it to real life. In contrast, the next lessons by teachers 3 and 1 might be viewed as different from the two lessons already presented, as the teachers emphasised learning-centred lessons, demonstrated a translation of instructional time as dependent on mathematics as a particular type of knowledge and created environments in which they were able to engage learners in routine and non-routine problem solving involving real-life situations. The lessons are presented below.

Translation of instructional time and mathematics as a particular type of knowledge

Teacher 3's lesson

In the lesson, the teacher deliberately took learners through the process of counting the faces (physically) of 3D objects while at the same time being assisted to conceptualise (abstractly) through imagination how the objects could be identified in the examination paper. When looking at the face of the object, it was not possible to see the opposite side of that face, so in real life, they could turn it around to see the face but in the case of mathematics, the learners had to be helped to learn how to abstractly (imagine) construct the 3D figure with its complete faces. The teacher provided them with opportunities as a platform to interactively and intersubjectively amplify mathematics content through relevant real-life activities that were familiar to them. The excerpt below is an example of what Teacher 3 did in her lesson:

> ***EXCERPT 2:*** *The teacher started by attending a staff briefing meeting which took long and the lesson that was to start at 7h30 did not start. Class started at 7h52 after the meeting. The teacher came to class with various boxes (shoe, toothpaste, washing powder soap, chalk box), prisms and pyramids as teaching aids. The class size was 62 learners, 29 boys and 33 girls. The teacher greeted learners and introduced the topic of the lesson as Geometry of 3D objects. The lesson proceeded and the following was part of it.*
>
> ***Teacher 3:*** *In the 3D objects we can find our faces as a flat surface, or we can also find our faces as curved. I will show you the examples of the faces (uses the boxes to explain the terms to learners). Here I am having the box, you can say it is a box of tools, or you can also say it is a box of sweets, or you can also say it is a box of pieces of chalk, with this box we need to identify the faces that we are talking about, do you think this box has a flat or a curved surface?*
>
> ***Learners:*** *Flat surface.*

Teacher 3: *Yes, if it is a flat surface, it means we need to know how we can identify those faces. I remember I told you last time that when we talk about the face, it is like your front side as a person. So, when you look at this box, please bear with me, in your question paper, the box that you are seeing now, in a question paper you won't see it the way it is now, but it is the same box. So, when we identify the faces, we need to think, we use what we call our imagination. You think beyond this other part that you are not seeing. You can see that when I'm holding the box like this, you are able to see the front one only. But as a mathematician, you need to think, as you know that the box has got how many sides … How many sides do we have here?*

Learners: *Six sides.*

Teacher 3: *So, this is how we are going to count the faces. …….. we also have another shape, yes, another shape, do you think this shape is different from this one? (Showing two different boxes)*

Learners: *No. (lesson continues)*

Teacher 3's pedagogic practices seemed to contradict the challenges that are highlighted in various studies (see Adler & Sfard, 2017; Brodie, 2010; Chirinda & Barmby, 2018; Dhlamini et al., 2019; Mabotja et al., 2018; Mogari, 2014; Savides, 2017; Venkat & Spaull, 2015), that some mathematics teachers fall short of teaching approaches that can facilitate the achievement of outcomes that are outlined in the CAPS. The teacher's ability to employ teaching strategies that were proper to address the recognition and realisation rules relevant to the content that they taught, revealed the expertise they had as a teacher of mathematics. Similar ability of using real-life experience of learners to facilitate understanding of concepts, dependent and independent variables, were observed in Teacher 1's lesson. The teacher used the distances that learners travel to school from their homes versus the time they spend on travelling, showing why the distance always stays the same but the time varied depending on the pace at which learners travelled.

Teacher 1's lesson

The strategies employed by Teacher 1 dealt with both the recognition and actualisation rules (Bernstein, 2003b). The recognition was addressed by encouraging the learners to describe concepts based on their experiences of travelling to school while the realisation of the concepts taught, dependent and independent, was taught by asking the learners to reflect and relate the time they spent going to school on different days – where at times they had to run to be at school on time – resulting in different times spent on the way while the distance stayed the same. According to the teacher, time had to be understood as the independent variable as its flow would be affected constantly by an event, while the distance covered would be a constant dependent on the manipulation of time. However, with the given mathematics problem, the distance was a constant while average speed was the dependent variable that changed as the independent variable was manipulated as different durations taken by learner to walk or run to school on different school days. As pointed out by Klette (2016), good questioning techniques are an attribute of high-quality teaching and optimal use of socio-instructional time. The pedagogic communications that the teacher used made it possible for learners to engage and collectively think of the concepts being taught. When the learners showed that they were struggling to grasp the meaning of dependent and independent variables, the teacher ought to have explained time in hours and minutes to illustrate that it can be manipulated (varied) and should thus be placed on the *x*-axis of the graph to show that it is an independent variable.

DISCUSSION

Mathematics is learnt through doing (Zhu & Simon, 1987), so the communication and activities that are meant to create learning contexts are not supposed to distort the essence (Cotton, 1988) of what is taught. Even though teachers have to facilitate meaningful learning (DBE, 2011) by encouraging the learners to share their knowledge and interact through discussing subject content in relation to their everyday knowledge, such sharing and interactions have to happen while observing the suitable rules for recontextualising the subject content within the limitations of the instructional time stipulated in the CAPS for doing so. Teachers in this study were aware and considered the instructional time as prescribed in the CAPS when presenting their lessons. However, the classroom observation revealed that they required curriculum expertise to develop pedagogic discourses that promoted logically an understanding and appreciation of relationships and patterns in the information provided to help learners develop relational understanding mathematically. Two teachers' curriculum practices (2 and 4) confirmed Dick and Dalmau's (2014) argument that there is always a gap between the espoused theories and theories-in-use. They were only able to promote recognition with their teaching and seemed unaware that it was important to aid the learners to extract concepts from the excerpts that had to be used to scaffold the understanding of how to analyse and interpret from a "… given description of a problem situation" (DBE, 2011, p. 65).

Teachers 2 and 4's strategies made the learners respond to questions during the lessons without any opportunities for sharing and discussing the experiences, ideas, and views on which responses were based. When teaching mathematics, it is always useful to provide opportunities for building meaningfully new knowledge on previous knowledge to unravel the complexity in concepts. Strong framing as suggested by (Bernstein's, 2003a) theory on the pedagogic devise, with a highly controlled classroom environment, tasks, and relationships ought to have encouraged intersubjective engagements that drew from the learners' material and social experiences and developed an understanding of mathematical concepts meaningfully through abstraction. The teachers' ability to convert the instructional time into social time through formal language and symbolic representation (Benis-Sinaceur, 2014) was important to create a classroom environment in which the learners' physical and social experiences could be drawn on within the limited time to identify and clarify both materially and abstractly mathematical concepts and thus enhance chance of relational understanding in the subject.

Abstraction takes place as learners construct mathematical knowledge through the insertion of a new discourse alongside existing concepts or images. However, the ways in which teachers 2 and

4 presented the prescribed content using real-life examples did not provide opportunities for learners to think beyond their realities and learn a new discourse of explaining the examples. The absence of more authentic mathematical problem-solving strategies (Adler & Ronda, 2014) in teachers 2 and 4's lessons was thus seen as being partly responsible for underplaying the importance of mathematical abstraction in their teaching. Makonye (2017) views such inability to adhere to the constructs in the CAPS, which include using proper approaches to teach concepts and skills and implement the curriculum as expected, as a reflection of teachers' inadequate capacity, leading to poor performance by learners. Mabotja et al. (2018) also explicitly link learners' lack of mathematical skills to the inadequate capacity of teachers to teach concepts in a way that facilitates meaningful understanding. They argue that teachers need to understand that knowledge is constructed socially, and therefore collectively and individually (see Jackson, 2014) as a cognitive enterprise; hence, the argument for the importance of using social time to guide engagement and interactions through anthropomorphic activities that promote the development and acquisition of new knowledge intersubjectively. The strategies or approaches, namely, problem-solving, investigative, project-based learning, cognitively guided instruction, and cooperative learning, are also promoted in the CAPS, Senior Phase Mathematics Participant's Orientation Manual 2013 (DBE, 2013, p. 23). For example, using a cooperative approach is described as allowing the grouping of learners for them to learn to listen to one another, share ideas and perspectives, give and receive assistance, seek ways to resolve difficulties, and actively work to construct new understanding and learning (Gillies, 2007).

Bansilal (2013) and Umugiraneza et al. (2018) argue that teachers in their studies struggled with interpreting and implementing the CAPS. They showed that teachers were comfortable with using teacher-led instruction as a method when teaching mathematics as opposed to progressive approaches advocated in the CAPS. Such a method fell short of innovative strategies that promote thinking, reasoning, and the construction of self-knowledge by learners. Many studies (see Mentz & Goosen, 2007; Mogari, 2014; Muthukrishna, 2013; Rhodes & Roux, 2004; Savides, 2017; Spaull, 2013; van der Walt & Maree, 2007; van Wyk, 2002) referred to when contextualising this study, view teacher knowledge as the main challenge to mathematics learning and teaching and the social contexts within which teaching takes place as also having a direct bearing on pedagogic discourses (Adler & Davis, 2006; Lazarides & Rubach, 2017; Mentz & Goosen, 2007; Prinsloo, 2007; Uys et al., 2007). Additionally, in this study, social contexts such as class size (Rice, 1999) and shortages of resources (Kaya et al., 2015) were also important aspects in understanding the challenges teachers faced in developing suitable curriculum practices. As Klette (2016) argues, even though social contexts such as socioeconomic background, class size, and teachers' formal education and experience contribute to the style or type of learning, teaching practices contribute significantly to such learning. In short, teachers 2 and 4 needed to improve their pedagogical content knowledge.

Modiba (2011) as well argues that even the best teachers need adequate subject matter knowledge, that is, professional knowledge and skills to develop pedagogic discourses that promote an understanding and appreciation of relationships and logical patterns in mathematical concepts. In her view, teachers with adequate subject content knowledge would be able to create instructional practices that meet situational demands. They would be responsive to learners' cultural and historical background knowledge by employing pedagogic communication or framing that addresses the potential discursive gap between the mundane and the esoteric knowledge (Bernstein, 2003a) while emphasising the importance of rules, principles, or procedures that have to be taught or are required for conceptual development. Therefore, teachers 2 and 4 could be described as having created learning environments that eased the recognition of subject content but fell short of facilitating its realisation. It was critical for learners to be helped to understand the procedures and principles (realisation rule) that were needed for abstract mathematical understanding. Inferences to real-life situations helped with similarity recognition (recognition rule) but not the development of relational understanding.

Mathematics as a knowledge area operates at an abstract level and is concerned with generalisation (Benis-Sinaceur, 2014; Mitchelmore & White, 2004; Scheiner, 2016) and this requires teaching to go beyond real-life examples and enable learners to understand the subject content conceptually. To varying degrees, Teachers 1 and 3 used learners' interactions to ease understanding of the legitimate processes of developing such understanding. The interactions focused on them and promoted thinking that probed the mundane and scientific content used in the lessons. The questioning used made intersubjective exchanges amongst the learners and teachers and the learners themselves possible. Responses were thought about and in diverse ways, the learners were helped to think about what was conceptually important in the mundane content. Teachers 1 and 3 deliberately focused on culturally relevant or responsive teaching. As Savides (2017) would argue, they seemed the better-educated teachers in the study.

As Stroh (2015) asserted, people as social beings cannot escape the process of interactions. It is in interacting that they develop the empathy that enables them to understand one another's perspectives and share experiences, which in the case of teachers 1 and 3, resulted in intersubjectivities that learners drew on to understand how to relate and solve mathematical problem situations by using their mundane experiences as a foundation or base. This enabled the learners to develop learning styles that eased sharing of experiences intersubjectively and through collective reflections on these experiences, mathematical concepts were interrogated and clarified amongst themselves. As Winfield (2014) pointed out, understanding mathematical concepts requires the availability of other learners in order to provide space or opportunities for reflecting on socially constructed knowledge and to reshape, remap, and reconstruct it as evidence of relational understanding. In contrast to teachers 2 and 4 who seemed to prioritise solipsistic individual perspectives (Stroh, 2015), teachers 1 and 3 used social time.

CONCLUDING REMARKS

Two (teachers 2 and 4) out of the four Grade 7 mathematics teachers in the study were not able to translate the CAPS' prescribed instructional time into effective teaching and learning styles. They seemed unable to understand what was needed to help the learners develop an abstract understanding of mathematical concepts and only managed to facilitate recognition of the responses to questions posed. In contrast, teachers 1 and 3, seemed to know what was necessary to translate the prescribed instructional time into social time. They used teaching styles or strategies that helped the learners develop learning styles that eased the sharing of experiences and how to collectively reflect on those experiences and identify what was conceptually significant about them that could, in turn, be used to explain mathematical concepts to each other. Therefore, the conclusion is that teachers 2 and 4 seemed unaware of the importance of the realization rules in teaching and learning mathematics. The other teachers, 3 and 1 clearly made efforts to lay the foundations that would be useful for conceptual understanding by using physical time in ways that encouraged learners to examine the subject content with the help of real-life examples and develop a relational understanding of both subject content and the examples. Admittedly, the small sample size of the study imposes limitations. A larger study in the future is recommended and likely to strengthen the conclusion.

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Enhancing the Learning of Limits of Functions Using Multiple Representations

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Abstract: This is an exploratory study considering instructional methods in mathematics education. The study examined misconceptions arising from selected assignments undertaken by sixty-five students. The students were attending a calculus mathematical course unit on limits of functions, in different academic areas offered at selected tertiary institutions in Western Uganda. In this study, forms of expressing ideas on limits of functions are considered as far as they facilitate access to underlying mathematical principles. We explored the use and application of tools for representing mathematical ideas to enhance students' conceptual understanding and problem-solving skills. A small-scale pilot of interventions using readily available GeoGebra dynamical computer software was applied. The diagnostic assignment on limits of functions was used for data collection. The main objective was to examine whether or not students preferred the application of multiple representations to the analytic approach. The semi-structured interview protocol was also conducted to probe further and correlate students' responses and their problem-solving abilities. The results showed that multiple representations-based instructions significantly changed students' understanding of the limits of functions. The results promise better access and understanding of more abstract mathematical concepts and may support students' problem-solving abilities. The semi-structured interviews conducted indicated that multiple representations supported and, therefore, enhanced students' understanding and solving of the limits of functions. This study highly recommends that mathematics educators should adapt multiple representations-based instructions to enhance students' critical thinking, and problem-solving.

INTRODUCTION

Research in the 21st-century education system on learning mathematics requires that educators should adapt or adopt effective learning approaches. This is aimed at ensuring that students appreciate the usefulness of mathematics and subsequently apply it in new contexts. Sometimes and most often the learning of mathematics is taking a differential trend. Mathematics teaching practices require that educators explicitly use and connect mathematical representations to learning challenging and more complex concepts. Accordingly, "effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem-solving" (NCTM, 2014 p.10). Thus, different representations help students to communicate mathematically, reason insightfully and build procedural fluency from conceptual understanding through the process of problem-solving (Mainali, 2021). The learning of mathematics should be structured in such a way that students are involved in making sense of mathematics tasks. To do this, educators should apply varied strategies and representations, justifying solutions, making connections to prior conceptual knowledge and understanding, and linking mathematical representations to underlying ideas and other representations. This may help to evaluate students' mathematical reasoning and explanations. Educators can, therefore, select tasks that promote students' reasoning and problem-solving.

Representations are inevitably unique and inherent in supporting the learning of mathematics. They are intended to visualize presumably harder scientific tasks. They help to simplify complex and abstract concepts so that they can be concretized, and make mathematics more attractive and interesting. According to Ainsworth et al. (2006), multiple external representations support different ideas and processes, constrain interpretations and, promote a deeper understanding of the subject-specific domain. Generally, representations are aimed at mitigating and simplifying challenging mathematical concepts. Representations may take two forms: internal or external(Mainali, 2021). The former is all about cognitive configurations of mathematical thinking and problem solving perceived as mental images while the latter refers to structured physical situations that can be seen as embodying mathematical ideas. External representations are therefore used to demonstrate and communicate mathematical relationships visually. Representational modes include verbal descriptions, videos, tabular forms, dynamic graphical representations, and the building of models (equations), animations, and simulations (Ainsworth, 2008). These representational modes are applied to help learners understand and solve complex forms of mathematical concepts. The objective is to develop learners' understanding of basic mathematical ideas, concepts, or principles and use them to support problem-solving strategies. External representations are the main focus of this study.

Multiple representation learning practices can effectively be used to support specific mathematics content (NCTM, 2014). This involves educators' competencies in delivering the conceptual approach, relational understanding, and adaptive reasoning of the subject matter (Kathirveloo & Marzita, 2014). This knowledge component is what Hill et al. (2008) referred to as mathematical knowledge for teaching (MKT) and the mathematical quality of instruction (MQI), the unique knowledge that intersects with the specific subject teacher characteristics to produce effective and meaningful instruction. According to Hill, "teachers with weak MKT would have teaching characterized by few affordances and many deficits". Hill further noted elements for MQI as those that involve dealing with students' mathematical errors, responding to students appropriately, connecting classroom practice to mathematics in real life, mathematical language, and richness of mathematics (p. 437).

Unfortunately, learners often fail to exploit benefits accruing from the application of appropriate combinations of multiple representations. Janvier (1987) was among the pioneers in exploring problems of multiple representations in teaching and learning mathematics. The author reiterates the common tendency of educators in underestimating the teacher's role in the representation system in the standard curriculum. Mainali (2021) supports this claim since multiple representations would enhance learning, hence supporting students' understanding of mathematical concepts and constructing mental relationships. This is vital in communicating mathematical concepts, providing arguments, critical thinking, and a sign of understanding. Consequently, learners apply mathematical concepts in solving societal realistic problems situations through the process of modeling. Thus, the translation modes of multiple representations (e.g., symbols, signs, characters, diagrams, objects, pictures, or graphs) are important for learners in developing their cognitive skills to be more proficient in learning the limits of functions. To adequately understand these concepts, Arnal-palacián and Claros-Mellado (2022) highlight the significance of the teachers' specialized content knowledge and advanced mathematical thinking. The teacher's role is to carefully help learners to apply them successfully and effectively.

Some empirical studies conducted in different settings and contexts have demonstrated the significance of multiple representations in enhancing students' understanding of science and mathematics (Adadan, 2013; Ainsworth, 2006, 1999; Ainsworth et al., 2006; Desai & Bush, 2021; Dreher et al., 2015; Kozma et al., 2000; Kuntze et al., 2018; Mainali, 2021; Meij & Jong, 2006; Rosengrant et al., 2005; Vogt et al., 2020). In understanding the learning of mathematics and the limits of functions, in particular, some studies (e.g., Liang, 2016; Tall & Vinner, 1981) show that the topic is challenging and that students have limited conceptual understanding. Some studies (e.g., Arnal-palacián & Claros-mellado, 2022) report on limits and infinite limits in particular. In their study, prospective teachers failed to understand the notion of infinite limits and its algorithmic procedures. Thus, they applied the wrong graphical representation system. Some of the factors that account for students' challenges in this topic include those ranging from analytic to graphical. Tall and Vinner (1981) investigated students' concept images and their cognitive structure regarding the limits of functions. The author noted that students' differing concept images from the formal definitions of a mathematical theory cause cognitive conflict since mathematical concepts, rules, and principles are defined accurately. Multiple representations are likely to help boost their ability to explicitly hold presumably harder concepts in their mind and to mentally retrieve and manipulate them to suit any context.

Thus, the limits of some functions are best evaluated using graphical methods for better visualization. The causes of students' learning challenges in evaluating the limits of functions are enormous and mainly stem from students' preconceptions of solving equations, functions, and inequalities. Students, on one hand, fail to understand the relationship between equations, functions, and inequalities, while educators, on the other hand, have not adequately applied students' flawed conceptions with suitable approaches to address the causes and sources of students' learning challenges. Thus, this study aims to use multiple external representations using limits of various functions to compare and contrast analytical approaches of evaluating the limits of a function, and graphical representations to visually examine the convergence of limits.

The Conceptual Framework and Literature Review

This research is situated on the PCK conceptual framework based on Shulman (1986). Shulman conceptualized that "pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons" (p. 9). According to Shulman, effective learning strategies involve teachers' integration of students' preconceptions and misconceptions held previously and how these preconceptions relate to subsequent learning. In supporting students' mathematical thinking and understanding, Taşdan & Çelik (2016) developed a framework for examining mathematics teachers' PKC. The framework is important in enhancing teachers' PCK (e.g., the use of graphics, and manipulatives) with the main objective of understanding students' mathematical thinking.

Indeed, the above theoretical framework aligns with the five strands of mathematical proficiency. Kilpatrick, Swafford, and Findell (2001) proposed a multidimensional five interwoven and interdependent strands of mathematical proficiency teachers should target during classroom instruction. These strands are conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (NCTM, 2014). Kilpatrick, Swafford, and Findell have argued: "that proficiency should enable them to cope with the mathematical challenges of daily life and enable them to continue their study of mathematics in high school and beyond" (p. 116). These five interrelated strands are inevitable for learning mathematics in the sense that they support, foster, and promote students' identification and acquisition of conceptual knowledge, procedural knowledge, and problem-solving abilities, which are all supported by the cognitive load theory. This is what is referred to as conceptual change, and is aimed at understanding and connecting previous knowledge to new knowledge (Merenluoto & Lehtinen, 2002; Trumper, 2006; Vamvakoussi et al., 2007; Vosniadou, 2007).

However, educators should ask themselves the effective ways learners can be motivated to represent and connect prior knowledge and understanding and effectively use it deeply and broadly during problem-solving. According to NCTM (2014), students' effective learning "depends fundamentally on what happens inside the classroom as teachers and learners interact over the curriculum" (p. 8).To support this theoretical stance, "the cognitive linking of representations creates a whole that is more than the sum of its parts … Cognitive flexibility theory highlights the ability to construct and switch between multiple perspectives of a domain as fundamental to successful learning" (Ainsworth, 2008 p.198). Other studies conducted by Adelabu et al. (2022)

Yimer (2022) and Yimer and Feza (2019) show that students' attitude, conceptual knowledge, and understanding is influenced by varying classroom instructional methods integrated with multiple representations.

The Concept of a Limit of Functions

Definition (Limit)
From Contemporary calculus 1 (Hoffman, 2012), the limit of a function is defined below.
Let $S \subseteq \Re$ *and* $f : S \to \Re$ be a function. A real number L is said to be a limit point of *f* at point $a \in S$ if given any $\in > 0\ \exists\ \ a\ \ \delta > 0$ such that if $x \in S$,
$\lim_{x\to a} f(x) = L$: For every given number $\varepsilon > 0$ there is a number $\delta > 0$ so that if x is within δ units of a (and $\neq a$) then $f(x)$ is within ε units of L. The symbol " $\to$ " means "approaches" or "gets very close to."
Equivalently: $\left|f(x) - L\right| < \varepsilon$ Whenever $\left|x - a\right| < \delta$ for $0 < |x - a| < \delta$.
Note that *f* may or may not be defined as $x = a$
We say that $f(x) \to L\ as\ x \to a$ and write

$$\lim_{x\to a} f(x) = L. \quad (1)$$

This above definition does not apply to the one side (left and right) limits. For the left limit, as x approaches α of $f(x)$ is L if the values of $f(x)$ get as close to L as possible when x is very close to and left of α, $x < \alpha$: $\lim_{x\to\alpha^-} f(x) = L$. Conversely, the right limit, written with $x \to \alpha^+$, requires that x lies to the right of α, $x > \alpha$. Hence, the **One–Sided Limit Theorem states that:**
$\boldsymbol{\lim_{x\to\alpha} f(x) = L}$ **iff** $\lim_{x\to\alpha^-} f(x) = \lim_{x\to\alpha^+} f(x) = L$.

Properties of limits

a) If $\lim_{x\to a} f(x)$ exists then it is unique. (1)
b) If $\lim_{x\to a} f(x) = L_1$, and $\lim_{x\to a} g(x) = L_2$ then (2)
c) $\lim_{x\to a}[f(x) \pm g(x)] = L_1 \pm L_2$ (3)
d) $\lim_{x\to a}[f(x)\, g(x)] = L_1\, L_2$ (4)
e) $\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x) = kL$ (5)
f) $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{L_1}{L_{2_1}}$ provided $g(x) \neq 0\ \forall x$ and $L_2 \neq 0$ (6)
g) $\lim_{x\to a}\{f(x)\}^n = \left\{\lim_{x\to a} f(x)\right\}^n = L^n$ (7)

h) $\lim_{x \to a}\left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\left[\lim_{x \to a} f(x)\right]} = \sqrt[n]{L}$ (If L > 0 when n is even). (8)

i) $\lim_{x \to a} k = k$ (9)

j) $\lim_{x \to a} x = a$ (10)

k) For polynomial and rational functions, If $P(x)$ and $Q(x)$ are polynomials, and a is any number, then $\lim_{x \to a} P(x) = P(a)$ and $\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$ provided $Q(a) \neq 0$ (11)

All the above properties can be proved. However, this is outside the scope of this study.

Calculus has played a significant role in the history of mathematics and mathematical analysis in particular. The subject of calculus has been integrated with learning science, technology, and engineering (Rasmussen et al., 2014). The concept of limits has been investigated for the last twenty decades (Tall & Vinner, 1981), and it is the most important topic in learning calculus. The definition and application of limits provide prerequisite knowledge for learning other advanced mathematics topics (e.g., differentiation, integration, and sequences and series) (Palacián et al., 2020). Understanding the concept of limits may guarantee a further grasp of other concepts like functions. However, as Juter (2005a) and Juter (2005b) note, many mathematicians have found learning the concept of limits and related concepts challenging with multiple misconceptions. Moreover, other empirical findings (e.g., Juter, 2003) on the learning of limits at university mathematics support this claim. The learning at the university level is structured and formally presented in textbooks and as lectures. The concepts in these textbooks need thorough conceptual understanding to minimize misconceptions and errors.

In this research, multiple representations were applied to examine students' understanding of the limits of functions. Our own experience, as university educators have shown that students' learning of limits of functions is demanding in terms of time and conceptual understanding as compared to other course units. We examined the significance of external multiple representations in enhancing students' understanding of the limits of functions. External multiple representations are those used to symbolize, describe and refer to the same mathematical entity. They are used to understand, develop, and communicate different mathematical features of the same object or operation, as well as connections between different mathematical properties and principles. This research provides insight and adds knowledge to other empirical findings on students' conceptual understanding of the limits of functions. The findings will also provide additional knowledge on the usefulness of external multiple representations to both learners and educators aimed at enhancing the learning of mathematics generally. This study aims to answer the research question of whether or not external multiple representations enhance students' mental representations of limits of functions.

METHODOLOGY

The Sample

The sample consisted of 65-year one university students (21 female and 44 male). The students' average age was aged 19.45 (S.D=0.95). The students had been admitted to sampled universities in western Uganda to pursue a Bachelor of Science with education and were in their first year, the first semester in a mathematics class with calculus 1 as a course unit. The duration for this course was16 weeks, and all students were in a full-time program. Two lectures in calculus were conducted weekly each with a duration of two hours. The total teaching time was 48 hours. Limits of functions were taught to provide prerequisite knowledge to other related courses (derivatives, integrals, sequences, and series). The students were taught both analytically and later used external multiple representations. Specifically, the two approaches were compared and contrasted. Finally, a written follow-up test was administered and students who presented ambiguous solution sketches were interviewed to examine their understanding and preference of the two approaches.

Diagnostic Tasks

The students were given two tasks on limits of functions to be solved analytically. Later, the same tasks were solved by the use of external multiple representations. Students received the questions as an assignment immediately after the topic was fully covered. The questionnaire contained tasks about evaluating the limits of functions (questions 1 and question 2). Based on the theory of constructivism (Czarnocha, 2020), The students were also asked to explain the necessary prior knowledge for learning concepts of limits of functions since it was their first year, first-semester university course. After limits had been fully covered, as a course unit, a second questionnaire with similar tasks on limits of functions at different levels of difficulty was administered to check their conceptual understanding and problem-solving abilities. The main objective was to examine students' preference for the two methods of computing the limit of a function, and their ability to explain what they did. The students consented before participating in a focus group and individual interviews. Of the 65 students, 52 students consented to participate in a semi-structured interview. By taking into consideration gender differences, 20 students out of 52 were systematically selected for individual interviews. Each interview session per student lasted for 20 minutes. Students were asked about definitions of a limit of a function and solved tasks on the limit of a function with various levels of difficulty. They were expected to reveal specific and general knowledge on their solution sketches to clarify and/or justify their answers to the given tasks. Students' responses to the questions were analyzed. The interviews were transcribed verbatim. We specifically examined how students presented solutions to the tasks.

Instruments

The questionnaire contained two tasks on the limits of functions. They were intended to test students' understanding and misconceptions when evaluating the limits of functions using L'Hopitals' rule and by rationalization. The solutions to these tasks can be obtained graphically. The tasks were solved analytically. Later, students' solution sketches were compared with the graphical solutions to challenge students to come up with alternative solutions. These tasks were challenging and, therefore, some students did not present substantial solutions. Others left these tasks unsolved. The following tasks were given to the students as an assignment. The students solved the tasks about the limits of functions and later submitted them for marking. They were asked to evaluate the limits of the following functions:

(a) $\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2}$

(b) $\lim_{x \to 16} \frac{\sqrt{x}-4}{x-16}$

The above tasks were designed to explore students' understanding of the limits of functions (including students' misconceptions) for analytical or external multiple representations.

Procedure

Students were individually interviewed to examine their understanding of the limits of functions. Interviews were conducted at their respective colleges or faculties and lasted for one hour during regular school working hours. Each interview was designed to cover specific aspects of the limits of functions to correlate with students' preconceptions. We are particularly concerned with the evaluation of limits of functions to L' Hopital's rule and rationalization using multiple representations. All interviews conducted were recorded and transcribed verbatim to support multiple representations in fulfilling the purpose of the present study. To do this, each student was provided with a paper-and-pen assignment test, as a questionnaire. Students thought aloud and justified their solutions to the stated questions.

RESULTS

There are three methods for evaluating the limit of a function. These are the algebraic method, the tabular method, and the graphical method. The purpose of this research is to use visual representation to compare and contrast these methods. The algebraic method involves the simplification of algebraic functions before evaluating their limit. This may take the form of factoring and dividing, although often more complicated algebraic and/or serious trigonometric functions with inherent steps are needed. Normally, the steps are difficult to handle algebraically or the algebraic properties of such functions are not known to the learners. The tabular and graphical methods are used to evaluate a limit of a function f(x) as x approaches a given value say α. This method involves calculating the values of f(x) for many values of x very close to α so that we can algebraically determine which value f(x) approaches α. If f(x) converges very first, we may not need many values of x. Important to note is that this method may be used to evaluate the limits of some complicated functions, mainly those that require learners to rationalize. This is done by evaluating f(x) for many values of x.

The graphical method is closely related to the tabular method. However, a graph of the given function is drawn, and then the graph is used to determine which value f(x) approaches α. (see Table 1 and Table 2). The choice of the method to apply depends on the difficulties inherent in the question. However, each of these methods serves as an alternative to the other. Moreover, graphing the function or evaluating it at a few points using tabular form provides learners with the skills to visualize and verify the solutions obtained algebraically. We now visualize the solutions to tasks (a) and (b) and together with the students' challenges.

(a) For $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}$, some students computed $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}$ directly and obtained either 0 or ∞. That is to say, $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2} = \lim_{x\to 4}\frac{4-4}{\sqrt{4}-2}$ (6)

$$= \lim_{x\to 4}\frac{0}{0}$$
$$= \infty$$

Some students stopped here and submitted their work for marking. However, those who had grasped the concept of limit applied L'Hopital's rule to evaluate the limit of the above function since $\frac{0}{0}$ is the prerequisite step for using L'Hopital's rule.

$$=\lim_{x\to 4}\frac{2(1-0)}{-\sqrt{x}-0} \qquad (7)$$
$$= \lim_{x\to 4} 2\sqrt{x}$$
$$= 2\sqrt{4}$$
$$= 2(2)$$
$$= 4$$

Hence, $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2} = 4$

Differentiation was also another hurdle for some students especially functions with fractional indices. Some students did not differentiate correctly from which the limit could be computed.

We need to investigate the values of $f(x)$ when x is close to 4. If the $f(x)$ values get arbitrarily close to or even equal to some number L, then L will be the limit. One way to keep track of both the x and the $f(x)$ values is to set up a table and pick several x values which are closer and closer (but not equal) to 4. We can pick some values of x that approach 4 from the left, say x = 3.91, 3.9997, 3.999993, and 3.9999999, and some values of x which approach 4 from the right, say x = 4.1, 4.004, 4.0001, and 4.000002. The only thing important about these particular values for x is that they get closer and closer to 4 without equaling 4. This is illustrated in the table below to confirm that the limit is convergent whenever $x \to 4$.

x	$x-4$	$\sqrt{x}-2$	$f(x)=\frac{x-4}{\sqrt{x}-2}$	x	$x-4$	$\sqrt{x}-2$	$f(x)=\frac{x-4}{\sqrt{x}-2}$
3.9100000	-0.0900000	-0.0226280	3.9773720	4.100000	0.100000	0.024846	4.024846
3.9997000	-0.0003000	-0.0000750	3.9999250	4.004000	0.004000	0.001000	4.001000
3.9999930	-0.0000070	-0.0000018	3.9999983	4.000100	0.000100	0.000025	4.000025
3.9999999	-0.0000001	0.0000000	**4.0000000**	4.000002	0.000002	0.000000	**4.000001**

Table 1: Values of $f(x)=\frac{x-4}{\sqrt{x}-2}$ as values of x tends closer and closer to 4 (from – and +).

As the x values get closer and closer to 4, the $f(x)$ values are getting closer and closer to 4. We can get $f(x)$ as arbitrarily close to 4 as we want by taking the values of x sufficiently close to 4. Hence, $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}=4$. This answer is the same as that obtained by the graphical method in Fig. 2.

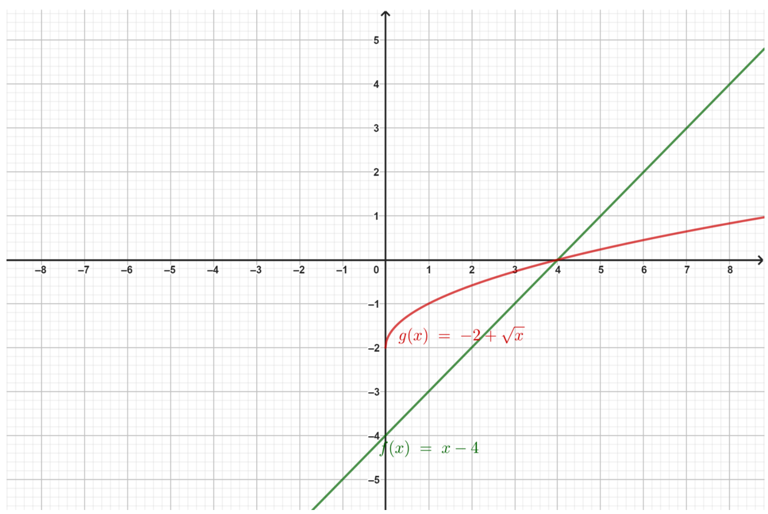

Figure 1: The graph of $f(x)=\frac{x-4}{\sqrt{x}-2}$

From Figure 1 and Table 1, it can be visualized that $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}=4$. The same result is obtained when the graph of $f(x)=\frac{x-4}{\sqrt{x}-2}$ and $2\sqrt{x}$ (by recognizing that $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}=\frac{0}{0}$ and applying L' Hopital's rule i.e. $\frac{\mathbf{f'(x)}}{\mathbf{g'(x)}}$) are plotted. This is visualized in Figure 2 below.

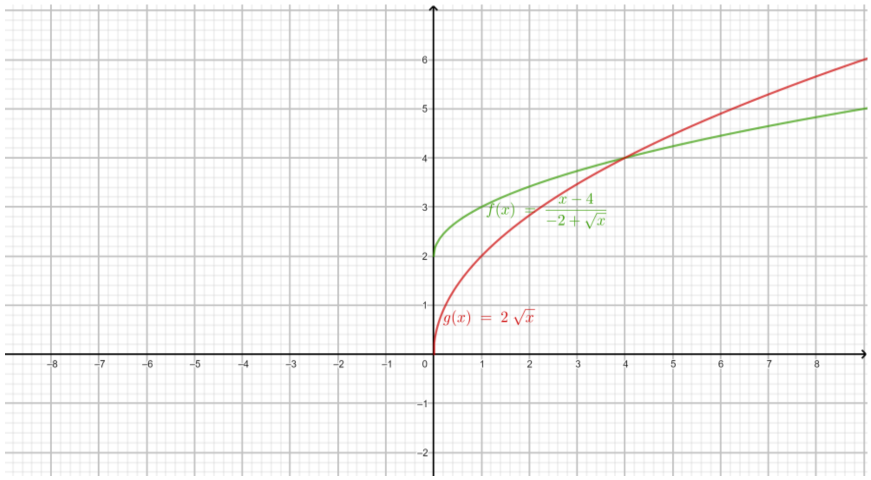

Figure 2: The graph of $\frac{f(x)}{g(x)} = \frac{x-4}{\sqrt{x}-2}$ and $\frac{f'(x)}{g'(x)} = 2\sqrt{x}$

(b) $\lim_{x\to 16} \frac{(\sqrt{x}-4)}{(x-16)}$
$= \lim_{x\to 16} \frac{(\sqrt{16}-4)}{(16-16)}$
$= \lim_{x\to 16} \frac{(4-4)}{(16-16)}$
$= \frac{0}{0}$
$= \infty$

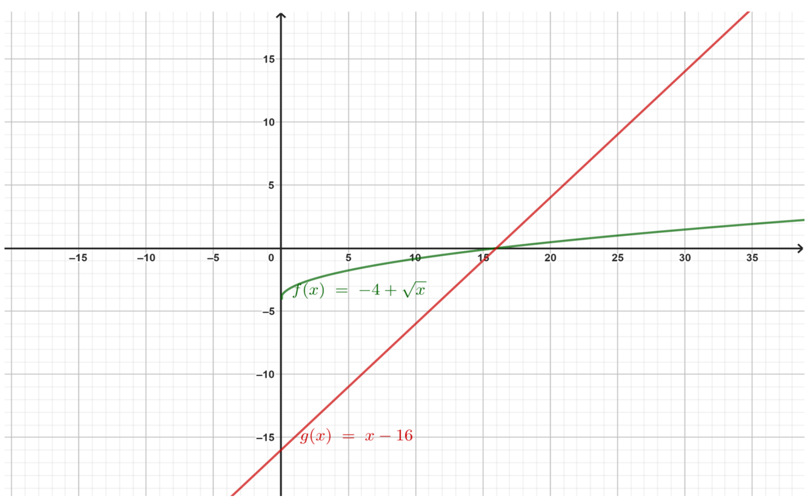

Figure 3: The graph of $\boldsymbol{f(x)} = \frac{(\sqrt{x}-4)}{(x-16)}$

The above graph of $\boldsymbol{f(x)} = \frac{(\sqrt{x}-4)}{(x-16)}$ leads to incorrect solution ($\lim_{x\to 16} \frac{(\sqrt{x}-4)}{(x-16)} = 16$). Yet, $\lim_{x\to 16} \frac{(\sqrt{x}-4)}{(x-16)}$, the rationalized function yields the correct solution (see Figure 4).

Recognizing $(\sqrt{\boldsymbol{x}}-\mathbf{4})$, $\lim\limits_{x\to 16} \frac{(\sqrt{x}-4)}{(x-16)}$ can be evaluated by rationalization. To do this, we introduce an innocent 1 by multiplying the numerator and denominator by $(\sqrt{\boldsymbol{x}}+\mathbf{4})$, the conjugate of the numerator $(\sqrt{\boldsymbol{x}}-\mathbf{4})$.

$= \lim\limits_{x\to 16} \frac{(\sqrt{x}-4)*(\sqrt{x}+4)}{(x-16)*(\sqrt{x}+4)}$

$= \lim\limits_{x\to 16} \frac{(x-16)}{(x-16)*(\sqrt{x}+4)}$

Clearly, $\frac{f(x)}{g(x)} = \frac{1}{\sqrt{x}+4}$ for $x \neq 16$

$= \lim\limits_{x\to 16} \frac{1}{(\sqrt{x}+4)}$

$= \lim\limits_{x\to 16} \frac{1}{(\sqrt{16}+4)}$

$= \lim\limits_{x\to 16} \frac{1}{(\sqrt{x}+4)}$

$= \lim\limits_{x\to 16} \frac{1}{(4+4)}$

$= \frac{1}{8}$

Hence, $\lim\limits_{x\to 16} \frac{(\sqrt{x}-4)}{(x-16)} = 0.125$

The same wrong answer is obtained when the graph of $f(x) = \frac{1}{(\sqrt{x}+4)}$ is plotted (Figure 4). This means the rationalized function does not converge.

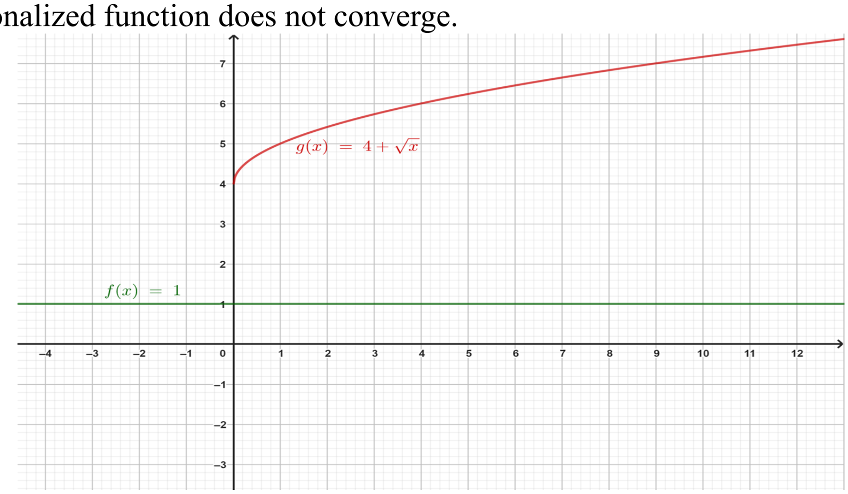

Figure 4: The graph of $f(x) = \frac{1}{(\sqrt{x}+4)}$

However, when the graphs of $f(x) = \frac{(\sqrt{x}-4)}{(x-16)}$ and $g(x) = \frac{\mathbf{f'(x)}}{\mathbf{g'(x)}} = \frac{1}{(2\sqrt{x})}$ were plotted, the limit was easily visualized. From Figure 5, $\lim_{x \to 16} \frac{(\sqrt{x}-4)}{(x-16)} = 0.125$ (visualized as 0.1).

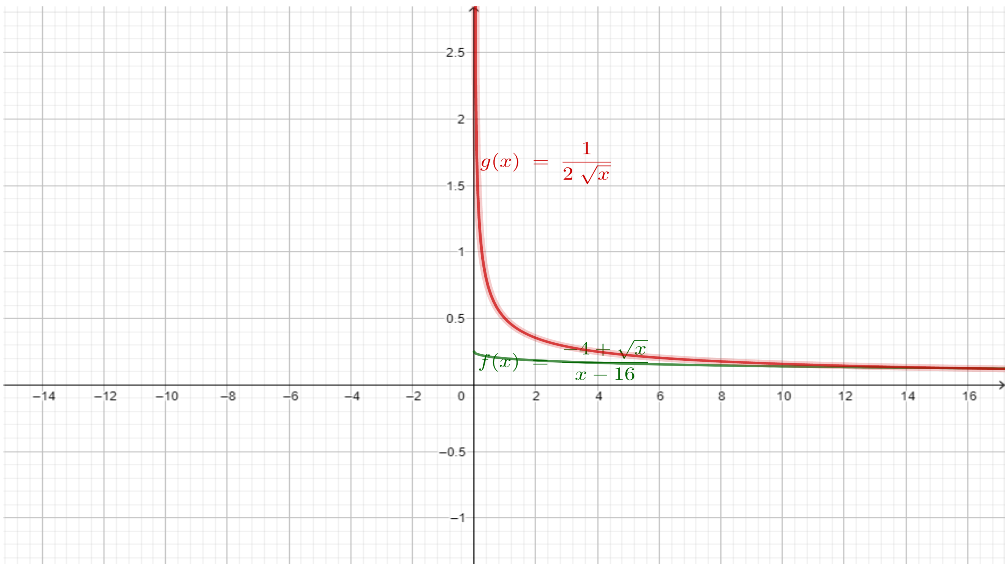

Figure 5: Graph of $f(x) = \frac{(\sqrt{x}-4)}{(x-16)}$ and $g(x) = \frac{\mathbf{f'(x)}}{\mathbf{g'(x)}} = \frac{1}{(2\sqrt{x})}$

Similarly, using the tabular method for (b) $\lim_{x \to 16} \frac{(\sqrt{x}-4)}{(x-16)}$,

x	$(\sqrt{x}-4)$	$(x-16)$	$f(x) = \frac{(\sqrt{x}-4)}{(x-16)}$	x	$(\sqrt{x}-4)$	$(x-16)$	$f(x) = \frac{(\sqrt{x}-4)}{(x-16)}$
15.910000	-0.011266	-0.090000	0.125176	16.10000	0.01248	0.10000	0.12481
15.991000	-0.001125	-0.009000	0.125018	16.00500	0.00062	0.00500	0.12499
15.999930	-0.000009	-0.000070	0.125000	16.00020	0.00002	0.00020	0.12500
15.999999	0.000000	-0.000001	0.125000	16.00002	0.00000	0.00002	0.12500

Table 2: Values of $f(x) = \frac{(\sqrt{x}-4)}{(x-16)}$ as values of x tends closer and closer to 16 (from – and +). As the x values get closer and closer to 16, the $f(x)$ values are getting closer and closer to 0.125. In fact, we can get $f(x)$ as arbitrarily close to 16 as we want by taking more values of x sufficiently

close to 16. Hence, $\lim_{x\to 16}\frac{(\sqrt{x}-4)}{(x-16)} = 0.125$. This answer is approximately the same as that obtained by graphical method in Fig. 5.

Students' conceptualization of Task a and Task b

Table 2 below shows percentages of students who solved the tasks correctly and partially. There were 65 students who participated in this study.

Tasks	Correct	Partially Correct	Incorrect
Task 1	25	31	52
Task 2	34	44	69

Table 1: Percentage of students who solved tasks correctly, partially correct and incorrectly.

The students presented solutions to the tasks with lots of misconceptions and errors. This explains why most students obtained incorrect solutions (52% in task 1 and 69% in task 2 respectively). One very prominent misconception peculiar in task 2 was the concept of $\frac{0}{0}$ and where most students got 0 instead of ∞. Others got ∞ but could not justify it by going ahead to rationalize the expression correctly. Below are students' incorrect vignettes of task 1 and task 2:

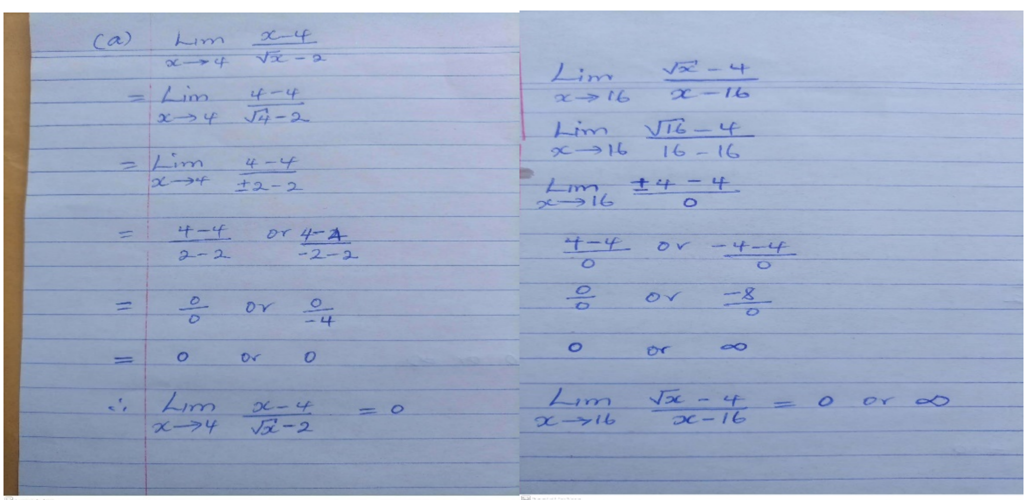

Figure 1: Students' wrong vignettes for task 1 and task 2

From figure 1 above, It appears that students had not fully understood how to evaluate $\frac{\mathbf{0}}{\mathbf{0}}$ or $\frac{\boldsymbol{a}}{\mathbf{0}}$. Many students obtain ∞ as a prerequisite step for using L' Hopital's rule. Yet, a prerequisite condition for applying L'Hopital's rule is to get $\frac{\mathbf{0}}{\mathbf{0}}$ analytically when evaluating the limit of a function. This was obtained by differentiating or multiplying by "the conjugate" of the numerator (also called the rationalization process). This was similarly done in task 2 confirming that most students lacked prior conceptual knowledge and understanding of basic concepts relating to the learning or limits of functions.

As indicated in Table 2 and Figure 1, Task 2 was very difficult and perhaps more challenging for students to conceptualize and solve. Analytical method was the most commonly used method meaning that the students had not fully grasped the solution (s) of limits of functions using L' Hopitals' rule and by rationalization. The results further show students' confidence and capability in answering the two tasks.

Tasks	**Confident**	**Partially Confident**	**Unconfident**
Task 1	25(31)	44(51)	59(64)
Task 2	22(36)	51(62)	63(75)

Table 2: Number and (%) of correct solutions to Task 1 and Task 2.

In Table 2 above, students' confidence in the two tasks has been compared and contrasted. Most students were partially confident (44(51), and 51(62)) in task 1 and task 2 respectively, and unconfident (59(64) and 63(75)) in task 1 and task 2 respectively. This shows again that the students were more confident in answering task 1 compared to task 2. Worth noting is that some students (5(3.25)) did not solve any of the above tasks while others solved just one ((16(24.62)) for task 1 and (34(22.1)). Generally, the overall students' confidence in evaluating the limits of functions in calculus was weak.

Tasks	Task 1	Task 2	Both
Number (%)	16(24.62)	34(22.1)	5(3.25)

Table 3: Number of students who did not solve Task 1 or Task 2 or both

Comparing and contrasting the results of Table 2 and Table 3, task 2 was seemingly harder for most students to conceptualize. This has an implication on the learning process. Educators should devise suitable learning strategies to enhance the learning of the limits of functions.

The Interview Notes

The interviews conducted with twenty students revealed that the students generally had a weak conceptual understanding and perhaps had developed a negative attitude towards the limits of functions. The observed lecture sessions in their small discussion groups further revealed that students' might not have fully understood the concepts of limits which might subsequently hinder their understanding of other topics. The students were fully engaged in task-solving during lectures. They solved from the blackboard as other peers observed and critiqued or paper and pen in their small groups. However, due to the complexity of task 2, some students lacked prior conceptual and procedural understanding and were unable to complete the tasks. They instead kept on requesting fellow students to complete them or their lecturers to solve the would-be students' problems. Students were asked whether or not they found evaluating the limit of functions harder than other tasks in calculus, and the response was in affirmative.

When students were asked if they found specific concepts on limits challenging, the students agreed and asked if there were alternative approaches to evaluating limits of functions than the analytical approach. In this case, students sought other approaches to the analytical method. Specifically, the application of L' Hopital's rule and rationalization were presumably harder procedures for answering the two tasks involving multiple calculations. To visualize and evaluate the limit of functions of the two tasks, external multiple representations were used. Students were amazed and excited to observe and realized the same answers that were obtained through the analytical approach.

DISCUSSION

The present study investigated the significance of external multiple representations in enhancing the learning of limits of functions. The above results support the conceptual and theoretical framework. The results of the present study confirmed our hypothesis that students face learning challenges in the limits of functions. The results, thus, address the stated research question of whether or not multiple representations may support students' conceptual understanding of the limits of functions. There is a connection between students' prior conceptual understanding, positive attitude, and confidence in learning mathematics. Juter (2005b) investigated students' attitude towards solving the limit of functions. The results revealed a positive relationship between students' confidence and attitudes towards mathematics and their ability to solve tasks on limits of functions. Indeed, students with positive attitudes performed better in solving tasks on limits. The author further noted the importance of a favorable student learning environment as this may offer and support varied opportunities for discussion and problem-solving.

Some students, however, applied previous conceptual knowledge and understanding to consolidate their knowledge thereby enhancing their problem-solving skills. Some students claimed they worked excessively hard to understand concepts of limits of functions that they had not grasped previously. When multiple graphical representations were applied, students were able to visualize the limits of functions and compare and contrast the solutions. Some students who had solved the two tasks analytically in their small discussion groups or individually during problem-solving sessions quickly noted that the external graphical representations supported the problem-solving strategies. This helped to demystify students' fear that the limits of functions were hard to conceptualize. Consequently, students' confidence and abilities were enhanced.

The fact that many students had weak conceptual mappings, and answered partially or eluded one or all the two tasks in the present study indicates that understanding mathematical concepts require several approaches and not just one. In this study, the analytical approach seemed to have yielded negative results. Multiple representations perhaps enhanced the learning of limits of functions.

This is also emphasized in Ainsworth's (2008) recommendations on the application of multiple representations as diagrams, graphs and equations may bring unique benefits. This is because mathematics is not all about solving problems analytically.

The interviews conducted with selected students revealed that some students did not solve the two tasks. The tasks were either too difficult for them to solve or the reasons for not solving the two tasks fully might be attributed to a lack of interest and confidence. The problem-solving sessions confirm this claim as some students were observed using the wrong approaches or failing to solve completely. Students who solved tasks in class or in their small groups showed a lack of confidence and positive attitudes towards the limits of functions. This was further revealed through face-to-face interactions with students' peers. The students were aware that the new approach worked. Although many of them still lacked confidence in applying information and telecommunications technology (ICT) gadgets. This means that the effective application of available gadgets in computer laboratories may boost the application of multiple representations to adequately answer tasks on limits of functions.

About half of the students were revealed to have learned by rote, meaning that multiple representations consequently enhanced their conceptual understanding. If students are learning the limits of functions by rote, educators may not guarantee specific concepts they may remember since many students find the limits of functions difficult to understand (Desai & Bush, 2021; Dreher et al., 2015). “If students are unable to understand the concept's critical features, then they do not know what to learn by heart” (Juter, 2005b). Multiple representations may help students to learn and remember, for example, the application of rationalization of functions and L' Hopital's rule in evaluating limits of functions. This is because the understanding of the limits of functions requires time and effort for most students to fully understand. Table 1, Table 2, and Table 3 indicate that the limits of functions are hard for students to understand. Therefore, educators should try as much as possible to vary approaches that cultivate and develop students' positive attitude towards the limits of functions and mathematics generally. If students successfully construct their cognitive representations, the use of external multiple representations and a positive attitude towards the graphical representation of limits of functions may be guaranteed (Liang, 2016; Juter, 2005b).

CONCLUSIONS

This study investigated the significance of external multiple representations in enhancing the learning of limits of functions. To answer the research question of whether or not multiple representations enhanced students' understanding of the limits of functions, it was observed that most students learned by rote. Indeed, multiple representations enhanced students' critical thinking and problem-solving strategies. Limits of functions were regarded to be one of the most difficult topics to understand. The implication is that most students either worked hard or applied cram work to answer tasks on the limits of functions. Yet, the limits of functions provide prerequisite knowledge for understanding differentiation, integration, sequences, and series. Therefore, the integration of new previous knowledge and understanding of the existing concept images is significant in ensuring students' conceptual understanding.

The study conducted by Aguilar and Telese (2018) revealed that procedural fluency, conceptual understanding, and problem-solving strategies enhances students' understanding of non-routine mathematical tasks. The fact that limits of functions are important in learning subsequent topics (e.g., differentiation, integration, and sequences and series), students are encouraged to apply several approaches including multiple representations to confirm, compare and contrast the solutions to several tasks. In so doing, students' conceptual understanding, procedural fluency, and attitude towards the limits of functions can be enhanced. The results from the present study also provide educators with evidence of students' flawed concepts. This points to the importance of prior knowledge and understanding and applying it in subsequent learning. This is vital in conceptualizing the limits of functions and may trigger their attitudes towards the topic.

We, therefore, recommend future studies in different or similar settings and contexts, and in different mathematics topics with the diversity of methods of multiple representations to compare and contrast our findings, and to gain deeper and broader insights into students' understanding and their attitude towards limits of functions and multiple representations generally. Students' attitudes point to issues related to their latent constructs for learning mathematics. Specifically, to gain more insight, this research recommends that future researchers should apply multiple representations to investigate other properties of limits not covered in this study (e.g., tangent lines as limits, use of squeezing theorem to compare limits of functions, and functions whose limit does not exist). This is a potential area for a further investigation aimed at improving the instructional strategies, the teachers' pedagogical content knowledge, and mathematical knowledge for teaching. To achieve this, the teachers' may routinely come together to hold their professional development programs aimed at emphasizing content knowledge and pedagogical content knowledge of learning LP.

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The Problem Corner

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The Purpose of **The Problem Corner** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

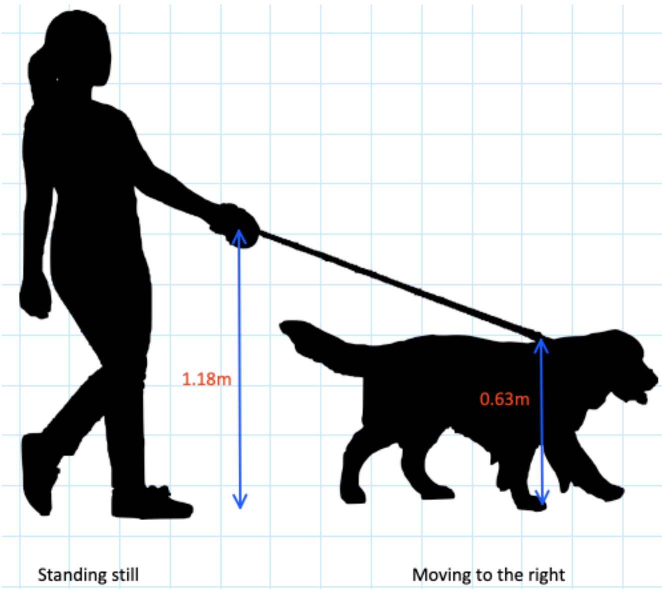
Greetings, fellow problem solvers!
I am delighted to announce that I have received solutions to both Problem 12 and Problem 13, and I am pleased to report that they were all correct, as well as fascinating and innovative. By showcasing what I deemed to be the most outstanding solutions, I aim to enrich and elevate the mathematical understanding of our global community.

Solutions to **Problems** from the Previous Issue

Interesting "Dog walking" problem.

Problem 12

Proposed by Ivan Retamoso, BMCC, USA.
Eva is standing still holding her dog via an extendable leash which she keeps at the heigh of $1.18\ m$ above the ground as shown in the figure below, suddenly her dog walks to the right at a constant speed of $0.9\frac{m}{s}$, at what rate is the leash extending when the end of the leash is $3m$ horizontally away from Eva?



Solution to problem 12

By Phuong Uy Nguyen, Borough of Manhattan Community College, Vietnam.

This efficient solution employs carefully chosen variables to establish an equation based on the Pythagorean theorem, ensuring its validity throughout Eva and her dog's motion. By isolating the main variable and utilizing differentiation with respect to time, incorporating the chain rule, our problem solver determines the rate at which the leash extends for the specified distance.

(Q) Let z = The length of the leash
x = Horizontal distance between Eva and the end the leash.

Using the hint: $0.55^2 + x^2 = z^2$

$\Rightarrow z = \sqrt{(0.55^2 + x^2)}$

Take derivatives with respect to time (t) of both side equation

$$\frac{dz}{dt} = \frac{1}{2}(0.55^2 + x^2)^{\left(\frac{-1}{2}\right)} \cdot (2x)\left(\frac{dx}{dt}\right)$$

$$\Rightarrow \frac{dz}{dt} = \left(\frac{x}{\sqrt{0.55^2 + x^2}}\right) \cdot \left(\frac{dx}{dt}\right)$$

Let $x = 3m$ and $\frac{dx}{dt} = 0.9 m/s$

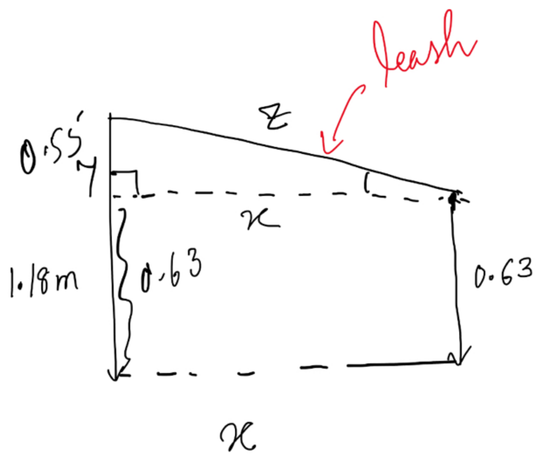
$$\frac{dz}{dt} = \left(\frac{3}{\sqrt{0.55^2 + 3^2}}\right) \times (0.9) = 0.885 \text{ m/s}$$

Conclusion, the leash is extending at a rate of 0.885 m/s when the end of the leash is 3m horizontally away from Eva

Second Solution to problem 12

By Aradhana Kumari, Borough of Manhattan Community College, USA.

This second solution adopts a different approach by employing implicit differentiation. After finding the derivative with respect to time for both sides of the equation, which holds true throughout Eva and her dog's motion, the required rate of change is determined. Each step of the solution is thoroughly justified, ensuring clarity and accuracy. Furthermore, a diagram is included to enhance visualization and provide a clearer understanding of the problem.
Solution: As per question we have the below diagram



From the above diagram
When x=3, y= (1.18-0.63= 0.55) we have
$x^2 + y^2 = z^2$.............(1)
$3^2 + (0.55)^2 = z^2$
$9 + 0.3025 = z^2$
Hence $z = \sqrt{(9.3025)}$
$z = 3.05$
Differentiate the equation given by (1) with respect to time we get
$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$
$x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$ ....(2)
Substituting the value x=3, $\frac{dx}{dt} = .9$, z = 3.05, and $\frac{dy}{dt} = 0$ (Since the dog is moving with the constant speed on the x-direction) in equation in given by (2) we get
$3\times(0.9) = 3.05\frac{dz}{dt}$
$\frac{dz}{dt} = \frac{3\times(0.9)}{3.05} = .88524590163 \approx .89$ m/s
Hence the rate at which the lease extending when the end of the lease is 3m horizontally away from the Eva is approximately .89 m/s.

"Intermediate Value Theorem" problem.

Problem 13

Proposed by Ivan Retamoso, BMCC, USA.

Prove that the equation $x^3 - 14x + k = 0$ where k is any real number, has at most one real number solution in the interval $[-2,2]$.

Solution to problem 13

By Jesse Wolf, Borough of Manhattan Community College, USA.

This comprehensive step-by-step solution utilizes various mathematical tools to ensure a thorough analysis. By incorporating the first derivative, the second derivative test, and The Intermediate Value Theorem, this solution covers all possible cases and provides rigorous justifications for each step.

$y = x^3 - 14x + k$
$y' = 3x^2 - 14$
$y'' = 6x$

$y' = 0 \implies x = \pm\sqrt{\frac{14}{3}} \approx \pm 2.2.$

$y''\left(-\sqrt{\frac{14}{3}}\right) < 0;\ y''\left(\sqrt{\frac{14}{3}}\right) > 0.$

Second Derivative Test $\implies$ there exists a relative max at $x = -\sqrt{\frac{14}{3}}$ and a relative min at $x = \sqrt{\frac{14}{3}}$.

So, $y\left(-\sqrt{\frac{14}{3}}\right) > y\left(\sqrt{\frac{14}{3}}\right)$ and is strictly decreasing on $\left(-\sqrt{\frac{14}{3}}, \sqrt{\frac{14}{3}}\right)$ which contains $[-2,2]$.

1:

If $y\left(-\sqrt{\frac{14}{3}}\right) > 0$ and $y\left(\sqrt{\frac{14}{3}}\right) \geq 0 \Rightarrow$ there exist 0 zeros of y on $\left(-\sqrt{\frac{14}{3}}, \sqrt{\frac{14}{3}}\right)$ and thus 0 zeros on $[\mathbf{-2,2}]$.

2:

If $y\left(-\sqrt{\frac{14}{3}}\right) > 0$ and $y\left(\sqrt{\frac{14}{3}}\right) < 0 \Rightarrow$ there exists at most one zero of y on $[-2,2]$.

2a:

The fact that y is continuous and strictly decreasing on $\left(-\sqrt{\frac{14}{3}}, \sqrt{\frac{14}{3}}\right) \Rightarrow$ (via the Intermediate Value Theorem) that there exists a unique zero on that interval.

If the zero occurs on $\left(-\sqrt{\frac{14}{3}}, 2\right)$ or $\left(2, \sqrt{\frac{14}{3}}\right)$ then there exist 0 zeros on $[\mathbf{-2,2}]$.

2b:

If the zero occurs on $[-2,2]$ there exist 1 zero on $[\mathbf{-2,2}]$.

3:

If $y\left(-\sqrt{\frac{14}{3}}\right) \leq 0$ and $y\left(\sqrt{\frac{14}{3}}\right) < 0 \Rightarrow$ there exists 0 zeros of y on $\left(-\sqrt{\frac{14}{3}}, \sqrt{\frac{14}{3}}\right)$ and thus 0 zeros on $[\mathbf{-2,2}]$.

QED

Second Solution to problem 13

By Ivan Retamoso (proposer), Borough of Manhattan Community College, USA.

This alternative solution employs a distinct methodology. It initiates by establishing the negativity of the derivative of the left side of the equation within the given interval. Consequently, it deduces that the left side of the equation exhibits strict monotonicity, specifically, it must be strictly decreasing over the provided interval.

Let $f(x) = x^3 - 14x + k$
Then $f'(x) = 3x^2 - 14$
For all x in $[-2,2]$

$-2 \leq x \leq 2$
$|x| \leq 2$
$|x|^2 \leq 2^2$
$x^2 \leq 4$
$3x^2 \leq 12$
$3x^2 - 14 \leq -2 < 0$
Then $f'(x) < 0$ for all x in $[-2,2]$
Then $f(x)$ is strictly decreasing on $[-2,2]$, since $f(x)$ is continuous then it means that the graph of $f(x)$ will intersect the x axis at most once.
Therefore, the equation $x^3 - 14x + k = 0$ where k is any real number, has at most one real number solution in the interval $[-2,2]$.

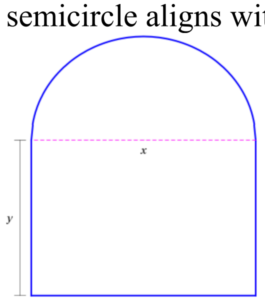
Dear fellow problem solvers,

I have confidence that your experience in solving problems 12 and 13 was not only enjoyable but also resulted in valuable insights. Now, let's progress to the next set of challenges, as I am genuinely thrilled to present you with the following two problems.

Problem 14

Proposed by Ivan Retamoso, BMCC, USA.

Let's imagine a scenario where a corral is being enclosed using 130 ft of fencing. The corral is in the shape of a rectangle, and it has a semicircle attached to one of its sides. The diameter of the semicircle aligns with the length of the rectangle, as depicted in the figure provided.



Determine the values of x and y that will result in the corral having the largest possible area.

Problem 15

Proposed by Ivan Retamoso, BMCC, USA.

x, y, and z are real numbers such that $x + y + z = 17$ and $\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} = \frac{4}{15}$ find the exact value of $\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$.