The Problem Corner

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The Purpose of **The Problem Corner** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem and its solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Greetings, fellow problem solvers!

As the editor of The Problem Corner, I am delighted to announce that we have received accurate solutions for both Problem 24 and Problem 25. These submissions not only met the criteria for correctness but also exemplified effective strategic application. Our primary aim is to showcase what we believe are the best solutions to foster and elevate mathematical knowledge within our global community.

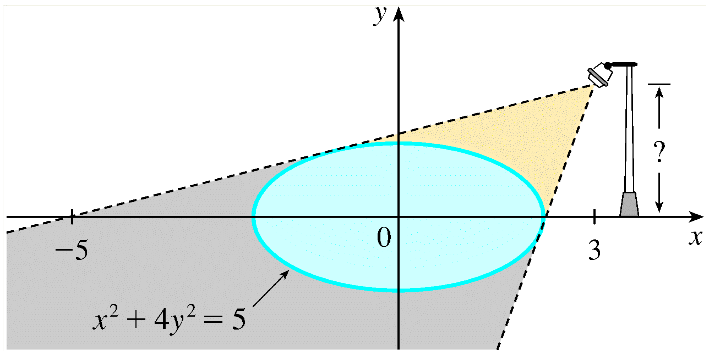
Solutions to **Problems** from the Previous Issue.

Deciphering the Puzzle of Lamp Height.

Problem 24

Proposed by Ivan Retamoso, BMCC, USA.

The diagram illustrates a lamp positioned three units to the right of the y-axis and casting a shadow due to the elliptical region defined by the equation $x^2 + 4y^2 \leq 5$. Given that the point $(-5, 0)$ lies on the shadow's edge, how far above the x-axis is the lamp located?

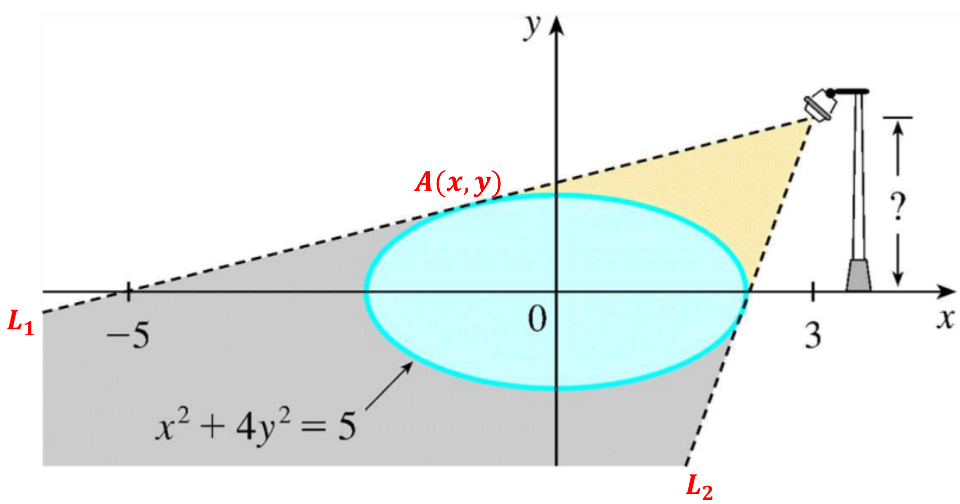


First solution to problem 24

By Dr. Abdullah Kurudirek, Mathematics Ed. Department, Tishk International University.

This efficient solution combines the power of implicit differentiation from Calculus with Analytic Geometry to compute the slope of a line tangent to the ellipse in two different but equivalent ways. The solution to the problem then smoothly follows.

As can be easily seen in the given figure, there are two tangent lines shown with dashed lines to the elliptical area. Let's denote them as L_1 and L_2. Additionally, let's call the point tangent to the ellipse on the left side by $A(x, y)$. We can find the slopes of these lines with the help of the implicit function's differentiation and two points that lie on a line.



Slope of L_1 can be calculated as $2x + 8y\frac{dy}{dx} = 0$, by using implicit differentiation. So, $m_{L1} = -\frac{x}{4y}$
Slope of L_1 can be calculated by using the points $A(x, y)$ and $(-5,0)$ on this line. So, $m_{L1} = \frac{y}{x+5}$
These slopes have both x and y so they can be equated as follows:

$$\frac{y}{x+5} = -\frac{x}{4y}$$

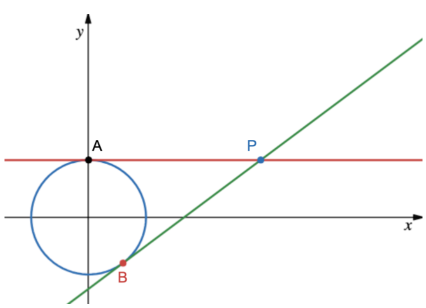
After cross-product we get $4y^2 = -x^2 - 5x$ and also knowing that $x^2 + 4y^2 = 5$. After substituting the variable y for x we get $x = -1$ and then $y = 1$. Then calculate the $m_{L1} = -\frac{x}{4y} = \frac{1}{4}$ and the equation of line L_1 will be $y - 1 = \frac{1}{4}(x + 1)$. The location of the lamp $(3, h)$ must satisfy the line L_1.
Therefore, $h - 1 = \frac{1}{4}(3 + 1)$ and $h = 2$.

The Challenge of Finding Coordinates for a Point Outside a Circle.

Problem 25

Proposed by Ivan Retamoso, BMCC, USA.

The blue circle $x^2 + y^2 = 25$ has tangent lines at the points A and B.



The point B has x-coordinate 3.
The tangent lines meet at the point P.
Find the coordinates of the point P.

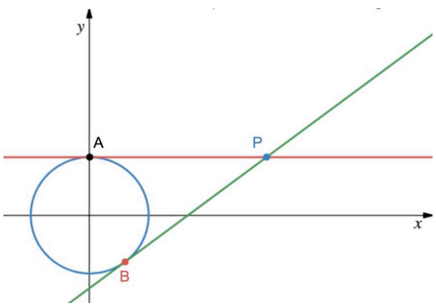
First solution to problem 25

By Dr. Hosseinali Gholami, University Putra Malaysia, Serdang, Malaysia.

Our solver presents two different approaches to solve this problem: one using the distance formula and the other using derivatives from Calculus. Labels and equations are added to facilitate the explanations.

Solution 1:

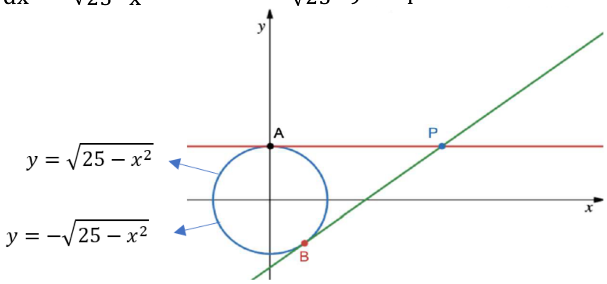
We know the point $B(3, m)$ is on the graph of the circle $x^2 + y^2 = 25$. Therefore, the equation $3^2 + m^2 = 25$ gives $m = \pm 4$. Based on the following figure, only $m = -4$ is acceptable and the coordinates of B are $(3, -4)$.



Using the points $A(0, 5)$, $P(n, 5)$ and the relation $PA = PB$ we obtain the following equation:
$n = \sqrt{(n-3)^2 + 9^2} \Rightarrow n^2 = (n-3)^2 + 9^2 \Rightarrow 6\text{n} = 90 \Rightarrow \text{n} = 15.$
Therefore, the coordinates of point P are $(15,5)$.

Solution 2:

$\boldsymbol{x^2} + \boldsymbol{y^2} = 25 \Rightarrow \text{y} = \pm\sqrt{25 - x^2}$. We consider two functions $y = \sqrt{25 - x^2}$ and $y = -\sqrt{25 - x^2}$ separately. The point B is on the graph of the function $y = -\sqrt{25 - x^2}$.
$\frac{\text{dy}}{\text{dx}} = \frac{\text{x}}{\sqrt{25-\text{x}^2}} \Rightarrow a_{PB} = \frac{3}{\sqrt{25-9}} = \frac{3}{4}$.



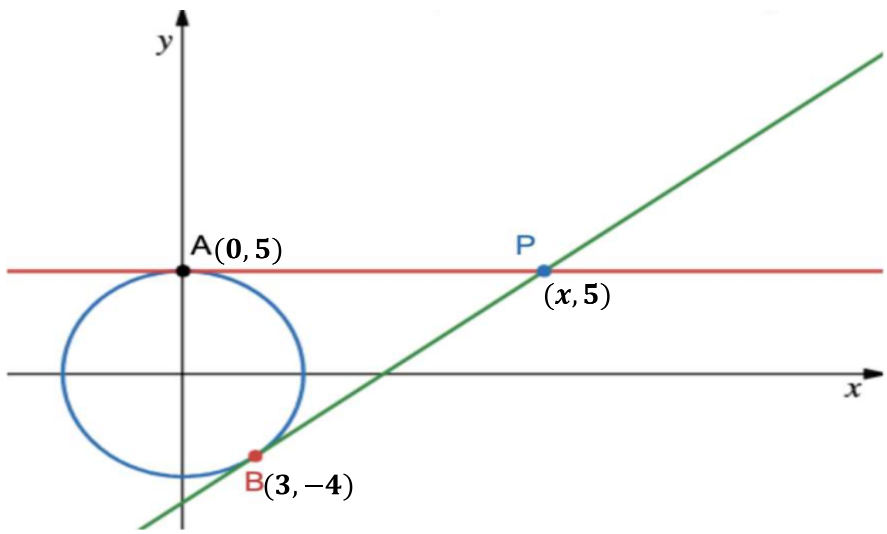
It is clear that the coordinates of B are $(3, -4)$. The equation of tangent line PB is as below.
$y = ax + b \Rightarrow \text{y} = \frac{3}{4}x + b \Rightarrow -4 = \frac{3}{4} \times 3 + b \Rightarrow \text{b} = \frac{-25}{4} \Rightarrow \text{y} = \frac{3}{4}x - \frac{25}{4}$.
The point $P(m, 5)$ is on the tangent line PB. Thus, we have:
$y = \frac{3}{4}x - \frac{25}{4} \Rightarrow 5 = \frac{3}{4}m - \frac{25}{4} \Rightarrow \text{m} = 15.$
It means the coordinates of point P are $(15, 5)$.

Second solution to problem 25

By Dr. Abdullah Kurudirek, Mathematics Ed. Department, Tishk International University.

This alternative solution combines differentiation from Calculus with the slope formula from Analytic Geometry.

We can easily determine the coordinates of the points $A(0,5)$, $B(3,-4)$, and $P(x,5)$ in the given figure below.



By using the points $B(3,-4)$, and $P(x,5)$ the slope of the green line can be calculated as $m_{green} = \frac{9}{x-3}$.

The green line is also a tangent line to the circle at the point B. The slope of this line can be calculated by differentiation as $\frac{dy}{dx} = m_{green} = -\frac{x}{y} = \frac{3}{4}$.

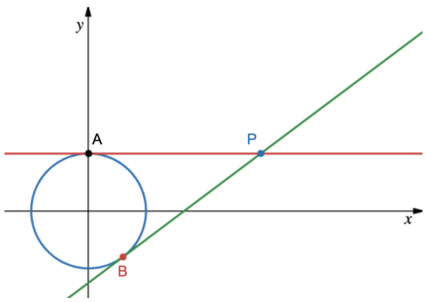
These slopes must be the same so, $\frac{9}{x-3} = \frac{3}{4}$ then $x = 15$ and the coordinates of the point $P(15,5)$.

Third solution to problem 25

By Dr. Aradhana Kumari, Borough of Manhattan Community College, USA.

This solution meticulously details the process of finding the coordinates of point B through the equation of a circle in Cartesian geometry. It then employs implicit differentiation from calculus *to pinpoint the coordinates of point P. If you enjoy thorough explanations and a methodical approach, you'll find this solution quite rewarding.*

Solution: Consider the below Circle



From the figure above, A lies on the y- axis. To find the coordinate of A which lies on y-axis we substitute $x = 0$ in the equation given by (1) we get

$$y^2 = 25$$

Hence $y = 5$ *or* $y = -5$. As from the figure y is positive hence y is 5.

Therefore, the coordinate of A is (0,5).

The equation of line passing through A which is tangent to the circle at A is

y = 5.

From the figure above the y-coordinate of the point P is 5.

From the above figure x- coordinate of point B is 3 and point B lies on the circle hence it satisfies the equation of circle therefore

$$3^2 + y^2 = 25$$

Therefore

$$y^2 = 25 - 9 = 16$$

$$y^2 = 16$$

Hence $y = 4$ or $y = -4$. Since B lies on the fourth quadrant hence the y- coordinate of B is -4. Therefore, the coordinate of B is $(3, -4)$.

To find equation of tangent line at B. We differentiate the equation given by (1). We get

$\frac{d}{dx}(x^2 + (y))^2 = \frac{d}{dx}(25)$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

Hence $\frac{dy}{dx}$ at $(3, -4)$ is $\frac{-2(3)}{2(-4)} = \frac{3}{4}$

The equation of line which is tangent to the circle at the point B with slope $\frac{3}{4}$ and passes through the point B $(3, -4)$ is $y = \frac{3}{4}x + b$

Substituting the coordinate of point B $(3, -4)$ we get $-4 = \frac{3}{4}(3) + b$

Hence b = $\frac{-25}{4}$. Hence the equation of the line which is tangent to the circle at the point B is the line $y = \frac{3}{4}x - \frac{25}{4}$ ....(2)

To find the x-coordinate of the point P we substitute $y = 5$ we get

$5 = \frac{3}{4}x - \frac{25}{4}$

Hence $x = 15$.

Hence the coordinate of the point P is (15, 5).

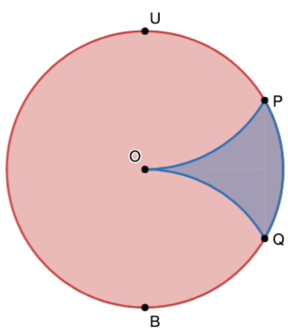
Dear fellow problem solvers,

I'm pleased you enjoyed solving problems 24 and 25 and have gained valuable new strategies for your mathematical toolkit. Let's proceed to the next set of problems to further sharpen your skills.

Problem 26

Proposed by Ivan Retamoso, BMCC, USA.

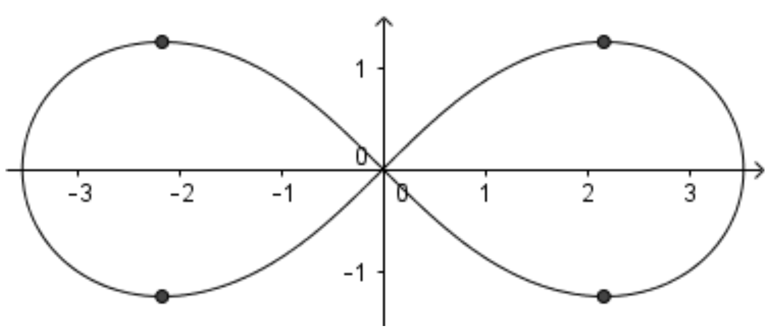
The diagram illustrates a circle with a radius of 6 inches and center O, with UB as its diameter. Points P and Q are positioned on the circle so that OP and OQ are arcs of circles with a radius of 6 inches and centers at U and B, respectively. Determine, in exact form, the area of the "blue" region OPQ.



Problem 27

Proposed by Ivan Retamoso, BMCC, USA.

Consider the lemniscate curve $2(x^2 + y^2)^2 = 25(x^2 - y^2)$.



a. Find the slope of the tangent line to the lemniscate in terms of the variables x and y.

b. The four points on the lemniscate where the tangent line is horizontal are all on the intersection of the lemniscate with circle $x^2 + y^2 = k$, find the value of k.