The Problem Corner

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The Purpose of **The Problem Corner** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem and its solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Greetings, fellow problem solvers!

I'm pleased to share that I've obtained answers for both Problem 16 and Problem 17. It brings me great joy to report that not only were all of them accurate, but they were also remarkably captivating and inventive. My main objective in highlighting what I believe to be the most exceptional solutions is to enhance and elevate the mathematical knowledge within our worldwide community.

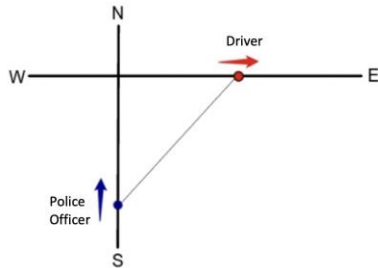Solutions to **Problems** from the Previous Issue.

Interesting "Police Officer and Driver" problem.

Problem 16

Proposed by Ivan Retamoso, BMCC, USA.

Let's consider a situation where a police officer is situated $\frac{1}{2}$ mile to the south of an intersection. This officer is driving northwards towards the intersection at a speed of $35\ mph$. At the exact same time, there is another car located $\frac{1}{2}$ mile to the east of the intersection, and it is moving eastward, away from the intersection.

a) Let's assume that the officer's radar gun displays a speed of $20\ mph$ when aimed at the other car. This reading indicates that the straight-line distance between the officer and the other car is increasing at a rate of $20\ mph$. What, then, is the speed of the other car?

b) Now, let's consider a different scenario where the officer's radar gun displays $-20\ mph$ instead. This indicates that the straight-line distance between the officer and the other car is decreasing at a rate of $20\ mph$. What is the speed of the other car in this situation?

Note: Round yours answers to three decimals places.

Solution to problem 16

By Aradhana Kumari, Borough of Manhattan Community College, USA.
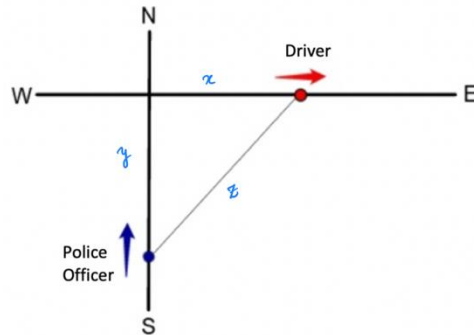
Our solver initiates with precise variable labels to represent evolving distances, then employs the Pythagorean theorem to establish their relationships. Finally, implicit differentiation is applied

to uncover the relationship between the rates of change in distances over time, leading to the final answers.

Solution: a) Consider the diagram below

We have $x^2 + y^2 = z^2$ ....(1) (using the Pythagorean triangle)

In our given problem $x = \frac{1}{2}, \ y = \frac{1}{2}$

Using equation (1) we have

$$z = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{2}{4}} = \sqrt{\frac{1}{2}} \ \text{.......} \ (2)$$

Differentiating the equation given by (1) we get

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt} \ \text{.......}(3)$$

As per question $x = \frac{1}{2}, \ y = \frac{1}{2} , \frac{dz}{dt} = 20, \frac{dy}{dt} = -35$

Since y is increasing in the north direction, going towards the intersection y is decreasing hence we have to $\frac{dy}{dt} = -35$

$$z = \sqrt{\frac{1}{2}} \quad \text{(from (2))}$$

From equation given by (3) we have

$$\frac{dx}{dt} = \frac{z}{x} \frac{dz}{dt} - \frac{y}{x} \frac{dy}{dt}$$

$$\frac{dx}{dt} = \frac{\sqrt{\frac{1}{2}}}{\frac{1}{2}} \ 20 - \frac{\frac{1}{2}}{\frac{1}{2}} \ (-35) = 63.284 \ mph$$

b) As per question we have $\frac{dz}{dt} = -20$

$$\frac{dx}{dt} = \frac{z}{x}\frac{dz}{dt} - \frac{y}{x}\frac{dy}{dt}$$

$$= \frac{\sqrt{\frac{1}{2}}}{\frac{1}{2}}(-20) - \frac{\frac{1}{2}}{\frac{1}{2}}(-35)$$

$$= \sqrt{2}\,(-20) + 35$$

$$= 6.716\ mph$$

"Looking for a pattern" problem.

Problem 17

Proposed by Christopher Ingrassia, Kingsborough Community College (CUNY)

Brooklyn, NY, USA

Suppose $n \times n$ matrix A and $n \times 1$ vector x are defined as follows:
$$A_{i,j} = \begin{cases} 1, & i \geq j \\ 0, & otherwise \end{cases}$$
$$x_i = 1$$
Describe, in words, the vector $A^k x$, where $k \geq 0$.
Find an expression for the quantity $x^T A^k x$ in terms of n and k (x^T is the transpose of vector x).

Solution to problem 17

By Aradhana Kumari, Borough of Manhattan Community College, USA.

In this clearly articulated solution, our solver step by step and systematically demonstrates how a basic matrix operation generates recognizable patterns, ultimately leading to a broader conclusion derived from inductive reasoning based on the diagonals of Pascal's triangle.

Solution: For k = 0 we have

A^k (x) = x.

For k ≥ 1, A^k x calculation is below

$$A^1_{1\times1}(x) = [1]\,x = x \quad , \quad A^2_{1\times1}(x) = x$$

$$A^1_{2\times2}(x) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A^2_{2\times2}(x) = A_{2\times2}A_{2\times2}(x) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1\times1 \\ 1\times1 + 1\times2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2c_0 \\ 3c_1 \end{bmatrix}$$

$$A^3_{2\times2}(x) = A_{2\times2}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1\times1 \\ 1\times1 + 1\times3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1+3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3c_0 \\ 4c_1 \end{bmatrix}$$

$$A^4_{2\times2}(x) = A_{2\times2}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1\times1 \\ 1\times1 + 1\times4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1+4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 4c_0 \\ 5c_1 \end{bmatrix}$$

$$A^K_{2\times2}(x) = \begin{bmatrix} Kc_0 \\ K+1c_1 \end{bmatrix}$$

$$A^1_{3\times3}(x) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1\times1 + 1\times1 \\ 1\times1 + 1\times1 + 1\times1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1c_0 \\ 2c_1 \\ 3c_2 \end{bmatrix}$$

$$A^2_{3\times3}(x) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1\times1 + 1\times2 \\ 1\times1 + 1\times2 + 1\times3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 2c_0 \\ 3c_1 \\ 4c_2 \end{bmatrix}$$
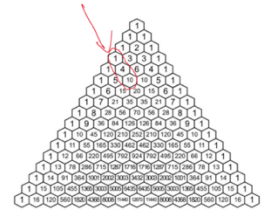
$$A^3_{3\times3}(x) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1\times1 + 1\times3 \\ 1\times1 + 1\times3 + 1\times6 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 10 \end{bmatrix} = \begin{bmatrix} 3c_0 \\ 4c_1 \\ 5c_2 \end{bmatrix} =$$

$$A^K_{3\times3}(x) = \begin{bmatrix} Kc_0 \\ K+1c_1 \\ K+2c_2 \end{bmatrix}$$

$$A^1_{4\times4}(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1\times1+1\times1 \\ 1\times1+1\times1+1\times1 \\ 1\times1+1\times1+1\times1+1\times1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1_{C_0} \\ 2_{C_1} \\ 3_{C_2} \\ 4_{C_3} \end{bmatrix}$$

$$A^2_{4\times4}(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1\times1+1\times2 \\ 1\times1+1\times2+1\times3 \\ 1\times1+1\times2+1\times3+1\times4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1+2 \\ 1+2+3 \\ 1+2+3+4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} 2_{C_0} \\ 3_{C_1} \\ 4_{C_2} \\ 5_{C_3} \end{bmatrix} =$$
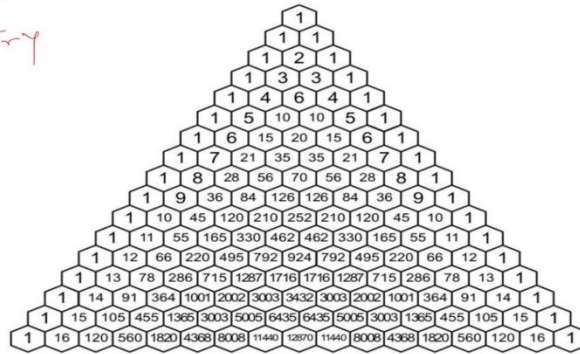
$$A^3_{4\times4}(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 1\times1+1\times3 \\ 1\times1+1\times3+1\times6 \\ 1\times1+1\times3+1\times6+1\times10 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 3_{C_0} \\ 4_{C_1} \\ 5_{C_2} \\ 6_{C_3} \end{bmatrix} =$$

$$\vdots$$

$$A^K_{4\times4}(x) = \begin{bmatrix} K_{C_0} \\ K+1_{C_1} \\ K+2_{C_2} \\ K+3_{C_3} \end{bmatrix}_{4\times1}$$

$$A^K_{m\times n}(x) = \begin{bmatrix} K_{C_0} \\ K+1_{C_1} \\ K+2_{C_2} \\ \vdots \\ K+n-1_{C_{n-1}} \end{bmatrix}_{n\times1} =$$

1st entry
2nd entry

n^{th} entry of $A^K_{n\times n}(x)$
$= (n-2)^{th}$ entry from the Pascal's triangle

counting 1 as the first entry
& $(K+1)$ as the second entry

Expression for $x^T A^k x$

For k = 0 we have $x^T A^k x = 1$.

For k ≥ 1, $x^T A^k x$ calculation is below

$\underline{x^T A x}$, A and x as defined in the problem

$A = [1]$, $x^T A x = 1 \times 1 \times 1 = 1$

$A_{2\times2} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $x^T \underset{2\times2}{A^1} x = \underset{1\times2}{[1\ 1]} \underset{2\times2}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

$= \underset{1\times2}{[1\ 1]} \underset{2\times1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} = 1\times1 + 1\times2 = 1+2 = {}^1c_0 + {}^2c_1$

$A^2_{2\times2} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $x^T \underset{2\times2}{A^2} x = [1\ 1] \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [1\ 1] \begin{bmatrix} 1 \\ 3 \end{bmatrix} = (1\times1) + (1\times3) = {}^2c_0 + {}^3c_1$

$x^T \underset{2\times2}{A^3} x = [1, 1] \begin{bmatrix} 1 \\ 4 \end{bmatrix} = (1\times1) + (1\times4) = 1+4 = {}^3c_0 + {}^4c_1$

$\vdots$

$x^T \underset{2\times2}{A^k} x = {}^kc_0 + {}^{k+1}c_1$

$A^1_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $x^T \underset{3\times3}{A^1} x = [1\ 1\ 1] \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = [1\ 1\ 1] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$= (1\times1) + (1\times2) + (1\times3)$
$= 1+2+3 = {}^1c_0 + {}^2c_1 + {}^3c_2$

$x^T \underset{3\times3}{A^2} x = [1\ 1\ 1] \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} = (1\times1) + (1\times3) + (1\times6) = 1+3+6 = {}^2c_0 + {}^3c_1 + {}^4c_2$

$x^T \underset{3\times3}{A^3} x = [1\ 1\ 1] \begin{bmatrix} 1 \\ 4 \\ 10 \end{bmatrix} = (1\times1) + (1\times4) + (1\times10) = 1+4+10 = {}^3c_0 + {}^4c_1 + {}^5c_2$

$x^T \underset{3\times3}{A^4} x = [1\ 1\ 1] \begin{bmatrix} 1 \\ 5 \\ 15 \end{bmatrix} = (1\times1) + (1\times5) + (1\times15) = 1+5+15 = {}^4c_0 + {}^5c_1 + {}^6c_2$

$\vdots$

$x^T \underset{3\times3}{A^k} x = {}^kc_0 + {}^{k+1}c_1 + {}^{k+2}c_2$

$$A_{4 \times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad x^T A^1_{4 \times 4} x = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = (1 \times 1) + (1 \times 2) + (1 \times 3) + (1 \times 4)$$

$$= 1 + 2 + 3 + 4$$

$$= {}^1 c_0 + {}^2 c_1 + {}^3 c_2 + {}^4 c_3$$

$$x^T A^2_{4 \times 4} x = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 6 \\ 10 \end{bmatrix} = 1 \times 1 + 1 \times 3 + 1 \times 6 + 1 \times 10$$

$$= 1 + 3 + 6 + 10$$

$$= {}^2 c_0 + {}^3 c_1 + {}^4 c_2 + {}^5 c_3$$

$$x^T A^3_{4 \times 4} x = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 10 \\ 20 \end{bmatrix} = 1 \times 1 + 1 \times 4 + 1 \times 10 + 1 \times 20$$

$$= 1 + 4 + 10 + 20$$

$$= {}^3 c_0 + {}^4 c_1 + {}^5 c_2 + {}^6 c_3$$

$$\vdots$$

$$x^T A^K_{4 \times 4} (x) = {}^K c_0 + {}^{K+1} c_1 + {}^{K+2} c_2 + {}^{K+3} c_3$$

$$\vdots$$

For $n \times n$ matrix A, where

$$A_{ij} = \begin{cases} 1 & i \geq j \\ 0 & \text{otherwise} \end{cases}$$

$$x_i = 1$$

We have

$$x^T A^K_{n \times n} x = {}^K c_0 + {}^{K+1} c_1 + {}^{K+2} c_2 + \cdots + {}^{K+n-1} c_{n-1}$$

Second Solution to problem 17

By Christopher Ingrassia (The proposer)
Kingsborough Community College (CUNY)
Brooklyn, NY, USA

Matrix A has the form $\begin{bmatrix} 1 & & \cdots & & 0 \\ & 1 & 0 & 0 & \\ \vdots & 1 & 1 & 0 & \vdots \\ & 1 & 1 & 1 & \\ 1 & & \cdots & & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}$.

Because the zero power of a matrix is the identity, $A^0 x$ simply yields x, a column of ones.

When $k > 0$, A^k can be thought of as a linear operator which computes a cumulative sum of the

elements of x: $A^1 x$ produces $\begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{bmatrix}$, $A^2 x = \begin{bmatrix} 1 \\ 3 \\ 6 \\ \vdots \\ \frac{n(n+1)}{2} \end{bmatrix}$, $A^3 x = \begin{bmatrix} 1 \\ 4 \\ 10 \\ \vdots \\ \sum_i (A^2 x)_i \end{bmatrix}$, $\dots$

These are the diagonals of Pascal's triangle. If we number the diagonals starting with 0 for the diagonal of all ones, $A^k x$ yields the first n entries of the kth diagonal.

The product $x^T A^k x$ simply sums the elements of vector $A^k x$. This value is found on Pascal's triangle one row below and just to the left of the nth entry of the kth diagonal. **This value, in terms of n and k, is** $C(n + k, n - 1) = \binom{n + k}{n - 1}$.

Because each entry of Pascal's triangle represents a binomial coefficient, this quantity may be expressed as the sum of the first n entries of diagonal k:

$\binom{n + k}{n - 1} = \sum_{i=0}^{n-1} \binom{k + i}{i}$

Dear fellow problem solvers,

I believe that solving problems 16 and 17 not only brought you pleasure but also offered valuable

insights. Now, let's move on to the next two problems to maintain this fulfilling journey of exploration and learning.

Problem 18

Proposed by Ivan Retamoso, BMCC, USA.

What are the dimensions of the cylinder that can be placed inside a right circular cone measuring 5.5 feet in height and having a base radius of 2 feet to maximize its volume?
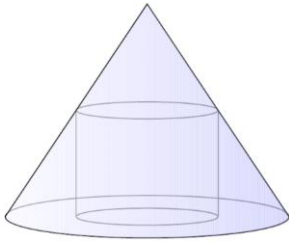
Note: Round yours answers to three decimals places.

Problem 19

Proposed by Dr. Michael W. Ecker, (retired) Pennsylvania State University, USA.

Prove that the diameter of a circle is the largest possible size of a chord of said circle.