The Problem Corner

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The Purpose of **The Problem Corner** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem and its solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Greetings, fellow problem solvers!

As the editor of **The Problem Corner**, I'm thrilled to announce that I've successfully received answers for both Problem 22 and Problem 23. I'm proud to report that all solutions were not only accurate but also demonstrated the effective application of strategies. My primary goal is to present what I believe are the best solutions to contribute to enhancing and elevating mathematical knowledge within our global community.

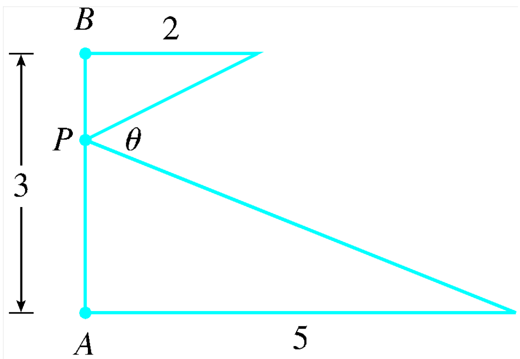
Solutions to **Problems** from the Previous Issue.

Interesting "optimization" problem.

Problem 22

Proposed by Ivan Retamoso, BMCC, USA.

In the illustration below, at what distance from B should point P be positioned to maximize the angle θ?

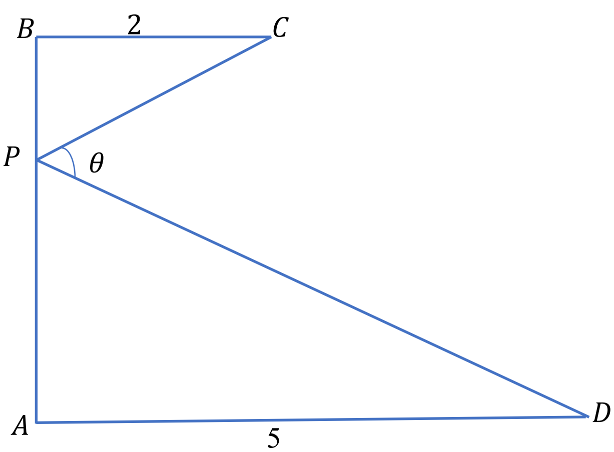


First solution to problem 22

By Dr. Abdullah Kurudirek, Mathematics Ed. Department, Tishk International University.

Our initial solution cleverly utilizes trigonometry, specifically employing the tangent function along with its inverse and derivative, to maximize the value of the angle θ.

Solution.
$CB \perp AB$ and $DA \perp AB$.
$AB = 3$.



First of all we should remember that this is an optimization question!
According to the given information, to make our operations understandable, we can use some labels as follows.
$\angle BPC = \beta$, $\angle APD = \alpha$, and for the segments $AP = x$ so $BP = 3 - x$ where $AB = 3$.
Since α, β and θ are on the same straight line the sum of them must be π. So, $\alpha + \beta + \theta = \pi$ and now let us solve this for θ

$$\theta = \pi - (\alpha + \beta) \qquad (1)$$

By using trigonometry in these right angled triangles we get
$tan\alpha = \frac{5}{x}$ then (by using inverse trigonometric function) $\alpha = arctan\left(\frac{5}{x}\right)$ and in a similar manner $tan\beta = \frac{2}{3-x}$ then $\beta = arctan\left(\frac{2}{3-x}\right)$. Let substitute these in the equation (1) and get
$\theta = \pi - arctan\left(\frac{5}{x}\right) - arctan\left(\frac{2}{3-x}\right)$ now differentiate both sides w.r.t x and get
$\frac{d}{dx}[\theta] = \frac{d}{dx}\left[\pi - arctan\left(\frac{5}{x}\right) - arctan\left(\frac{2}{3-x}\right)\right]$

$$0 = 0 + \frac{\frac{5}{x^2}}{1+\left(\frac{5}{x}\right)^2} - \frac{\frac{2}{(3-x)^2}}{1+\left(\frac{2}{3-x}\right)^2}$$

After necessary algebraic operations, we get

$$\frac{5}{x^2+25} - \frac{2}{(3-x)^2+4} = 0$$

and when we solve this equation
$x^2 - 10x + 5 = 0$ then $x_{1,2} = 5 \pm 2\sqrt{5}$ but $x = 5 + 2\sqrt{5} > 3$ so there is left just $x = 5 - 2\sqrt{5}$ and when we check this on the behaviour table we see that θ reaches its maximum.

x		$5-2\sqrt{5}$	
θ'	↗		↘

Hence, $AP = x = 5 - 2\sqrt{5}$ and $BP = 3 - x = 2\sqrt{5} - 2$

Obtuse isosceles triangle inside a circle problem.

Problem 23

Proposed by Ivan Retamoso, BMCC, USA.

Calculate the radius of the circle in which an isosceles triangle, with a base of 24 inches and legs each measuring 15 inches, is inscribed.

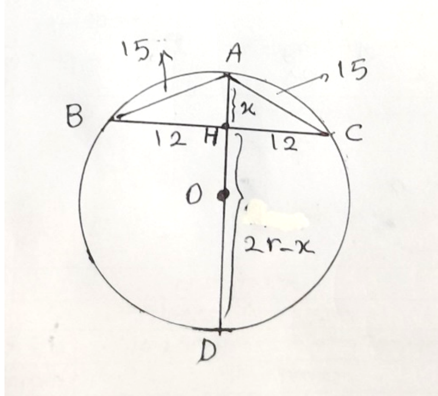
First solution to problem 23

By Dr. Hosseinali Gholami, University Putra Malaysia, Serdang, Malaysia.

Remarkably, our solver provides two simple and elegant solutions, complemented by graphs illustrating the obtuse isosceles triangle with its circumcenter located outside the triangle.

Solution 1:
As respect to the following shape, we obtain:

$$AH^2 = AC^2 - HC^2 \Rightarrow AH^2 = 15^2 - 12^2 = 81 \Rightarrow AH = 9.$$



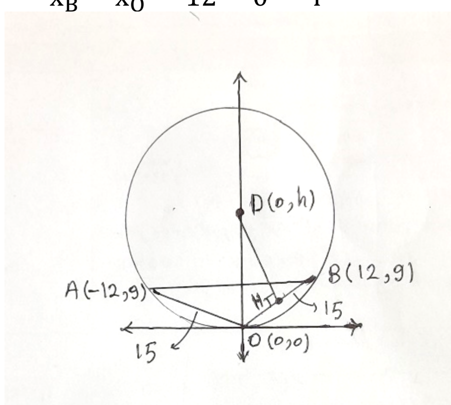
According to the Intersecting Chords Theorem we have:

$$AH \times HD = BH \times HC \Rightarrow x(2r - x) = 12^2 \Rightarrow 9(2r - 9) = 144 \Rightarrow r = 12.5.$$

Solution 2:

Based on the figure below, the point $H(6, 4.5)$ is the middle point of segment OB.

$$a_{OB} = \frac{y_B - y_O}{x_B - x_O} = \frac{9 - 0}{12 - 0} = \frac{3}{4} \Rightarrow a_{DH} = -\frac{4}{3}.$$



The equation of DH is $y = -\frac{4}{3}x + b$.

The coordinates of H satisfy in the above equation.

$$4.5 = \frac{-4}{3} \times 6 + b \Rightarrow b = 12.5.$$

The point D is located on the segment DH thus we have:

$$h = \frac{-4}{3}(0) + 12.5 \Rightarrow h = 12.5.$$

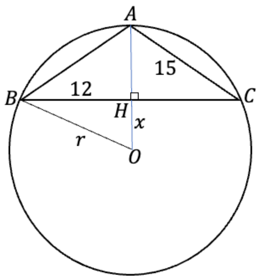
Therefore, it is clear that $r = OD = 12.5$.

Second solution to problem 23

By Dr. Abdullah Kurudirek, Mathematics Ed. Department, Tishk International University.

This alternative solution relies solely on the Pythagorean theorem and symmetry, as clearly demonstrated in the figure provided by our solver.

Solution.
We are given that the base of an isosceles triangle is 24 inches its legs are 15 inches and it is inscribed in a circle.
Let us draw a perpendicular that passes through the center of the circle to the base of ΔABC as follows.



The illustration shows that OA is the radius of the circumcircle. Since H is the middle of the base BC (where ABC is an isosceles triangle), therefore
$BH = HC = \frac{24}{2} = 12$ inches and knowing that $AB = AC = 15$ inches.
Now, using the Pythagoras theorem in ΔABH or ΔACH we get $15^2 = 12^2 + AH^2$ then $AH = 9$ inches.
Knowing that $AO = AH + HO$ where $AO = r$ and then get $OH = x$ so, $r = 9 + x$ and apply the Pythagoras theorem one more times in right triangle OHB and get $r^2 = 12^2 + x^2$. Let's reduce the unknown number to one by writing $r - 9$ instead of x. So, after all necessary algebraic operations we get $r = \frac{25}{2}$ inches.

Third solution to problem 23

By Irfan Rahman, Borough of Manhattan Community College, Bangladesh.

This alternative approach takes a distinct path by first employing the Pythagorean theorem. It then incorporates the triangle's area and links it with the radius of the circle through a specific formula.

Solution.
We're dealing with an isosceles triangle where:

- The base b is 24 inches.

- The two equal legs l are 15 inches each

Now we need to find the Height of the Triangle:

To find the height h, we'll split the triangle into two right triangles by drawing a line from a vertex at the base to the opposite vertex. This splits the base into two equal parts, each measuring $\frac{b}{2} = \frac{24}{2}$ = 12 inches.

using the Pythagorean theorem $a^2 + b^2 = c^2$ where c is the hypotenuse we have:

$h^2 + 12^2 = 15^2$

Solving for h now, $h =$ 9 inches

Now, we need to calculate the area of triangle. The area A of the triangle can be found using the formula:

A = ½ * base * height
A = ½ * 24 * 9 = 108 square inches
Finally, we need to find the Radius of the Circumscribed Circle. For a triangle inscribed in a circle, the radius R of the circumscribed circle can be found using the formula:
$A = \frac{abc}{4R}$
where a, b, and c are the sides of the triangle. Here a = 15 inches and b = 15 inches (because they both are the legs) and c = 24 inches (the base)

$R = \frac{abc}{4A}$ = (15 * 15 * 24) / (4*108) = 12.5 inches

The radius of the circle in which this isosceles triangle is inscribed is 12.5 inches.

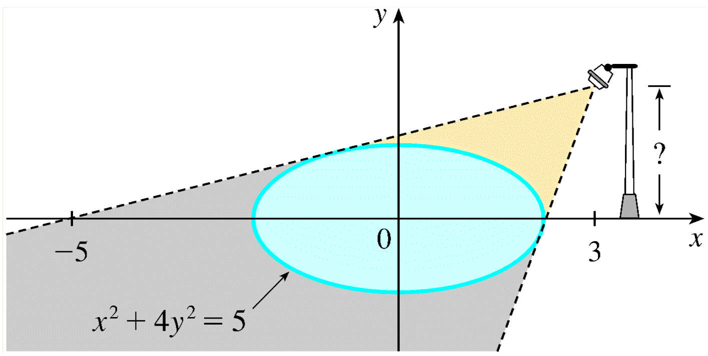
Dear fellow problem solvers,

I'm glad you found solving problems 22 and 23 enjoyable and that you've picked up some new strategies for your mathematical toolkit. Let's now transition to our next pair of problems to continue enhancing your skills.

Problem 24

Proposed by Ivan Retamoso, BMCC, USA.

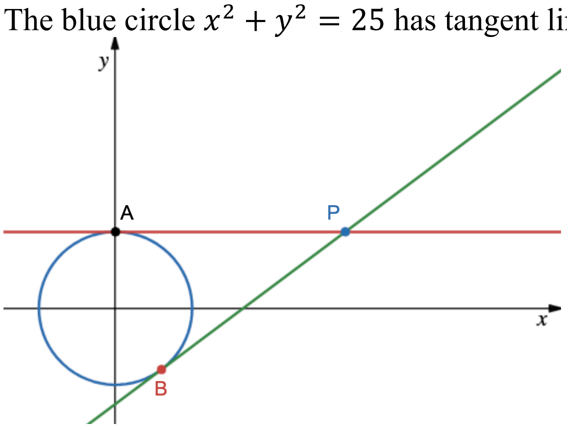
The diagram illustrates a lamp positioned three units to the right of the y-axis and casting a shadow due to the elliptical region defined by the equation $x^2 + 4y^2 \leq 5$. Given that the point $(-5, 0)$ lies on the shadow's edge, how far above the x-axis is the lamp located?



Problem 25

Proposed by Ivan Retamoso, BMCC, USA.

The blue circle $x^2 + y^2 = 25$ has tangent lines at the points A and B.



The point B has x-coordinate 3.
The tangent lines meet at the point P.
Find the coordinates of the point P.