Editorial of the Golden Fall 2023 issue 48, Vol 15 No 4

William Baker, Issue 48 Editor, Vol 15(4) MTRJ
Hostos Community College, CUNY, Bronx, NY

Editorial by William Baker

This issue contains teaching research articles from a variety of countries in the global community of mathematics teacher-researchers investigating student learning and reasoning with math concepts, at a wide range of educational levels. At the elementary level, it includes a teaching research article on children struggling to learn fractions in the Philippines, at higher levels of secondary education it includes teaching-research into the difficulty students have with trigonometry in Nepal, as well as the benefits and drawbacks of calculator usage in Vietnam.

At the undergraduate level, this issue includes teaching research on the difficulties student have in proving the irrationality of certain radical numbers in Afghanistan, and from Vietnam there is an article on the pedagogical technique referred to as the flipped classroom combined with the software GeoGebra to promote problem solving, also from Vietnam is a paper on the construction of a rubric for the analysis of medical student's use for statistical reasoning. An article from Columbia represents an excellent example of teaching-research that employs a cyclical approach to improving mathematical pedagogy through design-strategy, classroom implementation, followed by quantitative-qualitative assessment alongside the design of artifacts for undergraduate instruction by integration of mathematical modeling into STEM education. A teaching-research article on the use of optimization within the instruction of undergraduate Engineering students from Malaysia demonstrates the pedagogical importance of modeling real-life problem-solving situation in that field.

Alongside these teaching-research articles this issue also contains two articles focused on teacher development, the first investigates to what extent pre-service teachers in Turkey can assimilate constructivist pedagogy focused on the Realistic Mathematics Educational approach (RME). The second, examines the exposure and attitude existing secondary education teachers in the Philippines have towards educational research.

Content

This teaching research article is an investigation into the effects of the use of calculators in the educational system in Vietnam. The view of educators and teachers in Vietnam is that the use of technology has a positive effect on student affect through encouraging discovery and more conceptual approaches to problem solving yet, many educators are concerned with a possible decline in computational skills that may hinder mathematical thinking, and thus have a negative view of the use of calculators. The authors note that access to graphing calculators is very uneven in Vietnam and that teaching material to implement such technology is lacking. These authors' research into their own teaching with calculators supports both such positive aspects as well as reduced emphasis on basic computational skills. They conclude that professional development with an emphasis on how to effectively use calculators is critical for promoting conceptual learning within the problem-solving process is needed in Vietnam.

An Analysis of Realistic Mathematics Education Activities of Pre-service Teachers Trained with a Constructivist Approach
Emel Çilingir Altıner, Halil Önal, Alper Yorulmaz (Turkey) p.26

This article represents research to analyze preservice teachers' proficiency with a Constructivist foundation referred to as Realistic Mathematics Education RME, in Turkey. In this study of 137 preservice primary education teachers, the authors note that these teacher candidates, although educated in RME, a constructivist-based methodology, often struggled to overcome more traditional patterns of instruction. They conclude that a change from an approach to math education where every answer is either right or wrong based upon a calculation to one of discovery and construction of meaning would require system wide implementation beginning at the undergraduate level. The authors suggested that more professional development and training, a system wide mentorship program and a platform for teachers to share their resources, ideas, and experiences with each other would support such efforts.

Building framework for assessing students' statistical reasoning in solving real-life medical problems
Hien Tran Thuy, Son Le Phuoc (Vietnam) p.45

This study looks at the statistical learning and reasoning needed for the medical professions. This is clearly an important area of research given the increasing need for strong statistical understandings in the medical fields. In this paper the authors construct a framework to analyze medical student's statistical reasoning based upon Bloom's taxonomy, and PISA mathematics literacy classification. This framework was then used to assess student competency in solving real-life problems in the medical field.

Combining flipped classroom and GeoGebra software in teaching mathematics to develop math problem-solving abilities for secondary school students in Vietnam
An Thi Tan Nguyen, Hung Nguyen Thanh, Cuong Le Minh, Duong Huu Tong, Bui Phuong Uyen, Nguyen Duc Khiem (Vietnam) p.69

This teaching-research study examines the effect of using a flipped classroom teaching model with the software GeoGebra on learning outcomes, and student problem-solving. The authors employ an experimental-control group methodology with statistical analysis on pre and post test results.

Assessing the Implemented Research Lesson Using Mathematical Quality of Instruction
Zyrine Joy Morillo, Kim Gabrielle Del Puerto, Joyce Ann Alag, Julius Christopher Doolittle, and Minie Rose C. Lapinid (Philippines) p.98

This teaching research article focuses on fractions, a difficult foundational concept in elementary education. These authors review and analyze teachers' craft proficiency and content knowledge using the Mathematical Quality of Instruction Rubric. The authors make practical suggestions for improvement of fraction pedagogy including: the ability to communicate with students and explain content knowledge, making good connections between the different fraction representations, exposure to pattern generalization to prepare for algebra, and the importance of teaching and modeling explanations that go beyond short one-, or two-word responses.

Problematic and Supportive Aspects of Indirect Proof in Afghan Undergraduate Students' Proofs of the Irrationality of $\sqrt{3}$ and $\sqrt{5/8}$
Ahmad Khalid Mowahed, Jawed Ahmad Mayar, (Afghanistan) p.124

In this teaching research article, the authors investigate the difficulties and challenges their undergraduate student faced in reasoning about and specifically proving the irrationality of different radicals through extending or generalizing the proof of the irrationality of radical two. The authors identified the difficulties student had making appropriate connections necessary to generalize the proof of irrationality of two to composite numbers as well as radicals of rational numbers, they also proposed and explored alternates methods for proving irrationality.

Mathematical Modelling, Integrated STEM Education and Quality of Education for Linear Algebra and Vector Calculus Courses
Sandra Barragán, Orlando Aya, Camilo Soto (Columbia) p.136

This teaching-research article concerned an integrated approach to STEM education through mathematical modelling conceived of from a variety of vantage points, to improve student performance in advanced undergraduate classes of Linear Algebra and Vector Calculus in Columbia. This research was conducted in a cycling manner involving strategy-design, classroom-implementation, and then quantitative assessment, and produced classroom material as artifacts.

This article investigates what 211 High School mathematical teachers in Philippines think about research on teaching, in particular, whether it can enhance their experience with professional development. The methodology is through qualitative interviews, and the authors conclude that while High School teachers have very limited exposure to research, yet within a supportive environment they are willing to participate and recognize the importance of research for their teaching craft.

The teaching research paper is motivated by teachers' difficulty in explaining trigonometry and students' difficulty in understanding trigonometry in Nepal. This paper borders on action research in that it relies heavily upon the teacher-researcher's personal experience in the classroom, as such it is qualitative research designed to improve the educational experience, specifically, to promote higher order thinking in the math classroom.

This teaching-research article aims to establish a relationship between Calculus real-life optimization problems, and suitable and practical problem-solving for engineering students, as opposed to studying mathematics in an abstract vacuum. Data was gathered through surveys and subsequently analyzed using frequency analysis. This article incorporates feedback and insights from both students and professors obtained during the research, and should be of interest to educators, especially those of Calculus and civil engineering students.

The police officer and driver problem, proposed I. Retamoso, solved A. Kumari
The looking for a pattern problem, proposed C. Ingrassia, solved A.Kumari.
Two new problems, one by I. Retamoso, and another by M. Ecker.

The Use of Calculators in Teaching Mathematics: A Survey in Vietnam

Nam Danh Nguyen [1], Hung Van Nguyen [2]

[1] Thai Nguyen University, Tan Thinh Ward, Thai Nguyen, Vietnam

[2] Thai Nguyen Department of Education & Training, 185 Luong Ngoc Quyen, Thai Nguyen, Vietnam

danhnam.nguyen@tnu.edu.vn

Abstract: The purpose of this paper is to evaluate the reality of teaching and learning with the support of a calculator, examine the benefits and challenges of using a calculator in teaching mathematics in high school. The study is based on the survey data from math teachers, educational managers, and students who were selected from 24 high schools in seven provinces representing different regions of Vietnam. Initial research results show that using calculators for teachers and students in teaching and learning mathematics has been widely developed in recent times in Vietnam, especially since the implementation of the curriculum general education 2006. Calculators can support from knowledge discovery, conceptual approach to problem solving. For teachers, calculator have a positive impact on the views and attitudes of information technology applications in teaching, promoting innovation of teaching and learning methods. However, many teachers and educational managers used to worry about using calculator like affecting the students' computing skills and hindering the goal of developing mathematical thinking. In terms of organizing teaching with calculators, the main difficulties are uneven equipment, lack of teaching documents, and no mechanism on financial support teachers and students.

Keywords: Calculator, handheld calculator, mobile learning, teaching mathematics

INTRODUCTION

In high schools in Vietnam, the combination of theory teaching and practical calculation has not been promoted. Instructing students to use calculators creatively during math learning is still limited. In general, most students only use hand-held calculators at the level of performing simple calculations, but have not applied them to higher levels such as predicting results, creative reasoning to solve problems. At the same time, there is a lack of documents and research on the use of hand-held computers to support the discovery, fostering and development of mathematical competence for students at the high school level. Therefore, the use of handheld computers in teaching math in high schools should be fully and scientifically researched in order to promote the highest efficiency in using to improve the quality of teaching and learning.

Since the invention of electronic calculators more than 40 years ago, handheld calculator technology has grown very rapidly, from simple handheld calculators to scientific calculators and graphics calculators. The main advantages of a calculator are its small size, compact size, portability, and ease of access, as well as the price of a calculator that is relatively cheaper than the cost of buying a desktop computer or laptop (Kemp, Kissane & Bradley, 1995; Kissane, 2000; Wang, 2016). A calculator is a tool to support mathematical discovery, a tool that helps teachers and students solve math problems by exploiting and using the available functions of the calculator. Many studies suggest that the introduction of calculators has profoundly affected mathematics teaching and learning (Dunham & Dick, 1994; Drijvers & Doorman, 1996; Mao et al., 2017). Regarding the benefits of using a calculator in teaching mathematics, several studies recorded the benefits that may arise from the mathematical functions installed in calculators and the rational use of calculators in the teaching process, helping teachers to innovate teaching methods, positively impacting the attitude of teachers and students and improving the effectiveness of learning mathematics (Campbell & Stewart, 1993; Carlson, 1995; Dunham, 1993; 1995; Graham & Thomas, 2000; Hembree & Dessart, 1986). The challenges of using calculators in teaching are also mentioned by some studies, such as challenges that may arise from the inappropriate method of using calculators (Barling, 1991; Barnes, 1994; Bowman, 2018). The challenge arises from a situation where students are not equipping calculators equally in the classroom or not being used regularly for long periods is also an obstacle to using a calculator in teaching and learning (Penglase & Arnold, 1996; Floris, 2017). Besides, some studies also show challenges such as lack of teaching materials, untrained teachers, training on the use of calculators, the lack of time to invest in teaching plans of teachers also affects the efficiency of using calculators (Ruthven, 1990; Clarke & Leary, 1994; Martinovic, 2018).

In Vietnam, there has not been much empirical research on the use of calculators in teaching mathematics, mainly the instructions on the use of calculators, some documents mentioning techniques of using calculators to solve math problems. Some documents related to the use of calculators in teaching, such as instructional materials for teaching mathematics at the high school, manuals for using calculators for the mathematics (Ta & Pham, 2008), a book for training pedagogical skills for teacher students including referring to teaching maths with scientific calculators (Nguyen et al., 2014). In his research, Le (2011) confirmed that the calculator is a "powerful and fast" calculation tool, replacing numbers tables, facilitating the integration of new content into high school math program. Moreover, a calculator is a pedagogical tool that helps to build teaching situations that match the characteristics of active teaching methods (Ta & Pham, 2008). Teaching mathematics with the support of calculators are favorable to apply because of the popularity of calculators in high schools. Therefore, the study on using of calculators in teaching mathematics is necessary, as a practical basis for successful implementation of the new general education curriculum in Vietnam.

Moreover, the research results show the important role of hand-held calculators, including graphical calculators, in supporting teachers to organize discovery activities to help students learn and orientate solutions to problems. Based on these results, calculators could be encouraged to be used more in schools, including allowing students to use graphic calculators in math exams. Therefore, this research has great significance in the strong application of technology in math classrooms, thereby contributing to the development of technology skills for students.

METHOD

We have conducted a survey in 24 high schools of seven provinces throughout Vietnam including Thai Nguyen, Bac Kan, Bac Giang, Bac Ninh, Phu Tho, Hoa Binh, and Khanh Hoa. The high schools selected in the survey represent different types of schools from seven provinces in the country. These are provinces with different socio-economic conditions, with schools in rural, mountainous, remote areas and even schools in urban areas. Participants of the survey were 260 teachers, 40 educational managers, and 367 high school students. Teachers and students were randomly selected from high schools to participate in the survey. Questionnaires and interviews are used to collect data from these schools. The student questionnaire consists of two parts. The first part is general information about students, schools, and classes. The second part consists of open-ended questions for students to express their ideas. The teacher's questionnaire is also designed in two parts. The first part is general information about teachers. The second part contains specific information on the use of calculators in teaching, methods of using graphic calculators, assessing and evaluating the benefits, efficiency, and challenges when using a handheld calculator, including question items with open questions, answers on a five-level Likert scale, which are converted into points from 1 to 5 for each response.

Some teaching situations with graphic calculators are selected and sent to a math teacher for design. The research team conducted observing many lesson hours of math classes, conducted interviews, tested knowledge and skills using calculators for students and teachers. The research team asked the Departments of Education and Training and the principals of the schools to participate the survey to collect data. We also send the questionnaire to the corresponding respondent and conduct face-to-face interviews with students, teachers, and educational managers. Therefore, the collected data includes quantitative and qualitative data. Quantitative data is collected by using questionnaires designed for research purposes, using statistical software to obtain results on frequency, quantity, and percentage. Qualitative data comes from answering questions and interviews, categorized, and recorded in reports.

RESULTS

Using calculators in teaching and learning

We have studied and analyzed the Vietnamese general education programs from 1990 to the 2020 to clarify the role of calculators and the progress of using it in teaching and learning at schools.

For the general education program before 2000: The educational reform began in elementary school in 1980. In the primary education reform program, calculators first appeared in grade 5, to check the results of calculating when students learn about decimals. Following the elementary school program, the middle school program was implemented in 1986 and calculators continued to appear in grade 6, but with the role of support calculation. However, calculators were just introduced students to learn, not focus on skills to use a calculator in solving problem. The education reform program in lower secondary school in 1990, calculators were "forgotten" in the program.

For the adjusted program in 2000: In this program, calculators were completely forgotten. The "Mathematics Instruction Manual 10" stated: "The use of a pocket calculator to solve calculations for errors, equations, and inequalities with decimal coefficients, is very common". However, not all students can afford to buy a device, so they only rely on subjects like Physics so that students can practice. In other words, calculators are not considered a teaching content of high school level. The calculators were used to check the results of calculations.

For the program and textbooks of 2006: The high school math program was more interested in calculators to increase the use of handheld devices to mitigate unnecessary calculations. In the math textbooks 6, 7, 9, the number of explicit exercises using a calculator has increased compared to the previous books. In this program, the calculator was considered as a tool to support calculation and appeared the type of approximation task by the calculator. Currently, the use of calculators in schools is not common, even banned in exams. However, calculators are an indispensable tool for businesses, scientists, in the activities of many agencies.

For the program and textbooks of 2018: Calculators are used to handle with data, the method of using graphic calculator as a tool for calculation and solve mathematical problems. Students need to appreciate the way use these tools and methods of learning math in exploration, discovery and problem solving. It is essential that students are taught how to use the calculator. Therefore, in many categories of textbooks, they have presented how to use a calculator to calculate, solve equations and solve many other problems.

Thus, since 2006, the math program in high school has encouraged the use of graphic calculators. In the high school mathematics program, calculators have been included in the textbooks, which require practical content for students. To encourage the use of calculators in teaching and learning, the Ministry of Education and Training (MoET) has allowed the use of some kinds of calculators in tests, in exams, including national high school exams.

The approach to use of calculators

Most teachers have confirmed the use of calculators (especially graphic calculators) in teaching mathematics since the implementation of the general education program in 2006 because of the content of math textbooks that require the practice of solving math problems on calculators. Regarding the form of teacher access, equipped with the knowledge and skills to use calculators: 63% are self-studying by teachers, 15% teachers are studying from the universities, 19% teachers are taking some training courses and 3% from other forms.

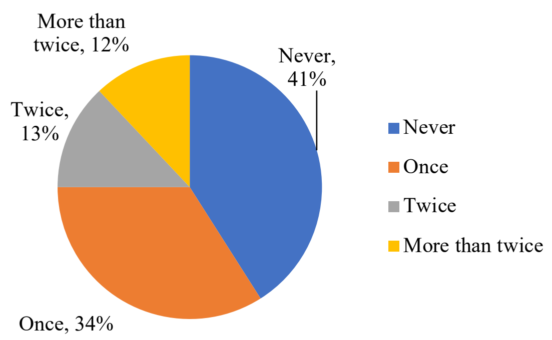


Figure 1. Percentage of teachers participating in training on calculators

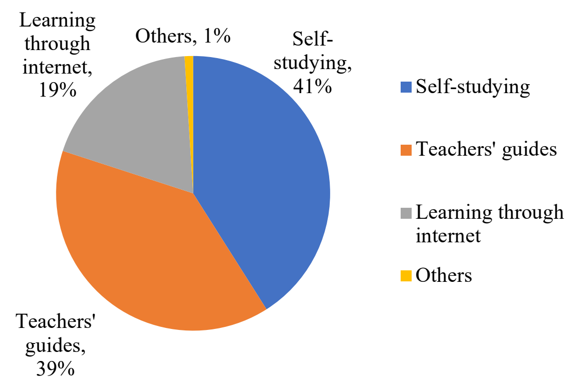


Figure 2. The form of access to a student's calculator

Regarding fostering and training for teachers using calculators in teaching, 41% of teachers have never attended refresher training courses on using calculators organized by different levels. There are very few teachers (12%) participating in training courses at least two times (see Figure 1). Regarding the form of access to calculators for learning: 41% of students are self-study, 39% of students are guided by teachers, 19% learn from the internet, and 1 % from other sources (see Figure 2).

Equipping calculators and methods of using calculators

According to the feedback of teachers, students, and assessments of educational managers of surveying schools, most math teachers are equipped with calculators to teach. Calculators are arranged in the equipment room for sharing. For students, 100% of students have a calculators to study and they buy calculators for themselves. For the type of calculators being used: 97.7% of teachers and 98.1% of students use conventional scientific calculators (this kind of calculator allowed by the MoET in exams). There are no teacher uses a calculator that has a more advanced graphing function.

Most of the teachers interviewed confirmed that they often use a calculator in teaching maths (80.5%), but 19.5% of self-assessing tablets are not regularly used; 77.8% of teachers rated that they were confident in exploiting and using calculators, 21.4% were not confident or limited their confidence when using calculators in teaching. For students, 95% of students confirmed that they often use a calculator to study mathematics, 74% felt confident when using it (proficiently exploiting the functions of calculators) (see Figure 3 and Figure 4).

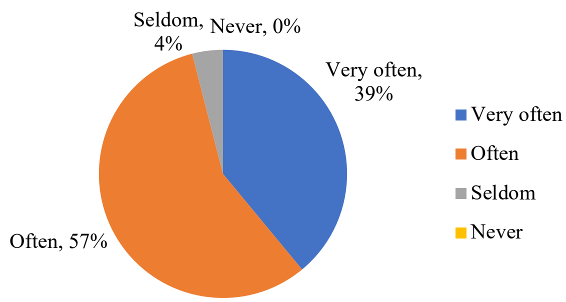


Figure 3. Frequency of students' use of a calculator

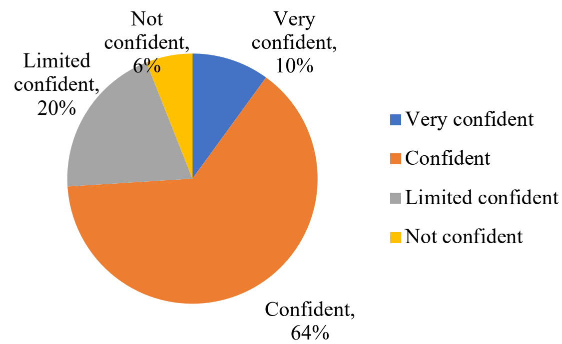


Figure 4. Students' confidence level when using a handheld calculator

Results of conducting surveys on teaching situations using calculators show that most teachers and students said that the calculators used effectively. Results in teaching situations, learning exercises, in tests and examinations. However, 100% of the teachers' opinions were asked without any confirmation that the calculator is effective for teaching mathematical concepts and theorems (see Figure 5 and Figure 6).

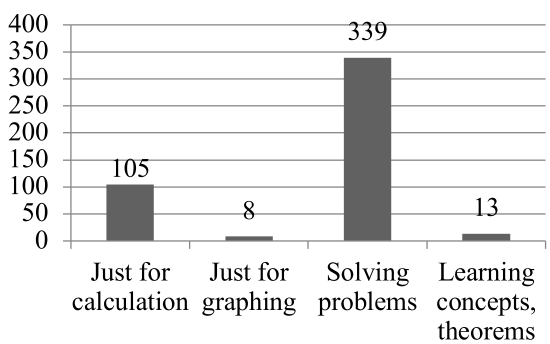


Figure 5. Using a calculator in students' learning situations

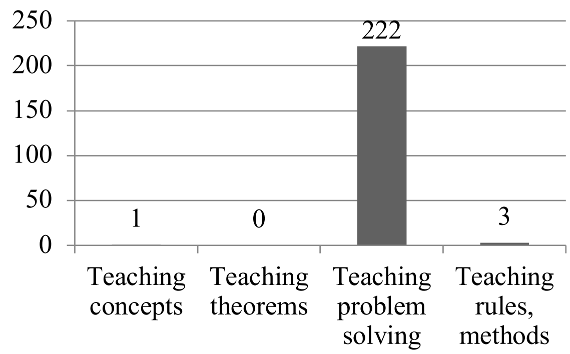


Figure 6. Using a calculator in a teacher's situation

The benefits of using a calculator in learning and teaching mathematics

This study has analyzed the benefits of using calculators in teaching and learning mathematics, including access to a calculator, the attitude of teachers and students, teaching, learning, and performance methods of the teaching process when using calculators. According to student and teacher feedback, the following statistics (see Table 1 and Table 2) show the benefits of using a calculator in the process oflearning and teaching.

Benefit from using a calculator in learning mathematics	Total	Agreed numbers	Ratio (%)	Mean score	Dev. Std.
1. Helping to calculate quickly and accurately	367	355	96.7%	4.43	0.23
2. Create excitement to study better	367	269	73.1%	3.78	0.19
3. Understand mathematics problems better	367	296	80.5%	3.43	0.13
4. Facilitates knowledge discovery	367	248	67.5%	3.70	0.17
5. Help to discuss and exchange groups better	367	204	55.6%	3.48	0.14
6. Helping the test / exam achieve higher results	367	326	88.9%	4.22	0.20

Table 1. The benefits of using a calculator according to student assessment

Benefit from using a calculator in teaching mathematics	Total	Agreed numbers	Ratio (%)	Mean score	Dev. Std.
1. Create opportunities for students to share ideas, allowing students to interact with challenges	260	200	76.9%	3.95	0.19
2. Bringing a new way of working for both teachers and students	260	200	76.9%	3.85	0.23
3. Not only support calculations but also support for discovering, discovering, and solving math problems	260	195	75.0%	3.87	0.21
4. Helping the conversion of mathematical representation and mathematical modeling to be performed more smoothly	260	162	62.3%	3.60	0.19
5. There is more time to focus on mathematical problems, not time on algebraic manipulations	260	195	75.0%	3.83	0.20
6. Solve some math problems that are hard to reach using algebraic techniques	260	189	72.7%	3.78	0.22
7. Students who use a handheld computer have better academic results than those not using it	260	212	81.5%	3.69	0.18

Table 2. The benefits of using a calculator according to the teacher assessment

The feedback shows that regular use of calculators helps teachers and students to be more creative in solving math problems (Walton & Wines, 1994; Ross, 2017). Teachers and students can look for alternative methods to solve a problem, thus avoiding a lot of work with paper and pens. Students can also experiment with different ways of expressing mathematical ideas while discussing with other students. Handheld calculators support the learning of math in students, such as students can calculate a large number of calculations in a given time because of the faster computing speed than calculating by paper or pen. Besides, handheld calculators are considered a common tool that students can use to save time by having to manipulate calculations, spending more time focusing on solving math problems (Hamrick, 1996; Tan, 2015; Bescherer, 2020).

Another benefit was discovered that the attitude of teachers and students also improved significantly when using a calculator in teaching and learning. Teachers' comments show that students have a positive attitude in sharing ideas, working collaboratively, make the classroom atmosphere more lively. This result is consistent with a study by Dunham (1995) confirms that using a calculator helps teachers and students feel more positive and have better attitudes in the process of teaching and learning. Regarding the effectiveness of the use of a calculator, in this study, 88.9% of students rated the use of a calculator to help the tests achieve higher results; 81.5% of teachers rated students using calculators to have better academic results than not using them. Hembree & Dessart (1993) conducted research between 1990 and 1992, the test results of 24 students using calculators were compared with those of students not using, showing that calculators had a positive impact on the improvement. High academic results for students at all levels from third through twelfth grade. This is also consistent with the results of some recent studies on the use of graphical calculators in teaching and learning mathematics (Noraini Idris, 2006; Graham & Thomas, 2000; Horton, Storm & Leonard, 2004; Berry & Graham, 2005). Empirial research by Noraini Idris (2006) investigated the efficiency of using graphical calculators in high school mathematics teaching, showing that students in the experimental group had higher results than those in a control group, confirming the use of graphing calculators in teaching and learning math which is effective in improving students' academic achievement.

The challenge of using a calculator in teaching mathematics

The requirements of mathematics education not only memorize procedures, events, formulas, and algorithms, but require creativity, thinking ability, and computational competence in learners (Kissane, 2000; Le, 2011; Nguyen & Trinh, 2015). Students need to have manipulations of thinking, reasoning, calculation, estimation, using calculation tools and measuring instruments, and proficient use of the calculator. Consistent with these requirements, this research seeks to identify the challenges faced by teachers and learners in the use of calculators in teaching and learning. The following answers to the challenges recorded by teachers and school administrators are shown in Table 3 and Table 4.

Challenges in using calculators	Total	Agreed numbers	Ratio (%)	Mean score	Dev. Std.
1. Students have not actively used calculators	260	146	56.2%	3.39	0.17
2. The ability to use a calculator in students is limited	260	190	73.1%	3.74	0.23
3. Teachers are inexperienced in using calculators	260	159	61.2%	3.56	0.20
4. The teacher has not taken time to design a lesson using the calculator	260	214	82.3%	3.25	0.16
5. Many mathematical content is difficult to use on calculator	260	183	70.4%	3.72	0.23
6. Many calculators are also prohibited from using in exams	260	135	51.9%	3.42	0.18

Table 3. Challenges in using calculators according to teachers

The challenges faced by students are the ability to exploit the functions of calculators to support mathematical discovery and problem-solving activities. The majority of students (95.5%) said that they only use calculators for normal calculations and directly use calculator functions to solve problems (such as solving equations, calculating functional values, limits, integrals, etc.), and also show that students still have difficulty using a calculator to support learning activities such as conceptual and theoretical learning (see Table 4). This was also pointed out by Ruthven (1992) who pointed out concerns about the use of calculators: students may become dependent on calculators, the use of calculators can lead to their laziness. The availability of mathematical functions of calculators can limit a student's mathematical skills. Besides, some other obstacles are the problem of equipping students with calculator skills in students: 56.2% of teachers said that students still lack calculators to practice, 73.1% of teachers' opinions that students' skill of using calculators (especially graphic calculators) are still limited.

Challenges in using calculators	Total	Agreed numbers	Ratio (%)	Mean score	Dev. Std.
1. Teachers are afraid to innovate for teaching	40	12	30.0%	2.95	0.15
2. Lack of documentation for teachers and students	40	19	47.5%	3.33	0.12
3. Teachers are rarely trained and fostered about using calculators in teaching	40	32	80.0%	3.44	0.14

4. Mathematics curriculum does not have specific requirements for using a calculator	40	18	45.0%	3.08	0.19
5. There is no policy to support and equip calculators for teachers and students	40	37	92.5%	3.69	0.15

Table 4. Challenges in using calculators according to educational managers

Challenges are evaluated by many teachers such as the problem of not spending much time on preparing lessons using calculators, the training and training are not regular, 70.4% of teachers commented that many mathematical contents were difficult to use with a calculator. This is in line with a Horton et al. (2004) study of lesson planning by high school teachers. Horton's research shows that poor teachers' preparation of lesson plans leads to ineffective use of facilities, affecting the quality of teaching. Some teachers said that the experience of using a calculator is also a challenge when using this medium in teaching, which leads to the ability to exploit the functions of computers to support activities of searching, exploring knowledge, and designing teaching situations still face many difficulties, especially in teaching concepts and mathematical theorems.

Another challenge also assessed by school administrators is the difficult and slow attitude of teachers in using information technology in teaching. In the surveyed schools, there are still about 20% of teachers who are not proficient in using calculators, so they have not exploited the benefits of calculators in teaching. Besides, the issue of calculators for students and teachers along with other policies such as the regulations of the MoET on the use of calculators in exams is also a challenge when bringing calculators to schools on a large scale. The survey results show that the use of calculators in students is entirely self-procured by students, not yet received the support of the state and calculator manufacturers. This has led to a part of students who do not have a calculator, do not have access to some advanced scientific calculators (such as graphical calculators) that are used by many countries in teaching. At the same time, some calculators are prohibited from using in exams, which limits the daily use of students. In particular, some graphic calculators are banned from national examination for high school students. In other words, the students could not use these calculators for representing graphs and mathematical modelling.

Designing some teaching situations

With the approach, calculator is the means used to analyze mathematical situations, in addition to the usual functions such as the calculator that supports calculations (algebra, trigonometry, exponential, logarithmic, etc.), solve some types of equations, systems of equations, inequalities, limits, integrals. Exploiting the functions of calculators, we offer some possibilities that calculators hands to help teachers and students discover in teaching and learning mathematics with following situations: (1) using calculators to support students in calculating and predicting rules; (2) discovering mathematical laws; (4) making and testing some mathematical hypotheses; (5)

modeling mathematical problems; and (6) exploring mathematical representations. In this study, we designed some examples to use calculators in calculations, problem solving, concept development, pattern recognition, data analysis, and graphing. Moreover, calculators are use to explore and test mathematical ideas such as predicting, finding rules, testing, proving, and disproving hypotheses.

Example 1. When solving the equation $x^2 + x = \frac{1}{x} + 2$ using the equation solver function of graphic calculator, the graph can be presented as follows:

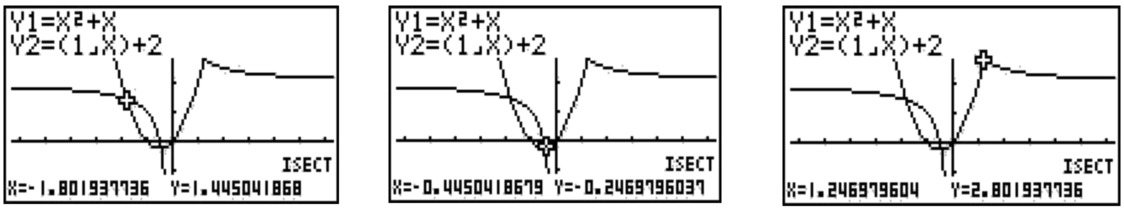


Figure 7. Approximate solution of the equation on a calculator Casio fx-9860

The following example describes a teaching task in data representation and modeling of real-world problems.

Example 2. With a graphical calculator it is possible to describe the modeling process. For illustration, choose Australia's famous Sydney Harbor Bridge. The requirement of the problem is: Consider how the shape of the bridge is a parabola with the equation?

To solve this problem, a graphic calculator was used for representation and modeling to refine the original model as follows (see Figure 8):

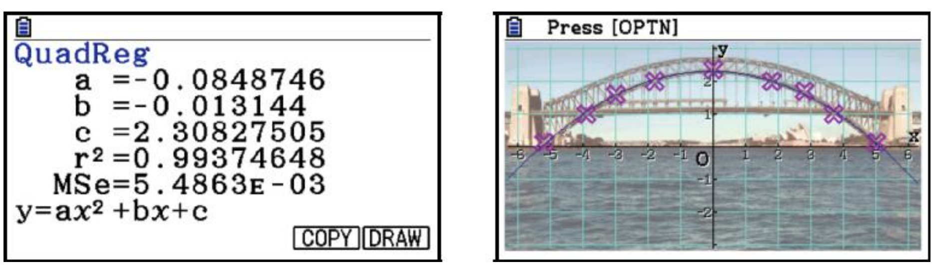


Figure 8. Modeling the situation on a calculator Casio fx-CG 20

With the error is very low, the results on the calculator indicating that the fit is very good. Therefore, a parabolic appropriate model has the equation $y = -0.08x^2 - 0.01x + 2.31$.

We also used a calculator to help students focus on developing problem-solving processes (not just computation involving problems). Robova (2002) asserted that using graphical hand-held calculators in teaching mathematics brings new working methods, especially the ability to predict and model mathematical problems and graphical support of results obtained by algebraic operations.

Example 3. When studying the development of the city's hotel system (Y) in a number of years (X) of the twentieth century. The collected the data presented in the table:

Year (X)	1920	1928	1938	1951	1957	1964	1966	1968	1972	1982
Number of hotel (Y)	15	20	17	25	29	42	53	47	75	88

The question of the problem is the growth of the city's number of hotels can be described by (or approximated) by what function? Based on that model, predict the city's hotel development in the future?

We used a calculator Casio fx-9860 to help students solve the problems as follows:

- Step 1: Rewrite the data, with 20 being the year 1920, the order of taking the last two numbers of the years in the table, x being the variable indicating the year.
- Step 2: Re-create the data table:

Year x	Number of hotel $f(x)$
20	15
28	20
38	17
51	25
57	29
64	42
66	53
68	47
72	75
82	88

←

Year (X)	The city's number of hotels (Y)
1920	15
1928	20
1938	17
1951	25
1957	29
1964	42
1966	53
1968	47
1972	75
1982	88

- Step 3: Using a calculator to plot a scatter plot from a data table (see Figure 9).
- Step 4: Modeling the problem to explore the relationship between x and $f(x)$, here is to find functions that fit the data. Enter an initial guess for a function that might fit the data, for example $y = x$.

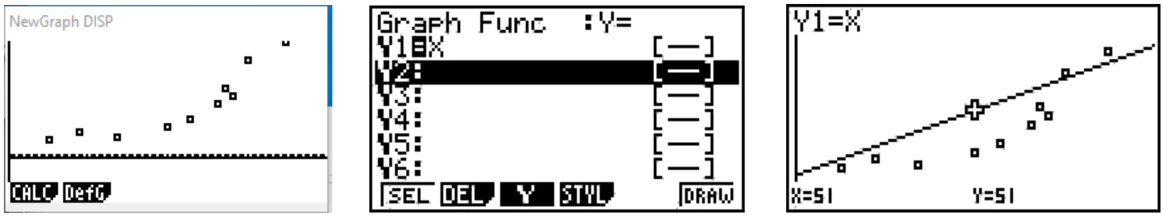


Figure 9. Approximate solution of the equation on a calculator Casio fx-9860

In the screens in Figure 9, the function $y = x$ is not good, since most of the points are below the line. A better guess might be for a line with a slightly reduced slope. The chart in Figure 10 shows one such prediction with $y = 1.2x - 20$. Using graphic calculator to calculate and plot a regression line (line of best fit through the data points). The screens below show a linear regression through the data in the form $y = ax + b$.

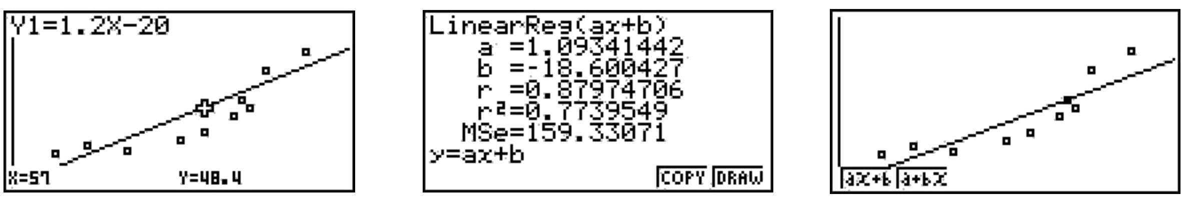


Figure 10. A linear regression through the data on a calculator Casio fx-9860

The regression coefficients a and b show that the best fitting line is given by the equation $y = 1.0934x - 18.600$, which is close to the previous prediction, but by observation still shows that the straight line is inconsistent with those this data. The student can make another choice to find a fitting data. In this case, the data suggest that a quadratic model may be the better choice:

Figure 11. The quadratic function model $y = 0.03x^2 - 1.95x + 45.5$ fits the data

- Step 5: Use the model to solve the original problem. For example, it is predicted that by 2020 (corresponding to $x = 120$) the number of hotels in city Y will be 243.

Example 4. Vietnam's population development issue: The following table records the population of Vietnam in the twentieth and early twenty-first centuries.

a) Plot the scatter plot, noting that a linear model would not be appropriate.

b) Find a natural exponential that models population growth. Plot the graph of the function you found along with the scatter plot. How well does the model fit the data?

c) Use the model you have found to predict the population of Vietnam in 2025.

We use a calculator Casio fx-9860 to help students solve this problem as follows:

Firstly, rewrite the data with 0 being 1990, 10 being 2000, x being the year variable, then for example 1995 corresponds to $x = 9.5$.

Year x	Population $f(x)$: (million people)
0	16.7
1	18.2
2	19.9
3	22.3
4	24.7
5	26.4
6	33.5
7	42.6
8	54.3
9	68.0
10	79.9
11	87.9
12	97.6

←

Year (t)	Vietnam's polulation (million people)
1990	16.7
1910	18.2
1920	19.9
1930	22.3
1940	24.7
1950	26.4
1960	33.5
1970	42.6
1980	54.3
1990	68.0
2000	79.9
2010	87.9
2020	97.6

Students used a calculator to plot the scatter plot from the data table (see Figure 12). The scatter plot of the data points does not lie on a straight line, so the linear model will not be suitable. The scatter plot shows that the data increases rapidly so a natural exponential model would be appropriate.

Secondly, we have a natural exponential model with the equation. The scatter plot of the points is close to the natural exponential graph found, so it fits the data (best fit the data):

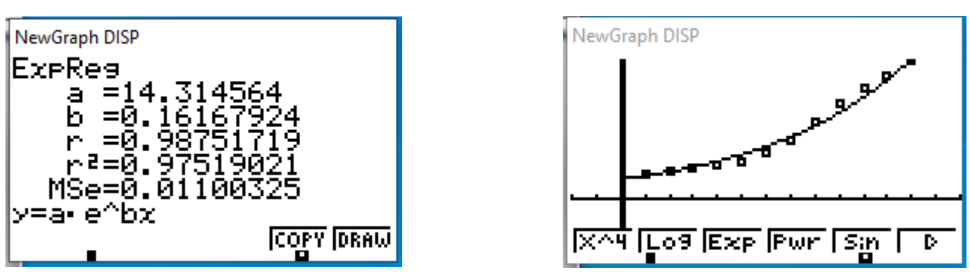


Figure 12. The quadratic function model y = $0.03x^2 - 1.95x + 45.5$ fits the data

Thirdly, the natural exponential function as a model for Vietnam's population growth is $f(x) = 14.31 \times e^{0.161x}$. In 2025, corresponding to $x = 12.5$, the population of Vietnam in 2025 is predicted to be: $f(x) = 14.31 \times e^{0.161 \times 12.5} = 107.07$ million people.

Thus, in a hand-held calculator environment, teaching and learning activities of teachers and students become more active. Hembree and Dessart (1986) have shown that students who use hand-held calculators have better learning attitudes and better ability to self-study mathematics than students who do not. Besides, Dunham and Dick (1994) also asserted that hand-held calculators with graphical functions can enable students to better solve problems, facilitate changes in student roles and teachers, leading to interactive and exploratory learning environments. In addition, Dunham and Dick (1994) and some other researchers argue that the benefits when students use graphic calculators in learning like: more success in problem-solving tests, take a more flexible approach to problem solving, willing to participate in problem solving and remember problems longer, focused on problems of mathematics and did not spend much time on algebraic transformations.

DISCUSSION

Through researching the real-life situation, acknowledging some recommendations and suggestions from teachers, educational managers, and students to improve the use of calculators in teaching. Teachers need a positive change in the use of graphic calculators. It should foster and train teachers on the use of calculators and calculators should be institutionalized in the subject curriculum. Moreover, course content, tests and assessments should be redesigned to suit the use of teaching media and digital technology. In particular, pedagogical colleges and universities need to equip students with knowledge and skills of using teaching facilities. Students need to receive instructions on how to use the calculator more from teachers and use calculators in exams and tests. Students have the freedom to use calculators whenever it feels appropriate and use more modern calculators (such as graphic calculators). Students also are encouraged to take the mathematics exam on the calculator as well as participated in forums, clubs about using calculators.

The proposed examples have ensured a close connection between theory and practice, each situation contains at the same time the training of skills in using calculators and the acquisition of students' knowledge. It creates opportunities for students to demonstrate their own abilities to an increasingly high level after a process of learning, stimulating initiative and creativity in discovering and put the acquired knowledge into practice. The proposed methods of organizing teaching by examples need to be implemented flexibly, regularly and continuously in the teaching process and can be extended to other teaching contents such as geometry, trigonometry, and statistics. Teaching means can also be replaced by graphic calculators such as using software with similar functions, using smart phones with app calculators or using mathematical software on computers. The creative application of the teachers ensures the suitability with the students and the actual conditions of the classroom. In this study, we proposed some typical potentials of using calculators in learning and teaching high school mathematics:

Firstly, the teachers can use graphic calculators as a tool to support math teaching. Studies and practices have confirmed the benefits, roles and effects of calculators in teaching. Calculators should be considered as a tool for discovering knowledge, as a means of analyzing mathematical situations. With advantages in calculation, visual display, fast and accurate data analysis, calculators make identification and prediction easier, and at the same time, from the obtained data, suggest proving predictions, exploring and discovering new problems. In other words, the use of calculators can be viewed as an external activity (using graphs, tables, and numbers to manipulate mathematical concepts), and then transformed into an internal activity (understanding the nature of mathematical concepts).

Secondly, the teachers can use graphic calculators in teaching mathematical content knowledge. Specifically, calculators are used to analyze mathematical situations, not just to perform calculations or just use the built-in functions of the calculator to perform mathematical calculations. However, it is necessary to avoid the abuse of calculators to practice calculation skills and to avoid over-expectancy and dependence on calculators in students.

For high school students, the basic requirements of calculation (arithmetic, algebra, geometry) and skills in calculation such as mental calculation, expression transformation, equation solving, simple inequalities, etc. has been equipped from the lower classes, so the use of graphic calculators to support normal calculations in math learning activities will help students get accurate results, save time for mathematical exploration, discovery and problem-solving activities.

Thirdly, high schools should organize training courses for teachers and students on the use of calculators in teaching and learning. Teachers are practitioners of instruction who need to determine their ability and confidence in using calculators and accept the positive advantages of calculators in order to promote the effective use of technology as a tool for teaching mathematics. For these reasons, teachers need to strive to embrace technology and make it a regular part of their classroom practice. Currently, in pedagogical universities calculators have been included in the curricula for pedagogical students, but the time and practice requirements are not much, in practice, many teachers still lack the necessary knowledge for effective use of calculators in teaching a lesson. Therefore, teachers need to be trained on how to use calculators effectively. Hence, it is important to train math teachers with skills in exploiting the functions of calculators, designing teaching situations in the environment of using calculators.

Fourthly, MoET needs to develop mechanisms and policies to develop a teaching environment using graphic calculators. It is necessary to develop the course curriculum to better exploit the potential of calculators. The mechanisms for curricula to adapt to the availability of new technologies depend on the structure of the education system in each country. In countries that have used calculators in teaching, the curriculum is designed to adapt to the technology by making changes to some mathematical content, such as redesigning the content to ensure that the requirements are met theory and practice, redesigning test content, and methods of assessing students. Teachers play an important role in classroom organization using graphic calculators. Therefore, teachers are encouraged to make good use of calculators before they are confident to use these devices in their classroom. Moreover, in order to bring graphic calculators into teaching in Vietnam, it is necessary to have a strategy of cooperation and companionship between educational management agencies, scientists and manufacturers to produce new products with mathematical functions suitable for subject education programs, with prices suitable to socio-economic conditions. Besides, educational administrators need to monitor the effectiveness of calculators use in the teaching of mathematics and develop timely interventions to address the challenges that may encounter with implementation.

In summary, using a graphic calcultor discovery case benefits students and teachers alike. However, the abuse of calculators can bring about negative effects such as reducing basic calculation skills in students, or being too dependent on calculators in the learning process. Therefore, rational use of graphic calculators, combined with paper and pens in the classroom is necessary, and at the same time, the use of graphic calculators in teaching mathematics needs to be further researched, especially research to build teaching situations to promote the benefits of graphic calculators, improve the quality of teaching mathematics in high schools.

CONCLUSION

The requirement for using media to support math teaching in high schools is an urgent issue in the context of current fundamental and comprehensive educational renovation. Teaching mathematics with graphic calculators has not been paid enough attention by teachers in high schools for many different subjective and objective reasons such as: only considering graphic calculators as the tool only has a calculation function, has not yet exploited the graphic calculators to become a tool to access knowledge, a means to analyze mathematical situations, teachers have not yet designed teaching situations in the environment for using graphic calculators.

Although the use of graphic calculators in the classroom is still being debated by many educators, the results of this study confirm that the use of calculators has many benefits for students and teachers. However, the misuse of calculators can have negative effects such as reducing basic computing skills in students or being too dependent on calculators in the learning process. Therefore, the rational use of calculators, combined with paper and pens in the classroom is a necessity, and the use of calculators in teaching mathematics should be further studied to promote the benefits of calculators, improve the quality of teaching mathematics. Further study is to examine how to use a calculator in teaching situations such as designing teaching situations for solving problems, teaching mathematical concepts, theorems, rules, and laws. It is necessary to continue proposing measures to put a calculator in a new general education curriculum such as the design of textbooks and school teaching materials. In particular, MoET should promulgate policies to connect scientists, educational managers, and businesses to produce calculators suitable to the financial capacity of teachers, students; support mechanism for students who are in difficult circumstances and who cannot yet equip graphic calculators themselves.

It can be said that the appearance of graphic calculators has had an impact on teaching and learning mathematics. It has been more than 40 years since many countries with advanced education systems such as the United States, Australia, Canada, United Kingdom, etc. have allowed it. Using graphic calculators in high schools and increasingly widely used in schools, along with the continuous change and improvement of technology, the calculators are increasingly meeting the needs of students in learning. However, in the process of studying the use of graphic calculators in teaching mathematics, we found that basically the studies only focused on affirming the benefits or challenges that calculators can bring during the teaching process. Therefore, the paper still opens the judgments about the effects of graphic calculators in teaching mathematics without much in-depth research on how to use calculators. It is more necessary to examine how to be highly effective in teaching, how are teaching situations designed to use calculators as a means of helping to explore, discover and solve mathematical problems.

References

[1] Barling, C. (1991). Calculus with a graphical calculator. In J. O'Reilly, S. Wettenhall (Eds.), Mathematics: Inclusive, dynamic, exciting, active, stimulating (pp.380-384). Melbourne: Mathematical Association of Victoria.

[2] Barnes, M. (1994). Using graphics calculators to motivate study of parabolas and quadratic functions. In C. Beesey & D. Rasmussen (Eds.), Mathematics without limits (pp.140-144). Melbourne: Mathematical Association of Victoria.

[3] Berry, J., Graham, T. (2005). On high-school students' use of graphic calculators in mathematics. Zentralblatt für Didaktik der Mathematik, 37, 140-148. https://doi.org/10.1007/s11858-005-0003-7

[4] Bescherer, C. (2020). Technologies in mathematics education. In: Tatnall A. (Eds) Encyclopedia of Education and Information Technologies. Springer, Cham. https://doi.org/10.1007/978-3-030-10576-1_27

[5] Bowman, E. W. (2018). Using graphing calculators to graph quadratics. In: Ball L., Drijvers P., Ladel S., Siller HS., Tabach M., Vale C. (Eds), Uses of Technology in Primary and Secondary Mathematics Education. ICME-13 Monographs. Springer, Cham. https://doi.org/10.1007/978-3-319-76575-4_24

[6] Campbell, P., Stewart, E. L. (1993). Calculators and computers. In R. Jensen (Ed.), Early Childhood Mathematics, NCTM Research Interpretation Project, 251-268.

[7] Carlson, M. P. (1995). A successful transition to a calculator integrated college algebra curriculum: Clues, surveys, and trends. In P. Bogack (Managing Ed.), E. D. Fife, & L. Husch (Eds.), Proceedings of the seventh Annual International Conference on Technology in Collegiate Mathematics, 73-77.

[8] Clarke, K., Leary, B. (1994). Calculus with a graphical calculator. In C. Beesey & D. Rasmussen (Eds.), Mathematics without limits (pp.151-155). Melbourne: Mathematical Association of Victoria.

[9] Drijvers, P., Doorman, M. (1996). The graphics calculator in mathematics education. The Journal of Mathematical Behavior, 15(4), 425-440. https://doi.org/10.1016/S0732-3123(96)90027-9

[10] Dunham, P. H. (1995). Calculator use and gender issues. Association for Women in Math. Newsletter, **25**(2), 16-18.

[11] Dunham, P. H., Dick, T. P. (1994). Research on graphing calculators. Mathematics Teacher, **87**(6), 440-445.

[12] Floris, R. (2017). Pocket calculator as an experimental Milieu: Emblematic tasks and activities. In: Aldon G., Hitt F., Bazzini L., Gellert U. (Eds), Mathematics and Technology. Advances in Mathematics Education. Springer, Cham. https://doi.org/10.1007/978-3-319-51380-5_9

[13] Graham, A. T., Thomas, M. O. J. (2000). Building a versatile understanding of algebraic variables with a graphic calculator. Educational Studies in Mathematics, **41**(3), 265-282.

[14] Hamrick, K. B. (1996). The history of the hand-held electronic calculator. The American Mathematical Monthly, 103(8), 633-639. https://doi.org/10.1080/00029890.1996.12004799

[15] Hembree, R., Dessart, D. J. (1986). Effects of hand-held calculators in precollege mathematics education: A meta-analysis. Journal for Research in Mathematics Education, **17**(2), pp.83-99. https://doi.org/10.2307/749255

[16] Horton, R. M., Storm, J., Leonard, W. H. (2004). The graphic calculator as an aid to teaching algebra. Contemporary Issues in Technology and Teacher Education, **4**(2), 152-162.

[17] Idris, N. (2006). Exploring the effects of TI-84 plus on achievement and anxiety in mathematics. Eurasia Journal of Mathematics, Science and Technology Education, **2**(3), 66-78. https://doi.org/10.12973/ejmste/75465

[18] Kemp, M., Kissane, B., Bradley, J. (1995). Assessment and the graphics calculator. In Anne, R. (Ed.), FLAIR: Forging links and integrating resources (pp.235-241), The Australian Association of Mathematics Teachers.

[19] Kissane, B. (2000). Technology and the curriculum: The case of the graphics calculator. In M. O. J. Thomas (Ed.), Proceedings of TIME 2000: An international conference on Technology in Mathematics Education (pp.60-71). Auckland, New Zealand.

[20] Le, T. B. T. T. (2011). Application of information and communication technologies in teaching mathematics and the advantages of calculators. Journal of Science, Ho Chi Minh City University of Education, **9**, 51-58.

[21] Mao, Y., White, T., Sadler, P. M. (2017). The association of precollege use of calculators with student performance in college calculus. Educational Studies in Mathematics, **94**, 69-83. https://doi.org/10.1007/s10649-016-9714-7

[22] Martinovic, D. (2018). Calculating aids in mathematics education before the advent of electronic calculators: Didactical and technological prospects. In: Volkov A., Freiman V. (Eds), Computations and Computing Devices in Mathematics Education Before the Advent of Electronic Calculators. Mathematics Education in the Digital Era, **11**, Springer, Cham. https://doi.org/10.1007/978-3-319-73396-8_17

[23] Nguyen, D. N., Trinh, T. P. T. (2015). Empirical research on the use of mobile phones to support students' mathematics learning. International Journal of Learning, Teaching and Educational Research, **12**(1), 133-141. https://www.ijlter.org/index.php/ijlter/article/view/83

[24] Penglase, M., Arnold, S. (1996). The graphics calculator in mathematics education: A critical review of recent research. Mathematics Education Research Journal, **8**, 58-90. https://doi.org/10.1007/BF03355481

[25] Robova, J. (2002). Graphing calculator as a tool for enhancing the efficacy of mathematics teaching. The 2nd International Conference on the Teaching of Mathematics.

[26] Ross, A. (2017). The graphing calculator. In: Allen G.D., Ross A. (Eds), Pedagogy and Content in Middle and High School Mathematics. SensePublishers, Rotterdam. https://doi.org/10.1007/978-94-6351-137-7_45

[27] Ruthven, K. (1990). The influence of graphic calculator use o translation from graphic to symbolic forms. Educational studies in mathematics, **21**, 431-450. https://www.jstor.org/stable/3482551

[28] Ruthven, K. (1992). Personal technology and classroom change: A British perspective. In T. Fey & C.R. Hirsh (Eds.), Calculators in mathematics education (pp.91-100). Reston VA: National Council of teachers of Mathematics.

[29] Tan, H. (2015). Gender and technology: A case of graphics calculators in the Singaporean mathematics curriculum context. In Bishop A., Tan H., Barkatsas T. (Eds), Diversity in

Mathematics Education, Mathematics Education Library, Springer. https://doi.org/10.1007/978-3-319-05978-5_5

[30] Walton, G., Wines, C. (1994). Using graphics calculators to help students understand plotting linear and quadratic equations. In C. Beesey, D. Rasmussen (Eds.), Mathematics without limits (pp.240-242). Melbourne: Mathematical Association of Victoria.

[31] Wang, W. (2016). Using the calculator. In: Absolute Beginners Guide to Computing. Apress, Berkeley, CA. https://doi.org/10.1007/978-1-4842-2289-8_25

An Analysis of Realistic Mathematics Education Activities of Pre-service Teachers Trained with a Constructivist Approach

Emel Çilingir Altıner[1], Halil Önal[2], Alper Yorulmaz[3]

[1] Çukurova University, Faculty of Education, Adana, Turkey, [2]Mehmet Akif Ersoy University, Faculty of Education, Burdur, Turkey, [3] Muğla Sıtkı Koçman University, Faculty of Education, Muğla, Turkey,

cilingire@gmail.com , halilonal@mehmetakif.edu.tr , alperyorulmaz@mu.edu.tr

Abstract: Due to the nature of RME, which sees mathematics as a human activity, there is never a fixed and complete theory for mathematics education. Therefore, it is seen as an approach that still needs to be developed. The aim of this research is to analyze the RME (Realistic Mathematics Education) activities prepared by the pre-service primary school teachers who were educated with a constructivist approach. Participants were 137 student teachers who have been trained with a constructivist approach for about 15 years and will teach with this approach. It was determined that the activities prepared by the pre-service primary school teachers were generally at a medium-level in terms of compliance with RME. The results of the research show that no matter how much education they had, the effects of constructivist education could not overcome some traditional patterns in pre-service teachers. In addition, the training they had about RME was not enough to break these patterns. It has been observed that pre-service teachers are insufficient in preparing activities for the basic principles of constructivism with RME. This shows that the disruption in the education system started at the undergraduate level. With this beginning, an endless cycle is formed. In order to avoid the basic patterns brought by traditional education, learning approaches such as the RME approach in addition to the constructivist approach should also be employed at lower grade levels.

INTRODUCTION

Constructivism defines people as active beings who seek to understand the world around them and explore their environment. By interacting with their physical and cultural environment, people embed a newly experienced situation into their existing schemas or revise them and take a step towards learning. They need an active and constructive teaching process. However, this process takes longer than traditional teaching with knowledge transfer. Teachers have noticed that students who prefer to learn in a short way are successful in the exam, but they forget this information easily (Quintero & Rosario, 2016). Therefore, it is seen that there are still university students who cannot make sense of fractions (Akbaba-Dağ & Kılıç-Şahin, 2019; Çetin, 2020) and the number line (Cumhur & Korkmaz, 2020).

In teaching a discipline in the constructivist approach, it is important to provide students with the opportunity to explore, make assumptions, discuss them and gradually develop active knowledge. In this approach, the context and problem or issue that leads to the topic we want to address is introduced, and then students are allowed to develop their own solutions. Because of such features, constructivism is very similar to the Realistic Mathematics Education (RME) approach. Zulkardi (1999) stated that there are three types of constructivist approaches in mathematics education. With radical constructivism, there is the construction of knowledge in the mind and students develop their own meanings, where students lack the social dimension. In social constructivism, it is argued that a social process is effective in constructing knowledge. A socio-constructivist, on the other hand, emphasized that mathematics should be taught through problem solving, students should develop their own strategies, and the importance of student-teacher and student-student interaction. Adopting a socio-constructivist perspective provides a collectivist perspective on teaching and learning that allows students to share and present their activities within the classroom community (Gravenmeijer, 2020). In this sense, it is the socio-constructivist perspective that is closest to RME among the constructivism types (Van den Heuvel-Panhuizen, 2003). RME and socio-constructivism are not only compatible but also complementary approaches (Gravenmeijer, 2020). Moreover, Gravenmeijer (2020) stated that RME can be seen as an integral part of a socio-constructivist approach because it will not be possible to apply RME in norms suitable for traditional mathematics education. Before the socio-constructivist, RME had a more individual, psychological perspective, but with the socio-constructivist approach, the interaction and cooperation roles between students were added and it was effective in the formation of the interaction principle, an important principle for RME (Inharjanto & Lisnani, 2018). Socio-constructivist makes an important contribution to the implementation of RME in the classroom in that it draws attention to classroom culture. In addition, it can be stated that RME also makes an important contribution to socio-constructivism in structuring mathematical knowledge and supporting students (Cobb & Yackel, 1996).

As in constructivism, it is emphasized in RME that students should learn from the contexts they have experienced (Author, 2015). Because daily life situations are more effective than formulas in making sense of the subject. For example, in the teaching of fractions, when the occupancy rate in a theatre is told, it can be provided to visualize the region model (for example, rectangle) and say how many of them can be full. The most distinctive feature of RME is that it gives an important place to rich "realistic" situations in the learning process. Realistic situations are important in terms of answering the student's question "what will this do for us" on the one hand, and interesting problems that need to be solved on the other hand. Therefore, the problems presented to students in RME may come from the real world, as well as from the world of fairy tales or the formal world of mathematics, as long as the problems are empirically real in the mind of the student (Van den

Heuvel-Panhuizen & Drijvers, 2020). From this, it can be concluded that RME is a combination of real-life problems and socio-constructivist mathematics teaching offered by the teacher (Rabbani & Muftianti, 2020).

In cases where the constructivist and RME approach is effective, first of all, the classroom culture should be organized, the social norms of the classroom should be changed and reorganized should be tried to make it more useful in the classroom environment. For this, teachers should identify students' reasoning situations and create teaching activities that support students in expanding and developing their current thinking styles (Gravenmeijer, 2020). It is stated that, thanks to the teachers' preparation for the lesson using RME, they enable their students to understand and deal with mathematics more (Dickinson & Hough, 2012).

Teaching activities are a set of systematically organized materials, both written and unwritten, to create an environment that allows students to learn. Good teaching activities - materials prepared according to the RME approach range from building student knowledge based on daily life experiences to find a mathematical concept (Rabbani & Muftianti, 2020). Activities are an important component in learning as they are used to help learn about the subject (Dickinson & Hough, 2012). The structured and systematic activities required to be prepared in this research and the grade level, learning area, sub-learning area, duration, learning outcomes, skills and values, learning and teaching process (strategy, method, technique, learning environment, materials used), process steps and evaluation stages are discussed.

Wahyudi, Joharman and Ngatman (2017) follow the active learning model as activities to be prepared by RME enable learning by doing, the student-centred learning model for students to solve problems themselves under teacher guidance, and the guided learning model as students need to invent and reinvent mathematical concepts and principles. They stated that the inquiry-based learning model should have supported the contextual learning model in terms of including the problems students encounter in their daily lives, and the constructivist learning model as students are directed to rediscover their mathematical knowledge by solving and discussing problems.

The features of the activities developed according to the RME approach and following constructivism should be as follows:

- Students need to develop hypothetical learning frameworks that include anticipating their mental activities and thinking about how they relate to learning goals.
- Students should be encouraged to think for themselves and explain their thoughts.
- It is of great importance that students who are willing to do mathematics and without fear of failure participate in the studies.
- Teachers should be encouraged to avoid judging students by external standards or comparing them to their classmates, and instead to consider students' personal development as an evaluation criterion (Gravenmeijer, 2020).

Despite the 50-year history of RME, it is still a developing approach that requires further work, especially in classroom applications (Van den Heuvel-Panhuizen & Drijvers, 2020). RME considers mathematics as a dynamic human activity and is not seen as a fixed theory. Therefore, studies on RME are valuable and contribute to the literature. Teaching activities are essential for creating meaningful learning, and it is important to examine the competencies of teachers and pre-service teachers in preparing activities for primary school students. The study analyzes the RME activities prepared by pre-service primary school teachers trained with a constructivist approach to reveal their proficiency in creating RME activities.

METHOD

Procedure and Participants

This study, which was conducted to determine the preparation of RME activities by pre-service primary school teachers educated with a constructivist approach, was carried out using the descriptive method. The reason for using the descriptive method is to determine the current situation of pre-service primary school teachers, who are expected to have a constructivist education approach, while preparing RME activities without any intervention of researchers. Descriptive studies allow describing a given situation as precisely and carefully as possible (Büyüköztürk, Çakmak, Kılıç, Karadeniz & Demirel, 2011). Descriptive research aims to describe, compare, classify and analyze the parts that make up the event in order to reveal what it is (Cohen, Manion & Morrison, 2000).

Purposeful sampling method was used because some criteria were taken into consideration in the determination of the study group (being a pre-service teacher, being in the third grade, having taken educational science courses such as Instructional Technologies, Teaching Principles and Methods, Basic Mathematics in Primary School, Mathematics Teaching I). While determining the study group, the condition of starting their primary school education in the 2005-2006 academic years was sought as a criterion. The study group consists of 137 pre-service primary school teachers who are studying in the third grade in the 2020-2021 academic year from three different universities to ensure maximum diversity. The reason for studying with pre-service primary school teachers in the third grade is that they have taken educational sciences courses such as Instructional Technologies, Teaching Principles and Methods in the previous semesters, as well as a lecture on how the constructivist approach and RME approaches are used in mathematics education in the "Mathematics Teaching I" course for one semester. In addition, since the 2005-2006 academic years in Turkey, education in which the constructivist approach is adopted has begun to be given. Pre-service teachers in the third grade have just started primary school in these years. All educational lives of the pre-service teachers participating in the research were shaped according to the constructivist approach. In this sense, it is expected that the activities that students will create will be designed in accordance with the constructivist approach. The volunteering and willingness of the pre-service teachers included in the study group were taken into consideration.

Data Collection Tool

The steps in the model of realistic mathematics education in developing a learning environment include giving contextual problems, group discussions involving horizontal and vertical mathematization processes in rediscovering mathematical concepts, giving other problems with the material, and presentations. Pre-service primary school teachers were asked to prepare activity plans. Therefore, activity plans suitable for Realistic Mathematics Education prepared by pre-service teachers were used as a data collection tool. In this plan, some instructions were given to pre-service teachers. In line with these guidelines, pre-service teachers prepared an activity plan for any acquisition they chose from the "natural numbers and operations" learning field at the second grade level of primary school.

Data Collection Process

Pre-service primary school teachers took the "Mathematics Teaching 1" course, which is included in their curriculum, and they had theoretical knowledge about RME within the scope of this course. At the same time, sample activity plans suitable for RME were shown to pre-service teachers. Pre-service teachers were informed before the study and documents showing that they approved to participate in the study were collected. The activity templates for the activities that they will prepare in the last week of the term and the instruction stating the acquisitions they should choose from the "natural numbers and operations" learning field were shared with the pre-service teachers over the distance education programs. Pre-service primary school teachers were given a 2-week process and were asked to send their activity plans at the end of the process. It was stated that the activity plans made by the pre-service teachers did not have a scoring regarding the content of the course, and they were also asked to prepare the activities originally without plagiarizing from any source. The activity plans sent by the pre-service primary school teachers were analysed and the activity plans, all of which were completed, were included in the analysis process.

Data Analysis

The qualitative data obtained from the activities prepared by the pre-service primary school teachers in line with RME were analyzed using the quantitative analysis approach. Analyzes made by quantifying qualitative data are included in the literature, and analysis was carried out using semantic content analysis and quantitative analysis methods in line with the approaches put forward by Abeyasekera (2005) regarding the quantitative analysis of qualitative data. The semantic content analysis includes areas and sub-areas related to the subject to be analysed and includes indicators related to these areas. In the study, four main categories were revealed by using the categories prepared by Wahyudi, Joharman, and Ngatman (2017) to evaluate the activities prepared by pre-service teachers for the RME approach. The fifth category is the category of reasoning, this category was not included in the research because the activities were not implemented in the classroom. The categories and indicators determined for the RME activities prepared by the pre-service primary school teachers are given in Table 1.

Category	Indicator
Understanding the daily problem / content	Teachers pose contextual problems and ask questions to guide students' understanding of the problem.
Explaining the contextual problem	Explaining the problem to the students about the given problem.
Solving contextual problems	Directing students to solve problems in groups or individually. Allowing students to use different ways of solving the problem. Using the activity sheet to enable students to work on the problem and solve problems of different difficulty levels. To motivate students to solve problems in their own way by providing direction in the form of questions and motivation. Reflecting the RME's vertical mathematization tool (use of the model) and relevance (use of the relationship).
Comparing and discussing answers	To facilitate discussion and to compare and discuss answers to problems in groups. Utilizing student contributions and interaction between students.

Table 1: Evaluation Categories and Indicators for Realistic Mathematics Education Activities

The qualitative data obtained from the RME activities prepared by the pre-service primary school teachers were transformed into quantitative data in line with the specified categories and indicators. While converting to quantitative data, the scoring scale prepared by the researchers was used in scoring the categories, and the scoring and sample activity statements are given in Table 2.

Category	Sample Activity Statements		
	Suitable (2 points)	Partially suitable (1 point)	Not suitable (0 points)
Understanding the daily problem / content	(Activity 43) Before entering the math lesson, the teacher asked the students, "What floor is your house on, and how many steps do you use to go up and down until you reach the house? Does anyone know how many steps are there? Did you ever pay attention to the steps of the stairs at school? How many steps are there? I want you to count how many steps there are when going up the stairs at the break." He starts the lesson by giving examples from daily life.	(Activity 103) In this course, students are told that they will learn to recognize the rules of number patterns and how to find the missing number. They are asked the difference between the numbers 1-3-5-7-9 written on the zebra. After the answers, they are asked to think about what the pattern rule will be and they are allowed to find the +2 rule.	(Activity 96) The teacher instructs the students to move to a seating arrangement where everyone can see each other easily. The teacher asks the students questions about the topic.
Explaining the contextual problem	(Activity 114) The teacher presents the animation and educational video content that he has prepared with Web 2.0 tools to the students. In line with the videos watched, the students are asked questions and the concepts related to the subject to be learned are associated.	(Activity 58) At the beginning of the activity, the students are divided into two different groups in terms of their success levels. The groups sit together so that they can see the teacher. The teacher selects any card and shows it to the groups. Cards are the form of expressions such as whole, half and quarter. He says that the group with the highest score will be rewarded.	(Activity 61) The problem is transferred to the board by the students in the form of mathematical operations. Then the teacher explains the relationship of the numbers and the subject of comparison.
Solving contextual problems	(Activity 46) He asks students how they put in order the pictures and gives time to each group for the students to talk about the images. The teacher tells the students the correct order once. The teacher checks the studies of each group.	(Activity 105) He asks students to write down the two best selling foods on the blank carton for Questions 3 and 4. During this whole process, the teacher moves between the groups and answers the students' questions and asks them to be careful when using scissors.	(Activity 52) The question (''Is there a relationship between the numbers ?'') is asked to the students. As a result of the activity, the number patterns that the groups complete their missing numbers are shared with the class.

Comparing and discussing answers	(Activity 14) A discussion environment is created in the class and other friends say that there is no need to jump on both sides, but the student is told that this approach makes sense. Student's enthusiasm is unbreakable. In this way, the student discovers shortcuts, makes practice, and provides rediscovery in new contextual situations. They implement this in the next step. Students arrive at a generalization here. It concludes that the number is rounded to whichever side it is closer to.	(Activity 125) Students are shown pictures of kangaroos, frogs, rabbits and grasshoppers. Students are asked to tell their friends what they know about these animals. It is asked if they have a common feature like nutrition, movement, etc.	(Activity 87) The activity process is explained to the students. If students have questions about the activity, they are answered. Finally, the teacher collects and evaluates the completed worksheets.

Table 2: Scoring for The Evaluation of Realistic Mathematics Education Activities

In Table 2, the activities prepared by the pre-service teachers were evaluated. Each category scores between 0-2 points. By adding the scores given to the categories, the pre-service teachers' preparations for RME activities were determined. Quantitative scores obtained were analyzed using descriptive statistical methods, frequency and arithmetic mean. Levels were determined to make sense of the mean scores. In this context, for each category and RME, 0-0.66 points indicate "low", 0.67-1.33 points indicate "medium" and 1.34-2.00 points indicate "high" level.

In the study, the RME activities prepared by the pre-service primary school teachers were evaluated according to the determined categories, and each researcher independently analysed and evaluated 35 activity plans to ensure the reliability of the analysis. Later, the researchers came together and evaluated the activities together according to the RME. In order to ensure the reliability of the coding carried out, "peer review" was carried out at the last stage. Peer review is an external control mechanism to ensure the reliability of research data (Lincoln & Guba, 1985). The evaluations made according to the indicators within the determined categories were presented to the opinions of two independent experts who had a doctorate in mathematics education. All of the evaluations made following the indicators in the categories were analysed by experts, and the evaluations with consensus and disagreement were calculated using the formula [Reliability = Consensus / (Consensus + Disagreement)] determined by Miles and Huberman (1994). After the calculations, the reliability value was found to be 0.79, and it can be said that the analysis was reliable because this value was higher than 0.70 according to Miles and Huberman (1994).

FINDINGS

According to the findings obtained in line with the purpose of the research, the findings regarding the pre-service primary school teachers' realization of RME activities are given in Figure 1.

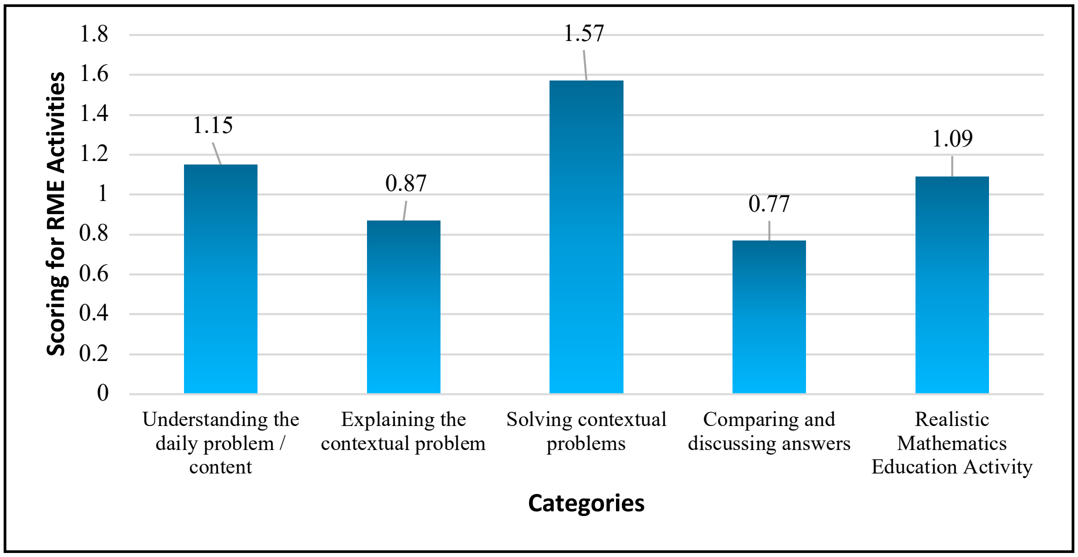


Figure 1. Average scores of pre-service primary school teachers regarding RME activities

When the situations of preparing the RME activities of the pre-service primary school teachers in Figure 1 are analyzed, it is seen that the RME activity score average ($\bar{X}$=1.09) is at the "medium" level. It was determined that the mean score of "understanding the daily problem/content" ($\bar{X}$=1.15), and the mean score of the "solving contextual problems" dimension ($\bar{X}$=1.57) is at a "high" level. From this point of view, it can be said that pre-service primary school teachers use contextual problem solving more frequently in the RME activities they prepare.

The findings regarding the dimension of "understanding the daily problem/content" in the RME activities prepared by the pre-service primary school teachers are given in Figure 2.

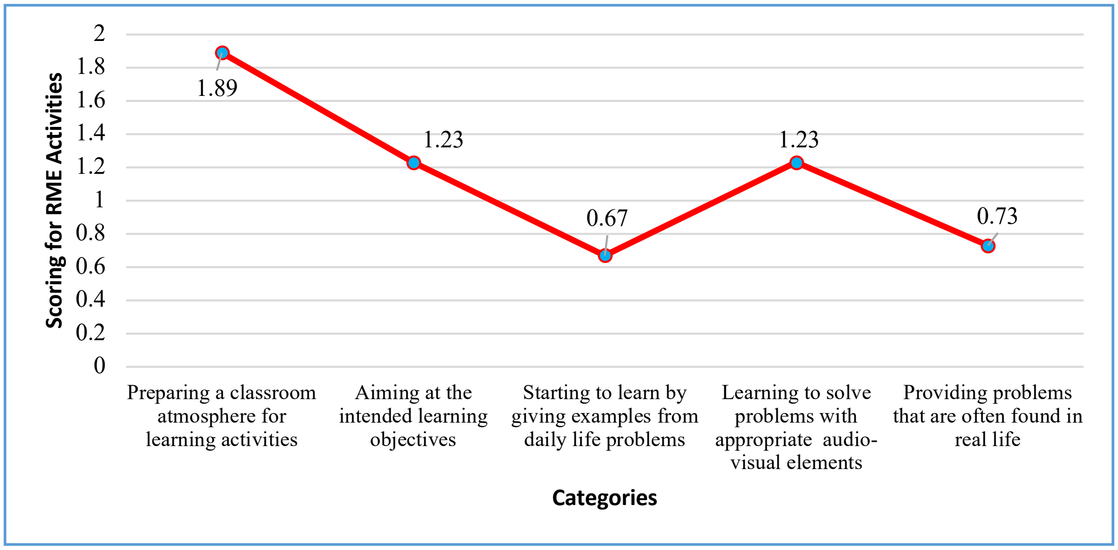


Figure 2. The average scores of the pre-service primary school teachers regarding the dimension of "understanding the daily problem/content" in RME activities

In Figure 2, it is seen that the item "preparing a classroom atmosphere for learning activities" ($\bar{X}$=1.89) in the dimension of "understanding the daily problem/content" in the RME activities prepared by the pre-service primary school teachers has the highest average score. It was determined that the item "starting learning by giving examples from the problems in daily life" ($\bar{X}$=0.67) had the lowest average score. It can be said that pre-service primary school teachers reflect a classroom environment that can provide learning activities to their activities. However, it can be stated that pre-service primary school teachers who prepare RME activities have difficulty in reflecting on their activities to initiate the learning process by giving problems from daily life.

The findings regarding the dimension of "explaining the contextual problem" in the RME activities prepared by the pre-service primary school teachers are given in Figure 3.

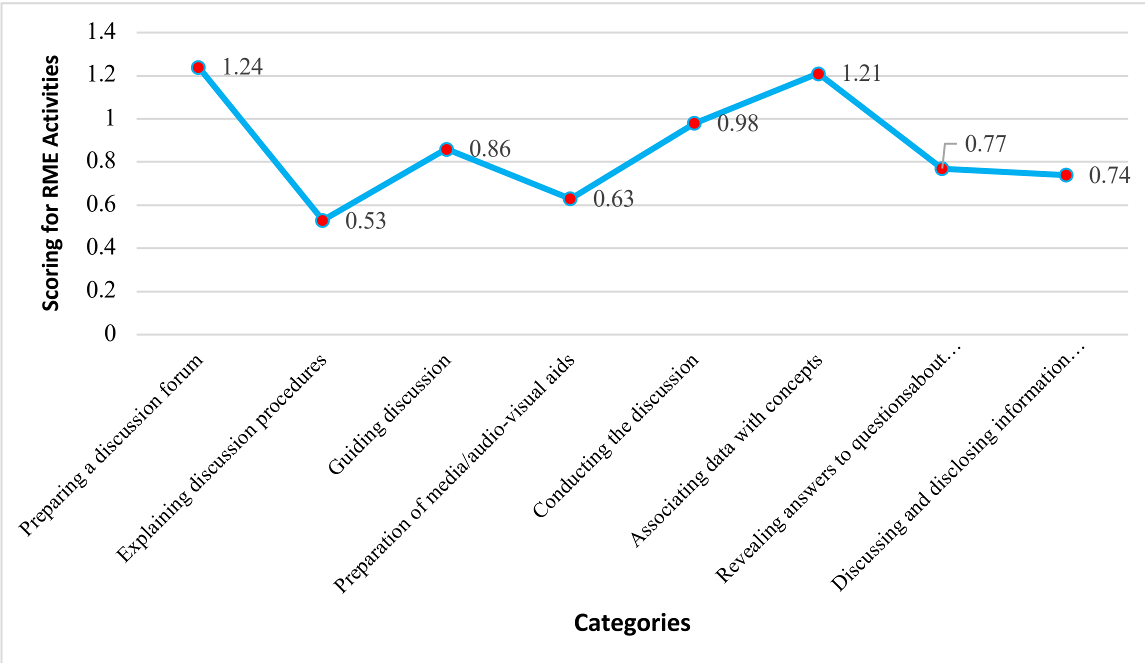


Figure 3. The average scores of the pre-service primary school teachers regarding the daily "explaining the contextual problem" dimension in RME activities

In Figure 3, it is seen that the item "preparing a discussion forum" ($\bar{X}$=1.24) in the dimension of "explaining the contextual problem" in the RME activities prepared by the pre-service primary school teachers has the highest average score. It was found that the item "explaining the discussion procedures" ($\bar{X}$=0.53) had the lowest average score. It can be said that the pre-service primary school teachers include discussion in the activities they carry out, and they cannot reveal the procedure for how these discussions will be conducted.

The findings regarding the dimension of "solving contextual problems" in the RME activities prepared by the pre-service primary school teachers are given in Figure 4.

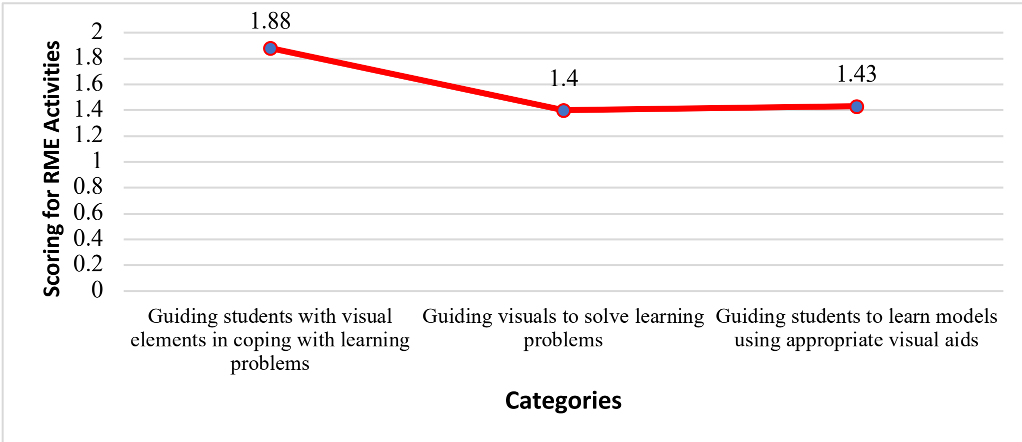


Figure 4. The average scores of the pre-service primary school teachers regarding the daily "solving contextual problems" dimension in RME activities

In Figure 4, it is seen that the item "guiding students with visual elements in coping with learning problems" ($\bar{X}$=1.88) in the dimension of "solving contextual problems" has the highest average score in the RME activities prepared by the pre-service primary school teachers. The item "guiding visuals to solve problems in learning" ($\bar{X}$=1.40) has the lowest average score. As a result of this finding, it can be said that pre-service primary school teachers are guiding students by using visual elements in solving contextual problems in the RME activities they prepare. However, it can be said that the pre-service teachers do not direct students to use the visuals used to solve the problem in their activities, and it can be said that a deficient situation can be created for the students.

The findings regarding the dimension of "comparing and discussing the answers" in the RME activities prepared by the pre-service primary school teachers are given in Figure 5.

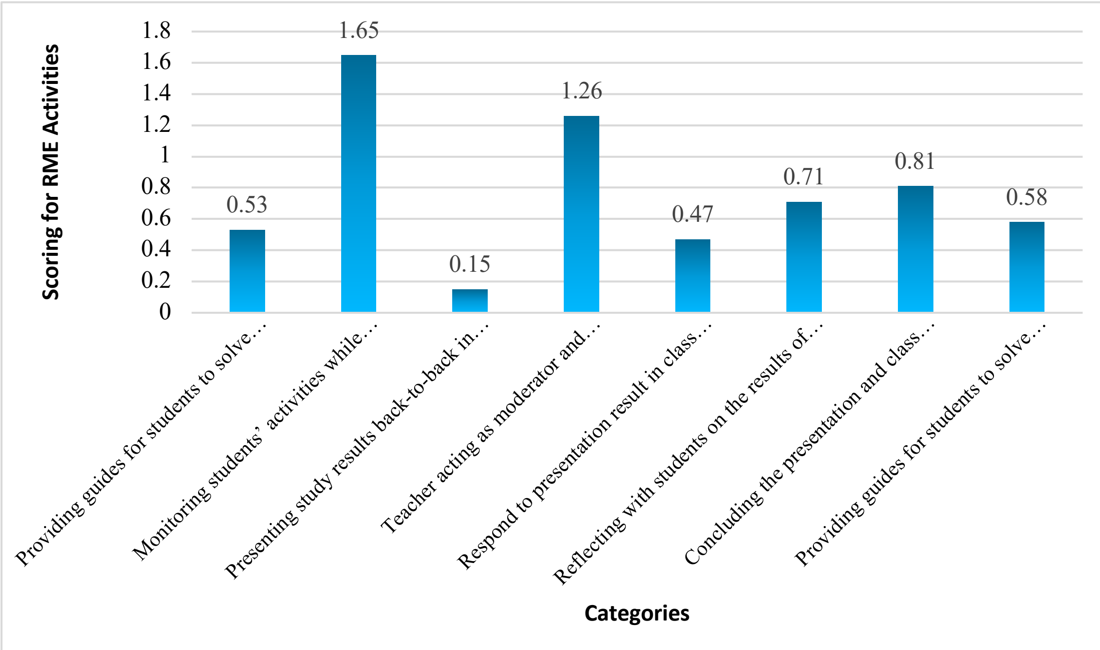


Figure 5. The average scores of the pre-service primary school teachers on the daily "comparing and discussing answers" dimension in RME activities

In Figure 5, it is seen that the item "monitoring students' activities while solving problems" ($\bar{X}$=1.65) in the dimension of "comparing and discussing the answers" in the RME activities prepared by the pre-service primary school teachers has the highest average score. It was determined that the item "presenting the results of the study in mathematics education consecutively in the classroom" ($\bar{X}$=0.15) had the lowest average score. As a result of this finding, it can be said that the pre-service primary school teachers closely followed the students' activities in comparing and discussing the answers in the RME activities they prepared and stated this situation in their activities. In addition, it is seen that they are insufficient in presenting the results of the study conducted in the classroom while the mathematics learning takes place in the written activities.

DISCUSSION

In recent years, with the adoption of the constructivist approach in education, the importance of the RME approach in mathematics has increased in terms of enabling the transition to a more constructive and open-minded attitude. In the study, the activity plans prepared by the pre-service primary school teachers for natural numbers and operation learning areas suitable for the second grade level of primary school were analysed. While analysing the RME activities prepared by the pre-service teachers, it was expected to see the traces of the constructivist approach. In this direction, when the RME activities scored under the dimensions of understanding the daily problem/content, explaining the contextual problem, solving the contextual problems, comparing the answers and discussing were analysed, it was determined that the activities prepared by the pre-service primary school teachers were generally at a medium level in terms of compliance with RME. In addition, it can be said that the RME activities prepared by the pre-service primary school teachers are at a higher level in the contextual problem solving category than in other categories. From a constructivist perspective, mathematical concepts are expected to be contextual, as they arise from human activities in a particular context. According to Arsoetar and Sugiman (2019), an approach using contextual problems is required to construct students' mathematical knowledge (Arsoetar & Sugiman, 2019). Considering the constructivist approach, contextual problems are affected by the social environment in the learning process. In this study, it can be said that pre-service teachers' use of problems in the context of daily life in RME activities is one of the most effective reflections of the constructivist approach.

When the category of "understanding the daily problem/content" in the RME activities prepared by the pre-service primary school teachers is analysed, it can be said that the participants reflect a classroom environment that can provide learning activities to their activities. However, it was determined that they could not reflect on their activities to initiate the learning process by giving examples of problems from daily life. In this case, it shows that there are problems in establishing context in the teaching process and that mathematization cannot be done. In addition, it has been seen that the pre-service teachers are not sufficient in the stage of using real-life problems frequently. However, the contextual problems used in RME need to be both realistic and based on the context of daily life. Therefore, the establishment or preference of problems suitable for RME forms the basis of the lesson. Thus, students can find their own answers to real-life problems, and students can develop and apply their knowledge by discussing the results of their answers with their peers (Sholikhah & Rasmita, 2020). In other words, knowledge must be made into a real life situation. Therefore, it is of great importance to use daily life problems both for RME and for the constructivist approach. This situation can also be explained by the principle of relative to the student and the principle of near to far in teaching principles.

Explaining the problem to the students regarding the problem given in the RME activities was included in the category of "explaining the contextual problem". In this category, it was observed that although the pre-service teachers prepared discussion forms, they could not present the procedure and media / audio-visual materials on how to conduct discussions in these forms. Discussion and reflection play an important role in supporting student development. It is important for students to share their ideas with each other. This is also one of the requirements of the constructivist approach. In the report presented by Searle and Barmby (2012), many teachers stated that it is acceptable for students to express wrong ideas and conflicts arise, and that discussion plays an important role for them. In addition, students can reveal their misconceptions in the solved problems by explaining their strategies to each other through discussion. Therefore, while preparing activities suitable for the constructivist approach, teachers should organize the classroom and classroom environment so that classroom discussions can be held. However, in the study, it was seen that the pre-service teachers could not put forward the procedure and media / audio-visual materials on how to conduct the discussions. It is necessary to support the problem and discussions with audio-visual materials in order to concretize the given problem and make it more understandable. The activities prepared in accordance with RME should be shown with multiple representations in the form of visual, auditory, verbal and media, and should be used continuously by the teacher as an alternative to improve the quality of mathematics learning at school (Muhtarom, Nizaruddin, Nursyahidah & Happy, 2019).

In the RME activities they prepared, pre-service primary school teachers provide the opportunity for students to determine their own methods by directing the problem-solving phase to cooperation in the "solving contextual problems" phase, valuing different ways, studying on problems at different levels, RME's vertical mathematization tool (use of the model) and being relevant to the subject (the use of the relationship) are important in terms of reflecting its characteristics. In this category, it was observed that although the pre-service teachers used visual elements to solve problems, they did not support the students to use visual materials on their own. However, in the constructivist approach, the teacher should provide rich learning environments and provide students with the opportunity to test the accuracy of the information they have previously constructed in their minds, correct their mistakes, and even create their own models by giving up their previous knowledge and replacing it with new ones (Yaşar, 1998). Similarly, in RME, students should be given the opportunity to use and develop their own materials and models in the problem-solving process.

As the last dimension of the activities analysed, in the category of "comparing and discussing the answers", the process of making use of students' contributions and interaction between students was analysed, as well as skills such as facilitating discussion and comparing and discussing the answers to the problems in groups. It was observed that the pre-service teachers closely followed the activities of the students and stated this in their activities. In addition, it has been revealed that they are inadequate in presenting the results of the study done while learning mathematics in the written activities, evaluating the results of the presentations made according to the planned discussion forms, and guiding the students to solve the problems based on their own experiences. Both the RME and the constructivist approach emphasize the guided rediscovery process (Gravemeijer, 2020). For this reason, considering the results obtained, it can be said that the constructivist approach that shapes the education process of pre-service teachers is insufficient at this stage of preparing RME activities.

CONCLUSION

RME and constructivist approach emerged as a challenge to traditional education. Although the constructivist education approach has been implemented in Turkey since 2005, the RME approach has remained only at the research and examination stage. The pre-service primary school teachers participating in the research started primary school in 2005. This shows that pre-service teachers receive education in accordance with the constructivist approach from primary school. On the other hand, the first encounter of pre-service teachers with RME was in the third year of university. This situation made us think that those who train pre-service teachers are still not able to get out of the traditionalist understanding.

Unfortunately, this study did not include a specific discussion on possible strategies to address the issue of deplorable practices of in-service teachers. However, based on the findings of the study, the authors could suggest some potential strategies that may be helpful.

- First, it may be useful to provide additional training and professional development opportunities for in-service teachers on the RME approach, particularly in terms of preparing contextual problem-solving activities and incorporating daily life problems. This can help improve their understanding and implementation of the RME approach in their classroom teaching practices.
- Second, it may be helpful to establish a mentorship program for in-service teachers where experienced RME practitioners can mentor and support less experienced teachers in their implementation of the RME approach. This can provide opportunities for collaboration, reflection, and continuous learning for both mentors and mentees.
- Third, it may be useful to create a community of practice for RME teachers where they can share resources, ideas, and experiences with each other. This can help create a sense of shared responsibility and commitment towards improving the quality of math education and promoting the use of RME practices.

In conclusion, these strategies, as well as others, may be helpful in addressing the issue of deplorable practices of in-service teachers and promoting the use of the RME approach in math education.

References

[1] Abeyasekera, S. (2005). Quantitative analysis approaches to qualitative data: why, when and how? In J. D. Holland & J. Campbell (Eds.), *Methods in development research; combining qualitative and quantitative approaches* (pp. 97-106). Warwickshire: ITDG Publishing.

[2] Akbaba-Dağ, S., & Kılıç-Şahin, H. (2019). Sınıf öğretmeni adaylarının kesirlerle çıkarma işlemine yönelik kurdukları problemlerin incelenmesi [Examining the problems posed by pre-

service primary school teachers in subtraction with fractions]. *Dumlupınar University Journal of Educational Sciences Institute,3*(1), 12-23.

[3] Arsoetar, N., & Sugiman, S. (2019, December). Development of student worksheets based on Realistic Mathematics Education (RME) oriented to mathematical reasoning. In *Journal of Physics: Conference Series The 6th International Conference on Research, Implementation, and Education of Mathematics and Science, Yogyakarta, Indonesia*, *1397*(1), p. 012091, IOP Publishing. https://10.1088/1742-6596/1397/1/012091

[4] Büyüköztürk, Ş., Çakmak, E. Kılıç, A., Özcan, E., Karadeniz, Ş. & Demirel, F. (2011). *Bilimsel araştırma yöntemler [Scientific research methods],* Ankara: Pegem Academy Publications.

[5] Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational psychologist*, *31*(3-4), 175-190. https://doi.org/10.1080/00461520.1996.9653265

[6] Cohen, L., Manion, L., & Morrison, K. (2000). *Research methods in education* (5th Ed.). London: Routledge Falmer. ISBN 9780203224342, https://doi.org/10.4324/9780203224342

[7] Cumhur, F., & Korkmaz, E. (2020). Sınıf öğretmeni adaylarının kesirlerde toplama işlemi ile ilgili öğretim stratejilerinin incelenmesi [Examination of primary school pre-service teachers' teaching strategies about the sum of fractions]. *Anemon Mus Alparslan University Journal of Social Sciences, 8(*6), 1841-1854. https://doi.org/10.18506/anemon.765378

[8] Çetin, H. (2020). İlköğretim matematik öğretmen adaylarının kesir kavramına ilişkin tanımlarının incelenmesi [Investigation of the definitions of pre-service elementary mathematics teachers on fraction concept.]. *Eurasian Journal of Teacher Education*, *1*(3), 172-185. Retrieved from https://dergipark.org.tr/en/pub/ejte/issue/59069/795482

[9] Dickinson, P., & Hough, S. (2012). *Using realistic mathematics education in UK classrooms*. Centre for Mathematics Education, Manchester Metropolitan University, *Manchester, UK*. Retrieved from https://mei.org.uk/files/pdf/rme_impact_booklet.pdf

[10] Gravemeijer, K. (2020). A socio-constructivist elaboration of realistic mathematics education. In M. V. den Heuvel-Panhuizen (Ed.), *National reflections on the Netherlands didactics of mathematics: Teaching and learning in the context of realistic mathematics education* (pp. 217–233). Springer International Publishing. https://doi.org/10.1007/978-3-030-33824-4

[11] Gravemeijer, K. P. E. (2001). Fostering a dialectic relation between theory and practice. In J. Anghileri (Ed.), *Principles and Practice in Arithmetic Teaching: Innovative approaches for the primary classroom* (pp. 147-161). Open University Press.

[12] Inharjanto, A., & Lisnani, L. (2019, April). Implementing Realistic Mathematics Education for Elementary Schools in Indonesia. In *International Conference on Educational Sciences and*

Teacher Profession (ICETeP 2018) (pp. 188-191). Atlantis Press. https://doi.org/10.2991/icetep-18.2019.47

[13] Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry.* Newburry Park, CA: Sage.

[14] Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook* (2nd ed.). Newbury Park, CA: Sage.

[15] Muhtarom, M., Nizaruddin, N., Nursyahidah, F., & Happy, N. (2019). The effectiveness of realistic mathematics education to improve students'multi-representation ability. *Infinity Journal of Mathematics Education*, *8*(1), 21-30. https://doi.org/10.22460/infinity.v8i1.p21-30

[16] Quintero, A. H., & Rosario, H. (2016). *Math makes sense!: A constructivist approach to the teaching and learning of mathematics*. World Scientific, Singapore, Singapore: Imperial College Press..

[17] Rabbani, S., & Muftianti, A. (2020). Implementation of teaching materials using a realistic mathematics education approach in primary student mathematics communication. *PrimaryEdu-Journal of Primary Education*, *4*(2), 230-240. https://doi.org/10.22460/pej.v4i2.1942

[18] Searle, J., & Barmby, P. (2012). *Evaluation report on the realistic mathematics education pilot project at Manchester Metropolitan University*. Durham: Durham University. Retrieved from www.mei.org.uk/files/pdf/RME_Evaluation_final_report.pdf on 3/4/12

[19] Sholikhah, O., & Rasmita, R. (2020). Realistic mathematics education based on virtual network in increasing the understanding of geometry concept. J*urnal Iqra': Kajian Ilmu Pendidikan, 5*(1). 85-93. https://doi.org/10.25217/ji.v5i1.785

[20] Streenfland, L. (1993). The design of a mathematical course. A theoretical reflection. *Educational Studies of Mathematics*, 25, 109–135. https://doi.org/10.1007/BF01274105

[21] Taşkın, S. Z. & Özer, M. B. (2006). *İlköğretim matematik dersi 6. sınıf öğretmen kılavuzu [Primary 6th grade math lesson teacher's guide].* Ankara. Taşkın Publishing.

[22] Van den Heuvel-Panhuizen, M., & Drijvers, P. (2020). Realistic mathematics education., S. Lerman (ed.), *Encyclopedia of Mathematics Education,* 713-717. https://doi.org/10.1007/978-3-030-15789-0

[23] Van den Heuvel-Panhuzen, M. (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics*, *54,* 9-35. https://doi.org/10.1023/B:EDUC.0000005212.03219.dc

[24] Wahyudi, M., Joharman, M., & Ngatman, M. (2017- October). The development of Realistic Mathematics Education (RME) for primary schools' prospective teachers. *Proceedings of the*

International Conference on Teacher Training and Education 2017 (ICTTE 2017), 814–826. https://doi.org/10.2991/ictte-17.2017.83

[25] Yaşar, Ş., (1998). Yapısalcı kuram ve öğrenme-öğretme süreci [Structuralist theory and the teaching-learning process]. *Anadolu Üniversitesi Eğitim Fakültesi Dergisi, 8*(1-2), 68- 75.

[26] Yilmaz, R. (2020). Prospective mathematics teachers' cognitive competencies on realistic mathematics education. *Journal on Mathematics Education*, 11(1), 17-44. http://doi.org/10.22342/jme.11.1.8690.17-44.

[27] Zulkardi, Z. (1999). How to design mathematics lessons based on the realistic approach. http://repository.unsri.ac.id/id/eprint/6362

Building framework for assessing students' statistical reasoning in solving real-life medical problems

Hien Tran Thuy[1], Son Le Phuoc[2]

[1]University of Medicine and Pharmacy, Hue University, Hue City, Viet Nam

[2]University of Law, Hue University, Hue City, Viet Nam

tthienyd@hueuni.edu.vn, sonlp@hueuni.edu.vn

Abstract: The goal of this research was to provide a general assessment framework of students' statistical reasoning in medicine, then build three scales to assess students' statistical reasoning ability in solving practical medical problems, including Description, Interpretation and Prediction. On that basis, the study designed a set of tools to assess students' statistical reasoning ability in solving practical medical problems. Through the analysis of the students' performance, assessments of the students' statistical reasoning in medicine were done. These research results were suitable for student assessment in medical statistics courses, useful for faculty and students in teaching and learning medical statistics.

INTRODUCTION

In quality assessment, we think of low-quality and high-quality learning. According to Vui (2018), it can be said that high-quality learning is characterized by nurturing individual ability to gain knowledge and to understand, and then to apply them to real-life situations to make well-informed decisions and also enhance the individuals' ability, actively share ideas with others. To measure and assess the quality of learning, it is necessary to classify educational goals, classify students' thinking or perceptions in a unified way so that they can be exchanged and discussed among educators. There are many different classifications in terms of thinking, understanding, and goals, but in general, the classifications are hierarchical from low to high and provide educators with a basis for lesson design and appropriate learning quality assessment tools. Bloom's revised Taxonomy is presented in "A Taxonomy for Learning, Teaching, and Assessment: A Revision of the Bloom's Taxonomy of Educational Objectives" (Anderson et al., 2001). Here, we are interested in the cognitive process dimension structure: The number of original categories is still six, but there are important changes, from six levels of Knowledge, Comprehension, Application, Analysis, Synthesis, Evaluation to Remember, Understand, Apply, Analyze, Evaluate and Create. Three categories were renamed, the order of two categories was switched, and the names of the main categories were changed from noun to verb to match the way they are used in the objective: The Knowledge Category is changed to Remember, Comprehension is changed to Understand. This is explained by the fact that because taxonomy reflects different forms of thinking and thinking is an active process, the use of verbs is more appropriate. The math assessment task hierarchy abbreviated as MATH (referred to as MATH classification of thinking or MATH taxonomy) is designed and proposed by Smith et al. (1996). The MATH taxonomy is designed to assist in the development and construction of advanced math assessments to ensure that learners are assessed on a variety of knowledge and skills (Darlington, 2013). The Program for International Student Assessment (PISA) of the Organization for Economic Cooperation and Development (OECD) assesses students from participating countries on their ability to apply their learned knowledge and skills in real life, including four areas of Reading, Math, Science, and Problem Solving. PISA arranges cognitive activities to classify math ability in 3 clusters: Reproduction, Connections, Reflection. Each such cluster comprises 2 levels. Reproduction level 1, 2 is considered low competency; Connections level 3, 4 indicates average competency, and Reflection level 5, 6 is assessed as high competency. The combination of low-to-high competencies including Reproduction, Connections and Reflection in the PISA classification is understood as "basic math skills" in the sense of understanding specific math knowledge and applying it to real-world situations (OECD, 2009). We studied Bloom's revised taxonomy, and MATH taxonomy together with the classification of Mathematical Literacy of PISA to develop scales to evaluate students' statistical reasoning when solving real-life medical problems.

THEORETICAL FRAMEWORK

Statistical Reasoning in Medicine

"Statistical reasoning may be defined as the way people reason with statistical ideas and make sense of statistical information. This involves making interpretations based on sets of data, representations of data, or statistical summaries of data. Statistical reasoning may involve connecting one concept to another (e.g., center and spread), or it may combine ideas about data and chance. Reasoning means understanding and being able to explain statistical processes and being able to fully interpret statistical results" (Dani Ben-Zvi, 2006). So, statistical reasoning is a type of cognitive activity and can be found at the stages in the human thinking process where learners are required to make sense of, explain, or justify a conclusion from the statistical results. Garfield & Gal (1999), then Garfield (2002) identified six types of Statistical Reasoning that direct cognitive activities to statistical tasks, including: Reasoning about data, Reasoning about representations of data, Reasoning about statistical measures, Reasoning about uncertainty, Reasoning about samples, Reasoning about associations. According to delMas (2002), to develop statistical reasoning, the types of statistical tasks may be required to perform such as explaining why or how results are produced, explaining why a conclusion is reasonable, well-founded. Regarding Statistical Reasoning in Medicine (SRiM), Moyé (2006) describes it as follows: "The essence of statistical inference in medicine is the process of determining whether research findings based on a sample can be extended, generalized for the whole or not". Thus, statistical inference in Medicine can be viewed as statistical inference in the field of medicine, which focuses on the fundamentals of statistical inference in medicine to decide when and how to derive results from a sample and apply them to the whole health care research field.

Competency is a rather abstract concept, OECD (2002) has defined "competency as the ability of individuals to meet complex demands and successfully perform the task in a particular context". Accordingly, when it comes to competence, it means the ability of an individual to be associated with a particular task and the conditions to accomplish that task effectively. To evaluate the students' statistical reasoning competency in medicine, it is necessary to clarify specific statistical reasoning abilities in medicine, first, we consider the model of activities that demonstrate that competency. When studying the statistical reasoning development model of high school students, Jones et al. (2001), Mooney (2002) introduced four key statistical processes that characterize statistical reasoning, including data description, data organization, data representation, data analysis, and interpretation. However, we believe that medical students, must be equipped with the thinking methods of scientists, so the requirements for statistical processes must be expanded, and the level of statistical reasoning needs to be advanced. According to Riffenburgh (2012) "A scientist usually seeks to develop knowledge in three stages: The first stage is to describe a class of scientific events, the second stage is to explain these events and the third stage is to predict the occurrence of events". Thus, the accumulation of knowledge in medicine also follows these three stages and in each stage, statistical reasoning in medicine is required. We propose three key medical statistical processes that demonstrate the medical students' statistical reasoning ability including *Description*, *Interpretation*, and *Prediction,* and these processes are described as follows:

(1) *Description:* the stage in which we seek to describe the data generating process in cases for which we have data from that process. For example, in the description phase, the cause of the disease or the health status of the community is recorded. This stage generates data and generates scientific hypotheses to be tested. Thus, the process of Description also includes the process of describing data, organizing and representing data.

(2) *Interpretation:* the phase in which we seek to reason characteristics of the (overall) data generation process when we have only part (usually a small part) of the obtainable data. In the explanation phase, data are evaluated to explain disease-related problems. At this stage, scientific hypotheses are tested. The explanation often takes the form of statistical hypothesis testing. Thus, the process of Explanation also includes the process of analyzing and interpreting data.

(3) *Prediction:* The stage in which we seek to make predictions about a characteristic of the data-generating process and establish a mathematical model based on newly taken related observations is newly performed. In which, there is the integration of test results and disease progression models are formed. Predictions are used in the diagnosis and prevention of diseases and even in the evaluation of the effectiveness of a treatment or disease prevention or control measure. This allows decisions to be made regarding a disease treatment or prevention or control

to change the chances of an event. Prediction involves establishing a mathematical model (called a regression model) of the correlation between the predicted (dependent) variable and the predictor (independent) variable.

Accordingly, specific SRiM competencies including *Description, Interpretation, Prediction* can be regarded as the ability to efficiently and scientifically perform the tasks in the medical statistical processes, respectively, the process of *Description, Interpretation, Prediction.*

The Bloom's taxonomy, **Math Assessment Task Hierarchy and Classifying Mathematical Literacy according to PISA for Statistical Reasoning in Medicine**

Research on math assessment (Brown, 2010; Tan, 2011) has shown that because learning math is often seen as mastering a set of skills, procedures, and formulas, traditional math essay assessments generally focus on skills to be mastered, by assessing students' computational skills or their ability to recall information about procedures and formulas in memory. The questions that appear in the written test mainly test the individual skills to solve a problem while not testing whether the student understands the math concept or can integrate math knowledge to solve the problem or whether or not they can communicate using the language of math. Because the purpose of Math education changes and aims at a broader goal such as developing students' ability to think mathematically, to apply their knowledge to solve real-life problems, it is no longer appropriate to assess students' knowledge by simply asking them apply formulas and calculate answers, but solve real-world problems and make mathematical reasoning. In order to develop the objectives of the medical statistics module to meet the output standards of the medical doctor training program and in the direction of SRiM-focused innovation, and to build a medical statistical reasoning competency rating framework corresponding to the goals, we study the applications of the Bloom's revised taxonomy in the assessment of Mathematics. Bloom's taxonomy represents a hierarchy from low to high thinking, this idea is applied in the design of tools to assess learners' learning level for each specific subject, including Mathematics. Question compilation at the lower levels of the Bloom taxonomy: Remember, Understand, Apply is often appropriate to assess preparation and understanding; diagnose strengths, weaknesses and failures; review or summarize the content. Higher-level questions: Analyze, Evaluate, Create are often appropriate to encourage more critical and deeper thinking; problem-solving, encourage discussion and communication; Motivate learners to find information on their own. In researching math assessment, Smith et al. (1996) showed that Bloom's taxonomy is good for structuring program objectives and assessment tasks, but there are limitations when used for mathematics. The thing research group interested in is the skills required to complete a particular math task and is geared toward developing advanced math assessments across a wide range of knowledge and skills. The MATH taxonomy identifies 8 descriptors of skills and knowledge, arranging them into 3 groups A, B, C. These descriptors are arranged according to the nature of the activity required to complete the task well. Groups A, B, C corresponding to 3 levels are arranged in the order from low to high appropriately according to the context.

Level A: Reproduction Familiar processes			**Level B: Connection** Use your existing math knowledge in new ways		**Level C: Reason** Apply conceptual knowledge to construct mathematical arguments		
A1	A2	A3	B1	B2	C1	C2	C3
Factual knowledge	Comprehension	Routine use of procedures	Information Transfer	Applications in new situations	Justifying and interpreting	Application, conjectures and comparisons	Evaluation

Table 1: MATH Taxonomy (Smith et al., 1996)

Level A (Reproduction) involves recalling events, formulas and recognizing familiar situations and calculations and applying given algorithms. Level B (Connection) continues with the classification of a mathematical object, application in a situation or an answer, and the ability to design a plan or select features to perform an independent assignment. Level C (Reason) involves reasoning, justifying, counterexamples, arguing or proving, stating or discovering patterns, constructing an example, or extending a concept. The test questions for tasks in group A are mainly about recalling formulas to solve familiar problems. The math skills associated with group C are "those that we associate with practicing mathematicians and problem solvers" (Pountney et al., 2002). Some of the limitations of the MATH taxonomy are as popular as most other taxonomy, that is, some tasks may involve the use of more than one type of knowledge or activity. Even more advanced skills have procedural parts. In the assessment of mathematics literacy, PISA selects cognitive activities to classify math skills into three clusters: *Reproduction, Connections, Reflection*, each of which corresponds to two levels: Reproduction level 1, 2 is considered low capacity; Connections level 3, 4 indicates the average capacity and Reflection level 5, 6 is assessed high capacity. The combination of low-to-high competencies including Reproduction, Connections and Reflection in the PISA classification are presented in Table 2.

Competencies	**Clusters**	**Levels**	**Description**
High	*Reflection*	5, 6	*Be proficient and apply math literacy in all situations:* Set and solve complex problems. Reflective and insightful. Access to original math. Complex methods. Generalization.
Average	*Connections*	3, 4	*Know math knowledge and can apply it:* Modeling. Displacement and interpretation,

			standard problem-solving. The methods are well defined in various respects.
Low	*Reproduction*	1, 2	*Not proficient in math knowledge:* Standard presentation and definitions. Computations, procedures, familiar problem-solving.

Table 2: Classification of Math Literacy in PISA (OECD, 2009)

Developing a general framework for assessing Statistical Reasoning in Medicine

In our research on statistical literacy in medicine, SRiM and statistical thinking in medicine, statistical literacy in medicine is considered as the development of basic skills and knowledge and provides the necessary foundation for the development of SRiM and statistical thinking in medicine. With that in mind, when compared with Bloom's revised taxonomy of thinking levels, with 6 levels from low to high, we assume that SRiM may be suitable with the level of Understand and some aspects of the higher levels than Apply and Analyze. The mathematics literacy in classification PISA can also be used to develop an SRiM competency assessment. According to the description of the clusters of reproduction, connections, and reflection capability of PISA, these clusters have a relatively consistent correspondence with MATH's three-level cognitive classification. The overall framework for assessing SRiM competency that we propose as shown in Table 3 consists of six levels from low to high, corresponding to three clusters of competencies *Reproduction, Connections and Reflection*. Similar to the use of "verbs" to describe in Bloom's taxonomy, each task level is described in detail and begins with verbs such as recall, recognize, identify, describe, explain, implement, apply, use, and verify. Corresponding to the levels of the MATH taxonomy, PISA's mathematics literacy classification, in the rating scale we propose, level 1, 2 correspond to the *Reproduction* cluster, level 3, 4 correspond to the *Connections* cluster and level 5, 6 correspond to the *Reflection* cluster.

Clusters	Levels	Task Description
Reflection	6	*Generate new results through mathematical modeling to use medical statistical reasoning to solve real-world problems:* Explain statistical processes and can fully explain statistical results. Use a higher level of thinking and reasoning skills in a statistical context to create mathematical representations of real medical situations. Use insight, reflection, and reasoning to accurately communicate results.

	5	*Apply medical statistical reasoning to analyze and interpret data and draw overall-related conclusions from the data:* Applying statistical knowledge in complex medical situations, which in a sense somewhat structured and mathematically represented clearly explained. Use deductive insight to explain given information. Verify or explain a given result in a medically factual context. Deduct an application or hypothesis for a real-world medical situation. Reflect on their activities, establish and communicate their explanations and reasoning.
Connections	4	*Routine use of procedures and conceptual understanding in solving unfamiliar medical problems:* Selecting suitable statistical formulas and methods for a new situation for a specific context. Apply a concept to a real-life situation or other concepts. Apply basic statistical and probability concepts combined with logical reasoning in less familiar situations. Use reasoning based on the explanation of data, data representation, and statistical summaries. Constructing and communicating explanations and arguments: Explaining text, transferring written descriptions into statistical problems.
	3	*Understand statistical concepts, formulas, and procedures:* Interpret information and data. Explain information from the representation. Link different sources of information. Convert information from one form to another, from one concept to another. Explain familiar statistical concepts, formulas, and procedures in a process sense. Use basic reasoning with simple statistical and probability concepts. Forming an argument from a specific problem with a medical context.
Reproduction	2	*Recall formulas, use familiar procedures:* Practice some basic statistics skills such as organizing data, building, and representing tables, and working with different representations of data. Retrieve simple formulas, algorithms, and step-by-step procedures. Explain or read the results of a simple statistical procedure. Implement a statistical procedure or algorithm in a familiar, similar medical context.
	1	*Retrieve knowledge:* Recall information, facts, formulas and recognize situations, perform familiar calculations.

		Identify statistical information presented in familiar graphical formats. Identify and use basic probability ideas in simple and familiar medical-related experimental situations. Apply the given statistical formulas and algorithms.

Table 3: Overall framework for assessing statistical reasoning in medicine

For medical students, we define levels 1, 2, 3 as lower thinking levels and levels 4, 5, and 6 as higher thinking levels. Levels 1, 2, 3 in the new assessing scale only show the students' performance in remembering formulas, procedures, statistical calculations and basic statistical understanding, level 4 and higher clearly show the student's performance on SRiM.

Model for assessing Students' Statistical Reasoning in solving real-life medical problems

The objective of the PISA assessment is to examine students' ability to flexibly apply knowledge and skills in fundamental areas of expertise to new contexts and problems encountered in real life. PISA focuses on mastering processes, understanding concepts, and being able to handle different situations in each field. We find that this point of view of PISA fits the goal of assessing medical students' statistical reasoning in solving real-life problems. Learning Math for life is a PISA perspective. According to the OECD (PISA, 2010), assessment in the field of mathematics literacy includes students' abilities to analyze, reason, and communicate mathematical ideas effectively as they place, formulate, solve, and interpret mathematical solutions to math problems in a variety of contexts. Assessment involves Mathematical Content (overarching ideas: space and shape, relationships, uncertainty, Mathematical Process (defined by mathematical competencies) and Context where math is used. Uncertainty is one of four overarching ideas, proposed in two related topics, data and chance, corresponding with the topics of probability and statistics, respectively. Thus, in a sense, mathematics literacy includes the ability to make statistical inferences. Applying PISA's assessment of mathematics literacy, we propose a model to assess students' SRiM competency when solving real-life problems as shown in Figure 1.

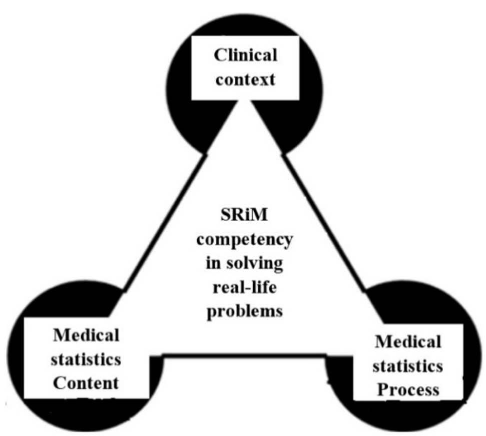


Figure 1: Model for assessing students' SRiM competency in solving real-life problems

In the diagram (Figure 1), medical statistics Content includes basic ideas used to solve problems, medical statistics Process is defined by three SRiM competencies, which are SRiM competence *Description, Interpretation, Prediction* and clinical context that uses SRiM. In a real-life context relevant to clinical medicine, it requires the ability to apply relevant knowledge and skills in medical statistics and medical statistical reasoning in a less structured context, students have to make decisions about what knowledge might be involved, what processes or procedures will lead to a possible solution, and have to use medical statistical reasoning in various degrees. Medical statistics content should focus on the big ideas of medical statistics, including Data; Distribution; Trend; Variability; Paradigm; Relationship; Samples and sampling; and Statistical Reasoning.

Framework for assessing students' Statistical Reasoning in solving medical real-life problems

The SRiM competencies are described according to three cognitive capability clusters as shown in Table 4. In the general assessment scale of SRiM competency we have proposed, six levels from low to high corresponding to the three clusters of reproduction, connections and reflection abilities. In which level 1, 2 corresponds to the reproduction cluster, level 3, 4 corresponds to the connections cluster and level 5, 6 corresponds to the reflection cluster. The cognitive capability levels of each SRiM competency are specifically described through three frameworks for assessing SRiM competency *Description, Interpretation, Prediction*.

Cognitive abilities cluster / SRiM competencies	*Reproduction*		*Connections*		*Reflection*	
	Level 1	Level 2	Level 3	Level 4	Level 5	Level 6
Description	–	–	–	–	–	–
Interpretation	–	–	–	–	–	–
Prediction	–	–	–	–	–	–

Table 4: Two-dimensional matrix of three SRiM competencies and three ability clusters

We consider the SRiM competency *Description* as the ability to efficiently and scientifically perform the tasks in the medical statistical processes *Description*. Accordingly, the SRiM competency *Description* can be summarized by three key components: (1) Data description, organization and representation; (2) Description of the data generation process: description of the sampling process, sampling distributions; (3) Generating scientific hypotheses which need to be justified: forming preliminary overall predictions. As such, the SRiM competency *Description* can relate to various types of medical statistical reasoning including reasoning about data, reasoning about data representations, reasoning about statistical quantities. Accordingly, we build a framework for assessing SRiM competency *Description* which describes in detail each assessing level.

The SRiM competency *Interpretation* as the ability to efficiently and scientifically perform the tasks in the medical statistical processes *Interpretation*. Accordingly, SRiM competency *Interpretation* can be summarized by four key components: (1) Recognizing trends in data, using descriptive statistics from samples to make inferences and predictions about overall characteristics; (2) Justifying scientific hypotheses: generalizing conclusions about the population based on samples. Usually, first of all is the formulation of statistical hypotheses, often in a form different from the clinical hypothesis. Sample data are used to justify these hypotheses. The justification procedure is a special case of two-choice decision theory, where the decision strategy contains only one error, which is the risk (probability) of making the wrong decision. More specifically, the error is measurable, controlling for the probability of each of two possible types of error: choose hypothesis H_0 while hypothesis H_1 is true and choose hypothesis H_1 when hypothesis H_0 is true; (3) Reasoning for the plausibility of the results based on ideas about the correlation between the samples and the population, the representativeness of the sample with respect to its population; (4) Interpret the likelihood of outcomes using an understanding of the ideas of randomness, chance, and uncertainty. As such, SRiM competency *Interpretation* can involve various types of medical statistical reasoning including reasoning about statistical quantities, samples, uncertainty and correlation. Accordingly, we build a framework for assessing SRiM competency *Interpretation* which describes in detail each assessing level.

The SRiM competency *Prediction* is the ability to efficiently and scientifically perform the tasks in the medical statistical processes *Prediction*. SRiM competency *Prediction* can be summarized by four key components: (1) Determining the existence of a correlation between variables, assessing the degree of association between variables; (2) Description of the correlation between variables: The description is based on a hypothetical model of the correlation, with at least one variable being an independent variable, i.e. hypothetical causal factor, predictor outcome and at least one dependent variable, the predictor outcome factor. This model can describe the fit, how good or bad the relationship is, how good the fit of the model is. If the model is incomplete, another suitable model can be found that better describes this correlation; (3) Testing the quality of describing the correlation between variables: using the test results to assess the nature of the correlation; (4) Predict the value of outcomes based on causal values: when the model is decided based on an understanding of the underlying physical or physiological factors that make the association, regression allows predict the value of outcomes arising from a specific clinical indicator of the independent variable. Thus, SRiM competency *Prediction* can involve inferring skills about uncertainty, reasoning about correlations. Within the scope of this article, we present in more detail frameworks for assessing SRiM competency *Prediction* in Table 5.

Clusters	Levels	Task Description
Reflection	6	Explain statistical processes and can fully and deeply interpret statistical results when considering correlations between variables, making predictions about variables.

		Use high-level thinking and reasoning skills in a statistical context to create mathematical representations of real medical situations. Use insight, reflection, and reasoning to accurately communicate results.
	5	Apply a deep understanding of statistical knowledge related to correlation analysis, univariate and multivariable linear regression analysis to complex medical research situations that are in some sense structured and somewhat clearly mathematically represented. Using reasoning insights to interpret given information, test, and interpret a given outcome about the correlation between variables in a real medical context. Create or work effectively with hypotheses for a real medical situation, with complex models. Reflect on their activities, form and communicate interpretations and reasoning.
Connections	4	Apply understanding of key concepts such as correlation coefficients, intercepts and regression coefficients, coefficients of determination, univariate and multivariate regression models to describe and interpret the correlations between variables through the data set. Use reasoning to explain statistical processes and fully interpret statistical results: interpret the reasonableness of the results about the correlation between variables; interpret the reasonableness of the established regression equation; make predictions about the outcome variable based on the predictor variable. Construct and communicate interpretations and reasoning: Interpreting descriptions included in an unfamiliar scientific situation; transferring written descriptions into statistical problems.
	3	Interpret information and data. Link different sources of information. Transfer information from one form to another, from one concept to another. Explain the meaning of concepts, formulas about correlation coefficient, intercept and regression coefficient, coefficient of determination, univariate and multivariable linear regression model, sample regression function. Explain the difference between correlation and regression. Use basic inference with simple statistical concepts in describing correlations between variables. Form an argument from a specific problem in a medical context. Select and apply options to solve simple problems by correlation analysis methods, univariate or multivariable linear regression.

Reproduction	2	Recall the formulas, algorithms, rules and conventions in linear regression analysis, correlation analysis: the formula for determining the Pearson correlation coefficient; univariate and multivariable linear regression models; estimation of the parameters in the regression model, the sample regression function; assumptions are reasonable. Interpret or read the results of a simple statistical procedure applied in situations where no more than a direct conclusion is required: read the results of the Pearson, Spearman correlation test; read the results of the conformity testing procedure of the regression model; read the results of the test procedure for the assumptions in the correlation analysis; Read the scatter plots in residual analysis to test assumptions in regression analysis. Implement simple procedures and algorithms in linear regression analysis, correlation analysis in the familiar, similar medical context.
	1	Directly apply the given formulas and algorithms to determine the estimated parameters in the regression model (intercepts and slopes or regression coefficients, coefficients of determination, correlation coefficients for simple experiment) Recall the meaning of correlation coefficients, parameters of the regression model, and coefficients of determination to evaluate the correlation between variables through the data set. Realize the correlation between variables through data tables, scatter plots.

Table 5: Framework for assessing SRiM competency *Prediction*

Build Test questions for assessing students' Statistical Reasoning in solving medical real-life problems

The general competency assessment scale and the detailed described SRiM competency scale *Description*, *Interpretation, Prediction* are an important basis for us to formulate assessment questions. Another basis for us to formulate assessment questions is the Course Alignment Map, with 5 course objectives and 29 objectives corresponding to 7 lectures, of which 21 are related to medical statistics. According to the SRiM competency assessment scale, each objective is identified corresponding to an SRiM competency cluster. Research on PISA assessment structure shows that PISA'S mathematical literacy assessment is done through a combination of the following question types: multiple-choice question (MCq), close-ended question (CEq) and open-ended question (OEq). According to lessons learnt in developing and using assessment questions for PISA exams, MCq is generally considered the most suitable for evaluating reproduction and connections capacity clusters. For some higher purposes and more complex processes, such questions as CEq or OEq will often be preferred. The main feature of OEq is that it allows students to prove their abilities by providing different solutions at various levels of mathematical complexity. To ensure a certain preset weight (such as a score weight) the percentage (%) of the questions in the *Reproduction: Connections: Reflection* of approximately 25: 50: 25 is appropriate.

Example 1. Matrix of test questions with a combination of both forms of objective multiple-choice questions (MCqs) and essay questions (CEq and OEq). The percentage (%) of the score weight of the questions in the cluster *Reproduction: Connections: Reflection* was ensured to be approximately 25: 50: 25, respectively. Table 6 is a matrix of SRiM competency assessment tests after students finish the module medical statistics (Matrix 1).

Course Objectives	*Reproduction*		*Connections*		*Reflection*		**Sum**		
	MCq	**Essay**	**MCq**	**Essay**	**MCq**	**Essay**	**MCq**	**Essay**	**Scores**
Objective 3	3		1	1 *(1)*			4 *(1.6)*	1 *(1)*	*(2.6)*
Objective 4	3		2	1 *(1.5)*		1 *(1.5)*	5 *(2)*	2 *(3)*	*(5)*
Objective 5			1	1 *(1)*		1 *(1)*	1 *(0.4)*	2 *(2)*	*(2.4)*
Sum	6 *(2.4)*		4 *(1.6)*	3 *(3.5)*		2 *(2.5)*	10 *(4)*	5 *(6)*	*(10)*
	(2.4)		*(5.1)*		*(2.5)*				

Table 6: Matrix of test questions for assessing SRiM competency (Matrix 1)

The ratio between MCqs and essay questions is 4: 6, the total test time is 60 minutes so the corresponding time is 24 minutes for MCqs (4 points) and 36 minutes for Essay questions (6 points). The set of questions includes 10 MCqs, 4 CEqs and 1 OEq. Each MCq has the same score weight of 0.4 and the average time to complete is about 2.4 minutes. The Essay question is numbered in accordance with the intended time of completion and the complexity of the reasoning intended to be evaluated, where the Essay question has a weighted score of 1 corresponding to an average time of completion of about 6 minutes. Weighted points corresponding to each course objective from 3 to 5 is 2.6: 5: 2.4. Weighted score corresponding to each level of reasoning *Reproduction: Connections: Reflection* is 2.4: 5.1: 2.5 which means that the percentage of 75% at the level of reproduction and connections, 25% at the reflection level, is in line with the ratio of the set goals. In Table 6, in each cell: the digit outside the bracket is the number of questions; the digit inside the bracket is the corresponding point weight.

We develop Test 1 corresponding to matrix 1. A set of 15 questions is the SRiM competency assessment level *Description, Interpretation, Prediction* and one score weight (Table 7).

	PROBLEM	**Question**	**Question type**	**Clusters**	**SriM levels**	**Scores**
1	Coronary vessels	Question 1	MCq	*Reproduction*	2	0.4
2		Question 1	MCq	*Reproduction*	2	0.4

3	Protein concentration	Question 2	MCq	*Connections*	3	0.4
4	Hemoglobin concentration	Question 1	MCq	*Reproduction*	2	0.4
5	ALT concentration	Question 1	CEq	*Connections*	4	1.0
6	Cholesterol concentration	Question 1	MCq	*Reproduction*	2	0.4
7	Beta-Crosslaps concentration	Question 1	MCq	*Reproduction*	2	0.4
8		Question 2	CEq	*Connections*	4	1.5
9	Drug addiction treatment	Question 1	MCq	*Connections*	3	0.4
10	Sanspeed group	Question 1	MCq	*Reproduction*	2	0.4
11		Question 2	MCq	*Connections*	3	0.4
12	Prostate cancer	Question 1	CEq	*Reflection*	5	1.5
13	Carotid intima-media thickness	Question 1	CEq	*Connections*	4	1.0
14	Relationship between Glucose and Hormone	Question 1	MCq	*Connections*	3	0.4
15		Question 2	OEq	*Reflection*	5	1.0

Table 7: Describe the test questions corresponding to matrix 1 (Test 1)

Example 2. Considering the question 5 in Test 1. In terms of describing the problem, this is the Question 1 of the problem "Alt concentration".

PROBLEM: ALT CONCENTRATION

The ALT levels (U/L) of people with chronic hepatitis B who visit a hospital do not follow normal distribution. The distribution of ALT yeast concentrations is asymmetrical and deviates right (skewed towards higher values) with the mean of 301.8 and the standard deviation of 363.9.

Question 1. (Level 4) One research team generates random samples of 60 chronic hepatitis B patients each. Describe and interpret the distribution of the sample average ALT yeast concentrations.

Analysis. Question type: CEq. Medical statistic content: Description of the sampling distribution of means. SRiM competence: *Description*.

Clinical context: The question builds in the context of a study describing the ALT index (U/L), which is produced by the liver and usually has a fixed content in blood, when the liver is damaged, the ALT enzyme content will increase. The ALT enzyme index helps doctors in diagnosing liver disease, as well as for monitoring the treatment of patients with liver disease, tracking the progression of the disease with treatment.

Cluster of competence: *Connections* – level 4. The above question is aimed to assess the SRiM capacity of students in relation to the description of the sampling distribution, namely, description of the sampling distribution of means. A sampling distribution is a frequency distribution or frequency of a statistical sample based on random samples drawn from the population. This question is consistent with level 4 of our SRiM competency scale. Students have to solve a problem in an unfamiliar situation but involves familiar elements. Students need to think and reason to understand the problem statistically, make arguments and present arguments appropriately. To solve the problem, students need to determine that this is a descriptive problem of the sample mean distribution in the case of a random sample of which sample size $n = 60$ drawn from the whole that does not satisfy the standard distribution, which has finite expectations and variances. Then, knowledge of Central limit theorem is applied to describe the whole variability of the sample mean distribution including shape, center, and dispersion. The central limit theorem ensures that when samples are large enough, the probability distribution of the sample mean $\overline{X}$ will approximate the normal distribution even if the samples are generated from a whole that does not satisfy the normal distribution and $E\overline{X} = \mu$, $Var\overline{X} = \sigma^2/n$.

The question requires students to: Read and interpret a more complex context including overall identification, random variables, random samples, sample sizes, and distribution characteristics of the whole (non-normal distribution, expectation $\mu = 301.8$ và standard deviation $\sigma = 363.9$); To apply the central limit theorem to determine the sample mean distribution characteristic; Infer and present the results of this process.

For this question, the successful student is the student who fully describes the distribution characteristics of the sample average ALT yeast concentrations: the approximately normal distribution shape, with the sample mean of approximately 301.8, the sample standard deviation of approximately 46.98, and provide appropriate interpretations based on the central limit theorem.

Example 3. Considering the question 15 in Test 1. In terms of describing the problem, this is the Question 2 of the problem "Relationship between Glucose and Hormone".

PROBLEM: RELATIONSHIP BETWEEN GLUCOSE AND HORMONE

Study of the dependence of the content of Glucose (blood glucose) and Hormone. The blood glucose (mmol/L) and Hormone (IU/mL) levels of a sample of 35 patients are identified and stored in the file Data.sav. The obtained result of data processing and the linear regression model includes: Hormone = –0.44. Glucose + 5.77; $R^2 = 0.56$; *t*-test of the coefficient of regression (slope) has Sig. = 10^{-4}.

Question 2. (Level 5) From the obtained results of the data sample, write a report that makes well-founded statements for the above study. From there, provide an example of predicting a patient's hormone levels when knowing that patient's blood glucose index.

Analysis. Question type: OEq. Medical statistic content: Simple linear regression analysis. SRiM competence: *Prediction*

Clinical Context: The question builds on the context of studying the association between the levels of Glucose and Hormone. Blood glucose is an important indicator that helps doctors assess patients' body ability to control blood glucose, so that he can determine whether the patients have glucose-related diseases (diabetes), and can also assess whether the patients with diabetes are responding to the ongoing treatments or not. Hormones are substances produced by the endocrine glands that have a tremendous effect on the body's processes. They affect growth and development, mood, sexual function, reproduction and metabolism.

Cluster of competence: *Reflection* – Level 5. The above question was asked to assess the *Prediction* SRiM capacity of student relation to simple linear regression analysis. This question is consistent with level 5 described in the SRiM competency assessment scale we developed (the general assessment scale and the reflection SRiM competency assessment scale). Based on the provided information, writing a report for this study means a fully explained description of the association between glucose and hormone levels, students need to have a deep understanding of the statistical concepts in univariate linear regression analysis, must know how to integrate concepts and apply inferential insight to verify and interpret information and make well informed statements.

The question requires students to: Understand the question in detail. Write reports for the study includes: Read and interpret in a context to determine whether or not there exists an association between glucose and hormone, how close the association is; Whether the univariate linear regression model is suitable for describing the association between glucose and hormone, whether changes in glucose significantly account for hormone levels; A sample regression function is an estimate of the overall regression function from a sample of data, which is meant to interpret and predict the correlation between two variables. After the suitability of the sample regression model through the results of the *t*-test coefficient of regression (slope) and determination coefficient R^2, a specific example of the significance of the regression model is provided which allows to predict the value of the outcome arising from a specific clinical indicator of the variable glucose. Being able to infer flexibly, establish interpretation and effectively present the results of this process.

This is an open-ended question, to which there can be various answers. Successful students are students who know how to fully exploit 3 given information: With the given information "$R^2 = 0.56$", this is the fixed coefficient of the regression model, determining the relevance of the sample regression function to the data, indicating that in 100% of the entire deviations of the data from the mean 56% is due to glucose, 44% is due to random errors and other factors (if any) that we don't include in this model to consider. Given that "*t*-test coefficient of regression (slope) has Sig. = 10^{-4}", slope b other than 0 is statistically significant ($p = 10^{-4} < 0{,}01$), there is an association between glucose and hormone, a linear regression model is appropriate to describe this association; Given the information "Hormone = –0.44. Glucose + 5,77" is a sample regression function, which is an estimate of the overall regression function, where an estimate is $b = -0.44$ và $a = 5.77$, showing an inverse correlation between glucose and hormone, within a consistent limit of glucose content, when glucose increase by 1 (mmol/L) the hormone decreases by 0.44 (IU/mL). The sample regression function allows the prediction of hormone levels in the blood for patients with specific glucose levels, for example, for patient with a blood glucose index of 6.5 (mmol/L), predicted blood hormone levels is then 2.91 (IU/mL).

The test scoring chart is developed in detail right from the compilation of the test matrix. Basing on the matrix of test questions to determine the score scale corresponding to SRiM levels and the test score (Table 8).

Clusters	SriM levels	Scores
Reflection	5, 6	(7.5; 10]
Connections	4	(4; 7.5]
	3	(2.4; 4]
Reproduction	1, 2	(0; 2.4]

Table 8: The score scale corresponding to SRiM levels of matrix 1

METHOD

Researches on math education focus on high school students, but in comparison to recognizing the shortcomings of assessment, new trends in assessment goals, and basic principles of math assessment, we think that there is a consistency with the assessment in teaching medical statistics to medical students. Therefore, we apply the theories in the assessment of math education as a reference for the research of assessing the medical statistical reasoning competency of medical students. Firstly, we apply a combination of Bloom's taxonomy, MATH taxonomy and PISA's mathematics literacy classification to propose an overall framework for assessing SRiM competencies. After that, applying PISA's assessment of mathematics literacy, we built a model to assess students' SRiM competency when solving real-life problems. From that, we built a framework for assessing students' SRiM competency in solving real-life problems. The framework is an important basis for designing a set of evaluation questions, we built Test 1 of MCqs, CEqs and OEqs. We conducted experiments with the self-developed assessment toolkit, analyzed the results and drew conclusions. We experimented on a classroom of $N = 103$ first-year students of the university medicine and pharmacy in Viet Nam. After completing the medical statistics course, the students took Test 1, in 60 minutes with a pen and paper. We collected student exam answers for Test 1 and use quantitative analysis methods combined with qualitative to analyze experimental results.

RESULTS

Test 1 score processing results help us determine the percentage (%) of students who reach the SRiM levels shown in Table 9. As can be seen in Table 9, among the surveyed students, the percentage of students who only achieve a low level of competence (reproduction) occupies only 7.8%, a majority of students who reach the competence levels 1, 2, 3 occupies 57.3%. Most students achieve competency levels at the levels of 3, 4 (connections) which occupies 79.6%, at higher levels of 5, 6 (reflections) is still low, occupying 12.6%. This proves that, SRiM level of students is rather alike, most of them are well equipped with basic statistical skills. Achieving the reproduction capacity level is not difficult for most students, according to the descriptive rating scale for the level of reproduction, that is, the task requires the export of knowledge, recall the formula, using familiar processes is not difficult for most students. Nevertheless, students meet difficulty in achieving a reflection level of competence, high levels of competence require students to know how to creatively apply known knowledge to produce new things, to deal with unfamiliar problems using unknown forms or apply SRiM to solve practical medical problems.

Cluster	SRiM level	Scores	Percentage (%) (N = 103)
Reflection	5, 6	(7.5; 10]	12.6
Connections	4	(4; 7.5]	30.1
	3	(2.4; 4]	49.5
Reproduction	1, 2	(0; 2.4]	7.8

Table 9: Percentage (%) of students achieving the levels of SRiM for Test 1

To know more about the students' SRiM capacity, we analyze students' answers to each question and found that a majority of students have good answers to questions at a reproduction level, for 6 questions requiring higher competency at the reproduction level (Question 1, 2, 4, 6, 7, 10) in Test 1, statistical data show that the percentage (%) of students who answers each question correctly at this level reaches 87.4% or more. Nevertheless, regarding the two questions at reflection level, for the percentage of students who achieves maximum score is 8.7% regarding Question 12 while only 1.9% regarding Question 15.

Example 4. Considering the students' answers to Question 5 in Test 1. Statistical data of students' responses to this question are shown in Table 10.

As can be seen from Table 10, among the participants, 62.1% of students gave the correct answers on the distribution characteristics of the mean ALT enzyme concentration of samples.

$N = 103$	Scores		
	0	0.5	1.0
Percentage (%)	38.8	23.3	37.9

Table 10: Results of answers corresponding to question 5 of Test 1

However, only 37.9% of students are successful with the question at this *Connections* level. Figure 2, Figure 3, Figure 4 are the work of three students (S1, S2 and S3) to this question.

The students' works show that, S1 can summarize the problem by converting information into symbols, but fails to answer the raised question;

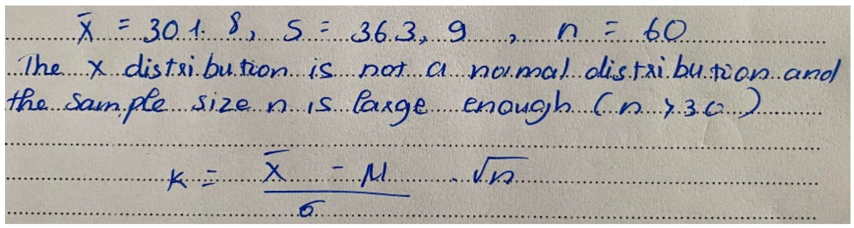


Figure 2: Student S1' answers to Question 5 of Test 1

S2 can identify the distribution shape of the sample mean ALT yeast concentration but can't provide an interpretation;

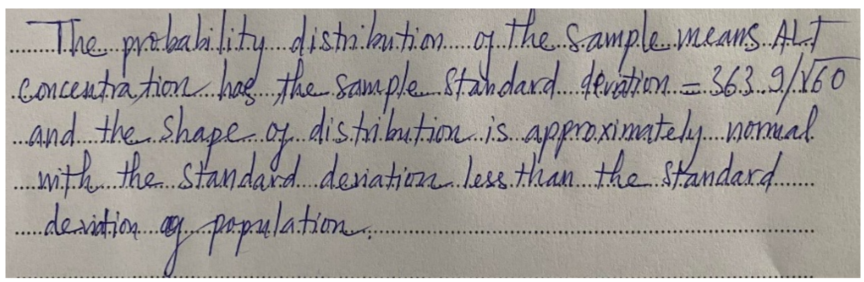


Figure 3: Student S2' answers to Question 5 of Test 1

S3 can define distribution shape, distribution characteristics, and provided an interpretation using the central limit theorem, however the interpretation is still incomplete.

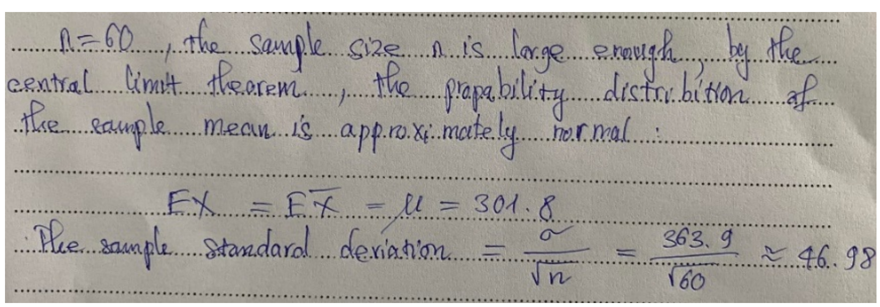


Figure 4: Student S3' answers to Question 5 of Test 1

This question allows us to assess students' SRiM capability *Description* at the level of *Connections*. Through our analysis of the students' work on this question, we found that although students are able to accurately predict the characteristics of the sampling distribution, it is clear that they have difficulty in fully integrating all information to make arguments.

Example 5. Considering the students' answers to Question 15 in Test 1.

Statistical data of students' responses to this question are shown in Table 11.

$N = 103$	Scores		
	0	0.5	1.0
Percentage (%)	77.7	19.4	2.9

Table 11: Statistical results of answers to Question 15 of Test 1

The results of the students' work show that three students successfully complete accounting for only 2.9%, most students fail this question, 77.7% of students leave the answer blank, the rest either answers but don't have any correct answer to any part of the question, or can give an example of predicting hormone levels with patients with specific blood level index but can't explain why linear regression model is appropriate to describe the association between glucose and hormone. Except the three students who have maximum score for this question, most students only stop at the calculation level, can provide a specific glucose value and input into the sample regression function to identify hormone levels; however, none of the students can come to a conclusion and fully explain given the information "*t*-test of the coefficient of regression (slope) has Sig. = 10^{-4}". For example, Figure 5 and Figure 6 are the work of two students (S4, S5) to this question.

The students' works show that, student S4 can give an example of predicting hormone levels with patients with specific blood glucose index, but has not previously tested whether the linear regression model is appropriate to describe the association between glucose and hormone. S4 just stop at calculation, have not done SRiM, have not achieved SRiM levels.

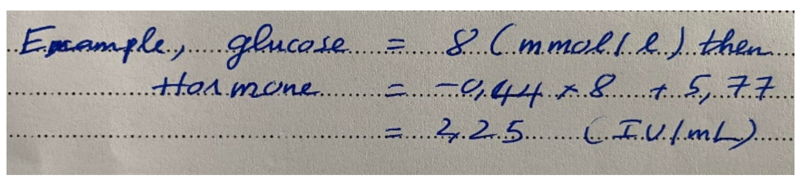


Figure 5: Student S4' answers to Question 15 of Test 1

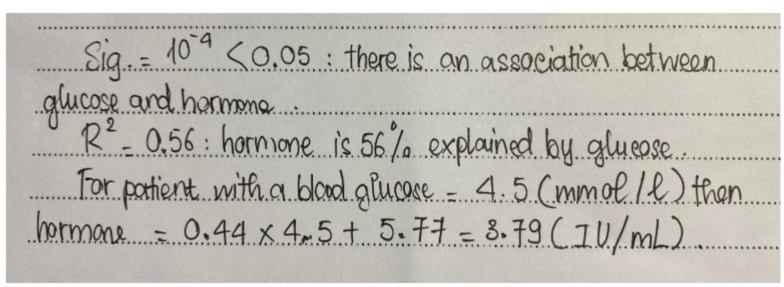


Figure 6: Student S5' answers to Question 15 of Test 1

Student S5 gives an example of predicting hormone levels for patients with specific blood glucose index, before considering and evaluating the relationship between glucose and hormone based on the coefficient of determination $R^2 = 0.56$, however, fully explain given the information "t-test of the coefficient of regression (slope) has Sig. = 10^{-4}", not fully explained to conclude that the linear regression model is suitable to describe the relationship between glucose and hormone. S5 achieved competency at the level 3 (*Connections*) in the cluster *Prediction.*

We conducted experiments and analyzed empirical results based on the assessment scale and assessment questions set (Test 1). The assessment scale is both the basis for building assessment questions and the basis for analyzing students' responses, from which assessments of students' SRiM competency when solving real-life problems. Analysis of initial empirical results reveals the effectiveness of the assessment scale and assessment questions set, allowing an overview of students' SRiM competency, through analyzing students' responses to each question, as well as analyzing each student's responses, we can assess this student's SRiM capacity, realize the difficulties students face in the process of reasoning, as well as the current situation of medical statistics teaching. Through the analysis of the results of the surveyed students' work, we obtained an overview of the student's SRiM capacity when solving practical problems: Most students achieved competencies at the levels 1, 2, 3, the number of students at the levels 4, 5, 6 are still very low. That is, most of them have achieved good results in basic statistical skills and understanding (shown at the levels 1, 2, 3), which is the foundation for the development of SRiM, medical statistical thinking, but it is clear that medical statistics teaching has not fully promoted that potential of students. Students can only solve problems that are partially or fully mathematically modeled. When facing essay problems in real world medical contexts, they can't successfully apply mathematical models to solve, especially meet difficulty with open-ended questions, have limited ability to create effective solutions to solve unknown problems, lack of flexibility in planning to solve unfamiliar practical medical problems. The limitation of our experiment is that we have only conducted on 103 first year students of 1 university, not yet diverse target study participants from many other medical universities in the country to have a comparison of SRiM competencies of different target groups, more empirical evidence to support the effectiveness of the assessment scale and assessment question set.

We conducted experiments and analyzed empirical results based on the assessment scale and assessment questions set (Test 1). The assessment scale is both the basis for building assessment questions and the basis for analyzing students' responses, from which assessments of students' SRiM competency when solving real-life problems. Analysis of initial empirical results reveals the effectiveness of the assessment scale and assessment questions set, allowing an overview of students' SRiM competency, through analyzing students' responses to each question, as well as analyzing each student's responses, we can assess this student's SRiM capacity, realize the difficulties students face in the process of reasoning, as well as the current situation of medical statistics teaching. Through the analysis of the results of the surveyed students' work, we obtained an overview of the student's SRiM capacity when solving practical problems: Most students achieved competencies at the levels 1, 2, 3, the number of students at the levels 4, 5, 6 are still very low. That is, most of them have achieved good results in basic statistical skills and understanding

(shown at the levels 1, 2, 3), which is the foundation for the development of SRiM, medical statistical thinking, but it is clear that medical statistics teaching has not fully promoted that potential of students. Students can only solve problems that are partially or fully mathematically modeled. When facing essay problems in real world medical contexts, they can't successfully apply mathematical models to solve, especially meet difficulty with open-ended questions, have limited ability to create effective solutions to solve unknown problems, lack of flexibility in planning to solve unfamiliar practical medical problems. The limitation of our experiment is that we have only conducted on 103 first year students of one university, not yet diverse target study participants from many other medical universities in the country to have a comparison of SRiM competencies of different target groups, more empirical evidence to support the effectiveness of the assessment scale and assessment question set.

CONCLUSIONS

In this article, we have applied the theories of assessment in math education including Bloom's taxonomy, MATH taxonomy, and PISA's mathematics literary classification to build an overall framework and a model for assessing medical students' SRiM competency in solving real-life problems. An overall assessing framework of SRiM competency includes six levels from low to high, corresponding to three clusters of Reproduction, Connections and Reflection. The model is to evaluate medical students' SRiM competency of solving practical problems, in which the assessment is considered regarding three aspects including the content of medical statistics, the process of medical statistics demonstrating SRiM competency, occupational clinical context, the medical statistical process is defined by the three SRiM competencies Description, Interpretation, Prediction. On that basis, we built the framework for assessing SRiM competency Description, Interpretation, Prediction including 6 levels corresponding to 3 cognitive competencies clusters Reproduction, Connections and Reflection. The development of assessment models and frameworks provided us a basis to design toolkits to assess SRiM competency of medical students when solving real-world problems. These research results on assessing students' SRiM competence in solving real-life problems have a significant effect on the innovation of medical statistics teaching, contributing to the innovation of assessment methods in teaching and learning medical statistics in particular, contribute to improving the quality of teaching medical statistics for medical students in general. Medical students have a good foundation in basic math skills, good foundation in statistical literacy in medicine, high level skills related to the application of statistical rules and procedures, which are the foundation for the development of SRiM, statistical thinking in medicine. Students could do well in *Reconstruct*, 87.4 percent or more. They had more difficulty but still did well in *Connect* with 79.6 percent. In specific, they appeared to understand what formula-tests were needed in most exercises and could apply them quite well. The reflection stage was where the matters appeared, only 12.6 percent could successfully make the required deductions and fewer could justify or provide satisfactory answer to how their work reflected upon the patients' condition. Which means medical statistics teaching should not just be teaching students to recognize, understand concepts, or apply procedures, processes and perform statistical calculations, but more importantly, students must be able to apply medical statistics knowledge to solve real-life problems in professional practice. Medical statistics teaching needs innovation to maximize the great potential of the students' foundation of basic math skills to the fullest, to motivate students to use basic statistical skills in solving real-life medical problems that require a high level of SriM. Medical statistics teaching must undergo a real and synchronous innovation, from its objectives, curriculum, contents, teaching materials, teaching methods and assessment methods. Innovation in the direction of integration, further connection of medical statistics with basic medicine, clinical medicine and medical research.

REFERENCES

[1] Anderson, L. W., Krathwohl, D. R., & Bloom, B. S. (2001). A taxonomy for learning, teaching, and assessing: a revision of Bloom's taxonomy of educational objectives. *New York: Longman*, page 352.

[2] Brown, T. (2010). Truth and the renewal of knowledge: The case of mathematics education. *Educational Studies in Mathematics*, 75 (3), (pages 329–343). https://doi.org/10.1007/s10649-010-9259-0

[3] Dani Ben-Zvi, J. G. (2006). The Challenge of Developing Statistical Literacy, Reasoning and Thinking. In *Kluwer Academic Publishers* (Vol. 1999, Issue December).

[4] Darlington, E. (2013). The Use of Bloom's Taxonomy in Advanced Mathematics Questions. *British Society for Research into Learning Mathematics*, (pages 7–12). www.bsrlm.org.uk/IPs/ip33-1/BSRLM-IP-33-1-02.pdf

[5] delMas, R. C. (2002). Statistical literacy, reasoning and learning: A commentary. *Journal of Statistics Education*, 10 (3). https://doi.org/10.1080/10691898.2002.11910679

[6] Garfield, J. (2002). The challenge of developing statistical reasoning. *Journal of Statistics Education*, *10*(3). https://doi.org/10.1080/10691898.2002.11910676

[7] Garfield, J., & Gal, I. (1999). Teaching and assessing Statistical reasoning. *Developing Mathematical Reasoning in Grades K-12*, 207–219.

[8] Jones, G. A., Langrall, C. W., Thornton, C. A., Mooney, E. S., Wares, A., Jones, M. R., Perry, B., Putt, I. J., & Nisbet, S. (2001). Using students' statistical thinking to inform instruction. *Journal of Mathematical Behavior*, 20 (1), (pages 109–144). https://doi.org/10.1016/S0732-3123(01)00064-5

[9] Mooney, E. S. (2002). A Framework for Characterizing Middle School Students' Statistical Thinking. *Mathematical Thinking and Learning*, 4 (1), (pages 23–63). https://doi.org/10.1207/s15327833mtl0401_2

[10] Moyé, L. A. (2006). Statistical Reasoning in Medicine. *Statistical Reasoning in Medicine*. https://doi.org/10.1007/978-0-387-46212-7

[11] OECD. (2002). Definition and Seclection of Competencies (DeSeCo): Theoretical and Conceptual Foundation. *OECD, Paris, France*.

[12] OECD. (2009). The PISA 2009 Assessment Framework–Key competencies in reading, mathematics and science. *OECD, Paris, France*.

[13] PISA. (2010). *Learning Mathematics for Life*. Programme for International Student Assessment.

[14] Pountney, D., Leinbach, C., & Etchells, T. (2002). The issue of appropriate assessment in the presence of a CAS. *International Journal of Mathematical Education in Science and Technology*, 33 (1), (pages 15–36). https://doi.org/10.1080/00207390110086616

[15] Riffenburgh, R. (2012). Statistics in Medicine. In *Statistics in Medicine* (Vol. 10, Issue 4). Academic Press. https://doi.org/10.1016/C2010-0-64822-X

[16] Smith, G., Wood, L., Coupland, M., Stephenson, B., Crawford, K., & Ball, G. (1996). Constructing mathematical examinations to assess a range of knowledge and skills. *International Journal of Mathematical Education in Science and Technology*, 27 (1), (pages 65–77). https://doi.org/10.1080/0020739960270109

[17] Tan, K. (2011). Assessment for learning in Singapore: Unpacking its meanings and identifying some areas for improvement. In *Educational Research for Policy and Practice* (Vol. 10, Issue 2). https://doi.org/10.1007/s10671-010-9096-z

[18] Vui, T. (2018). Bringing Mathematics Education into the global orbit to develop thinking, logic and creativity in solving realistic problems with closed-open approach. *Vietnam Journal of Education*, 5, (pages 28–33).

Combining flipped classroom and GeoGebra software in teaching mathematics to develop math problem-solving abilities for secondary school students in Vietnam

An Thi Tan Nguyen[1], Hung Nguyen Thanh[2], Cuong Le Minh[3],
Duong Huu Tong[4*], Bui Phuong Uyen[4], Nguyen Duc Khiem[4]

[1]Hue University of Education, Hue University, Vietnam,
[2] The University of Danang – University of Science and Education, Vietnam,
[3] Center for Training Partnership and Professional Development of the Dong Thap University, Vietnam,
[4] School of Education, Can Tho University, Vietnam
[*] Corresponding Author: dhtong@ctu.edu.vn

Abstract: Flipped classroom is one of the teaching models that swaps students' learning space between learning in and before class. GeoGebra software is a dynamic math software that positively affects math teaching, especially geometry, and develops students' problem-solving abilities. The study was conducted to test the effect of combining flipped classrooms and GeoGebra in teaching math on students' outcomes, learning attitudes and problem-solving abilities. This study involved 74 students in 7th grade, including 37 students in the experimental group and 37 in the control group. Results from qualitative and quantitative data of pre-tests, post-tests, classroom observations and surveys show that students in the experimental group who learned with flipped classroom and GeoGebra have better problem-solving ability, results and learning attitudes. Specifically, with the significance level $\alpha = 0.05$ *and degrees of freedom* $df = 72$*, the observed significance level (Sig. 2-tailed) is 0.010, an independent samples t-test of two groups in post-test indicates that the results of the experimental group were significantly higher than that of the control group. Besides, with the significance level* $\alpha = 0.05$*, the observed significance level (Sig. 2-tailed) is 0.000, and the paired t-test results reveal that the experimental group has a higher mean score in the post-test. The influence level (ES) is close to 0.64, showing that the combination of the flipped classroom and GeoGebra in the teaching of this study has a positive impact on learning outcomes and students' problem-solving ability. On the other hand, the student survey results are observed that students have a positive learning attitude toward this teaching process. In addition to the obtained results, the study also points out the remaining limitations and proposes new research directions.*

Keywords: Academic achievement, Flipped classroom, GeoGebra software, Mathematical problem-solving ability

INTRODUCTION

The application of innovative teaching approaches with the support of information technology in mathematics education is an important need of the 21st century (Cevikbas & Kaiser, 2020). In particular, during the period of the global outbreak of the COVID-19 pandemic, the need for decentralized learning made the role of technology and online teaching methods fully exploited by teachers. Flipped classroom is a modern teaching method that ensures student participation through face-to-face and online learning during the pandemic (Cevikbas & Kaiser, 2022). For mathematics education, the development of digital technologies has changed the content of mathematics teaching in schools and promoted the development of students' knowledge and understanding of mathematics (Heid, 2005; Olive et al., 2009; as cited in Cevikbas & Kaiser, 2020). Nowadays, technologies have revolutionized math education by providing platforms that enable 2D and 3D graphics simulation, so teachers can teach math concepts visually and promote students' interest in learning (Chivai et al., 2022). In particular, GeoGebra is a commonly used dynamic geometry software in geometry teaching, designed to create teaching materials with various mathematical representations (Nzaramyimana et al., 2021). However, the effectiveness of using GeoGebra in teaching mathematics depends a lot on the learning content, the amount of time in class and the conditions of the facilities (computer, network connection) (Manganyana et al., 2020).

Many studies have shown that through learning geometry, students could develop deductive and reasoning thinking, logical thinking, analysis, systematization, critical thinking and creative thinking, and improve visualization and spatial thinking. At the same time, students could develop mathematical problem-solving abilities by applying mathematical knowledge and skills in solving problems (Osman et al., 2018). Many practical studies have demonstrated that developing students' ability to solve math problems is one of the important goals of education worldwide (OECD, 2019). Nonetheless, much research has not been done on using GeoGebra and flipped classrooms in math instruction to improve students' problem-solving skills. For these reasons, the study investigated the effectiveness of combining flipped classrooms and GeoGebra in teaching geometry to develop students' ability to solve math problems.

LITERATURE REVIEW

Flipped classroom

The flipped classroom is a modern teaching approach that promotes digital transformation in mathematics education (Cevikbas & Kaiser, 2022). According to Cevikbas and Kaiser (2020), the flipped classroom is a blended learning approach in which students learn theoretical content online, and face-to-face classroom time is used to cope with math problems. This approach provides more opportunities for student-centered activities such as teamwork and math problem-solving (Birgili, 2021).

Flipped classrooms allow teachers to create interactive and flexible learning environments (Cevikbas & Kaiser, 2020; Ramadhani et al., 2022). Pre-class activities like watching lecture videos and reading materials are assigned to students for independent study. In class, students engage in deep learning through discussions, problem-solving, and queries, but adopting flipped classrooms requires adjustments to traditional learning habits (Nielsen, 2020). Key requirements include well-structured out-of-class study time, teacher evaluation of pre-class activities, collaborative activities during class, and teacher support and feedback to create an organized learning environment.

With the above characteristics, the flipped classroom has gained wide popularity recently, with numerous studies highlighting its effectiveness for students and teachers. The flipped classroom enhances students' core competencies, including mathematical thinking (Cevikbas & Kaiser, 2020), critical thinking (Shaikh, 2022; Voigt et al., 2020), problem-solving competence (Cevikbas & Kaiser, 2022) and boosts student engagement, motivation, time management, teamwork, and positive learning attitude, ultimately improving student achievement (Lo & Hew, 2017; Cevikbas & Kaiser, 2022; Nugraheni et al., 2022).

Besides the undeniable benefits of math education, flipped classrooms pose many challenges for teachers and educational institutions. For teachers, the basic ability to use information technology in teaching is a fundamental requirement (Cevikbas & Kaiser, 2020; Moreno et al., 2020). Besides, the transition from traditional education to an innovative teaching method such as flipped classroom poses for teachers paradigmatic obstacles as the ability to design interactive teaching connect between out-of-class and in-class learning content and be able to encourage active social interaction in the classroom (Cevikbas & Kaiser, 2022; Swart et al., 2022). More importantly, due to the nature of the flipped classroom, teachers must regularly prepare new teaching content (e.g., lecture videos, online practice exercises, and other teaching materials), which increases the workload in the teacher's near-fixed-time budget. On the other hand, numerous studies have also mentioned technical and infrastructure challenges, such as issues with internet access, a lack of digital tools and devices, the need to create new learning materials, and the absence of guidelines for students, parents, teachers, and educational institutions (Cevikbas & Kaiser, 2022).

GeoGebra

GeoGebra is an interactive math software with features of an algebraic and dynamic geometry calculator, designed by Markus Hohenwarter, which can be accessed at https://www.geogebra.org (Ogbonnaya & Mushipe, 2020). Many studies have indicated the outstanding advantages of this software for teaching mathematics. Technically, GeoGebra (1) is free, open-source and regularly updated software, (2) can be used on many different operating systems, adapting to a variety of devices that can connect to the internet, (3) can represent a variety of mathematical representations (charts, equations, tables), (4) has a friendly, lively, easy interface, and (5) has a public community of users who can share experiences and products created when using the software. With the above characteristics, GeoGebra allows students to make a mathematical investigation and exploration easily and interactively, helping them become active actors in the knowledge-building process.

Thus, these benefits enhance interest in learning, activeness and independence in students' learning (Saputra et al., 2019). Moreover, many empirical studies have verified the effect of using GeoGebra in teaching mathematics on developing higher-order thinking skills (Ramlee et al., 2019; Wijaya et al., 2019), spatial visualization and especially mathematical problem-solving skills (Tran et al., 2014; Hernández et al., 2020). These merits make GeoGebra a highly effective teaching tool for enhancing students' math learning outcomes (Lognoli, 2017). In particular, the study of Chivai et al. (2022) emphasized the positive effect of GeoGebra on teaching concepts in geometry.

Nonetheless, teachers, students and educational institutions must overcome certain challenges to use GeoGebra in teaching math effectively. Students must be equipped with basic information technology skills to use GeoGebra software in learning effectively. On the other hand, exploratory learning with GeoGebra requires students to have independent learning skills, self-discipline and active learning participation, which can be difficult for some learners with low math achievement (Manganyana et al., 2020). For teachers, skills in using information technology and proficiency in lesson design with GeoGebra are two basic requirements. Also, factors such as lesson preparation time, choice of teaching methods, positive attitudes, and teachers' confidence in using GeoGebra are important challenges (Aliyu et al., 2021). Furthermore, the conditions of facilities - information technology, teaching materials, and teachers' professional knowledge and skills in using GeoGebra are also challenges for educational institutions and teacher training organizations (Ogbonnaya & Mushipe, 2020; Zengin, 2017).

Problem-solving abilities

Problem-solving is crucial in mathematics education (Purnomo et al., 2022). In general, problem-solving is a cognitive process to achieve some goal when the subject does not have a solution. Specifically, problem-solving is the application of existing knowledge and higher-order thinking skills such as visualization, relation, abstraction, understanding, application, reasoning and analysis (Nafees, 2011); to deal with new, unfamiliar situations through asking questions, analyzing situations, converting results, illustrating results, graphing and using trial and error. Problem-solving competence is the ability to apply mathematics to various mathematical situations (Osman et al., 2018). According to Osman et al. (2018), problem-solving in mathematics gives students experience using mathematical knowledge and skills to solve real-world problems.

According to Niss and Højgaard (2011), mathematical problem-solving competence is reflected in the ability to identify, formulate, delimitate, specify, and solve different types of mathematical problems such as "pure" or "applied", "open" or "closed" that do not have algorithms available, as well as analyze and evaluate how to solve their own or others' problems. In the Vietnamese educational program, MoET (2018) has clearly stated the requirements of mathematical problem-solving capacity as follows: (1) Identify and detect problems that need to be solved by mathematics, (2) Select and propose solutions to solve problems, (3) Use relevant mathematical knowledge and skills (including tools and algorithms) to solve the posed problem, (4) Evaluate the proposed solution and generalize to the same problem.

Thus, through mathematical problem-solving, learners can practice reasoning and analytical skills, form and evaluate problem-solving strategies (Osman et al., 2018) and especially contribute to developing other mathematical competencies. Besides, problem-solving helps students become more aware of problem-solving processes and strategies and improves students' skills in selecting and putting problem-solving strategies into practice (Hoon et al., 2013).

Many studies have investigated the challenges teachers and students face when teaching to develop mathematical problem-solving abilities. The main difficulties for students are understanding, analyzing, and converting the problem into a mathematical one. These difficulties often stem from students' lack of confidence, carelessness in calculations, and weak background of knowledge and experience (Angateeah, 2017; Yayuk et al., 2020). For teachers competency-based teaching, in general, requires teachers with professional knowledge and skills to design various learning activities that facilitate students to solve unfamiliar problems, real-world problems and practice exercises (Greiff et al., 2017). Therefore, a flipped classroom with a two-phase division of online and face-to-face learning can help overcome the challenge of teaching time. Additionally, it is possible to use the online learning environment to combine the use of GeoGebra to improve the effectiveness of concept teaching and students' ability to recognize, reason and solve problems. As a result, research was conducted to answer the following questions: (1) Is there a significant difference in learning outcomes between students learning with a combination of the flipped classroom and GeoGebra (experimental group) and students learning with traditional methods (control group)? (2) Is there a significant difference in students' learning outcomes in the experimental group before and after the intervention? (3) Is there any improvement in the math problem-solving ability of students learning with the combination of the flipped classroom and GeoGebra? (4) What is the attitude of students in the experimental group towards the combined learning environment between flipped classrooms and GeoGebra?

Context of the study

The research was conducted on the topic of converging lines in triangles in grade 7 in the General Education Program of Vietnam. The General Education Program in Mathematics (2018) has clearly stated the teaching content and requirements for studying this topic. Regarding the teaching content, the topic of the concurrent lines in triangles in the 7th-grade curriculum of the textbook includes the following contents: (1) the median in the triangle, the properties of the three medians of a triangle and the ratio of the distance from the vertex to the centroid to the corresponding median; (2) property of bisectors of an angle and property of three bisectors in a triangle; (3) properties of the perpendicular bisector of a line segment, property of three orthogonal lines of a triangle; and (4) properties of the altitudes in a triangle and the theorems about altitudes, medians, orthogonal, and bisectors from the vertex of an isosceles triangle (MoET, 2018). Regarding the requirements to be met, after studying this topic, students need to: (1) recognize the special lines in a triangle (median, altitude, bisector, orthogonal) and congruence. The rules of those particular lines; and (2) express geometric arguments and proofs in simple cases (e.g., argue and prove congruent line segments, equal angles from initial conditions involving triangles) (MoET, 2018).

Besides, the order of teaching organization when combining flipped classrooms with problem-solving teaching used in this study is described in the following chart (Figure 1):

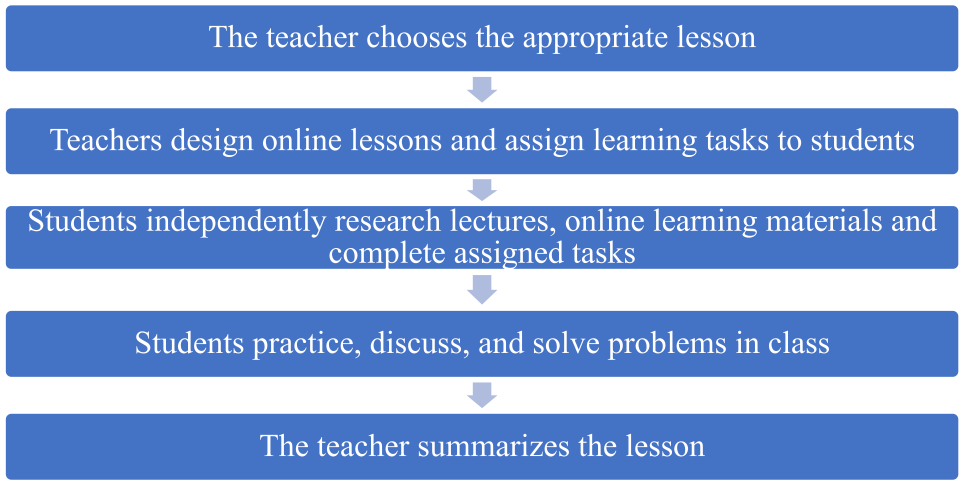


Figure 1: Teaching process

In the second step, the teacher makes requirements for solving learning problems, stating the knowledge goals to be achieved so that students have research orientation. After the learners fulfill the requirements in the third and fourth steps, the problems of the lesson are explored, discussed, resolved and commented upon by the whole class.

METHOD

The experiment was conducted to test the effectiveness of using a flipped classroom model and GeoGebra software to teach congruent lines in triangles in Vietnamese 7th-grade math textbooks to develop students' math problem-solving abilities. The experiment was conducted from April 10, 2022, to May 14, 2022, at Cao Thang secondary school, in Vinh Long City, Vietnam, for 74 students, including 37 students of the experimental group studied with flipped classrooms and GeoGebra and 37 students from the control group studied with traditional methods. Then, the data collected from the pre-test, post-test, classroom observation and student survey were thoroughly analyzed through different data analysis methods. The study was conducted based on the consent of the Ethics Council at Can Tho University, the Board of Directors of the junior high school and the consent of parents and students at Cao Thang junior high school in Vinh Long City, Vietnam.

Design

A quasi-experimental study with a control group was conducted to test the effectiveness of the combination of flipped classrooms and GeoGebra in teaching to develop math problem-solving abilities for students. In experimental designs, a pre-test was performed on the experimental and control groups to ascertain the participants' entry scores before treatment and, at the same time, verify the equivalence between the two groups. The experimental group was taught lessons in the flipped classroom combined with GeoGebra in a problem-solving orientation, while the control group was taught with conventional instruction. Then, a post-test was performed in both groups to measure student performance when learning with the new methods (Cresswell, 2012). This experimental design was used by many previous studies on the effectiveness of flipped classrooms as well as GeoGebra in math education (Adelabu et al., 2022; Kusumah et al., 2020), and there are similarities with some studies on mathematics education in Vietnam (Tong et al., 2021). With the above design, the experimental process occurred in the following order. Before teaching experimentally, the experimental group and the control group were selected based on the results of the pre-test scores of the two groups. Based on students' learning results in the experimental and control groups, a teaching plan was prepared based on applying flipped classroom and GeoGebra and teaching traditionally.

Based on the Mathematics general education curriculum requirements (Ministry of Education and Training, 2018), a scale was developed according to each level of math problem-solving ability to evaluate students; this scale is presented in Table 1. Research by Tong et al. (2021) on teaching to develop students' mathematical competence was also achieved through the design of scales when assessing the performance of students' abilities.

Math problem-solving ability	Behavioral index	Levels	Criteria
1. Ability to recognize and understand problems	1.1 Identify the problem	1	Recognize some information about the problem but do not realize the problem.
		2	Recognize most of the information about the problem but do not understand the whole problem.
		3	Get to know the whole problem.
	1.2 Identify and interpret information	1	Identify some initial information related to the objective of the task, but the relationship between those information has not been determined.
		2	Identify the majority of information relevant to the task's objective, and understand the value of that information.
		3	Identify sufficient information relevant to the task's objectives, and understand and interpret the value and relationships between such information.
2. Set up the problem space	2.1 Select and connect information with known mathematical knowledge	1	Select and connect a small amount of task information with known mathematical knowledge.
		2	Select and correctly match most of the task's information with known math knowledge.
		3	Make accurate, complete, logical connections of task information with known mathematical knowledge

	2.2. Choose a solution to solve the problem	1	Establish a partial solution to the problem
		2	Set up most of the solution to the problem, but not exactly logically.
		3	Establish a clear, specific solution to the problem.
3. Ability to plan and present solutions	3.1. Set the execution process	1	Partially built up the implementation process.
		2	Build up the majority of the implementation.
		3	Build logical process, perfect.
	3.2. Present the solution	1	Present only some ideas of the solution, but it is incomplete or lacks logic.
		2	Present most of the logical solutions but do not solve the problem.
		3	Present complete, accurate, logical steps according to the correct solution to solve the problem.
4. Ability to evaluate and reflect on solutions	4.1. Evaluate and comment on solutions	1	Initially know how to comment on the solution, it is not accurate and right to the point.
		2	Comment and evaluate the correctness of the solution.
		3	Comment and evaluate the solution with logical and persuasive arguments.
	4.2. Reflect on the value of the solution, discover new problems	1	Know how to reflect and identify some knowledge gained from the problem-solving process.
		2	Reflect on knowledge gained from problem-solving, and suggest alternatives to similar problems.
		3	New problems can be discovered through generalization and specialization from the resolved problem.

Table 1: Scale to evaluate students' ability to solve mathematical problems

After that, the research team conducted experimental teaching with the teaching plans designed for the experimental group and taught according to the traditional method for the control group. The research team made non-public observations for the experimental and control groups during the teaching process. The content of classroom observations was analyzed according to criteria including the teacher's teaching method, the student's learning method, acquired skills, classroom atmosphere and especially the expression of competence regarding solving math problems of students in the experimental group and the control group in two time points before and after the experiment. Finally, both the experimental and control groups of students were given a post-test to assess the impact on the development of their mathematical problem-solving abilities.

Furthermore, the study surveyed students' opinions in the experimental group to assess their attitudes. A set of questions was designed as multiple-choice questions on the Likert scale with five levels of Totally disagree - Disagree - Neutral - Agree - Totally agree (Likert, 1932) to collect data on the attitude, learning motivation, interest and receptivity of experimental group students when learning with experimental designs.

Regarding the validity and reliability of instruments, the experimental teaching lesson plan was moderated by experts in mathematical education methods at Can Tho University, and the tests were verified by colleagues who were teachers. High school staff conducted experiments to ensure the lesson objectives were specified in the program. After completing the adjustments according to the experts' recommendations, the tools were confirmed to be appropriate in terms of academic content and ability to assess students' abilities and could be used when conducting experiments. Moreover, the reliability of Cronbach's Alpha was used to test the reliability of the questionnaires used in the post-test. The student attitude survey and the Pearson correlation coefficient will be calculated to determine the correlation between the scores of the experimental group.

Data collection and analysis

Data was collected from pre-test (multiple-choice test) and post-test (descriptive test), class observation results and student opinion survey results after the experiment. The collected data were analyzed quantitatively with SPSS 26 software and qualitatively. This experimental method was conducted in many studies on developing mathematical competence for students. The experimental process is depicted in Table 2 as follows:

Groups	Pre-test	Intervention	Post-test	Opinion survey
EG	x	x	x	x
CG	x	-	x	-

Table 2: Experimental design

RESULTS

Pre-test results

The study used a test consisting of 20 multiple-choice questions to test the correlation between the experimental and control group's math learning levels. The data processing results demonstrate that the test scores of the two groups are normally distributed. The results of the Shapiro-Wilk test for the observed significance level of both classes are greater than 0.05, so it can be confirmed that the pre-test scores of both groups are normally distributed. The obtained results are found in Table 3.

	Statistic	df	Sig.
EG	0.968	37	0.356
CG	0.958	37	0.173

Table 3: Shapiro-Wilk test results of pre-test scores

The independent t-test method was used to test the hypothesis that the mean pre-test scores of the experimental and control groups were not significantly different. Tables 4 and 5 indicate the results of descriptive statistics and t-tests of mean pre-test scores of the experimental group and control groups.

	N	Mean	Std Dev	Median	Minimum	Maximum
EG	37	6.135	1.553	6.000	3	9
CG	37	5.987	1.991	6.500	2	10

Table 4: Descriptive statistics of scores before the intervention

Statistical results from Table 4 are observed that the average score of 37 students in the experimental class is 6.315, and that of 37 students in the control class is 5.987. The data dispersion of the experimental class (standard deviation) is 1.553. The standard deviation of the control class is 1.991; in addition, the mean and median scores in both groups are almost the same. Also, an independent t-test was used to test the hypothesis of equality of pre-test mean scores of both classes. The test results are presented in Table 5.

Levene test	t-test			
Sig.	df	t Stat	Sig. (2-tailed)	Mean Difference
0.104	72	0.358	0.721	0.149

Table 5: Results of independent Levene and t-test of pre-test scores

Levene test results give an observed significance level of 0.104 (greater than 0.05), proving that the two groups' scores have the same variance. An independent t-test was used to test the significance of the mean difference between the experimental group and the control group. Accordingly, with a significance level of 0.05 and degrees of freedom df = 72, the critical value (Sig.) equals 0.721 (greater than 0.05). Correspondingly, there was no difference in the mean score between the experimental and control groups. In other words, the test results reveal that the qualifications of the two groups are similar.

Post-test results

The study used three essay questions to test the difference in mean scores on the experimental and control groups' post-test scores. The data processing results indicate that the test scores of the two groups are normally distributed. The results of the Shapiro-Wilk test in Table 6 show that the observed significance level of both groups is greater than 0.05, so it can be confirmed that the post-test scores of both groups are normally distributed.

	Statistic	df	Sig.
EG	0.958	37	0.178
CG	0.980	37	0.739

Table 6: Shapiro-Wilk test results of pre-test scores

The independent t-test method was used to test the hypothesis that the difference in mean post-test scores between the experimental and control groups was statistically significant. Tables 7 and 8 demonstrate the results of descriptive statistics and independent t-tests of mean post-test scores of the experimental control groups.

	N	Mean	Std Dev	Median	Minimum	Maximum
EG	37	7.588	1.630	7.500	3.75	10
CG	37	6.615	1.520	6.500	3	9.75

Table 7: Descriptive statistics of scores after intervention

Statistical results of post-test scores from Table 7 show that the mean score of students in the experimental group is 7.588 and that of students in the control group is 6.615. The data dispersion of the experimental group (standard deviation) is 1.630, and the standard deviation of the control group is 1.520; in addition, the mean and median scores in both groups are almost equal. The scores of the experimental group were 7.588 and 7.5, and in the control group, 6.615 and 6.500. This demonstrates that the mean scores of the two groups are no longer similar. An independent t-test was used to test the hypothesis of the post-test mean equality of scores of both groups. The test results are presented in Table 8 below:

Levene tets	t-test			
Sig.	df	t Stat	Sig. (2-tailed)	Mean Difference
0.569	72	2.655	0.010	0.973

Table 8: Levene test results and independent t-test of post-test scores

Levene test results give an observed significance level of 0.569 (greater than 0.05), proving that the two groups' scores have the same variance. An independent t-test was used to test the significance of the mean difference between the experimental group and the control group. Accordingly, with a significance level of 0.05 and degrees of freedom df = 72, the critical value (Sig.) equals 0.010 (less than 0.05). From that, it is concluded that the mean score difference between the experimental and control groups is statistically significant. Additionally, the experimental group's mean score in Table 7 was higher than the control group's, leading to the conclusion that the experimental group's mean score in the post-test results was higher than the control group's.

Moreover, the calculated standard mean difference (SMD) is 0.64, which is in the mean (from 0.5 to 0.79) according to the Cohen influence scale (2011), so it can be concluded that the teaching process with the combination of the flipped classroom and GeoGebra software has a moderate impact on the academic performance of students in the experimental group. On the other hand, a paired test was performed to test the improvement in the group's learning outcomes after the intervention. Figure 2 shows that the scores before and after the experimental group have a positive linear correlation; the scores are distributed in a straight line before and after the treatment, so the correlation can be quite high. Besides, a correlation test was run to confirm the reliability of the scores before and after the intervention.

	N	Correlation	Sig.
Pair of scores before and after intervention	37	0.970	0.000

Table 9: Results of the correlation test regarding the scores of the experimental group before and after the intervention

The results obtained from Table 9 give an observation value of 0.000 (less than 0.01), so the calculated Pearson correlation coefficient (0.97) is statistically significant. In other words, the correlation between scores before and after the intervention is very large. A paired-sample t-test was performed; the results are presented in Table 10. The obtained critical value is 0.000 (less than 0.05), indicating that the score difference between the experimental group before and after the impact is statistically significant. Specifically, the difference between the mean scores before and after the intervention was calculated to be 1.453. It can be revealed that the math learning results of students in the experimental group are better than before the treatment.

	Mean	df	Sig.
Pair of scores before and after intervention	37	36	0.000

Table 10: Results of t-test by pair of scores of the experimental group before and after the intervention

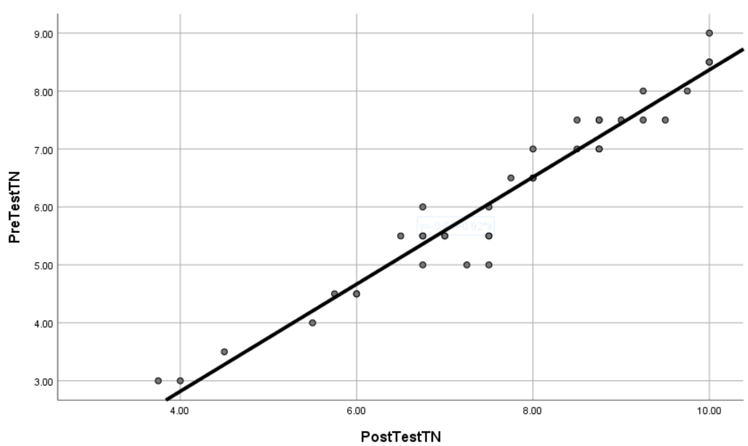


Figure 2: Scatter plot of points before and after the experiment of the experimental group

Assessment of math problem-solving abilities

Assessing the ability to solve math problems through the process of teaching in the classroom

Teachers in the experimental group carried out the teaching process in accordance with each topic, using the same teaching methodology for all topics as follows.

Step 1. Send an online lecture on the properties of the three medians of a triangle for students to self-study.

Step 2. Students go to class to discuss the content they have learned.

Step 3. The teacher concludes the problem and corrects it.

Step 4. Apply new knowledge.

Through the experimental lessons, students could identify the problem that needed to be resolved in most of the exercises; they also pointed out the information related to the goal to be directed and identified which was suitable. Nevertheless, in the proof problems, calculating the length of the line segment, some students have not been able to determine the appropriate information for the task. Most students selected and connected useful information with newly learned knowledge to solve assigned tasks when working in groups. Many students came up with most solutions to the problem, but they were not logical and precise. Some students were keen on formulating solutions and presenting them to the whole group. In the group discussions, many solutions were proposed, but there were rejections because they were not logical or did not reach the set goals.

Through the experimental process, it can be observed that students had been forming behavioral indicators in the scale of ability to cope with mathematical problems. Most students recognized the problem and identified the information, and many chose the information to resolve the problem. Many students learned and came up with the process of solving problems, knowing how to argue and refute the arguments of other members. From here, the mentioned teaching model helped students form and develop their problem-solving skills.

Assessment of the ability to solve math problems through the results of students' work

By teaching flipped classroom processes combined with GeoGebra software, students developed math problem-solving abilities; these competencies were analyzed based on a scale to assess math problem-solving abilities.

Math problem-solving competence	Behavioral Index	Level	Experimental group size	Control group size
1. Ability to recognize and understand problems	1.1 Identify the problem	1	0	0
		2	3	7
		3	34	30
		1	0	0

2. Set up the problem space	1.2 Identify and interpret information	2 3	4 33	7 30
	2.1 Select and connect information with known mathematical knowledge	1 2 3	3 1 33	7 0 30
	2.2. Choose a solution to solve the problem	1 2 3	3 1 33	7 0 30

Table 11: Analysis of students' ability to solve math problems in Question 1

Question 1 was given to measure students' ability to recognize and understand problems; establishing the problem space, the results of the experimental group show that most of the students of the experimental group and the control group fully understood the problem and applied the property of three medians in the triangle to fill in the blanks and identify most of the information and connect this information correctly when finding the segment length. Nonetheless, according to Table 11, three students in the control group still did not connect the information with the general knowledge; these students could not find the length of the line from the property of the three medians. This shows no significant difference in students' problem recognition in the two groups.

Math problem-solving competence	Behavioral Index	Level	Experimental group size	Control group size
1. Ability to recognize and understand problems	1.1 Identify the problem	1 2 3	0 3 34	1 7 29
	1.2 Identify and interpret information	1 2 3	3 15 19	11 18 8
2. Set up the problem space	2.1 Select and connect information with known mathematical knowledge	1 2 3	3 16 18	12 17 9
	2.2. Choose a solution to solve the problem	1 2 3	3 31 3	10 25 2
3. Ability to plan and present solutions	3.1. Set the execution process	1 2 3	7 27 3	11 24 2
	3.2. Present the solution	1 2	5 29	13 23

		3	3	1

Table 12: Analysis of students' ability to solve math problems in Question 2

Question 2 was asked to assess students' ability to recognize and understand problems, how to set up problem spaces, and ability to plan and present solutions. Most of the experimental group's students knew of the problem and could name the details important to achieving the task's goal. They also selected and connected most of the information correctly, but only three students presented the solution in the most specific and clear way. The setting up and presentation of the students' solutions were very satisfactory, but there were still some small shortcomings in the students' work. For students in the control group, only 29 students were able to identify the problem, and 26 could identify and interpret the information at levels 2 and 3. As for the identification and interpretation of the information at levels 2 and 3, only eight students in the control group could be identified; this number was quite different from the experimental group of 19 students; this shows that the experimental group had a better ability to identify the problem than the control group.

Regarding the selection of information connection and problem-solving, the experimental group only had three students at level 1; the control group had about ten students at specific level 1; 12 students selected and concluded information and ten students chose the solution to solve the problem. The control group had many students who could not select the appropriate information and connect them to handle the problem. In the ability to plan and present solutions, the experimental group had three students present their work well, while the control group only had one student present the complete solution. Table 12 indicates that the number of students who established the process and presented the solution at level 1 of the control group was still too high (over ten students).

Math problem-solving competence	Behavioral Index	Level	Experimental group size	Control group size
1. Ability to recognize and understand problems	1.1 Identify the problem	1	5	6
		2	22	23
		3	10	8
	1.2 Identify and interpret information	1	5	6
		2	22	23
		3	10	8
2. Set up the problem space	2.1 Select and connect information with known mathematical knowledge	1	7	6
		2	18	21
		3	12	10
	2.2. Choose a solution to solve the problem	1	6	6
		2	19	24
		3	12	7

3. Ability to plan and present solutions	3.1. Set the execution process	1	7	15
		2	27	21
		3	3	1
	3.2. Present the solution	1	10	25
		2	24	17
		3	3	0

Table 13: Analysis of students' ability to solve math problems in Question 3

Question 3 was given to assess students' ability to recognize and understand problems, how to set up problem spaces, and ability to plan and present solutions. Most students in both groups recognized most of the information but did not fully understand the problem. These students identified and understood the information in the problem but did not fully identify the relevant information. Since then, many students have been unable to connect this information accurately and completely. The number of students who selected the right information to use and set up the solution of both groups was also quite similar; most of them still could not connect the information logically and establish it with clear and specific solutions. In sentences c and b of Question 3, many students in the experimental group built up the process of doing the test and presented the solution, but the logic was lacking, or the process did not overcome the problem raised. For the control group, most of the children could not present the process or only presented some ideas of the solution but still had many shortcomings. According to Table 13, three students in the experimental group presented well and logically all the requirements of this problem, but in the control group, none of the students in the control group presented fully, accurately and logically according to the correct solution to the problem.

Analysis of student work

Below we present the score distribution of the control group and the group experiment and the level achieved on each pre-test question.

Classification	Poor 0.0-3.4	Weak 3.5-4.9	Average 5.0-6.4	Good 6.5-7.9	Very Good 8.0-10.0
EG	0	3	4	13	17
	0.0%	8.11%	10.81%	35.14%	45.94%
CG	2	3	9	15	8
	5.41%	8.11%	24.32%	40.54%	21.62%

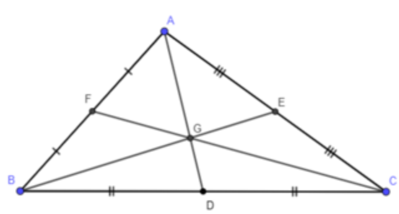
Table 14: Results of grading pre-test scores

Test scores were classified into five levels, including poor, weak, average, good and very good. In the experimental group, the excellent grade accounted for the highest percentage, including 17 out of 37 students (accounting for 45.94%). Poor and weak grades accounted for a very low percentage; specifically, there were no students with poor grades (rate of 0.0%), and average grade accounts for 8.11%, corresponding to 3 students. Thus, the total number of students below the average was three, accounting for 8.11%. Statistical results show that there were 13 students with good grades (accounting for 35.14%), an average of 4 students also accounting for 10.81%, and good grades 17 students, accounting for the highest percentage of 45.94 %. This indicates that most of the students in the experimental group achieved quite good results after this post-test. Based on statistical results from Table 14, the experimental group's post-test results were quite good, and the group's score classification was mostly good and very good.

In the control group, there were two students with poor grades, accounting for 5.41%, and three with weak grades, accounting for 8.11%. Eight students, or 21.62%, or less than half of the experimental group, were considered to be good students. Most of the students' scores were concentrated at a good average.

The two groups had a difference in the percentage of grades and the distribution of scores; specifically, the group was at a lower level than the experimental group, but there were two students in the control group, accounting for 5.41%. The number of students with weak grades in both experimental and control groups was three students, accounting for 8.11%. The experimental group's mean was 13.51% less than the control group. The good grades in the experimental and control groups were nearly equal, with 13 and 15 students accounting for 35.14% and 40.54%. In terms of good ranking, the experimental group outperformed the control group, the difference between the experimental group and the control group was nine students, accounting for 24.32% of the total students. The following are the results of the post-test for each specific question.

Câu 1 (3,5 điểm). Cho hình vẽ



a. Điền số thích hợp vào chỗ chấm

$AG = ...AD$ $GE = ...BE$

$CG = ...CF$ $FG = ...GC$

b. Biết $AD = 6cm, BE = 9cm$. Tính độ dài các đoạn thẳng DG và BG.

(Translate into English: Question 1 (3.5 marks). For drawing:

a. Fill in the blank with the correct number.
b. Knowing AD=6cm, BE=9cm. Calculate the lengths of the line segments DG and BG.)

Figure 3: Question 1 of the post-test

Classification	Weak 0.0 – 0.5	Average 0.6 -1.75	Good 1.75-2.75	Very Good 2.75-3.5
EG	0	0	3	34
	0.0%	0.0%	8.11%	91.89%
CG	0	0	7	30
	0.0%	0.0%	18.92%	81.08%

Table 15: Statistics on the results of students' work for Question 1

Question 1 in Figure 3 asked students to identify and understand the properties of the triangle's centroid. Table 15 demonstrates that 91.89% of the experimental group's students earned good grades; this shows that most of the students in the experimental class recognized the centroid of the triangle and understood the three properties of the median line in the triangle, and knew how to apply this property to calculations. Nonetheless, 8.11% of students could not apply the calculation but only filled in the appropriate proportion in the blank. A typical correct answer involves identifying the centroid of the triangle and using the information given in the exercise to calculate the lengths of the line segments. For the control group, the number of students with good scores was higher than in the experimental group (18.92% in the control group and 8.11% in the experimental group). Many students in the control group did not use their understanding of the properties to complete the calculation, as shown in Table 15. Through Question 1, most students in both groups could recall the property of the three medians in a triangle and use it to perform particular calculations.

The analysis of the students' work in the two groups for Question 2 of the post-test is summarized below.

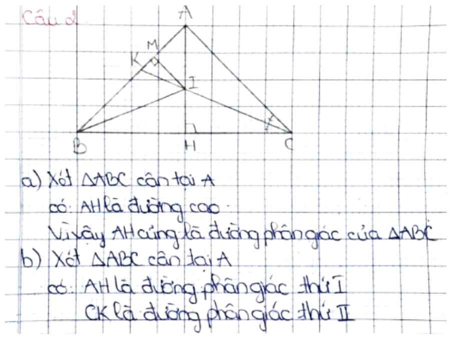
Question 2 (3.0 marks): Let ABC be an isosceles triangle at A with altitude AH and bisector CK (with K on AB). Let I be the intersection of AH and CK. IH=2cm. (a) Is AH a bisector of triangle ABC? Why? (b) Prove that BI is the bisector of angle ABH. (c) Draw IM perpendicular to AB. Calculate IM length.

Figure 4. Question 2 of the post-test

Classification	Weak 0.0 – 0.5	Average 0.6 -1.0	Good 1.1-2.0	Very Good 2.1-3.0
EG	0	3	19	15
	0%	8.11%	51.35%	40.54%
CG	1	10	18	8
	2.7%	27.03%	48.65%	21.62%

Table 16: Statistics on the results of students' work for Question 2

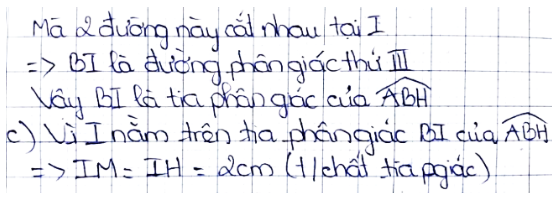
Question 2 in Figure 4 required students to apply the knowledge they have learned about concurrent lines in triangles to reason and solve the problem of geometric proofs. The majority of the experimental group's students (51.5% and 40.54%, respectively) had successful outcomes, as shown in Table 16. In the experimental group, there were no students with weak points, and the number of students with average scores accounted for a low rate (8.11%). For the control group, the student who achieved the weak point was one student, accounting for 2.7%. The number of average and good students in the control group accounted for the highest percentage (27.03% and 48.65%). In general, the results of the experimental and control groups have a clear difference through Question 2, which shows that in the control group, students did not apply the theorems and properties effectively.



(Translate into English: a) Consider triangle ABC isosceles at A; triangle ABC has AH as altitude, also the bisector of triangle ABC.

b) Consider triangle ABC isosceles at A; triangle ABC, where AH is the first interior bisector, and CK is the second interior bisector.)

Figure 5: Correct answer to Question 2 by a student in the experimental group



(Translate into English: Since these two lines intersect at I, BI is the third bisector.

c) Since the point I lies on the bisector BI of angle ABH, IM=IH=2cm.)

Figure 6: Correct answer to Question 2 by a student in the experimental group

Figure 5 and Figure 6 are students' answers in the experimental group, who captured and analyzed important information in assumptions and arguments to resolve problems. This student also used the properties of concurrent lines in calculations, but the reasoning was unclear and still needed more practice.

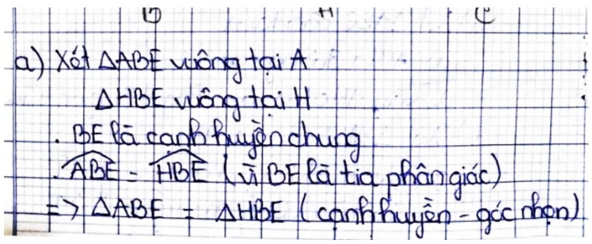
Question 3. Let ABC be a right triangle at A, where BE is the bisector of angle ABC; point E lies on AC. Draw AH perpendicular to BC, with point H lying on BC. Let K be the intersection of AB and HE. (a) Prove that triangle ABE is equal to triangle HBE. (b) Compare the length of line segment AB and the length of line segment BH. (c) Is BE the perpendicular bisector of line segment AH? Why? (d) Prove that BE is perpendicular to KC.

Figure 7: Question 3 of the post-test

Classification	Weak 0.0 - 0.5	Average 0.6 -1.0	Good 1.1-2.5	Very Good 2.6-3.5
EG	1 2.7%	3 8.1%	23 62.16%	10 27.03%
CG	1 2.7%	3 8.1%	32 86.49%	1 2.7%

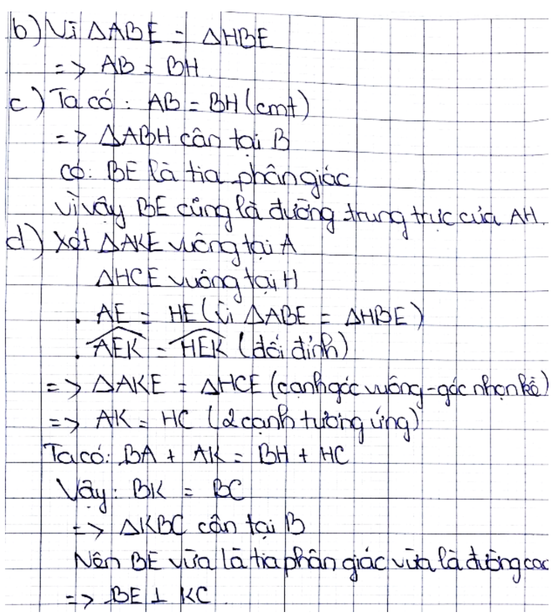
Table 17: Statistics on the results of students' work for Question 2

In Question 3 (Figure 7), the problem asked students to apply the definition and properties of the perpendicular bisector and altitude in a triangle to solve the problem. Students needed to manipulate triangles that were congruent and, from that, compared the lengths of the line segments related to two similar triangles. Students used the property of the perpendicular bisector of the line segment, the property of the three altitudes. in a triangle, or students could employ the properties of an isosceles triangle to deal with the exercise. According to Table 17, both groups' weak and averagely graded assignments for this question account for the same percentages of 2.7% and 8.11%, respectively. The control group had only one student with good scores, and the experimental group had ten students with good scores (accounting for 2.7% and 27.03%). Thus, most of the students in the experimental group could apply new knowledge in reasoning and problem-solving, and the experimental group's problem-solving ability was different from that of the control group.



(Translate into English: a) Consider triangle ABE right-angled at A; triangle HBE right-angled at H; BE is the common hypotenuse. So angle ABE is equal to angle HBE. Therefore, triangle ABE is equal to triangle HBE.)

Figure 8: Correct answer to Question 3 by a student in the experimental group



(Translate into English:

a) Because triangle ABE is equal to triangle HBE, AB=BH

c) We have AB=BH, so triangle AABH is isosceles at B; BE is the bisector. So, BE is also the perpendicular bisector of AH.

d) Triangle AKE is right angled at A; Triangle HCE is right angled at H; AE=HE; angle AEK is equal to angle EHK; so AK=HC; we have: BA+AK=BH+HC, so BK=BC. Hence, triangle KBC is isosceles at B, so BE is both the bisector and the altitude; infer that BE is perpendicular to KC).

Figure 9: Correct answer to Question 3 by a student in the experimental group

Figure 8 and Figure 9 are examples of the correct work of students in the experimental group, who proved that the triangle is congruent, deducing the pair of equal sides in question (b). In questions

(c) and sentences (d), students applied their knowledge of perpendicular and bisectors in an isosceles triangle to prove the orthogonal line.

Classroom observation results

After teaching the lessons on the converging line in the triangle, the results of the experimental group and the control group's observations were analyzed and compared based on the factors of the teaching methods, learning methods, acquired skills, learning content and students' interest in learning. The observed results are shown in Table 18.

Factors	Experimental group	Control group
Teaching methods	- Ask questions related to the lesson in videos sent to students. Class activities are discussed and explained, and do exercises based on the teacher's suggested questions. - Student-centered. - Teachers use GeoGebra software to form concepts and properties with visual images and perform measurements. - The teacher presents the combined slide show using GeoGebra. Use the visualization software GeoGebra to comment on the theorem. - The teaching process follows the process of the reverse classroom combined with GeoGebra software.	- Explain and present. - Teacher-centered. - The teacher presents concepts and properties; proves properties. - The teacher presents the board. - The teaching process starts from explaining concepts and properties to solving examples and exercises in textbooks.
Learning methods	- Discuss together, work in groups, collaborate, observe, predict, present, and critique. - Work individually and in groups. - Learn new knowledge through problem-solving. Explore knowledge under the guidance of teachers and self-study about new knowledge before going to class.	- Work individually, listen to the teacher's questions and express the opinions - Expressing opinions and working individually. - Acquiring new knowledge imparted by the teacher.
Acquired skills	- Achieve teamwork, communication, questioning, problem-solving, presentation, analysis, prediction, and criticism skills.	Ability to comment, adjust math solutions, and prove formulas.
Learning content	- Know and draw a triangle's medians, medians, and bisectors.	- Know and draw a triangle's medians, medians, and bisectors.

	- Understand and apply properties of concurrent lines to calculate angle measures and find side lengths. - Understand the properties of concurrent lines in triangles and apply them to complex problems. - Know several practical problems related to concurrent lines to apply to solving real-life problems.	- Understand and apply properties of concurrent lines to find the measure of an angle, the length of the side, and the ratio of the sides.
Student attitude	- Students actively participate in group activities and exchange ideas. - Students enthusiastically expressed, commented and absorbed ideas.	Students are still passive; a few students actively voice their opinions.

Table 18: Classroom observation results between the experimental group and control group

Classroom observation results show that this teaching and learning method positively impacted students' skills, learning content and learning attitude in the experimental group. When learning new knowledge with the support of GeoGebra, students in the experimental group had access to real things and visual images to new knowledge, helping students to be more excited about new content. Because the students in the experimental group were studied before coming to class, they were more confident when expressing their opinions, asking their questions as well as commenting on the opinions and questions of others. Besides, the length of lessons devoted to problem-solving was relatively large, enabling students in the experimental class to practice the steps of problem-solving in learning mathematics. Thereby forming the quality of confidence and self-control for students in problem-solving was observed.

Results of the survey of students' opinions after the experiment

After completing the lessons in the experimental group, the research team surveyed the opinions of students in the experimental group through a set of multiple-choice questions, according to the Likert scale. The survey was conducted to investigate the students' attitudes towards the combined learning method of the flipped classroom and GeoGebra and the students' assessment of the learning effectiveness and capacity building in math problem-solving after the experiment. Statistical results of the responses are as follows.

Items	Totally disagree	Disagree	Neutral	Agree	Totally agree
1. I like the lessons about concurrent lines in triangles.	0 0%	0 0%	1 2.7%	13 35.1%	23 62.2%
2. I find that the learning process following these lessons helps me study more effectively.	0 0%	0 0%	3 8.1%	14 37.8%	20 54.1%

3. I find that the learning process following these lessons helps me access new knowledge more effectively and easily.	0 0%	0 0%	2 5.4%	13 35.1%	22 59.5%
4. I find that the learning process following these lessons helps me to solve problems similar to those approached before.	0 0%	1 2.7%	4 10.8%	15 40.5%	17 46.0%
5. I want to take similar lessons in another lesson.	0 0%	0 0%	1 2.7%	11 29.7%	25 67.6%

Table 19: Students' feedback on questions of the survey

In Item 1, 100% of students answered fully; specifically, 23 students (accounting for 62.2%) agreed that they would like to learn lessons about convolutions using a combination of the flipped classroom and GeoGebra software, 13 students agreed (accounting for 35.1%), and one student (accounting for 2.7%) chose the normal answer, no student disagreed or totally disagreed with the above question. Based on the above data, Table 19 shows that students were interested and interested in experimental lessons in class when learning the topic of congruent lines in triangles by combining flipped classroom model and GeoGebra software.

In Item 2, most of the students totally agreed and agreed that organizing the activities of the experimental class helped them learn more effectively (accounting for 54.1% and 37.8%), and besides, three students chose the normal answer to the question (accounting for 8.1%). No student chose to disagree or completely disagree. Based on the data mentioned above, it can be concluded that students are more adaptable to a new teaching model that incorporates GeoGebra software and flipped classrooms than they are to the traditional classroom-based teaching method. It also contributed to motivating teachers when applying other teaching methods.

Item 3 was given to survey students' opinions about finding that accessing new knowledge by combining the flipped classroom model and GeoGebra software was more effective and easier to achieve the expected results. Specifically, 22 students (accounting for 59.5%) completely agreed with the given statement, 13 students (accounting for 35.1%) agreed, and two students (accounting for 5.4%) gave the opinion that it was normal. In this question, no student disagreed or completely disagreed with the statement. Thus, through collecting opinions from students, research shows that this teaching method is one of the methods that can help students acquire knowledge effectively and easily.

Item 4 was interested in whether the students themselves saw an improvement in computation, resolved problems similar to those approached before, and obtained the following results: there were 17 students (accounting for 46.0%) out of 37 students who completely agreed with this, 15 students (accounting for 40.5%) agreed, four students (10.8%) had no opinion, and one student (accounting for 2.7) %) disagreed with the statement given. Thus, most students saw progress when learning with the combined learning method of flipped classroom model and GeoGebra software, and besides, there were still some students who did not notice any progress in learning practice with this method.

In this last question, students were asked if they would like to take similar lessons in other lessons instead of only learning in lessons on the topic of concurrent triangles and congruences. The results were as follows: There were 25 students (accounting for 67.6%) who totally agreed, 11 students (accounting for 29.7%) who agreed, and one student (accounting for 2.7%) who was normal (Table 19). Thus, most students wanted to learn other lessons by this method and could extend to other subjects.

DISCUSSIONS

The experimental results answered the research questions posed. Regarding the learning results, the two groups before the experiment had similar qualifications, and following the experiment, there was a difference between the mean scores of the experimental control groups. Specifically, with the significance level of 0.05 and degrees of freedom df = 72, the critical value (Sig.) equal to 0.010 shows that the mean score of the post-test results of the experimental group is higher than that of the control group. Also, the influence level of 0.64 according to the Cohen influence scale (2011) shows that the pedagogical experiment has an average impact on students' learning outcomes in the experimental group. Besides, the experimental group's test of two sets of scores before and after the treatment reveals that, at a significance level of 0.05, the critical value is obtained at 0.000, and the difference between the mean scores before and after the intervention's impact is 1.453, indicating that the math learning outcomes of the experimental group's students improved since before the experiment. Thus, the experiment has confirmed the effectiveness of applying the combined teaching process of the flipped classroom and GeoGebra software in teaching the converging lines in the triangle for students' learning outcomes. This result is similar to the conclusions of (Cevikbas & Kaiser, 2022; Birgin & Acar, 2020). Regarding the development of students' ability to solve math problems through the built-in scale, the research team analyzed students' work based on how well students achieved the criteria of the scale and based on student performance in class. Through the analysis of the post-test work of the students in the experimental group, it can be seen that the majority of students in the experimental group performed well in the requirements of problem recognition and understanding (over 90%), choosing and connecting information with the learned knowledge and selecting problem solutions (over 70%), and planning and presenting solutions to solve problems (over 60%).

Moreover, the classroom observation results show that with the combined teaching method of the flipped classroom and GeoGebra, students had access to real things and visual images for new knowledge, helping students to be able to learn more excited about new content. At the same time, because the students in the experimental group were studied before coming to class, students were more confident when expressing their opinions and offering solutions to solve problems because the duration of the sessions was devoted to their problem-solving relatively much, creating conditions for students in the experimental group to practice skills in the process of problem-solving. Thus, it can be concluded that integrating GeoGebra into the flipped classroom in teaching math contributes to training students' problem-solving abilities. This result is consistent with studies on the effectiveness of GeoGebra (Thapa et al., 2022; Tran & Nguyen, 2020) and flipped classrooms (Cevikbas & Kaiser, 2020; Lo & Hew, 2017) for the development of math problem-solving. Besides, the survey results of students in the experimental group show that students had a positive attitude towards the combined teaching method of the flipped classroom and GeoGebra and realized that they made progress in learning. with this method (Lo & Hew, 2017; Cevikbas & Kaiser, 2022).

During the experiment, the research also recorded positive results when applying the teaching process that combines flipped classrooms and GeoGebra software into teaching. Experimental results show that students grasped the knowledge of the lesson content and applied knowledge to problems with a high degree of application. Indeed, they learned new knowledge with the combined teaching process of the flipped classroom and GeoGebra software to help students retain knowledge more deeply because the knowledge was thoroughly researched at home with the support of teachers and was reinforced in classroom learning, evident in classroom sessions.

Nonetheless, applying flipped classrooms and GeoGebra in teaching mathematics had certain difficulties. Firstly, students did not have the same level of knowledge and skills in using information technology; some did not learn new knowledge at home, making it difficult to keep up with other students. Secondly, teaching according to a process different from the traditional method made it difficult for some students who could not adapt in time and could not yet study independently. Third, teachers had to prepare new learning content thoroughly in advance, which took time and increased teachers' workload (Cevikbas & Kaiser, 2022). Consequently, in order to overcome the challenges above, the study recommends that educational researchers and teachers thoroughly equip themselves with the knowledge and abilities to use information technology for students, guide students about the learning process with flipped classrooms, and organize training for teachers on subject matter expertise as well as methods of organizing teaching models (Cevikbas & Kaiser, 2022). Furthermore, teachers can track students' learning progress during the teaching process by letting students use GeoGebra math software to answer questions in the middle of lessons and collect statistics on lesson outcomes. Teachers can also create lesson-related questions using other online tools so that students can complete practice exercises and assess the information they learned at home.

Besides the obtained results, the study still has many limitations. With a small experimental sample and short experimental time, the experimental results may be partial and not comprehensive due to public health issues in preventing COVID-19. Hence, the study's conclusions may be more representative, and the experimental influence may be investigated more deeply if the sample size is larger and the experimental time is longer. At the same time, requiring students to be exposed to a new learning environment that is different from what is done in a regular classroom makes some students unable to adapt to the classroom activities. The post-test results in the experimental class were quite high, but the study has shown a moderate impact effect; some students still did not achieve good results. In addition, the tools to assess the ability to solve math problems do not reflect students' achievement levels. On the other hand, the study has not yet explored the difficulties teachers and students face in the flipped classroom's combined learning environment and GeoGebra, focusing on developing math problem-solving abilities. However, this information also contributes to deepening the conclusions of the study.

CONCLUSIONS

Results from this experimental study show that combining flipped classrooms and GeoGebra improves students' math problem-solving abilities, learning outcomes and attitudes. Analysis of the post-test results found that the experimental group had significantly higher scores than the control group (Sig. 2-tailed=0.010 with $\alpha = 0.05$ and degrees of freedom $df = 72$). Additionally, the mean score after the intervention in the experimental group was higher than before (paired t-test showed a significance level of 0.000). The combined approach had an effect size (ES) of 0.64, positively affecting learning outcomes and mathematical problem-solving abilities.

For new studies in the future, the research team proposes several related research directions such as (1) applying a combination of the flipped classroom and GeoGebra in teaching various math topics and in order to develop other math competencies for students; (2) researching and applying a combination of the flipped classroom and GeoGebra with other active teaching methods such as problem-based learning and project-based learning; (3) research on the influence of some factors on the development of student's ability to handle math problems such as academic levels of students; and (4) study the long-term effects of using a combination of the flipped classroom and GeoGebra. On the other hand, in terms of research design, the research team proposes new studies organized with large sample sizes and long observation time to assess the effectiveness and challenges of combined flipped classrooms comprehensively and GeoGebra in teaching math.

REFERENCES

[1] Adelabu, F. M., Marange, I. Y., & Alex, J. (2022). GeoGebra software to teach and learn circle geometry: Academic achievement of grade 11 students. *Mathematics Teaching-Research Journal, 14*(3), 2-16.

[2] Aliyu, J., Osman, S., Daud, M. F., & Kumar, J. A. (2021). Mathematics teachers' pedagogy through technology: A systematic literature review. *International Journal of Learning, Teaching and Educational Research, 20*(1), 323-341. https://doi.org/10.26803/ijlter.20.1.18

[3] Angateeah, K. S. (2017). An investigation of students' difficulties in solving non-routine word problem at lower secondary. *International Journal of Learning and Teaching, 3*(1), 46-50. https://doi.org/10.18178/ijlt.3.1.46-50

[4] Birgili, B., Seggie, F. N., & Oğuz, E. (2021). The trends and outcomes of flipped learning research between 2012 and 2018: A descriptive content analysis. *Journal of Computers in Education, 8*(3), 365-394. https://doi.org/10.1007/s40692-021-00183-y

[5] Birgin, O., & Acar, H. (2020). The effect of computer-supported collaborative learning using GeoGebra software on 11th grade students' mathematics achievement in exponential and logarithmic functions. *International Journal of Mathematical Education in Science and Technology, 53*(4), 872-889. https://doi.org/10.1080/0020739x.2020.1788186

[6] Cevikbas, M., & Kaiser, G. (2020). Flipped classroom as a reform-oriented approach to teaching mathematics. *ZDM, 52*(7), 1291-1305. https://doi.org/10.1007/s11858-020-01191-5

[7] Cevikbas, M., & Kaiser, G. (2022). Can flipped classroom pedagogy offer promising perspectives for mathematics education on pandemic-related issues? A systematic literature review. *ZDM, 55*(1), 177-191. https://doi.org/10.1007/s11858-022-01388-w

[8] Chivai, C. H., Soares, A. A., & Catarino, P. (2022). Application of GeoGebra in the teaching of descriptive geometry: sections of solids. *Mathematics, 10*(17), 3034. https://doi.org/10.3390/math10173034

[9] Cohen, L., Manion, L., & Morrison, K. (2017). Mixed methods research. *Research Methods in Education*, 31-50. https://doi.org/10.4324/9781315456539-2

[10] Hernández, A., Perdomo-Díaz, J., & Camacho-Machín, M. (2019). Mathematical understanding in problem solving with GeoGebra: a case study in initial teacher education. *International Journal of Mathematical Education in Science and Technology, 51*(2), 208-223. https://doi.org/10.1080/0020739x.2019.1587022

[11] Hoon, T. S., Kee, K. L., & Singh, P. (2013). Learning mathematics using heuristic approach. *Procedia - Social and Behavioral Sciences, 90*, 862-869. https://doi.org/10.1016/j.sbspro.2013.07.162

[12] Kusumah, Y. S., Kustiawati, D., & Herman, T. (2020). The effect of GeoGebra in three-dimensional geometry learning on students' mathematical communication ability. *International Journal of Instruction, 13*(2), 895-908. https://doi.org/10.29333/iji.2020.13260a

[13] Lo, C. K., & Hew, K. F. (2017). A critical review of flipped classroom challenges in K-12 education: possible solutions and recommendations for future research. *Research and Practice in Technology Enhanced Learning, 12*(1). https://doi.org/10.1186/s41039-016-0044-2

[14] Lognoli, D. (2017). The area of the disk in middle school grade by GeoGebra. *International Journal of Emerging Technologies in Learning (IJET), 12*(11), 28-40. https://doi.org/10.3991/ijet.v12i11.6834

[15] Manganyana, C., Van Putten, S., & Rauscher, W. (2020). The use of GeoGebra in disadvantaged rural geometry classrooms. *International Journal of Emerging Technologies in Learning (IJET), 15*(14), 97-108. https://doi.org/10.3991/ijet.v15i14.13739

[16] Ministry of Education and Training (2018). *Mathematics general education curriculum.* Hanoi, Vietnam. https://data.moet.gov.vn/index.php/s/m6ztfi7sUIIGQdY#pdfviewer

[17] Moreno, D., Palacios, A., Barreras, Á., & Pascual, V. (2020). An assessment of the impact of teachers' digital competence on the quality of videos developed for the flipped math classroom. *Mathematics, 8*(2), 148. https://doi.org/10.3390/math8020148

[18] Nielsen, K. L. (2020). Students' video viewing habits during a flipped classroom course in

engineering mathematics. *Research in Learning Technology, 28*. https://doi.org/10.25304/rlt.v28.2404

[19] Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9-28. https://doi.org/10.1007/s10649-019-09903-9

[20] Nugraheni, M. M. Y., Suryaningrum, P., & Rudito, M. A. (2022). Analysis of the students' engagement in the process of learning mathematics using a flipped-classroom approach to trigonometry. *Journal of Mathematics and Mathematics Education, 12*(2), 52-60. https://doi.org/10.20961/jmme.v12i2.64420

[21] Nzaramyimana, E. (2021). Effectiveness of GeoGebra towards students' active learning, performance and interest to learn mathematics. *International Journal of Mathematics and Computer Research, 9*(10), 2423-2430. https://doi.org/10.47191/ijmcr/v9i10.05

[22] OECD. (2019). PISA 2018 Mathematics Framework. https://doi.org/10.1787/13c8a22c-en

[23] Ogbonnaya, U. I., & Mushipe, M. (2020). The efficacy of GeoGebra-assisted instruction on students' drawing and interpretations of linear functions. *International Journal of Learning, Teaching and Educational Research, 19*(9), 1-14. https://doi.org/10.26803/ijlter.19.9.1

[24] Osman, S., Che Yang, C. N. A., Abu, M. S., Ismail, N., Jambari, H., & Kumar, J. A. (2018). Enhancing students' mathematical problem-solving skills through bar model visualisation technique. *International Electronic Journal of Mathematics Education, 13*(3), 273-279. https://doi.org/10.12973/iejme/3919

[25] Purnomo, E. A., Sukestiyarno, Y. L., Junaedi, I. & Agoestanto, A. (2022). Analysis of problem solving process on hots test for integral calculus. *Mathematics Teaching-Research Journal, 14*(1), 199-214.

[26] Ramadhani, R., Bina, N. S., & Syahputra, E. (2022). Flipped classroom assisted autograph in calculus learning for engineering students: A Rasch measurement study. *Mathematics Teaching-Research Journal, 14*(4), 36-55.

[27] Ramlee, N., Rosli, M. S., & Saleh, N. S. (2019). Mathematical HOTS cultivation via online learning environment and 5E inquiry model: Cognitive impact and the learning activities. *International Journal of Emerging Technologies in Learning (IJET), 14*(24), 140-151. https://doi.org/10.3991/ijet.v14i24.12071

[28] Shaikh, S. S. (2022). Practice research on mathematics critical thinking ability cultivation strategy based on flipped classroom. *Journal of Educational Research and Policies, 4*(8), 178-181. https://doi.org/10.53469/jerp.2022.04(08).35

[29] Thapa, R., Dahal, N., & Pant, B. P. (2022). GeoGebra integration in high school mathematics: An experiential exploration on concepts of circle. *Mathematics Teaching-Research Journal, 14*(5), 14-32.

[30] Tong, D. H., Uyen, B. P. & Quoc, N. V. A. (2021). The improvement of 10th students' mathematical communication skills through learning ellipse topics. *Heliyon, 7*(11), e08282.

https://doi.org/10.1016/j.heliyon.2021.e08282

[31] Tran, T., & Nguyen, N. G. (2020). Teaching geometry according to the discovery method with GeoGebra software: a case study in Vietnam. *Proceedings of the 3rd International Conference on Research of Educational Administration and Management (ICREAM 2019)*. https://doi.org/10.2991/assehr.k.200130.166

[32] Voigt, M., Fredriksen, H., & Rasmussen, C. (2020). Leveraging the design heuristics of realistic mathematics education and culturally responsive pedagogy to create a richer flipped classroom calculus curriculum. *ZDM, 52*(5), 1051-1062. https://doi.org/10.1007/s11858-019-01124-x

[33] Wijaya, T. T., Zhou, Y., Ware, A. & Hermita, N. (2019). Improving the creative thinking skills of the next generation of mathematics teachers using dynamic mathematics software *International Journal of Emerging Technologies in Learning*, *16*(13), 212-226. https://doi.org/10.3991/ijet.v16i13.21535

[34] Yayuk, E., Purwanto, P., Rahman, A., & Subanji, S. (2020). Primary school students' creative thinking skills in mathematics problem solving. *European Journal of Educational Research, 9*(3), 1281-1295. https://doi.org/10.12973/eu-jer.9.3.1281

[35] Zengin, Y. (2017). The effects of GeoGebra software on pre-service mathematics teachers' attitudes and views toward proof and proving. *International Journal of Mathematical Education in Science and Technology, 48*(7), 1002-1022. https://doi.org/10.1080/0020739x.2017.1298855

Assessing the Implemented Research Lesson Using Mathematical Quality of Instruction

Zyrine Joy Morillo, Kim Gabrielle Del Puerto, Joyce Ann Alag, Julius Christopher Doolittle, and Minie Rose C. Lapinid

De La Salle University, Manila, Philippines

zyrine_morillo@dlsu.edu.ph, minie.lapinid@dlsu.edu.ph

Abstract: Fractions remain one of the most difficult topics to convey to students. Thus, employing professional development programs known for improving the teaching and learning process, such as lesson study, is deemed necessary. This study aims to assess and analyze the strengths and weaknesses of the implemented research lesson using the Mathematical Quality of Instruction (MQI) rubric. This standardized observation tool evaluates the quality of a mathematical instruction. Results indicated that the strengths of the implemented research lesson were demonstrated in the fluent utilization of mathematical language under the domain Richness of Mathematics and the minimal instances of teacher's error and imprecision. On the other hand, weaknesses were observed in student participation under the domains of 'Common Core Aligned Student Practice' and 'Working with Students' in Mathematics. Implications for educators and the various educational processes include among others the importance of lesson preparation, teaching students to articulate ideas explicitly and the continuous use lesson study as a professional learning community.

Keywords: Lesson study, Mathematical Quality of Instruction, Operations on Fractions

INTRODUCTION

The Philippines, as mentioned by OECD (2020), is one of the countries reported to be unprepared for online distance learning, where 30% of 15-year-old students do not have conducive homes for home study and technology inadequacy. After a year of closure, most countries returned to the normal classroom setup; however, the Philippines took two years to conduct in-person classes gradually indicative of educational opportunities in the country are disrupted more severely by the pandemic.

As most schools reopened in 2022, it is interesting to observe how learners engage in mathematics learning amid the transition stage. Online learning, teaching, evaluation, and engagement have all garnered positive and negative responses from students since the introduction of online classrooms during the pandemic (Stoian et al., 2022). Some research, however, has shown that online students lack a sense of community and hence struggle to learn from one another (Hollister et al., 2022), resulting in an "absence of emotional closeness" (Taufik & Effendy, 2022). Since interaction is one of the most essential variables in student experience, isolation and disconnectedness may lead to a lack of interest in learning (Bolliger et al., 2010). Teachers were also reported to be dissatisfied due to a lack of student-teacher interaction, which is essential for effective instruction (Sulaimi, 2022).

The unfavorable effects of the health crisis and school lockdowns are also beginning to reemerge. Students who had previously taken math online but had returned to traditional classroom settings performed poorly because they lacked the foundational knowledge necessary to succeed (Meniano & Tan, 2022). The study of Engzell et al. (2021) supports this claim since students make very little progress through remote learning; therefore, the learning losses are projected to be notable in countries that took a while to reopen schools. Students' performance in the most recent grading period revealed that a percentage of the learners failed to meet at least 75% of the learning competencies set for them. The existing knowledge gap in students' mathematical understanding has increased, and new issues concerning mathematics learning are being introduced into the education system. Public schools suffered from this negative trend especially those who have no access to internet and gadgets, and simply relied on the printed learning modules sent to their homes with parents as their home teachers during the pandemic. Teachers claimed many students struggled to learn on their own and failed to complete assignments because they could not understand the directions (Dangle & Sumaoang, 2020). In addition, most students report having difficulties with the printed learning modules adding to the existing stigma on mathematics being one of the most difficult subjects.

To lessen the disastrous effects of the pandemic, governments and stakeholders are urged to "reimagine" education and to transform the teaching and learning process as policy solutions (United Nations, 2020) especially that the resumption of traditional classroom instruction requires thoughtful consideration of the many factors involved in this shift.

Difficulty in fractions

Considering this situation before the pandemic, a far worse scenario can be gleaned as learners had to learn mathematics independently. When it comes to mathematics, fractions are among the most difficult concepts for pupils to grasp and thus pose a significant barrier to their academic success (Punzalan & Buenaflor, 2017). To this day, the ideas behind fractions remain one of the most difficult to convey to students throughout mathematical education (Africa et al., 2020). A growing body of evidence highlights the problems and hurdles children experience as they learn and comprehend fractions (Hansen et al., 2017; Lortie-Forgues et al., 2015), even though learning and teaching fractions have been regularly highlighted in a variety of mathematical standards. A nationwide study of algebra instructors in the US found that students' weakness in understanding fractions was the second most significant factor in their struggle with algebra and higher level mathematics (Hoffer et al., 2007).

Students' struggles in learning fractions and the lack of procedural fluency may be traced back to a lack of conceptual understanding, according to UNESCO's International Bureau of Education (IBE) (Africa et al., 2020). Teaching for conceptual and procedural fluency in fractions in kindergarten through eighth grade is one of the most difficult tasks for teachers. These students' difficulties in learning fractions and other mathematical content posed an emerging pedagogical challenge among teachers.Learning fractions requires students to be able to express those fractions mathematically so that they can recognize the symbols, solve the problems, and get acquainted with the procedure of solving problems (Saskiyah & Putri, 2020). The three operations on fractions—multiplication, subtraction, and division—between them posed the greatest challenge for the learners. Consequently, high school students struggle with evaluating fractions, decimals, integers, and rational expressions because they were not adequately prepared in elementary school (Punzalan & Buenaflor, 2017; Lubienski & Lubienski, 2006; Africa et al., 2020).

To help students who are having difficulty, Namkung and Fuchs (2019) recommend teachers directly instruct struggling students on how to create high-quality explanations by modeling such explanations, outlining key elements of an explanation, and giving students practice assessment and applying them in problem solving. It has been proven that the strategies and representations used in problem-solving are related to students' success in learning (Copur-Gencturk & Doleck, 2021). There is no one best way to teach, but rather a wide range of approaches, each with its own set of advantages and disadvantages, that may lead to long-term gains in students' mathematical competence. Examining education from the viewpoint of the interactions between instructors, students, and course content yields the most insightful results (Kilpatrick et al., 2001).

Lesson Study for Professional Development

Teachers must have the knowledge and teaching skills representative of the standards' depth to prepare learners to meet the necessities of educational standards successfully. Professional development and continuous professional growth help educators to improve their essential skills and learner achievement (Darling-Hammond, 1997). One of the ways teachers can participate to professional development is being part of a professional learning community such as the lesson study.

Lesson study is a collaboration-based approach originating from Japan (Murata, 2011) and is one of the growing professional development programs in the past two decades. It encourages individuals to collaborate and learn from one another's experiences and observations in one's area of specialization in teaching (Gholami, 2022; Gholami et al., 2021). A lesson study comprises teachers and educators who regularly meet to work on planning, teaching, observing, evaluating, and improving "research lessons" (Rock & Wilson, 2005). This is a cyclical process where, if desired and needed, the research lesson is revised and reimplemented for a new group of students. The principle behind the lesson study is to improve teaching, and it would best be in the context of a classroom lesson (Elipane, 2011). By observing in such a manner, multiple aspects of teaching may then be improved, such as teaching instruction, the lesson content, and the learning experiences.

With the lesson study's cyclical design, teachers can develop the research lesson continuously and indefinitely, as well as improve their teaching practices. Teachers in a lesson study also gain insights to support student learning by observing students' responses to the lesson and teaching methods. While lesson study focuses on student learning, it also has the strong involvement of teachers in the design process, which allows the design to be based on different experiences and expertise (Jansen et al., 2021). Gholami et al. (2022) observed that teachers may significantly improve their grasp of the subject matter through collaborating and sharing their experiences in the classroom. Educators develop skills in generating new challenges, debating students' misconceptions about certain subjects, enhancing their knowledge of how those topics may be used, and developing problem-based research lessons.

Teachers tend to work alone in a classroom setting, where their experiences are only obtained by themselves, and as such only improve on their own. Lesson study is an avenue for both experienced and inexperienced teachers to learn. Gholami et al. (2021) recommended using the lesson study method as a mandatory professional development strategy for teachers. The cooperative preparation, observation, and analysis of several practitioners can serve as a comparison between the point of view of a teacher who is teaching, and the points of view of teachers who are observing. With this, the teacher may become conscious of things they normally would not be conscious of due to habit (Dudley, 2011).

Mathematics Quality of Instruction

The Mathematical Quality Instruction (MQI) is a framework developed by the Heather Slope and associates at the College of Michigan (Center for Education Policy Research, n.d.). The MQI was created to provide a balanced and multidimensional perspective on mathematics instruction.

As presented in Figure 1, the mathematics quality instruction is based on the different *segments* and the entire or *whole lesson*. *Segment codes* consist of the following domains: *classroom work is connected to mathematics*, *richness of the mathematics*, *working with students and mathematics*. The *whole lesson codes* consists of *errors and imprecision*, and *common core aligned student practices*. The domain of *classroom work is connected to mathematics* measures whether the majority of time spent in the classroom was spent on activities that develop mathematical concepts.

The area of the *richness of the mathematics* is divided into two components. The first is on meaning making which includes clarifications of numerical thoughts and relating various math concepts. Whereas the other is observing the different mathematical practices, such as solving using procedures and making generalizations.

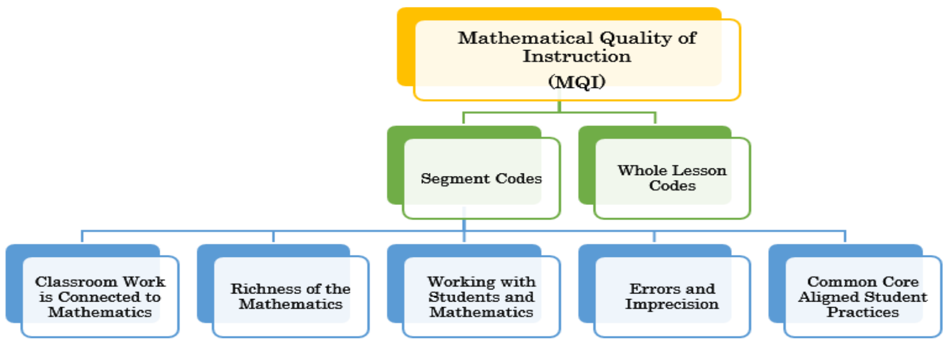


Figure 1: Domains of Mathematical Quality of Instruction

The next domain *working with students and mathematics*, focuses on how the teacher uses students' math ideas and can identify whether it is correct or not to perform immediate remediation. The subsequent domain, *errors and imprecision*, targets to check if there are math errors done by the teacher, which affects the mathematical instruction process. And lastly, *common core aligned student practices* captures how students interact with mathematical material and ideas.

Conceptual Framework

This study makes use of the Mathematical Quality of Instruction (MQI) rubric to assess and analyze a research lesson on fractions. The MQI tool does not only analyze the content of the lesson but also the interactions and performances of the teacher and the students. This helps identify which aspects of the lesson needs improving in terms of instructional strategies effective in supporting students to understand the lessons. Adkins (2017) observed that teachers are expected to be knowledgeable of their mathematics and are skillful in providing explanations and examples of their topic. However, their ability to guide the students in making generalizations and reasoning mathematically is a relative weakness. By identifying what are the outstanding practices as well as the ones that need improvement through the lesson study approach, professional learning and development can be tailored to enhance the excellent practices and mend the weak practices. With this in mind, the purpose of this investigation was to examine and evaluate the research lesson using the Mathematical Quality of Instruction by answering the following research questions:

1. What are the strengths of the implemented research lesson based on the MQI observation tool?

2. What are the weaknesses of the implemented research lesson based on the MQI observation tool?

METHODOLOGY

Research Design

This participatory action research used an explanatory sequential mixed method design. The qualitative data consisting of observations, remarks, and transcripts offer a more in-depth explanation obtained from the MQI ratings as a means of ensuring the rigor and accuracy of the research. The combination of qualitative and quantitative research enhances the comprehension of the research problems and yields greater insight than the use of either method alone (Creswell & Plano Clark, 2017).

Participants

The study was conducted in a select public secondary school in Mabalacat City, Pampanga, particularly in a grade 7 math class. It is a heterogenous, intact class consisting of 32 students. The study was conducted during the school year when face-to-face classes were first re-implemented, and all the students were coming from two years of distance learning. The locale was chosen since the researcher is one of the division's instructors and is in charge of the aforementioned class, as well as based on the availability of the teacher-implementer. Recording of the in-person lesson implementation was done for later observations.

Instrument

This study utilized the Mathematical Quality of Instruction (MQI) rubric. This rubric is a standardized observation tool that evaluates the different domains of the quality of mathematical instruction. The MQI comprises segment and whole lesson codes as shown in Figure 1. Individual codes within each of the five domains contain score points that classify instruction into various quality levels. In addition, each code covers different elements for each domain to be observed. The MQI scale is specifically designed for recorded video lessons rather than in-person observations. The recorded class session was broken up into 7-minute and 30 seconds sections. The MQI rubric's descriptions and sample situations for each element were used to identify present elements of instructions in each segment.

Rating and Analysis

The segment codes are classified into five domains, each with elements to be observed, while the whole lesson codes have ten elements. Since this study focused more on determining the instructional strengths and weaknesses of the implemented research lesson rather than estimating its overall quality, the overall and whole lesson codes were excluded from the analysis. The recorded lesson lasted for about 80 minutes, yielding ten segments of 7 and a half minutes, and one segment came with 5 mins and 28 seconds (refer to Table 1). Segmenting the recorded class enables the observers to evaluate events as they occur by not simply recalling what happened at the end of the video. Each part was assigned randomly to two teacher-raters (See Table 1 for reference of the rater assignments), choosing among the members of the lesson study group except the teacher-implementer. Descriptions were provided for each code, which rates teacher and student actions as not present, low, middle, or high, except for the first domain, *classroom work is connected to mathematics* which only has a yes or no rating.

Table 1

Segment Codes and their Assigned Raters

Segments	Time Duration	Teacher Rater
1	0:00:00 - 0:07:30	Teachers A & B
2	0:07:30 - 0:15:00	Teachers B & C
3	0:15:00 - 0:22:30	Teachers A & C
4	0:22:30 - 0:30:00	Teachers A & B
5	0:30:00 - 0:37:30	Teachers B & C
6	0:37:30 - 0:45:00	Teachers A & C
7	0:45:00 - 0:52:30	Teachers B & C
8	0:52:30 - 1:00:00	Teachers A & B
9	1:00:00 - 1:07:30	Teachers A & C
10	1:07:30 - 1:15:00	Teachers B & C
11	1:15:00 – 1:20:28	Teachers A & C

Procedures

There were three distinct stages to the research process: pre-implementation (planning), implementation (execution), and post-implementation (evaluation).

Pre-implementation

Following the lesson study process (Stepanek et al., 2006), the researchers began by familiarizing themselves with the whole lesson study cycle, procedures, and application methods. The lesson study group came up with an overarching research subject that encapsulates the learning targets for the learners. The group identified a research subject and a content area topic around which to center the lesson. Upon examining the lesson study schedule and comparing it to the K to 12 Mathematics Curriculum, one of the possible topics to be taught during the observation and debriefing is fractions (Africa et al., 2020; Namkung & Fuchs, 2019). Considering the student assessment data from the past school years, the curriculum gaps, and the existing literature on the topics that are found to be challenging to teach and learn in mathematics (Hiebert et al., 2002; Hill, Rowan, & Ball, 2005), the team decided to focus on the operations on fractions. The student assessment data obtained from the school head and mathematics department head of the participating junior high school revealed that fractions were among the least mastered skills in Grade 7 mathematics last school year 2021-2022.

Also, the period by which the lesson was implemented and the target learning competencies (Department of Education, 2019) for the quarter were considered in finalizing the research lesson topic. In the K to 12 Curriculum Guide for Mathematics, fractions were introduced as early as the 1st-grade level and this concept is being further developed across other grade levels. For instance, in grade 7 math, the focus was on mastery of the operations of fractions. Members of the group paid careful attention to the progression of ideas as they were introduced and expanded upon throughout the unit. That includes the research lesson (operation on fractions) itself as well as all lessons leading up to and after it. Since the greatest common factor (GCF) and the least common multiple (LCM) are both prerequisites of the topic of interest, the researchers decided to include them both in the review. Moreover, the researchers anticipated students having certain knowledge gaps brought by the last two years of modular distance learning. Thus, the concept of similar and dissimilar fractions was included before moving on to learning how to add and subtract fractions.

After deciding on a topic for the research lesson, the group collaborated and devised a comprehensive lesson plan noting learning goals, activities, expected student responses, and assessment questions. The lesson plan and the teaching materials were developed and crafted collaboratively by the teacher-researchers. These materials underwent evaluation and validation by four mathematics teacher experts and were revised according to the recommendations provided for the enhancement of the lesson.

The researchers secured the approval of the school head on the conduct of the study. Upon the principal's approval, students and their guardians were asked to sign an informed consent form indicating willingness to participate in the research.

Implementation

The teacher started the lesson by checking the assignment on finding GCF and LCM. Students would raise their hands if they wanted to share their answers on specific items. Although the teacher did not ask students to write their solutions on the board, the PowerPoint presentation revealed the correct answers one by one.

After checking assignments, examples of similar and dissimilar fractions were shown on the screen (see Figure 2). The students were instructed to analyze each set of examples and then characterize and differentiate similar and dissimilar fractions. To check students' understanding of these concepts, sets of fractions were presented on the screen where students would classify whether the fractions were similar (if so, they would stand up) or dissimilar (they would remain seated).

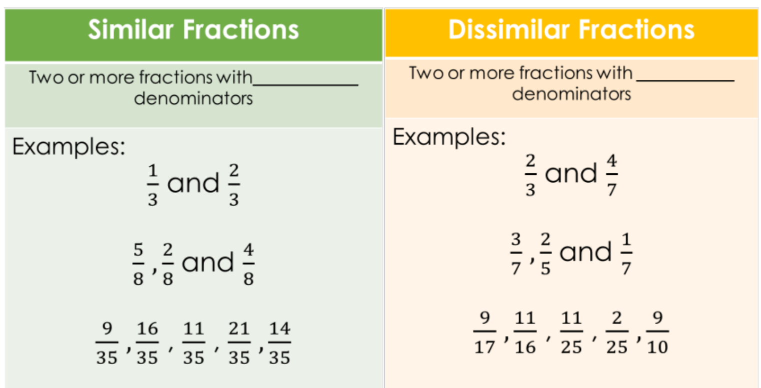


Figure 2: Similar and Dissimilar Fractions

The teacher shared a real-life word problem among the students to present instances of the new lesson. Students were asked for the possible operation or solution to the problem, to which they correctly answered that they needed to add the given fractions. Consequently, the teacher presented addition and subtraction of similar fractions task, which the teacher and the students solved together using the area models (see Figure 3).

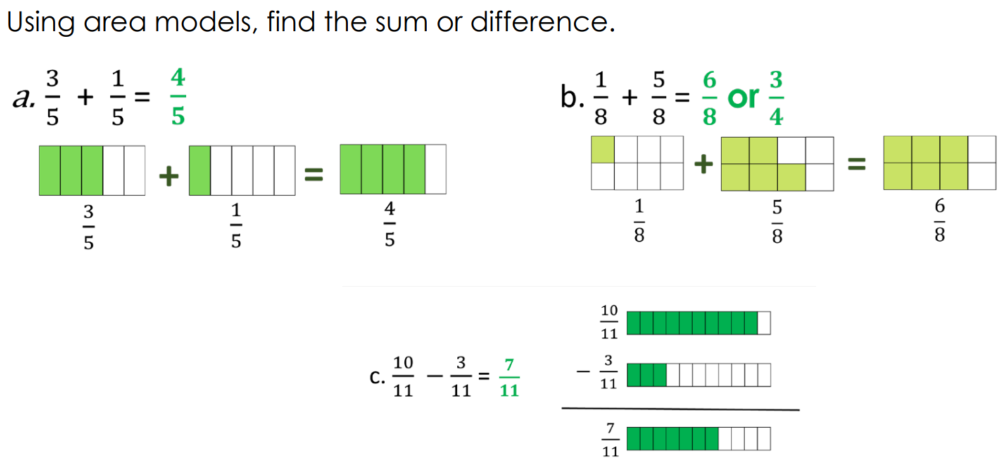


Figure 3: Adding and Subtracting Similar Fractions Using Area Models

The students were then tasked to observe these examples. Afterward, they were prompted to generalize how to add or subtract similar fractions without using area models. The area model approach in adding and subtracting dissimilar fractions (see Figure 4) was also discussed before going over the procedural process. Students worked on practice exercises to build mastery.

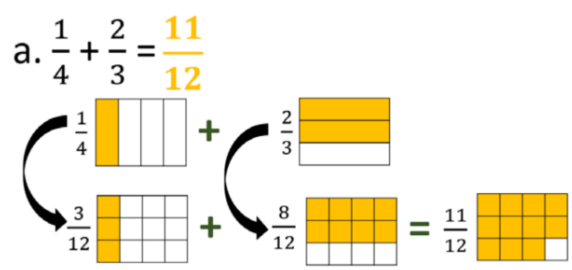


Figure 4: Adding and Subtracting Dissimilar Fractions Using Area Models

The word problem at the start of the lesson was presented again by the teacher. The students then applied the concepts and skills they learned to solve the mathematical problem. Two students presented and wrote their solutions on the board. To wrap up the lesson, students were instructed to fill in the blanks with the correct words to complete the steps in adding or subtracting fractions. For their assignment, students were tasked to create and solve their own word problem involving the addition or subtraction of fractions.

Post-implementation

After the implementation, the lesson study group convened via an online conferencing platform. Their main agenda was to discuss the procedures on how to utilize the MQI rubric tool in analyzing video recordings of the lesson study implementation. Following the MQI procedures, the randomly assigned lesson study members rated in each segment. These raters were given time to accomplish the MQI analysis of their designated segments individually. Each rater was unaware of the other rater's score. After the evaluation, the raters convened again to compare their ratings with their assigned partners. In cases of the assigned pairs' different ratings for a particular coded segment, all three lesson study raters discussed it further and then arrived at a final agreed rating. This manner of addressing disagreements on ratings was based on the MQI administering procedures found in other studies (Hill et al., 2012; Hill et al., 2011).

After identifying and rating the present MQI elements, the researchers recorded the frequency of each domain element's occurrences across the 11 segments. This frequency data was then organized and presented graphically to display trends not explicitly seen in the MQI analysis table. It is also important to note that the number of elements that make up each domain varies, leading to variation in each domain's highest possible frequency. For example, the domain of *richness of mathematics* has six elements, whereas there are 11 segments. Hence, 66 is its highest possible domain frequency. On the other hand, the domain of *working with mathematics* is comprised of 3 elements, leading to its highest possible frequency of 33.

RESULTS

Classroom Work is Connected to Mathematics

Among the domains of MQI, this area is different in terms of the manner of rating. Instead of rating as low, mid, or high, this domain simply asks whether half or more of the segment observed was related to the mathematical content. The rating shows that all the instructional time was utilized on activities to develop mathematical ideas. This further implies that the instructional time was used productively.

Richness of the Mathematics

This domain aims to determine how in-depth mathematics was offered to the students. As Figure 5 shows, most segments obtained a high level. It covers more than half of this domain's total recorded frequencies (34 out of 66). Moreover, mathematical language stands out among the elements since it gained the most frequent high-level rating. This implies that the teacher was able to use mathematical language throughout the class fluently. Meanwhile, the remaining elements still gathered a recurrence rate in the other segments rated as low, mid, and high. However, it cannot be concluded that in these elements, the quality of math richness is lacking; it is just that some of the elements can only occur in specific parts of the lesson.

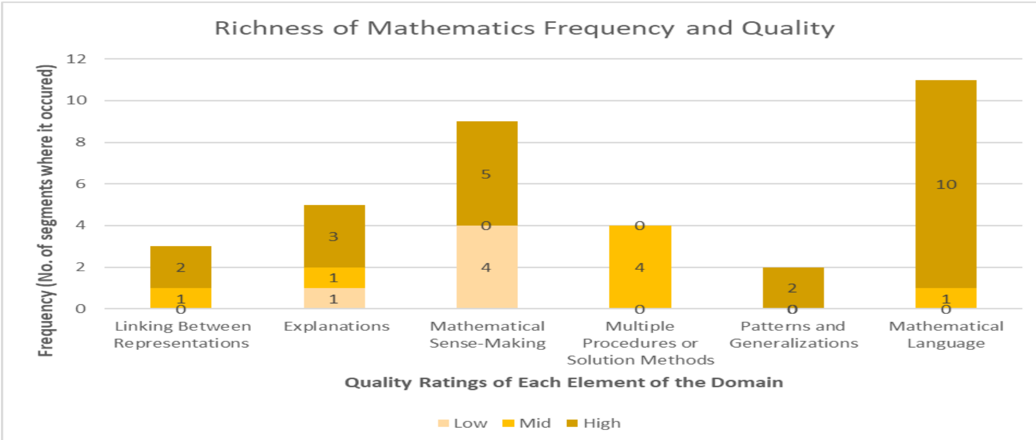


Figure 5: Richness of Mathematics Frequency and Quality

For instance, linking between representations was present since the teacher was able to use area models to represent the addition of dissimilar fractions; however, it can only be seen in segments 5 and 6 of the implemented lesson. Furthermore, this paved the way for developing the students' way of explaining contents and mathematical sense because they were able to identify and differentiate fractions. In addition, the solutions and generalizations can mostly be exhibited once the lesson is taught, which supports how it can only be present in some parts of the lesson.

Working with Students and Mathematics

A high rating is very rare among the elements of *working with students and mathematics* as can be seen in Figure 6. It was only noticed for two segments which shows that the teacher made an effort to provide remediation in addressing students' errors and difficulties. Moreover, it was observed that more segments got a low rating than a mid-score. This means that the teacher mainly interacts with students in a usual or pro forma way and further elaboration of the content was not always evident.

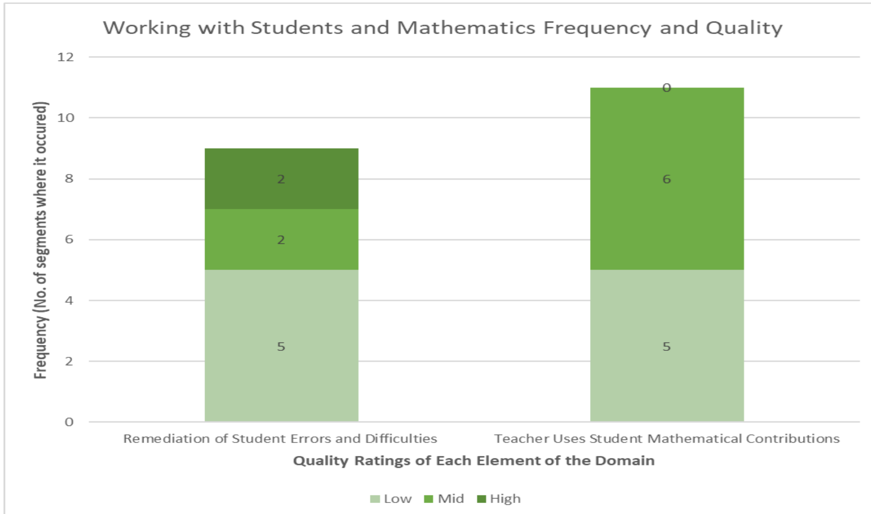


Figure 6: Working with Students and Mathematics Frequency and Quality

Here is a portion of the recorded class session that shows an example dialogue in which the students responded incompletely without providing an explanation, resulting in the teacher primarily explaining the ideas in a standard usual way.

Excerpt 1:

During the discussion on adding similar fractions…

*Teacher: But kagaya nung sinagutan natin kanina. Sa b. 6/8 at ginawa natin siyang ¾. Ano ang tawag natin sa fraction na iyon? (**Translation:** Just like the previous exercise, we simplify 6/8 to ¾. What do we call that type of fraction?)*
*Student A: Dinivide po natin (**Translation**: We divided)*
*Teacher: Dinivide natin siya by? (**Translation:** We divided by what?)*
Student B: 2
*Teacher: Dinivide natin siya by two which is the GCF. kapag nadivide na natin siya by two, nakakakuha tayo ng tinatawag na… (**Translation**: We divided it by two which is its GCF. If we divided it by two, the process is called?)*
Student C: Lowest term
*Teacher: So ano ang tawag natin sa process na yun? Lowest term or? (**Translation**: So, what do we call that process? Lowest term or?)*
Teacher: Starts with the letter s
Students: Simplify
Teacher: After you add the numerators and copy the denominator, check if you can still get the lowest term or the simplified fraction.

*Teacher: Those are the steps, tama po. … we first need to add or subtract, depende on the operation. And then we need to copy the same denominator. And the last step is… **(Translation**: Yes, that's correct those are the steps. We first need to add or subtract depending on the given operation then we need to copy the same denominator. And the last step is…)*
Students: Simplify
*Teacher: Yes, when necessary. Not all the time class na you need to simplify. Because there are times you cannot simplify anymore. For example dun sa assignment ninyo. The GCF of 3 and 7 is 1. So kung 1 lang ang GCF nila, wala ka nang isisimplify. Tama po ba? (**Translation**: Yes when necessary. It is not all the time that we need to simplify because there are times that you cannot simplify anymore. For example, on your assignment, the GCF of 3 and 7 is 1, then if 1 is their GCF, you do not have to simplify. Is that right?)*
Students: Opo (Translation: Yes)
*Teacher: So, it depends. Kaya nga sabi diyan if necessary. (**Translation**: So, it depends, that is why it said only when necessary.)*

Thus, this shows that since students commonly use one or two-word answers, which is why most of the time the teacher is the one who completes the idea during the discussion.

Errors and Imprecision

Figure 7 summarizes the frequency for each element of the *teacher's error and imprecision* domain. Aside from having the lowest recorded frequency among all MQI domains, This domain also receive a consistently low rating for all of its elements which is a good indicator of quality instruction. It means that the minor content error, a few slips in language, and a brief lack of clarity did not obscure the mathematical teaching and learning in those instances.

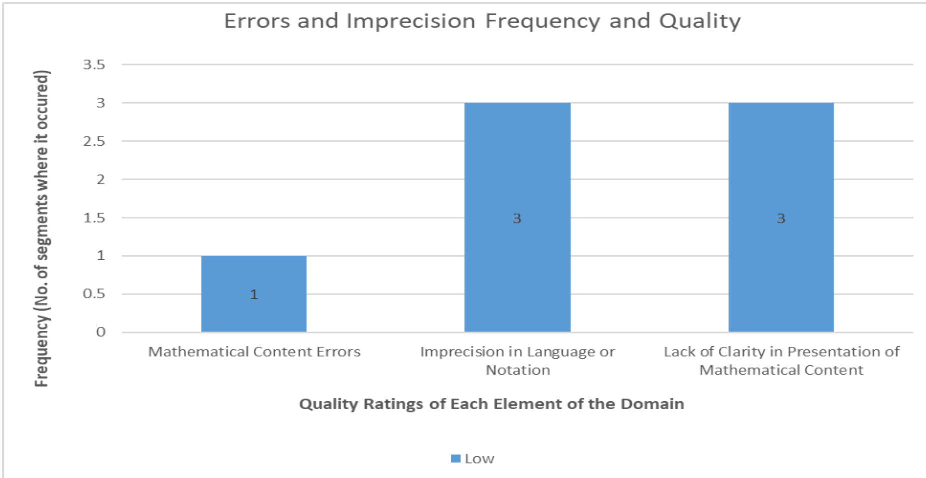


Figure 7: Errors and Imprecision Frequency and Quality

Below is an excerpt of the recorded class session, revealing an instance of teacher mathematical content error and language imprecision.

Excerpt 2:

During the checking of assignment…

Teacher: What is the LCM of 4,8, 12, and 16?
Student A: 96
Teacher: Do you agree?
Other Students: No
Teacher: Why not? (asks one among the students who want to answer)
Student B: 48 (did not answer why. Just simply stated 48)
Teacher: Is it 48? (asking the class)
Students: yes
Teacher: That is correct. 48 is the LCM. When we talk about LCM, it should be the least or the lowest common multiple of the given numbers. 96 is also correct but it is not the least.

Although the teacher's elaboration and emphasis on the concept of 'least common multiple" is correct, the error specifically occurred when she said, *"96 is also correct, but it is not the least"*. What she originally meant to say is that *"96 is also a multiple, but it is not the least"*. However, she had a spur-of-the-moment careless slip of the tongue. It resulted in a statement that might confuse learners and give them the wrong impression that 96 is acceptable while 48 is the best answer.

Common Core Aligned Student Practices

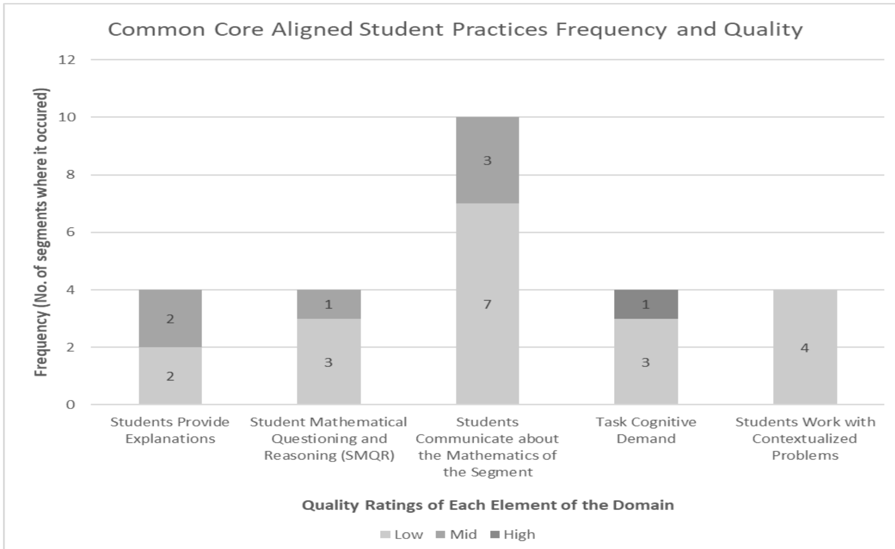


Figure 8: Common Core Aligned Student Practices Frequency and Quality

Common core aligned practices is the MQI domain that aims to capture evidence and describe the extent of students' meaningful engagement in "doing" mathematics during the instruction. Figure 8 shows that elements of students' involvement and participation were indeed present, but the majority of these instances leaned towards a low rating. Low ratings were given in students' explanation, questioning and reasoning, and communication of math due to the brevity of students' responses and talks in one-or-two-word answers. There were few instances of complete sentence responses, yet the delivery of ideas still needs improvement.

The teacher tried to elicit students' explanations through various strategies, such as asking why and how questions about an idea, procedure, or solution. However, either the students did not respond to these questions (refer to Excerpt 2) or they only provided single word or brief phrase and sentence-length explanations. There were some instances of more sustained student explanations (rated as mid-quality) but these were only obtained through the teacher's follow-up questions.

Mathematical questioning refers to instances where students ask questions that lead to the development of mathematical ideas for the lesson. Although the teacher often told the class just to ask her in case there were questions about the lesson, they did not ask anything. Hence, the teacher was the one who raised questions to elicit students' ideas and reasoning in moving forward with the lesson. An example of this case is when the teacher asks a series of probing questions to lead students in making a generalization about adding or subtracting similar fractions without the use of area models. Please refer to the excerpt below.

Excerpt 3:

a. $\frac{3}{5} + \frac{1}{5} = \frac{4}{5}$

b. $\frac{1}{8} + \frac{5}{8} = \frac{6}{8}$ or $\frac{3}{4}$

c. $\frac{10}{11} - \frac{3}{11} = \frac{7}{11}$

d. $3\frac{6}{7} - 1\frac{2}{7} = 2\frac{4}{7}$

Teacher: Without using the area models, how would you get the sum or difference of similar fractions?
Teacher: Do you see any patterns? What generalization can we make out of this pattern that you noticed?
Student C: Ipag-a-add po natin ang numerators and then ika-copy po natin ang denominator. (Translation: We will add the numerators of the given fractions and then copy their common denominator.)

When a contextualized problem was used to introduce the lesson about adding and subtracting fractions, the teacher asked the students, "what mathematical operation are we going to use to solve the problem?". They answered correctly, but they did not solve this problem yet. They only went back to solve this problem after the lesson discussion. This one contextualized problem covered 4 out of the 11 lesson segments. However, all of these instances were rated low. This is due to their rote or routine nature of learning and heavy scaffolding by the teacher. It is a far-fetched opposite of students performing the cognitive load of solving problems with greater autonomy and less to no help from the teacher - the characterizing feature for the high-quality manifestation of students' work on contextualized problems.

Most of the tasks involved were undemanding activities such as applying procedures discussed previously or within the current segment. Some activities were not routine but hints were mostly provided by the teacher. The only highly cognitively demanding task was when the students were asked to observe the examples and make conjectures based on patterns they noticed.

DISCUSSION AND CONCLUSION

The research has uncovered several significant findings and highlighted some aspects deserving of further investigation.

Strengths of the Implemented Research Lesson

Among all the domains of MQI, all segments under *error and imprecisions* obtained a consistently low rating. Errors and ambiguity made by educators in solving problems, defining concepts, introducing tasks, and using the correct notation are all within the scope of this domain. The low remarks indicate rare instances of teacher errors and imprecision, which is ideal and highly desirable in the teaching and learning process.

Moreover, it is also notable within this domain that *mathematical content errors* are the least common among the recorded data. Teacher content-related errors in the classroom is a sign that they are unsure about the material they are teaching, revealing a lack of mathematical competence (Center for Education Policy Research, n.d.). Thus, the low scores achieved under this segment imply that the teacher committed few content errors in instruction and highlights the level of conceptual understanding and proficiency of the teacher-implementer of the subject matter.

Hill et al. (2008) emphasize the favorable correlation between a teacher's level of mathematical knowledge for teaching and the quality of the mathematics education students receive. However, the results obtained in this undertaking are not sufficient to support this claim. On the other hand, Ronda & Adler (2019) explained that teachers' subject-matter knowledge was connected to teachers' present instructional practice but not to its development; rather, an increase in instructional quality was associated with teachers' vision of excellent teaching. This illustrates how well the educator-implementer of the topic knows the material and how well she can explain it to students.

Notable content errors occurred once when the instructor failed to correctly state that only one definition could be met by a given value. Thus, it was recorded by the raters as an error and imprecision. Being specific helps communicating correct mathematical ideas with pupils since it eliminates room for misunderstanding (Bieda & Voogt, 2019). Mathematical mistakes instructors make during education, especially if persistent, may expose knowledge gaps. Overall, even though there were some momentary and minor mistakes, the teacher-implementer did not make it hard to understand the concept of adding or subtracting fractions in the segments.

Another remarkable result obtained based on the coded data is that the teacher-implementer's *mathematical language* received a consistently high rating for almost all the segments. Thus, this implies that the teacher's skills in appropriately using mathematical terms to explain and discuss the content with the students are commendable. The ability to think critically, a solid understanding of the English language, a well-built number sense, and a strong background in mathematical content and pedagogy are all necessary for mathematical communication (Riccomini et al., 2015). Also, as Smith (2017) mentioned, proficiency in the instructional language is essential for comprehending the contents being taught. Therefore, being able to identify that the teacher-implementer's mathematical language with a high rating indicates quality of instruction.

Aside from mathematical language, other elements of richness, such as linking representations, explanations, sensemaking, and generalizing patterns, notably received high ratings for quality. These specific elements were coded in segments 2 through 6, where the concept-building part of the lesson occurred. Thus, reflecting the level of mathematical depth explored during the development of the lesson.

In the research lesson, area models were explicitly used to represent the addition and subtraction of similar and dissimilar fraction problems. Several studies revealed that using multiple representations improves students' learning of mathematical content such as fractions (Flores et al., 2019; Fyfe & Nathan, 2019). However, teachers must also help students understand how these representations are connected and elaborate on when relationships exist between these representations (Blanton, 2008). Furthermore, exposing learners to pattern generalization tasks does not only help them make sense of the operational procedures, it also develops their mathematical thinking in general and, more specifically, their algebraic thinking later on (Demonty et al., 2018).

Weaknesses of the Implemented Research Lesson

On the other hand, the recorded data resulted low on the *common core aligned student practices*. This means the students did not contribute much to meaning-making or reasoning during the lesson. From obtaining a mostly low rating for this domain, it reflects that the students were not participating at a moderate or high cognitive level. Some students provided brief answers, and their responses were not substantial. It is not to say that the students were not engaging or participating, but rather their responses were mostly at an undemanding cognitive level, including routine and procedural responses. Although there were tasks that were not completely routine, it was heavily scaffolded by the teacher. The common core was made to improve the standards for student achievement (Marchitello & Wilhelm, 2014). Lynch (2015) also said that the *common core state standards* were designed to give students a more profound conceptual understanding of mathematics, not just procedural literacy. In this case, students are expected to develop and express conceptual ideas and understanding instead of just reporting procedural steps in solving equations or problems.

There were also limited occurrences of high ratings in the domain of *working with students in mathematics*. This domain pertains to how the teacher understands and responds to student's mathematical contributions or mathematical errors. There exists comprehensive research evidence to support that interactive dialogue between students and teachers can be a great source of drive for the development of student performance (Mercer & Howe, 2012). Several of the ratings were low, as the teacher mostly provided brief conceptual and procedural remediations, re-demonstrating procedures, or asking the students again so they might be able to correct themselves. The use of student contributions was also minimal, mostly because they were answering calculations or giving short definitions. Solomon and Black (2008) claimed that student participation is significant for learners in building their understanding through verbal interactions with the teacher as well as their fellow students during the mathematical discourse. There were occasions of extensive remediation or great use of student contributions, highlighting said contributions to develop the mathematics during the discussion. However, although the teacher and student interactions go beyond pro forma, there was a mix of high and low elements, which resulted in ratings not qualifying as high.

The level of teachers' use of students' contributions might be attributed to the quality of students' contributions. If students are only providing one-to-two-word answers or just simply enumerating the steps, this does not give sufficient ideas for teachers to use in moving the lesson forward. Classroom discourse is the major setting in most mathematics classrooms, and some studies show that communication inside the classroom can help improve student achievements (Thompson, 2007). However, as the student's contributions lacked additional ideas to advance the discussion, the teacher had to put the ideas in to proceed with the lesson. The teacher provided ample opportunities for the students to give their thoughts, explanations, and justifications, with only several students able to speak their minds with responses suitable to further the discussion.

Limitations

The research lesson's MQI ratings and analysis were administered by the lesson study group members except the teacher-implementer. However, they were also the ones who collaborated in conceptualizing the lesson's instructional plan. Hence, this study's findings are limited to the group's self-evaluation. Nonetheless, the MQI observational tool helped the group to thoroughly scrutinize several aspects of mathematics instruction, a vital part of improving teaching practices.

Since the lesson study aims to improve the education process, other areas, such as student academic performance, can be explored to measure the effectiveness of a lesson study. Lesson study that is well-executed and useful for teachers boosts their expertise in both subject and pedagogy, which eventually benefits students' educational experiences.

Lastly, the results may not be generalizable to different contexts, given that this research examined a single video lesson involving a specific mathematics subject, instructor, and learners. Nonetheless, suggestions for improvement in the research lesson particularly in addressing its weaknesses could be considered in the next cycle of the research lesson revision and implementation.

Implications

Despite some limitations, the findings of this research have implications for educators and the educational system.

First, committing minimal mathematical errors and imprecision illustrate the teacher's level of content mastery and how well the teacher can facilitate learning among the students.

Communicating with students more clearly and precisely helps students learn because it leaves little opportunity for ambiguity (Bieda & Voogt, 2019). Teachers' skills in appropriately using mathematical terms to explain and discuss the content with the students are important in promoting high-quality instruction.

Second, in terms of the research lesson on operations on fractions, teachers must explain to students the connections that may exist between the various fraction representations and aid them in recognizing those links when they are present (Blanton, 2008). Students benefit in more ways than one by being exposed to pattern generalization problems beyond just better understanding of the functional methods. It prepares pupils for subsequent success in algebra by strengthening their ability to reason mathematically (Demonty et al., 2018).

Third, the quality of student work may influence how teachers draw ideas. Getting learners' one- or two-word responses is not helpful for teachers in advancing the subject. It is suggested that teachers explicitly teach students how to provide high-quality explanations. Students must create and articulate concepts and knowledge instead of just describing the processes they used to solve equations or problems. Explicit teaching entails modeling high-quality explanations, emphasizing key aspects of the explanation, and giving challenging practice exercises.

Finally, since the MQI domains highlight the importance of lesson preparation, we hope that teachers seek opportunities to work together to improve mathematics education in their classrooms and beyond.

References

Africa, M., Borboran, A. M., Guilleno, M.-A., Mendiola, M., Portento, K. M., Rodriguez, R., Torrefranca, J., Viernes, R. M., & Conde, R. (2020). A Lesson study on using the Concrete-Pictorial-Abstract (CPA): Approach in addressing misconceptions in learning fractions. *SABTON: Multidisciplinary Research, 2*(1), 1–8.

Adkins, A. (2017). Instructional strategies used by effective mathematics teachers in Adventist Elementary Schools in Florida: An MQI analysis (10689066) [Doctoral dissertation] https://www.proquest.com/dissertations-theses/instructional-strategies-used-effective/docview/1984977548/se-2

Bieda, K. N., & Voogt, K. J. (2019). Mathematical errors when teaching: A case of secondary mathematics prospective teachers' early field experiences. *Proceedings of the 22nd Annual Conference on Research in Undergraduate Education*. https://par.nsf.gov/biblio/10101301-mathematical-errors-when-teaching-case-secondary-mathematics-prospective-teachers-early-field-experiences

Blanton, M. L. (2008). *Algebra and the elementary classroom: Transforming thinking, transforming practice*. Portsmouth, NH: Heinemann.

Bolliger, D. U., Supanakorn, S., & Boggs, C. (2010). Impact of podcasting on student motivation in the online learning environment. *Computers & Education*, *55*(2), 714–722. https://doi.org/10.1016/j.compedu.2010.03.004

Center for Education Policy Research at Harvard University. (n.d.). *Mathematical quality of instruction domains.* Retrieved December 15, 2022, from https://cepr.harvard.edu/mqi

Copur-Gencturk, Y., & Doleck, T. (2021). Linking teachers' solution strategies to their performance on fraction word problems. *Teaching and Teacher Education*, *101*, 103314. https://doi.org/10.1016/j.tate.2021.103314

Creswell, J. W., & Plano Clark, V. L. (2017). *Designing and conducting mixed methods research*. SAGE Publications.

Dangle, Y. R. P., & Sumaoang, J. D. (2020). The implementation of modular distance learning in the Philippine secondary public schools. *Proceedings of the 3rd International Conference on Advanced Research in Teaching and Education*. https://www.doi.org/10.33422/3rd.icate.2020.11.132

Darling-Hammond, L. (1997). *Doing what matters most: Investing in quality teaching*. National Commission on Teaching and America's Future.

Department of Education. (2019). *DepEd PISA 2018 National Report of the Philippines*. https://drive.google.com/file/d/1n7Sb_JND8B-nBuGLigiwM4T8gC0S6kJw/view?usp=embed_facebook

Demonty, I., Vlassis, J., & Fagnant, A. (2018). Algebraic thinking, pattern activities and knowledge for teaching at the transition between primary and secondary school. *Educational Studies in Mathematics, 99*, 1–19. https://doi.org/10.1007/s10649-018-9820-9

Dudley, P. (2011). *Lesson Study*: *A Handbook*. Retrieved December 9, 2022, from https://lessonstudy.co.uk/wp-content/uploads/2012/03/Lesson_Study_Handbook_-_011011-1.pdf

Elipane, L. E. (2011). Incorporating lesson study in pre-service mathematics teacher education. *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education, 2*, 305-312.

Engzell, P., Frey, A., & Verhagen, M. D. (2021). Learning loss due to school closures during the COVID-19 pandemic. *Proceedings of the National Academy of Sciences*, *118*(17). https://doi.org/10.1073/pnas.2022376118

Flores, Inan, F. A., Han, S., & Koontz, E. (2019). Comparison of algorithmic and multiple-representation integrated instruction for teaching fractions, decimals, and percent. *Investigations in Mathematics Learning, 11*(4), 231–244. https://doi.org/10.1080/19477503.2018.1461050

Fyfe, E. R. & Nathan, M. J. (2019). Making "concreteness fading" more concrete as a theory of instruction for promoting transfer. *Educational Review*, *71*(4), 403–422. https://doi.org/10.1080/00131911.2018.1424116

Gholami, H., Ayub, A. F. M,, & Yunus, A. S. M. (2021). Improving mathematics lecturers' content knowledge through lesson study. *Mathematics Teaching Research Journal, 13*(2), 85-106.

Gholami, H. (2022). Improving the performance of mathematics teachers through preparing a research lesson on the topic of maximum and minimum values of a trigonometric function. *Mathematics Teaching Research Journal*, *14*(2), 26-43.

Hansen, N., Jordan, N. C., & Rodrigues, J. (2017). Identifying learning difficulties with fractions: A longitudinal study of student growth from third through sixth grade. *Contemporary Educational Psychology*, *50*, 45–59. https://doi.org/10.1016/j.cedpsych.2015.11.002

Hiebert, J., Gallimore, R., & Stigler, J. W. (2002). A knowledge base for the teaching profession: What would it look like and how can we get one? *Educational Researcher*, *31*(5), 3–15. https://doi.org/10.3102/0013189X031005003

Hill, H. C., Rowan B., Ball D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of Instruction: An Exploratory Study. *Cognition and Instruction, 26*(4), 430–511. https://doi.org/10.1080/07370000802177235

Hill, H. C., Kapitula, L., & Umland, K. (2011). A validity argument approach to evaluating teacher value-added scores. *American Educational Research Journal*, *48*(3), 794–831.

Hill, H. C., Umland, K., Litke, E., & Kapitula, L. R. (2012). Teacher quality and quality teaching: Examining the relationship of a teacher assessment to practice. *American Journal of Education*, *118*(4), 489–519.

Hoffer, T. B., Venkataraman, L., Hedberg, E. C., Shagle, S., Banfield, M., & Flawn, T. (2007). *Final report on the national survey of algebra teachers for the National Math Panel.* National Opinion Research Center. https://www.researchgate.net/profile/Eric-Hedberg/publication/228513219_Final_report_on_the_national_survey_of_algebra_teachers_for_the_National_Math_Panel/links/55364e710cf20ea35f11ca6d/Final-report-on-the-national-survey-of-algebra-teachers-for-the-National-Math-Panel.pdf

Hollister, B., Nair, P., Hill-Lindsay, S., & Chukoskie, L. (2022). Engagement in online learning: Student attitudes and behavior during COVID-19. *Frontiers in Education, 7*. https://www.frontiersin.org/articles/10.3389/feduc.2022.851019

Jansen, S., Knippels, M.-C. P. J., & van Joolingen, W. R. (2021). Lesson study as a research approach: A case study. *International Journal for Lesson & Learning Studies, 10*(3), 286–301. https://doi.org/10.1108/ijlls-12-2020-0098

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. National Academy Press. https://doi.org/10.17226/9822

Lortie-Forgues, H., Tian, J., & Siegler, R. S. (2015). Why is learning fraction and decimal arithmetic so difficult? *Developmental Review, 38*, 201–221. https://doi.org/10.1016/j.dr.2015.07.008

Lubienski, S. T., & Lubienski, C. (2006). School sector and academic achievement: A multilevel analysis of NAEP mathematics data. *American Educational Research Journal, 43*(4), 651–698.

Lynch, J. (2015). The effects of implementing the Common Core State Standards on student achievement [master's thesis, Northwest Missouri State University].

Marchitello, M., & Wilhelm, M. (2014). The cognitive science behind the Common Core. Washington, D.C.: Center for American Progress.

Meniano, K. R., & Tan, R. G. (2022). Challenges in studying mathematics using self-learning module during COVID-19 Pandemic. *American Journal of Educational Research*, *10*(4), 182–187. https://doi.org/10.12691/education-10-4-4

Mercer, N., & Howe, C. (2012). Explaining the dialogic processes of teaching and learning: The value and potential of sociocultural theory. *Learning, Culture and Social Interaction, 1*(1), 12–21. https://doi.org/10.1016/j.lcsi.2012.03.001

Murata, A. (2011). Introduction: Conceptual overview of lesson study. In Hart, L., Alston, A., Murata, A. (Eds.) *Lesson study research and practice in mathematics education* (pp. 1–12). Springer, Dordrecht. https://doi.org/10.1007/978-90-481-9941-9_1

Namkung, J., & Fuchs, L. (2019). Remediating difficulty with fractions for Students with mathematics learning difficulties. *Learning Disabilities: A Multidisciplinary Journal, 24*(2), 36–48. https://doi.org/10.18666/LDMJ-2019-V24-I2-9902

Organisation for Economic Co-Operation and Development. (2020). *PISA 2018 results (Volume VI): Are students ready to thrive in an interconnected world?* OECD Publishing. https://doi.org/10.1787/d5f68679-en

Punzalan, E. M. D., & Buenaflor, R. F. (2017). Modeling the relationship between the basic computational and problem-solving skills of Fourth-Year high school students in the Division of Zambales, Philippines. *International Journal of Education and Research, 5*(1), 173-184.

Riccomini, P. J., Smith, G. W., Hughes, E. M., & Fries, K. M. (2015). The language of mathematics: The importance of teaching and learning mathematical vocabulary. *Reading & Writing Quarterly, 31*(3), 235-252. http://dx.doi.org.ezp.waldenulibrary.org/10.1080/10573569.2015.1030995

Rock, T. C., & Wilson, C. (2005). Improving teaching through lesson study. *Teacher Education Quarterly, 32*(1), 77–92. http://www.jstor.org/stable/23478690

Ronda, E., & Adler, J. (2019). Subject matter knowledge and the quality of mathematics made available to learn: Some hypotheses. *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education, 3.* Pretoria, South Africa.

Saskiyah, S. A., & Putri, R. I. I. (2020). Mathematical representation on fraction operation for seventh-grade students using collaborative learning. *Journal of Physics: Conference Series*, *1663*. https://iopscience.iop.org/article/10.1088/1742-6596/1663/1/012001

Smith, O. S. (2017). *The influence of language on the teaching and learning of mathematics* [Doctoral dissertation, Walden University]. https://scholarworks.waldenu.edu/dissertations/4682/

Solomon, Y., & Black, L. (Eds.) (2008). *Talking to learn and learning to talk in the mathematics classroom*. SAGE Publications Ltd. https://doi.org/10.4135/9781446279526.n5

Stepanek, J., Appel, G., Leong, M., Mangan, M. T., & Mitchell, M. (2006). *Leading Lesson Study: A Practical Guide for Teachers and Facilitators*. Corwin Press.

Stoian, C. E., Fărcașiu, M. A., Dragomir, G.-M., Gherheș, V. (2022). Transition from Online to Face-to-Face Education after COVID-19: The Benefits of Online Education from Students' Perspective. *Sustainability, 14*(19), 12812. https://doi.org/10.3390/su141912812

Sulaimi, S. A. (2022). Teacher-student rapport in emergency remote teaching: Autoethnography. *Studies in Technology Enhanced Learning*. https://doi.org/10.21428/8c225f6e.8851078e

Taufik, M., & Effendy, M. B. (2022). "The absence of student-teacher emotional closeness", rural and urban english teachers' voices during Covid-19 pandemic. *Journal of English Language Teaching and Linguistics, 7*(1), 43–61. https://doi.org/10.21462/jeltl.v7i1.707

Thompson, L. (2007). The effects improving student discourse has on learning mathematics. *Action Research Projects*. 23. https://digitalcommons.unl.edu/mathmidactionresearch/23

United Nations. (2020). *Policy brief: Education during COVID-19 and beyond*. https://www.un.org/development/desa/dspd/wp-content/uploads/sites/22/2020/08/sg_policy_brief_covid-19_and_education_august_2020.pdf

Problematic and Supportive Aspects of Indirect Proof in Afghan Undergraduate Students' Proofs of the Irrationality of $\sqrt{3}$ and $\sqrt{5/8}$

Ahmad Khalid Mowahed[1], Jawed Ahmad Mayar[1]

[1] Mathematics Department, School of Education, Balkh University, Balkh, Afghanistan

khalid.mowahed@gmail.com & jawed.amayar@gmail.com

Abstract: *In this study, we aimed to find problematic and supportive issues in Afghan undergraduate students' proofs of the irrationality of* $\sqrt{3}$ *and* $\sqrt{5/8}$ *while using the indirect proof method. Collecting and analyzing produced proofs of 30 sophomore and 48 senior undergraduate students on the irrationality of* $\sqrt{3}$ *and* $\sqrt{5/8}$*, respectively, revealed that the majority were not able to extend indirect proof beyond showing the irrationality of* $\sqrt{2}$ *due to not being able to apply the supportive 'multiples of an integer' notion in their reasoning process of the irrationality of* $\sqrt{3}$*, and this notion was not supportive in the case of showing irrationality of* $\sqrt{5/8}$*, and finally we propose the method of finding nonzero integral solutions to the resulted Diophantine equation and using the rational roots theorem to prove that* $\sqrt{p/q}$ *is irrational.*

INTRODUCTION

Theoretical Background: We trace back how the concept of irrational numbers is understood in high school considering the works of prominent researchers to form a theoretical background for the current study and determine how this mental image of irrational numbers may affect students' understanding of irrationality at the university level. The important notions that may influence students' understanding of rational and irrational numbers are repeating and nonrepeating decimal representations. We start with a decimal representation of some rational numbers at the school level, usually starting beyond grade 7 in Afghanistan. For example, there are rational numbers that their decimal representation is finite or ends such as $\frac{1}{2} = 0.5, \frac{2}{5} = 0.4, \frac{1}{8} = 0.0125$ and so on.

There are some rational numbers that their decimal representations does not end, rather it periodically repeats infinitely, such as $\frac{1}{3} = 0.333\cdots, \frac{1}{6} = 0.16666\cdots, \frac{5}{11} = 0.454545\cdots$, $\frac{3079}{9900} = 0.31282828\cdots$ and so on. Conversely, if we let $x = 0.333\cdots$, compute $10x = 3.333\cdots$, and subtract both sides of the former equality from the latter, we obtain $9x = 3$ and solving for x yields $x = \frac{1}{3}$. This observation indicates that there are two types of rational numbers: rational numbers with finite/ending decimal representations and those with repeating non-ending decimal representations. At some point in school, the teacher puts the results of this observation into the following proposition without providing a general proof: every rational $\frac{a}{b}$, provided $b \neq 0$ and a are integers, can be written as a finite/ending decimal representation or repeating infinite/non-ending decimal representation; conversely, every finite or periodically infinite decimal expansion is equal to a rational number.

There is another interesting observation while dealing with $\frac{1}{3} = 0.333\cdots$; that is, multiplying both sides of $\frac{1}{3} = 0.333\cdots$ gives $1 = 0.999\cdots$, which is counterintuitive for some students. This counterintuitive observation is actually correct, but why? Suppose we have the decimal $0.99999\cdots$ and is a rational number. Suppose $x = 0.99999\cdots$ and multiplying both sides by 10, gives $10x = 9.99999\cdots$and subtraction gives $9x = 9$ or $x = 1$. As a final result of these few above observations, we knew two types of rational numbers $\frac{a}{b}$ in its reduced form: those $\frac{a}{b}$ in which the only factors of b are 2 and 5 (which can be written as finite or infinite decimal representation such as in $\frac{1}{2} = 0.5 = 0.499999\cdots$) and the rest of rational numbers that can only be written as infinite decimal representations such as in $\frac{1}{3} = 0.33333\cdots$ (Niven, 1961).

According to Tall (2013) theory of 'three worlds,' all of us learn mathematics by building on our previous knowledge and experiences that may be supportive and facilitate generalization of ideas in new contexts or may be problematic and impede our understanding of mathematical ideas. A met-before is "a mental structure we have now as a result of experiences we have met before." Supportive met-befores give the pleasure to learn more, whereas problematic met-befores cause frustration and force us to quit learning. For instance, the met-before "take away leaves less" is true while working in the set of positive integers and finite sets, but this met-before is no longer true in the set of negative integers and infinite sets. Students who are capable of making sense of new ideas will develop confidence in responding to problematic met-befores, address difficulties in learning mathematical concepts, and build increasingly knowledge structures latter on, whereas those students who are unable to deal with new situations becoming increasingly sophisticated may feel detached from learning new ideas, which in turn may lead to mathematical anxiety, causing them to quit learning new ideas just because they do not cope with problematic met-befores. In addition, a student's success in learning mathematical concepts at one stage may be impeded by problematic met-before in future learning. According to the goal-oriented theory of Skemp (1979), this problematic met-before deviates students from the goal of understanding a mathematical concept to the goal of learning procedures to solve routine problems. Learning procedures for solving standard problems is not a bad thing itself because it can enable a student to be successful initially, but overemphasizing it may not facilitate learning in new contexts.

Students' mental images of repeating and non-repeating decimals learned in school may be supportive in high school, but problematic met-before at university level mathematics, which in turn may pose difficulties for students in the process of transitioning from rational to irrational numbers. Tall (2013), for instance, points out that though students encounter irrational numbers such as $\sqrt{2}$, π and e in school being on the number line filling the gaps between rational numbers, they do not know what really these irrational numbers are. However, infinite decimals are conceived as one of its finite approximations, in which the precision is improved by adding one more digit after the decimal point. This perception is characterized by Kidron and Vinner (1983) as "the dynamic perception of the infinite decimal" (p. 306). On the other hand, Durand-Guerrier and Tanguay (2016) contend that students entering university have limited knowledge of real numbers and their conception of a number is dependent on how it is written; that is, they sometimes say π is not a "true" number but rather seen as a "sign," and only those numbers have a real number status when they are written in the decimal representation. Furthermore, Fischbein et al. (1995) claimed that little attention has been paid to clarifying and discussing irrational numbers in school mathematics, as well as not conveying mathematics as a coherent and structurally organized body of knowledge to students. One reason for this shortcoming may be the complex epistemological process of extending rational numbers to real numbers. For instance, in modern algebra, extending integers to rational numbers is easier to understand than extending rational numbers to real numbers.

Method of Indirect Proof: The method of indirect proof is prevalent in all mathematics. Indirect proof includes both methods of proof by contraposition and proof by contradiction. Proof by contraposition, put in simple terms, constitutes a process in which we first negate the hypothesis H and conclusion C and then prove the implication $\neg C \rightarrow \neg H$ for a mathematical claim or theorem in the form of $H \rightarrow C$ whenever it seems difficult or impossible to be proved by direct proof; the main reason behind allowing us to carry out such a process is that in the language of propositional logic we have $H \rightarrow C \equiv \neg C \rightarrow \neg H$ (Koshy, 2007). Proof by contradiction includes a process in which we first negate the conclusion C of a theorem $H \rightarrow C$ while imposing and maintaining specific conditions in the process and reaching a contradiction at the end, and this contradiction at the end of careful reasoning process proves that the theorem is correct; again the laws of propositional logic allows us to carry out this process because we have $H \rightarrow C \equiv [H \wedge (\neg C)] \rightarrow F$. Sometimes, both are needed in an indirect proof, and sometimes one is sufficient, depending on the nature of the theorem or mathematical claim.

Although the processes of indirect proof, proof by contraposition, and proof by contradiction may seem explicit according to the laws given above, it has been stated in the literature that indirect proof is difficult and problematic for students to carry out in proving mathematical claims (Harel & Sowder, 1998; Leron, 1985; Robert & Schwarzenberger, 1991; & Tall, 1979). The literature reports several reasons why students have difficulty with indirect proof. For example, difficulty in negating statements (Wu Yu et al., 2003) influenced by natural daily language in the form of being opposite of the given statement such as increasing function is seen as the negation of decreasing function, considering several possibilities for the given statement such as negating 'f is strictly decreasing' as 'g increasing, g is constant, or g is decreasing but not strictly decreasing' (Antonini, 2001), emotional barriers (Reid & Dobbin, 1998), and disliking indirect proof (Antonini & Mariotti, 2008) are some to name. On the other hand, the logical representation of proof by contradiction, that is, $H \to C \equiv [H \land (\neg C)] \to F$, is problematic mode of proving because one must simultaneously stick to the falsehood of the conclusion to be true and argue why it is false under a tremendous state of stress (Tall, 1979).

To reduce such difficulties and stress, we may consider Leron's (1985) suggestion of preferring to reorganize proof by contradiction such that it initially involves some steps of construction in the process of proving and postpone the contradiction to the end. For example, in proving the infinity of prime numbers, one may initially consider constructing some new primes from old ones in few construction steps for $N = p_1 p_2 \cdots p_k + 1$ as $2 \times 3 + 1, 2 \times 3 \times 5 + 1, 2 \times 3 \times 5 \times 7 + 1, \cdots$ and then deducing that if there were a finite number of primes, then we would always get another prime or a composite involving a new prime, leading to a contradiction. An alternative technique that has been employed is the use of generic proof (a generic example or particular case that is neither trivial nor very complicated) in which the irrationality of $\sqrt{2}$ is proved by first proving that if one squares a rational number where the denominator and numerator are factorized into different primes, then its square has an even number for each prime factor in the numerator or denominator. Thus, one can deduce that $\sqrt{2}$ cannot be rational because its square is 2, which only has an odd number of occurrences of prime 2. In similar approach, Tall (1979) presented the generic proof that $\sqrt{5/8}$ is irrational because 5/8 contains an odd number of 5s and also an odd number of 2s in its prime factorization, while the square $\frac{5}{8} = \frac{r^2}{s^2} = \frac{(p_1 \cdots p_i)^2}{(q_1 \cdots q_j)^2}$ has even number of each prime factor in the numerator or denominator, leading to a contradiction. He contends that this generic proof has explanatory power and can be generalized easily, whereas proof by contradiction is both problematic and not easily generalized by students. Malek and Movshovitz-Hadar (2011) also conducted an extensive study with first-year Israeli students taking linear algebra to investigate the use of generic proofs and found that their students benefitted from generic proof by "transformable cognitive structures related to proof and proving" (Male and Movshovitz-Hadar, 2009, p. 71). Looking at the benefits of using generic proofs as a starting point to proof by contradiction, the author also recalls his class experiences, showing that generalizing proof by contradiction is problematic for students beyond showing the irrationality of the square root of square free positive integers and fractions. The author asks: Why is this generic proof still unable to penetrate into textbooks so that students have the opportunity to easily access the generalizability of proof by contradiction, at least in the case of irrationality of the square root of non-zero rational numbers?

The alternative to proof by contradiction in the form of $H \to C \equiv [H \land (\neg C)] \to F$ may seem promising, but turning all contradiction proofs to more direct proofs of the nature explained earlier may not always be helpful because proof by contradiction in its form of

$H \rightarrow C \equiv [H \wedge (\neg C)] \rightarrow F$ is central in all mathematics, whether hypothesis H is explicitly or implicitly stated. Dreyfus and Eisenberg (1986) contended that mathematicians prefer the proof of irrationality of $\sqrt{2}$ by contradiction proof over alternative proofs because proof by contradiction of the form $H \rightarrow C \equiv [H \wedge (\neg C)] \rightarrow F$ does not require complex prerequisite knowledge and is very appropriate for teaching.

Although the literature focuses on many aspects of students' difficulties in indirect proof, there is little discussion on how students can extend indirect proof beyond proving the irrationality of $\sqrt{2}$ in the context of Afghanistan. Thus, the purpose of this study is to address this problem; it specifically focuses on addressing the following research questions:

- Which methods of proof do Afghan undergraduate students employ to prove the irrationality of $\sqrt{3}$ and $\sqrt{5/8}$?
- How supportive or problematic is the indirect proof for Afghan undergraduate students when employing it beyond proving the irrationality of $\sqrt{2}$?

METHOD

The main objective of this research was to determine whether indirect proof can become supportive or problematic for undergraduate Afghan students once they know what indirect proof is and learn to carry out the method considering its logical form of $H \rightarrow C \equiv [H \wedge (\neg C)] \rightarrow F$ on the specific example showing $\sqrt{2}$ is irrational, and then apply the same method to prove the irrationality of $\sqrt{3}$ and $\sqrt{5/8}$. The research method used in this research is qualitative in nature, seeking to find supportive and problematic aspects of employing indirect proof in proving the irrationality of $\sqrt{3}$ and $\sqrt{5/8}$. The main data source for the current research was students' written works about proving the irrationality of $\sqrt{3}$ and $\sqrt{5/8}$. The sample consisted of 30 second-year and 48 fourth-year undergraduate Afghan students who took a modern algebra1 course with the researcher as their fulfillment of their degree in mathematics at the Mathematics Department, School of Education, Balkh University, Afghanistan. Second-year students were asked to prove the irrationality of $\sqrt{3}$ and fourth-year students were asked to prove the irrationality of $\sqrt{5/8}$. As embedded in the curriculum of modern algebra, the course must discuss the methods of proofs in mathematics. The main methods of proofs to be discussed in this course are direct and indirect proofs as well as the method of proof by mathematical induction. This is required because the later concepts discussed in modern algebra require these proof methods to deduce further results in group theory and ring theory; thus the foundations must be laid so that students can easily understand the process of deducing results from axioms related to concepts in group and ring theory.

RESULTS

The main objective of this research was to determine whether indirect proof can become supportive or problematic for undergraduate Afghan students once they know what indirect proof is and learn to carry out the method considering its logical form of $H \rightarrow C \equiv [H \wedge (\neg C)] \rightarrow F$ on the specific example showing $\sqrt{2}$ is irrational and then apply the same method to prove the irrationality of $\sqrt{3}$ and $\sqrt{5/8}$. These questions were asked to see whether students could apply reasoning similar to that discussed for the case of $\sqrt{2}$ to prove that $\sqrt{3}$ and $\sqrt{5/8}$ are irrational.

The written works of 30 students proving the irrationality of $\sqrt{3}$ were analyzed. Eleven out of the 30 students did not provide any answers. The remaining 19 students produced proofs that differed; however, their various proofs can be categorized into two categories. The first category included 17 students who could only provide the following reasoning, were unable to go beyond this stage, and simply concluded that $\sqrt{3}$ is irrational. Their reasoning includes the following steps:

Suppose $\sqrt{3}$ is rational. Then we will have $\sqrt{3} = \frac{m}{n}, n \neq 0$ and $\frac{m}{n}$ is in irreducible terms. Hence, $3 = \frac{m^2}{n^2}$ or $m^2 = 3n^2$, so $\sqrt{3}$ is irrational.

This category students' produced partial proofs show that they did not had any problem with negating the conclusion C of the theorem in the way mentioned in the literature, however, they were not able to continue the process by showing that if the square of the integer a is a multiple of 3, then a must be a multiple of 3. This could be done easily if these students followed the process of reasoning applied in the case of irrationality of $\sqrt{2}$, using contrapositive proof to show that if the square of an integer is even, then the integer must be even. When asked why some of these students could not go beyond this stage, students mentioned that the main barrier that hindered their reasoning to go beyond this stage was that they did not know what to call the right-hand side of $m^2 = 3n^2$ and forgot the definition of 'multiples of an integer,' because everywhere the word 'even' is used in the case of proving the irrationality of $\sqrt{2}$. The author believes that the main problem in the case of proving the irrationality of $\sqrt{n}$, n a positive and square free integer, is knowing how to deal with the equation $a^n = cb^n$ for a fixed integer c to find nonzero solutions $a, b \in \boldsymbol{Z}$ or knowing how to use the definition of multiple of an integer and contraposition; which of these two methods can facilitate students' reasoning process can be determined in a comparative study, which is not the concern of this study.

On the other hand, only two students could provide more evidence beyond the above reasoning to show that $\sqrt{3}$ is irrational, but none of them proved that if the square of integer a is a multiple of 3, then a must be a multiple of 3. They simply took this for granted from the proof of the irrationality of $\sqrt{2}$.Their reasoning is as follows:

Suppose $\sqrt{3}$ is rational. Then we will have $\sqrt{3} = \frac{m}{n}, n \neq 0$ and $\frac{m}{n}$ is in irreducible terms. Hence, $3 = \frac{m^2}{n^2}$ or $m^2 = 3n^2$. Now $m = 3r$. Substituting it in $m^2 = 3n^2$ gives $m^2 = (3r)^2 = 9r^2 = 3n^2$ or $n^2 = 3r^2$ which implies that$n = 3s$ which contradicts the irreducibility of $\frac{m}{n}$. Thus, $\sqrt{3}$ is irrational.

In addition, the written works of 48 students proving the irrationality of $\sqrt{5/8}$ were analyzed. None of them provided complete proof of the irrationality of$\sqrt{5/8}$, but their written works can be categorized as follows. 34 students were able to negate the conclusion of the theorem and follow up to find the equality $8a^2 = 5b^2$ and simply concluded that it was irrational. Their work can be summarized as follows:

Suppose $\sqrt{5/8}$ is rational, that is $\sqrt{5/8} = \frac{a}{b}$ where the numerator and denominator have no common factor. Then squaring both sides gives $\frac{5}{8} = \frac{a^2}{b^2}$, from which we have $8a^2 = 5b^2$.

Although all of them presented their partial proofs, as above, they had different types of reasoning. For instance, some of these students wrote that no non-zero integer values for a and b satisfy $8a^2 = 5b^2$ by giving specific values.

Solution: suppose $\sqrt{5/8}$ is rational. Then we have:

$$\sqrt{5/8} = \frac{a}{b}$$

$$\left(\sqrt{5/8}\right)^2 = \left(\frac{a}{b}\right)^2$$

$$\frac{5}{8} = \frac{a^2}{b^2}$$

$$5b^2 = 8a^2$$

$$b^2 = \frac{8}{5}a^2$$

$$b = \pm\sqrt{8/5}\,a$$

Figure 1. A student's written work on showing the irrationality of $\sqrt{5/8}$

(translated from Dari to English)

And in the second category of 14 students, 2 students computed $\sqrt{5/8} \approx 0.790569415\cdots$ and said that since this decimal number neither ends nor repeats at some decimal place, then $\sqrt{5/8}$ must be irrational, and the rest just simplified $\sqrt{5/8}$ algebraically using laws of radicals.

$$\sqrt{5/8} = 0.790569415\cdots$$

If it is in this state, it is rational. Then

$$\sqrt{5/8} = \frac{a}{b}$$

$$\frac{5}{8} = \frac{a^2}{b^2}$$

$$8a^2 - 5b^2 = 0$$

Since Δ is not defined in the field of real numbers, then $\sqrt{5/8}$ is not rational. Therefore, $\sqrt{5/8}$ is irrational.

Figure 2. A student's written work on showing the irrationality of $\sqrt{5/8}$

(translated from Dari to English)

This indicates that using the definition of multiples of an integer, as in the case of $\sqrt{2}$ or $\sqrt{3}$, could not help students decide on $8a^2 = 5b^2$ to show that $\sqrt{5/8}$ is irrational. They recalled a met-before that a fraction either ends or repeats in decimal representation, but none of them showed whether it ends or not.

DISCUSSION

This study aimed to investigate the supportive and problematic aspects of indirect proof and any other met-before concept that may influence students ability to determine the irrationality of $\sqrt{3}$ and $\sqrt{5/8}$ using their knowledge of the irrationality of $\sqrt{2}$ as proved by the indirect proof method. The results showed that the majority of participants were not able to extend the method of indirect proof beyond proving the irrationality of $\sqrt{2}$. Their written works showed that while some students were able to provide partial proof of the irrationality of $\sqrt{3}$ and $\sqrt{5/8}$, most of them were unable to reach a contradiction and prove the irrationality using the method of indirect proof. The main problem reported by students not being able to apply indirect proof was that they forgot or were unable to connect the definition of multiples of 3 in deducing by contraposition that if a^2 is a multiple of 3, then so is a.

The study also found an important problematic aspect that hindered students application of indirect proof, as they did not know how to go beyond step $a^2 = 3b^2$ in proving the irrationality of $\sqrt{3}$ and reaching a contradiction. One thing that can be drawn from this is that the most problematic step in the process of indirect proof for proving the irrationality of $\sqrt{n}$, for positive nonzero and square free integer n, is to reach a contradiction from $a^2 = nb^2$. Indeed, once the student understands well the supportive notions of multiples of an integer and contrapositive proof, he/she will have little difficulty showing that the square root of any positive non-zero and square free integer is irrational.

However, latter in this study it was found that this supportive notions of 'multiples of an integer and proof by contrapositive' in the case of$\sqrt{n}$, n is positive nonzero and square free integer, may not help students in other situations such as in proving the irrationality of $\sqrt{5/8}$ or generally in $\sqrt{p/q}$, where p/q is irreducible. The study found that most of the senior undergraduate students who attempted to prove the irrationality of $\sqrt{5/8}$, using multiples of an integer, were unable to reach a contradiction using indirect proof. This was because of the complexity of the equation and the presence of different multiples on both sides of the equality.

The study also found two other interesting issues in the students' written works on the proof of the irrationality of$\sqrt{5/8}$. The first issue was related to the infinite and non-repetitive decimal expansion of $\sqrt{5/8}$, which some students approximated using a calculator and then deduced that is irrational because the decimal expansion is non-repetitive, this finding is consistent with Patel and Varma's (2018) study (as cited in Rafiepour et al., 2022). However, it is unclear how the students knew that the decimal expansion did not end, and whether they used any other previous knowledge such as the following theorem: "every rational a/b, provided $b \neq 0$ and a are integers, can be written as a finite(ending) decimal representation or repeating infinite (non-ending) decimal representation; conversely, every finite or periodically infinite decimal expansion is equal to a rational number." Whatever previous knowledge is used, there is still a challenge in calculating all decimal digits of an irrational number because hand calculators cannot compute it beyond 11$^{\text{th}}$ digit, as found in Rafiepour et al.'s (2022) study.

The second issue is related to the theoretical analysis of finding nonzero solutions to Diophantine equations as an alternative to indirect proof, which may help undergraduate students continue until the last step is reached to show the irrationality of $\sqrt{p/q}$ where p/q is irreducible. There is one legitimate question to be asked: what would cost both the teacher and student to search for nonzero integer solutions of $8a^2 = 5b^2$ or generally of $qa^2 = pb^2$ using university mathematics? The empirical search for answering this question requires working with students and many Diophantine equations, but working with the polynomial of two variables $f(x, y) = 8x^2 - 5y^2 \in \boldsymbol{Z}[x, y]$ and finding its nonzero integral solutions will tell us whether $8a^2 = 5b^2$ has nonzero integral solutions, which in turn will determines whether $\sqrt{5/8}$ can be written as a rational p/q. Thus, theoretical analysis of finding nonzero solutions to such equations as an alternative to indirect proof may shed light and help undergraduate students continue until the last step is reached to show the irrationality of $\sqrt{p/q}$ where the rational p/q is irreducible. There are also advanced methods to show that $f(x,y) = 8x^2 - 5y^2 \in \boldsymbol{Z}[x,y]$ has no nonzero integer solution such as solving the Diophantine equations in number theory.

Finally, this study proposed a simple and clever method to determine the irrationality of such numbers. That is, if we set $x = \sqrt{2}$, then squaring both sides of the equality yields $x^2 = 2$ or $f(x) = x^2 - 2$. Now, we know that $f(x) = x^2 - 2 \in \boldsymbol{Q}[x]$ is a polynomial in the ring of polynomials over the field of rational numbers $\boldsymbol{Q}$. Suppose $f(x)$ is reducible in $\boldsymbol{Q}[x]$. Then $x = x_0$ must be rational; otherwise, it is irrational. The irreducibility of the polynomial can be verified using the rational root theorem. The rational root theorem states that:

> Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be a polynomial in $\mathbf{Z}[x]$, where $a_0 \neq 0$ and $a_n \neq 0$. Then every rational root of $f(x)$ has the form $\frac{c}{d}$, where $c|a_0$ and $d|a_n$(for proof, see Nicholson, 2012, p. 211).

We now apply the rational root theorem. Because $a_0 = -2$ and its factors are $\pm1, \pm2$; $a_n = 1$ and its factors are ±1. Then the values for $\frac{c}{d}$ are: $\pm1, \pm2$. Now, we can see that $f(\pm1) \neq 0$ and $f(\pm2) \neq 0$. Thus, according to the rational root theorem, $f(x) = x^2 - 2$ is irreducible over $\boldsymbol{Q}$. Therefore, $\sqrt{2}$ is irrational. The same method can be applied to $\sqrt{3}$, $\sqrt{5/8}$ and any other number in the form $x_0 = \sqrt[n]{t}$, where $n \in \boldsymbol{Z}^+_{>1}$ and $t > 0 \in \boldsymbol{Q}$, to prove its irrationality.

CONCLUSION

In conclusion, the study provided valuable insights into the supportive and problematic aspects of indirect proof and related concepts that may impact students' ability to determine the irrationality of square roots. This study highlighted the challenges faced by undergraduate students in extending the method of indirect proof beyond proving the irrationality of $\sqrt{2}$. The study identified the main problem faced by students in applying indirect proof as forgetting or being unable to connect the definition of multiples of an integer in deducing by contraposition. The study also highlighted the complexity of the equation and the presence of different multiples on both sides of equality as a significant challenge in proving the irrationality of $\sqrt{5/8}$ and other similar square roots. The study proposed alternative methods to indirect proof, such as the theoretical analysis of finding nonzero solutions to Diophantine equations; and a simple and clever way to determine the irrationality of numbers in the form of $\sqrt[n]{t}$. This study acknowledges the limitations of the sample size and calls for further research to investigate the generalizability of the findings to other populations and to explore other instructional strategies suggested here for teaching indirect proof.

Based on the findings of this study, there are several pedagogical and methodological approaches that educators can adopt to effectively teach the topic of proving the irrationality of square roots using indirect proof. First, educators should focus on developing students' understanding of the supportive concepts of multiples of an integer and proof by contrapositive, which are crucial for applying indirect proof. Second, educators should adopt a problem-solving approach to teach this topic. This involves presenting students with challenging problems and guiding them through a problem-solving process, emphasizing the importance of logical reasoning, critical thinking, and perseverance. Finally, educators should consider using alternative methods for indirect proof, such as the rational root theorem. These methods can provide students with alternative strategies to approach complex problems and enhance their understanding of underlying concepts.

REFERENCES

[1] Antonini, S. (2001). Negation in mathematics: Obstacles emerging from an exploratory study. In M. van den Heuvel-Panhuizen (Eds.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 49–56). Utrecht: Freudenthal Institute.

[2] Antonini, S., & Mariotti, M. A. (2008). Indirect proof: What is specific to this way of proving. *ZDM – The International Journal on Mathematics Education*, *40*,401–412.

[3] Dreyfus, T., & Eisenberg, T. (1986). On the aesthetics of mathematical thoughts. *For the Learning of Mathematics, 6* (1), 2–10.

[4] Durand-Guerrier, V., & Tanguay, D. (2016). Introduire les réels à la transition lycée-université: Questions épistémologiques et horizon didactique. In E. Nardi, C.Winsløw, & T. Hausberger (Eds.), *Proceedings of the first conference of the international network for didactic research in university mathematics* (INDRUM) (pp. 177–179). Montpellier, France: University of Montpellier and INDRUM.

[5] Fischbein, E., Jehiam, R., & Cohen, C. (1995). The concept of irrational number in high school students and prospective teachers. *Educational Studies in Mathematics*, *29*, 29–44.

[6] Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research on collegiate mathematics education III* (pp. 234–283). Providence, RI: American Mathematical Society.

[7] Ireland, K., & Rosen, M. (1990). *A Classical Introduction to Modern Number Theory*. NY: Springer.

[8] Kidron, I., & Vinner, S. (1983). Rational numbers and decimals at the senior high level: Density and comparison. In R. Hershkowitz (Eds.), *Proceedings of the 7th conference of the International Group for the Psychology of mathematics education* (pp. 301–306). Rehovot: Weizmann.

[9] Koshy, Thomas. (2007). *Elementary Number Theory with Applications.* MA: Elsevier.

[10] Leron, U. (1985). A direct approach to indirect proofs. *Educational Studies in Mathematics*, *16*(3), 321–325.

[11] Malek, A., & Movshovitz-Hadar, N. (2009). The art of constructing a transparent proof. *Proceedings of the ICMI Study* (Vol. 2, pp. 70–75). Taipei: National Taiwan Normal University.

[12] Malek, A., & Movshovitz-Hadar, N. (2011). The effect of using transparent pseudo-proofs in linear algebra. *Research in Mathematics Education, 13* (1), 33–57.

[13] Niven, I. M. (1961). *Numbers: Rational and Irrational*. USA: Mathematical Association of America.

[14] Rafiepur, A., Abdolahpour, K., & Farsani, D. (2022). Developing conceptual understanding of irrational numbers based on technology through activity system. *Mathematics Teaching Research Journal*, *14*(1), 114-137.

[15] Reid, D., & Dobbin, J. (1998). Why is proof by contradiction difficult? In A. Olivier & K. Newstead(Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 41–48). Stellenbosch, South Africa: University of Stellenbbosch.

[16] Robert, A., & Schwarzenberger, R. (1991). Research in teaching and learning mathematics at an advanced level. In D. Tall (Eds.), *Advanced mathematical thinking* (pp. 127–139). Dordrecht, Netherlands: Kluwer.

[17] Tall, D. (1979). Cognitive aspects of proof, with special reference to the irrationality of √2. In D. Tall (Eds.), *Proceedings of the third international conference for the psychology of mathematics education* (pp. 206–207). Warwick, UK: Warwick University Mathematics Education Research Centre.

[18] Tall, D. (2013). *How humans learn to think mathematically. Exploring the three worlds of mathematics*. New York: Cambridge University Press.

[19] Wu Yu, J., Lin, F., & Lee, Y. (2003). Students' understanding of proof by contradiction. In N. A. Pateman, B. J. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 2003 joint meeting of PME and PMENA* (vol 4, pp. 443–449). Hawai'i : University of Hawai'i.

Mathematical Modelling, Integrated STEM Education and Quality of Education for Linear Algebra and Vector Calculus Courses

Sandra Barragán[1], Orlando Aya[2], Camilo Soto [1]

[1]Universidad de Bogotá Jorge Tadeo Lozano, Colombia, [2]Universidad Pedagógica Nacional, Colombia

Sandra.barragan@utadeo.edu.co, oaya@pedagogica.edu.co, camiloe.sotos@utadeo.edu.co

Abstract: Articulated pedagogical and didactic strategies allow improving academic quality and strengthen individual and institutional cognitive processes such as student retention. Mathematical modelling, articulated with complementary actions, contributes to teaching and learning processes within situations that go beyond the classroom since they are permeated by social, political, and cultural environments that globally define pedagogical work. Therefore, the objective of this article was to visualize the progress of the implementation of the mathematical modelling of integrated Science-Technology-Engineering-Mathematics (STEM) education and classroom teaching to improve the academic quality of Linear Algebra and Vector Calculus courses at Universidad de Bogotá Jorge Tadeo Lozano (Utadeo). Through a mixed methodology composed of three phases – one qualitative, one implementation, and one quantitative– the theory and practice of academic work were guided, and the strategy, classroom implementation, and work materials were designed. The results obtained, in terms of appreciation of the work in the classroom and publications, evidence the articulation of various efforts and institutional, disciplinary, personal, technological, and theoretical resources in the context of changing realities.

INTRODUCTION

The etymology of the word "quality" defines it as an abstract concept not applicable to something substantial, it is an intrinsic attribute of something that has certain characteristics. For Daros (2014), this attribute can be based on an entity, be it a person, thing, or event, which will be given a qualification related to its functional mode of being. The Aristotelian conception of quality defines it through the category of substance, stating that it cannot be spoken of without a subject, which performs or undergoes an action, whether of quality or quantity. Plato in the Protagoras, the Republic, and the Laws defines education as an essential instrument for the development of society and the State, implying that education must be of quality. Ballén (2010) highlighted that the starting points for the construction of the virtuous man are education and the academic world.

Critical, sensitive, and liberal citizenships are born from a social state of law that guarantees education, among others, as a fundamental right through public investment and State management of curricular content. At a specific moment, the actor that fosters and guides the content of educational policy through reforms and modifications is usually the leading government (Vázquez, 2015). These reforms seek to meet the educational needs of society as a priority, joining efforts that serve to expand coverage in remote regions of a nation. Likewise, its purpose is to expand intellectual capital, focusing exhaustively on the quality of education to achieve an advanced technological, industrial, environmental, and agrarian society (Cristia & Pulido, 2020; Castellar & Uribe, 2004).

The contribution of education to the development of these fields has been fundamental and it is, without a doubt, unquestionable if said growth is broken down into the educational levels of each population at the national, regional, departmental, or municipal level. In relation to this, Loaiza & Hincapié (2016) indicated that economic theory has been adjusting to the so-called economics of education, through public policies that converge in its development and promotion. In this sense, Martín (2018) asserted that business practice is based on quality standards related to market expectations. That is why the quality of education, in this area, is understood as the measurement of results. Likewise, the concept of quality comes from the neoliberal conviction that the primary task of educational institutions is not predominantly focused on education, but rather on training people to perform business functions of immediate application (Martín, 2018).

When referring to the quality of education, the concept of evaluation cannot be ignored, because there must be a measure that assigns non-generalized or homogenized standards, in specific social contexts, to the way in which said measure will be extended as a reference to the community. However, it is important to note that these standards present a bias when the actors involved seek to meet their expectations. As Cano (1998) states it,

> the concept of quality is subjective above all since each consumer or user has a different idea of what it means. However, everyone agrees that we talk about quality when we see all our expectations met, whether it is by a product or a service (p.31).

Similarly, Mosquera (2018) emphasized that the concept of quality of education has roots in the business environment, in which quality must be evaluated according to production. However, Mosquera indicates that when comparing the manufacture of products with the educational training of people, and by linking both activities as if one were the consequence of the other, the true goal and purpose of education is obscured.

The Colombian Ministry of National Education (MNE) defined the concept of quality in higher education academic programs

> as the set of articulated, interdependent, dynamic attributes, built as references by the academic community, which respond to social, cultural, and environmental demands. Such attributes allow internal and external evaluations of the institutions to promote their transformation and the permanent development of their training, academic, teaching, scientific, cultural and extension work (2019, p.4).

In Colombia, quality of education is measured with the results of standardized government tests, among others, (Melo-Becerra et al., 2021) and with student dropout rates (Ministerio de Educación Nacional, 2020). The standardized tests are called Saber tests and are applied to all students in the country under the same conditions –instrument, time, and number of questions– in accordance with their educational level (Instituto Colombiano para la Evaluación de la Educación, 2021). Particularly, at the end of basic education (primary, secondary, and mid-secondary) the Saber 11 tests, whose theoretical range of results goes from 0 to 500, are applied. The tests have a math component whose maximum score to achieve is 100 points (Instituto Colombiano para el Fomento y Evaluación de la Educación Superior, 2020). On the other hand, at the end of higher education, the Saber Pro tests, whose results range is between 0 and 300, are applied. Table 1 summarizes the scores of the two tests applied annually at the national level for the period 2018 to 2021 (Instituto Colombiano para la Evaluación de la Educación, 2022) and it shows that the overall and the mathematics results are close to the middle of the maximum possible score, which is evidence of the academic capital that students have at the beginning of higher education.

Table 1: Average results in Saber 11 for the math component, the overall score and number of students per application.

	Math Score	**Global Score**	**Number of students**
2021 2	49.66	246.02	532,979
2021 1	61.23	301.82	15.527
2020 2	51.17	249.01	504,871
2020 1	58.62	289.43	15,434
2019 2	50.52	245.34	546,211
2019 1	56.90	281.56	21,082
2018 2	50.19	250.77	549,933
2018 1	63.79	319.06	12,527

Additionally, the dropout rate by cohort is monitored to measure the long-term dropout of students who entered in the same period (Ministerio de Educación Nacional, 2009; Ministerio de Educación Nacional, 2015). This indicator for the 2016-2020 period was 9.83% and 8.38% for public and private universities, respectively (Asociación Colombiana de Universidades, 2022).

Consequently, both the initial academic conditions and the dropout rates are a challenge for any Higher Education Institution (HEI). In this sense, there are many parties interested in providing possible academic, administrative, and analytical alternatives to face the challenges (Guzmán et al, 2021; Rincón & Espitia, 2020; Radinger et al., 2018; Ministerio de Educación Nacional, 2009, 2015). The MNE (2015) recommended improving academic quality by strengthening training spaces, academic activities, and teaching resources in accordance with the specific characteristics of the institutional population (Ministerio de Educación Nacional, 2015).

Additionally, Decree 1330 of 2019 of the MNE modified the conceptual bases for the training processes of higher-level students, moving from a system based on capabilities to one based on outcomes, under the premise that learning outcomes are unequivocal statements of what a student is expected to know and demonstrate by the time they complete their academic program (Ministerio de Educación Nacional, 2019, p. 4). To act accordingly, HEIs must work on: 1) A broad conceptual understanding of learning outcomes; 2) The appropriation and organizational incorporation of the writing of the learning outcomes; 3) The design and the adaptation of syllabus; 4) Formulation and proposal of curricular and extracurricular academic activities according to the learning levels; and 5) provision of assessments throughout the training process from admission to completion with the aim of verifying progress (Ministerio de Educación Nacional, 2020).

The institutional research in Utadeo that arose in this context has as a broad research question: how can the quality of education be improved through modelling the dynamics associated with the indicators of student dropout and the evaluation focused on learning outcomes? This article is part of the research, and its objective is to visualize the implementation progress of the mathematical modelling of integrated STEM education and classroom teaching to improve academic quality at Utadeo mathematics courses.

To fulfil the objective of this article, following this introduction, the reference framework of the integrated STEM education approach, the mathematical modelling, and its relationship with academic quality are described. Next, the implemented methodology is presented, sectioned into three non-consecutive phases: qualitative phase, implementation phase, and quantitative phase considering the courses of Linear Algebra and Vector Calculus of the Academic Area of Basic Sciences and Modelling of Utadeo. Subsequently, the results are detailed in terms of educational practice and quantitative outcomes. Finally, both the discussion and the conclusions obtained are presented.

REFERENTIAL FRAMEWORK

Conceptual Definition of Integrated STEM Education

Considering simultaneously the academic entry conditions, the dropout rates, and the learning outcomes in the context of each HEI, perspectives such as those provided by the STEM approach take a leading role (Clavijo, 2021; Moreno, 2019; García et al., 2019; Vásquez, 2014).

For Moore et al. (2014) it is essential to understand STEM education as an effort to integrate science, technology, engineering, and mathematics through a holistic and complementary approach aimed at finding connections between the topics of these disciplines and the real world. Moore et al. (2014) indicated that it is essential to understand STEM education as an effort to combine and integrate science, technology, engineering, and mathematics through a holistic and complementary approach. STEM education seeks connections between the topics of these disciplines and the real world.

The emergence of the STEM education approach, stated Moreno (2019), is underpinned by the need to work on projects related to topics in these fields of science because they potentiate industrial and economic development at different levels. It should be noted that the National Administrative Department of Statistics (DANE, in Spanish), in its technical bulletin of the training for work module in Colombia for 2021, showed a percentage reduction of 0.9 points compared to the bulletin of the year 2019, and 2.8% in comparison to 2013 in the participation of people in training areas related to engineering (Departamento Administrativo Nacional de Estadística, 2021). These data show a decrease in the interest shown by students in these training programs. The relationship that exists today between academia, industry, and the government makes it a priority to modify the curricular contents to direct them to these issues and thus promote them through the STEM approach. With this, educators would receive the necessary tools to develop and promote these skills in their students, considering the processes of modernization and industrialization, and the global boom in software development (Vásquez, 2014). The digital age which fosters STEM education justifies the implementation of computational tools, both primary and complementary, to improve the quality of education. With them, knowledge is directed to students through diverse forms and spaces that facilitate the handling of information and formulate it in a generalized and concise manner, relate it to current problems and issues and, in turn, allow it to be studied in different contexts making use of numerous platforms and tools which enable progressive digital literacy and the reception of the information contained herein (López et al., 2020).

Additionally, the STEM approach makes it possible to relate social and educational problems, such as student dropout at all levels, and understand them in various geographic, economic, and political contexts (Barragán & Guzmán, 2022). For Santamaría et al. (2020), the empowerment of women derived from the expansion of education makes STEM a driver in the development of societies and the participation of women for this purpose. An educated woman, therefore, will have greater influence in decision-making, allowing her to be part of the construction of an advanced society.

Conceptual Definition of Mathematical Modelling

Modelling is an essential practice for the study of systems and is based on the description of the properties that characterize them (Blanchard et al., 1998). It is defined as an activity that cuts across all disciplines and has different types, such as verbal, physical, mental, and mathematical models (Landriscina, 2013). The latter are worked analytically, that is, due to their complexity they require a computerized system to carry out simulations applying numerical methods (Sayama, 2015). On the other hand, modelling is an essential tool in different social, political, economic, and environmental fields, whose dynamics need to be studied (Landriscina, 2013).

The principle of mathematical modelling is to relate the macroscopic magnitudes, both extensive and intensive, typical of each system (Urquía & Martín, 2013). Such relations are mostly based on mechanistic laws or fundamental laws of nature that can become simple or complex models, depending on the different characteristics of the system such as its capacity for self-organization, its emergent properties, and the level of detail that it provides (Velten, 2009).

Oliva (2019) presents five definitions of modelling that are summarized in Figure 1. To this article, it is appropriate to understand modelling as a teaching strategy.

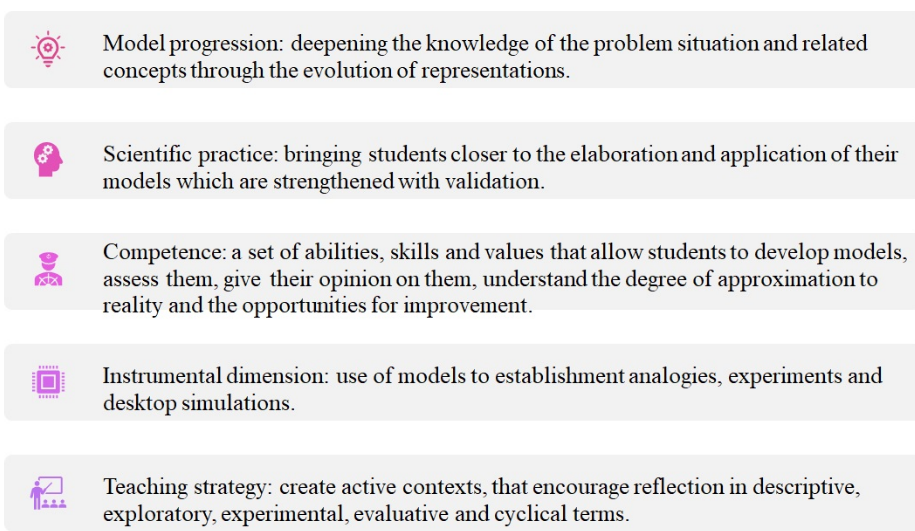


Figure 1: Modelling meanings

Source: Prepared by the authors according to Oliva (2019).

Thus understood, mathematical modelling aims to provide the student with various tools that work to integrate different areas of knowledge. Modelling will promote the understanding of concepts and their relationship with their actual environment, will improve reading and writing skills, enhance their understanding and problem-solving (Purnomo et al., 2022; Villalobos, 2021), and will provide students with skills such as multidisciplinary work and research through constructivist methods applied inside classrooms (Oliva, 2019; Zaldívar et al., 2017). To formulate and apply modelling, Vergel et al. (2020) recommended considering the progressive development of mathematical thinking, based on the training and contribution of teachers to the variational thinking of students, understanding it as a mechanism of dynamic reasoning, whose purpose is to produce relationships between variables that covary through cognitive actions that distinguish between variable and constant magnitudes. Likewise, Cabezas & Mendoza (2016) identified it in the epistemological study of mathematical notions originating in infinitesimal calculus.

Quality of Education, Academic Quality, and Learning Outcomes in Linear Algebra and Vector Calculus Courses at Utadeo

In Colombia, social gaps, inequity, and inequality are the result of factors such as corruption, poor management and unequal distribution of land, lack of State presence, the concentration of wealth, centralization of resources, low institutional capacity, and deplorable levels of education in areas and regions that are far from capital cities (Cristia & Pulido, 2020). Sánchez (2017) stated that the adverse effects of the armed conflict –forced displacement, the extension of unproductive land, and drug trafficking, among others– make Colombia one of the most unequal countries in the Western Hemisphere.

Cataño (1984) highlighted that social inequality is synonymous with inequality of educational opportunities. He emphasizes the differences between classes within our society and the differences in opportunities in each of them, which has a significant impact on the country. However, in compliance with the Millennium Development Goals (MDGs), Colombia has been on a path of transformation for decades towards educational efficiency and quality, safeguarded by the Political Constitution of Colombia of 1991, which is a guarantor of this inalienable right and grants it the necessary benefits for its sustainability and promotion (Congreso de la República, 1991).

Currently, one of the factors of social inequality –understood as inequality of educational opportunities– that was added to the other factors known until now is the COVID-19 pandemic. Herrera (2020) stated that mandatory isolation intensified social disparity for vulnerable and low-income communities, since it made it impossible for them to access education, among others, due to their lack of the technological and communications infrastructure necessary to continue with the training process of girls, boys, youth, and adults at all educational levels.

In accordance with the context described above, Utadeo has provided a wide set of tools for its students to face the difficulties of Colombian society and thus deliver them a quality education that follows the guidelines of the MNE (Universidad de Bogotá Jorge Tadeo Lozano, 2022, 2011). Based on this idea and in the specific case of the Linear Algebra and Vector Calculus courses –which are part of the Basic Sciences and Modelling Area– STEM education has been implemented within the engineering and economic-administrative sciences programs. Given the recognized educational benefits of this approach, diverse technological tools were applied in the courses to facilitate communication with and among students and allow them to access the contents and material designed.

These courses were structured in accordance with the experience acquired in various contexts, including the COVID-19 pandemic. Compared to previous dynamics, this design involved the recognition and exploration of the technological tools that can be used to expand the coverage of the subject's content, diversify the ways in which knowledge is approached, strengthen the cognitive skills of the students regarding the program's content and foster the ability to address everyday problems with an emphasis on modelling and simulation.

METHODOLOGY

The methodology designed to achieve the objective of this article included three non-consecutive phases in line with the STEM educational practice, at the higher level, circumscribed to the Colombian context.

Qualitative face has two stages: Initially, a collection of literature, experiences, and good practices was made to build a frame of reference and a state-of-the-art. Subsequently, the documentation of the progress of the process was conducted along with the selection of an approach within the range of the STEM movement (Tsupros et al., 2008; Oliva, 2019; Margot & Kettler, 2019) so that it responded to the observed characteristics of the population to which it was addressed and to the institutional expectations (Arango et al., 2019; Universidad de Bogotá Jorge Tadeo Lozano, 2011, 2022).

Implementation phase: in this phase, the way in which the classroom teaching was conducted in the Linear Algebra and Vector Calculus courses of the Academic Area of Basic Sciences and Modelling of Utadeo was recorded. These classroom teaching practices were compiled both in a general way and with examples. This work was carried out on several fronts: 1) Updating the teaching skills of the teachers in charge since they require specialized knowledge in multiple aspects (e.g. pedagogy, didactics, and evaluation) (Kelley & Knowles, 2016; Margot & Kettler, 2019; Gros & Cano, 2021); 2) Designing activities, materials and resources in accordance with the cognitive and conceptual domains of each course (Mullis et al., 2009) (teaching practice); and 3) Documenting and recording the teaching and learning process in articles, book chapters, and books (teaching research).

Quantitative phase: in this phase, the pedagogical proposal with the STEM approach, the evolution of the approval rates of each course, the evaluation made by the students of the teaching practice (Universidad de Bogotá Jorge Tadeo Lozano, 2011), as well as the academic production of teachers according to the terms of the Ministry of Science and Technology, were followed and monitored. According to the ministry, the products of the Groups and Researchers are the results that they obtain in the processes of research, technological development, or innovation, and respond to work plans, lines of research, and projects (Ministerio de Ciencia Tecnología e Innovación, 2020, p. 56). Such products must be part of one of the following categories: Generation of New Knowledge, Technological Development and Innovation, Social Appropriation of Knowledge, and Public Dissemination of Science and Training of Human Resources in Science, Technology, and Innovation (Ministerio de Ciencia Tecnología e Innovación, 2020). It should be noted that university professors oversee different activities according to their missionary functions within their HEIs: direct and indirect teaching or research and social projection (Cosenz, 2022; Congreso de Colombia, 1992).

RESULTS

This section presents the results, initially focused on the implementation phase and then on the quantitative phase of the described methodology. It is very important to highlight that the incorporation of the STEM approach and mathematical modelling as its expression in the courses of Linear Algebra and Vector Calculus of the Academic Area of Basic Sciences and Modelling of Utadeo has been conducted for four and two years, respectively, and it is still an ongoing process.

Educational Practice: Implementation of the STEM Approach and Mathematical Modelling in Linear Algebra and Vector Calculus Courses at Utadeo

The implementation of modelling in the courses of Vector Calculus and Linear Algebra –which are part of the training axis of different engineering majors and the economic-administrative sciences of Utadeo in the foundation cycle– is conducted from different perspectives, theoretical approaches, and epistemological conceptions. Two of them will be presented below so that, without exhausting the topic, they will offer an overview of the strategies that support not only the modelling process itself but also its articulation with the general and specific objectives included in the academic syllabus as well as with the purposes of professional training that are based on the Pedagogical Model of the university (Universidad de Bogotá Jorge Tadeo Lozano, 2011). The approaches presented are generically called articulate and sedimentary to describe what happens when one of them is performed. A comparative analysis between some learning styles and performance in mathematics can be expanded in sources such as Ma (2014).

The first approach to be described is articulation. It has been named like this because it considers a general model for a specific phenomenon that articulates the concepts and processes of the discipline and the objects of study. To illustrate the how and why of this approach, the Vector Calculus course is described, and an example of the Cobb-Douglas function is used. This function is explained from different perspectives, over approximately five weeks, using the support of a sequence of seven videos available on the YouTube channel (Aya, 2021), associated with the playlist Vector Calculus - Calculus in Several Variables.

The microeconomic object is articulated for two fundamental reasons:

(1) Contribute to the economic education of citizens (basic understanding of a production model centred on capital and labour) as a priority issue in the treatment of socially relevant or sensitive concepts.

(2) Articulate the specific object of a discipline or context considered as extra-mathematical with various mathematical objects presented in the course (Dolores & García, 2017).

The first reason is understood as part of the comprehensive training of future professionals as citizens and that will allow them to understand and account for some of the social, political, economic, and cultural phenomena of their environment and of global society in general. It will also allow them to reflect on the social and cultural role of STEM education, which has been widely questioned in academic settings. Specifically, they will be able to understand that productivity depends, among other factors, on labour, and capital and that one of them is not enough. That is, they will focus on the tensions between capital (injection of money-resources) and the available labour (staff, job qualification, wages).

The second reason is based on exploring concepts typical of the economic-administrative sciences and taking advantage of the property of the Cobb-Douglas function as a function of the logarithmic-linear type in several variables (particularly ln-linear. Considering P, the total production; L, the amount of labour; and K, the amount of capital invested, thus, lnP is a linear function of lnK and lnL. In a first video, the initial development of the model and the theoretical assumptions that support it. In this process, what is proposed for the modelling phases is addressed through eight steps implemented in the format designed for the courses (which will be mentioned later). Also, technological elements associated with the use of software, such as Excel, are incorporated to perform the least squares adjustment to the proposed model. Within this phase, an analytical process is conducted on what it means to make an adjustment to a model and an analysis from basic elements of the theory of error such as absolute, relative, and percentage error that allow a reflection on the importance of considering aspects of numerical validation of the models and how it broadens the conceptual tools beyond those provided by calculus and which could well require elements of statistics or other analytical disciplines such as those offered by specialized software.
The calculation of the images serves to analyse the behaviour of the Cobb-Douglas function, when varying both capital and labour by a factor C, and shows that productivity is also modified by a factor C to break the intuitive process. If each of the variables is doubled, the function will quadruple since it is in terms of two independent variables. Contrary to what one might think, even in these courses the idea of linearity and direct proportionality is rooted.

The level curves are approached to show that, for fixed productivity, the proportionality between labour, and capital is inversely proportional and of the type

$$K \propto \frac{1}{L^{\alpha}} \qquad (1)$$

taking advantage to deal with the technical conditions of the model. To support the visualization and conceptualization of the mathematical object, the use of the GeoGebra software is relevant in this approach. The partial derivatives are treated from the marginal productivities with respect to labour and capital and are interpreted considering the construction of the model and the assumptions that support it.

The conceptual development is directed towards the chain rule and an application is presented in the Cobb-Douglas model to determine the variation of productivity under the assumption that both capital and productivity are functions that depend on time. To conclude with the mathematical objects associated with the study of the function, two examples of Lagrange multipliers are discussed. The first refers to a single restriction on the function associated with capital (linkage of the system); the second deals with the situation in which productivity is constrained. In both cases, some implicit properties of the gradient and the relationship between the gradient of the objective function and that of the constraint are emphasized. It is highlighted that these treatments are not made on specific cases or values, since a generalized treatment of both the microeconomic model (Cobb-Douglas function) and the mathematical model (Lagrange multipliers method) is intended. Figure 2 presents a synthetic diagram of the developed process.

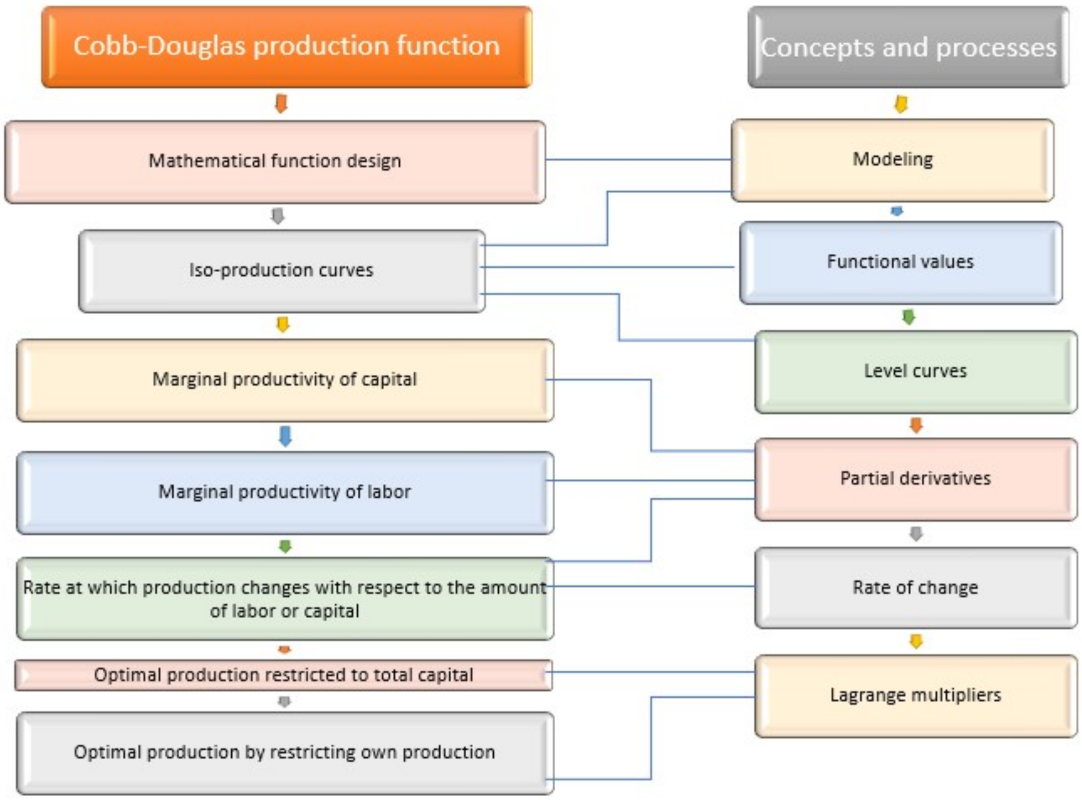


Figure 2: The microeconomic object (model to be analysed), the mathematical concepts and processes.

The second approach illustrated here is called sedimentary and is exemplified for Linear Algebra. It is understood as an action with which a series of processes and concepts are built that serve as the basis or substratum to approach a model in the long term, since its methodology, analysis and interpretation require the stabilization of certain cognitive actions –not only characteristic of human thought– such as generalizing, associating, interpreting, or visualizing. Likewise, they require a procedure that allows the analytical maturation of processes and associated concepts for the construction of tools that enable an informed approach to a situation to be modelled (specific to the area). These processes occur, according to López (2011), after a maturation that, in the case of class work, is associated with aspects such as regularity in working with processes and concepts. Of course, while working with them, an identity is formed and mediated by relationships that are established in various fields such as conceptual, procedural, or analytical. Some of these relationships may go through a state of inactive dormancy that may or may not be beneficial.

In case there is inactivity and this is of little benefit, the actions proposed in the book for Linear Algebra are activated, such as exercises solved in detail (step by step), guided exercises (with steps and key indications), proposed exercises (to be solved by the student, without guidance), use of computer and technological resources (GeoGebra®, Wolfram Alpha®, online calculators), presentations in PowerPoint format (prepared by the teachers on of the subjects for each of the class sections that are in accordance with the syllabus), support videos on two YouTube channels designed by teachers in the area for the specific purposes of the courses, and spaces such as the collaborative forum between students in the Moodle classrooms of the Tadeísta Virtual Learning Environment (Avata).

The continuous reactivation phase of the processes and concepts occurs at the end of each cycle or at the end of the academic term through two specific actions. The first is a partial exam that synthesizes the central work axes related to the mathematical objects addressed and the second focuses on the modelling process.

In the sedimentation phase, the basic elements are built with two structures: the one given by the mathematical objects and the one generated by the modelling process. For the first structure, vectors, transformations in general, linear transformations and their properties, the matrix representation of a transformation, the characteristic polynomial of a transformation, the characteristic equation, the eigenvalues, and the eigenvectors associated with a transformation are addressed. In parallel, for the case of transformations of R^n in R^n, the theorem that formulates the equivalence between: the non-zero determinant of the associated matrix, the existence of the inverse matrix, unique solution for the non-homogeneous system, trivial solution for the homogeneous system, the equivalence of the matrix by rows or columns to the identity matrix, the echelon form of the matrix has n pivots, linear independence of the rows and columns of the matrix, nullity of the transformation zero, the rank of the transformation n, and eigenvalues not null has been established.

For the second structure, it has been emphasized that the approach to both problem situations and modelling itself is intended to be systematized in classes, activities specifically aimed at modelling, as well as in videos of examples, in which the methodology of eight basic steps for the modelling process is developed. For example, in the Linear Algebra course, the work is carried out with a situation that aims to conclude a population model for a species of birds in the orientation of the Leslie Model (Grossman, 2012). For this, a discussion is made about the central hypotheses of the model (closed system, does not admit migrations, unlimited resources in the system for sustenance and survival, the predominant sex for the preservation of the species are females and particularly those that are in fertile age, the reproductive rates of young females per adult female and the survival rate from one period to the next are constant, the population of the next period depends fundamentally on the population in the current period as in the Malthusian Model). In this phase the conditions of the model are highlighted, and its limitations are evidenced to determine its scope to understand what a model is, and the context of its validity is an essential competence for the development of scientific thought, science, and technology in themselves.

Subsequently, in a second step, the unknowns, variables, and parameters are defined, as well as the technical conditions. That is, symbolize the populations of young females and adult females in each period, and in the previous period with non-negative integer values, survival rate for young and adults from one period to the next and number of new young females for each adult female (assumed constant). Next, the matrix A, associated with the transformation of the population, is constructed emphasizing the composition of the population of females, and specifically, the fertile ones, since they are the ones that essentially determine the development of the populations of the species. The composition of the population. The composition of the population constitutes a fundamental aspect of the biological sciences and of the general behaviour of dynamic population systems. With these conceptual elements, the model is established based on the assumptions associated with exponential growth, that is, in

$$P_{n+1} = A^n P_n \quad (2)$$

The third step corresponds to a population calculation process from a given population P_0 and it is suggested to carry out some initial calculations to strengthen algorithmic processes and with this give way to projections and simulations using the software. To this end, a template was designed that, based on a population of young and adult females and the corresponding matrix A, can be modified within the technical conditions established by the model, allows the population to be measured for several successive periods and yield some results of interest to the model such as the population of total females T_n, and the ratios of the population of young females to adult females in each period

$$\frac{P_{j,n}}{P_{a,n}} \quad (3)$$

and the ratio of the total population of females in one period to the total population of females in the previous period

$$\frac{T_{n+1}}{T_n} \quad (4)$$

In the fourth step, two calculation processes are involved, one related to the eigenvalues associated to A and the other to the corresponding eigenvectors. This process can be done manually since, due to the order of the matrix, it leads to a quadratic equation. However, the incorporation of the use of the designed resource in Wolfram Alpha or Symbolab® is suggested. It is included here as it makes it possible to analyse and discuss the mathematical power of the software. Once these values have been evaluated, an analysis of the long-term evolution of the system and its terminal behaviour with the trend of the logistic growth curves is established. The fundamental elements were summarized in an explanatory video that is suggested at the end of the interaction work (Barragán, 2021). The analysis can be extended to a slightly more complex Leslie Model in which different factors are considered, such as the types of populations (infertile youth, fertile youth, fertile adults, non-fertile adults, with their respective survival and reproduction rates) or extended to new phenomena such as predator-prey models or competition models. Any expansion of the analysis has effects on the dimension of the associated matrix and on the use of software, among others, for both the calculation process and the analysis of results (Figure 3).

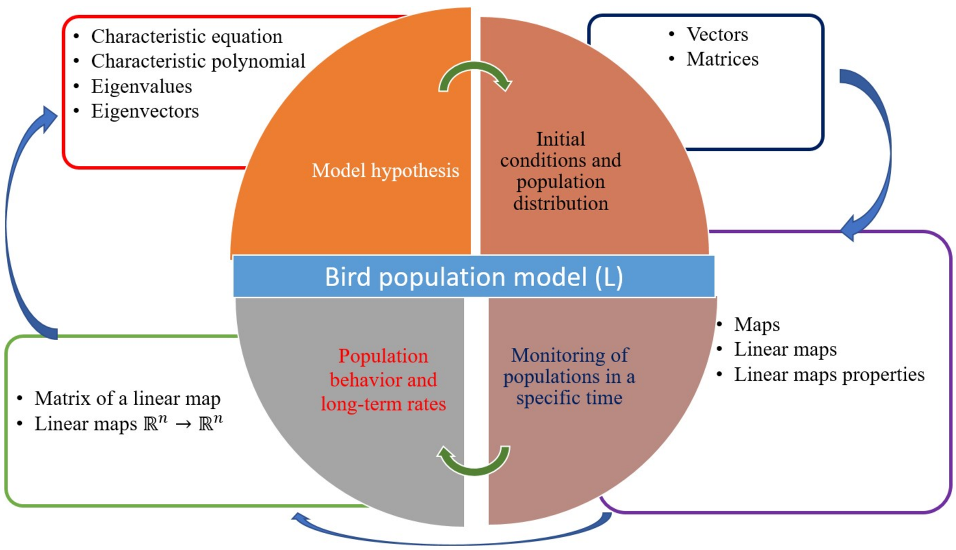


Figure 3: Synthesis of the treatment of the bird population model.

It is essential to mention that the evaluation, within its holistic process, refers to:

(1) The level of academic achievement of students, measured with assessments applied by teachers. A total of sixteen evaluations are applied per academic term with different exam formats, such as quizzes on platforms like Socrative and Kahoot and modelling workshops. Mock tests are used to familiarize students with the settings and types of questions. The questions used in the midterms are standardized so that the level of demand for all groups is the same.

(2) The developed academic processes, measured with instruments designed by administrative instances, filled out by the students about the teachers. Throughout the academic period, Utadeo applies two evaluations, one called Early Evaluation to timely identify aspects to improve in teaching, methodology and evaluation. The second is applied at the end of the course and allows students to evaluate teachers through a structured survey. The results are systematized, and the corresponding administrative instances deliver the results to the teachers, who can access them at any time.

(3) The academic facilities and the support provided by the professors as well as the monitors linked to the teaching practice, through structured formats. The teacher-coordinator jointly evaluates the performance of the teachers in their practice and that of the outstanding students who support them through academic monitoring.

(4) The syllabus and resources used in each of the courses. During and at the end of each academic period, teachers jointly review the achievements made, contrasting them with what is proposed in the syllabus and the relevance of the resources used, fundamentally seeking not only to improve them but also to adjust them to the changing needs and the dynamic conditions of teaching and learning processes.

As mentioned at the beginning of this section, the diversity of strategies or approaches transcends what is summarized here. Such diversity is reflected in the curricular designs, the training and information objectives, the learning approaches, the materials, and the available resources, as well as in the capacity for innovation and reflection on each constitutive element of the pedagogical proposal, which is enriched with the incorporation of articulated elements of the STEM approach to teaching mathematics at university.

Classroom Teaching: Virtual Educational Spaces and Evaluation

As previously indicated, various tools were applied during the development of Vector Calculus and Linear Algebra courses, such as 1) Videos that are specifically designed for the courses, 2) training and peer evaluations conducted on Socrative and Kahoot platforms, 3) modelling situations designed with specific objectives, 4) evaluations implemented in the Moodle platform in the virtual learning environment. The tools were articulated in each specific class session in accordance with the syllabus or following the consensus reached among professors.

Next, the way in which a sequence of classes was developed in the virtual spaces in which the proposal was implemented is described. The case of a Linear Algebra class is presented as an example. Initially, several virtual educational spaces were set in the Avata (a virtual learning environment). The first was the Class material space (Figure 4), in which the general presentations were published four days in advance to guide the students in the topics that would be addressed each week (students had four weekly hours of class accompanied by the professor). The idea was to encourage the students to follow the principle of autonomy.

Figure 4: General scheme of the structure of the course in the Avata and a prototype of a presentation prepared for each of the class sessions.
Note: The course was delivered in Spanish.

A second space, named Support materials (Figure 5), included a base of 51 problems from various application contexts (both interdisciplinary and mathematics). Some were solved as class examples, others in workshops, and others individually by students.

21. Determine el polinomio $p(x) = ax^2 + bx + c$ que satisfaga las condiciones $p(1) = f(1), p'(1) = f'(1)$ y $p''(1) = f''(1)$, donde $f(x) = e^{-5x}Cos(\pi x)$.

12. Un dietista desea planear cierta dieta con base en tres tipos de alimentos. Una onza de cada tipo de alimento contiene unidades de proteínas, carbohidratos e hierro como aparecen en la siguiente tabla.

Unidades de	Alimento I	Alimento II	Alimento III
Proteínas	2	3	2
Carbohidratos	3	2	1.5
Hierro	0.5	2	1

Si la dieta debe proporcionar exactamente 25 unidades de proteínas, 24 unidades de grasa y 21 unidades de carbohidratos, ¿cuántas onzas de cada tipo de alimento deben utilizarse?

Figure 5: Interdisciplinary or mathematical context problems prototypes.
Note: The course was delivered in Spanish.

The space named Computer resources (Figure 6) is the inventory of videos that are used for three purposes: 1) Exemplifying specific aspects of the mathematical objects addressed, 2) supporting students and help them deepen in some respects, and 3) supporting the modelling process. The activity that uses the inventory is conducted following the scheme referred to above. The activity is used to collect the written production of the students and help them develop skills and competencies for modelling in different contexts and problem situations.

Figure 6: Elements of the proposal: Modelling process and resources available on YouTube.
Note: The course was delivered in Spanish.

Figure 7 shows an open question exercise for a linear transformation that produces changes in the polygon appearance and the answer elaborated by a student. The answer to the question required calculations, graphs, interpretation, and the student's understanding of the context.

Primera situación problema:

Se tiene una transformación

$$T:\mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{pmatrix}x\\y\end{pmatrix} \to \begin{pmatrix}-2 & 0\\0 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

y se tiene un polígono cuyos vértices son $A(3,2), B(4,0), C(-2,3), D(-4,0), E(-1,-1)$.

Determine e interprete los eigenvalores y los eigenvectores asociados a la transformación. Realice la transformación aplicada al polígono e interprete los resultados obtenidos.

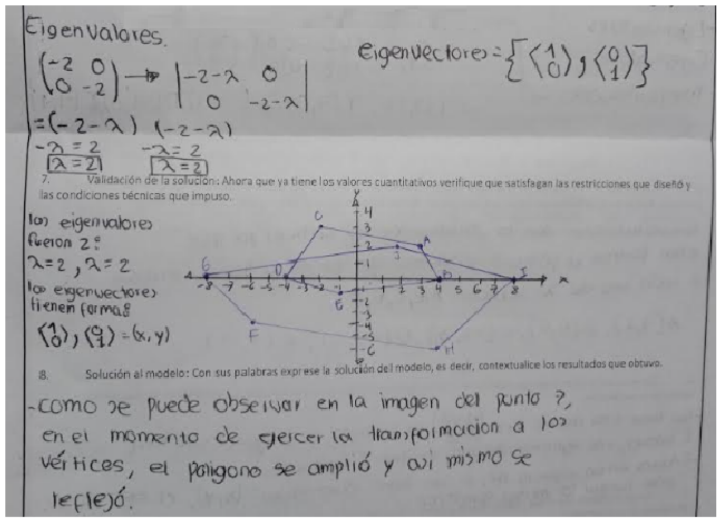


Figure 7: Example of a modelling phase with a fragment of a student's production. Note: The course was delivered in Spanish.

Classroom Teaching Results

The implemented methodology received high levels of approval. In 2022, 242 students from first and second semesters enrolled in the Linear Algebra course (176) and in the Vector Calculus course (66). 218 approved (90%) and 24 failed (10%). The average grades for these two categories were 3.8 and 1.6 correspondingly, on a 0 to 5 scale in which the minimum passing grade for each course was 3.0.

To understand the levels of approval of the courses, it is essential to discriminate the characteristics of the academic results. According to Figure 8, in the Linear Algebra course, the students achieved an average grade of 3.8 in the two semesters of 2022. Also, in both semesters, 25 % of the students obtained a grade below 3.5. In the case of the Vector Calculus course, a significantly different behaviour was identified: in the first semester of 2022, an average grade of 3.7 was observed, while in the second semester, the average grade was 4. The students who failed some of the courses were classified as outliers and were located below the respective interquartile ranges in each semester.

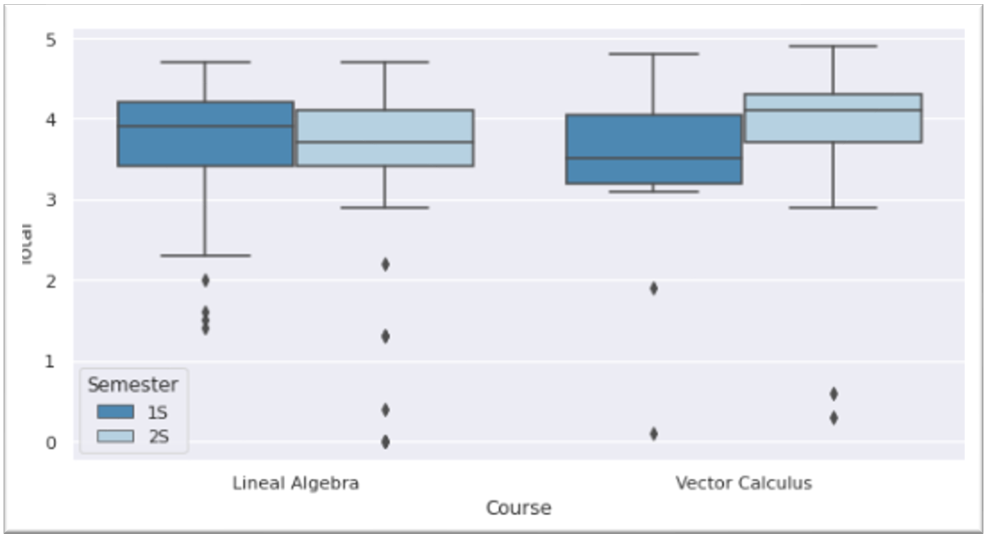


Figure 8: Grades distribution in 2022.

Using the STEM approach oriented to mathematical modelling has made it possible to condense the set of results obtained in the approval category. Figure 9 shows the significant relationship between the grades of the modelling test and the results obtained in each course. A correlation was observed in most cases that was higher than 50% in the first semester of 2022. An analogue behaviour was observed in the second semester. Such appropriation of the concepts makes it possible to relate reaching the general objective of the mentioned courses as the acquisition of knowledge applied to a real-life problem.

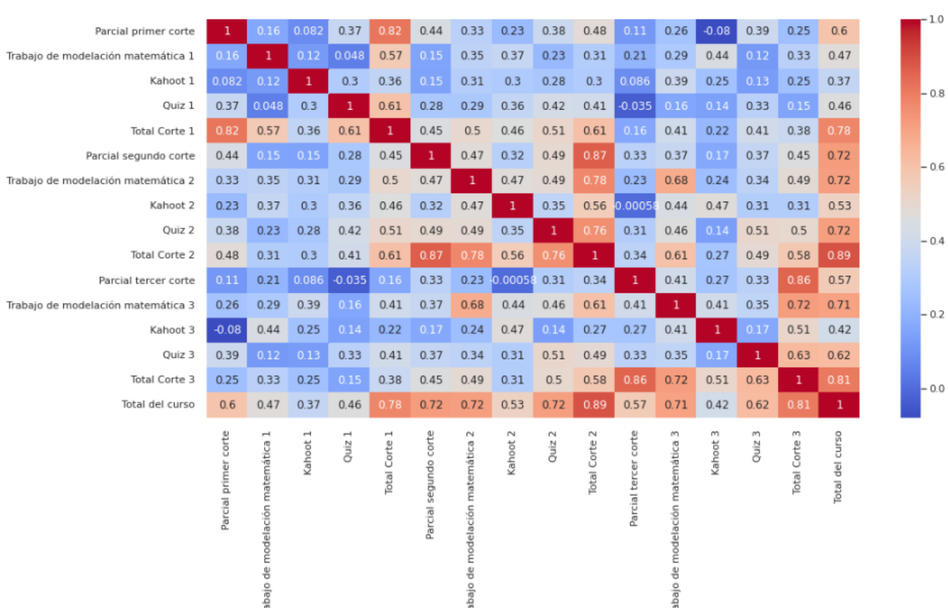


Figure 9: Correlation matrix for evaluation activities in the first semester of 2022.
Note: The course was delivered in Spanish.

Results in Terms of Didactic Material Production and in Pedagogical Research

Table 2 summarizes the results of the preparation of the material that condenses the pedagogical proposal and methodology from the STEM approach and from the adopted expression that is mathematical modelling. It is important to emphasize that this material has several direct interlocutors such as the students, the professors who oversee the courses, the monitors, the teaching interns, and the subject coordinators, who support the development of the academic work and therefore have different orientations and scopes.

Table 2: Results of the preparation of the material on the STEM approach.

Material produced	**Linear Algebra**	**Vector Calculus**
Slide presentations	13	13
Instructional brochure for assessment	9	9
Assessment item on institutional Learning Management System	352	273
Assessment test	9	9
Textbook	1	0
Protocol for professors	2	2
Test through platforms (Kahoot or Socrative)	11	8
Files with simulations, projections, or animations	3	65
Explanatory videos	93	102
Virtual classrooms	1	1

On the other hand, the academic production of the professors of the courses that underpins and supports all the academic work from the point of view of pedagogical research is gathered in Table 3.

Table 3: Results of the pedagogical research.

Academic production	**Total**
Published research articles	2
Submitted research articles	2
Textbooks	1
Published research book chapters	2
Research projects	3
International conferences	2

Regarding the evolution of the approval percentages for the first academic period of 2022, consisting of 16 weeks of classes, it was found that 83.09% of the Linear Algebra students passed the course with an average of 3.9 on a scale of 0.0 to 5.0. While, in the Vector Calculus course, the passing rate was 84.61% with an average passing rate of 3.7. To these results it can be added that, in the first academic period of 2022, 226 hours of extracurricular academic support were offered for each course, which corresponds to 14 hours per week on average.

The evaluation of the academic and general processes developed by the professors was answered on average by 61.6% of the students. The average rating was 4.27. The following comments from students were highlighted:

1. Real life problems are studied, in which we applied what we learned in class.

2. Studying algebra with "real" life problems is a different way of understanding algebraic problems.

3. We did many exercises and we tried to make the most difficult, this allows us to understand a little better the cases in which the problems are very complex and to develop the simplest ones more easily.

CONCLUSIONS

At Utadeo, an eight-step sequence applicable to exercises related to modelling, and simulation was developed, guided by the application and development of conceptual, analytical, and computational procedures. This sequence provides students with a mechanism that allows them to systematically break down the proposed application problems, aimed at achieving the general and specific objectives established in the Linear Algebra and Vector Calculus courses.

Implementing an integrated STEM approach has led to the applying graphical and algorithmic tools that have fostered or developed new skills and competencies in students oriented toward the learning outcomes of these subjects. Academic work in this approach is not a finished matter, it requires constant updating of teaching skills and the design of appropriate pedagogical and didactic material and educational research products that support the process. Therefore, the teaching and learning processes are carried out in the long term and in a sustained manner.

Additionally, many of the stages of the teaching and learning process require specific and timely evaluation methods that offer an opportunity for improvement at the right time. Thus, the holistic evaluation within the academic process has made it easier for all interested parties to recognize the degree of progress and identify the achievements to be attained. The way in which the knowledge imparted in the Linear Algebra and Vector Calculus courses is evaluated has highlighted the relevance of using virtual tools and environments for these processes, since they diversify the structure of the items, their scope, and the possible answers, which, therefore, has led to a satisfactory evolution of the evaluation of academic results.

What has been described so far demonstrates the fulfilment of the objective of visualizing the progress of the implementation of the mathematical modelling of integrated STEM education to improve academic quality in Utadeo mathematics courses. At the same time, it was evidenced that the implementation of mathematical modelling is not a completed stage but rather requires academic work that must continue to be strengthened because it requires the maturation of teaching skills and students' competencies and learning outcomes. The integrated STEM approach has not only resulted in a decrease in dropout rates and repetition but also, by integrating various methodological and strategic elements, has allowed students to integrate the interdisciplinary knowledge articulated by math skills. Simultaneously, this approach has increased teaching sensitivity towards their practice and towards written academic production, with which their role as university professors and researchers has been strengthened.

REFERENCES

[1] Arango, C., Camelo, C., Huertas, M., Rodríguez, C., Sánchez, C., Sánchez, V., & Sojo, J. (2019). *Centennials: generación sin etiquetas*. Universidad de Bogotá Jorge Tadeo Lozano Sancho BBDO.

[2] Asociación Colombiana de Universidades. (March 6th, 2022). *Análisis de la sostenibilidad financiera de la educación superior en Colombia*. https://ascun.org.co/documentos/analisis-de-la-sostenibilidad-financiera-de-la-educacion-superior-en-colombia/.

[3] Aya, O. (August 2021). Orlando Aya Corredor. *Cálculo vectorial (Cálculo en Varias Variables).* [Video file]. Youtube.https://www.youtube.com/channel/UCYFwpMhDywUufL540iT0mLw.

[4] Ballén, R. A. (2010). La pedagogía en los diálogos de Platón. *Diálogos De Saberes*, (33),35-54. https://revistas.unilibre.edu.co/index.php/dialogos/article/view/2001.

[5] Barragán, S. Sandra Patricia Barragán Moreno. (2021). *Crecimiento de una población de pájaros Álgebra Lineal*. [Video file]. https://www.youtube.com/watch?v=uGmu9tynnEM&list=PLb_2seQUX7WalwtDAv0OfODWrpo3nLsSC.

[6] Barragán, S., & Guzmán, A. (2022). Conceptual Model for Assessment in STEM Subjects in Higher Education. *TECHNO REVIEW. International Technology, Science and Society Review*, 11(1), 41-55. https://doi.org/10.37467/gkarevtechno.v11.3085.

[7] Blanchard, P., Devaney, R., & Hall, G. (1998). *Ecuaciones Diferenciales*. International Thomson Editores.

[8] Cabezas, C., & Mendoza, M. R. (2016). Manifestaciones emergentes del pensamiento variacional en estudiantes de cálculo inicial. *Formación Universitaria*, 9(6), 13-26. https://dx.doi.org/10.4067/S0718-50062016000600003.
[9] Cano, E. (1998). *Evaluación de la calidad educativa*. La Muralla, S.A.
[10] Castellar, C. E., & Uribe, J. I. (2004). Capital humano y señalización: evidencia para el área metropolitana de Cali, 1988-2000. *Revista Sociedad y Economía*, (6), 51-79. https://www.redalyc.org/comocitar.oa?id=99617648002.
[11] Cataño, G. (1984). Educación y diferenciación social en Colombia. *Revista Colombiana de Educación*, (14), 1-11. https://doi.org/10.17227/01203916.5108.
[12] Clavijo, G. (octubre de 2021). *El reto de Colombia ante la formación STEM*. https://www.universidad.edu.co/el-reto-de-colombia-ante-la-formacion-stem-galo-adan-clavijo-oct-21/.
[13] Congreso de Colombia. (28 de diciembre de 1992). Ley 30 de 1992. Por la cual se organiza el servicio público de la Educación Superior. https://www.mineducacion.gov.co/1621/article-86437.html.
[14] Congreso de la República. (1991). Constitución Política de Colombia. Bogotá, Colombia. https://dapre.presidencia.gov.co/normativa/normativa/Constitucion-Politica-Colombia-1991.pdf.
[15] Cosenz, F. (2022). *Managing Sustainable Performance and Governance in Higher Education Institutions. A Dynamic Performance Management Approach*. Springer.
[16] Cristia, J., & Pulido, X. (2020). *La educación en América Latina y el Caribe: segregada y desigual. En M. Busso, & M. Julian*, La crisis de la desigualdad: América Latina y el Caribe en la encrucijada, 167-193. Banco Interamericano de Desarrollo. http://dx.doi.org/10.18235/0002629.
[17] Daros, W. R. (2014). La calidad educativa y la problemática de las competencias. *Revista Internacional de Estudios en Educación*, 14(2), 119-133. https://doi.org/10.37354/riee.2014.145.
[18] Departamento Administrativo Nacional de Estadística. (November 20th, 2021). *Formación para el trabajo*. https://www.dane.gov.co/index.php/estadisticas-por-tema/educacion/formacion-para-el-trabajo.
[19] Dolores, C., & García, J. (2017). Conexiones Intramatemáticas y Extramatemáticas que se producen al Resolver Problemas de Cálculo en Contexto: un Estudio de Casos en el Nivel Superior. UNESP - Universidade Estadual Paulista, *Bolema Boletim de Educação Matemática*, 31(57), 158-180. https://doi.org/10.1590/1980-4415v31n57a08.
[20] García, A., Camacho, A., & García, F. (2019). La brecha de género en el sector STEM en América Latina: una propuesta europea. V Congreso Internacional sobre Aprendizaje, Innovación y Competitividad. CINAIC 2019. *Zaragoza: Servicio de Publicaciones Universidad de Zaragoza*, 704-709. http://dx.doi.org/10.26754/CINAIC.2019.0143.
[21] Gros, B., & Cano, E. (2021). Procesos de feedback para fomentar la autorregulación con soporte tecnológico en la educación superior: *Revisión sistemática. RIED. Revista*

Iberoamericana de Educación a Distancia, 24(2), 107-125. https://doi.org/10.5944/ried.24.2.28886.
[22] Grossman, S. (2012). *Álgebra Lineal (Séptima ed.)*. McGraw Hill.
[23] Guzmán, A., Barragán, S., & Cala, F. (2021). Rurality and Dropout in Virtual Higher Education Programmes in Colombia. *Sustainability*, 13(9), 1-21. https://doi.org/10.3390/su13094953.
[24] Herrera, D. (2020). El modelo de la alternancia y la desigualdad educativa territorial en la educación en Colombia. *Revista internacional de pedagogía e innovación educativa*, 1(2), 61-84. https://doi.org/10.51660/ripie.v1i2.38.
[25] Instituto Colombiano para el Fomento y Evaluación de la Educación Superior. (2020). *Clasificación de establecimientos y sedes*. Bogotá. https://www2.icfes.gov.co/web/guest/marcos-de-referencia-examen-saber-11%C2%B0.
[26] Instituto Colombiano para la Evaluación de la Educación. (2021). *ICFES, 50 años de historias. Bogotá D.C*. https://www.icfes.gov.co/50-icfes.
[27] Instituto Colombiano para la Evaluación de la Educación. (June 2nd, 2022). *Dataicfes*. https://www2.icfes.gov.co/data-icfes.
[28] Kelley, T., & Knowles, G. (2016). A conceptual framework for integrated STEM education. *International journal of STEM education*, 3(11), 1-11. https://doi.org/10.1186/s40594-016-0046-z.
[29] Landriscina, F. (2013). *Simulation and Learning. A model-centered approach*. Springer.
[30] Loaiza, O., & Hincapié, D. (2016). Un estudio de las brechas municipales en calidad educativa. *Science Direct*, (34), 3-20. http://dx.doi.org/10.1016/j.espe.2016.01.001.
[31] López, M. (2011). Sedimentación del sentido y tradición (Überlieferung). Eikasia. *Revista de Filosofía*, (36), 89-120. http://www.revistadefilosofia.com.
[32] López, V., Couso, D., & Simarro, C. (2020). Educación STEM en y para un mundo digital: el papel de las herramientas digitales en el desempeño de prácticas científicas, ingenieriles y matemáticas. *Revista de Educación a Distancia*, 20(62), 1-29. http://dx.doi.org/10.6018/red.410011.
[33] Ma, X. M. (2014). A comparative analysis of the relationship between learning styles and mathematics performance. *International Journal of STEM Education*, 1(3),1-11. https://doi.org/10.1186/2196-7822-1-3.
[34] Margot, K., & Kettler, T. (2019). Teachers' perception of STEM integration and education: a systematic literature review. *International Journal of STEM education*, 6(2), 1-16. https://doi.org/10.1186/s40594-018-0151-2.
[35] Martín, J. (2018). Calidad educativa en la educación superior colombiana: una aproximación teórica. *Sophia*, 14(2), 4-14. https://doi.org/10.18634/sophiaj.14v.2i.799.
[36] Melo-Becerra, L., Ramos-Forero, J., Rodríguez, J., & Zárate-Solano, H. (2021). *Efecto de la pandemia sobre el sistema educativo: El caso de Colombia*. https://repositorio.banrep.gov.co/bitstream/handle/20.500.12134/10225/be_1179.pdf.

[37] Ministerio de Ciencia Tecnología e Innovación. (2020). Convocatoria nacional para el reconocimiento y medición de grupos de investigación, desarrollo tecnológico o de innovación y para el reconocimiento de investigadores del sistema nacional de ciencia, tecnología e innovación – 2021 [PDF file]. https://minciencias.gov.co/sites/default/files/upload/convocatoria/anexo_1_-_documento_conceptual_2021.pdf.
[38] Ministerio de Ciencia, Tecnología e Innovación. (25 de febrero de 2021). Convocatoria nacional para el reconocimiento y medición de grupos de investigación, desarrollo tecnológico o de innovación y para el reconocimiento de investigadores del Sistema Nacional de Ciencia, Tecnología e Innovación - SNCTI 2021. https://minciencias.gov.co/convocatorias/fortalecimiento-capacidades-para-la-generacion-conocimiento/convocatoria-nacional-para.
[39] Ministerio de Educación Nacional. (2009). Deserción estudiantil en la educación superior colombiana. Metodología de seguimiento, diagnóstico y elementos para su prevención. Imprenta Nacional de Colombia.
[40] Ministerio de Educación Nacional. (2015). Guía para la implementación del modelo de gestión de permanencia y graduación estudiantil en instituciones de educación superior [PDF file]. http://www.mineducacion.gov.co/1759/articles-356272_recurso.pdf.
[41] Ministerio de Educación Nacional. (2018). Referentes de calidad: una propuesta para la evolución del sistema de aseguramiento de la calidad. Ministerio de Educación Nacional [PDF file]. https://www.mineducacion.gov.co/1759/articles-369045_recurso.pdf.
[42] Ministerio de Educación Nacional. (July 25th, 2019). Decreto 1330: Por el cual se sustituye el Capítulo 2 y se suprime el Capítulo 7 del Título 3 de la Parte. https://www.mineducacion.gov.co/1759/articles-387348_archivo_pdf.pdf.
[43] Ministerio de Educación Nacional. (May 8th, 2020). Índice Sintético de Calidad Educativa - ISCE. https://www.mineducacion.gov.co/1759/w3-article-397385.html?_noredirect=1.
[44] Ministerio de Educación Nacional. (November 19th, 2020). Resolución 21795 de noviembre de 2020. Bogotá, Colombia. https://diario-oficial.vlex.com.co/vid/resolucion-numero-021795-2020-852735682.
[45] Moore, T. J., Stohlmann, M. S., Wang, H. H., Tank, K. M., Glancy, A. W., & Roehrig, G. H. (2014). Implementation and integration of engineering in K-12 STEAM education. *In Engineering in Pre-College Settings: Synthesizing Research, Policy, and Practices*, 35-60. Purdue University Press.
[46] Moreno, N. (2019). *La Educación STEM/STEAM*. Apuestas hacia la formación, impacto y proyección de seres críticos. Fondo Editorial Universitario Servando Garcés de la Universidad Politécnica Territorial de Falcón. https://biblioteca.ucatolica.edu.co/cgi-bin/koha/opac detail.pl?biblionumber=79668.

[47] Mosquera, D. (2018). Análisis sobre la evaluación de la calidad educativa en América Latina: Caso Colombia. *Revista Iberoamericana de Evaluación Educativa*, 11(1), 43-55. https://doi.org/10.15366/riee2018.11.1.003.
[48] Mullis, I., Martin, M., Ruddock, G., O'Sullivan, C., & Preuschoff, C. (2009). *TIMSS 2011 Assessment Frameworks. Boston: International Association for the Evaluation of Educational Achievement (IEA)*. https://timssandpirls.bc.edu/timss2011/downloads/TIMSS2011_Frameworks.pdf.
[49] Oliva, J. (2019). Distintas acepciones para la idea de modelización en la enseñanza de las ciencias. *Enseñanza de las Ciencias*, 37(2), 5-24. https://doi.org/10.5565/REV/ENSCIENCIAS.2648
[50] Purnomo, E., Sukestiyarno, Y., Junaedi, I. & Agoestanto, A. (2022) Analysis of Problem-Solving Process on HOTS Test for Integral Calculus. *Mathematics Teaching Research Journal.* 14(1), 199-214. https://commons.hostos.cuny.edu/mtrj/wp-content/uploads/sites/30/2022/04/v14n1-Analysis-of-Problem-Solving.pdf
[51] Radinger, T., et al. (2018), OECD Reviews of School Resources: Colombia 2018, OECD Reviews of School Resources, OECD Publishing, https://doi.org/10.1787/9789264303751-en.
[52] Rincón, C., & Espitia, A. (2020). La educación superior de Colombia en riesgo: ¿Dónde están los estudiantes? *Ecos de Economía*, 24(51), 4-28. https://doi.org/10.17230/ecos.2020.51.1.
[53] Sánchez, R. M. (2017). Desigualdad del ingreso en Colombia: un estudio por departamentos. *Cuadernos de Economía*, 36(72), 139-178. http://dx.doi.org/10.15446/cuad.econ.v36n72.65880.
[54] Santamaría, V., Carabal, M. Á., Taroncher, J. M., & Moreno, E. (2020). El Art Thinking como catalizador del pensamiento divergente. *Repositorio Universitat Politècnica de València*. 1, 1080-1091. http://dx.doi.org/10.4995/INRED2020.2020.11980.
[55] Sayama, H. (2015). *Introduction to the Modeling and Analysis of Complex Systems*. Open SUNY Textbooks.
[56] Tsupros, N., Kohler, R., & Hallinen, J. (2008). *STEM Education in Southwestern Pennsylvania. Pennsylvania*. https://www.cmu.edu/gelfand/documents/stem-survey-report-cmu-iu1.pdf.
[57] Universidad de Bogotá Jorge Tadeo Lozano. (2011). *Modelo Pedagógico*. Universidad Jorge Tadeo Lozano.
[58] Universidad de Bogotá Jorge Tadeo Lozano. (2022). *Plan Estratégico 2022-2026*. https://www.utadeo.edu.co/sites/tadeo/files/collections/documents/field_attached_file/plan-estrategico_20220525.pdf.
[59] Urquía, A., & Martín, C. (2013). *Modelado y simulación de eventos discretos. Universidad Nacional de Educación a Distancia*. https://books.google.es/books?hl=es&lr=&id=BZBGAgAAQBAJ&oi=fnd&pg=PP1&dq=modelado+y+la+simulacion&ots=WPa5i7PBCq&sig=uzv7TzzogvHgMNpma1OzOB7F_sc#v=onepage&q=modelado%20y%20la%20simulacion&f=false.

[60] Vásquez, A. (2014). *Hacia un perfil docente para el desarrollo del pensamiento computacional basado en educación STEM para la media técnica en desarrollo de software*. Tesis de maestría. Medellín, Colombia. https://repository.eafit.edu.co/bitstream/handle/10784/5139/AlbertoV%C3%A1squezGiraldo_2014.pdf?sequence=2&isAllowed=y

[61] Vázquez, M. G. (2015). La calidad de la educación. Reformas educativas y control social en América Latina, *Latinoamérica. Reformas educativas y control social en América Latina*. (60), 93-124. https://www.redalyc.org/articulo.oa?id=64039200004.

[62] Velten, K. (2009). *Mathematical Modeling and Simulation*. WILEY-VCH Verlag GmbH & Co. KGaA.

[63] Vergel, M., Gallardo, H. d., & Márquez, J. F. (2020). Percepción en la planificación de cursos de física: relaciones en entornos sincrónicos. *Revista boletín REDIPE*, 9(12), 93-100. https://doi.org/10.36260/rbr.v9i12.1137

[64] Villalobos, G, (2021). The Calculus Concept Inventory Applied to the Case of Large Groups of Differential Calculus in the Context of the Program "Ser Pilo Paga" in Colombia. *Mathematics Teaching Research Journal.* 13(3), 208-223. https://commons.hostos.cuny.edu/mtrj/wp-content/uploads/sites/30/2021/10/v13n3-The-Calculus-Concept-Inventory.pdf

[65] Zaldívar, J. D., Quiroz, S. A., & Medina, G. (2017). La modelación matemática en los procesos de formación inicial y continua de docentes. *Revista de investigación educativa de la REDIECH*, 8(15), 88-107. https://doi.org/10.33010/ie_rie_rediech.v8i15.63.

Bridging the Gap Between Theory and Practice: The Research Productivity and Utilization of Research Outcomes Among Secondary Mathematics Teachers

Roldan S. Cardona, PhD
cardona.rs@pnu.edu.ph
Philippine Normal University North Luzon, Alicia, Isabela, Philippines

Abstract: Mathematics teaching is viewed as an inquiry process and a powerful context and practice for professional development with the goal of providing greater access, challenge, and support for every learner. This paper stems from a larger research project that investigates the research productivity and processes of integrating research in the curriculum delivery of high school mathematics. This descriptive work through survey, interview and documentary analysis involved 211 high school mathematics teachers in the quantitative component and four purposively selected in the qualitative section. Findings show that mathematics teachers demonstrate suboptimal level of research productivity but have shown promising potential for growth. Research is employed in various layers but typically as a mean to revisit teaching practices, as a basis of a teaching strategy, as a source of another research, and as a motivation for a research-oriented mindset. Research is strongly linked to the mathematics teaching and learning process and thus, have policy implications on nurturing and sustaining mathematics teacher-researchers. It is recommended to develop professional development plans that enhances the productivity and incorporation of research in the teaching-learning process.

Keywords: Professional Development, Research Integration, Teaching and Learning Math, Utilization of Research

INTRODUCTION

The 'teacher as a researcher' movement has been of interest to mathematics education because of the impact it can create. It highlights the concept of teachers as researchers and claims key roles for teachers in the production of knowledge about teaching (Adler, 1997). The mirror of co-learning agreement asserts teachers as active counterparts in the process of inquiry, and the practice of being a researcher (Ruthven, 2002). It is seen as a vital link in improving teaching quality and instructional delivery (Carlsson, Kettis, & Soderholm, 2013; Ayala & Garcia, 2013; Jusoh & Abidin, 2012; Cabral & Huet, 2011; van den Akker, 2010; Prince, Felder, & Brent, 2007), institutional changes and administrative policies (Ayala & Garcia, 2013), job satisfaction (Silver, 2009; Thomas & Harris, 2001). Without research, teaching would be little more than a trial-and-error process (Cabral & Huet, 2011; Thomas & Harris, 2001).

Linking research and practice in mathematics education is necessary for addressing critical issues of mathematics teaching and learning (NCTM, 2012; NCTM, 2011; Clements, 2007) especially in an educational system where the vision for *No Child is Left Behind* is thriving. High quality mathematics research plays a central role in any effort to improve mathematics learning (Battista et al., 2009; Goos, 2008; Carnine & Gersten, 2000) and improvement of school performance (Goos, 2008; Chval, Reys, Reys, Tarr, & Chavez, 2006). The results of research could also be a basis for policy-making (van den Akker, 2010; Craig, 2009) which shows that the problems of school mathematics necessitate the value of research given the right human energies and other resources (Ball, 2003).

In the Philippine basic education system, teachers are required to conduct researches specifically action researches or AR. Most authorities (e.g., Afify, 2008; Chant, Heafner, & Bennett, 2004) describe AR as a systematic, cyclical, and educative activity which is problem-focused, content-specific, and future-oriented involving a change intervention aimed at understanding, improvement, and involvement. The emphasis of AR or problem-centered approach as originally proposed by Kurt Lewin (Burnes, 2004) lies on the idea of knowledge generation as it provides a way for basic education teachers to investigate issues of interest perpetuating the classroom or the school environment. In essence, based on Afify (2008), AR attempts to solve problems and bring about change. It is a crucial part of school's performance review (Gao, Barkhuizen, & Chow, 2011) and links theory with praxis (Megowan-Romanawicz, 2010) which ignites a change or solves a problem in a social system (Crespin, Miller, & Batteau, 2005).

Relationally, the nexus between research and teaching has stirred continuous conversations in the country much more when the Department of Education (DepEd) has strengthened and institutionalized research-related policies such as the Basic Education Research Agenda, Research Management Guidelines, Basic Education Research. Local studies have shown that there is a very low proportion of teachers in the elementary and secondary level who engaged in research (Tupas, 2019; Ulla, 2018; Gepila, Rural, Lavadia, Nero, Palillo, & Besmonte, 2018; DO 43, s. 2015). Limited knowledge on research, voluminous academic and non-academic works, limited resources and opportunity for mentoring and networking were some of the reasons divulged in research findings (Cardona, 2020; Oestar & Marzo, 2022; Ulla, 2018). Capability building programs to addressed some of the concerns were limited on conceptual understanding of AR instead of its articulation and integration in the actual setting (Tupas, 2019; Basilio & Bueno, 2019).

Thus, two important points emerged for the case of Filipino secondary teachers: knowledge generation *through* action researches and knowledge transfer *out* of action researches. The former deals on teachers' research involvement and constraints they face in conducting ARs while the latter focuses on the process of integration and utilization of research in the learning delivery. However, the connections between research and practice are perplexing because of the lack of available research findings that are easy to grasp, context-dependent, lack of interest to research results, tremendous pressure to focus on short-term results that contradict what research talks about how children learn mathematics, and increasing narrow understanding of research (Seeley, 2005).

This study aims to contribute on the body of knowledge by probing on ways and means how mathematics teachers used research in their profession and how research can enhance their professional development. Also, the findings of this study contribute to development of policies on the professional development programs for teachers. This article is focused on answering two core questions: 1) what is the level of research productivity (RP) of secondary teachers, and 2) how do secondary teachers utilize research in the mathematics teaching-learning process.

Literature Review

There are three known perspectives regarding the relationship of research and teaching - complementary, antagonistic and disconnection. The Convention Wisdom Model (Neumann, 1992) and G Model (Hattie & Marsh, 1996) both alluded that teaching and research are positively associated. Not only do teachers perceive the benefits of action research and research in general but also do students (Kinash, 2015; Carrlson et al., 2013; Ayala & Garcia, 2013; Jusoh & Abidin, 2012; Hughes, 2004) who can be better constructors of knowledge if exposed to empirical analysis (Robles, 2016; Lerman, 1990). It facilitates the enthusiastic interest in teaching of up-to-date courses and promotes deeper understanding of relevant topics (Artes, Pedraja-Chaparro, & Salinas-Jimenez, 2017; Duff & Marriot, 2017; Carrlson et al., 2013), and self-improvement as well (Silver, 2009). The ripple effect prevails when the excitement and involvement in research is connected to students and they can see knowledge as constantly growing. The inclusion of research findings and evidences in teaching can also be performed as a mode of linking research in teaching (Markides, 2007) or utilizing past research experiences as an entry basis for teaching a subject or a basis for another research (Pawar, 2015). Navarro and Santos (2011) also highlighted that by using research as a basis, teachers can be able to change some aspects of their teaching, proceed to self-monitor the effectiveness of the changed strategy, and deepen their understanding towards the adoption of the changed strategy.

In the basic education of the country, the second key result area (KRA) of the Basic Education Sector Research Agenda (BESRA) mandates teachers to enhance their contribution to learning outcomes by coming up with informed decisions through action or applied researches. Action research is regarded as the convergence between theory and praxis (Afify, 2008) that transforms teacher attitude and approach to instruction (Bonner, 2006), pushes for personal theorizing (Chant, Heafner, & Bennet, 2004), and ultimately, brings about change (Afify, 2008; Evitts, 2004). This is strengthened with the creation of Basic Education Sector Reform Agenda (BESRA), Policy Development Process (PDP), Research Management Guidelines (RMG), and the provision for

Basic Education Research Fund (BERF) through DepEd Order (DO) Nos. 24 s. 2010, 13 s. 2015, 43 s. 2015, 4 s. 2016, 39 s. 2016, and 16 s. 2017. These efforts of the agency which is tasked to supervise the basic education are all gearing towards the engagement of teachers in problem-solving within their classroom environment.

METHOD

Design and Sample

This descriptive study took place in the northeastern part of Luzon, Philippines using multistage sampling and purposive sampling.

In the survey phase, five out of eight cluster-schools' divisions in Cagayan Valley region were chosen. Then, a cluster sampling was utilized with the Grade 10 mathematics teachers as group of samples. In the next stage, another cluster sampling was conducted with specific schools as basis. All mathematics teachers in those schools were provided with the questionnaire which resulted to a response rate of 80 percent or 211 samples. The sample consisted of 151 from Isabela province, 15 from Cauayan City, 12 from Santiago City, 22 from Quirino province and 11 from Batanes province.

In the interview, four grade 10 public high school mathematics teachers, one from each division except Batanes, were purposively selected using maximum variation approach. A set of inclusion-exclusion criteria was utilized to determine the key informants. This resulted to the selection of two females and two males who are on average 32 years old. Sample documents such as the actual research outputs were also gathered.

Instrument

Research Productivity Tool. The mathematics teachers reported their RP for the last three years. The sources of data include: articles published in refereed or non-refereed journals which circulate locally or internationally; published reviews of books, conference proceedings, book abstracts or compendiums, articles, and chapters, textbooks, monographs, and any other types of books; research presentations or technical reports at professional meetings like conferences, trainings, other research-based innovative projects, programs and workshops; speakership or being a trainer in any professional development programs, memberships in mathematical and research-related associations/societies; and professional services like research consultation, conduct of language editing and statistical analyses. The scope of RP is based from the conducted literature review and was validated by experts using Content Validity Index (CVI). The CVI showed acceptable levels of validity.

Interview Guide. The researcher prepared an interview guide, validated by experts and its results were analyzed using CVI. Both the item- and scale-levels CVI are 1.0.

Procedure

Permission from the regional office of the DepEd and consent from the participants were sought. The survey information was gathered personally or through mail. The interview was conducted either personally or through phone. The researchers depended on the self-reported information of teachers regarding their research production. Teachers who did not responded immediately with the invitation were asked until three attempts. Enumerators were also asked to float the questionnaire in unreachable places.

Data Analysis

The researcher developed a weighting scheme to quantify RP. He distinguished whether the activity was performed in solo or jointly (Abouchedid & Abdelnour, 2015). For joint RP, there are no sub-classifications for second, third and succeeding authors though. The researcher separated single presenter versus multiple presenters, participant versus trainer/speaker versus organizer, international level against local, among others. Greater weights were also assigned to manuscripts that were completed, juried, published, DepEd and externally funded, and competitive research awards than on materials that were just proposed, non-refereed and non-competitive awards. The raw score can go limitless but the weighted score only allowed each teacher a maximum of 100 points. The highest possible source of RP is in publication. The data generated were processed using descriptive statistics.

The qualitative data generated from the interview were thematized (Clarke & Braun, 2013) using inductive approach (Nowell et al., 2017). It includes the following phases: familiarizing with the data, generating initial codes, searching for themes, reviewing themes, defining and naming themes, and producing the reports. Documentary analysis was also performed to validate claims of respondents (Nowell et al., 2017).

FINDINGS

Research Productivity of Mathematics Teachers

The involvement in research-related activities of Grade 10 mathematics teachers is generally in the early development phase and highly skewed. They are primarily engaged in research-related trainings, seminar-workshops, and conferences (raw mean = 1.65, weighted mean = 0.25). Second on the list is writing of research manuscripts which only manifests that some are also writing researches (proposals, on-going, and completed) with raw score mean of 0.60 (SD = 1.62) and weighted mean of 0.15 (SD = 0.41). Joining mathematics and research-related professional organizations (raw mean = .07, weighted mean = 0.00) is not on their priority list. The skewness and kurtosis indices display that accomplishment on research publications is badly skewed to the right (Sk = 9.33) and leptokurtic (Ku = 96.55) which means that only few them are performing as teacher-researchers.

Table 1. Research Productivity of Grade 10 Mathematics Teachers

RP Indicators	Raw		Weighted		Sk	Ku	Remarks	Rank
	Mean	SD	Mean	SD				
Research publications (35%)	0.37	2.48	0.13	0.87	9.33	96.55	Beginning	3
Research manuscripts (25%)	0.60	1.62	0.15	0.41	3.02	9.45	Beginning	2
Research presentations (10%)	0.10	0.52	0.01	0.05	5.53	33.20	Beginning	5.5
Research- and math-related affiliations (5%)	0.07	0.35	0.00	0.02	4.72	20.98	Beginning	7
Research- and math-related trainings (15%)	1.65	2.47	0.25	0.37	2.07	4.87	Beginning	1
Research- and math-related awards (5%)	0.11	0.52	0.01	0.03	4.66	20.65	Beginning	5.5
Research-related professional services (5%)	0.49	1.47	0.02	0.07	4.21	20.33	Beginning	4

Ave Weighted RP Index: 0.57 (Low) SD: 1.26 Max: 11.93 Min: 0.00 Sk: 5.82 Ku: 43.25

On Research Publications

The research publications revolve on the performance of students in the National Achievement Test (NAT) as a basis for strategic intervention and the impact of mathematics drills on the students' proficiency in integer operations. These two materials were published by a single teacher who acted as a co-author. There were also other teacher-respondents who submitted their articles for publication. These materials were on curriculum, educational leadership, mathematics learning, academic performance, and math manipulatives. Notably, only four teacher-respondents were able to publish these 11 outputs.

On Research Manuscripts (Proposed, On-going, Completed)

Almost all topics and titles self-reported are action researches (ARs) delving on teaching and learning issues across all learning areas.

According to Teacher Charlie, his students cannot cope in algebra specifically in multiplying binomials. If one were to analyze, this concept is a pre-requisite to major concepts in math. Struggling with it means a greater problem in the advance courses and topics. This is probably a triggering factor why majority of the papers' criterion variable is academic performance. Their initiatives target a better performance across mathematics competencies like in polynomials, word problem solving, inscribed and central angles, intersecting secants, and integers, among others. For instance, some researchers evaluated students' performance in problem solving through Newman's Error Analysis and some devised interventions like using jigsaw model, tutorials, drills, differentiated instruction, peer tutoring, and collaborative reading interventions to support at-risk students. One even considered team games tournament as a strategy in improving the performance of his students in polynomials. Some also explored and embraced the concept of lesson study (LS).

Relationally, Teacher Beth showed her research paper and as far as the background of the study is concerned, she initially presented the elegance of mathematics. As the paper unfolds, she placed it in the context of the Trends in International Mathematics and Science Study 2003 and 2008 results where the Philippines was a participating country and cited observations and experiences in her actual classrooms with respect to mathematics learning before clarifying the proposed intervention and objectives of her study.

Meanwhile, in the research where Teacher Angelo was a co-researcher, the paper commenced with:

> Mathematics is considered by many as a difficult subject. Many students had a negative perception about math and end up disliking the subject up to the extent of falling (2018, p. 1).

The discussion was followed by the relevance of a support system to help students become engaged in mathematical tasks. Then, they logically presented the idea of their intervention which is also considered by the researchers as a support to enhance students' critical thinking skills and collaborative interaction.

Clearly, the research topics and the way they were conceptualized reflect the situation G10 mathematics teachers are facing in their classrooms and they were conceived because there is a need to improve classroom learning. On the other side, however, the number of research proposals both on-going and completed researches illustrate a very low turnout, which means that only few are participating in research production tasks This affirms studies like Ramos' (2017) as well as the DepEd report in the Research Management Conference 2018 and the DO No. 43, s. 2015.

On Innovation Projects. While the respondents feel responsible of their students' learning as proven by their intervention programs in their researches, they also believe in holistic learning as some of them were engaged in environmental-related projects. One teacher also demonstrated her skills in programming when she developed an electronic system for the checking of attendance. Evidentially, they perform multiple tasks in the appreciation of the fact that being a math teacher does not only reside in the mathematical formulas, problems and calculations, but also on solving real-life situations.

On Research Presentations

The presented papers are the same researches and projects that were completed by the teacher-respondents. Analyzing further, all are ARs except for the performance in NAT as a basis for an intervention which is presumed to trigger an AR. Likewise, it was established that completed researches both funded and self-initiated were given opportunities to be presented in the district, division, and even in the regional conferences usually conducted by DepEd.

On Research- and Mathematics- Related Affiliations

The teachers are limited to Mathematics Teachers Association in the Philippines (MTAP) as their professional national organization. This is due probably to the fact that MTAP has programs like the MTAP-Metrobank-DEPED Math Challenge that is conducted annually as well as scholarship programs for teachers. A lone mathematics teacher is connected to SCAAP, an organization consisting of science club advisers in the Philippines.

When asked about the limited membership with mathematics- and research-related organizations, teacher Beth had a striking response, "*Wala, bakit ako magme-member? Ano ba iyong pwedeng maging advantage o benefit ng pagiging member mo?* ("*None.*" "*Why would I do so?*" "*What advantage or benefit will I gain from being a member?*")

Teacher Beth commented further, "*Ire-require iyong time ko eh wala na nga akong time para sa sarili ko.*" ("*It [being a member of a professional organization] requires my time, but I barely have time for myself.*")

Apparently, the teacher cannot accommodate active participation in a professional organization because of the volume of other priorities.

On Research- and Mathematics-Related Trainings

The math teachers were immersed in trainings mandated or endorsed by the DepEd especially matters on K to 12 curriculum. These trainings focused on the content and pedagogical content knowledge needed by them. These include the Regional Mass Trainings in Grade 10 and in senior high school which were cascaded down to teachers who were not accommodated in the major training. They called this as training for untrained teachers. Aside from these, trainings on localization, contextualization, spiral curriculum, calculus, and effective and contemporary strategies of teaching mathematics in the 21st century were provided.

The four key informants appreciated the importance of trainings in their professional development activities. One of the trainings was on LS to which Teacher Angelo was a participant. LS is a Japanese practice of sharing teaching practices by conducting systematic inquiry into their pedagogical practices (Fernandez, 2002). Teacher Angelo described the program:

> *Iyong lesson study for students, iyon kasi, gagawa kayo ng lesson plan na mas appropriate sa mga bata. Kung paano mo iniisip iyong higher order thinking and skills nila doon.* (In lesson study, you will develop lesson plans appropriate for the students by reflecting on their higher-order thinking skills.)

As an output, they were tasked to develop a lesson plan. They were closely monitored by external and internal partners until they accomplished one. They documented their practices and were able to craft research out of the training outputs. They were as well given the chance to present their outputs for public verification and dissemination.

Also, AR-related activities were organized that aim to propel the teacher-researcher role like training workshops on research, statistics, probability, quantitative and qualitative researches, basic education researches, and action researches. In the interview, some of them claimed that the research trainings were geared towards conceptual development only. Further, they said that sustainable technical assistance is needed as well as decongesting some school activities to help them translate their learning into an output. Some remarked about voluminous trainings which are not related to their field of specialization, but certainly a component of teacherhood. Aside from the use of SPSS, they had trainings on the use of Microsoft Excel and a special kind of scientific calculator.

On Research- and Mathematics-Related Awards

It is shown that the teachers' efforts also paid off as they achieved awards like best in research proposal, poster presentation, and innovation. The researchers of these papers received funding from the DepEd and even the outstanding teacher awardees had researches listed in their vitae. Also, few of them were accorded distinguished secondary math teachers.

On Research-related Professional Services

Mathematics teachers are also data analysts, research consultants and advisers, and language editors. The number of clients served however, reveals that they are more into performing data analysis than being research consultants and language editors. From the informal interview, most of these *pro bono* professional services were intended to help senior high school students taking up Practical Research 1 and 2. Some were sought for the graduate studies and by their colleagues as well.

Link of Research Productivity and Mathematics Teaching and Learning Process

It is found out that teachers utilize research in various layers and levels. The link of RP and mathematics teaching-learning process is presented for each teacher and then were summarized into themes thereafter.

Teacher Angelo

He stressed that participating in research-related activities is driven by his will to improve himself and as well as his students. He stated, "*...to improve the class and the students.*". For instance, his research being a co-author about lesson study (LS) allowed him to thoroughly design lesson plans that integrate higher order thinking skills and how to smoothly scaffold these to the learning situations. In other words, the way teacher Angelo integrates research in teaching and learning situations is by considering the intervention as a teaching strategy. After the research, he used the concept in developing his succeeding lesson plans. As he pointed out, *"I used it in planning my class especially in problem solving situations."*.

Consequently, he claimed he was able to improve the content, presentation of his lessons, and elicitation of responses from his students explaining that, "*I usually lectured my topics but in LS, I use problems that are considered of a higher level.*

There was also an admission that through research-related activities, it became instrumental in updating himself in content and pedagogies. He shared that during research- and mathematics-related seminars, they are taught on how to develop instructional materials and according to him, he utilized those materials inside his classes.

Teacher Beth

She acknowledged that participating in research-related activities are avenues to improve herself as she opined*, "I attend trainings to strengthen my knowledge and skills on research and in math."*

Not only did she improve herself through research-related activities but also her students. She further said*, "I conduct researches so that I have knowledge to impart to my students."*. She explained that the learnings she gained from research were also applied in the class when these are proven effective. In a way, research becomes a mode of reflecting on her own teaching practices. She exemplified their Program LOVE which is related with child protection policy. They continued it until it became a part of their school culture. This is also like her peer learning strategy. She also stressed that to teach is to read and find ways on what techniques are best suited to the learners and this never ends.

Further, the product of a research can be another research. She cited for instance her plan on her peer learning strategy which is action research. She acknowledged that the strategy can become a research as well in other learning areas and other learning situations.

Teacher Charlie

In the interview, he confessed that he has yet to explore the beauty of research because he perceives it as interesting and beneficial to his students. He only attended few mathematics-related trainings that allowed him to have a better understanding of the K-to-12 curriculum. He acknowledged that research- and mathematics-related trainings helped him with a variety of strategies that can be used in different teaching and learning situations. He elaborated, "*I became* *aware of what should be learned. During trainings, I become updated and see myself a critical thinker.".*

Like teacher Beth, he reads to supplement things that he learns from other venues. Apparently, the transcripts showed that for Teacher Charlie, the connection of teaching and research or the integration of research and teaching is unclear. There were several points in the interview where he could hardly clarify his ideas in the way he integrates research-related ideas in teaching. The teacher-researcher role is therefore, a thinking yet to be achieved in reality. Teacher Charlie is about to propose his first ever research paper after almost a decade of teaching.

Teacher Dindin

Among the four, together with Teacher Beth, she was strong in acknowledging the benefits of research particularly as a way of determining the problems encountered by her students. In her words:

> What I like in action research is that we identify the problems of the students for us to address those problems.

According to her, research-related activities such as trainings and research writing helped her in revising instructional materials. So, other than helping her students to learn through her enhanced learning materials as by-products of research-related activities, she also got updated and developed the mindset of a researcher. Essentially, her being a research teacher made her more vigorous to learn and become skilled in research to enable her to develop the same skills among her students. She mentioned, "*Because what they learn from us are also the things, we teach them.".*

She sees to it that research findings related to classroom discussions are spilled over her to her class to supplement the ideas coming from the books. She averred,

> What we learn from research and other related activities are utilized inside the classroom. We become updated, we become critical, we explore our learning environment, and we become reflective.

Like Teacher Beth, she never stops learning new things. Moreover, the recommendation of the previous researcher, be it herself or other researchers, can become again a basis of an intervention which in turn, can become another action research.

From the individual cases discussed, the following themes were generated. The themes are presented visually in the next figure.

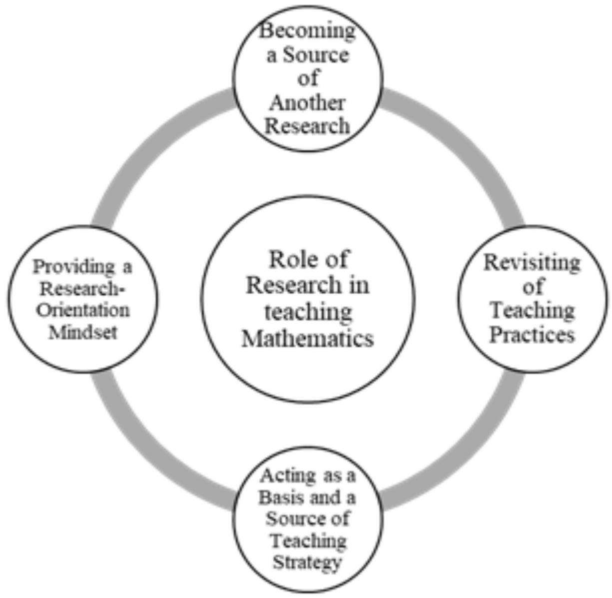


Figure 1. Integration of Research in Teaching-Learning of Mathematics

Acquiring a Research-Oriented Mindset

The four teachers were engaged in different research-related activities because they sought to update themselves and stimulate deeper understanding in terms of content and processes in their field (Artes et. al., 2017; Carlsson, et al., 2013; Hughes, 2004). For instance, Teacher Charlie shared his views of the present curriculum as compared to the previous curriculum. Because of trainings, although he yearns for more trainings, he distinguished the expectations of both curricula. Accordingly, they gave importance to reading as an important element to teaching. By reading, he grew professionally. Hence, through readings, trainings and other research-related activities, commitment to learning is refreshed (Duff & Marriot, 2017) which in turn allows them to become critical thinkers. Teacher Charlie explains it this way:

> I become more aware of what competencies ought to be learned that have application in real life. During my trainings, I am updated and I became a critical thinker.

In Grade 10 mathematics, one expected outcome in the last quarter is a mini-research. Students according to Teacher Beth craft investigations that encourage them to apply their knowledge in math. She believes that her knowledge in research motivates her to imbue research thinking among his students to carry out their investigation. Teacher Dindin shared the same viewpoint as Teacher Beth in her class in statistics. Because of her knowledge on the power of research, she wanted her students to develop a similar research mindset. She declared:

> In my statistics class, I required them to develop quantitative researches. I told them to identify problems in the school and propose so that the awareness of the students will improve. That is what we wanted for our researches.

She further remarked that she infused research findings in her class to supplement the information given in the books. This reduces assumptions in the classroom (Magidson, 2005) and the tendency to work towards facts and information. This finding concurs with Markides (2007) and Burke & Rau (2010, as cited by Pawar, 2015) as a mode of linking the two. Teacher Beth aptly concludes, "*You become intelligent if you do research*." The teachers' critical thinking and commitment for continuous learning is sustained because of research.

Revisiting of Teaching Practices

The multiple data revealed that teachers utilized research in teaching and learning situations to monitor their own teaching practices (Ulla, 2018; Navarro & Santos, 2011; Segal, 2009; Kane, Sandretto & Heath, 2004; Lerman, 1990).

Their proposed and completed classroom researches as well as innovation projects unravels the urge of teachers to determine the impact of their interventions to learning outcome. Their trainings allow them to self-revisit their practices especially in the mathematical pedagogical content knowledge (MPCK). Teacher Charlie consistently seeks for more MPCK trainings and research orientations as opportunities to revisit his teaching practices.

The key informants also believed that research is an avenue to assess their teaching practices if effective or needs some calibration. Teacher Beth remarked, "*What we do is to apply what we have discovered to determine if it is effective or not.*".

Further, because of an LS training program, Teacher Angelo and his colleagues utilized the concept of LS in developing their math lessons. After the first implementation of the first teacher, the group sat down and analyzed the implementation to see what improvement can be done in the next execution. After a thoughtful reflection, one can determine what works and what does not work in a classroom set-up leading to new perspectives (Afdal & Spernes, 2018). They used the knowledge gained from the first execution to polish their practices in teaching a particular lesson. He further expressed the value of research processes in teaching, when he said, "*Because you know how to teach a particular lesson. Knowing how to teach it guarantees understanding.*". It can be said then that he was ensured of self-improvement in terms of pedagogy because of his participation in research and research-related activities (Silver, 2009) which may also guarantee students' success. He pointed out, "*If I attend, my lesson and lesson presentation are also improved.*".

In the case of Teacher Beth, she conducted the $5x + y$ shepherding style to investigate whether pairing of high- and low-ability students in mathematics help improve the academic performance. Through this, it allows the teacher to do self-introspection on the effectiveness of the strategy. Teacher Dindin affectingly points out, *"In teaching, I am able to assess whether my strategy is effective and, in that way, we become more reflective."*.

Generating a Basis and Source of a Teaching Strategy

If a researcher finds evidences that a particular strategy is effective, then it gives him or her confidence to adopt such strategy. In this case, trainings and seminar become relevant grounds for teachers to understand the K-to-12 Curriculum and to explore ways and means to support every learner (NCTM, 2016; Willis, 1995).

Teacher Beth was driven to conduct the research by her desire to determine a strategy that will impact the learning of her students because of the perennial struggles in mathematics. This similar view was raised by NCTM (2012), Rasmussen et al. (2011), Battista et al. (2009), Goos (2008), Clements (2007), NRC (2001) and Carnine and Gersten (2000) on the link of teaching and research. Before implementing the shepherding style, one must meet the rigors of reading to gain knowledge on the strategy and what can be done with such a technique. After finding out that the $5x$ + y shepherding style is effective, Teacher Beth adopted it in her remediation activities to students who are considered at-risk in mathematics. It also gave the opportunity to high-ability students to master the content of the lesson through the shepherding act. Not only did she effect learning in her class, but she was able to produce knowledge about teaching (Adler, 1997) by aligning the learning opportunities and learning outcomes (Cai, Morris, Hohensee, Hwang, Robinson, & Hiebert, 2017). According to her, this is also the same concern of her planned Project LOVE. If it will provide evidence of a favorable impact, she will consider the strategy in her actual instructional process.

For Teacher Dindin, she puts emphasis on instructional materials as a possible by-product of research activities. Understanding the students shapes the way a teacher develops instructional materials. According to her:

> For example, today's age is technological so we maximize the use of cellphone. We need to keep abreast on things we think will help them.

Unstated explicitly, Teacher Angelo to a certain degree has this same intention for why he conducts research. During the interview, he repetitively sent the message that his students could hardly grasp higher order thinking competencies. When asked of the add-on learning research provided him, he responded, "The problem-solving based teaching.".

So, sustaining his participation in LS programs can be regarded as a way of attuning his mathematics lesson to achieve higher order thinking skills among his students.

Becoming a Source of another Research

Teacher Dindin believes that research is a cyclical process pointing out that research becomes a basis for another research. The recommendation section of a paper signifies that much must be done on a certain investigation. The research of Teacher Beth on child protection policy became the foundation of Program LOVE – a program on values integration which aims to change the untoward attitudes of the students.

Teacher Angelo immersed himself on the concept of LS and designs such program in some math lessons. The same program encouraged him to repeat conducting one. According to Pawar (2015), past researches become a basis of another research.

DISCUSSION AND CONCLUSION

Generally, the high school mathematics teachers are considered beginners or have limited experiences in terms of research production, presentation, and publication (Besmonte et al., 2018) and therefore, have an immense potential room for growth. As opined by teacher Angelo, "*Sa tingin ko hindi pa ako ganoon ka-productive. Kasi for the sake lang na umattend and maki-participate."* ("I think I am not that productive because I only attend for the sake of attending and participating.") Unfortunate it may seem for Teacher Angelo, but teacher Beth's words are promising:

> *Productive siguro, in the sense na nag-iistart na ako ng research. This SY, I started two researches. Unang una, about bullying and iyong isa is learning strategies in mathematics na kaka-submit ko sa DO. About pairing of students para in terms of academic achievement para mai-share ko sa ibang teachers kasi I think hindi lang siya sa math applicable.* (Perhaps, I am productive in the sense that I am starting to conduct researches. This SY, I started two researches. The first is about bullying and the other one is about learning strategies in mathematics which I just submitted to the DO. It [referring to the latter] is about pairing of students so that in terms of academic achievement, I will be able to share to other subject teachers [research results].)

These results substantiate various studies (e.g., Abramo, et al., 2017; Milburn and Brown, 2016; Tagaro, 2015) as well as local reports like those of Dullas (2018) Ulla (2017), Ramos (2017) and from DO No. 43 s. 2015 stating the right skewed distribution in participating in research related activities which means that only few of the teachers are actively participating in research-related tasks. The low number of publication manifests either or both of the following: firstly, few are joining the rank of teacher-researchers; and secondly, it is still true that researches are made for incubation instead of presentation and publication. Hence, studies on what enable or disable teachers to conduct should be studied thoroughly.

Further, it seems clear that the level of integration of research in teaching varies from one teacher to another which is primarily a function of the prior knowledge and experience in research as well as the level of constraints each of them felt in the conduct and participation in research-related activities. Hence, they have demonstrated varied, personal, and non-linear ways of integrating, linking, and utilizing research in the learning delivery of mathematics curriculum. The disparities of integration between respondents relies heavily on the orientation and previous experiences of a teacher in the realm of research and teaching. The integration of research commenced with the teachers' process of growing as a better facilitator of learning and with the terminal point for student development. It transforms teachers more reflective of their teaching and their students' learning, and effective in understanding and addressing the needs of students.

Teacher Charlie who was limited to only few mathematics- and research-related trainings spoke only of the add-on learning he derived in those activities and how those activities helped him in teaching mathematics. He could not speak of the fourth theme (Becoming a source of another research). Meanwhile, Teachers Beth and Dindin who had experiences on research writing were both evolving in ideas and outspoken about the potential benefits of research in teaching. Teacher Beth's experience on action research was however, more solid than teacher Dindin's, whose immersion was more on basic types of research. Teacher Angelo, a millennial, had some uncertainties in the integration process. For him, together with Teacher Charlie, there is a mixed notion of research as a source of knowledge production and symbolic act of compliance.

The flurry of disempowering conditions tends to skew their action towards accommodating the idea of research. The promising aspect however, is that while they lean on the idea that their participation is a function of the organization's expectation there are strong contentions that their wanting for professional development pushed them to slowly embrace the role of a teacher-researcher. Provided with a favorable institutional and leadership atmosphere, their commitment to participate in research-related activities to impact their professional development and the students in general is expected.

References

[1] Abouchedid, K., & Abdelnour, G. (2015). Faculty research productivity in six Arab countries. *International Review of Education*, *61*(5), 673-690. doi: 10.1007/s11159-015-9518-5

[2] Adler, J. (1997). Professionalism in process: Mathematics teacher as researcher from a South African perspective. *Educational Action Research, 5*(1), 87-103. https://doi.org/10.1080/09650799700200016

[3] Afdal, H. W., & Spernes, K. (2018). Designing and redesigning research-based teacher education. *Teaching and Teacher Education, 74*, 215-228. https://doi.org/10.1016/j.tate.2018.05.011
[4] Afify, M. (2008). Action research: Solving real-world problems. *Tourism and Hospitality Research*, *8* (2), 153-159. https://doi.org/10.1057/thr.2008.13
[5] Artes, J., Pedraja-Chaparro, F., & Salinas-Jimenez, M. D. (2017). Research performance and teaching quality in the Spanish higher education system: Evidence from a medium-sized university. *Research Policy, 46*, 19-29. https://doi.org/10.1016/j.respol.2016.10.003
[6] Ayala, C. J., & Garcia, B. H. (2013). Research productivity and utilization in higher education institutions in the CALABARZON region. MSEUF Research Studies, 15(1). Available at: http://ejournals.ph/form/cite.php?id=6751
[7] Ball, D. L. (2003). *Mathematical proficiency for all students towards a strategic research and development program in mathematics education.* Available at: https://www.rand.org/pubs/monograph_reports/MR1643.html
[8] Basilio, M. B., & Bueno, D. C. (2019). Research skills and attitudes of master teachers in a division towards capability training. In *19th CEBU Philippine International Conference on Economics, Education, Humanities and Social Sciences* (pp. 163-171).
[9] Battista, M., Boerst, T., Confrey, J., Knuth, E., Smith, M. S., Sutton, J., White, D., & Quander, J. (2009). Research in mathematics education: Multiple methods for multiple uses. *Journal for Research in Mathematics Education, 40* (3), 216-240.
[10] Besmonte, M., Gepila, E. C., Jr., Lavadia, M.K., Nero, J.M., Palillo, D. C., & Rural, J. D. (2018). Research skills assessment of the selected DepEd teachers in Metro Manila*.* *American Scientific Publishers, 24* (11), 8248-8253. Retrieved from https://tinyurl.com/yxjpslyb
[11] Burnes, B. (2004). Kurt Lewin and the planned approach to change: A re-appraisal. *Journal of Management Studies*, *41*(6). https://doi.org/10.1111/j.1467-6486.2004.00463.x
[12] Cabral, A. P., & Huet, I. (2011). Research in higher education: The role of teaching and student learning. *Procedia – Social and Behavioral Sciences, 29*, 91-97. https://dx.doi.org/10.1016/j.system.2014.03.003
[13] Cai, J., Morris, A., Hohensee, C., Hwang, S., Robinson, V., & Hiebert, J. (2017). Clarifying the impact of educational research on learning opportunities. *Journal for Research in Mathematics Education, 48* (3), 230-236. https://doi.org/10.5951/jresematheduc.48.3.0230
[14] Cardona, R. S. (2020). The enablers and outcomes of research productivity among junior high school mathematics teachers: A structural model. *EURASIA Journal of Mathematics, Science and Technology Education*, *16*(11), em1901.
[15] Carlsson, H., Kettis, A., & Soderholm, A. (2013). *Research quality and the role of the university leadership*, 1-48. Stockholm, Sweden: The Swedish Association of Higher Education (SUHF)/ Experts' Committee on Quality. Available at: https://goo.gl/9YD938
[16] Carnine, D., & Gersten, R. (2000). The nature and roles of research in improving achievement in mathematics. *Journal for Research in Mathematics Education, 31*(2), 138-143. https://doi.org/10.2307/749748

[17] Chant, R., Heafner, T., & Bennett, K. (2004). Connecting personal theorizing and action research in preservice teacher development. *Teacher Education Quarterly*, *31*(3), 25-42. Available at: https://files.eric.ed.gov/fulltext/EJ795251.pdf

[18] Clarke, V., & Braun, V. (2013) Teaching thematic analysis: Overcoming challenges and developing strategies for effective learning. *The Psychologist*, *26* (2), 120-123. Available at: http://eprints.uwe.ac.uk/21155/3/Teaching%20thematic%20analysis%20Research%20Repository%20version.pdf

[19] Clements, D. (2007). Curriculum research: Toward a framework for "research-based curricula". *Journal for Research in Mathematics, 38* (1), 35-70. Available at: http://www.jstor.org/stable/30034927

[20] Craig, C. J. (2009). *Teacher research and teacher as researcher*. International Handbook of research on Teachers and Teaching, 61-70. https://doi.org/10.1007/978-0-387-73317-3_4

[21] Crespin, P., Miller, C., & Batteau, A. (2005). Ethnographic research methods. In R. A. Swanson & E. F. Holton (Eds.), *Research in organizations: Foundations and methods of inquiry* (pp. 281-294). San Francisco: Berrett-Koeler Publishers.

[22] DepEd Order No. 43, s. 2015. *Revised guidelines for the basic education research fund (BERF).* Pasig City, Philippines.

[23] Duff, A., & Marriot, N. (2017). The teaching-research gestalt in accounting: A cluster analytic approach. *The British Accounting Review, 49*, 413-428. Available at: https://dx.doi.org/10.1016/j.bar.2017.05.001

[24] Gao, X., Barkhuizen, G., & Chow, A. (2011). 'Nowadays, teachers are relatively obedient': Understanding primary school English teachers' conceptions of and drives for research in China. *Language Teaching Research*, *15* (1), 61-81. https://doi.org/10.1177/1362168810383344

[25] Gepila, E. C., Rural, J. D., Lavadia, M. K., Nero, J. M., Palillo, D. C., & Besmonte, M. (2018). Research Skills Assessment of the Selected DepEd Teachers in Metro Manila. *Advanced Science Letters*, *24*(11), 8248-8253.

[26] Goos, M. (2008) Researcher-teacher relationships in mathematics education. *Proceedings of the 31st Annual Conference of the Mathematics Education Research Group of Australia,* 227-236.

[27] Hattie, J., & Marsh, H. W. (1996). The relationship between research and teaching: A meta-analysis. *Review of Educational Research, 66* (4), 507-542. https://doi.org/10.2307/1170652

[28] Hughes, C. A. (1998). Factors related to faculty publishing productivity. *Proceedings of the IATUL Conferences.* Paper 14. Available at: https://goo.gl/bGFmmp

[29] Jusoh, R., & Abidin, Z. Z. (2012). The teaching-research Nexus: A study on students' awareness, experiences and perceptions on research. *Procedia Social and Behavioral Sciences, 38*, 141-148. https://doi.org/10.1016/j.sbspro.2012.03.334

[30] Kane, R., Sandretto, S., & Heath, C. (2004). An investigation into excellent tertiary teaching: Emphasising reflective practice. *Higher Education, 47* (3), 283-310. https://doi.org/10.1023/b:high.0000016442.55338.24

[31] Kinash, S. (2015). 8 steps to improving learning and teaching through research. *Teaching the Bond Way, 3*, 1-6. Available at:

https://pure.bond.edu.au/ws/portalfiles/portal/27948077/Teaching_the_Bond_Way_8_steps_to_improving_learning_and_teaching_through_research.pdf

[32] Lerman, S. (1990). The role of research in the practice of mathematics education. *For the Learning of Mathematics, 10* (2), 25-27. Available at: https://flm-journal.org/Articles/2DFA8E98A93079AAF8B16D6462927.pdf

[33] Magidson, S. (2005). Building bridges within mathematics education: Teaching, research, and instructional design. *Journal of Mathematical Behavior, 24*, 135-169. https://doi.org/10.1016/j.jmathb.2005.03.004 Markides, C. (2007). In search of ambidextrous professors. *Academy of Management Journal*, *50*, 762-768. https://doi.org/10.5465/amj.2007.26279168

[34] Markides, C. (2007). In search of ambidextrous professors. *Academy of Management Journal*, *50*, 762-768.

[35] National Research Council. (2001). *Adding it up: Helping children learn mathematics*. J. Kilpatrick, J. Swafford, and B. Findell (Eds.). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press. Available at: https://alearningplace.com.au/wp-content/uploads/2016/09/Adding-It-Up_NAP.pdf

[36] Navarro, R. L., & Santos, R. G. (2011). *Research-based teaching and learning.* Quezon City, Philippines: Lorimar Publishing, Inc.

[37] NCTM Research Committee, 2011. *Trends and Issues in High School Mathematics: Research Insights and Needs*. Journal for Research in Mathematics Education, *42*(3), 204. Available at: https://doi.org/10.5951/jresematheduc.42.3.0204

[38] NCTM Research Committee. (2012). Linking mathematics education research and practice. *Journal for Research in Mathematics Education,* 1-2. Available at: https://www.nctm.org/uploadedFiles/Standards_and_Positions/Position_Statements/Linking%20Research%20and%20Practice.pdf

[39] NCTM Research Committee. (2016). High expectation in mathematics education. *Journal for Research in Mathematics Education.* Available at: https://www.nctm.org/uploadedFiles/Standards_and_Positions/Position_Statements/High%20Expectations%200816.pdf

[40] Neumann, R. (1992). Perceptions of the teaching-research nexus: A framework for analysis. *Higher Education, 23*, 159-171. https://doi.org/10.1007/bf00143643

[41] Nowell, L. S., Norris, J. M., White, D. E., & Moules, N. J. (2017). Thematic analysis: Striving to meet the trustworthiness criteria. *International journal of qualitative methods*, *16*(1), 1609406917733847.

[42] Oestar, J., & Marzo, C. (2022). Teachers as Researchers: Skills and Challenges in Action Research Making. *International Journal of Theory and Application in Elementary and Secondary School Education*, *4*(2), 95-104.

[43] Pawar, B.S. (2015). Enhancing research-teaching link in organizational behavioral: Illustration through an actual example. *The International Journal of Management Education, 13,* 326-336. https://doi.org/10.1016/j.ijme.2015.10.005

[44] Prince, M. J., Felder, R. M., & Brent, R. (2007). Does faculty research improve undergraduate teaching? An analysis of existing and potential synergies. *Journal of Engineering Education, 96* (4), 283-294. https://doi.org/10.1002/j.2168-9830.2007.tb00939.x

[45] Robles, M.M. (2016). (Ed). The relationship between academic research and instructional quality, rank. *Association for Business Communication 2016 Annual Conference Proceedings.*

[46] Megowan-Romanowicz, C. M. (2010). Inside Out: Action Research from the Teacher–Researcher Perspective. *Journal of Science Teacher Education, 21* (8), 993-1011. https://doi.org/10.1007/s10972-010-9214-z

[47] Ruthven, K. (2002). Linking researching with teaching: towards synergy of scholarly and craft knowledge. In L. English (Ed.), *Handbook of International Research in Mathematics Education* (581-598). New Jersey: Lawrence Erlbaum Associates.

[48] Seeley, C. L. (2005). *Using research to improve teaching.* Available at: https://www.nctm.org/uploadedFiles/News_and_Calendar/Messages_from_the_President/Archive/Cathy_Seeley/2005_04_usingresearch.pdf

[49] Segal, S. U. (2009). *Action research in mathematics education: A study of a master's program for teachers TEACHERS* (Dissertation). Available at: https://scholarworks.montana.edu/xmlui/bitstream/handle/1/2237/SegalS0509.pdf

[50] Silver, E. A. (2009). Some ideas on enhancing research productivity. *Int. J. Production Economics, 118*, 352-360. https://doi.org/10.1016/j.ijpe.2008.08.037

[51] Sleeter, C. (2014). Toward teacher education research that informs policy. *Educational Researcher, 43* (3), 146-153. https://doi.org/10.3102/0013189x14528752

[52] Thomas, R., & Harris, V. (2001). Exploring connections between teaching and research in hospitality management. *International Journal of Hospitality Management, 20*, 245-257. https://doi.org/10.1016/s0278-4319(01)00008-1

[53] Tupas, F. P. (2019). Close Mentoring Approach: Enhancing Action Research Skills and Knowledge of Science and Mathematics Teachers. *Journal of Educational and Social Research*, *9*(3), 107.

[54] Ulla, M. B. (2018). Benefits and challenges of doing research: Experiences from Philippine public school teachers. *Issues in Educational Research, 28* (3), 797-810. Available at: https://www.iier.org.au/iier28/ulla.pdf

[55] van den Akker, J. (2010). Building bridges: How research may improve curriculum policies and classroom practices. *Research and Development for Education Policy in Europe*, 175-195. Available at: https://ris.utwente.nl/ws/portalfiles/portal/5601607/Akker-building-YB+10++Beyond+Lisbon+2010-2.pdf

[56] Willis, S. (1995). Educators use 'action research' to improve practices. *Teachers as Researchers, 37* (3). Available at: http://www.ascd.org/publications/newsletters/education-update/mar95/vol37/num03/Teachers-As-Researchers.aspx

Students' Experience in Learning Trigonometry in High School Mathematics: A Phenomenological Study

Sandip Dhungana[1], Binod Prasad Pant[1] and Niroj Dahal[1]*

[1]Kathmandu University School of Education, Department of STEAM Education, Lalitpur, Nepal

*Corresponding Author: niroj@kusoed.edu.np

Abstract*: Trigonometry is an area of mathematics that students believe to be particularly difficult and/or abstract compared to other areas of mathematics. It is introduced as the concept in the right-angled triangle from the basic level, but the curriculum of Nepal introduced it from Grade nine onwards as a separate chapter. Its content area has a weighting of nearly one-fourth of the part in additional mathematics and around 10% in compulsory mathematics. This study aims to explore students' experiences in learning trigonometry. The data are extracted from twelve grade ten students who had chosen additional mathematics in their optional courses. Formal and informal interviews, diagnostic tests, and in-depth engagement with students in their classrooms were the major sources of data extraction. In this sense, this study adopted phenomenology as a methodological stance. The data collected from the diagnostic test were analyzed, and students' explanations of each question were discussed in this study. In doing so, this study concludes with some major findings. Students have difficulty learning trigonometry and have misconceptions about the basic concepts, producing obstacles and errors in solving trigonometric problems. The possible errors are in procedural knowledge, conceptual knowledge, or link between these two types of knowledge. It is also found that a teacher needs to incorporate the learners' everyday experiences using materials, diagrams, and equipment for meaningful learning and long-lasting knowledge. A teacher needs to be aware and responsible for students' activities inside the classroom. Healthy relationships between the teacher and the students can significantly contribute to learning trigonometry.*

Keywords: trigonometry, students' experience in trigonometry, engagement, long-lasting knowledge

INTRODUCTION

Upon reflecting on Sandip's journey in learning trigonometry, he used to belong in the category who wasn't likely found of trigonometry at the school level. Although being counted in the list of so-called good students in mathematics because of Sandip's high achieving marks/grades as compared with his classmates; trigonometry was always the most difficult area to learn. The abstract concepts, formulas, axioms, and theorems required in solving trigonometry were found straightforward in those days as we (Sandip including Sandip's friends) used to memorize most of the concepts easily and quickly. Sandip remembers his mathematics teacher who continuously repeats, "*Trigonometry is like a game of formulas unless you memorize everything in trigonometry, don't even dare to solve it*". Though as a learner in the Nepali context, where mathematics subject is considered a difficult subject (Luitel & Pant, 2019), Sandip was found to be solving problems in mathematics in those days, and normally he was much more comfortable in solving routine-based mathematics and trigonometry problems. Likewise, Sandip believes that "problem-solving ability is one of the soft skills that students should have in learning mathematics" (Nurmeidina, & Rafidiyah, 2019, p. 2). As the importance of mathematics cannot be separated from our real life, one cannot deny its importance in studying at the school level.

Next, school-level education (up to grade ten in Nepal) is considered mathematics as compulsory. The area of trigonometry as a chapter is introduced in grade nine in compulsory mathematics, though the concept of right-angled triangle and solving a right-angled triangle is also introduced in lower grades (e.g., grade seven and eight). In the case of secondary mathematics, "contents of trigonometry are taught in a compulsory mathematics course to some extent and in detail in the optional courses of mathematics" (Adhikari & Subedi, 2021, p. 90) in the Nepali school curriculum and is useful in solving mathematical problems related to height and distance, solving right-angled triangles, dealing with sine, cosine and tangent functions, measurement of angles in different measurements (degree, grade, and radian), trigonometric proof, among others.

Further, trigonometry has been considered a tough area for the majority of students; they struggle to understand the concept of trigonometry and its abstract nature (Gur, 2009; Tyata et al., 2021). The problems which Sandip had faced (as a first author) in learning trigonometry in school days were still there when Sandip started teaching at the same level. As a teacher, Sandip experienced that the problems were not only for the students but also for the teacher as they were also stuck in some points and steps in solving/proving trigonometric identities. Even we also experienced the same in our time at schooling on those days when we got trouble proving some problems in trigonometry and told students, *"I will solve it by tomorrow; please solve the next question",* and many times at the beginning of teaching career, Sandip solved the trigonometry questions first in the student's copy and later on the board. We found the problems in trigonometry which Sandip encountered as a student are not only the problems that Sandip encountered but the majority of the Nepali learners experienced in a similar pattern. For instance, many of our students also shared similar kinds of experiences in the earlier career of our teaching of mathematics it may be because of adopting a similar strategy to how trigonometry was taught, which can be considered as "teaching as a transmission of knowledge" (Pant, 2017, p. 16). At this stage, we found the delivery of knowledge in Nepali context from the teacher to his/her students is like transferring data from one electronic device to another (Dhungana, 2021). It will still be prolonged until teaching techniques won't improve.

Based on Sandip's experience, high school trigonometry is a problematic and detached area of mathematics from practical life in our context (Adhikari & Subedi, 2021, Asomah et al., 2023). Many teachers and educators might have tried different teaching methods and programs to familiarize students with trigonometry concepts, sometimes with success (Usman & Hussaini, 2017; Adhikari & Subedi, 2021, Kamber & Takaci, 2017). Arriving at this stage, this article explores secondary-level students' experiences in learning trigonometry. All of the above in responding to Sandip's experiences as a researcher, teacher, and student in mathematics education, we found most of the students in his context feel mathematics is a difficult subject. As a first author, Sandip, when involved himself in a deep investigation to find out the reason behind this difficulty, we found that many of the students feel difficulty in the chapters namely, geometry, algebra, trigonometry, and vector (to name but a few). While doing so, we also experienced most of the student's feeling difficulty in the geometry section (Adhikari & Subedi, 2021; Aminudin et al., 2019). Many of them had to repeat the same difficulty we faced as a student in our school life. They also have illusions about trigonometry's terminologies and get confused about the strategy for solving trigonometric problems (Tyata et al., 2021).

More so, trigonometry is considered the foundation of higher mathematics which connects peoples' day-to-day life with mathematics. The importance of trigonometry in the field of Arts and Engineering is unavoidable. Knowingly or unknowingly, the concepts of trigonometry are making people's lives easier, either in the form of carpentering or construction works or finding the height and distance without measuring the actual height. But the scenario of trigonometry learning is just the opposite of its application in our practices. Students are less motivated in learning trigonometry and are not able to show their conceptual understanding of it (Fauziyah et al., 2021, Tyata et al., 2021; Wilson et al., 2005). Orthodox pedagogies, the setting of classrooms, and the belief of teachers and students that grade points are everything could be a few examples that contribute to producing anxiety in trigonometry for students (Kamber & Takaci, 2017; Weber, 2005). In our experience, teaching and learning mathematics in Nepali schools is not as fruitful as expected. Teachers are teaching mathematics to fulfill their jobs and to complete the course context of the textbooks and the students are learning mathematics just to score marks and grades in the examinations. Believing in mathematics involves self-stabilization and seriousness, with subjective truths, and includes learning about oneself and the environment, influenced by internal and external factors (Kurniasih &Waluya, 2020). We think the teaching and learning strategies are less conceptual and less practice-oriented but more algorithm problem solving, which also raises a big problem in trigonometry learning and leads to less success in this portion of mathematics. As a result of this, achieving success in trigonometry is a nightmare for most students because of its abstractness and straightforward nature. Some problems in learning trigonometry in secondary classes might directly relate to the teachers' academic background, classroom practices, school management, and leadership. Similarly, other problems in learning trigonometry might concern the pre-knowledge of students, their learning environment, peers, and family background.

The beliefs indicating the poor performance of students in trigonometry (Gholami, 2022) underscore the importance of conducting this study. We, as a researcher, have gone through different studies like Aminudin et al. (2019), Arhin and Hokor (2021), Asomah et al. (2023), Gur (2009), Tyata et al. (2021), Mulwa (2015), Nurmeidina and Rafidiyah (2019), Usman and Hussaini (2017). In this situation, as researchers and teachers of mathematics, we found this topic relevant to study in our context. Guided by the research question--in what ways do secondary-level students learn trigonometry?, this study investigates the learning experience of grade ten students in trigonometry at the secondary level, including both the challenges and the joys they encountered. With this introduction, the paper covers methods, data analysis, interpretations, findings, and discussions.

METHODS

This section in this study gives a complete framework for this study. It also explains the design and the way through which we have designed this study. It includes a detailed description of how decisions have been made about the type of data and the data collection procedures. In this section, we have explained the method of this study with the research site and data analysis and interpretation procedure.

As a researcher in this study, we have collected the lived experiences of grade ten students in learning trigonometry. In doing this, Sandip sat together with the students in their class for two weeks, where he involved himself in extracting their real experiences of learning trigonometry. Twelve grade ten students were taken for the diagnostic test and selected from twenty-five students in the class. These twelve students are those who have taken additional mathematics as their optional courses from from grade nine. In Nepal, students have the options to choose optional subjects in grade nine, and the trigonometry section is found in additional mathematics. The selected twelve students were all the students who chose additional mathematics. Observing their copies, discussing with the students, and informal interviews were some possible strategies for data collection methods. In this sense, this study is a phenomenological study where we looked for every possibility of collecting the real experiences of the students in learning trigonometry. Phenomenology "aims to focus on people's perceptions of the world in which they live and what it means to them; a focus on people's lived experience" (Langdridge, 2007, p. 4). Likewise, Manen (1990) noted phenomenology as the appropriate method to explore the phenomena of pedagogical significance and elaborates phenomenology as a response to how one orient to lived experience and questions the way one experiences the world. In this regard, phenomenology as a research method is the best fit for this study.

This research project is qualitative. Qualitative research “aims to explore people’s perceptions and experiences of the world around them by synthesizing data from studies across a range of settings” (Ahmes et al., 2019, p. 2). Denzin and Lincoln (2000) argued that qualitative research is a situated activity that locates the observer in the world and consists of a set of interpretive, material practices that make the world visible. “Qualitative research often seeks to interpret, illuminate, illustrate, and explore meaning, context, unanticipated phenomena, processes, opinions, attitudes, actions, and to learn about people who are few or hard to reach” (Saini & Shlonsky, 2012, pp. 12-13). The argument is that a qualitative evidence synthesis provides an in-depth understanding of complex phenomena while focusing on the experiences and perceptions of research participants. In this context, we have found the qualitative manner in this study is the most appropriate research technique to find the real and expected results. This study is based on primary and secondary data sources. Students’ observations and the discussion with them are the primary data for us, and the literature from the pre-existing research is the secondary source of data. Semi-structured and unstructured questionnaires were used for the interview.

DATA ANALYSIS AND INTERPRETATIONS

This section in this study contains the analysis and interpretation of the collected data from the field which has addressed the need of the research question. Initially, after the orientation and purpose of the study, ten questions were handed out to a group of twelve students, who recorded their answers on loose sheets that were subsequently gathered. They were kept in such a way that one student in one bench and restricted from interaction during the question-solving time. Though the setting was like an examination, they were informed about the purpose and objectives of our study and were suggested to respond to the questions freely. After collecting all the worksheets, we sat individually with them to explore their experiences and assumptions on each particular problem of the question paper. In a few cases, we seat multiple times with them to conform to their ideas. In doing this, we asked about their understanding of the major concepts of that particular problem, the reason behind applying particular formula or concept, and why they were not able to solve a problem or partially solved it. The formal and informal interactions related to this study were collected in diaries, which we first coded and thematically analyzed. The diagnostic test paper was labeled from 1 to 12.

We sit together multiple times for the procedures of data analysis and interpretation. Finally, we have analyzed students' responses based on the particular problem. This section provides a depth understanding of the students on the particular problem. We have included all the student responses in each question. If they had not attempted the particular question, then it is mentioned as not an attempt. The examples presented in this study are errors committed by students, obstacles, and misconceptions that arose from secondary-level lessons in trigonometry.

FINDINGS

The present study is completely based on the student's responses to the diagnostic test paper and oral interviews. There are many errors, obstacles, and misconceptions in trigonometry, and these are given in this section.

Question 1: $sin^2A + cos^2A = 1$ Why? Explain in detail.

The list of students' writing of question 1 related to the "$sin^2A + cos^2A = 1$" was coded and presented below according to the various level of understanding. Students' justification is given below.

I. *Correct answer* (12 responses out of 12)
$(p/h)^2 + (b/h)^2 = (p^2 + b^2)/h^2 = h^2/h^2 = 1$ (If p, b, and h indicates the sides of right angled triangle where h is the longest side).

All students gave a mathematically valid explanation for why this equation was true. The alongside figure (fig.1) is a sample solution for this equation which is taken from one of the responses. Further, when we seat with them to

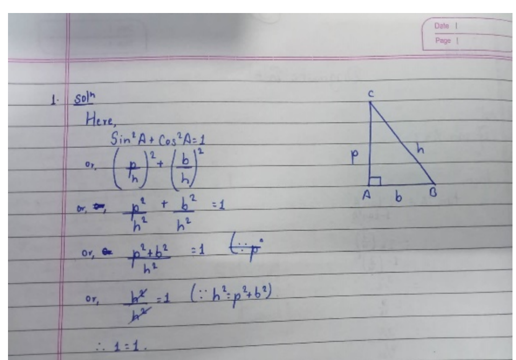


Figure 1: Sample solution for Q. No. 1

explore their understanding of it. Sandip asked them to give examples for this equation; 9 out of 12 were able to answer it, whereas 3 were stuck with their pen on their copy. All the examples were in a pattern like $sin^2 30^o + cos^2 30^o = \ldots = 1$. Even Sandip further asked each of them to prove this equation in any alternative ways. Five of them added $sin^2A + cos^2A = sin^2A + (1 - sin^2A) = 1$. Three of them, out of the remaining, accepted that they had also attempted this question in alternative ways but could not remember now. The remaining students replied that this is the ultimate way for them and do not have any ideas for alternative ways of solutions. The scenario shows that all were able to answer the stepwise solutions, but only a few of them were able to explain it in detail with examples and with alternative ways.

Question 2: $tanA = \frac{1}{cotA}$ or $tanA.cotA = 1$. Explain in detail.

The list of students' writing of question 2 related to the "$tanA.cotA = 1$" was coded and presented below according to the various level of understanding. Students' justification for this question is given below.

I. *Correct answer (7 responses out of 12)*

The students' written responses show that seven students had given the correct answers. Among them, five had used the concept of the Pythagorean ratios: $tanA.cotA = (p/b) \times (b/p) = 1$, one had respond $tanA.cotA = tanA \times (1/tanA) = 1$ and one had respond $(tanA.cotA = (sinA/cosA) \times (cosA/sinA) = 1)$. In addition, one student explains like this after the solution:"

"*cot A is in the form of divide (in this tanA=1/cotA) so if it is taken to another side it transforms into the multiplication (in this tanA.cotA=1)*"

Further, when we sat with them to know how they understood this concept, most of them replied that the product of two opposite trigonometric ratios like sine and cosecant, cosine and secant, tangent and tangent and cotangent, is always one. We found one interesting response; a boy from these seven only responds like "sirs, it is universal truth" then again, Sandip asked him why he is telling so he responded similar type of answer like others. They said that they have memorized this concept in the earlier days when they started learning trigonometry.

II. *Incorrect* (1 response out of 12)

One boy among all had responded unacceptable solution (See fig. 2 alongside) to the given idea. He performed the addition operation where he needed to use the product rule: $(p/b) \times (b/p) = (p^2b^2/pb) = \ldots$
But when Sandip asked him why he has done like this, he was surprised with his answer and said "*sir, mistakenly happened*". Then Sandip asked him to solve it on an additional page then he was able to solve it. From this, we found that students get panic during tests/exams and such types of settings. He accepted that he was hurried to finish all the questions and didn't revisit at last as he was confident with his answer then.

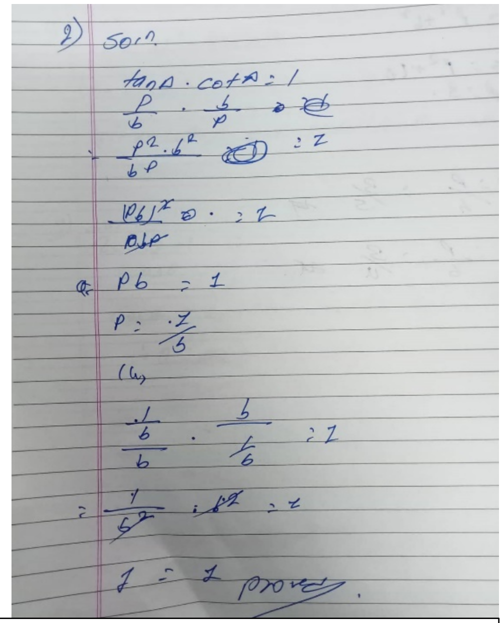


Figure 2: Sample solution for incorrect response

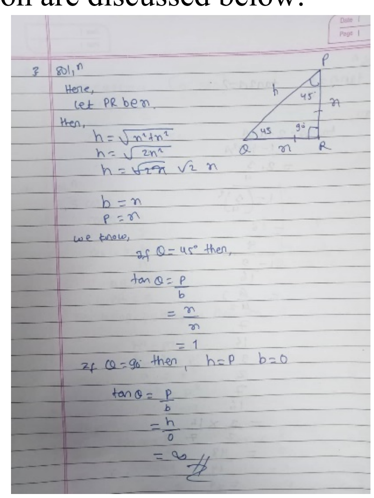
III. *No attempt (2 respondents out of 12)*

Two students out of twelve had not attempted this question. Though they had not responded to the question, Sandip tried to convince them that they couldn't solve it or there could be any other reason behind it. Both of them were able to answer correctly when Sandip provoked them by giving other examples like *sinA×cosecA=1,* and both of them accepted that they had forgotten this during the test time and also accepted that they had used these concepts many times in trigonometry lessons.

Question 3: $tan\ 45 = 1\ and\ tan\ 90$ is undefined. Explain in detail.

Common errors, obstacles, and misconceptions that students make with the above question are highlighted. Students' justifications and responses to this question are discussed below:

I. *Correct answer* (6 responses out of 12)



In responding to this question, six students answered correctly (See the sample solution in fig. 3 alongside), where all of them had used the concept of a right-angled isosceles triangle. All of them had drawn pictures with a reference angle of 45°.
tan45=p/b=p/p=1 where p and b represent the adjacent equal sides of a right-angled isosceles triangle.
In responding to the second part, four students used the concept of shortening the base to make the hypotenuse equal to perpendicular in a right-angled triangle. Then,
tan90=p/0=Undefined. Among the two of them had written the conclusion as well, which is like:

"The value of tan90 is p/0. If something is divided by zero then it is undefined."

Figure 3: Sample solution for question 3

The other two correct responses were:
Tan90=sin90/cos90=1/0=undefined (Any number divided by zero is undefined)
The interaction with these students also gives a clear picture of what they have presented in their solution.

II. *Partially correct* (1 response out 12)

One student among them responded: "*Here, the actual value of tan90 is 1/0 and if you divide anything y zero then it is undefined. So, the value of tan90 is undefined.*"
He just answered this for the above question and was not able to answer anything for *tan45=1*.

III. *No attempt* (5 respondents out of 12)

Five students among them had not responded to this question. When Sandip asked the reason behind this two of them replied that they had forgotten, whereas the remaining told him that they might not have heard the reasons behind those expressions. However, all of them accepted that they had applied those concepts in solving trigonometric expressions. Sandip interaction with those questions also shows that students have something for particular concepts, but they are not able to express it in systematic form.

Question 4: Simplify: $(\boldsymbol{sin^2 A})^2 - (\boldsymbol{cos^2 A})^2$ =?

The information collected for question four from the students' responses is presented below.

I. *Correct answer* (9 responses out of 12)

In responding to this question, nine students respond correctly where $(sin^2A)^2 - (cos^2A)^2$ = $(sin^2A+cos^2A)\ (sin^2A-cos^2A) = 1.\ (sin^2A-cos^2A) = (sin^2A-cos^2A)$ was the common method. Five among these nine ends with this answer, whereas the remaining four reached up to $(sin^2A-cos^2A)=(sinA+cosB)(sinA-cosB)$. See fig. 4 for a sample solution from one of the students.

II. *Incorrect* (3 responses among 12)

Three responses from the participants ends with this solution: $(sin^2A+cos^2A)\ (sin^2A-cos^2A) = (sinA+cosB)(sinA-cosB).(sin^2A-cos^2A)$

The above scenario shows how students have performed in their paper tests, but when we interacted with them, the scene

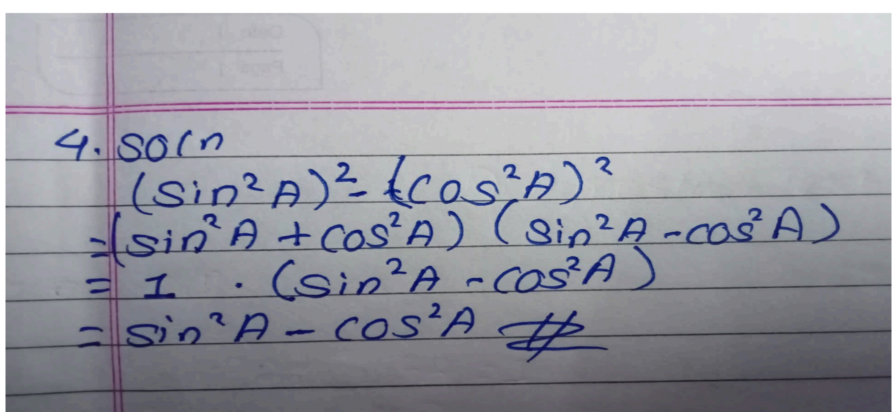


Figure 4: Sample solution for question 4

was different. One girl from these three who attempt incorrect answers was found clear with the concept and solved the question again in front of us without any hesitation and answered all the questions Sandip asked but the remaining two were found confused there as well. From this, we found that some students can not attempt the correct solutions in the examinations though they know it. Two among the nine who attempted the correct answer were not able to explain their solution and the remaining were found clear with their solution to this question.

Question 5: $tanA = \frac{3}{4}$ then $tan2A$ =?

Common errors, obstacles, and misconceptions that students make with the proof "$tanA = \frac{3}{4}$ *then* $tan2A$" equations are highlighted. Students' responses to this question are discussed below:

I. *Correct answers* (9 responses out of 12)

Nine responses were found with correct answers and the correct process for solving the above question (Question 5). They used the formula $tan2A=2tanA/1\text{-}tan^2A$ and then inserted the value of *tanA,* which gives the output of 24/7. See fig. 5 for a sample solution.

II. *Incorrect answer (2 responses out of 12)*

Two students were found to have the incorrect answer to the solution to the above question (Question 5). However, they have inserted the correct formula and correct values for tanA. Both of them attempt mistakes in subtracting during the calculations.

III. *No attempt* (1 response out of 12)

One student was found not attempting this questio[n] … test paper. In the interview round, when Sandip aske[d] … replied: "Sir, I forgot the formula".

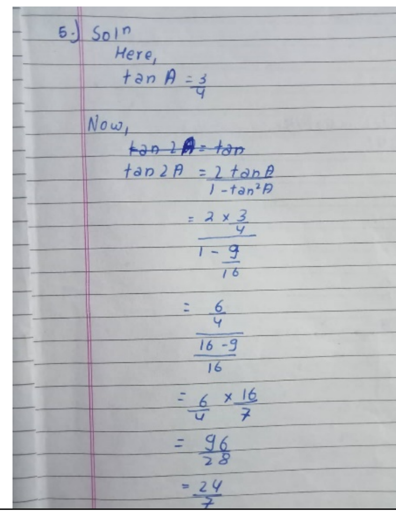


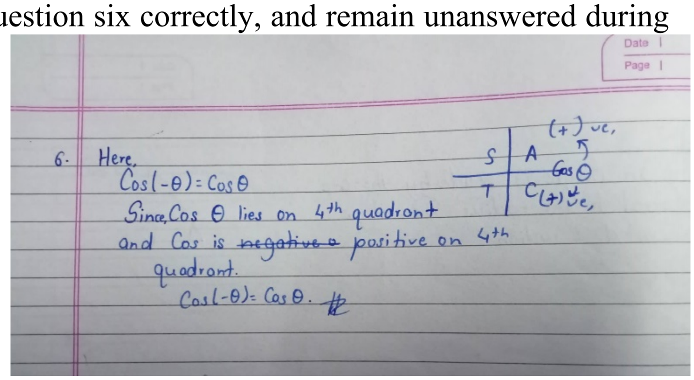
Figure 5: Sample solution for question

Question 6: $\cos(-\theta) = cos\theta$. Why?

I. *Correct answers* (0 responses out of 12)

None of the students could explain question six correctly, and remain unanswered during the interview also.

"Sir, I know this concept, and I have applied it many times, but I don't know its explanation", one of the students said on the interview. See fig. 6 for a sample solution.



II. *Incorrect answer* (4 responses out of 12)

Four students responded to this question but were not able to justify the question. Among them, three had responded like "*cos is positive in the fourth quadrant, so* $cos(-\theta) = cos\theta$" which is a partial understanding of this question. During the interaction also, they were not able to reply more than this.

III. *No response* (8 responses out of 12)

Eight students were found unanswered with question six. During the discussion and interview time, they replied: "Sir, no idea!" In addition, they added that they had applied this idea to solving/proving trigonometric concepts, this idea is not new to them, but they don't know how $\cos(-\theta) = cos\theta$?

Figure 6: Sample solution for question 6

Question 7: Can you identify the figure below? What is the y-intercept of this function?

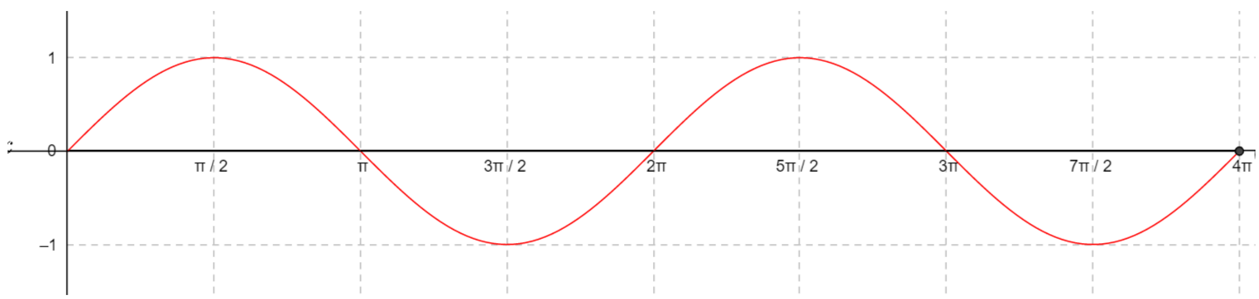


Figure 7: Sample picture of trigonometric curve

I. *Correct answer* (7 responses out of 12)

"*The curve represents the sine curve, and its y-intercept is 0*" (Student 10 replied in the interview). This is a student's response coded as 10 in his copy. Each of these seven students responded to those questions in their own way and language and was able to explain in the interaction as well. This shows their clear concept of trigonometric curves. See fig. 7.

II. *Incorrect answer* (3 responses out of 12)

One student among these three responded with an incorrect answer, whereas the remaining two responses were found with partial understanding. Among the two, one mentioned the correct name of the curve and a mistake in the y-intercept, whereas the next was vice-versa.

III. *No response* (2 responses out of 12)

Two students were found unanswered with this question. During the interview, they were also found confused with the sine and cosine graph. However, one of these two was able to tell the y-intercept but has not mentioned it in his copy. He replied that he was anxious while responding to these questions and forgot to mention this idea.

Question 8: In which quadrant lies the angle 480□?

The responses of the students the question 8 are discussed below:

I. *Correct responses* (8 responses out of 12)

It lies in the second quadrant because *480 = (6×90-60),* and counting from the first quadrant as 1, it only reaches the second quadrant at six and minus 60o lie there. One of the students explained her solution this way when Sandip requested that she explain question eight. Among these eight students, three were found without explanation in their copies but could explain during the interview.

II. *No responses* (4 responses out of 12)

Four students were found without a solution to this question. During the explanation, one could answer after counting the graph on the side of his copy and telling the second quadrant, but he was unsure about his answer. The remaining three remained silent and just said, *"I don't know sir", "I forgot sir".*

Question 9: Solve $2cos\theta + 1 = 0$ over the interval [0^0-360^0).

The list of students' responses related to "*Solve* $2cos\theta + 1 = 0$ *over the interval [0; 360)"* was coded and discussed below:

I. *Correct answer* (2 responses out of 12)

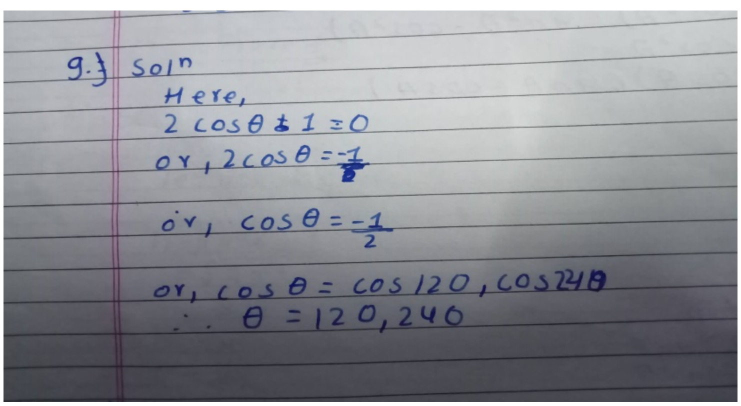


Figure 8: Sample solution for question 9

$2cos\theta+1=0$, $cos\theta=-1/2$,

Cosine is negative in the second and the third quadrant so we need to look for the possible values in the second and third quadrants. Thus, the correct answer is 120^o and 240^o. These two students ended up with a correct answer and could explain it during the interview. See fig. 8 for a sample solution.

II. *Partial Understanding* (8 responses out of 12)

This level of understanding contains the majority of the students' responses to this question, where they had forgotten to look at the given interval where they need to calculate the value of θ. They have resulted in 120^o, which is the correct answer but failed to see in the third quadrant (as cosine is negative in the second and third quadrants). During the discussion majority of them accepted that they had forgotten to see the interval; otherwise, the concept was familiar to them.

III. *Incorrect responses* (2 responses out of 12)

Out of the twelve responses, two are found incorrect as the value θ was not calculated in the right quadrant. The interaction with these students also shows their misconceptions regarding such ratios. They are confused with the negative and positive signs of the values, which they need to compare. Though they reached up to *-1/2* in the calculation but ended with 60^o and $(360-60)=300^o$ as the answer, which is not true.

Question 10: If 5 cosθ = 4, find the trigonometric ratios sin θ and tan θ.

The list of students' writing of the question "*If 5cosθ = 4, find the trigonometric ratios sin θ and tanθ"* was coded and discussed below under the different levels of understanding.

I. *Correct answer* (12 responses out of 12)

If 5cosθ = 4 then cosθ=4/5=b/h (b=base and h=hypotenuse)⟹b=4k and h=5k then perpendicular (p) = 3k (by using Pythagoras Theorem). Hence, sinθ=p/h=3k/5k=3/5 and tanθ=p/b=3/4. See fig. 9 for a sample solution.

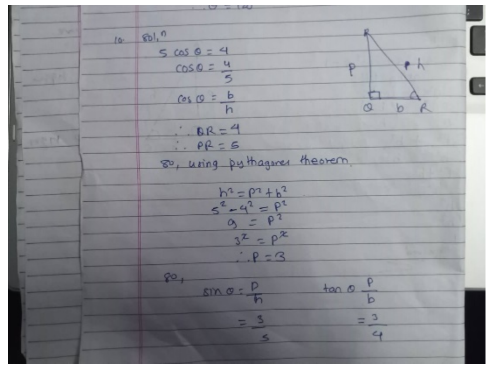


Figure 9: Sample solution for question

This was the common method in those correct responses. Ten students out of twelve had got the correct answer in this way whereas remain two had done like:

$sin\theta = \sqrt{1 - cos^2\theta}$ =3/5 and tanθ=$sin\theta/cos\theta$

During the interaction, all the students were able to explain their solution to this question and found a clear concept with such ratios in trigonometry.

DISCUSSION ON THE STUDENTS' EXPERIENCES

In serving the purpose and the need of the research question of this study, we have presented the findings of this study in the above section in descriptive form. The result of this study shows that students are having difficulty learning trigonometry. From the diagnostics test, we found that students have misconceptions about basic concepts and are even unable to explain what they had written in their copies. The formal and informal interactions with the students show that they are less confident with what they learned. They know the concepts like *$sin^2A+cos^2A=1$, $cos(-\theta)=cos\theta$,* but they are not able to answer how these concepts are derived. In analyzing the result section, we found that the students had problems finding the connections between the idea they learned and the context of the problem. They even share that they are not able to apply the right formula in the correct place though they know it. They know the chunk of formulas, but the connection among them is unclear for any of them. Do they know that *$cosec^2A-cot^2A=1$* but are not able to answer *$cosec^2A-1$=*? We found that there are considerable research of such kind where students' difficulties and errors are explored, and even though a teacher of each scenario might be aware of the difficulty and possible errors made by the students in their classroom but "students have not been encouraged to take advantage of errors as learning opportunities in mathematics instruction" (Gray & Tall, 1994, p. 166). It is also found that students' errors are the symptoms of misunderstanding (Lai, 2006). The causes of systematic errors may relate to students' procedural knowledge, conceptual knowledge, or links between these two types of knowledge (Manandhar et al., 2022). Errors can be mistakes, blunders, miscalculations, or misjudgments; such a category falls under systematic errors (Muzangwa & Chifamba, 2012). Even our discussion with the students and their solutions reveals different types of errors (reading error, comprehension error, transformation error, process skills error, and encoding error), as argued by Newman (1977). A teacher can significantly minimize such errors and familiarize students with trigonometric concepts.

In addition, some students during the interactions replied like; "*sir, we have learned it that I can remember but I cannot answer it*", "*sir has taught this to us but I forgot*", and "*I had attempted this question in last class/exam but now I forget it*". Such replies from the students showed us that the students cannot understand the concepts clearly; they just memorize them for the exam or classroom purpose and forget after that. Their learning might be limited at the cognitive level and produces lower-order thinking skills. At this moment, we remember a research Luitel (2019) where he concluded that "how people explain the nature of mathematics explains how they understand it" (p. 7). To minimize such scenarios, the teacher can engage students in activities and projects so that they can learn from their own experiences and produces higher-order thinking skills. I believe that engaging students in so-called progressive ideas in teaching and learning like collaborative learning, group work, and pair work can also give lifelong learning and understanding to the learners. Three students from this group were found less motivated toward trigonometry learning. They told us in an informal discussion, "*Sir, I never understand these trigonometric concepts and felt difficulty in solving even easy problems which make me uncomforted in front of my teacher and friends*" they even added that they can't ask about their difficulty with the teacher thinking that "*if the teacher asks any other simple idea, then we don't know even that*". Further, they added that the teacher offered the general rules, ideas, and concepts in any topic which the high-achievers can easily grab, and they continued solving the exercises. "*Though our teacher makes groups in our class where we low-achievers are kept with high-achievers but instead they guide us, they give their copies, and we copy from there*". As for the disadvantages of pair work and group work, we found such a scenario in this group of students. If a teacher is less active and does not give attention to the students' jobs and activities they are doing within themselves, it also creates learning problems. We found such students when we were with them in their classes; a student has a good mathematical understanding but needs to poke by the teacher each time for the learning. After listening to the students' experiences, We found that a teacher needs to be aware of the student's activities and figure out their individual learning styles. We found that engagement in classroom activities also might not be enough all the time, but effective engagement is required (Lamichhane & Dahal, 2021). The teacher needs to be aware of such scenarios and need to make students motivated and responsible for their learning.

Another area of weakness that students revealed during the interaction with them when we asked the reason behind their errors and misconception about trigonometry is the teacher-student relationship (Dahal, 2013; Dahal et al., 2019). The teacher's strict nature and appearance in the classroom prevent them from asking their queries. "*Though our math sir seems friendly and encouraging teacher in the classroom, his appearance and his voice vibrate my heart, I cannot even speak with him in general topics; mathematics is already tough for me*". Such statements from the students make us stubborn for some time. We remembered our school days when the scenario was similar. These students' experiences are not only those of the students from one particular institution but of most of the schools in Nepal. There is huge room for changes in teaching-learning activities in mathematics and other subjects, but in our opinion, teacher-student relations should be the first one. The research like Dahal (2020) concluded that "teacher-student bonding has to be considered a vibrant feature in the school hall of both remote and urban institutions in Nepal where Nepalese mathematics teachers have not been able to link bonding of teachers and students not as such" (p. 72). But at the same time, it should be considered that "the sociocultural viewpoint of both the teacher and students respectively influence the social and cultural interactions in a mathematics classroom" (Dahal et al., 2019, p. 119). Lack of healthy relations and good rapport between the teacher and students create communication gaps, and students cannot share their difficulties with the teacher/facilitator. The interaction with each student also results in the same issue, and all the students want their teacher to be frank in case of teaching-learning activities. Classroom communication in mathematics teaching and learning is thus an essential factor.

CONCLUSIONS AND IMPLICATIONS

This qualitative research paper focuses on Nepali students' challenges, hindrances, and misunderstandings while solving and/or studying trigonometry concepts. The impetus for this paper stems from the challenges teachers face in elucidating trigonometry contents and the difficulties students encounter in comprehending the abstract nature of trigonometric concepts. The above discussion concludes that students are facing difficulty in learning trigonometry. Errors, obstacles, and misconceptions are the significant domains for many students in trigonometry. They even have misconceptions about the basic concepts, leading to errors in learning/solving trigonometric problems. The possible errors might be in procedural knowledge, conceptual knowledge, or a link between these two types of knowledge. It is also found that students may feel comfortable learning trigonometry with materials, diagrams, and equipment. The connection of abstract context with the learners' context might be another possibility that students shared in the interview. This study highlights the significance of phenomenology in mathematics education. We as authors have developed guidelines based on their own teaching experiments, which can be utilized by other instructors when teaching trigonometry independently. The author, like Usman and Hussaini (2017), revealed that Africa's context seems almost similar to the Nepali context, i.e., students, regardless of their varying cognitive abilities, are prone to making errors when solving trigonometry problems. Unlike, Usman and Hussaini (2017) quantitative study, this qualitative study explored the errors that occurred both in conceptual and procedural understanding. A study like Rosjanuardi and Jupri (2022) also highlights the procedural errors in trigonometry and trigonometric functions, whereas this study concludes that both conceptual and procedural aspects are equally responsible.

This study also uncovers the limitation of collaborative learning in mathematics classrooms. Students are copying and engaging in unnecessary activities during the group/paired work. The study demonstrates that simply encouraging group or paired work among students may not always be sufficient and that regular supervision and periodic monitoring are also necessary. We also realized that rapport among the teachers and students is essential while teaching mathematics in general and trigonometry in particular. Teacher-student relationships can be another factor that contributes to learning difficulty in trigonometry. It is found that a healthy relationship between teacher/facilitator and students can create a harmonious and favorable environment for learning. Such an environment motivates students in the learning process and promotes meaningful learning.

LIMITATIONS AND FUTURE RESEARCH

This study was limited to twelve grade ten students as participants based on ten trigonometry questions from a single private school in Kathmandu, Nepal. Further research can be conducted in schools with many participants from private and public schools.

REFERENCES

[1] Adhikari, T. N., & Subedi, A. (2021). Difficulties of grade X students in learning Trigonometry. *Siddhajyoti Interdisciplinary Journal*, *2*(1), 90-99.

[2] Ames, H., Glenton, C., & Lewin, S. (2019). Purposive sampling in a qualitative evidence synthesis: A worked example from a synthesis on parental perceptions of vaccination communication. *BMC Medical Research Methodology*, *19*(1), 1-9.

[3] Aminudin, M., Nusantara, T., Parta, I. N., Rahardjo, S., & As'Ari, A. R. (2019). Engaging problems on trigonometry: Why were student hard to think critically? In *Journal of Physics: Conference Series* (Vol. 1188, No. 1, p. 012038). IOP Publishing.

[4] Arhin, J., & Hokor, E. (2021). Analysis of high school students' errors in solving trigonometry problems. *Journal of Mathematics and Science Teacher*, *1*(1).

[5] Asomah, R. K., Agyei, D. D., & Ntow, F. D. (2023). Developing in-service mathematics teachers' pedagogical content knowledge and skills to teach trigonometry: Using cooperative teaching and learning approach. *Contemporary Mathematics and Science Education, 4*(1), ep23001.

[6] Dahal, N. (2013). *Teacher-students relationship and its potential impact on mathematics learning: An autoethnographic inquiry* [Unpublished Master Dissertation]. Kathmandu University School of Education, Dhulikhel, Nepal.

[7] Dahal, N., Luitel, B. C., & Pant, B. P. (2019). Teacher-students relationship and its potential impact on mathematics learning. *Mathematics Education Forum Chitwan*, *4*(4), 35–53. https://doi.org/10.3126/mefc.v4i4.26357

[8] Dahal, N. (2020). Teacher-students bonding in mathematics learning: An exploration of past, present and future. *AU eJournal of Interdisciplinary Research, 5*(1), 63-76.

[9] Dahal, N., Luitel, B. C., & Pant, B. P. (2019). Understanding the use of questioning by mathematics teachers: A revelation. *International Journal of Innovative, Creativity and Change*, *5*(1), 118-146.

[10] Denzin, N. K., & Lincoln, Y. S. (2008). Introduction: The discipline and practice of qualitative research. In N. K. Denzin & Y. S. Lincoln (Eds.), *Strategies of qualitative inquiry* (pp. 1–43). Sage Publications, Inc.

[11] Dhungana, S. (2021). *Problems in learning geometric theorems in secondary schools: A mixed method study* [Unpublished master's dissertation]. Kathmandu University, Dhulikhel, Nepal

[12] Fauziyah, N., Asari, S., Ma'Rifah, U., Uchtiawati, S., & Husniati, A. (2021, February). Improving Students' Creativity through Sharing and Jumping Task in Mathematics Lesson Study Activity. *In Journal of Physics: Conference Series* (Vol. 1764, No. 1, p. 012100). IOP Publishing.

[13] Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A perceptualal" view of simple arithmetic. *Journal for Research in Mathematics Education*, *25*(2), 116-140.

[14] Gholami, H. (2022). Improving the performance of mathematics teachers through preparing a research lesson on the topic of maximum and minimum values of a trigonometric function. *Mathematics Teaching Research Journal*, *14*(2), 26-43.

[15] Gur, H. (2009). Trigonometry Learning. *New Horizons in Education*, *57*(1), 67-80.

[16] Kamber, D., & Takaci, D. (2017). On problematic aspects in learning trigonometry. *International Journal of Mathematical Education in Science and Technology, 5211*(August), 1– 15. https://doi.org/10.1080/0020739X.2017.1357846

[17] Kurniasih, M. D., & Waluya, S. B. (2020, April). The analysis of learning obstacle of mathematics belief. In *Journal of Physics: Conference Series* (Vol. 1511, No. 1, p. 012106). IOP Publishing.

[18] Lai, C.-F. (2006). Error analysis in mathematics. *Behavioral Research and Teaching*.

[19] Lamichhane, B. R., & Dahal, N. (2021). Engaged mathematics learning as/for a transformative praxis. *Mathematics Education Forum Chitwan*, *6*(6), 1–7. https://doi.org/10.3126/mefc.v6i6.42395

[20] Langdridge, D. (2007). *Phenomenological psychology: Theory, research and method*. Pearson.

[21] Luitel, L. (2019). Nature of mathematics and pedagogical practices. A conference paper. Proceedings of the *Tenth International Mathematics Education and Society Conference Hyderabad*, India, 2019

[22] Luitel, L., & Pant, B. P. (2019). Images of the mathematics curriculum held by school mathematics teachers: Mapping the road for transformative pedagogies. In *Empowering science and mathematics for global competitiveness* (pp. 510-516). CRC Press.

[23] Manandhar, N. K., Pant, B. P., & Dawadi, S. D. (2022). Conceptual and procedural knowledge of students of Nepal in algebra: A mixed method study. *Contemporary Mathematics and Science Education, 3*(1), ep22005. https://doi.org/10.30935/conmaths/11723

[24] Mulwa, E. C. (2015). Difficulties encountered by students in the learning and usage of mathematical terminology: A critical literature review. *Journal of Education and Practice*, *6*(13), 27-37.

[25] Muzangwa, J., & Chifamba, P. (2012). Analysis of errors and misconceptions in the learning of calculus by undergraduate students. *Acta Didactica Napocensia*, *5*(2), 1-10.

[26] Newman, A. M. (1977). Analysis of sixth-grade pupils' errors on written mathematical tasks. *Victorian: Institute for Educational Research Bulletin*, 31-43.

[27] Nurmeidina, R., & Rafidiyah, D. (2019, August). Analysis of Students' Difficulties in Solving Trigonometry Problems. In *PROFUNEDU 2019: Proceedings of the 4th Progressive and Fun Education International Conference, Profaned 2019, 6-8 August 2019, Makassar, Indonesia* (p. 9). European Alliance for Innovation.

[28] Pant, B. P. (2017). Doing, teaching, learning and thinking about mathematics–On becoming a transformative teacher. *Journal of Education and Research*, *7*(1), 11-24.

[29] Rosjanuardi, R., & Jupri, A. (2022). Epistemological obstacle in learning trigonometry. *Mathematics Teaching Research Journal, 14*(2), 5-25.

[30] Saini, M., & Shlonsky, A. (2012). *Systematic Synthesis of Qualitative Research*. OUP USA.

[31] Tyata, R. K., Dahal, N., Pant, B. P., & Luitel, B. C. (2021). Exploring project-based teaching for engaging students' mathematical learning. *Mathematics Education Forum Chitwan*, *6*(6), 30–49. https://doi.org/10.3126/mefc.v6i6.42398

[32] Usman, M. H., & Hussaini, M. M. (2017). Analysis of students' error in learning of trigonometry among senior secondary school students in Zaria Metropolis, Nigeria. *IOSR Journal of Mathematics*, *13*(2), 1-4.

[33] Van Manen, M. (1990). Beyond assumptions: Shifting the limits of action research. *Theory into Practice, 29*(3), 152-157.

[34] Weber, K. (2005). Students' understanding of trigonometric functions. *Mathematics Education Research Journal, 17*(3), 91-112.

[35] Wilson, P. S., Cooney, T. J., & Stinson, D. W. (2005). What constitutes good mathematics teaching and how it develops: Nine high school teachers' perspectives. *Journal of Mathematics Teacher Education, 8*, 83-111.

Teaching the Mathematical Optimization Concept to First-year Engineering Students Using a Practical Problem

Hosseinali Gholami[1], Nur Azam Abdullah[1], Adib Hamdani[1]

[1]Department of Mechanical Engineering, International Islamic University Malaysia, Kuala Lumpur, Malaysia

hossein_gholami52@yahoo.com

Abstract: Optimization is a vital mathematical concept widely applied in various engineering fields. However, teaching this subject to engineering students often involves abstract materials, making it challenging for them to grasp. To address this issue, a group of five mathematics lecturers collaborated on designing a practical lesson on optimization for first-year engineering students. The aim was to present the concept through a real-world problem, making it more accessible and applicable. The lesson focused on the design of water channels used for transferring water from sources to farms, showcasing how engineers can create optimal water channels with different geometrical cross-sections for agricultural purposes. This approach allowed students to see the direct application of mathematics in the real-world. The lesson was conducted as a three-hour workshop and attended by 38 volunteer first-year engineering students at a Malaysian university. Data were collected through observations and interviews and analyzed using the thematic analysis method. Feedback from both the lecturers and the participating students indicated that this teaching method significantly enhanced their conceptual understanding of mathematical optimization. Moreover, it fostered the development of problem-solving skills in real-world scenarios, bridging the gap between engineering and mathematics.

Keywords: Optimization, Optimal water channel, Geometrical cross-section, Mathematics

1. INTRODUCTION

Mathematics holds significant importance in various engineering fields, and engineering students must possess the necessary skills to solve mathematical problems effectively. In the realm of mathematics, tasks that present challenges and are unfamiliar to students, with unknown methods of solving, are referred to as mathematical problems (Xenofontos & Andrews, 2014). Otherwise, if the tasks are straightforward and already known methods can be applied to solve them, they are simply considered as mathematical exercises (Gholami, 2021). In engineering fields, any real-world problem solved using mathematical concepts is considered a practical problem (Gholami & Sathar, 2021; Savizi, 2007). In other words, practical problems exemplify the real-life application of mathematical concepts. For instance, the given problem provided below is a practical problem (Fatmanissa et al., 2019).

> *A rectangular thin metal sheet measures 140 cm in length and 120 cm in width. Its corners are cut in the form of identical squares. The remaining sheet is then folded to form a box without a top. Determine the maximum volume of the created box.*

Therefore, once engineering students fully comprehend the practical problem, they should aim to find an optimal solution for the given scenario. By seeking optimal solutions to real-world practical problems, they prepare themselves for addressing similar challenges in the future. Engaging in discussions about small real-world projects within the classroom setting helps engineering students develop problem-solving skills that are directly applicable to real-life situations. This preparation equips them to tackle human life's problems effectively in their future careers. In engineering fields, lecturers often resort to teaching mathematical concepts using abstract materials. Unfortunately, this approach sometimes fails to establish a clear connection between real-world problems in engineering and the corresponding mathematical concepts. According to a study conducted by Harris, engineering students who encounter challenges with mathematics tend to face difficulties in understanding the connection between engineering and mathematics (Harris et al., 2015). The main aim in learning engineering mathematics is not only to practice mathematical skills but also to develop mathematical thinking for solving real-world problems (Szabo et al., 2020). Lecturers should enhance students' ability to solve practical engineering problems by emphasizing the importance of mathematical modeling, a powerful tool for addressing real-world challenges in engineering (Pepin et al., 2021). Hence, incorporating practical problems and discussing their optimal solutions in engineering classrooms is essential and unavoidable.

Optimization theory finds wide application in engineering fields, providing solutions to numerous practical problems, as engineers can employ various methods to find mathematical optimization functions for solving diverse challenges (Tsai et al., 2014). Engineering optimization forms a crucial aspect of mathematical modeling, where optimization techniques are employed to tackle specific real-world challenges in engineering disciplines (Vagaska et al., 2022).The term "optimization" shares its root with "optimal," implying the quest for the best possible outcome. Mathematical optimization, a branch of applied mathematics, finds applications in various fields like mechanical engineering, marketing, manufacturing, and economics. To enhance students' proficiency in solving real-world optimization problems, calculus serves as a fundamental course (Retamoso, 2022). However, optimization theory is extensively utilized in engineering disciplines to address a wide range of practical challenges.

Engineering optimization involves the process of optimizing (maximizing or minimizing) a modeled mathematical function, which serves as the objective function, considering a set of constraints and input parameters (Alshqaq et al., 2022). The primary objective of engineering optimization is to identify the most suitable and feasible solution for a given practical problem, taking into account the problem's assumptions, in order to save both time and money (Dastan et al., 2022). The present study aims to introduce the concept of optimization to first-year engineering students through a practical problem related to the construction of open metal or concrete channels for agricultural purposes. This problem will serve as an illustrative example to enhance their understanding of optimization principles and their applications in real-world scenarios.

1.1. Theoretical framework

Utilizing constructivist principles in mathematics teaching for engineering students presents a significant opportunity for educators to delve deeply into lesson planning, emphasizing real-world problem-solving. According to Hoover (1996), adopting a social constructivist approach to teaching mathematics enables students to actively construct mathematical knowledge rather than passively receiving it from lecturers. As Hoover suggests, educators are encouraged to involve students in appropriate mathematical problem-solving activities, enhancing their teaching methods. Social constructivism further advocates for the collaborative nature of mathematical knowledge, emphasizing its improvement through interactions and sharing between lecturers and students (Gergen, 1995). In essence, mathematical skills are fostered through group interactions involving negotiations, reflections, discourse, and explanations. Therefore, lessons prepared collaboratively by a group of lecturers are supported by a robust theoretical foundation.

2. METHODOLOGY

In this study, five lecturers, including the first author, possessing a minimum of 9 years of experience in teaching mathematics to engineering students, participated in a collaborative effort to develop a lesson on the topic of optimization concepts. Drawing from their collective expertise and real-world applications of mathematics, they engaged in three extensive discussion sessions to plan and design the lesson. Ultimately, they chose a practical problem from the agricultural sector as the focal point for the lesson, despite initially considering various other practical problems during the first meeting. To ensure the quality and validity of the prepared lesson, it underwent content analysis and validation by two external professors not involved in the research. Upon incorporating their feedback, the refined lesson was conducted as a scientific workshop for 38 enthusiastic first-year engineering students from a public university in Malaysia.

The method of transferring the optimization concept to the students involved engaging them in group-based problem-solving activities centered on a practical problem from the agricultural sector. The lecturer facilitated the workshop using a student-centered approach, where the emphasis was on active participation and problem-solving. Students were encouraged to work collaboratively in groups, and the lecturer prompted some groups to share their methods and solutions with the rest of the class. This approach fostered a dynamic and interactive learning environment, allowing students to learn from each other's perspectives and approaches. Additionally, the lecturer made continuous observations and gathered feedback from the students during the workshop. This feedback was then used to enhance the quality of the lesson, ensuring that the content and teaching methods were better tailored to meet the needs and understanding of the students. Through this hands-on and participatory teaching approach, the optimization concept was effectively transferred, enabling students to grasp the subject matter more comprehensively.

Following the conclusion of the workshop, 38 students (referred to as s_1, s_2, …, s_{38} were interviewed, and their responses were digitally recorded and transcribed verbatim for analysis and reporting purposes. Thematic analysis was utilized to analyze the gathered data. Furthermore, all participants provided informed consent for their involvement in the research. The interview questions posed to the participants are as follows:

Are the contents of this lesson interesting to you?
How do you evaluate the relationship between this lesson and the real-world?
How do you evaluate the relationship between this lesson and engineering?
How do you evaluate the effect of this lesson in improving your ability to solve problems in the real world?
How does this lesson affect your understanding of solving real-world problems?
Do you have any other opinions about this educational program?
Through these interviews and thematic analysis, the study aimed to gain valuable insights into the students' perspectives on the lesson's content, its real-world applicability, its relevance to engineering, and its impact on their problem-solving abilities and understanding of real-world challenges.

2.1.Introducing the practical problem

Water wastage during the transfer of water from the source to farms through traditional channels and rivers is a significant challenge faced by the agricultural sector (Chojnacka et al., 2020). Efficient water management in agriculture, particularly in diverse climatic conditions, necessitates the optimization of water usage through scientific canalization (Vastila et al., 2021). To mitigate water losses, the adoption of metal and concrete channels is recommended by agricultural experts as a more effective alternative to traditional channels. As depicted in Figure 1 and Figure 2, the stark contrast between a traditional river and a concrete channel highlights the potential benefits of using modern canalization methods. The practical problem introduced to engineering students revolves around optimizing the design and implementation of metal or concrete channels for transferring water from sources to farms, aiming to enhance water efficiency and agricultural productivity. This practical problem is as follows:

> *Explain the step-by-step process that engineers can follow to design an optimal open water channel for agricultural use using a rectangular metal sheet. Additionally, discuss the significance of considering different geometrical shapes for the channel's cross-section and their impact on water volume pass.*

Figure 1: Traditional River Utilized for Agricultural Purposes
Source: https://www.fwi.co.uk/business/payments-schemes/sfi-farmers-alerted-to-good-reason-clause-in-agreements

Figure 2: Concrete Channel with Trapezoidal Cross-Section for Agricultural Use
Source: [https://www.123rf.com/photo_56488343_irrigation-ditch-in-the-plain-of-the- river-esla-in-leon-province-spain.html](https://www.123rf.com/photo_56488343_irrigation-ditch-in-the-plain-of-the-river-esla-in-leon-province-spain.html)

Engineers aim to create optimal water channels that efficiently supply the required water to agricultural farms. Achieving this goal depends on various factors, including the amount of water needed for field irrigation, the distance between the water source and the farm, the channel materials, the geometric shape of the channel cross-section, and the cost of channel construction.

In this educational article, a group of lecturers focuses on teaching the concept of optimization to engineering students through a practical problem. They discuss the design of water channels with different geometrical cross-sections, aiming to maximize water volume pass. The classroom exercise involves constructing these channels to efficiently pass the maximum amount of water. Although the channels are designed using metal sheets as examples, the results can be applied to concrete channels as well. Figure 3 showcases the engineering process of curving a metal sheet to produce a water channel, while Figure 4 illustrates an open metal channel with a rectangular cross-section. Through this educational approach, future engineers gain valuable insights into channel optimization, preparing them for real-world challenges in agricultural water management and irrigation.

Figure 3: Fabrication Process of a Metal Channel
Source: https://www.shutterstock.com/image-photo/bending-sheet-metal-hydraulic-machine-factory-1771306607

Figure 4: Completed Metal Channel with Rectangular Cross-Section
Source: https://www.ebay.co.uk/itm/271175304196

The lecturers began this lesson by posing the question, "What are the common geometrical shapes of open water channels for agricultural use?" They encouraged student participation and gathered opinions on the possible geometrical shapes of such channels. Subsequently, the lecturers elucidated that the most prevalent geometric cross-sections of water channels for agricultural purposes are the triangular cross-section, rectangular cross-section, and trapezoidal cross-section, as depicted in Figure 5.

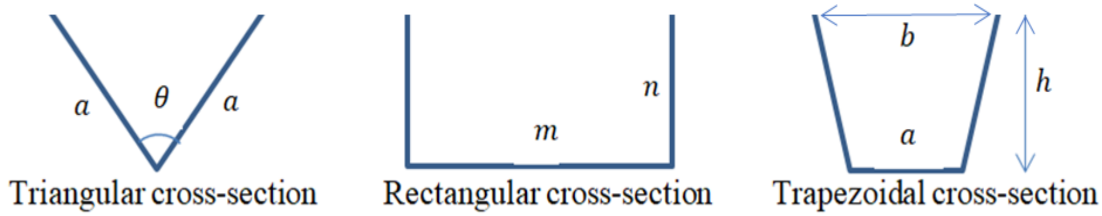


Figure 5: Common Geometrical Cross-Section Shapes for Water Channels

3. DISCUSSION ON SOLVIN THE PRACTICAL PROBLEM

In the classroom, the lecturers facilitated an interactive session where students had the opportunity to explore the design of optimal water channels with various geometrical cross-section shapes using a rectangular metal sheet with dimensions long "l" and wide "w" (Figure 6). Guided by the lecturers, students were encouraged to derive the optimal solutions for different cross-sectional shapes of water channels. Emphasizing a student-centered teaching approach, the lecturers actively discussed the optimally designed channels with the students, fostering a collaborative learning environment.

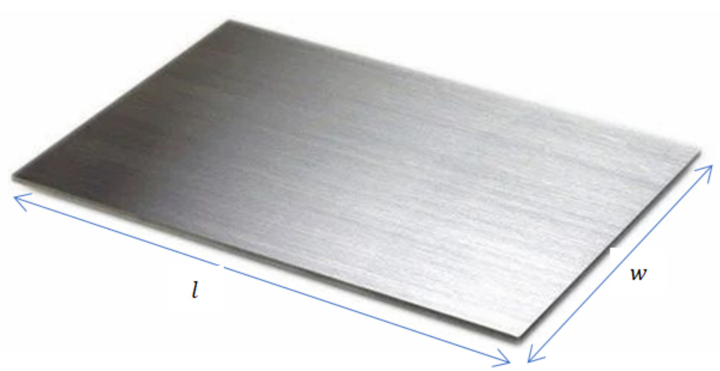


Figure 6: Rectangular Metal Sheet with Dimensions l and w

3.1.Designing an optimal water channel with triangular cross-section

Upon bending the metal sheet (Figure 6) from its wide form into a triangular channel with an angle θ, the cross-sectional area can be calculated as follows:

$$s = \frac{1}{2}\left(\frac{w}{2} \times \frac{w}{2}\right)\sin\theta \Rightarrow \mathrm{s}(\theta) = \frac{w^2}{8}\sin\theta \quad (1)$$

$$s'(\theta) = \frac{w^2}{8}\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = 90. \quad (2)$$

Therefore, the maximum area value for the triangular cross-section of the constructed channel is as follows:

$$\mathrm{s} = \frac{w^2}{8}\sin\theta \Rightarrow \mathrm{s} = \frac{w^2}{8}\sin 90 \Rightarrow \mathrm{s} = \frac{w^2}{8}. \quad (3)$$

As a result, the maximum volume value of the triangular channel constructed using a rectangular metal sheet with dimensions l and w is as follows:

$$v = sl \Rightarrow v = \frac{w^2 l}{8}. \quad (4)$$

Additionally, some students arrived at the same result for this part using an alternative method, as shown below:

It is evident that for any angle θ, the inequality $-1 \le \sin\theta \le 1$ holds true. Hence, we can derive the inequality $\frac{-w^2}{8} \le \frac{w^2}{8}\sin\theta \le \frac{w^2}{8}$. Consequently, $\frac{-w^2}{8} \le \mathrm{s}(\theta) \le \frac{w^2}{8}$. Therefore, when $\sin\theta = 1$, the function $\mathrm{s}(\theta) = \frac{w^2}{8}\sin\theta$ attains its maximum value, which is $\frac{w^2}{8}$. This implies that the angle of the constructed channel must be 90 degrees.

3.2.Designing an optimal water channel with rectangular cross-section

In this scenario, the cross-section of the constructed water channel, utilizing a rectangular metal sheet with dimensions l and w, forms a rectangle with length m and width n as depicted in Figure 5. Hence, we can establish the relationships $2n + m = w$ and $s = mn$, where s represents the area function, and it is expressed as follows:

$$s(n) = n(w - 2n) = nw - 2n^2 \quad (5)$$

$$s'(n) = w - 4n = 0 \Rightarrow \mathrm{n} = \frac{w}{4} \Rightarrow \mathrm{m} = \mathrm{w} - 2\left(\frac{w}{4}\right) = \frac{w}{2}. \quad (6)$$

The maximum area value of the rectangular cross-section for the constructed water channel is as follows:

$$s = nm \Rightarrow \mathrm{s} = \left(\frac{w}{4}\right)\left(\frac{w}{2}\right) = \frac{w^2}{8}. \quad (7)$$

Therefore, the optimal volume value for the water channel constructed using the given metal sheet is as follows:

$$v = sl \Rightarrow v = \frac{lw^2}{8}. \quad (8)$$

3.3.Designing an optimal water channel with trapezoidal cross-section

Figure 7 illustrates the cross-section of a trapezoidal water channel, constructed using a rectangular metal sheet with dimensions l and w. The perimeter of the constructed open channel cross-section (p) can be expressed in terms of θ as follows:

$$p = a + 2(\frac{h}{\sin\theta}) \Rightarrow \mathrm{w} = a + \frac{2h}{\sin\theta}. \quad (9)$$

The area value of this trapezoidal cross-section is obtained as follows:

$$s = \frac{h}{2}(2a + 2(\frac{h}{\tan\theta})) \Rightarrow \mathrm{s} = \mathrm{h}\left(\mathrm{a} + \frac{h}{\tan\theta}\right). \quad (10)$$

Thus, we obtain the following:

$$a + \frac{h}{\tan\theta} = \frac{s}{h} \Rightarrow \mathrm{a} = \frac{s}{h} - \frac{h}{\tan\theta}. \quad (11)$$

After combining the perimeter formula and the area formula of this cross-section, we have the following:

$$w = \frac{s}{h} - \frac{h}{\tan\theta} + \frac{2h}{\sin\theta} \quad (12)$$

$$\Rightarrow \frac{dw}{dh} = \frac{-1}{h^2}s - \frac{1}{\tan\theta} + \frac{2}{\sin\theta} \quad (13)$$

$$\Rightarrow \frac{dw}{dh} = 0 \Rightarrow \frac{-1}{h^2}s - \frac{1}{\tan\theta} + \frac{2}{\sin\theta} = 0 \quad (14)$$

$$\Rightarrow \frac{dw}{dh} = \frac{-1}{h^2}\left(h(a + \frac{h}{\tan\theta})\right) - \frac{1}{\tan\theta} + \frac{2}{\sin\theta} = 0 \quad (15)$$

$$\Rightarrow \frac{-a}{h} - \frac{2}{\tan\theta} + \frac{2}{\sin\theta} = 0 \Rightarrow \frac{-2\cos\theta}{\sin\theta} + \frac{2}{\sin\theta} = \frac{a}{h} \quad (16)$$

$$\Rightarrow \frac{2(1-\cos\theta)}{\sin\theta} = \frac{a}{h} \Rightarrow h = \frac{a\sin\theta}{2(1-\cos\theta)}. \quad (17)$$

Therefore, the area formula for the trapezoidal cross-section of the water channel is as follows:

$$\mathrm{s} = h\left(\mathrm{a} + \frac{h}{\tan\theta}\right) \Rightarrow \mathrm{s} = \frac{a\sin\theta}{2(1-\cos\theta)}(a + \frac{\frac{a\sin\theta}{2(1-\cos\theta)}}{\frac{\sin\theta}{\cos\theta}}) \quad (18)$$

$$\Rightarrow \text{s} = \frac{a\sin\theta}{2(1-\cos\theta)}\left(a + \frac{a\cos\theta}{2(1-\cos\theta)}\right) \quad (19)$$

$$\Rightarrow \text{s} = \frac{a\sin\theta}{2(1-\cos\theta)}\left(\frac{2a(1-\cos\theta)+a\cos\theta}{2(1-\cos\theta)}\right) \quad (20)$$

$$\Rightarrow \text{s} = \frac{a^2\sin\theta(2-\cos\theta)}{4(1-\cos\theta)^2}. \quad (21)$$

In fact, the area formula is a function of two variables, as shown below:

$$\text{s}(\text{a},\theta) = \frac{a^2\sin\theta(2-\cos\theta)}{4(1-\cos\theta)^2}. \quad (22)$$

The length of the metal sheet used in constructing the water channel with a trapezoidal cross-section is represented as l. Consequently, the volume value of the built channel can be calculated using the following two-variable function:

$$v(\text{a},\theta) = \frac{a^2 l\sin\theta(2-\cos\theta)}{4(1-\cos\theta)^2}. \quad (23)$$

The lecturers emphasized an important conceptual understanding for students that the dimensions of the optimal channel with a rectangular cross-section can be determined using the formula $h = \frac{a\sin\theta}{2(1-\cos\theta)}$. To find the value of a in terms of h, it suffices to consider the angle θ equal to 90 degrees, resulting in the following relationship:

$$h = \frac{a\sin 90}{2(1-\cos 90)} = \frac{a(1)}{2(1-0)} = \frac{a}{2} \Rightarrow \text{a} = 2h. \quad (24)$$

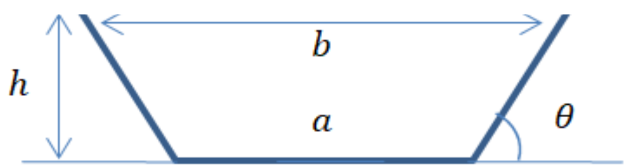


Figure 7: Cross-Section of a Trapezoidal Water Channel

3.4. Constructing an optimal water channel with trapezoidal cross-section using an optimal channel with rectangular cross-section

By altering the angle of the optimal channel walls with a rectangular cross-section, we can design an optimal channel with a trapezoidal cross-section that allows a greater amount of water to pass through. This method involves transforming the optimal water channel from a rectangular cross-section, as shown in Figure 8, into an optimal water channel with a trapezoidal cross-section.

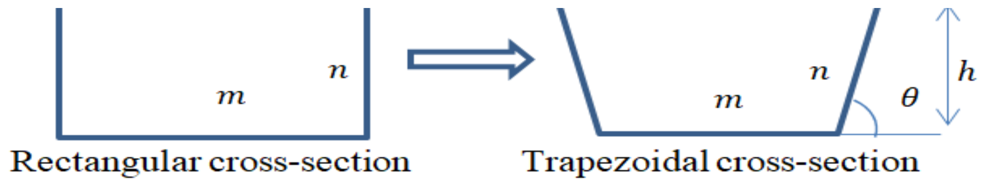


Figure 8: Transformation from Rectangular Cross-Section to Trapezoidal Cross-Section

According to Figure 8, the area value of the trapezoidal cross-section of the open channel is calculated as follows:

$$s = \frac{1}{2}(n \sin\theta)(2m + 2n\cos\theta) = \frac{1}{2}\left(\frac{w}{4}\sin\theta\right)\left(2\left(\frac{w}{2}\right) + 2\left(\frac{w}{4}\right)\cos\theta\right) \qquad (25)$$

$$\Rightarrow \mathrm{s} = \frac{w^2}{16}\sin\theta\,(2+\cos\theta). \qquad (26)$$

The trigonometric part of the area function is represented as $g(\theta) = \sin\theta\,(2+\cos\theta)$, and the maximum value of the function g determines the maximum value for the area function s.

$$g(\theta) = \sin\theta\,(2+\cos\theta) \Rightarrow g'(\theta) = \cos\theta\,(2+\cos\theta) - \sin\theta\,(\sin\theta) \qquad (27)$$

$$\Rightarrow g'(\theta) = 2\cos\theta + cos^2\theta - sin^2\theta = 2\cos\theta + cos^2\theta - (1 - cos^2\theta) \qquad (28)$$

$$\Rightarrow g'(\theta) = 2cos^2\theta + 2\cos\theta - 1 \qquad (29)$$

$$\Rightarrow g'(\theta) = 0 \Rightarrow 2cos^2\theta + 2\cos\theta - 1 = 0 \qquad (30)$$

$$\Rightarrow \cos\theta = \frac{\sqrt{3}-1}{2} \Rightarrow \theta = \mathrm{Arc}\cos(\frac{\sqrt{3}-1}{2}). \qquad (31)$$

The volume value of the water channel constructed using the given metal sheet is calculated based on the following formula:

$$v = \frac{lw^2}{16}\sin\theta\,(2+\cos\theta). \qquad (32)$$

Therefore, by utilizing the relation $\theta = \mathrm{Arc}\cos(\frac{\sqrt{3}-1}{2})$, the volume value of the optimal channel is calculated as follows:

$$v = \frac{lw^2}{16}\sin\theta\,(2+\cos\theta) \qquad (33)$$

$$\Rightarrow v = \frac{lw^2}{16}\sin\left(Arc\cos\left(\frac{\sqrt{3}-1}{2}\right)\right)\left(2+\cos(Arc\cos\left(\frac{\sqrt{3}-1}{2}\right)\right). \qquad (34)$$

We have $\cos(Arc\cos(\frac{\sqrt{3}-1}{2}))=\frac{\sqrt{3}-1}{2}$, and by considering $\alpha=\text{Arc}\cos(\frac{\sqrt{3}-1}{2})$, we can find the value of $\sin(Arc\cos(\frac{\sqrt{3}-1}{2}))$ as below.

$$\alpha=\text{Arc}\cos(\frac{\sqrt{3}-1}{2})\Rightarrow\cos\alpha=\frac{\sqrt{3}-1}{2}. \tag{35}$$

Therefore,

$$\sin(Arc\cos(\frac{\sqrt{3}-1}{2}))=\sin\alpha=\sqrt{1-cos^2\alpha}=\sqrt{1-(\frac{\sqrt{3}-1}{2})^2}=\frac{\sqrt{2\sqrt{3}}}{2}. \tag{36}$$

Thus, the maximum value of channel volume is as follows:

$$v=\frac{lw^2}{16}\left(\frac{\sqrt{2\sqrt{3}}}{2}\right)\left(2+\frac{\sqrt{3}-1}{2}\right)\Rightarrow v=\frac{\sqrt{2\sqrt{3}}(\sqrt{3}+3)lw^2}{64}. \tag{37}$$

3.5.Designing an optimal water channel with trapezoidal cross-section and equal dimensions

Various water channels can be designed using a metal sheet. In this section, the lecturers discussed the process of designing an optimal water channel with a trapezoidal cross-section and equal dimensions. As depicted in Figure 8, both values m and n are equal. Consequently, the area value of the channel cross-section can be calculated as follows:

$$s=\frac{1}{2}h(m+(m+2n\cos\theta))=\frac{1}{2}(n\sin\theta) \tag{38}$$

$$\Rightarrow s=\frac{1}{2}\left(\frac{w}{3}\sin\theta\right)\left(2\left(\frac{w}{3}\right)+2\left(\frac{w}{3}\right)\cos\theta\right) \tag{39}$$

$$\Rightarrow s=\frac{w^2}{9}\sin\theta\,(1+\cos\theta). \tag{40}$$

So, to determine the maximum value of the trigonometric function $h(\theta)=\sin\theta\,(1+\cos\theta)$, we need to calculate it in order to obtain the maximum value for the area function of the trapezoidal cross-section of the water channel.

$$h(\theta)=\sin\theta\,(1+\cos\theta)\Rightarrow h'(\theta)=\cos\theta\,(1+\cos\theta)+\sin\theta\,(-\sin\theta) \tag{41}$$

$$\Rightarrow h'(\theta)=\cos\theta+cos^2\theta-sin^2\theta\Rightarrow h'(\theta)=\cos\theta+cos^2\theta-(1-cos^2\theta) \tag{42}$$

$$\Rightarrow h'(\theta)=2cos^2\theta+\cos\theta-1 \tag{43}$$

$$h'(\theta)=0\Rightarrow 2cos^2\theta+\cos\theta-1=0\Rightarrow\cos\theta=\frac{1}{2}\Rightarrow\theta=60. \tag{44}$$

Therefore, the maximum volume value of the open channel with a trapezoidal cross-section and equal dimensions can be determined as follows:

$$v = sl = \frac{lw^2}{9}\sin\theta\,(1+\cos\theta) = \frac{lw^2}{9}\sin 60\,(1+\cos 60) = \frac{lw^2}{9}\left(\frac{\sqrt{3}}{2}\right)\left(1+\frac{1}{2}\right) \quad (45)$$

$$\Rightarrow v = \frac{3\sqrt{3}lw^2}{36}. \quad (46)$$

3.6.Designing Curved Water Channels: Optimization and Practical Considerations

The cross-sections of certain water channels used in agriculture are arcs of circles. Engineers utilize advanced machines to curve metal sheets and create these water channels. Figure 9 depicts a curved channel constructed using a metal sheet with dimensions l and w (Figure 6). According to Gholami and Sathar (2021), the circle's radius, in terms of the values h and k, can be calculated as $r = \frac{k^2+4h^2}{8h}$. The angle between two radiuses is denoted by α, and the cross-sectional area of the built open channel is computed as follows:

$$s = \frac{\alpha}{360}\pi r^2 - \frac{1}{2}r^2\sin\alpha = \frac{1}{2}r^2\left(\frac{1}{180}\pi\alpha - \sin\alpha\right) \quad (47)$$

$$\Rightarrow s = \frac{1}{2}\left(\frac{k^2+4h^2}{8h}\right)^2\left(\frac{1}{180}\pi\alpha - \sin\alpha\right). \quad (48)$$

Therefore, the volume value of the open channel constructed using a metal sheet (Figure 6) can be determined using the following formula:

$$v = sl \Rightarrow v = \frac{l}{2}\left(\frac{k^2+4h^2}{8h}\right)^2\left(\frac{1}{180}\pi\alpha - \sin\alpha\right). \quad (49)$$

In this formula, if $\alpha = 180$, then $k = 2r$ and $h = r$. Thus, the optimal volume value of the curved channel can be calculated as follows:

$$v = \frac{l}{2}\left(\frac{(2r)^2+4r^2}{8r}\right)^2\left(\frac{1}{180}\pi(180) - \sin(180)\right) \quad (50)$$

$$\Rightarrow v = \frac{lr^2\pi}{2} = \frac{(lr)(\pi r)}{2} = \frac{lrw}{2} \Rightarrow v = \frac{lrw}{2}. \quad (51)$$

In other words, the built channel has a semi-circular cross-section with a perimeter of w. Therefore,

$$\pi r = w \Rightarrow r = \frac{w}{\pi}. \quad (52)$$

The volume formula for the curved water channel is as follows:

$$v = \frac{lrw}{2} = \frac{l\left(\frac{w}{\pi}\right)w}{2} = \frac{lw^2}{2\pi} \Rightarrow v = \frac{lw^2}{2\pi}. \quad (53)$$

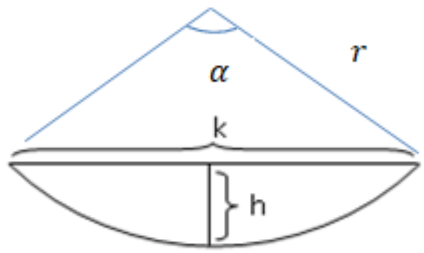


Figure 9: Cross-Section of a Curved Water Channel

4. COMPARING THE VOLUME VALUE OF OPTIMAL CHANNELS

This article delves into the diverse geometrical shapes achievable with a rectangular metal sheet of dimensions l and w. Table 1 presents the derived volume formulas for open water channels based on different sections of the article, including an illustrative example.

Geometrical cross-section	Volume formula	Volume value of a channel with $l = 3$ and $w = 1.8$ meters
Triangular	$v = \frac{w^2 l}{8}$	$1.215m^3$
Rectangular	$v = \frac{w^2 l}{8}$	$1.215m^3$
Trapezoidal ($m = 2n$)	$v = \frac{\sqrt{2\sqrt{3}}(\sqrt{3}+3)lw^2}{64}$	$1.336m^3$
Trapezoidal ($m = n$)	$v = \frac{3\sqrt{3}lw^2}{36}$	$1.402m^3$
Semi-circular	$v = \frac{lw^2}{2\pi}$	$1.547m^3$

Table 1: Comparison of Volume Values for Different Geometrical Water Channels

In Table 1, the optimal channel with a semi-circular cross-section allows the maximum water flow, but certain limitations hinder its widespread use. Interestingly, some students mistakenly assumed that changing a rectangular cross-section channel to a trapezoidal one automatically increases its volume value. To illustrate this, the lecturers considered an example: "In Figure 8, assuming $m = 2n$ and constructing a rectangular channel with a metal sheet having dimensions $l = 3m$ and $w = 1.8m$, converting it to a trapezoidal channel with $\theta = 45$ degrees. We can compare the volume values of the rectangular and trapezoidal cross-section channels." This example serves as a valuable exercise for students, promoting critical analysis and the identification of optimal solutions for real-world challenges. In this case, the volume value of the built channel with a rectangular cross-section is calculated as follows:

$$v = \frac{w^2 l}{8} = \frac{(1.8)^2 \times 3}{8} = 1.215m^3.$$

Whereas, the volume value of the built channel with a trapezoidal cross-section is determined as follows:

$$v = \frac{w^2 l}{16} \sin\theta\,(2 + \cos\theta) = \frac{(1.8)^2 \times 3}{16} \sin 45\,(2 + \cos 45)$$

$$\Rightarrow v = \frac{(1.8)^2 \times 3}{16}\left(\frac{\sqrt{2}}{2}\right)\left(2 + \frac{\sqrt{2}}{2}\right) = 1.162m^3.$$

The lecturers found that employing this practical problem focused on optimizing agricultural water channels was highly effective in helping students develop a profound understanding of the concept of optimization. By using a student-centered teaching method centered around problem-solving, the students were drawn to the subject due to the clear connection between engineering and mathematics. This approach demonstrated the significance of mathematics in solving real-world challenges through engineering methods. Furthermore, it fostered a deeper appreciation for how mathematical principles can be applied to practical engineering situations, showcasing the tangible benefits of mastering such skills. To further enhance the teaching approach, incorporating more real-world examples and hands-on experiences could help students grasp the practical applications of the concepts and further engage them in the learning process.

Collaborative work and knowledge-sharing among the lecturers significantly enhanced their teaching capabilities, particularly in designing mathematical lessons that bridge the gap between mathematics and real-world challenges. Through collaborative efforts, the lecturers were able to pool their diverse experiences and expertise, fostering a more comprehensive and engaging learning environment for students. This approach enabled them to incorporate practical examples and applications of mathematical concepts, making the lessons more relevant and applicable to real-life situations. As a result, students were better equipped to understand the importance of mathematics in addressing real-world problems and could perceive its direct relevance to their future careers in engineering and other fields. To further bolster this approach, continued professional development and ongoing sharing of best practices could lead to even more effective teaching strategies and improved student learning outcomes. One of the lecturers provided the following explanation:

In the past, I used to teach the concept of optimization to students using abstract materials from textbooks. However, being part of this collaborative program has greatly enhanced my teaching approach in conveying the optimization concept. By incorporating real-world problems that are well-suited to the students' abilities, I found that it is an effective teaching method that creates a strong connection between mathematics and engineering. Moreover, exploring various mathematical models for the practical problem discussed in this program has not only improved my lecturing skills but also enhanced the students' proficiency in solving real-world challenges. This experience has reinforced the importance of integrating practical applications into mathematical lessons, making the learning process more engaging and relevant for the students. As I continue to develop as an educator, I look forward to exploring more hands-on approaches and expanding my repertoire of real-world problem-solving activities to better equip my students for their future endeavors in various fields.

5. STUDENTS' PERSPECTIVES

According to the feedback from the students who participated in this study, they found the mathematical materials and teaching methods to be enjoyable. All students agreed that this workshop significantly enhanced their proficiency in solving real-world problems. Table 2 provides an analysis of the students' responses to the first five interview questions.

Question	Very little	Little	Not sure	Much	Very much
1	0	0	1	4	33
2	0	0	0	0	38
3	0	0	0	3	35
4	0	0	0	7	31
5	0	0	0	5	33

Table 2: Summary of Students' Responses to Interview Questions One to Five

Based on the data presented in Table 2, it is evident that the workshop had a highly positive impact on the participants. Approximately 97% of the students found the prepared lesson to be enjoyable, with 33 of them finding it very interesting and an additional 4 finding it interesting. All participants unanimously agreed (100%) that the lesson effectively connected with real-world applications, emphasizing optimization concepts through practical problems rather than abstract mathematical concepts. The primary objective of teaching engineering mathematics—to familiarize students with real-world problem-solving—was clearly met. An impressive 92% of the students recognized the deep connection between the lesson's contents and engineering.

Furthermore, 82% of the students acknowledged that the course not only enhanced their understanding of real-world problems but also improved their problem-solving skills under diverse assumptions. Remarkably, 87% of the participants believed that the teaching method greatly influenced their comprehension of real-world issues, helping them select appropriate strategies and consider necessary assumptions when tackling problems.

However, the overwhelming majority (90%) of students strongly agreed that this mathematical lesson was not only interesting and enjoyable but also highly effective in enhancing their abilities to solve real-world problems. The positive feedback from the participants underscores the workshop's success in achieving its goals and fostering an engaging and practical learning environment.

During the interview process, the sixth question stood out for its flexibility, allowing engineering students to freely express their opinions about the presented lesson without any specific constraints. The researchers used the thematic analysis method to carefully analyze the data collected from these interviews. They diligently identified recurring ideas and patterns in the interview transcripts, employing a six-step process of cognition, coding, producing themes, checking themes, naming themes, and writing up the report to discover general themes in the data analysis (Kiger & Varpio, 2020).

The obtained general themes from the interviews' transcripts highlight the role of this lesson in enhancing students' ability to solve real-world problems. Firstly, understanding the real-world problems emerges as a foundational aspect of effective problem-solving, enabling students to devise appropriate strategies using mathematical models. Secondly, the translation of real-world problems into mathematical scenarios facilitates systematic analysis and solution development. Thirdly, the importance of making reasonable assumptions in constructing mathematical models for real-world challenges is emphasized. Flexibility in employing assumptions, the fourth theme, is crucial to adapt to dynamic real-world conditions. Furthermore, the fifth theme emphasizes exploring diverse mathematical models to find different solutions for real-world problems, fostering creativity and broadening students' problem-solving capabilities. The sixth theme emphasizes the significance of critically evaluating and comparing the mathematical models developed by students to select the most appropriate one based on accuracy and applicability.

Regarding students' learning experience, the seventh theme highlights the effectiveness of practical, real-world problems in teaching optimization concepts, as they provide context and relevance. The eighth theme emphasizes the fundamental role of mathematics in addressing real-world challenges, demonstrating its practical applications to students. The ninth theme recognizes the value of errors in the learning process, promoting resilience and critical thinking in students when tackling real-world problems. Lastly, the tenth theme underscores the positive impact of the lesson on students' self-confidence. As students gain competence in solving real-world problems using mathematical tools, their self-assurance grows, motivating them to tackle more complex challenges confidently. Therefore, this study identifies a comprehensive set of factors contributing to students' improvement in solving real-world problems. Understanding these themes enables educators to design more effective lessons that empower students with essential skills and confidence to address future challenges successfully.

For example, student s_7 explained that:

> *The approach to teaching the concept of optimization is fascinating to me due to my practical experience in optimizing various models to address real-world challenges. By comparing these optimized models, I have gained a profound understanding of how to effectively tackle real-world problems.*

Student s_{16} stated that:

> *The design and implementation of agricultural water channel projects can be affected by errors arising from the geometric shape of their cross-section. Notably, the objective function for different cases of this problem is a one-variable function. However, when additional factors are taken into account, such as the distance of the water source to the farm, the type of water channel, and the construction cost, checking the error rate and optimizing the objective function becomes more complex. The materials presented in this lesson, along with the transferring method, proved to be highly engaging and interesting for me.*

6. CONCLUSION

Optimization is a crucial concept in engineering fields, empowering engineers to solve real-world problems based on specific conditions. Engineering students must grasp this concept to effectively connect mathematics with engineering. This article's results have been instrumental in aiding engineering students to design a variety of optimal open channels with different cross-sections, utilizing rectangular metal sheets for agricultural purposes. While the article focused on conceptually teaching the design of optimal agricultural water channels, students can readily extend these findings to build concrete channels. Moreover, the results can be incorporated into engineering classes for designing and constructing industrial concrete channels, especially for flood control purposes.

The prepared lesson tackles a practical problem, encouraging students to enhance their mathematical reasoning, writing, and project skills through collaborative interaction (Van Lierde, 2022). This approach to teaching mathematics in engineering classes not only fosters conceptual understanding but also enables students to establish connections between mathematics and engineering, thus equipping them to tackle real-world challenges with enthusiasm. Incorporating practical problems into engineering classrooms, rather than relying solely on abstract materials, proves to be invaluable as students gain essential skills in handling large engineering projects. As such, instructors are encouraged to embrace real-world problems in their teaching approach (Larina, 2016).

7. LIMITATIONS AND RECOMMENDATIONS

One limitation of this study is the relatively small sample size of 38 first-year engineering students from a single public university in Malaysia. While the participants' perspectives and feedback provided valuable insights into the effectiveness of the optimization lesson, the findings may not be fully representative of the broader population of engineering students. To enhance the generalizability of the results, future research could include a more diverse and larger sample of students from various universities or educational institutions. Additionally, the study's focus on a specific practical problem from the agricultural sector might limit the transferability of the optimization concept to other engineering domains. Exploring different practical problems and their applications in various engineering fields could offer a more comprehensive understanding of the lesson's adaptability and impact. Addressing these limitations would strengthen the study's significance and broaden its implications for mathematics teaching in engineering education.

Acknowledgments

The authors sincerely thank the dedicated lecturers and students who actively participated in this research, offering invaluable time, unwavering support, and thoughtful insights. Additionally, we extend our gratitude to the reviewers for their valuable and diverse feedback.

References

[1] Alshqaq, S. S. A., Ahmadini, A. A. H., & Ali, I. (2022). Nonlinear Stochastic Multiobjective Optimization Problem in Multivariate Stratified Sampling Design. *Mathematical Problems in Engineering*, *2022*, 1–16. https://doi.org/10.1155/2022/2502346

[2] Chojnacka, K., Witek-Krowiak, A., Moustakas, K., Skizypczak, D., Mikula, K., & Loizidou, M. (2020). A transition from conventional irrigation to fertigation with reclaimed wastewater: Prospects and challenges. *Renewable and Sustainable Energy Reviews*, *130*, 1–14. https://doi.org/10.1016/j.rser.2020.109959

[3] Dastan, M., Shojaee, S., Hamzehei-Javaran, S., & Goodarzimehr, V. (2022). Hybrid teaching-learning based optimization for solving engineering and mathematical problems. *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, *44*, 1–31. https://doi.org/10.1007/s40430-022-03700-x

[4] Fatmanissa, N., Kusnandi, & Usdiyana, D. (2019). Student difficulties in word problems of derivatives: A multisemiotic perspective. *Journal of Physics: Conference Series*, *1157*, 1–9. https://doi.org/10.1088/1742-6596/1157/3/032111

[5] Gergen, K. (1995). Social construction and the educational process. *In L. P. Steffe & J. Gale, Constructivism in Education (Pp. 17-39). N.J.: Lawrence Erlbaum Associates*.

[6] Gholami, H. (2021). Teaching the circle equation using a practical mathematical problem. *International Journal of Mathematical Education in Science and Technology*, 1–13. https://doi.org/10.1080/0020739X.2021.1979261

[7] Gholami, H., & Sathar, M. H. A. (2021). The Application of Circle Equation in Building

Composite Frontage. *Advances in Mathematics: Scientific Journal*, *10*(1), 29–35. https://doi.org/10.37418/amsj.10.1.4

[8] Harris, D., Black, L., Hernandez-Martinez, P., Pepin, B., & Williams, J. (2015). Mathematics and its value for engineering students: What are the implications for teaching? *International Journal of Mathematical Education in Science and Technology*, *46*(3), 321–336. https://doi.org/10.1080/0020739X.2014.979893

[9] Hoover, W. A. (1996). The practice implications of constructivism. *SEDL Letter*, *9(3)*, 1–2.

[10] Kiger, M. E., & Varpio, L. (2020). Thematic analysis of qualitative data: AMEE Guide No. 131. *Medical Teacher*, 1–9. https://doi.org/10.1080/0142159X.2020.1755030

[11] Larina, G. (2016). Analysis of Real-World Math Problems: Theoretical Model and Classroom Application. *Educational Studies Moscow*, *3*, 151–168. https://doi.org/10.17323/1814-9545-2016-3-151-168

[12] Pepin, B., Biehler, R., & Gueudet, G. (2021). Mathematics in Engineering Education: A Review of the Recent Literature with a View towards Innovative Practices. *International Journal of Research in Undergraduate Mathematics Education*, *7*, 163–188. https://doi.org/10.1007/s40753-021-00139-8

[13] Retamoso, I. (2022). Heuristic method for minimizing distance without using calculus and its significance. *Mathematics Teaching Research Jouenal*, *14*(4), 225–236.

[14] Savizi, B. (2007). Applicable problems in the history of mathematics: practical examples for the classroom. *Teaching Mathematics And Its Applications*, *26*, 51–54.

[15] Szabo, Z. K., Kortesi, P., Guncaga, J., Szabo, D., & Neag, R. (2020). Examples of Problem-Solving Strategies in Mathematics Education Supporting the Sustainability of 21st-Century Skills. *Sustainability*, *12*, 1–28. https://doi.org/10.3390/su122310113

[16] Tsai, J., Carlsson, J. G., Ge, D., Hu, Y., & Shi, J. (2014). Optimization Theory, Methods, and Applications in Engineering 2013. *Mathematical Problems in Engineering*, *2014*, 1–5. https://doi.org/10.1155/2014/319418

[17] Vagaska, A., Gombar, M., & Straka, L. (2022). Selected Mathematical Optimization Methods for Solving Problems of Engineering Practice. *Energies*, *15*(6), 1–23. https://doi.org/10.3390/en15062205

[18] Van Lierde, V. (2022). Using a discussion forum to encourage writing in a differential equations class. *Mathematics Teaching Research Jouenal*, *14*(3), 133–143.

[19] Vastila, K., Vaisanen, S., Koskiaho, J., Lehtoranta, V., Karttunen, K., Kuussaari, M., Jarvela, J., & Koikkalainen, K. (2021). Agricultural Water Management Using Two-Stage Channels: Performance and Policy Recommendations Based on Northern European Experiences. *Sustainability*, *13*, 1–26. https://doi.org/10.3390/su13169349

[20] Xenofontos, C., & Andrews, P. (2014). Defining mathematical problems and problem solving: prospective primary teachers' beliefs in Cyprus and England. *Mathematics Education Research Journal*, *26*(2), 279–299. https://doi.org/10.1007/s13394-013-0098-z

The Problem Corner

Ivan Retamoso, PhD, *The Problem Corner* Editor

Borough of Manhattan Community College

iretamoso@bmcc.cuny.edu

The Purpose of **The Problem Corner** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem and its solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Greetings, fellow problem solvers!

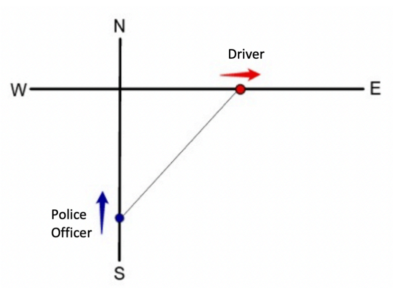
I'm pleased to share that I've obtained answers for both Problem 16 and Problem 17. It brings me great joy to report that not only were all of them accurate, but they were also remarkably captivating and inventive. My main objective in highlighting what I believe to be the most exceptional solutions is to enhance and elevate the mathematical knowledge within our worldwide community.

Solutions to **Problems** from the Previous Issue.

Interesting "Police Officer and Driver" problem.

Problem 16

Proposed by Ivan Retamoso, BMCC, USA.



Let's consider a situation where a police officer is situated $\frac{1}{2}$ mile to the south of an intersection. This officer is driving northwards towards the intersection at a speed of $35\ mph$. At the exact same time, there is another car located $\frac{1}{2}$ mile to the east of the intersection, and it is moving eastward, away from the intersection.

a) Let's assume that the officer's radar gun displays a speed of $20\ mph$ when aimed at the other car. This reading indicates that the straight-line distance between the officer and the other car is increasing at a rate of $20\ mph$. What, then, is the speed of the other car?

b) Now, let's consider a different scenario where the officer's radar gun displays $-20\ mph$ instead. This indicates that the straight-line distance between the officer and the other car is decreasing at a rate of $20\ mph$. What is the speed of the other car in this situation?

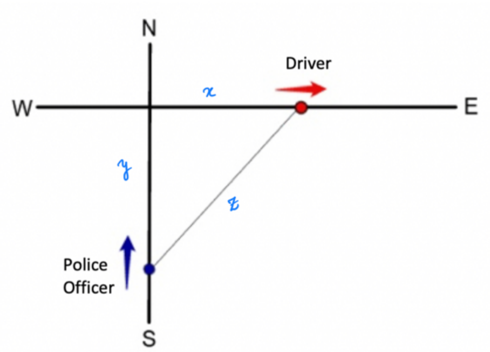
Note: Round yours answers to three decimals places.

Solution to problem 16

By Aradhana Kumari, Borough of Manhattan Community College, USA.

Our solver initiates with precise variable labels to represent evolving distances, then employs the Pythagorean theorem to establish their relationships. Finally, implicit differentiation is applied *to uncover the relationship between the rates of change in distances over time, leading to the final answers.*

Solution: a) Consider the diagram below



We have $x^2 + y^2 = z^2$ ....(1) (using the Pythagorean triangle)

In our given problem $x = \frac{1}{2}$, $y = \frac{1}{2}$

Using equation (1) we have

$$z = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{2}{4}} = \sqrt{\frac{1}{2}} \text{ .... (2)}$$

Differentiating the equation given by (1) we get

$$2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} = 2z\,\frac{dz}{dt} \text{ ....(3)}$$

As per question $x = \frac{1}{2}$, $y = \frac{1}{2}$, $\frac{dz}{dt} = 20, \frac{dy}{dt} = -35$

Since y is increasing in the north direction, going towards the intersection y is decreasing hence we have to $\frac{dy}{dt} = -35$

$z = \sqrt{\frac{1}{2}}$ (from (2))

From equation given by (3) we have

$$\frac{dx}{dt} = \frac{z}{x}\frac{dz}{dt} - \frac{y}{x}\,\frac{dy}{dt}$$

$$\frac{dx}{dt} = \frac{\sqrt{\frac{1}{2}}}{\frac{1}{2}}\,20 - \frac{\frac{1}{2}}{\frac{1}{2}}\,(-35) = 63.284\ mph$$

b) As per question we have $\frac{dz}{dt} = -20$

$$\frac{dx}{dt} = \frac{z}{x}\frac{dz}{dt} - \frac{y}{x}\frac{dy}{dt}$$

$$= \frac{\sqrt{\frac{1}{2}}}{\frac{1}{2}}(-20) - \frac{\frac{1}{2}}{\frac{1}{2}}(-35)$$

$$= \sqrt{2}(-20) + 35$$

$$= 6.716\ mph$$

"Looking for a pattern" problem.

Problem 17

Proposed by Christopher Ingrassia, Kingsborough Community College (CUNY)

Brooklyn, NY, USA

Suppose $n \times n$ matrix A and $n \times 1$ vector x are defined as follows:

$$A_{i,j} = \begin{cases} 1, & i \geq j \\ 0, & otherwise \end{cases}$$

$$x_i = 1$$

Describe, in words, the vector $A^k x$, where $k \geq 0$.
Find an expression for the quantity $x^T A^k x$ in terms of n and k (x^T is the transpose of vector x).

Solution to problem 17

By Aradhana Kumari, Borough of Manhattan Community College, USA.

In this clearly articulated solution, our solver step by step and systematically demonstrates how a basic matrix operation generates recognizable patterns, ultimately leading to a broader conclusion derived from inductive reasoning based on the diagonals of Pascal's triangle.

Solution: For k = 0 we have

$A^k (x) = x$.

For $k \geq 1$, $A^k x$ calculation is below

$$A^{1}_{1\times1}(x) = [1]\,x = x \quad , \quad A^{2}_{1\times1}(x) = x$$

$$A^{1}_{2\times2}(x) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A^{2}_{2\times2}(x) = A_{2\times2}A_{2\times2}(x) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1\times1 \\ 1\times1 + 1\times2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} {}^{2}C_0 \\ {}^{3}C_1 \end{bmatrix}$$

$$A^{3}_{2\times2}(x) = A_{2\times2}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1\times1 \\ 1\times1 + 1\times3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1+3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} {}^{3}C_0 \\ {}^{4}C_1 \end{bmatrix}$$

$$A^{4}_{2\times2}(x) = A_{2\times2}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1\times1 \\ 1\times1 + 1\times4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1+4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} {}^{4}C_0 \\ {}^{5}C_1 \end{bmatrix}$$

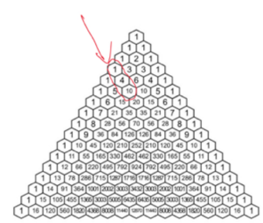
$$\vdots$$

$$A^{K}_{2\times2}(x) = \begin{bmatrix} {}^{K}C_0 \\ {}^{K+1}C_1 \end{bmatrix}$$

$$A^{1}_{3\times3}(x) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1\times1 + 1\times1 \\ 1\times1 + 1\times1 + 1\times1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} {}^{1}C_0 \\ {}^{2}C_1 \\ {}^{3}C_2 \end{bmatrix}$$

$$A^{2}_{3\times3}(x) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1\times1 + 1\times2 \\ 1\times1 + 1\times2 + 1\times3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} {}^{2}C_0 \\ {}^{3}C_1 \\ {}^{4}C_2 \end{bmatrix}$$

$$A^{3}_{3\times3}(x) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1\times1 + 1\times3 \\ 1\times1 + 1\times3 + 1\times6 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 10 \end{bmatrix} = \begin{bmatrix} {}^{3}C_0 \\ {}^{4}C_1 \\ {}^{5}C_2 \end{bmatrix} =$$

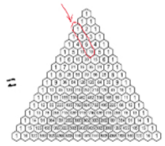


$$\vdots$$

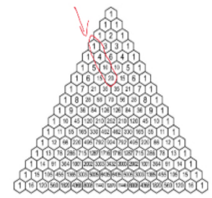
$$A^{K}_{3\times3}(x) = \begin{bmatrix} {}^{K}C_0 \\ {}^{K+1}C_1 \\ {}^{K+2}C_2 \end{bmatrix}$$

$$A^1_{4\times4}(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1\times1 + 1\times1 \\ 1\times1 + 1\times1 + 1\times1 \\ 1\times1 + 1\times1 + 1\times1 + 1\times1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} {}^1C_0 \\ {}^2C_1 \\ {}^3C_2 \\ {}^4C_3 \end{bmatrix}$$

$$A^2_{4\times4}(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1\times1 + 1\times2 \\ 1\times1 + 1\times2 + 1\times3 \\ 1\times1 + 1\times2 + 1\times3 + 1\times4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1+2 \\ 1+2+3 \\ 1+2+3+4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} {}^2C_0 \\ {}^3C_1 \\ {}^4C_2 \\ {}^5C_3 \end{bmatrix} =$$



$$A^3_{4\times4}(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 1\times1 + 1\times3 \\ 1\times1 + 1\times3 + 1\times6 \\ 1\times1 + 1\times3 + 1\times6 + 1\times10 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 10 \\ 20 \end{bmatrix} = \begin{bmatrix} {}^3C_0 \\ {}^4C_1 \\ {}^5C_2 \\ {}^6C_3 \end{bmatrix} =$$



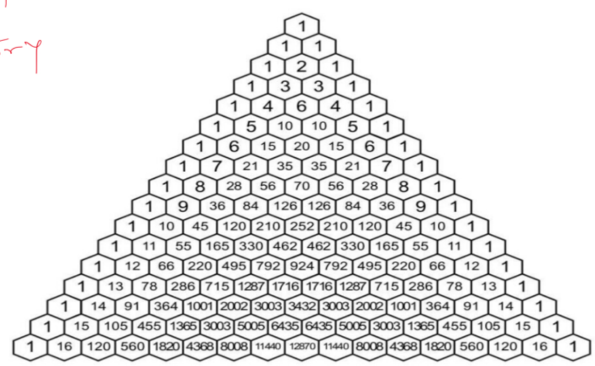
$$\vdots$$

$$A^K_{4\times4}(x) = \begin{bmatrix} {}^KC_0 \\ {}^{K+1}C_1 \\ {}^{K+2}C_2 \\ {}^{K+3}C_3 \end{bmatrix}_{4\times1}$$

$$A^K_{n\times n}(x) = \begin{bmatrix} {}^KC_0 \\ {}^{K+1}C_1 \\ {}^{K+2}C_2 \\ \vdots \\ {}^{K+n-1}C_{n-1} \end{bmatrix}_{n\times1} =$$

1^{st} entry

2^{nd} entry



1, K+1, …

n^{th} entry of $A^K_{n\times n}(x)$

$= (n-2)^{th}$ entry from the Pascal's triangle

counting 1 as the first entry & (K+1) as the second entry

Consider n × 1 A where A_{ij} is 1 if $i \geq 1$ *and* 0 otherwise. x is $n \times 1$ matrix where $x_i = 1$. In words $A^k x$ where $k \geq 1$ is the $n \times 1$ *coloumn* matrix where $A_{11} = 1$, $A_{21} = k+1$.... and A_{n1} entry is the

$(n-2)^{th}$ entry which we get if we go along the diagonal starting from k+1 in the pascal's triangle counting 1 as the first entry, k+1 as second entry as shown above.

Expression for $x^T A^k x$

For k = 0, x is a $n \times 1$ vector where $x_i = 1$ we have $x^T A^0 x = x^T x = n$,

For $k \geq 1$, $x^T A^k x$ calculation is below

$\underline{x^T A x}$, A and x as defined in the problem

$A = [1]$, $x^T A x = 1 \times 1 \times 1 = 1$

$A_{2\times2} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $x^T A^1_{2\times2} x = [1\ 1]_{1\times2} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}_{2\times2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{2\times1}$

$= [1\ 1]_{1\times2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{2\times1} = 1\times1 + 1\times2 = 1+2 = {}^1C_0 + {}^2C_1$

$A^2_{2\times2} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $x^T A^2_{2\times2} x = [1\ 1] \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [1\ 1] \begin{bmatrix} 1 \\ 3 \end{bmatrix} = (1\times1) + (1\times3) = {}^2C_0 + {}^3C_1$

$x^T A^3_{2\times2} x = [1, 1] \begin{bmatrix} 1 \\ 4 \end{bmatrix} = (1\times1) + (1\times4) = 1+4 = {}^3C_0 + {}^4C_1$

⋮

$x^T A^K_{2\times2} x = {}^KC_0 + {}^{K+1}C_1$

$A^1_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $x^T A^1_{3\times3} x = [1\ 1\ 1] \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = [1\ 1\ 1] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$= (1\times1) + (1\times2) + (1\times3)$

$= 1+2+3 = {}^1C_0 + {}^2C_1 + {}^3C_2$

$x^T A^2_{3\times3} x = [1\ 1\ 1] \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} = (1\times1) + (1\times3) + (1\times6) = 1+3+6 = {}^2C_0 + {}^3C_1 + {}^4C_2$

$x^T A^3_{3\times3} x = [1\ 1\ 1] \begin{bmatrix} 1 \\ 4 \\ 10 \end{bmatrix} = (1\times1) + (1\times4) + (1\times10) = 1+4+10 = {}^3C_0 + {}^4C_1 + {}^5C_2$

$x^T A^4_{3\times3} x = [1\ 1\ 1] \begin{bmatrix} 1 \\ 5 \\ 15 \end{bmatrix} = (1\times1) + (1\times5) + (1\times15) = 1+5+15 = {}^4C_0 + {}^5C_1 + {}^6C_2$

⋮

$x^T A^K_{3\times3} x = {}^KC_0 + {}^{K+1}C_1 + {}^{K+2}C_2$

$$A_{4\times4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad x^T A^1_{4\times4} x = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = (1\times1)+(1\times2)+(1\times3)+(1\times4)$$
$$= 1+2+3+4$$
$$= {}^1C_0 + {}^2C_1 + {}^3C_2 + {}^4C_3$$

$$x^T A^2_{4\times4} x = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 6 \\ 10 \end{bmatrix} = 1\times1 + 1\times3 + 1\times6 + 1\times10$$
$$= 1+3+6+10$$
$$= {}^2C_0 + {}^3C_1 + {}^4C_2 + {}^5C_3$$

$$x^T A^3_{4\times4} x = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 10 \\ 20 \end{bmatrix} = 1\times1 + 1\times4 + 1\times10 + 1\times20$$
$$= 1+4+10+20$$
$$= {}^3C_0 + {}^4C_1 + {}^5C_2 + {}^6C_3$$

$\vdots$

$$x^T A^K_{4\times4}(x) = {}^KC_0 + {}^{K+1}C_1 + {}^{K+2}C_2 + {}^{K+3}C_3$$

$\vdots$

For $n\times n$ matrix A, where

$$A_{ij} = \begin{cases} 1 & i \geq j \\ 0 & \text{otherwise} \end{cases}$$

$$x_i = 1$$

We have

$$x^T A^K_{n\times n} x = {}^KC_0 + {}^{K+1}C_1 + {}^{K+2}C_2 + \dots + {}^{K+n-1}C_{n-1}$$

Second Solution to problem 17

By Christopher Ingrassia (The proposer)
Kingsborough Community College (CUNY)
Brooklyn, NY, USA

Matrix A has the form $\begin{bmatrix} 1 & & \cdots & & 0 \\ & 1 & 0 & 0 & \\ \vdots & 1 & 1 & 0 & \vdots \\ & 1 & 1 & 1 & \\ 1 & & \cdots & & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}$.

Because the zero power of a matrix is the identity, A^0x simply yields x, a column of ones.

When $k > 0$, A^k can be thought of as a linear operator which computes a cumulative sum of the elements of x: A^1x produces $\begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{bmatrix}$, $A^2x = \begin{bmatrix} 1 \\ 3 \\ 6 \\ \vdots \\ \frac{n(n+1)}{2} \end{bmatrix}$, $A^3x = \begin{bmatrix} 1 \\ 4 \\ 10 \\ \vdots \\ \sum_i (A^2x)_i \end{bmatrix}$, …

These are the diagonals of Pascal's triangle. If we number the diagonals starting with 0 for the diagonal of all ones, A^kx yields the first n entries of the k^{th} diagonal.

The product x^TA^kx simply sums the elements of vector A^kx. This value is found on Pascal's triangle one row below and just to the left of the n^{th} entry of the k^{th} diagonal. **This value, in terms of n and k, is** $C(n+k, n-1) = \binom{n+k}{n-1}$.

Because each entry of Pascal's triangle represents a binomial coefficient, this quantity may be expressed as the sum of the first n entries of diagonal k:

$$\binom{n+k}{n-1} = \sum_{i=0}^{n-1} \binom{k+i}{i}$$

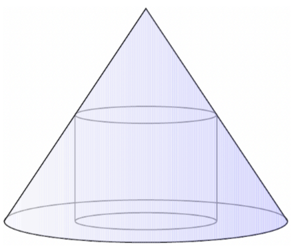
Dear fellow problem solvers,

I believe that solving problems 16 and 17 not only brought you pleasure but also offered valuable insights. Now, let's move on to the next two problems to maintain this fulfilling journey of exploration and learning.

Problem 18

Proposed by Ivan Retamoso, BMCC, USA.

What are the dimensions of the cylinder that can be placed inside a right circular cone measuring 5.5 feet in height and having a base radius of 2 feet to maximize its volume?



Note: Round yours answers to three decimals places.

Problem 19

Proposed by Dr. Michael W. Ecker, (retired) Pennsylvania State University, USA.

Prove that the diameter of a circle is the largest possible size of a chord of said circle.