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Which season is your country in right now? Summer or dry season. No matter what season you are currently going through, we can enjoy it together by reading a special issue in **Volume 15 Number 3**. It is my pleasure and pride to present ten articles from Indonesia as a part of MTRJ Mission in teaching-research and learning mathematics. Mathematics education in Indonesia cannot be separated from the historical journey of the curriculum, which can be grouped into four periods, namely traditional mathematics learning (before 1975), modern mathematics learning (1975 curriculum), current mathematics learning (1984 curriculum to 2013 curriculum), and Merdeka curriculum that regards mathematics as a mental activity. The Indonesian Ministry of Education implemented the Merdeka Curriculum in February 2022. The reason for the change in the curriculum is the results of the **Programme for International Student Assessment (PISA)** in recent years, which showed that 70% of 15-year-old students do not meet the minimum competency level in basic reading comprehension and the application of basic mathematical concepts. Therefore, the Merdeka curriculum places a strong emphasis on problem-solving skills as a one of fundamental objectives of mathematics education. In addition, the curriculum offers comprehensive support to schools, aiming to enhance the professionalism of educators themselves. It is essential for a wide spectrum of stakeholders to embrace the Merdeka Curriculum—this includes not only teachers, schools, and students, but also researchers, parents, and the learning community. To address the challenges in Indonesian mathematics education, this volume is divided into two learning themes within the overarching theme of ‘problem solving’: prospective mathematics teachers and students.

Prospective teachers are featured in the first four articles, each addressing different problems. Using Didactical Design Research (DDR) with tittle “**The Effectiveness of Didactic Designs for Solutions to Learning-Obstacle Problems for Prospective Mathematics Teacher Students: Case Studies on Higher-Level Derivative Concepts”**, Entit Puspita, et al, highlights the importance of addressing learning obstacles in mathematics education and suggests that the developed didactic design can contribute to improving the quality of learning in high-level derivative topics. The next research is motivated by low mathematics achievements of Indonesian students, attributed to conventional learning models. Using the Outdoor Learning Mathematics Project (OLMP) prospective teachers were actively engaged in diverse activities, enabling them to explore numerous mathematical concepts and apply them effectively to complete the project.

Another team, Didik Sugeng Pambudi, et al, conducted research under the title "**The Mathematics Prospective Teachers Activities When Solving Outdoor Learning Mathematics Projects in the Campus Garden**".

Continuing in the context of prospective teachers' problem-solving abilities in the title "**Prospective Teachers' Perspectives on Collaborative Problem Solving in Mathematics**" Fatmanisa, et al, conducted interviews with participants who had different perspectives toward creative problem solving. Despite having some similarities in their views on perseverance and interest in collaboration, there are noticeable distinctions in their attitudes towards openness to problems and the value of teamwork. Let your journey about pre-service teachers end with a newfound arising from the fruitful collaboration between Indonesian and Malaysian researchers. Yanuarto, et al, in their paper "**The Moderating Model of Teaching Anxiety on Teaching Beliefs and TPACK Effect to ICT Literacy Among Pre-Service Mathematics Teachers**" has resulted in a study using Structural Equation Modeling (SEM) to examine the relationships between teaching anxiety, teaching beliefs, and pre-services mathematics teacher's ability to integrate technology into education.

On the other hand, the process of learning mathematics typically takes place in a classroom environment, where students engage with mathematical content. This volume consists of six results regarding student activities in Indonesian mathematics learning. It begins with students' difficulties in understanding algebraic notation, particularly the interpretation of letters in algebra, are the focus of the paper titled **"Is 'Fruit Salad Algebra': Still a Favorite Menu in Introducing Algebra in Schools?"** by Lia Ardiansari and colleagues. In addition to algebraic content, Muslim et al, in the paper entitle **"Student Commognitive Analysis in Solving Algebraic Problems"** describe student cognition based on open-ended algebraic problem-solving. As we know, commognitive is a combination of communication and cognition. Recommendations for further research can be obtained from Kadir's article entitled **"Students' Mathematics Achievement Based on Performance Assessment through Problem Solving-Posing and Metacognition Level"**. Through a comprehensive study and supported by complete statistical data, Kadir's suggests that teachers should adopt a teaching style that includes a diverse set of questions during problem-solving and problem-posing tasks. Through case study method, Safarini et al, in their paper **"Students' Proceptual Thinking Outcomes in Learning Differentiability Using Desmos Classroom Activities Based on The Three Worlds of Mathematics Framework"**, analyze the conceptual thinking results of 25 students as they answer problems related to the differentiability. The given problem is challenging: "show that $f(x) = \begin{cases} x, x \leq 1 \\ x^2, x > 1 \end{cases}$ is continues but NOT differentiable at x=1". Another practical problem solving is demonstrated by Ivan Retamoso as **The Problem Corner Editor**. This section presents innovative solutions and challenges to address current issues in the field, offering fresh perspectives on tackling complex problems.

Towards the end of the page, the analysis using Mathematical Understanding Layers revealed original solutions in the students' problem-solving activities. Octavina Rizky Utami Putri, et al, nicely presented it in the article "**Problem-solving: Growth of Students' Students' Mathematical Understanding in Producing Original Solutions**". The issues closed with a study on problem solving by Kamariah, a woman from the easternmost region of Indonesia, Papua. Through an article entitled **"Exploring Students' Work in Solving Mathematics Problem through Problem-Solving Phases"**, it is shown that students' onto-semiotics in solving combinatoric problems provide insight into the variations in the formation of the use of mathematical objects in each problem-solving phase.

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The Effectiveness of Didactic Designs for Solutions to Learning-Obstacle Problems for Prospective Mathematics Teacher Students: Case Studies on Higher-Level Derivative Concepts

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Abstract: Various studies have concluded that many students have difficulty understanding concepts related to function derivatives. One of the concepts in function derivatives is higher-order derivatives, primarily the n^{th} derivative pattern. This study aims to: 1) identify various types of learning obstacles experienced by prospective mathematics teacher students on high-level derivative topics; 2) design alternative didactic designs based on the findings of learning obstacles; 3) identify the effectiveness of the didactic design developed for the learning obstacle solution. This study used a qualitative method with a Didactic Design Research (DDR). Involves 41 students who have received Differential Calculus courses and 43 second-semester students at one of the universities in Indonesia. Data from test results, interviews, and document studies were analyzed through identification, clarification, reduction, and verification techniques and presented narratively. The results showed that: 1) students were still experiencing epistemological, didactical, and ontological types of conceptual and instrumental learning obstacles; 2) the development of a didactic design based on learning obstacle findings refers to the Theory of Didactical Situation combined with the Socratic Questioning technique; 3) the alternative didactic design is quite effective in overcoming the initial learning obstacle, but a new obstacle appears, namely the ontological psychological type obstacle. The results of this study indicate that the phenomenon of learning obstacle findings and the developed didactic design can be used to improve the quality of learning on an ongoing basis.

Keywords: higher-level derivatives, didactic design, learning obstacles, prospective teachers

INTRODUCTION

There are problems in mathematics, physics, and many other branches of science that cannot be solved by ordinary geometry and algebra, so calculus is needed to solve them (Rohde et al., 2012). In addition, derivatives are one of the mathematical concepts in college needed to learn other concepts, subjects, or applications to solve real-world problems (Tall, 2012; Pepper et al., 2012; Tarmizi, 2010).

In the Differential Calculus course, the derivative is an essential prerequisite for several other concepts. In the Study Program that produces prospective mathematics teachers at one of the universities in Indonesia, Differential Calculus is a subject that is a prerequisite for several other courses. Puspita et al. (2020), in their research, concluded that there was a significant correlation between the achievement of the Differential Calculus course and the student's cumulative achievement index.

Various things related to learning Differential Calculus have become interesting studies for researchers. Many researchers investigate various causes of difficulties and topics in calculus that are considered problematic by students. For example, most undergraduate students still think derivatives are difficult to learn (Willcox & Bounova, 2004; Tarmizi, 2010; Pepper et al., 2012; and Tall, 2012). These findings strengthen the results of research that focus on the difficulties experienced by students in several sub-topics of derivative functions. For example, students still have difficulty in determining the derivative of rational functions and derivatives of functions using chain rules (Tokgöz, 2012), determining the extreme values of functions (Fatimah & Yerizon, 2019). In addition, students are still experiencing difficulties learning the concept of limit, a prerequisite for function derivatives (Kim et al., 2015; Wahyuni, 2017; Fatimah & Yerizon, 2019; Arnal-Palacián & Claros-Mellado, 2022). The lack of conceptual understanding (Tall, 2012; Denbel, 2015; Orton, 1983b, Dahlia et al., 2018), there is a relationship between difficulties with mathematical thinking processes and the complexity of mathematical objects (Quezada, 2020), and the level of student ability (Gray & Tall, 1991) are suspected to be the cause of the difficulties experienced.

The difficulties experienced by students, as revealed from the various studies above, are also expected to be experienced by prospective mathematics teacher students. Therefore, studying the difficulties experienced by prospective mathematics teachers is necessary. In turn, they will spearhead the success of a learning process later when they become teachers. In this study, learning constraints will be a guide in identifying students' learning difficulties; This means that the impact of didactic design on the knowledge construction of prospective mathematics teacher students will be the focus of the study. Furthermore, knowing the learning obstacles of prospective mathematics teachers will assist lecturers in developing a hypothetical didactic design to overcome learning obstacles.

The research will focus on prospective mathematics teacher students, particularly student learning obstacles on high-level derivative topics. This topic is important to study because students must master many concepts, including function derivatives, derivatives of the quotient of two functions, sequence patterns, factorial ideas, and exponential properties. Furthermore, Students must master the concepts they have acquired before high-level derivative concepts to make it easier to solve high-level derivative problems, especially the n^{th} derivative pattern. The objectives of this study are 1) to identify various types of learning obstacles experienced by prospective mathematics teacher students on high-level derivative topics; 2) to design alternative didactic designs based on the findings of learning obstacles; 3) to identify the effectiveness of the didactic design developed for the learning obstacle solution.

THEORETICAL FRAMEWORK

If you pay attention, the causes of various student learning difficulties revealed in previous research could come from internal or external factors. Difficulties caused by external factors are referred to as obstacles; if they are associated with learning, then the difficulties are called learning obstacles. According to the Theory of Didactical Situation (TDS), an obstacle is a learning obstacle caused by external factors (Brousseau, 2005). Strictly speaking, Suryadi (2019) said that the external factor that could cause obstacles was didactic design. According to the source, there are three types of learning obstacles, namely: ontological obstacles, didactical obstacles, and epistemological obstacles (Brousseau, 2002).

According to the Theory of Didactical Situations (TDS), learning has a tiered flow that allows students to construct knowledge. The learning flow starts from an action situation, a formulation situation, a validation situation, and ends with an institutionalization situation. This learning flow is in line with the results of Obreque & Andalon's (2020) research, which concluded that most teachers understand mathematics as a priori knowledge, which requires action to discover, interpret, and formalize it. Knowledge can be constructed using the Socratic Questioning Technique at each stage of the TDS. Learning is guided by questions posed to promote students' independent thinking. The Socratic Questioning technique has six categories of questions in learning, which allows students to know the importance of questions, clarify, investigate assumptions, investigate reasons, investigate alternative solutions, and investigate answer implications (Paul, 1990).

Several studies concluded that the Socratic Questioning Technique could facilitate teachers and students to get the best results. For example, Cojocariu & Butnaru (2014) concluded that using the Socratic Questioning technique in learning can facilitate teachers to provoke students to be directly involved in learning. Higher-order thinking skills occur when students think, discuss, debate, evaluate, and analyze concepts through their thinking and those around them (Elder & Paul, 2007; Roger D. Jensen Jr., 2015).

METHOD

The qualitative method chosen in this study was the Didactical Design Research (DDR) design. DDR started to develop in 2010 (Suryadi, 2019), which explores the characteristics of learning design and its impact on the development of students' thinking processes (Fuadiah et al., 2019). At the same time, Sidik et al. (2021) said that DDR is a form of educational innovation. The paradigm used in qualitative research is the interpretive paradigm (Suryadi, 2019; Denzin & Lincoln, 2018; Creswell, 2014). The interpretive paradigm underlies researchers in understanding didactic design problems of high-level derivative topics in textbooks, which are references for Differential Calculus courses. The research data is sourced from the Respondent's Ability Test (RAT), the results of interviews, and document studies. The research data is sourced from the Respondent's Ability Test (RAT), the results of interviews, and document studies. The data from the RAT on learning obstacles was carried out qualitatively. The analysis was carried out simultaneously through data reduction techniques and presented narratively. In addition, interviews and document studies were conducted to strengthen the analysis of RAT results.

The learning obstacle findings are then used to design alternative didactic designs on high-level derivative topics. The critical paradigm is the basis for designing the intended alternative didactic design. The didactic design developed is based on the Didactic Situation Theory with four stages of situations, each step guided by the Socratic Questioning Technique. Prospective mathematics teachers became the target of a didactic design trial developed based on the findings of the learning obstacle. Finally, based on the trial results data, the effectiveness of the didactic design will be seen in overcoming the findings of the identified learning obstacles.

The grouping of research participants are: 1) to find out the initial learning obstacle, RAT was given to 41 prospective mathematics teacher students who had received a differential calculus course after this, referred to as group one, and 2) after the didactic design was developed based on the findings of the leaning obstacle, alternative didactic designs were tested on students who were attending lectures Differential Calculus has as many as 43 people after this, referred to as group two. The trial aims to see the effectiveness of the didactic design in overcoming learning obstacles. The analysis will focus on seeing which learning obstacles can be overcome, which learning obstacles still arise, and the possibility of new learning obstacles emerging. Things that need to be known further were conducted by interviewing several students as an effort to clarify and carried out after the researchers gave tests related to high-level derivative topics. In simple terms, the stages of DDR are presented in Figure 1 below:

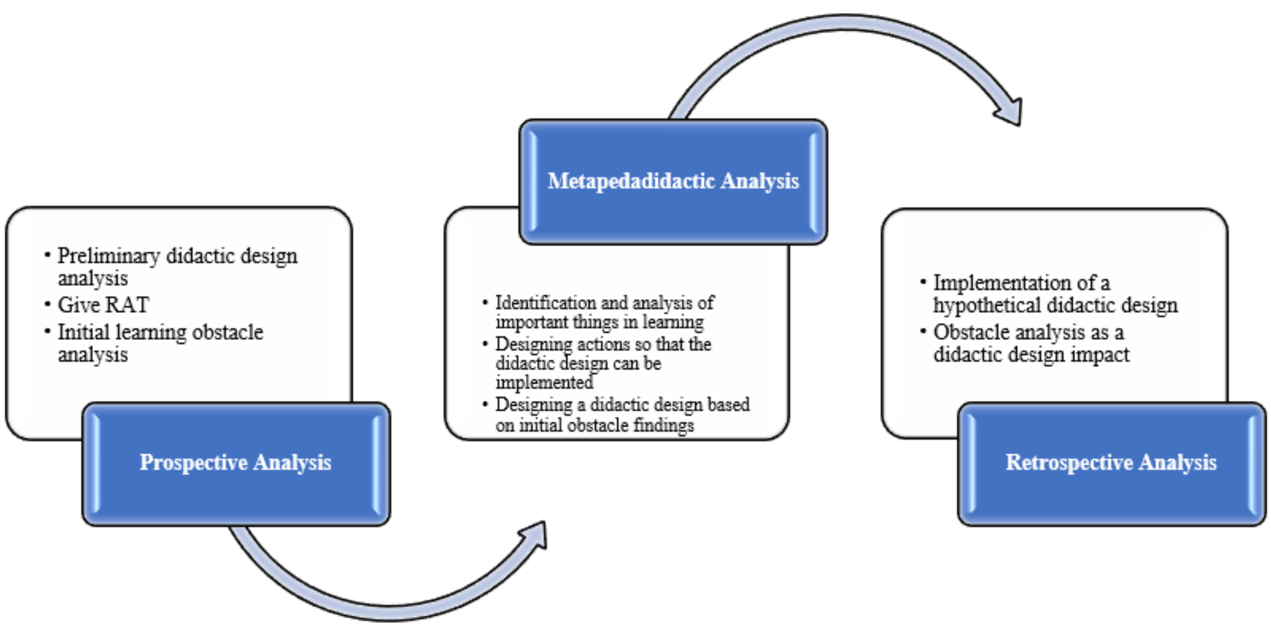


Figure 1: Stages of DDR

RESULTS

The ability of students to determine the higher-order derivative of a function depends on their understanding of various rules for finding the first derivative of a function. Students often have problems determining a higher-level derivative because they do not master the rules for finding derivatives, especially the derivative of the quotient of two functions. The reference book presents the concept of higher-order derivatives after the derivation search rules for different types of functions. In the sourcebook, examples and problems related to higher-order derivatives are generally for simple functions, one of which is a polynomial function. When high-level derivatives are associated with other concepts, for example, the derivatives of rational functions, exponential properties, sequence patterns, or others, students often experience difficulties. Based on these problems, the authors designed a test called the Responsive Ability Test (RAT) has the intention of identifying various learning barriers experienced by students in high-level derivative concepts, particularly determining the n^{th} derivative pattern of rational functions. The problem on the RAT is "Find the n^{th} derivative of $y = \frac{1}{x^2}$, present it in the simplest form!".

From the findings of the learning obstacle, a didactic design was developed to overcome these problems. Furthermore, the implementation of the didactic design aims to see its effectiveness in solving the problem of learning obstacles. Therefore, in addition to analyzing the solution to the problems found by the initial obstacle, it will also look at the possibility of the emergence of new obstacles, which will be the basis for further revision of the didactic design. The following sections describe the analysis of research findings for each research objective.

Findings of early learning obstacles

Group one students generally can determine the 1^{st}, 2^{nd}, ……, n^{th} derivatives of the given function. However, students still have difficulty determining the n^{th} derivative formula in the simplest form. Students can already use exponential properties from the answers given, so the power rule determines the function's derivative for various levels. Although some students use the derivative of the quotient without using the exponential property, the mistakes made are more than those using the exponential property.

The following figure presents some student answers regarding the derivative of a function and the general form of the n^{th} derivative:

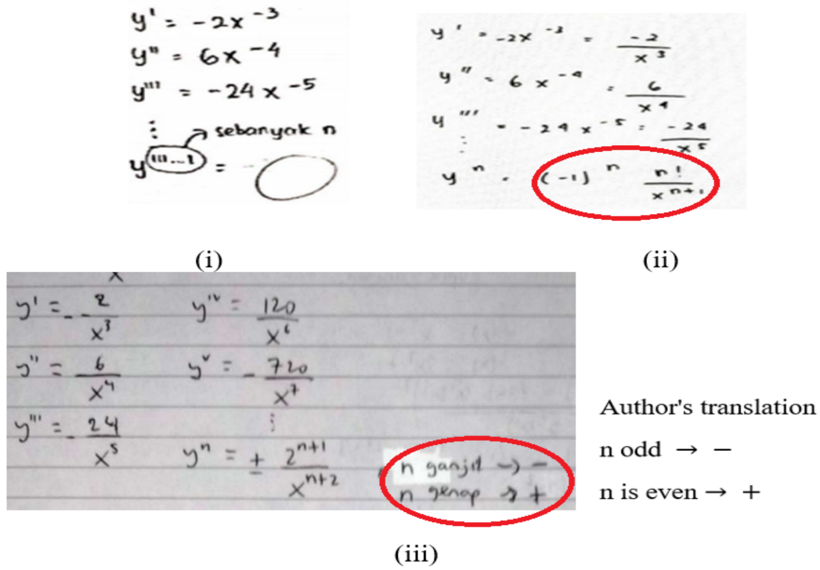


Figure 2: Student answers regarding high-level derivatives

From the answers given, some of the students' difficulties in determining the identified n^{th} term can be grouped as follows: 1) students cannot write the n^{th} term pattern (Figure 2. part (i)); 2) students can write the pattern of changing signs well, but an error occurs in determining the pattern by utilizing the factorial concept (Figure 2. part (ii)); 3) students cannot determine the pattern of changing signs appropriately, only writing positive or negative signs when n is odd or even (Figure 2 part (iii)).

The difficulties experienced by these students can be categorized as obstacles, with the following details:

a. Students experience an epistemological type obstacle because of the limited understanding and mastery of high-level derived concepts associated with their habits. For example, students are used to high-level derivatives for simple functions and never determine the pattern of the n^{th} derivative in a simple form; existing lecture notes and sourcebooks reveal these problems.
b. Students experience didactic-type obstacles caused by the existing didactic designs that are not following the continuity of students' thinking processes. In the sourcebook, it only asks students to determine the 1^{st}, 2^{nd}, ... and not asked to determine the general form of the n^{th} derivative. The functions given are generally simple functions, for example, polynomial functions. More complex functions, one of which is a rational function, are rarely used for the concept of higher-order derivatives.
c. Students experience the ontological obstacle of the instrument type; students cannot solve the problem entirely because they do not master technical matters. The technical thing in question is to determine the pattern of the sequence using the factorial concept.

To strengthen the findings of the obstacle, the authors conducted semi-structured interviews with one of the first group participants. The results of interviews with students reinforce the findings of these obstacles. The results of the interviews provide the following information: 1) Students cannot see the relationship between the 1st, 2nd, 3rd, and so on with the resulting exponents of -3, -5, -7, etc.; 2) Students cannot see patterns 2, 6, 24, and so on as regular patterns when associated with the factorial concept; 3) Students have difficulty presenting the pattern of alternating signs -, +, -, +, ... as a result of the 1st, 2nd, 3rd, etc.

In addition to these obstacles, students are also still experiencing conceptual type ontological obstacles. For example, students do not understand the concept of the quotient rule and cannot distinguish the n^{th} derivative from the value of the derivative of a function at time n. This is revealed from student answers as follows:

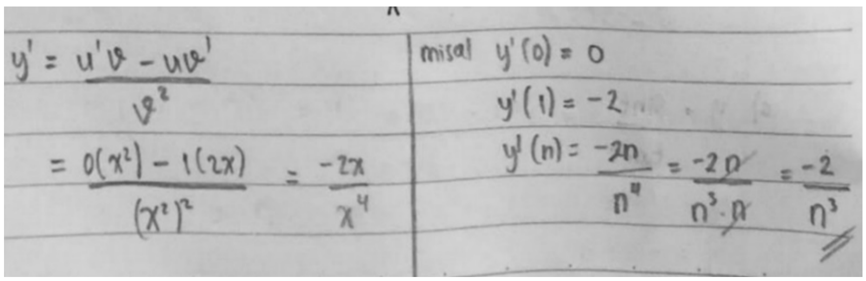


Figure 3 : Examples of conceptual type ontological obstacles

Based on the results of the analysis of various identified learning obstacles indicates the need for an alternative didactic design designed to overcome these learning obstacles.

Didactic design based on learning obstacle findings

The author designs a didactic design to overcome student learning obstacles. The didactic design developed on high-level derivative topics allows students to construct knowledge of the concept. The didactical design enables students to determine the n^{th} derivative and facilitates them to build the n^{th} derivative pattern. In addition, the stages in didactic design encourage students to relate the concepts they have learned to other concepts, such as the concept of sequences, exponential properties, and factorial concepts. The didactic design developed includes 1) topics and sub-topics, 2) predictions of student responses, 3) didactic and pedagogical anticipation, and 4) developed mathematical objects and abilities.

Topic and sub-topic components contain concepts that will be discussed and will become students' learning objects. Finally, the predictive element of student responses has various estimates of student responses that will appear during the learning process; this will be useful for lecturers in preparing various anticipations. Various student response predictions that the author identified are: being able to determine the nth derivative for all types of functions; can determine the nth derivative of a polynomial function but experiencing constraints for rational functions; experiencing problems in determining the nth derivative and the general form of the nth derivative of trigonometric functions or simple rational functions; be able to determine the nth derivative of trigonometric functions or simple rational functions but experience problems when determining the general form of the nth derivative of these functions; can apply the concept of high-level derivatives in solving a given problem.

The didactic and pedagogical anticipation components contain the stages of learning that follow the TDS stage. The TDS stages in question are: 1) action situations, in the form of presenting problems so that it helps students determine high-level derivatives, understand the meaning of function derivatives, and determine the general form of the n^{th} derivative of several types of functions; 2) formulation situations, students are directed to the formation of an understanding of high-level derivative concepts; 3) validation situations, if several students have different formulations or even come up with erroneous constructions, then a validation process is needed that allows improvements or reinforcement of the concept to be made; and 4) the institutionalization situation, at the end of the TDS design stage closed with problems, aiming to see students' abilities in applying high-level derivative concepts to other problems in different contexts. Through the Socratic Questioning Technique, students are guided in understanding problems, formulating, and validating derivatives for various levels of rational and trigonometric functions. The stages continue until students can determine the general pattern of the n^{th} derivative of the given function.

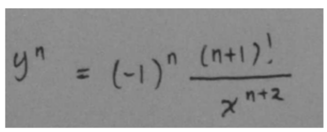
The didactic design developed by the researcher based on the findings of the initial learning obstacle contains four components. The topic components discussed are high-level derivatives; the student response prediction component contains various student response predictions, as explained in the second paragraph of this section. The other two components, namely the didactic and pedagogical anticipation components, contain various stages of TDS, while the components of the mathematical objects developed are high-order derivatives (including polynomial functions, trigonometric functions, and rational functions), general forms of high-order derivatives, and high-order derivative applications. The didactic anticipation and pedagogical design components contain situation stages in TDS, including (a) action situations in the form of problem presentations related to high-level derived notations and meanings that stimulate participants to think; (b) the formulation situation in the form of efforts made by researchers to lead to the formation of an understanding of high-level derivative concepts; (c) the validation situation contains case examples in the form of a rational function that encourages participants to carry out validation at various stages of determining the derivative and its n^{th} derivative pattern; (d) the institutionalization situation in the form of presenting various problems that encourage participants to apply high-level derivative concepts.

The Effectiveness of Alternative Didactic Designs on Learning Obstacle Solutions

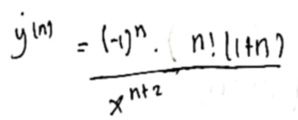
After the implementation of the didactic design in group 2 learning, the didactic design's effectiveness will be analyzed. For example, will it be investigated which early learning obstacles are overcome, which learning barriers are still emerging, or are new learning barriers emerging?

After implementing the alternative didactic design, it was found that most of the learning obstacles could be overcome properly. However, the findings indicate that there are still unresolved obstacles. The initial constraint findings still emerge after the implementation of the didactic design, although in a slightly different form. These problems will be considered in making improvements to the didactic design. This process continues as an effort to improve the quality of learning on an ongoing basis. In detail, the analysis results after implementing the developed didactic design are presented below.

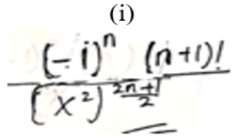
The developed didactic design has overcome various obstacles in group one. Students' answers in group two corroborate these findings. For example, some of the results related to determining the simplest form of the nth derivative are:



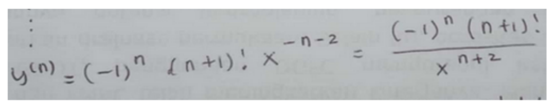
(i)



(ii)



(iii)

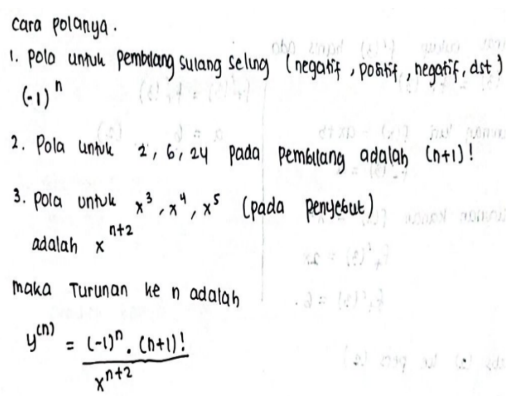


(iv)

Figure 4: Student answers after the implementation of the didactical design

Associated with problems related to the initial obstacles indicated, an analysis will be carried out on the various answers given after the implementation of the didactic design. From various student answers, the authors conclude that 1) Students can determine patterns by utilizing the factorial concept in the general form, namely $(n + 1)!$, even in a different form (flexibility) by presenting it in the form of $(n + 1)n!$; 2) Students already understand the concept of a sign change sequence and present it in a general form, and can even give it in a different condition.

Students can generally determine the available form of the nth derivative of the given problem. This is reinforced by the answer of one of the students who not only correctly determined the general form of the nth derivative but provided detailed processing steps. This phenomenon illustrates that students completely master the concepts of quotient derivatives, power derivatives, factorial concepts, and the concept of sign pronouns. The following picture presents student answers regarding the problem in question:



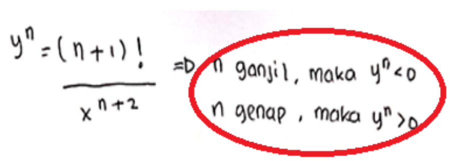
Author's translation:

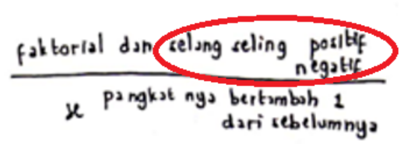
How to determine the pattern:

1. The pattern for alternating numerators (-, +, -, +) is presented in the form $(-1)^n$
2. The pattern for 2, 6, and 24 in the numerator is $(n+1)!$
3. The pattern for x^3, x^4, x^5 ... (in the numerator) is x^{n+2}
4. Then the nth derivative is $y^{(n)} = \frac{(-1)^n\,(n+1)!}{x^{n+2}}$

Figure 5: Evidence of complete mastery of derivative concepts

Although the didactic design has overcome most obstacles, some students still have difficulty determining the pattern of alternation of signs in a general form. For example, some students only associate the change of sign with the type of the nth derivative; the nth derivative will be negative when n is odd and positive when n is even. In addition, some students only mention the sign of the nth derivative, which will alternate negative and positive. The following student answers reveal the problem:





(i) (ii)

Author's translation

n is odd, then $y^n < 0$

n is even, then $y^n > 0$

$\frac{\text{factorial and sign alternating positive negative}}{\text{x is increased to the power of one from before}}$

(i) (ii)

Figure 6: Examples of psychological-type ontological learning obstacles

The writer categorizes this difficulty as an ontological obstacle of psychological type because mentally, students are not yet ready to receive knowledge. The students' weak desire to determine the line pattern without considering the possible value of the variable x indicates the student's unpreparedness to obtain knowledge.

The findings reveal that research participants still experience various learning obstacles in high-level derivative concepts. The causes of these learning obstacles stem from external factors, namely mental readiness to receive knowledge, the applied didactic design, and the limited context that students have. The obstacle experienced by prospective mathematics teacher-students is focused on the concept of high-level derivatives, especially determining the pattern of the n^{th} derivative. A weak understanding of rational function derivatives, factorial concepts, exponential concepts, and the concept of a sign-changing sequence is allegedly the cause of the obstacle. The results of the study are in line with the results of the study: 1) Tokgoz (2012), students still have difficulty in the concepts of rational function derivatives and function derivatives using the chain theorem; 2) Tarmizi (2010), Tall (2012), Pepper (2012), Hashemi (2014), and Dahlia et al. (2018), understanding of derivative concepts is still weak and is a concept that is considered difficult by students; 3) Orton (1983), still found fundamental errors in the concept of derivatives from students majoring in mathematics.

The study also concluded that students experienced learning obstacles included in the Didactical obstacle category. The initial didactic design did not accommodate students in constructing the general form of the n^{th} derivative. The function given to the concept of higher order derivatives was generally simple (eg a polynomial function), and no examples asked students to determine the nth sequence pattern. The author suspects the initial didactic design did not provide students sufficient experience constructing the nth derivative pattern. Learning obstacles experienced by students can be a consideration for lecturers in designing didactic designs. The didactic design developed can facilitate students to construct knowledge without experiencing significant obstacles. This statement is in line with the opinion of 1) Amzat et al. (2021), who said that the task of educators is not only to educate but also to develop a curriculum, including preparing didactic designs; 2) Arnal-Palacian & Claros-Mellado (2022), who concluded that the teaching method was one of the causes of the difficulties experienced by students.

CONCLUSIONS

Some of the prospective mathematics teachers in group one still experience learning difficulties in high-level derivative concepts. The various types of learning obstacles found were: 1) epistemological type obstacles, due to limited understanding and mastery of high-level derived concepts associated with their habits; 2) didactic learning obstacles arise because the didactic design does not follow the flow of students' thinking processes, 3) the instrument-type ontological obstacle, the student cannot solve the problem entirely because they do not master technical matters related to the sequence pattern using the factorial concept. The findings of learning obstacles experienced by prospective mathematics teacher students in determining the nth derivative pattern are caused by a weak understanding of the concept of rational function derivatives composite. Besides that, it also indicated that the learning barriers experienced are caused by the understanding of the concept of rational function derivatives which is still low, this is in line with the research of Tarmizi (2010), Tall (2012), Pepper (2012), Hashemi (2014), and Dahlia et al. (2018). After the implementation of the didactic design, which was developed based on the findings of the obstacles in group one, the results showed that most learning obstacles could be overcome properly. Therefore, the researcher concludes that the didactic design developed effectively overcame the learning obstacle problem. However, initial obstacles still arise after the implementation of the didactic design, although with a slightly different form. These problems will be taken into consideration to make improvements to a better didactic design. The lecturer has to develop a didactic design that can overcome various student learning obstacles because the lecturer, apart from being an educator, also acts as a curriculum developer, one of which is developing a didactic design; this statement is in line with Amzat et al. (2021).

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The Mathematics Prospective Teachers Activities when Solving Outdoor Learning Mathematics Projects in the Campus Garden

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Abstract: The facts show that the mathematics achievement of Indonesian students is still very low. One of the causes of this problem is that the learning model used by the teacher is still conventional. This study aims to describe the activities of Mathematics Prospective Teachers (MPT) in completing assignments/projects using the Outdoor Learning Mathematics Project (OLMP). The research subjects were 29 MPT at the Faculty of Teacher Training and Education University of Jember, Indonesia, in August 2022. The type of research was quasi-experimental using one group pretest-posttest design and a one-shot case study. Data were collected using observation sheets, worksheets, learning outcome tests, and questionnaires. Data analysis used descriptive qualitative statistics. From the results of the data analysis, it can be concluded that the OLMP model is effective in increasing the activities of MPT. This can be seen that all groups are very active in carrying out very varied activities during learning activities in the Campus Garden. They can find many mathematical concepts and apply them to complete projects. In addition, with an N-gain score of 0.79, the OLMP model is effective in improving the learning outcomes of MPT. The majority of the subjects (97.59%) have a very positive response concerning the application of the OLMP teaching model. In the future, further research needs to be carried out using more subjects from various fields of science, and in many places. It is hoped that MPT can be skilled in guiding students using the OLMP model when they are already teachers at school.

Keywords: Learning Activities, Learning outcomes, Mathematics Project, Outdoor Learning, Students' Response.

INTRODUCTION

The Mathematics Prospective Teachers (MPT) need to be guided in mastering skills using various innovative learning models. This is necessary so that when they become mathematics teachers in schools, they can use this innovative model in the mathematics learning process to improve students' mathematics achievement. Because the facts show that until now the mathematics learning achievement of Indonesian students is still very low. For example, the results of the PISA tests of fifth-grade elementary and 8 junior high school students in Indonesia for mathematics are still very low. The ranking of Indonesian students in 2012 was 64 out of 65 participating countries (OECD, 2014), in 2015, ranking 62 out of 70 countries (OECD, 2016), and in 2018 ranking 73 out of 79 countries (OECD, 2019).

The low mathematics achievement of students is caused by many teachers still using conventional learning models. The characteristic of conventional learning is that it is mechanistic which is oriented to the completeness of the material and is only carried out in the classroom. As a result, this kind of learning is less meaningful, less interesting, and boring for students and low learning achievement (Fauzia, et al., 2020; Fonseca, et al., 2020; Pambudi, 2022; Yeh, et al., 2019). Reflecting on these problems, the conventional model should immediately be replaced with an innovative learning model. Therefore, MoECRT RI (2022) recommends the use of a project-based learning model (PjBL) in schools.

So far, MPT only attend lectures in the classroom, so they need to be guided to attend lectures outside the classroom, for example in campus gardens. For learning activities using the OLM method in campus gardens to be effective, it needs to be integrated with the PjBL model, which we named the OLMP (Outdoor Learning Mathematics Project) model (Pambudi, et al., 2022). This means that MPT are allowed to carry out various activities to find mathematical concepts and complete tasks/projects related to the campus garden. There are several interesting things to study regarding the application of the OLMP model to MPT in campus gardens. The questions posed in this study are as follows:

1. How is the activity of MPT in learning mathematics using the OLMP model?
2. What mathematical concepts are found in mathematics learning activities using the OLMP model?
3. Is the OLMP model effective in improving the learning outcomes of MPT?
4. What is the response of MPT after doing the activity of finding mathematical concepts using the OLMP model?

The purpose of this study is to describe the activity, mathematical concepts, learning outcomes, and the response of MPT when they doing the activity of finding mathematical concepts using the OLMP model.

LITERATURE REVIEW

Project Based Learning (PjBL)

PjBL is a learning model that guides students to complete assignments/projects in groups. PjBL can be applied to various subjects to guide students to achieve abilities, such as collaborating, thinking creatively, critically, and communicatively, and improving student learning outcomes (Irham, et al., 2022; Kay, & Greenhill, 2011; Williams, & Charless-Ogan, 2016). The PjBL syntax is as follows: (1) The teacher selects a subject, topic, and context, (2) The teacher designs a project from the selected topic, (3) The teacher determines the project completion time, (4) The teacher motivates, and monitors project completion, (5) Report generation and group presentations, (6) Assessment: learning activities, attitudes, skills, and learning outcome tests, and (7) Reflection and follow-up (Wolpert-Gawron, 2016; Viro, et al., 2020; Haatainen, & Aksela, 2021; Markula, & Aksela, 2022; MoECRT RI, 2022).

Outdoor Learning in Mathematics (OLM)

In learning mathematics, the teacher not only guides students to learn abstract concepts in the classroom, but the teacher can complement it by using the Outdoor Learning method (Richmond, et al., 2017). Outdoor Learning in Mathematics (OLM) is the right method to guide students in learning mathematics by doing mathematics, namely finding concepts and applying them to solve problems outside the classroom. This method is widely applied in various countries, such as in Chile (Vásquez et al., 2020), Europe (Bearnes & Ross, 2010; Bilton, 2014; Fägerstam, & Blom, 2012; Waite, 2011; Zotes & Arnal-Palacián ,2022), USA (Feille, 2021; Moss, 2009), Australia (Laird, et al., 2021; Thomas, 2018); and Malaysia (Mohamed, et al., 2021; Samsudin, et al., 2021). OLM succeeded in increasing student motivation, students' learning outcomes (Cahyono, et al., 2020; Fägerstam, & Blom, 2012; Laird, et al., 2021; Pambudi, 2022; Widada, et al., 2019), and improving students' connections (Ernawati, & Amidi, 2022; Haji, et al., 2017; Pambudi, et al., 2022). Unfortunately, this method is still rarely used by mathematics teachers in Indonesia (Haji, et al., 2017; Pambudi, 2022; Widada, et al., 2019).

METHOD

Research Subject, Place, and Time

The research subjects were students of MPT, FKIP University of Jember (Unej) semester 3 from class C as many as 29 people (6 Male and 23 Female). Data collection was carried out in the campus garden on August 2022.

Research Types and Approach

The type of research is quasi-experimental using a one-group pretest-posttest design combined with a one-shot case study design (Arikunto, 2013; Creswell, 2014; Sugiyono, 2015; Pambudi, et al., 2022). The approach used is descriptive qualitative, to describe activities, findings of mathematical concepts, and responses to MPT in learning using the OLMP model.

Method of Data Collection

The research data consists of (1) research subject activities during the learning process using the OLMP model; (2) mathematical concepts found from activities using the OLMP model; (3) the subjects' responses to the application of the OLMP teaching model; (4) subjects' learning outcomes. The data collection methods used were observation, test, and questionnaire.

The instruments used are: (1) an observation sheet to observe the activities of research subjects, (2) a Students' Worksheet (SW) to record mathematical concepts found during learning, (3) the subject's response questionnaire to the OLMP model contains 10 questions according to the Likert scale, and (4) The written test contained 3 questions (Geometry, Trigonometry, and Differential Calculus) to determine the subject's learning outcomes. All of these instruments have been validated by 3 mathematics education experts, and are declared valid, with a score between 4.3 to 4.7 (scores ranging from invalid 0 to a maximum score of very valid 5.0). The Cronbach's Alpha are declared reliables, with a score between 0.75 to 0.85 (scores ranging from not reliable 0 to a maximum score of very reliable 1.0).

Through the OLMP model, students are given the task of observing, and designing projects related to mathematics in the campus garden of FKIP Unej. There are 3 projects agreed to be done by MPT.

Figure 1: Campus Garden

Project 1: measuring the perimeter and area of the campus garden (Figure 1).

a) Make a sketch, and measure the perimeter and area of the campus garden.
b) What is the area of the campus garden that does not include the road, Gazebo, and fountain pool in the middle of the garden?

Figure 2: Fountain Pool

Project 2: Calculating the speed related to the filling of water in the fountain pool (Figure 2). When water is put into the pool with a tap water hose at a uniform rate of 2 liters per minute. How fast is the water surface rising when the water depth is 20 cm?

Figure 3: Flagpole

Project 3: measuring the height of the flagpole (Figure 3).

a) Observe, sketch, and measure the height of the flagpole using the concept of Trigonometry.
b) Think of other ways to measure the height of the flagpole.

METHOD

The data from the observation of learning activities and mathematical concepts found by the subject during learning using the OLMP model were analyzed descriptively and qualitatively. The research subject's response questionnaire to the OLMP model consisted of 10 positive statements with answer choices of SA (Strongly Agree), A (Agree), N (Neutral), D (Disagree), and SD (Strongly Disagree). Score for answer choices SA = 5, A = 4, N = 3, D = 2, and SD = 1. A goodness-of-fit test of one-sample Kolmogorov-Smirnov test was employed to determine the uniformity of data by following the level of significance is 5%, using SPSS software version 21.

Students learning outcome test results were analyzed using an effectiveness test (N-gain score) on pre-test and post-test data (Hake, 1999; Sundayana, 2015), with the formula as follows.

$$N - gain\ score = \frac{posttest\ value - pretest\ value}{ideal\ value - pretest\ value}$$

The categories N-gain score (g) can be seen in Table 1.

N-Gain Score (g)	Category
$-1.00 < g < 0.00$	Occur drop
$g = 0.00$	Permanent
$0.00 < g < 0.30$	Low
$0.30 < g < 0.70$	Moderate
$0.70 < g < 1.00$	High

Table 1: Criteria for N-Gain Score (g)

RESULTS

The MPT Activities in Solving Projects in the Campus Garden

The MPT activities while working on projects vary widely, both physically and mentally. There is a member who holds the end of the roll meter, and another member pulls the roll meter to the end of the measurement. Other members record the data obtained at the SW. Other members observe and advise on the accuracy of measurements. Then they sat on a garden bench to sketch the campus garden, discuss the results of their work group, and complete the SW. Some MPT activities can be seen in Figure 4.

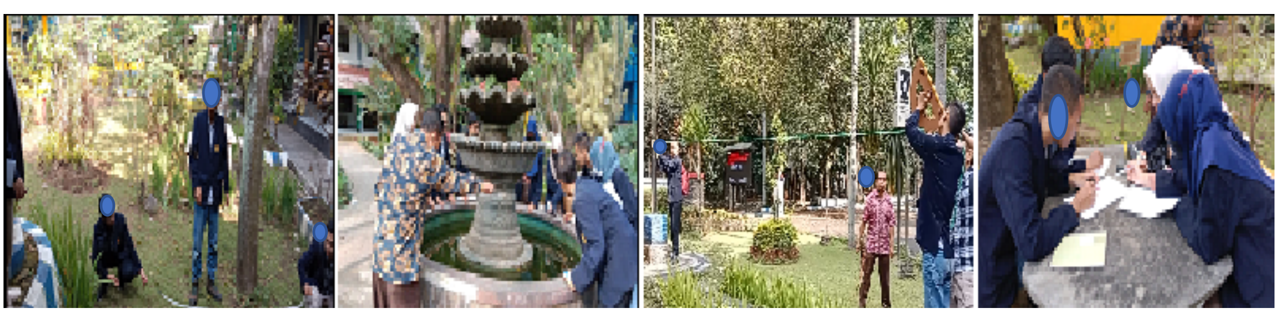


Figure 4: MPT Activities in Solving Mathematics Projects in the Campus Garden

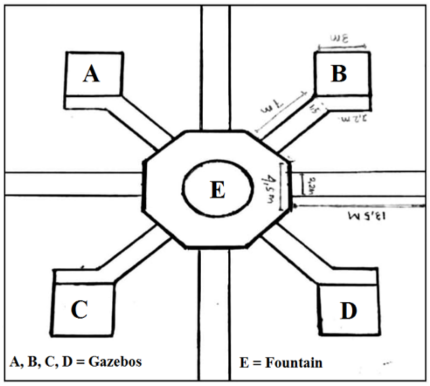


Figure 5: A sketch of FKIP Unej's campus garden

In the second project, after the MPT collected data directly on the radius ($r=1.25$ *m*), pool height (*h meters*), and pool volume ($V=\pi r^2 h$). After that, they used the concept of derivative and related rate ($dV/dt=2$ *liters/minute*), to calculate the rate of increase in water (dh/dt) which was introduced into the pond at a time $h = 20$ *centimeters*. After performing arithmetic operations on real numbers, MPT get the answer $dh/dt=0.0004$ *liters/minute.*

The third project is to measure the height of the flagpole using the concept of Trigonometry, and other ways. Here, all group members collaborated to carry out various activities, such as the first member who served as an observer observing the top of the flagpole with a clinometer. The second member measured the height of the observer and the distance of the observer to the flagpole. The third and forth members made a sketch and wrote the results of the experiment in SW. From the right triangles obtained on the flagpole, and the flag hoist, the group can find the formulas for Sine, Cosine, and Tangent on the sides of the right triangle. Next, they calculated the height of the flagpole using the tangent formula. Because the angle of the clinometer is 37^o, then the angle of elevation between the line from the top to the horizontal is $90-37^o=53^o$. By substituting the elevation angle data, the observer's distance to the flagpole ($b=7$ *meters*), and the observer's height of 1.6 meters, the flagpole height is close to 11 meters.

After all groups have completed the project, the lecturer asks a lighter question "Do you think there are other ways to calculate the height of the flagpole?" This question is directed to guide the MPT to think creatively and connect various mathematical concepts. Some of them answered the question in three other ways. The first method uses the concept of an isosceles right triangle (this concept was learned in elementary school). The concepts connected are the definition of an isosceles right triangle, the properties of an isosceles right triangle, right angles, elevation angles, and flagpole heights. By forming a right angle between the flagpole and the horizontal rope at the base of the flagpole and an elevation angle of 45^o, the height of the flagpole is equal to the distance from the flagpole to the observer. The second method uses similarity triangles (this concept was learned in junior high school). The concept that is connected is the concept of the similarity of triangles to the height of the flagpole. The third method uses the length of the shadow of an object due to sunlight (this concept was learned when studying science in junior high school).

Recapitulation of Mathematical Concept Findings by MPT

Next, from the results of working on 3 projects, MPT wrote down the findings of mathematical concepts in SW. The concepts they found are presented in Table 2.

No.	Projects/Topics/Courses	Concept Finding
1.	**Project 1**: the context of measuring the perimeter and area of the garden /**Topic/Course**: Geometry	Area and perimeter of rectangle (whole garden.; Square area (gazebo base); Area of a regular octagon (the base around the fountain); Radius, diameter, and circle on the fountain pool; Fountain pool height; Fountain pool volume, and Real number operations.
2.	**Project 2**: the context of related rate **Topic/Course**: Differential Calculus	Fountain pool volume; Derivative.; Related rate; Real number operations.
3.	**Project 3**: context measuring the height of the flagpole. **Topic/Course**: Trigonometry and related subjects	Rectangular; Right triangle; Pythagorean Theorem; Elevation angle; Sines, Cosines, Tangen; Comparison; Similarity of triangle; Length of the object's shadow due to sunlight; Real number operations.

Table 2: Mathematical Concepts Found

From collaborative activities, the MPT were able to rediscover many mathematical concepts to complete 3 projects on the topics of Geometry, Differential Calculus, and Trigonometry. On the topic of Geometry, they found the concept of a square, from the shape of a garden, then the area and perimeter of a square, the area, and perimeter of a rectangle, a circle, the area of a regular octagon, and the volume of a tubular fountain. On the topic of Trigonometry, they can develop creativity to calculate the height of the flagpole. Here, they can use the concepts of trigonometry, the similarity of triangles, isosceles right triangles, and use the length of the flagpole's shadow due to sunlight.

The MPT Learning Outcomes

Before the implementation of learning using the OLMP model, the MPT were given a Pre-Test, and after that, they were given a Post-Test. The results of the Pre-Test and Post Test can be seen in Table 3.

No.	Pre-test (n=29)				Post-test (n=29)			
	Geo	Cal	Tri	Average	Geo	Cal	Tri	Average
Maximum Score	80.00	70.00	100.00	78.33	100.00	100.00	100.00	100.00
Minimum Score	70.00	40.00	90.00	65.00	90.00	80.00	90.00	88.33
Average	77.24	50,17	93.28	73.56	97.24	87.59	98.79	94.54
St Deviation	3.43	6.19	4.28	3.20	3.68	6.76	2.55	3.05

Table 3: Pre-Test and Post-Test Results

From Table 3, it can be seen that the average Pre-Test score of 73.56 increased to 94.54 from the Post Test results. From this increase, if the N-gain score is calculated, the results are (94.54-73.56)/(100-73.56) = 20.98/26.44 = 0.79, which means that the OLMP model is effective (high category) in improving the learning outcomes of the MPT. An example of a MPT's work for a GEO (Geometry) problem is as follows.

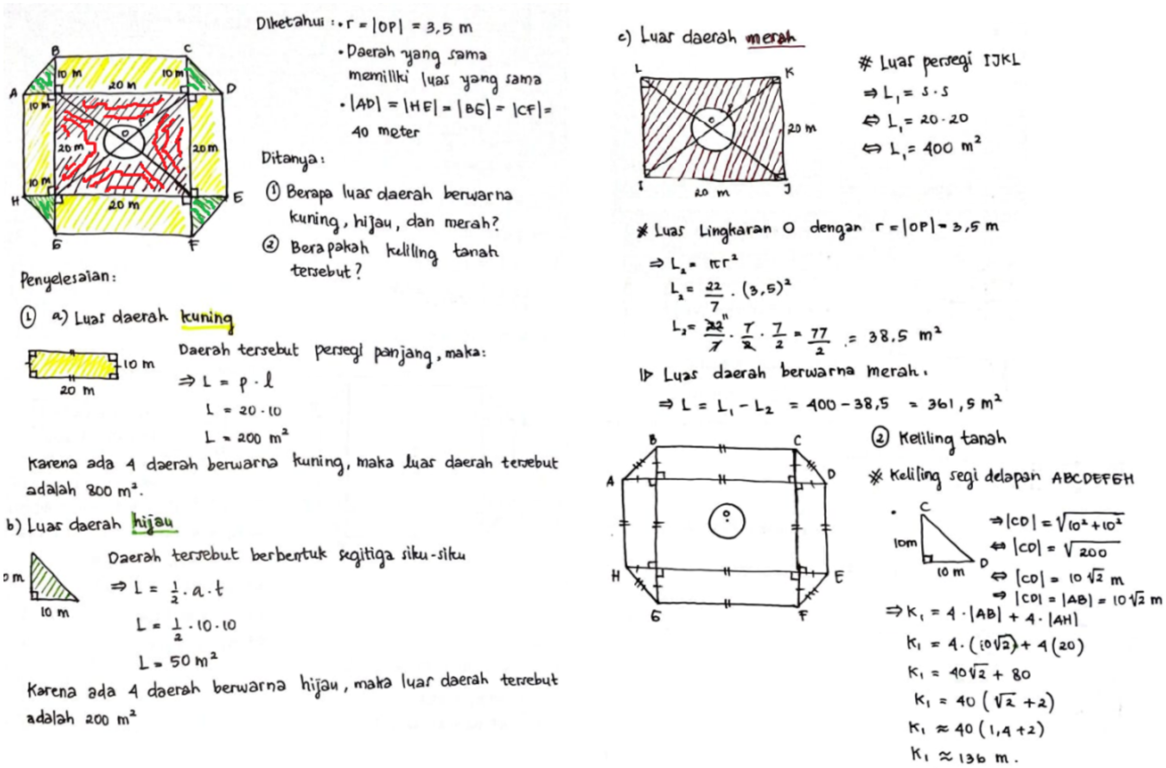
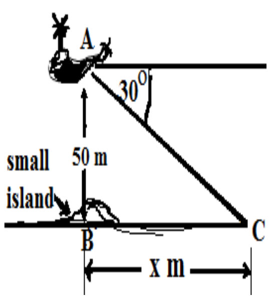


Figure 6: Test results on Geometry post-test questions

The post-test questions for Trigonometry can be seen in Figure 7. An example of an answer from a MPT can be seen in the same figure.



A helicopter pilot at an altitude of 50 meters looks directly at a small island (B) in the Bali strait. From the helicopter, the pilot also saw a place opposite the small island, namely, point C. If the angle of depression is 30°, what is the distance of the islet to point C?

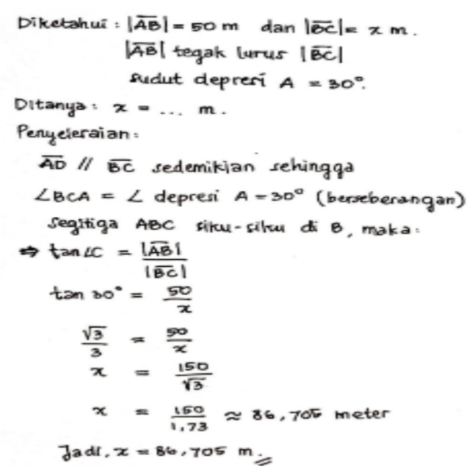


Figure 7: Post-test question and answer for Trigonometry

From their activity in working on projects in the campus garden, they found many mathematical concepts related to Geometry, Differential Calculus, and Trigonometry. This certainly encourages them to study harder to face the post-test. This high learning motivation is able to lead them to achieve optimal achievement in the post-test.

The MPT's response about the OLMP model

The MPT's response to the OLMP model was obtained from the results of filling out the questionnaire. To find out the subject's response to the OLMP model, a test was carried out using the Kolmogorov-Smirnov test. The test results for the 10 statements SA and A can be seen in Table 4.

From Table 4, it was concluded that 97.59% of the respondents agree and strongly agree with the application of the OLMP teaching model in the mathematics learning process. This means that the majority of the MPT have a very positive response concerning the application of the OLMP teaching model. MPT argue that the OLMP model has more benefits than the conventional model.

No.	Statements of Questionnaire	SA and A Chosen (%)	Kolmogorov Smirnov
1.	The OLMP model makes MPT more physically and mentally active in the mathematics learning process compared to conventional models	100.00	3.343
2.	The OLMP model makes MPT can improve collaborative, creative, and communicative abilities	100.00	3.157
3.	The OLMP model makes mathematics learning more meaningful than the conventional model	100.00	3,528
4.	The OLMP model makes mathematics learning more unforgettable compared to conventional models	100.00	3.343
5.	The OLMP model makes the learning atmosphere more relaxed, fun, and not boring compared to the conventional model	100.00	2,785
6.	Using the OLMP model, MPT can be taught the importance of environmental care attitudes	100.00	3.157
7.	The campus garden is always a comfortable, beautiful, and healthy place to study	100.00	2,971
8.	Learning mathematics in the campus garden can make prospective teachers have a positive attitude towards mathematics	100.00	3.343
9.	The OLMP model makes MPT a high motivation to learn mathematics compared to conventional models	100.00	3.157
10.	The OLMP model can make MPT achieve optimal learning outcomes compared to conventional models	75.86	1.857
	Average	97.59	

Table 4: Kolmogorov Smirnov test of Strongly Agree (SA) and Agree (A) on MPT's responses about the OLMP model

DISCUSSION AND CONCLUSIONS

From the above explanations, we can discuss about several interesting results. Firstly, the use of the OLMP model can increase the activity of the MPT in learning mathematics. This can be seen from the observation data that all groups (100%) are very active in carrying out very varied activities during learning activities. They do more activities when studying outside the classroom than when studying in the classroom. Secondly, the use of the OLMP model can improve MPT activities, and collaboration skills. This can be seen from all groups planning, and carrying out activities to complete projects together. These results are followed by the results of research by Laird, et al. (2021); Mygind, (2007); and Pambudi, et al. (2022). It is clear that the campus garden is effectively a place for MPT to improve collaborative skills and creativity in finding many mathematical concepts and applying them to complete projects given by lecturers. These results are followed by the opinion of Agusta, & Noorhapizah (2018); Auliadari, et al. (2019); Usmeldi, & Amini (2022); and Pambudi, et al. (2022). Thirdly, the use of the OLMP model can improve MPT learning outcome. These results are are followed by the opinion of Auliandari, et al. (2019); Fägerstam, & Bloom (2012); Fonseca, et al. (2020); Laird, et al. (2021); and Pambudi, et al. (2022). Thus, the OLMP model can improve activities, collaborative, creative, and communicative abilities. This strongly supports learning oriented toward improving life skills in the 21st century (Wijaya, et al., 2019; Irham, et al., 2022; Kay, & Greenhill, 2011; Mabitad, et al., 2021; Usmeldi, & Amini, 2022).

In addition, the OLMP model makes mathematics learning more meaningful and unforgettable. Because, using the OLMP model, the MPT can also be taught the importance of environmental care attitudes, such as maintaining cleanliness in campus gardens, and not littering, so that the garden is always a comfortable, beautiful, and healthy place to study. This is followed by the opinion of Agusta, & Noorhapizah (2018); Mann, et al. (2022); and Pambudi (2022). A comfortable and pleasant learning atmosphere in campus gardens has made the MPT have a positive attitude toward mathematics so motivation increases (Cameron, & McGue, 2019; Mackenzie, et al., 2018; Ryan, et. al, 2010). The more MPT learn, both independently and in groups, the more mathematical concepts are found and understood by them. Mastery of mathematical concepts and sufficient learning experience make the MPT succeed in taking the learning outcome test with optimal results. So, the OLMP model can improve student learning outcomes. These results are followed by the opinion of Auliandari, et al. (2019); Fägerstam, & Bloom (2012); Fonseca, et al. (2020); Laird, et al. (2021); and Pambudi, et al. (2022). However, some subjects argued that mathematics learning achievement was not only influenced by the learning model, but could be caused by the level of difficulty of the questions and others.

From the results of data analysis, it can be concluded that the OLMP model is effective in increasing activities of the mathematics teacher candidates (MPT). This can be seen that all groups are very active in carrying out very varied activities during learning activities. They can find many mathematical concepts and applying them to complete projects. In addition, with an N-gain score of 0.79, the OLMP model is effective improving the learning outcomes of MPT. The majority of the subjects (97.59%) have a very positive response concerning the application of the OLMP teaching model. MPT argues that the OLMP model has more benefits than the conventional model.

The limitation of the research is that the research subject only uses 1 class of mathematics education students. In the future, it is necessary to carry out further research in mathematics, science, technology, and engineering students so that they become STEM Outdoor Learning Projects. Research places also need to be expanded, and set up in various forms, such as math trails, field trips, camps, and others.

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Prospective Teachers' Perspectives on Collaborative Problem Solving in Mathematics

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Abstract: This study describes prospective teachers' perspectives on collaborative problem-solving (CPS) in mathematics. The study employed a 20-item questionnaire distributed to 47 prospective mathematics teachers in Indonesia. The questionnaire responses were scored, and two participants with the highest and lowest mean score were interviewed. Despite being quite similar in perspectives on perseverance and interest in collaboration, the prospective teacher with the highest mean score showed more openness to problems and the value of teamwork. Further research can be done by investigating how prospective teachers' perspectives relate to their knowledge of CPS or teaching practices.

Keywords: Collaborative problem solving, mathematics education, teacher education.

INTRODUCTION

Along with the rapid development of research related to the 21st century, attention to the skills students need to have in this century is also proliferating, one of which is collaborative problem solving (CPS). Collaborative problem-solving is a collective problem-solving skill where the individuals involved share the required knowledge and effort (OECD, 2017). The need for an investigation of students' CPS skills had been shown by PISA 2012's focus on interactive problem solving, where students faced problems that required them to interact with tools or media to obtain adequate information (OECD, 2013). The focus was changed to CPS in PISA 2015 (OECD, 2017). PISA 2015 focused its analysis on students' CPS for the first time by developing a student CPS assessment framework.

Several studies have been conducted related to CPS in Mathematics Education. Some of these studies focus on analyzing students' problem-solving processes by comparing individual and collaborative problem-solving (Barron, 2000; Kapur & Bielaczyc, 2012; Schmitz & Winskel, 2008; Stacey, 1992). Several other studies investigated how to properly assess students' CPS (Chan & Clarke, 2017; Harding et al., 2017). In addition, the analysis of teaching and learning practices in the classroom to improve students' CPS was also the concern of several studies (Chiu, 2008; Hähkiöniemi et al., 2016). Among the studies related to CPS, studies examining prospective Mathematics teachers are still very limited. One of them is the study by Bjuland (2007), which identified the geometrical reasoning of prospective teachers in the collaborative problem-solving process.

In mathematics education, the study of prospective teachers, both their knowledge and perspective on a concept, is critical. In particular, the prospective teacher's perspective on a concept will influence how they teach in the future. Thus, the prospective teacher's perspective on CPS is vital in determining how they facilitate students practicing collaborative problem-solving skills (Xenofontos & Kyriakou, 2017). This study aims to describe the perspective of prospective mathematics teachers on collaborative problem-solving in mathematics. This description of the perspective on CPS will help study how prospective teachers perceive CPS and its potential to facilitate its practice in the future.

THEORETICAL FRAMEWORK

Collaborative Problem Solving (CPS)

CPS, which contains two important 21st-century skills, i.e., problem-solving and collaboration, has been discussed and defined by various entities. The term “collaborative problem solving” was shown to be used formally in works of literature starting in 2015 (Fatmanissa et al., 2022). PISA defined CPS as "an individual's capacity to effectively engage in a process in which two or more agents seek to solve a problem by sharing the understanding and effort required to reach a solution and pooling their knowledge, skills, and efforts to reach that solution" (OECD, 2017). The need to investigate students' CPS has been seen in PISA 2015, which focused on CPS (OECD, 2017).

Another project focusing on CPS is the Assessment and Teaching of 21st Century Skills (ATC21S) project. The project defined CPS as “approaching a problem responsively by working together and exchanging ideas” (Griffin & Care, 2015). Further, it stated that CPS is a joint activity in which a group takes several steps to turn a problem condition into a desired goal. While PISA gave a content-dependent explanation of CPS, ATC21S divided CPS into content-free and content-dependent. The content-dependent CPS involves skills and knowledge of CPS in particular content such as mathematics and science. ATC21S assessed CPS skills by assessing two skills, i.e., problem-solving as a cognitive skill and collaboration as a social skill.

Understanding CPS in Mathematics required understanding problem-solving and collaboration, but they alone were insufficient. Understanding how both constructs relate to each other and how they could be integrated into the mathematics education context should be there as well (Kapur & Bielaczyc, 2012; Munson, 2019). Therefore, problem solving in mathematics is one thing, but putting it into a collaborative context is another. It led to the need for teachers, and thus prospective teachers, to have knowledge of it and positively consider its integration into practice. For example, van Leeuwen and Janssen (2019) highlighted the importance of teachers in giving more or less control, either socially or cognitively, while identifying an essential moment for students to be engaged in CPS. This control would determine how students' CPS process turned into meaningful learning. Another study by Haataja et al. (2019) investigated teachers' visual attention in scaffolding a successful CPS process. The study showed that the teacher's visual attention was targeted most dominantly to students' papers, to follow students' problem-solving process, and to students' faces to cater for their collaboration, acknowledging the importance of the two constructs in CPS.

Prospective Teachers' Perspectives

Mathematics teacher education is challenged to prepare prospective teachers to teach and constantly reflect their views and beliefs on the issue surrounding it (Columba & Stotz, 2016; Haug & Mork, 2021). Prospective teachers' views or perspectives on teaching and learning significantly impact how they plan, orient, and evaluate their own teaching (Beswick, 2012; Chapman, 2012; Clark et al., 2014; Middleton, 1999). Understanding prospective teachers' perspectives on CPS would benefit the understanding of their future practices surrounding it.

Perspective on CPS is related to how one views CPS. The 2015 PISA conceptual framework (OECD, 2017) divides students' perspectives on CPS into two main dimensions: their perspective on problem-solving and collaboration (Figure 1). The problem-solving dimension includes the perspective on persistence and openness to problems. In contrast, the collaboration dimension is divided into an interest in and value of collaboration.

The "perseverance" sub-dimension refers to how students perceive persistence in completing a task as determining the solution to a problem. It relates to how they view the problem as something that must be solved entirely or not. Openness to problems refers to how students are open to types of problems and the steps for solving them. This sub-dimension relates to how students perceive different problems and various problem-solving strategies. The sub-dimension of interest in collaboration refers to students' interest in collective work and the variety of perspectives that may occur, while the sub-dimension of the value of teamwork refers to how students view teamwork as an important aspect of solving math problems. Although intended for students, this framework is still relevant for being used with prospective teachers as participants. The CPS conceptual framework in PISA 2015 was used to develop a prospective teacher perspective questionnaire on CPS.

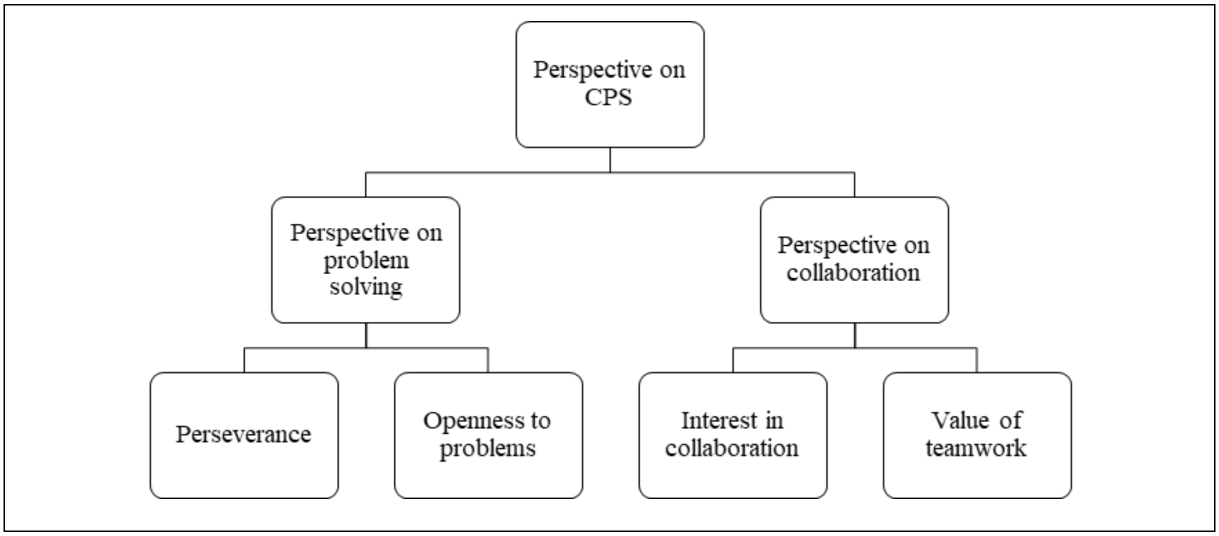

Figure 1: Dimensions of Perspective toward CPS adapted from PISA 2015

METHOD

This study utilized 47 responses from questionnaires and interviews with the selected two participants. Perspectives on CPS were identified using a questionnaire (the complete questionnaire is available upon request to the corresponding author) based on the framework from PISA 2015 (Figure 1). The questionnaire was constructed in Indonesian, and the framework for preparing it is presented in Table 1. Forty-seven prospective teachers were given a questionnaire with Likert scale options: strongly disagree, disagree, agree, and strongly agree.

Dimension	Description	# of Items	
		Positive	Negative
Perseverance	View persistence in completing tasks as determining the solution to a problem	3	3
Openness to problem	Be open to the types of problems and the steps to solve them	3	3
Interest in collaboration	Have an interest in collective work and a variety of perspectives	3	3
Value of teamwork	View teamwork as important in solving math problems	3	3

Table 1: Questionnaire construction

Each sub-dimension was translated into three pairs of positive and negative items. Each positive item contained a statement indicating a positive perspective on the corresponding dimension, while the negative item contained a negative perspective. On positive items, a larger scale indicated a more positive perspective on CPS, while on negative items, a larger scale indicated a more negative perspective on CPS. For example, participants who responded 'agree' on a negative item had a more negative perspective on CPS.

Two mathematics education experts checked the validity of the contents of the questionnaire. After the questionnaires were distributed, 47 participants' responses to the questionnaire were converted into numerical values. For positive items, the answers "strongly disagree", "disagree", "agree", and "strongly agree" were changed to scores of 1, 2, 3, and 4, respectively. For negative items, the responses were changed into a score of 4, 3, 2, and 1. Scores 3 and 4 indicate a tendency to have a positive perspective on CPS. Then, the scores of each participant are added up. An illustration of this process is given in Figure 2.

Participant A	Response	Score	
Item 1 (positive item)	Agree	3	Score conversion
Item 2 (negative item)	Strongly agree	1	
Item 3 (positive item)	disagree	2	
etc	…	…	
Total score	*30*		Total score calculation

Figure 2: Example of response scoring

The construct validity test was carried out through statistical tests by calculating the correlation between the scores of each item and the total score (Cohen et al., 2007; Creswell, 2012). If the correlation between the two was significant, the item was declared valid. The test used was the non-parametric correlation test, namely the Spearman test. The significance value or p-value in the Spearman test is less than 0.05 (significant) for all items (Table 2 and Table 3) except positive items no. 4, 6, 8, and 12. Thus, these items were discarded in further analysis.

Item No.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Correl. Coeff.	.452**	.337*	.513**	0.154	.616**	0.228	.437**	0.280	.465**	.369*	.594**	0.264

* or ** indicates a significant correlation

Table 2: Spearman Correlation Coefficient of Positive Items

Item No.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Correl. Coeff.	.485**	.549**	.467**	.383**	.534**	.541**	.453**	.659**	.354*	.525**	.655**	.291*

* or ** indicates a significant correlation

Table 3: Spearman Correlation Coefficient of Negative Item

The reliability test was done by checking the inter-rater reliability of the instrument, i.e., by calculating the Cronbach Alpha coefficient of the instrument (Creswell, 2012). The test is considered reliable if the Cronbach Alpha coefficient is high. The value of the Cronbach Alpha coefficient is 0.825 and is considered very reliable (Hendriana & Sumarmo, 2014).

Based on the mean scores on the questionnaire, two prospective mathematics teachers were selected to be interviewed, i.e., one prospective teacher with the highest mean score (initial AA) and one prospective teacher with the lowest mean score (initial JH). The two prospective teachers were chosen to help us contrast two different kinds of perspectives upon CPS. Both of them were senior-year students majoring in mathematics education at their university. Both prospective teachers had passed a course on problem-solving in teaching and several mathematics courses (e.g., geometry, calculus, discrete mathematics, etc.), expecting them to solve non-routine problems. At the beginning of their senior year, they also underwent a practice teaching program in a real classroom under the guidance of a mentor teacher.

To explore the perspective of prospective teachers more deeply, especially in each sub-dimensional perspective on CPS, semi-structured interviews were conducted with two selected prospective teachers. Each participant was given a separate interview schedule and interviewed individually to minimize the possibility of sharing information. The interview process began with confirming their willingness to be interviewed. The interview focused on extracting participants' opinions about mathematical problem-solving, collaboration, and collaborative problem-solving. The guiding questions included, but were not limited to: (1) what do you consider the most in solving mathematics problems? (2) what do you think about solving mathematical problems in groups? (3) As a prospective teacher, what do you think about implementing collaborative problem-solving activities in the classroom? All interviews were recorded and transcribed for analysis.

In the analysis process, interview transcripts were coded based on the identified sub-dimensions and further analysed to explain each prospective teacher's perspective. Statements expressing opinions on a particular sub-dimension were coded as in Table 1. The coding process referred to the description of each sub-dimension (Table 4) and then written in a memo. For example, opinions that considered the different perspectives of friends were included in the "interest in collaboration" sub-dimension (IC code). Code-based statements were collected and then categorized based on the similarity of perspectives presented in them. Analysis of the interviews was used to enrich the findings obtained from the questionnaire.

Sub-dimension	**Code**	**Sub-dimension**	**Code**
Perseverance	P	Interest in collaboration	IC
Openness to problem	OP	Value of teamwork	VT

Table 4: Coding scheme of interview transcripts

RESULTS

As indicated by the score distribution on each item and the total score, there was a tendency for positive perspectives toward collaborative problem-solving (Table 5). Prospective teachers' responses were broken down into each dimension, and it was shown that openness to problem and interest in collaboration had the most range of scores. Respondents' total scores ranged from 49 as the lowest to 76 as the highest. An interview was conducted with two prospective teachers to further understand prospective teachers' perspectives, especially on explaining the range of perspectives from the least favouring CPS and the most favoring CPS.

Dimension	Score Average	Score Maximum	Score Minimum
Perseverance	3.35	4.00	2.33
Openness to problem	3.15	4.00	2.00
Interest in collaboration	3.25	4.00	2.00
Value of teamwork	3.23	4.00	2.20
Total Score	*65.10*	*76.00*	*49.00*

Table 5: Item Score Distribution

Findings of the interview results revolved around two prospective teachers whose mean score was highest (participant AA) and lowest (participant JH) in the questionnaire. The two participants were chosen to represent two spectrums of perspective, i.e., perspective favoring CPS and perspective less favoring CPS. The description started by elaborating on participants' perspectives on each dimension, i.e., problem-solving and collaboration; then, findings were generated from the perspectives of collaborative problem solving. In general, the interview excerpts revealed apparent differences in some sub-dimensions of the perspectives.

Perspectives on Problem-Solving

On the problem-solving dimension, AA shows strong perseverance. It was revealed through his thorough explanation of how he pursued a solution to a problem he faced. His explanation when being asked about how he put effort into finding a solution was as follows:

> *We can try the solutions that have been done before with my own thoughts, so I'm sure why this method doesn't work. There must be a motivation to find out how we can solve the problem. If we are stuck, we don't know anymore, give us a break first, give our brains a break, for example, like listening to music, after cooling down, we still can't find a solution. So, like that, I usually search again for the material session, maybe on YouTube or the internet, or ask someone more expert.*

His explanation showed many ways to solve a problem, i.e., using a familiar strategy, reflecting on his thinking about why the strategy did not work, cooling his mind down, looking for other sources, or asking experts. His long and detailed procedures revealed his perspectives on how important perseverance is for solving problems. He further added that it bothered him whenever a problem remained unsolved, and he always wanted to find ways to solve it. Other excerpts revealed his openness to types and strategies of problems. Participant AA did not consider the explicit features of the problems, such as length and available picture, as the determining factors of how he thought about a problem. He also argued that it was important to construct new strategies to solve problems. His openness to new strategies could be inferred from an excerpt below after being asked how if he could not find the solution on the first try:

> *As for how to solve it, it can also be affected, so we can first try solutions that have been done in real life for this problem. From there, if that doesn't work, we'll just try to find a new way.*

Interestingly, AA further emphasized that the new strategies might not have been learned before and might not be usually taught at school. Acknowledging the need to write a solution formally, he nevertheless thought that guess-and-check was a valid strategy whenever the truth of the solution could be proven. AA's openness to solution strategies might be related to his perseverance, considering that various ways to pursue solutions are important.

Similar to AA, JH showed favour to perseverance to some extent. For example, she thought about giving up after several actions, i.e., she could not understand the material related to the problem, looked for other sources, or asked for help. Despite mentioning giving up while being asked about her ways of finding a solution to a problem, JH's description of pursuing a solution could be considered as not different from AA's. However, apparent differences came from her perspectives on the types of problems. JH considered the explicit feature of the problem, such as the length or the availability of supporting pictures. After being asked how she perceived a problem the first time reading it, she stated:

> *If I'm honest, the first time I see it, it's like this. It's going to be difficult when you see such a long question. So sometimes, it's more likely to help if the explanation of the problem is short and clear, accompanied by illustrations*.

Further, JH explained that she considered the mathematics topic related to the problem as an essential feature. She would determine the topic and thus would be able to figure out the solution using the concepts of the topic. Her reason lengthened her emphasis on mathematics topics, that it was difficult for her to solve problems whenever she did not understand the underlying mathematics concept. She usually found it easy when she mastered the topic related to it because the strategy to solve the problem would come from it. Her previous point supported it when she considered giving up when she could not understand the problem's material. It could be inferred that JH saw a problem as the extension of a particular mathematics topic and that the solution heavily relied on it. When she was followed up by a question about how she could describe mathematics, JH's view below could be the reason behind her less-open perspective on the problem:

> *Mathematics is an exact science, so if we look for an answer, it's right or wrong. So surely, we can say it's true or false because there must be a way or procedure to solve it*.

It could be synthesized from both prospective teachers that despite being quite similar in viewing perseverance to pursue a solution, they had different perspectives on problem-solving, especially on how open they were to problem types and strategies. AA did not consider the surface feature of the problem, contrary to JH, who considered the length and picture of the problem. He was more reflective (when he mentioned checking previously done strategies) in choosing strategies and accepted the use of guess-and-check to formulate a new problem-dependent strategy. On the other hand, JH saw a problem as content-dependent and that its solution relied heavily on the underlying topic.

Perspectives on Collaboration

Both prospective teachers revealed a view on collaboration and valuing teamwork to some extent. Both mentioned their preferences for working with peers who could communicate or give opinions well. Interestingly, related to mathematics competence, they have quite different perspectives. AA considered their peers to have "various" competence and that competence was not important in choosing teammates. He was asked about what he did the first time being in a group, and he stated:

> *Of course, within a group, there will be different competencies between us. So, ask first, what is your competence, in what part you are an expert, while where am I more? So, from there, if we don't understand the material, we will explore it again or the problem to be solved. It doesn't matter how competent (you are). But if you are invited to communicate or solve a problem, we can work together.*

In this case, AA perceived collaboration as groups sharing their expertise and effort to reach a solution. He saw it as a process in which 'exploring each competence as possible again. On the contrary, JH mentioned that her peers should "be able to understand the concept of the material". She further explained that understanding the concept and giving an opinion was essential to get a solution in a group. She also mentioned that she preferred collaborating when she did not understand the problem. She explained:

> *When I understand the problem given, I can do it myself, but when I don't understand, I sometimes share it with friends.*

JH might perceive collaboration as a way to learn from others. Her perspective that problem solution relied heavily on the concept brought her emphasis to the needs of someone who mastered it and felt enough when she was that someone. It was understandable for JH to have that opinion, as she mentioned 'feeling burdened' when their teammates relied on her.

Even though both participants preferred collaboration over individual work, they had different reasons behind it. The different perspectives were more exemplified in how they valued teamwork. When asked how to overcome communication problems created by a particular teammate, AA considered reminding his teammates and finding a solution by communicating the concern to that teammate. He spoke:

> *If during our education to become a teacher we are like this, what will happen when we become teachers. So, we can remind (the person) first if it doesn't work anymore, we can find another way to solve this, what to do so he can be active again. So, he can communicate well in group work. How if, how if next time I get a group with him again.*

He perceived teamwork as something to fight for as part of his exercise on becoming a teacher. He also acknowledged the possibility of being in a group with that person. His action to face the person and work it out together for the team might be better for future interactions. He did admit for the sake of the group that it was possible if the agreed solution was not the correct one, and that was okay as he mentioned, "convey ideas together, choose them together, and we get the results together". He perceived group decision was important, following the process within the group to obtain it. JH had a similar opinion about a group decision, yet she had a different opinion on handling communication problems created by a specific teammate. She thought being in the same group with such a person was difficult enough for her to collaborate with another group. She shared:

> *If you get a friend like that, maybe you can collaborate again with other groups if allowed, because you are looking for a solution. If, for example, it is allowed to collaborate with other groups for sharing, it can help, ma'am, because it's difficult If we are in a group with friends who do not communicate.*

JH perceived collaborating with a more communicative person was important, regardless of which group s/he belonged. It was understandable as she previously considered that she preferred collaboration more when she needed help, and communication problems were probably too difficult to face.

Perspectives on Collaborative Problem Solving

Prospective teachers' perspectives on problem solving and collaboration brought links to those of collaborative problem-solving. Despite being similar in perseverance, AA showed stronger openness to problems by showing a more flexible view of problem types and strategies. In contrast, JH tended to have a more static view of them. AA was more interested in a collective effort and how he valued teamwork, while JH seemed interested in beneficial interaction. When asked about their opinions on implementing CPS activities in their future classroom, both prospective teachers agreed to have the activities under similar conditions.

AA perceived collaborative problem-solving as an important activity to exercise communication and knowledge of students. Further, he thought:

> *The group is for in the beginning. For the children practice, (...) so it seems like they are more solid, that's how they understand their foundation. So, if you work together, remind each other, so when the material is solid, then they will be able to do it individually again.*

Like AA, JH liked having a CPS activity in her teaching, as she believed teachers no longer dominantly provide the information. She stated:

> *When we teach, it becomes more like a group. Why? Because when we are in a group, we can collaborate with friends and share. Right now, ma'am, the curriculum doesn't require dominant teachers, so they focus more on students. It suits students in groups so that they will look for information first, and then the teacher will correct them.*

During the interview sessions, some dialogues with prospective teachers were about assessment. Regarding assessing students' CPS, the two prospective teachers argued slightly differently. AA recommended assessing CPS in a project-like way, in which there was an explicit assessment scheme for communication alongside assessing students' mathematical understanding of the problem. He mentioned the reason that he understood the challenge of assessing individual performance in CPS activities and continued as follows:

> *It's usually not based on individual assessments for groups, so it's like a project. So, we can also see more about the child's skills. For groups, collaboration is (for example) communication (...). It usually depends on the percentage of the aspects of assessment.*

While being asked about the same challenge faced in assessing CPS, JH mentioned that it was possible to have only some of the group members contribute to the problem solution. She responded to this by reflecting on her experience as follows:

> *I admit too. Sometimes I'm like that too. My friends are like that too. But again, ma'am, from the student's point of view, he could seem to walk together. He could also learn, "Oh, it turned out like this" Even though the answer wasn't correct, at least he understood the concept of that answer, right? We learn that the process is more important than the grade.*

JH's opinion was more into the knowledge constructed by students through the process. She argued that the case was expected as she sometimes did it. While being asked further about how she would assess students' performance in this case, she deliberately repeated the same opinions. While listening to JH, it might be inferred that, according to her, assessing other skills outside mathematical concept mastery was not part of assessing students' performance. Thus, while having a CPS activity, she considered having students learn the concept was enough modal to assess their performance. In comparison, AA perceived communication as part of the assessment process, while JH seemed to disregard it for students' assessments. Both prospective teachers recognized CPS to be implemented in the classroom yet had distinct perspectives on assessing students.

DISCUSSION

It had been shown that AA, whose mean score was the highest, and JH, whose mean score was the lowest, demonstrated different perspectives upon most of the sub-dimensions of the perspectives. The noticeable difference was in the sub-dimension of openness to problems, where AA was more flexible in considering problem types and strategies. At the same time, JH was more inflexible by viewing problems based on their surface feature. Both prospective teachers' perspectives on openness to problems might be connected to their beliefs in mathematics. AA's perspectives on creating a new strategy based on the problem and that a problem might not be constituted by its surface feature or structure corresponded to the problem-solving view of mathematics (Ernest, 1989). Especially when he perceived finding a solution to a problem might need new exploration, showing that a solution or discussion result remained upon improvement. In contrast, JH's repeated utterances of a right or wrong solution, understanding of materials, and the exact mathematics corresponded to the instrumentalist view.

On the collaboration dimension, a clear distinction was on the sub-dimension of the value of teamwork. AA perceived teamwork and team members as inevitable, and what came between them should be faced to achieve the solution. On the other hand, JH considered team members as places to lean on, and thus problems that came within them could be the reason for her to look for other 'places' to achieve the solution. Both views became the debates in some literature defining a 'good' CPS skill (Chan et al., 2018). On one side, AA's perspective showed his willingness to handle problems during CPS and gather ideas from within his team, no matter who his teammates were. However, on another side, his perspectives on his team might be biased compared to his view of other teams. What JH argued to have other groups beyond her get help would be what she viewed as a proper collaboration and how she valued teamwork. As the collaboration dimension was more into how prospective teachers viewed social aspects, both perspectives could illustrate the subtle difference between the two.

AA's perspectives on assessing CPS were in line with how PISA assessed CPS, i.e., for problem-solving, the cognitive processes were still included, while assessment of social and collaborative skills, which are associated with noncognitive skills, was added (Greiff, 2013). JH did not consider social aspects, such as communication between the team, as an aspect to be assessed, which showed how she perceived the assessment of CPS or mathematics assessment in general.

Besides contrasting the difference between the two prospective teachers, the reason behind those perspectives was as noteworthy. While CPS is a necessary skill to face the complexity of 21st-century demands, some perspectives less favoring CPS (e.g., not exploring new perspectives, being inflexible in solving problems, etc.) might hinder the practice of improving CPS in the future (Graesser et al., 2018), especially when such perspectives are dynamic and to be developed through teacher education program (Evans, 2011). By understanding the prospective teachers' perspectives on CPS and why they had them, teacher education and preparation could be more meaningfully conducted (Marble et al., 2000).

CONCLUSION

This study has described several points differentiating two prospective mathematics perspectives toward CPS. Despite being quite similar in perspectives on perseverance and interest in collaboration, the prospective teacher favoring CPS showed more openness to problems and the value of teamwork than another prospective teacher less favoring CPS.

By understanding the characteristics of each perspective, it is hoped that more effort can be made to promote meaningful teacher education in promoting CPS. Each perspective can also contribute to the broader understanding of prospective teachers' perspectives, not only in CPS but also in the teaching practices or the inclusion of social aspects of mathematics in general. It may be too early to conclude that these perspectives might be directly related to what both participants will do in the future as teachers. For example, when the participant considered leaving the group due to uncommunicative peers, s/he might not recommend their students to do so in the future. Yet, the prospective teachers might position students facing the same situation as themselves and have different viewpoints on the uncommunicative person. By this, perspectives have been essential factors in driving teachers' practices.

Due to the small sample and limited duration used in this study, the findings cannot be generalized. Still, it provides valuable descriptions of prospective teachers' perspectives on CPS. Further research can be done by investigating how prospective teachers' perspectives relate to their knowledge of CPS or future teaching practices. A longitudinal study to investigate the changes in perspectives on CPS may be conducted to portray prospective teachers' education influence.

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The Moderating Model of Teaching Anxiety on Teaching Beliefs and TPACK Effect to ICT Literacy Among Pre-Service Mathematics Teachers

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Abstract: Research on teaching anxiety has increased dramatically during the previous two decades. Teachers who experience significant anxiety in the classroom are also more likely to lack confidence in their abilities. How teachers evaluate their pedagogical abilities in areas like TPACK and ICT literacy may potentially play a role in the development of teaching anxiety. We have 458 pre-service teachers in Indonesia and Malaysia who are immersed in a specific training program. To choose our samples, we relied on a criteria sampling strategy. Structural equation modeling (SEM) is a powerful statistical analysis method for theory creation because it allows researchers to test ideas about connections between observable and latent variables while also analyzing those associations. The result of the study state that pre-service teachers' levels of ICT literacy is influenced by the beliefs teachers hold about the relationship between TPACK and ICT literacy, as revealed by the structural equation model designed to examine this relationship. The author agrees that more research is always needed in this field but believes that the results of this study could be valuable to other researchers interested in beginning new studies or expanding existing studies linked to the ICT literacy paradigm.

Keywords: ICT Literacy, structural equation model, teacher's beliefs, TPACK, teaching anxiety

INTRODUCTION

Over the past 20 years, more and more study has been done on occupational anxiety in teaching. This reflects a growing interest in the topic worldwide (e.g., Bringula et al., 2021; Lea, 2019). Connie (2020) shows that 60% and 70% of teachers experience anxiety, and 30% feel burned out.

Also, Agus and Mastika (2018) say that teachers have one of the highest amounts of anxiety at work compared to other jobs. There is proof that teacher anxiety has adverse effects on teachers' mental health (Matoti & Lekhu, 2016), teachers' beliefs (Uysal & Dede, 2016), and student's academic performance (Omar et al., 2020). This has led to what some call a "teacher exodus" across the country (Fisher, 2019). It has been noticed that teachers who feel much anxiety in the classroom also tend to have less belief in their skills (Uysal & Dede, 2016).

Teaching anxiety has also been linked to how they see their skills in the classroom (Bringula et al., 2021), what they think about their abilities (Novak & Tassell, 2017), and how they feel about mathematics (Ramirez et al., 2018). Ersozlu (2019) examined the link between teaching anxiety and epistemological views among future teachers. In the same study, anxiety about content knowledge, attitude towards mathematics teaching, and pedagogical content knowledge were all found to be negatively correlated with constructivist beliefs, which are part of beliefs about mathematics teaching and learning. Anxiety about self-confidence was also negatively correlated with constructivist beliefs, but only at a low level. Wilson (2018), also did a similar study with mathematics teachers. Still, there are studies on different methods that could help with mathematics teaching anxiety and typical methods tend not to work (Connie, 2020; Omar et al., 2020).

Also, Eickelmann and Vennemann (2017) say that the greater use of technology in mathematics education over the past two decades has likely led to a decrease in teachers' anxiety about teaching mathematics. If this is true, then the goal of Barry (2017) and Bouzid et al. (2021) has been met. Technological Pedagogical Content Knowledge (TPACK) is one type of technology skill that teachers should have. Still, there aren't many studies on how TPACK and mathematics anxiety are related in real life. More study must give us a fuller picture of what's happening here. Experiments mentioned in the research on TPACK and anxiety have shown that worry about teaching mathematics goes down. Exercises based on WebQuest (Nejem & Muhanna, 2018) and GeoGebra (Eickelmann & Vennemann, 2017) have been found to make teaching mathematics less stressful. But there isn't much study on how technology affects mathematics anxiety in the classroom. Nisfah and Purwaningsih (2018) looked at how TPACK competencies and teachers' feelings about using technology in the classroom connect to mathematics teaching anxiety. In addition, mathematics teachers who used technology to help them with their work said they felt less stressed. Prestridge (2018) meta-analysis showed that using technology to teach mathematics made teachers less worried about mathematics. Also, learning experiences made with GeoGebra (Velázquez & Méndez, 2021) and content-based and technology-supported projects (Nejem & Muhanna, 2018) helped teachers feel less afraid of mathematics. Lastly, there have been no studies done that look into the link between TPACK and mathematics anxiety.

Naziri et al. (2019) argue that much work must be done to fully understand how students' technology pedagogical and subject knowledge, along with some information, communication, and technology (ICT) literacy, could affect how teachers teach. So, many programs that train teachers for the classroom include ICT literacy in their course modules or have separate units on how to use these tools to teach. While there has been more focus on digital problem-solving skills in society (Hatlevik & Arnseth, 2018), there has been more interest in how ICT can be used in educational settings. Because of this, mathematics teachers need to have a firm grasp of how to use ICT in the classroom and evaluate their students' progress in mathematics while using ICT themselves (Gurcay et al., 2018). Peters-burton (2019) says that teachers' knowledge, views, and attitudes have much to do with how well ICT is used and integrated with the classroom.

This study examined how teachers' anxiety is affected by how teacher training programs use ICT literacy to help teachers feel less anxious about teaching. On the other hand, few studies show how these three things are related, and teaching anxiety is a variable that can't be ignored in the relationship between TPACK and teachers' views about ICT literacy. So, this study aimed to examine how mathematics teaching anxiety works as a bridge between TPACK and teachers' beliefs. Then, there's the linked concept of how ready mathematics teachers are for ICT literacy. So, this study aimed to determine how mathematics anxiety affects the relationship between TPACK and teachers' views about ICT literacy. The researchers used structural equation modeling (SEM) because it can account for measurement errors and show both direct and indirect links between factors, which regression analysis can't do. An evaluation of preservice teachers in this area will likely add to the body of knowledge because these three factors significantly affect how students learn.

LITERATURE REVIEW

Teaching Anxiety

Several widely used strategies for teaching arithmetic have been connected to elevated stress levels among teachers. The teacher's aptitude or knowledge of the issue, the teacher's attitude toward the topic, the teacher's reaction to queries or clarification requests from students, and the teacher's public humiliation in front of the class are all factors. But when it comes to teacher stress, Bouzid et al. (2021) zero attention to the teachers' subjective experiences in the classroom, including how they feel, how they are evaluated, and where they choose to pursue their education. This indicates that their performance in the mathematics lesson reflects the teacher's anxiety about the topic. Potential classroom anxiety causes include a lack of self-assurance, tiredness from teaching, and concern about the student's learning ability. Students' self-efficacy, grade anxiety, future, in-class, and assignment factors are also used as indicators of anxiety and attitude in the mathematics classroom (Ersozlu, 2019).

Teacher's Beliefs

Teachers have strong opinions about the nature of the knowledge (Prestridge, 2018), what makes for good teaching, and how students should be trained (Ellerani & Gentile, 2018). To characterize these convictions, however, much more theoretical work is needed (Berger et al., 2018). According to Masibo and Barasa (2017), there are three distinct approaches to teaching mathematics: techniques that emphasize the students, methods that prioritize the transfer of knowledge, and methods that stress student performance. Teachers with a learner-centered vision see mathematics instruction as a group effort to build knowledge. In mathematics education, there are two primary schools of thought: those that emphasize conceptual understanding and those that emphasize the development of procedural fluency. Several authors contrast teachers who focus on school subjects and teachers who focus on student growth (Han et al., 2017).

TPACK

TPACK is the intersection of three types of expertise: technological, instructional, and subject matter. From this intersection, discussions expanding to include the overlap of PCK, TPK, and TPK were often launched (Koehler et al., 2014). A more recent moniker for TPCK is TPACK, which stands for the "whole package" of skills needed to effectively incorporate digital tools into mathematics education for both the teaching and learning (Suharyati et al., 2022). As a result, the relevance of the relationship between these concepts is being more widely acknowledged. Because of the ever-changing nature of technology, students, teachers, and classroom contexts, it is crucial that teachers have a flexible framework for understanding the knowledge teachers need to design curriculum and instruction that focuses on preparing students for mathematical thinking and learning with digital technologies (Sunman, 2022).

ICT Literacy

Researchers Hatlevik and Arnseth (2018) found a correlation between a person's computer literacy and ICT literacy (also known as computer fluency, computer competence, cyber literacy, digital literacy, or electronic literacy). This electronic, digital, or ICT literacy encompasses many skills, including reading, writing, exchanging information, and communicating. ICT literacy has been connected to the ability to use various software programs, including those used for word processing, spreadsheets, and presentations. These ideas allow us to evaluate teachers' familiarity with digital and computer technologies. The original "ICT literacy" concept heavily emphasized knowing and understanding the basics of utilizing a computer (Novita & Herman, 2021). Common computer hardware knowledge, program fluency, the ability to articulate one's computing preferences, and skill with application management are all indicators of a user's computer literacy level.

METHOD

Research Design

This study set out to answer the question, "How do teaching anxiety, teacher beliefs, TPACK skills, and ICT literacy all relate to one another?" Anxiety about the classroom was thought to influence the relationship between teachers' ICT literacy and their TPACK attitudes and practices. Hence, the researchers employed a causal survey approach to determine a possible relationship between variables. This method works within a cause-and-effect framework (Kline, 2017) to determine how various elements interact. By simultaneously testing hypotheses about correlations between observable and latent variables and analyzing those relationships, structural equation modeling (SEM) provides a potent statistical analysis tool for the theory development (Byrne, 2019). Also, SEM can be utilized to obtain more believable results because it calculates linear relations between variables more correctly than regression and path analysis (Bauldry, 2019). The resulting study plan is shown in Figure 1.

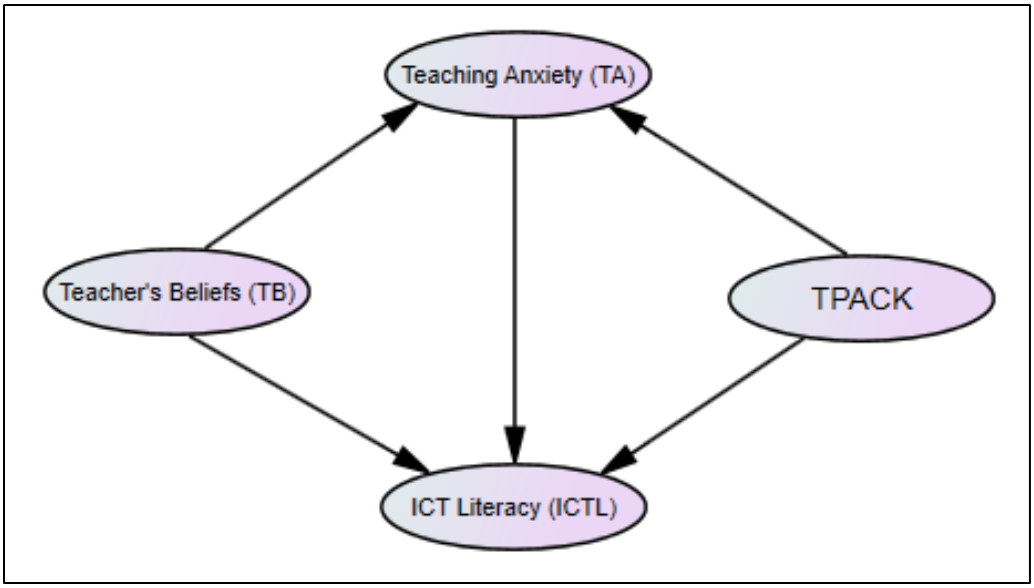

Figure 1: Model of Study

Meanwhile, these are the seven hypotheses that will guide this research: H1: Teachers' beliefs in their ability to use ICT literacy effectively; The use of ICTs is associated with TPACK, as stated in H2; Hypothesis 3: Teaching anxiety in the Classroom Reduces ICT Literacy. H4: Teacher views are inversely connected to anxiety when teaching; H5: Teacher TPACK is inversely associated with anxiety. Two hypotheses are tested here: H6: teaching anxiety moderates the connection between teachers' beliefs and ICT literacy; and H7: teaching anxiety moderates the connection between teachers' TPACK and ICT literacy.

Participants

To assess hypothesized correlations when there is even a slight measurement error, using the standard error of the mean necessitates a sample size greater than N (Kline, 2017). As a result, in Indonesia and Malaysia, 458 pre-service teachers are concentrating on a program tailored to the four categories (grade, gender, locations, and the use of technology per day). We used a criteria sampling method to pick our samples. Pre-service mathematics teachers must have completed coursework in analysis, geometry, algebra, statistics, probability, and computer-assisted mathematics education. Because the study took place towards the end of the 2020-2021 school year, participants could sign up for TPACK competencies and field courses. Table 1 shows a breakdown of the participants by four categories.

Category	Sub-category	Total	Percentage (%)
Grade	Third	48	10.48
	Fifth	131	28.60
	Seventh	279	60.92
Gender	Male	248	54.15
	Female	210	48.85
Location	Urban	292	63.80
	Sub-urban	166	36.24
The use of technology per day	< 3 hours	28	6.11
	3 – 5 hours	242	52.84
	> 5 hours	188	41.05

Table 1: The Participants of the Study

Data instruments

Mathematics Teaching Anxiety Rating Scale

Michael (2018)-created a scale that aspiring mathematics and elementary school teachers can use to assess their anxiety levels about teaching mathematics. The 23-item scale is divided into four categories, and respondents were given the option of selecting a response between 1 (strongly agree) and 5 (strongly disagree) on a Likert scale. Anxiety in the classroom can be broken down into two categories: mathematics teaching anxiety (KsM) and pedagogical content knowledge anxiety (KsP). Bayat (2018), reported a Cronbach's Alpha for this scale of 0.91; the current study confirmed that value, locating it at 0.90. The maximum possible score is 115, while the minimum is 23. The score reflects the degree to which a potential teacher is anxious about leading a mathematics lesson. The first ten items are listed in the wrong sequence. Exploratory factor analysis of the scale's validity reveals that the factor loadings for the 23 items range from 0.528 to 0.857, explaining 56.5% of the total variance.

Technological Pedagogical Content Knowledge Scale

To evaluate the TPACK skills of aspiring mathematics teachers, Valtonen et al. (2019) created a scale. There are nine groups, each including five items (5 = I am thoroughly competent, 1 = I am entirely inept) from the 5-point Likert scale. Examples include TK ("technological"), CK ("Content"), PK ("pedagogical"), PCK ("pedagogical content"), TCK ("technological content"), TPK ("technological pedagogical content"), TPCK ("technological pedagogical content knowledge"), and so on (for "technological pedagogical content"). For this study, researchers employed both online and offline TPK and evaluated the results using eight criteria. Scores range from a high of 295 down to a low of 59. No items on the scale have inverse-coded counterparts. Higher exam scores indicate more TPACK competence amongst preservice teachers. The present study confirmed the 0.97 value reported by Fitri (2019) for the Cronbach's Alpha of this scale.

Factor loadings for the 59 items on the scale ranged from 0.495 to 0.797, explaining 66.2% of the total variance, as shown by exploratory factor analysis of the scale's validity.

Teaching Beliefs Scale

Both the treatment and control groups filled out the Teaching Beliefs questionnaires in person. There are three components to the TB scale: the belief in mathematics (KpSM), the belief in teaching and learning (KpPM), and the belief in the use of technology (KpPT) (Connie, 2020). The survey includes 55 questions. Each respondent saw the same set of questions in the same order since the random number generator chose them. After recalibrating the scale using the information from both surveys, only 21 of the original 55 questions made it into the final survey.

ICT Literacy Scale

As the prevalence of ICT and digital technologies grows, Markauskaite (2019), presented a model based in part on the language competency model given by Hatlevik and Arnseth (2018). Current definitions of digital literacy stress a wide range of language skills, including procedural competence (LiPP), social-digital competence (LiES), digital discourse competence (LiPO), and strategic competence (LiPK). Many dimensions of digital competency are essential for efficient "diagnosis, analysis, and repair of learners' digital requirements" (Abdulteeef & Khateeb, 2017).

Data Collection

The pre-service teachers were initially briefed on the study's goals. Due to the technical issues regarding the two countries participants, 458 willing individuals took the tests online using Google Forms. Standard method bias can be avoided by collecting data from many sources or by collecting data on the dependent, independent, and moderator variables at intervals apart from one another (Hollweck, 2016). All three scales were completed by the same people, who received them one week apart. Each scale took the pre-service teachers five days to complete. The respondents took their time completing the questionnaires. We couldn't include 32 scale forms since they were either randomly filled out or extreme outliers.

Data Analysis

Path analyses were run on the model's measurement and structural components to determine whether AMOS fit the data well. SEM model fits were evaluated using several different metrics, including the chi-square to the degree of freedom (Chi-square/df) ratio, *root mean square error of approximation* (RMSEA), *standardized root mean square residual* (SRMR), *normed fit index* (NFI), *non-normed fit index* (TLI), and *comparative fit index* (CFI) (Harahap et al., 2020). The difference test was run after each round of error binding, and appropriate indices and chi-square significance were evaluated between the old and new models. ICT Literacy was tested as a dependent variable, and TPACK and Teachers' Beliefs are independent variables via the moderator variable of mathematics anxiety. There are three prerequisites for conducting a moderator analysis (Syafiq et al., 2022).

First, the independent and dependent variables must have a strong and direct relationship. Second, the independent and moderator variables should be related by linear regression. Lastly, when the effect of the moderator variable is removed, the link between the dependent and independent variables should weaken (absolute value) to demonstrate the moderating role in the moderator model. Field Bauldry (2019) states that the moderator variable may entirely or partially explain the underlying link between dependent and independent variables. When the moderator explains the entire relationship, it is termed the full moderator. When the moderating variable is included in an analysis of a full moderator, the relationship between the dependent and independent variables weakens and becomes statistically insignificant. Partial moderator occurs when the link between the dependent and independent variables is too complex for the moderator variable to capture fully. There is still a substantial association between the dependent and independent variables, but the effect coefficient and degree of significance have decreased. Following the advice of Preacher and Hayes, we conducted a bias-corrected bootstrapping approach using AMOS to examine the potential impact of Teacher Beliefs, ICT literacy, and TPACK competence on mathematics teaching anxiety. The confidence interval was secured by increasing the sample size to 5.000. For Bootstrap assessments of moderator effects to be valid, the 95% confidence interval must contain at least one value that is not zero (Kline, 2017).

The Bootstrapping for Moderated Model

To estimate a population parameter, statisticians use "bootstrapping," which involves drawing and replacing samples from the population many times. Estimating a parameter directly from the dataset using this method is impractical. Hence a sampling strategy is employed instead. The dataset stands in for the population in this scenario, and the samples are constructed to represent a range of possible outcomes in the actual population. Between one thousand and ten thousand samples are collected on average. The bootstrap method can produce confidence intervals for your statistical estimate, which could be helpful. This provides us with crucial information regarding the likely value of a parameter in contrast to the single number provided by the p-value, which evaluates the likelihood of our statistic under the null hypothesis.

Direct and indirect effects can be broken down and comprehended using the moderated model. People form opinions depending on their interactions with one another. The only things keeping these two nouns apart are the free modifier and the modifier. A free mod's immediate effect is modified by an intermediate mod (Masunga et al., 2021). An example of the intermediate effect at work explaining how two shapeshifters (Hair et al., 2014). If it can successfully shift the dynamic between the two main modifiers, we can call him a full moderator (complete moderator). If two modifiers are related, but not exclusively through an intermediary moderator, then there is partial moderator between them (Byrne, 2019).

RESULT AND FINDINGS

Participant Profile

Pre-service teachers from two universities in Indonesia and Malaysia comprised 458 participants. Whereas 248 (or 54.15%) of the students are men, only 210 (or 45.85%) are women. Of 279 students who have completed all seven semesters of their degree program, 28.60% are in their third semester, 48 are in their fifth semester, and 34.41 are in their seventh semester. Most students (63.80%) are in metropolitan regions, while just 166 are in rural areas. Just 28 students have access to the internet for less than 3 hours per day, whereas 242 have access between 3 and 5 hours per day, and 188 have access for more than 5 hours per day.

Reliability and Constructs Consistency

Constructs	Sub-Constructs	Factor loading	CR >0,6	AVE > 0.5	Alpha Cronbach >0.70	Decision
Teacher's Beliefs (TB)	KpSM	0.773	0.682	0.533	0.766	Achieved
	KpPM	0.795	0.782	0.554	0.722	Achieved
	KpPT	0.793	0.645	0.576	0.775	Achieved
Teaching Anxiety (TA)	KsM	0.883	0.663	0.658	0.856	Achieved
	KsP	0.739	0.637	0.522	0.738	Achieved
Technological Pedagogical Content Knowledge (TPACK)	PT	0.774	0.718	0.635	0.883	Achieved
	PP	0.772	0.692	0.656	0.875	Achieved
	PKP	0.794	0.634	0.585	0.826	Achieved
	PKT	0.733	0.786	0.567	0.778	Achieved
	PPT	0.723	0.751	0.644	0.752	Achieved
ICT Literacy (ICTL)	LiES	0.863	0.637	0.734	0.731	Achieved
	LiPP	0.773	0.678	0.674	0.889	Achieved
	LiPK	0.873	0.765	0.552	0.777	Achieved
	LiPO	0.753	0.643	0.641	0.754	Achieved

Table 2: The Cronbach Alpha, CR, AVE, and Factor Loading

The consistency with which a measurement represents a concept is an excellent indicator of its reliability; one way to quantify this consistency is with Cronbach's alpha. The reliability coefficient that results from such an exhaustive measure assessment can take any value between 0 and 1. If there is a strong relationship between the items on an infinite scale, the alpha will be close to 1, while if there is no relationship, the alpha will be close to 0. (i.e., they share no covariance). High alpha coefficients for items indicate their ability to measure the same underlying concept through common covariance. The high value of 0.8705 for Cronbach's alpha in this study shows that the items are highly consistent with one another, exceeding the level of acceptability set by the researchers (see Table 2 for details; keep in mind that a reliability coefficient of 0.7 or higher is generally considered "acceptable" in social science and humanities studies). However, for the rating to be credible, it must meet both the *Composite reliability* (CR) and *average variance extraction* (AVE) criteria, wherein the CR must be more excellent than 0.6, and the AVE must be greater than 0.5. The following CR, AVE, and Cronbach Alpha values were calculated after performing a Confirmatory Factor Analysis (CFA) TPACK analysis. Table 2 summarises the CR, AVE, and Alpha Cronbach values for the TPACK construct. This would indicate significant connections (correlations) among the elements under scrutiny.

Measurement Model of Teacher's Beliefs in ICT Literacy

The same holds for measurement models; they are only reliable if they are unidimensional and valid. To guarantee these three requirements are met, a pooled CFA should not be carried out until after the structural model analysis has been completed. When all the loading factors for the items and dimensions add up to more than 0.6, we have achieved dimensional consistency. One way to accomplish this is by using this method. The *Exploratory Factor Analysis* (EFA) analysis can reveal multiple types of validity, such as convergent validity, construct validity, and discriminative validity. In a minute, we'll delve deeper into each form of validity. We have achieved convergent validity when all of the measurement model's components either have a statistically significant value or can be independently confirmed using the Average Variance Extracted (AVE) value. Convergent validity refers to a measuring model in which all parts have the same numerical value. Construct validity was determined by the value of the fit indicator (*goodness-of-fit*/GOF), and discriminative validity was attained when the measurement model did not include any items measuring the same two things as the construct validity item. The goodness-of-fit indicator value was calculated for each type of validity. When comparing two exogenous constructs, a correlation value of less than 0,4 is used as evidence that discrimination is legitimate. Figure 2: A Model for Assessing Teachers' Beliefs Towards ICT Literacy.

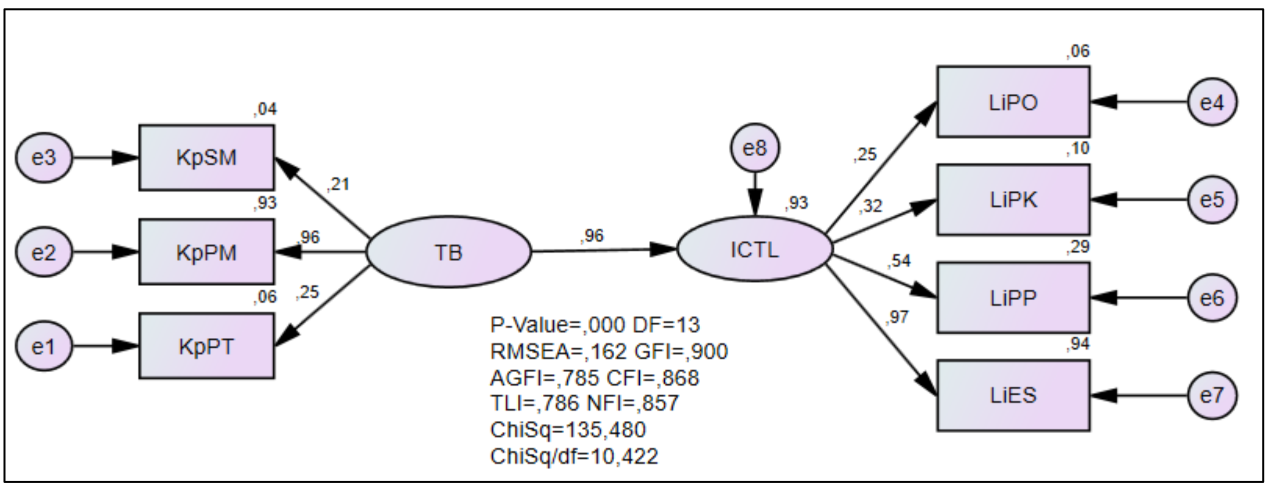

Figure 2: The Measurement Model of Teacher's Beliefs and ICT Literacy

For this investigation, we can classify the strength of the connection into three levels: small, simple, and strong, with small corresponding to values less than 0.10 and vital to values greater than 0.50. We found a moderate correlation (β = 0.010) in our analysis. A statistically significant correlation was found in the measurement analysis, with a p-value of 0.962. To identify the optimal measuring framework, several frameworks were investigated. Table 3 below provides the results of every conceivable set of correlations between the two variables.

Sub-constructs			*β*	SE	CR	p	Decision
KpSM	←	TB	0.682	0.544	1.453	0.059	Significance
KpPM	←	TB	0.683	0.572	2.054	0.000	Significance
KpPT	←	TB	0.643	0.591	3.164	0.004	Significance
LiPO	←	ICTL	0.763	0.574	0.867	0.656	Significance
LiPK	←	ICTL	0.757	0.644	2.943	0.016	Significance
LiPP	←	ICTL	0.663	0.641	2.457	0.019	Significance
LiES	←	ICTL	0.682	0.642	2.953	0.000	Significance
ICTL	←	TB	0.683	0.586	2.577	0.000	Significance

Table 3: The Measurement Analyze of Teacher's Beliefs and ICT Literacy

Measurement Model of TPACK to ICT Literacy

The validity of CFA was calculated using Cronbach's Alpha. Design the parameters for estimating validity on average and creating a CR score. Cronbach's Alpha > 0.7, Cronbach's Rho > 0.6, and AVE > 0.5 indicate that all three constructs are reliable. However, examining the measurement model distinguishes between two distinct types of TPACK and ICTL. The TPACK component was divided into five smaller factors: PKPT, PPT, PKT, PKP, and PP. The ICTL framework is divided into three sub-factors (LiPK, LiPP, and LiES). Figure 3: The outcomes of the measuring model.

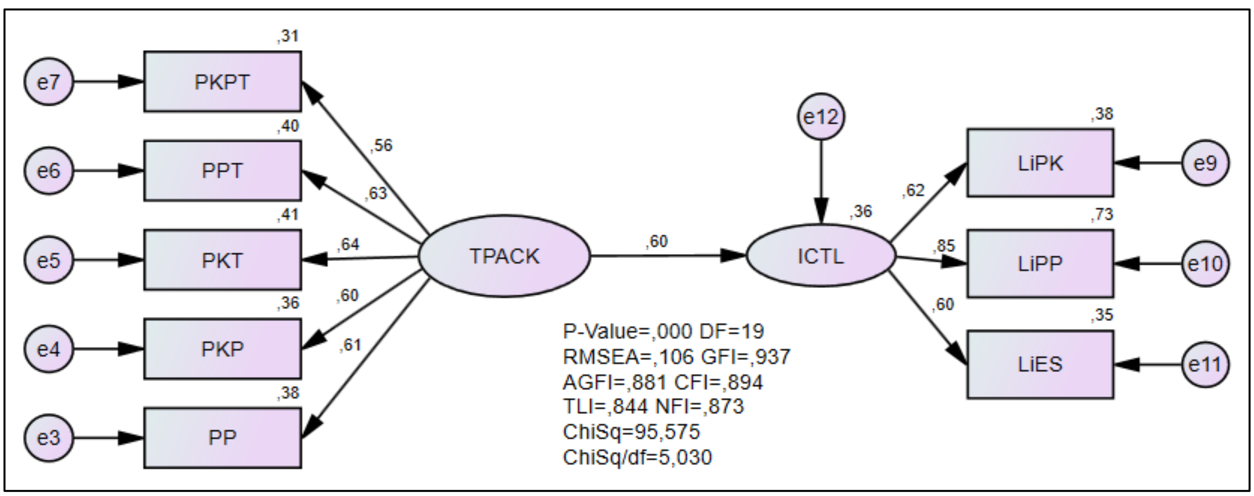

Figure 3: The Measurement Model of TPACJ to ICT Literacy

Figure 3 indicates a correlation between the two factors (β= 0.604). The potential measurement models were evaluated to select the most suitable one. Table 4 displays the findings of every possible correlation between the two variables.

Sub-constructs			*β*	SE	CR	p	Decision
PKPT	←	TPACK	0.644	0.674	2.909	0.000	Significance
PPT	←	TPACK	0.746	0.584	1.975	0.005	Significance
PKT	←	TPACK	0.635	0.678	1.487	0.004	Significance
PKP	←	TPACK	0.774	0.674	1.667	0.748	Significance
PP	←	TPACK	0.674	0.535	0.654	0.742	Significance
LiPK	←	ICTL	0.678	0.745	1.743	0.890	Significance
LiPP	←	ICTL	0.642	0.677	3.574	0.000	Significance
LiES	←	ICTL	0.654	0.564	2.631	0.000	Significance
ICTL	←	TPACK	0.635	0.674	1.435	0.000	Significance

Table 4: The Measurement Analyze of TPACK and ICT Literacy

The Moderating Model of Teaching Anxiety

Figure 4 depicts the three models used to analyze the connection between teacher beliefs and TPACK and ICT literacy. In all three models, we discovered results with statistical significance. Anxiety about both teaching and learning, as well as mathematics, played a moderating role. Teacher beliefs were shown to be negatively associated with teaching anxiety. At the same time, TPACK was found to be positively associated with teaching anxiety, teaching anxiety was found to be positively associated with ICT literacy, and teacher beliefs were found to be positively associated with ICT literacy.

Findings suggest that teacher beliefs and TPACK safeguard ICT literacy. The more frequently these two strategies are used the less intense the discomfort. Instead, ICT literacy was hampered by pessimistic mindsets and actions like pointing fingers and not caring. Figure 4 displays the results of five interactions between stressors and coping mechanisms, showing that teacher anxiety moderates the link between teachers' views and students' ICT literacy (β = 0.02; p = 0.000).

Figure 4 shows a positive correlation between TPACK and ICT Literacy, with Teaching Anxiety as a moderator. The upper line shows the effect of teachers' beliefs on students' ICT literacy (β = 0.961; p = 0.000), whereas the lower shows the direct correlation between TPACK and students' ICT literacy (β = 0.053; p = 0.000). Those who used it occasionally experienced a decrease in Teaching Anxiety, while those who frequently observed a moderate decrease in the intensity of their Teacher's Beliefs and TPACK linked with ICT Literacy. This evidence lends credence to the theory that using accommodation as a Teaching Anxiety strategy exacerbates the unfavorable effects of pre-service teachers' challenges on teachers' anxiety in the classroom.

Cronbach's residuum, alpha for internal consistency, and alpha for reliability were then calculated for the CFA of teachers' beliefs, TPACK, teaching anxiety, and ICT literacy (Table 5).

Constructs	Item	Factor loading	CR >0.6	AVE > 0.5	Alpha Cronbach >0.70	Decision
Teacher's Beliefs (TB)	KpSM	0.852	0.743	0.584	0.769	Achieved
	KpPM	0.744				
	KpPT	0.776				
TPACK	PT	0.746	0.683	0.649	0.737	Achieved
	PP	0.835				
	PKP	0.764				
	PKT	0.775				
	PPT	0.736				
Teaching Anxiety (TA)	KsM	0.757	0.763	0.673	0.784	Achieved
	KsP	0.866				
ICT Literacy (ICTL)	LiES	0.753	0.684	0.692	0.743	Achieved
	LiPP	0.734				
	LiPK	0.825				
	LiPO	0.757				

Table 5: The CFA of Constructs Values in CR, AVE, and Alpha Cronbach

The results of the moderated model study were used as the foundation for a comparative analysis to establish the superior model. The total worth of the deal can be broken down into three parts for the sake of this discussion. Interactions with values below 0.10 belong to the weak group, those between 0.10 and 0.50 to the simple group, and those above 0.50 to the solid group. Our findings indicate that the relationship between TPACK, TA, and ICTL is weak ($\beta = 0.023$), as is the relationship between TB, TA, and ICTL. However, when examining the direct link between TPACK and ICTL, we find a weak interaction ($\beta = 0.052$). A strong correlation ($\beta = 0.964$) exists between TB and ICTL. The moderated analysis showed no significant correlation between TB and TA ($\beta = -0.051$). As seen in Figure 4, this study uses a modified version of the Teaching Anxiety model.

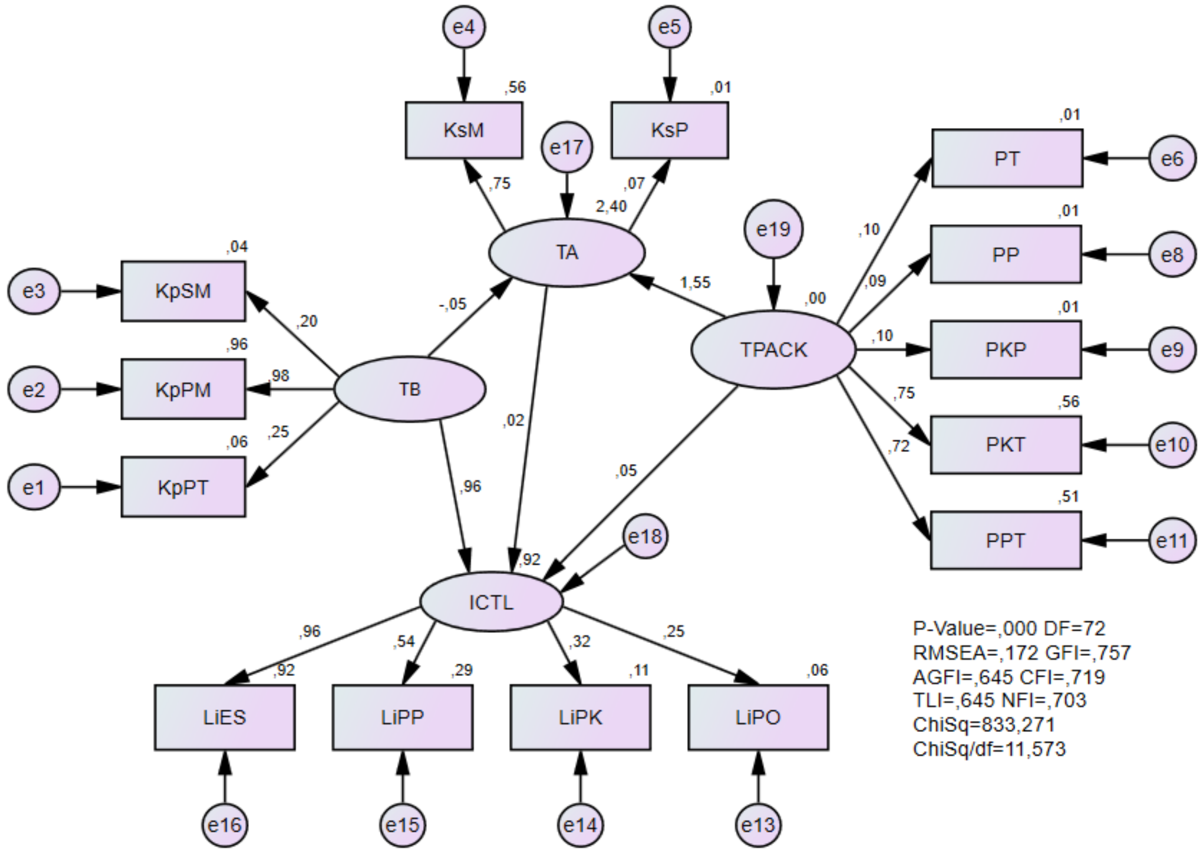

Figure 4: The Moderated Model of Teaching Anxiety

The Bootstrapping Analyse for Moderated Model

Reviewing the significance of the relationship is a crucial initial step in evaluating a moderator's effectiveness since it is the fundamental factor in determining whether the moderator's influence is complete or partial. The procedures provided by Hair et al. (2014) have been applied to determine if a full or partial moderator is present. Figure 4 helps judge the validity of the model—the immediate impact of teachers' TPACK on ICT literacy. Subsequently, a very high ($\beta = 0.963$) value of equivalence between teacher beliefs routes and literacy ICT (Kp→Li) is indicated by a decrease in chi-square (chi-square = 1.096; df = 15). The analytical decision is displayed in Table 4; it shows no effect of a complete moderator for this route budget since the direct impression of teacher beliefs in ICT literacy (Kp→Li) is substantial. Therefore, where there is evidence of a solid indirect belief in ICT literacy, the moderator can be utilized to identify a partial moderator. A glance at the bootstrapping result confirms that Kp→Li is sizable. The chain reaction from Kp to Ks to Li has a significant knock-on effect. Hair et al. (2014) state that if both direct and indirect impressions are essential, then a partial moderator exists. This suggests that the Ks, as a partial moderator, influence the Kp→Li pathway significantly. The results of bootstrapping the model are displayed in Table 6.

Aspect	Substantial indirect impact	Substantial direct effect
Bootstrapping value	Kp→Ks→Li (0.03) Kp→Li (0.08)	Kp→Ks→Li (0.03) Kp→Li (0.08)
Decision	Achieved	Achieved
Bootstrapping Model	Partial Moderator	Partial Moderator

Table 6: The Bootstrapping Value

DISCUSSION

This study employs a belief measurement paradigm built on beliefs in mathematical nature, the efficacy of education, and the value of technology in the classroom. All of these factors have substantial effects on the belief hidden variable. Covariance between constructs indicates the strength of the relationship between them (Hair et al., 2014).

The highest correlation coefficient is found between belief in the intrinsic worth of mathematics and belief in the educational potential of technology. Covariance can be thought of as a multiple of the Pearson correlation coefficient, as shown by the work of Creswell (2014). This long-held belief about mathematics' essential character may have survived because the high covariance value has provided mathematics teachers with a solid basis for developing their strategies for conveying mathematics ideas. Statements like "Mathematics is a set of rules that define how to solve problems, involving the reminder and use of Mathematical formulas, facts, and procedures" are supported by the teachers in this review because they refer to items that evaluate the consistency of Mathematical properties.

It follows that the mathematical components inherent in technology are likely to be associated with each other, as there is a significant correlation between the view that technology plays an essential role in education and the belief that mathematics is a core subject area. As a bonus, Novita and Herman (2021) shed light on this phenomenon by showing that a trait of beliefs in mathematics is adaptive and in step with the technological function. The importance of technology in teaching is intrinsically tied to this problem, as solving it effectively necessitates a solid grounding in mathematics. Since there are numerous approaches to solving mathematics problems, choosing objects is crucial. Technology in education, which has proven effective and efficient, is another tool for education.

Second, consider the degree to which various instructional theories are interconnected. The belief, like Mathematics, is discovered to have a more vital link with the belief in the use of technology in education. This happens because each teacher's views on the merits of technology are reflected in the various components of Mathematics instruction. This problem can be used by any mathematics teacher, regardless of their ideology or educational background. According to Liu (2019), Mathematics Teacher is an application that successfully combines cutting-edge pedagogical practices with substantive mathematics information.

Thus, attitudes about mathematics as a discipline are the weakest link in the confidence chain. This is due to the participants' emphasis on instilling traditional values in their young charges. The teacher is open to learning about mathematics' underpinnings but needs to put more stock into becoming an expert in the field herself. Lessons built on the items of the belief structure of mathematics education that were confirmed to be accurate in this research may increase student engagement and discussion. Therefore, teachers should ensure their teachings are as precise and comprehensive as possible. Therefore, there is a wiggle area for the message to be conveyed by trial and error. Calculus and other advanced mathematics classes are usually taught in a separate room so the teacher can focus on the argument. According to this metric, students' confidence in mathematics and their teacher's mathematics skill are unrelated. Stephen (2018) agrees with these findings, stating that a teacher is a transmitter of knowledge and that the content is not generated by the existence of impediments to confidence in the unique nature of Mathematics.

As a result, there is a robust and statistically significant relationship between LiPK and LiPP. This confirms the conclusion drawn by Juniati (2018), who found that the quality and professionalism of mathematics teachers improve when ICT is utilized in the classroom. The close relationship between LiPK and LiPP exemplifies the linked nature of the ICT components that comprise teachers' knowledge. As previously mentioned, ICT's indirect emphasis on professional teaching was used to establish that the roundtable members valued education highly.

Because of their extensive ICT, this problem is connected to teachers' work and classroom instruction. The existence or absence of ICT-friendly variables, such as an open classroom climate and a teacher's willingness to adapt teachings to students' ICT levels, can provide insight into this question. Additional study results include knowledge of effective pedagogical practices and the ability to select ICT, which increases lesson content. Then, using a simple technique, figure out how well LiPO and LiES are linked. The views of mathematics educators on their students' procedural and applied knowledge of ICT are available, as are aspects of social ecology and the instruction of the ICT (Kiswanto & Helsa, 2019). Please explain why selecting elements that depict the LiPO and LiES constructs significantly impacts their connection stability. Teachers with access to a wide range of ICT and are more likely to use pedagogical content knowledge to solve mathematics issues and have better success in the classroom are given adequate opportunities to teach with each type. When teachers decide to include ICT in the school, they want to challenge their students to reach higher levels of knowledge and skill acquisition (Sukmana et al., 2019).

In contrast, the analysis of moderator modifiers sheds light on a nuanced and shifting interaction (Byrne, 2019). Moderators are used to explaining freedom and leaning against one another of many modifiers in an interaction (Kline, 2017). The presence of a moderator in a model is currently more sought after in social science research and theory-building fields(Jackson, 2018), and this line of inquiry has been active for over twenty years. The authors of this study propose two possible routes: the first involves using belief moderators. In contrast, the second begins with limitless alternatives for modifying material pedagogical and technological expertise.

Problem-solving skills that integrate mathematics and ICT necessitate an active learning approach. As a result of its behavioural and direct cognitive nature, teaching ICT mathematics has been shown to increase ICT literacy. This is true because the standards for efficiency or appropriateness must coincide with the notion of belief. With ICT literacy, the moderator for teaching anxiety can determine the extent to which teachers feel anxious about the subject.

Despite this, ICT literacy (the intersection of subject pedagogy and ICT) adheres to a different route of concern than a moderator effect. This study's findings suggest that teachers may impact their students' ICT literacy levels by instruction in content pedagogy technological knowledge. Just how much do you fear failure in mathematics? In what ways do you fear instructing Mathematics? These two inquiries are linked to ICT literacy, as shown by the degree of worry. ICT proficiency may or may not be affected by the author's acquaintance with mathematical concerns related to educational technology. Teachers must be well-versed in cutting-edge subject pedagogy and how technology might influence their students' topic knowledge and skill development to prepare them for high-stakes mathematics assessments. Having a firm grasp of the content and pedagogy of mathematics within the context of ICT is crucial for students aiming for proficiency in advanced mathematics. As a result, students' ICT proficiency in content pedagogy may benefit from, or suffer from, a moderator's level of care.

The structural model created for this study supports the hypothesis that anxiety may have a secondary effect on ICT literacy. To restate, anxiety can affect ICT literacy and the relationship between belief in and knowledge of content pedagogical technology, the other two moderators. In line with these findings, Fu (2017) found that knowing how much to worry about something functions as a facilitator to change the moderator between educational technology material and a believing attitude towards it. This research shows that anxiety moderates belief and ICT competence, altering the nature of that relationship. It connects the fields of pedagogy, technology, subject matter, and ICT literacy. According to Hausfather (1996), technological competence is influenced by three interrelated elements, one of which is anxiety. Equally crucial to using and believing in technology is the ability to articulate one's anxiety. According to Brady and Bowd (2016) findings, this makes it difficult to evaluate the technological competence of teachers who suffer from high levels of anxiety. Vygotsky's theory of cognitive development characterizes anxiety as a mental state. The mathematics a teacher exhibits can affect students' technological ability (Hadley & Dorward, 2018). That is to say, and while some research supports Vygotsky's theory, other studies show that the associations between the variables are, at best, weak. There has yet to be a decision made. One possible reason for this constancy is the moderator-changing function played by teaching anxiety or other crucial modifiers (Lea, 2019).

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According to the results of this research, developing and refining belief as a set of skills is essential for enhancing ICT literacy. First, mathematics teachers need to believe that students' technological competence can be developed (Misfeldt et al., 2016). This is because jobs that require the use of technology are notoriously challenging. As a result, there is a compelling motivation for the professor to carve out time in the curriculum for students to work on expanding their capacity for original thought.

The findings also show how crucial it is to consult with professionals in the knowledge content pedagogical technology field. This is especially true for mathematics classrooms, where there is a direct correlation between ICT literacy and the implementation of technology-enhanced, content-based pedagogy. To this end, knowledge-content education and technological advancement must work together to encourage classroom mastery effectively. In addition, many believe that mathematics teachers should have a solid understanding of the technological competence content (Lamichhane, 2018).

CONCLUSION

Academic success, information (technological or content knowledge), content and pedagogy, and the cultivation of good values are all essential aspects of creating knowledgeable, inventive, and moral persons. However, teachers' ICT literacy did not increase, and widespread concern exists. This calls for swift action to take preventative steps. As a result, there needs to be a shift in how Mathematics is taught in the classroom, emphasizing giving teachers the tools they need to become more proficient with technological tools. The use of technology in the classroom can be managed in several ways; one is to encourage secondary school teachers to use their understanding of technology, the mathematics curriculum, and teacher pedagogy.

The developed structural equation model shows that the ICT literacy of pre-service teachers is affected by the interplay between the teacher's belief aspects to the TPACK and ICT literacy. This approach can help teachers, educational institutions, and the Ministry of Education by fostering a climate of belief and reducing anxiety among teachers. The researcher thinks that the results of this study could be helpful to other researchers interested in starting new studies or expanding on existing studies relating to the ICT literacy model and acknowledges that more research is always needed in this area.

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Is "Fruit Salad Algebra" Still a Favorite Menu in Introducing Algebra in Schools?

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Abstract: Students' difficulties in algebra are generally caused by the use of algebraic notation, the meaning of letters, as well as the types of relationships or methods used. Therefore, interpreting letters in algebra is one of the critical points in the transition from arithmetic to algebra. However, many students have misconceptions in interpreting letters in algebra. One of the most well-known misconceptions in interpreting letters in algebra is the "letter as object" where the term "fruit salad algebra" is sometimes used to name this misconception. The aim of this research is to explore further information about this misconception, its causes, and alternative solution. This study used the case study method with 35 grade 7th junior high school students as respondents. Data collection was carried out through written tests, interviews, and documentation. The results showed that students interpreted letters in algebra in various ways but tended to lead to "letters as objects", the "fruit salad algebra" approach is found in several textbooks and is rooted in the teaching culture, and there is a gap between the development of operations on letters as unknown and the idea of equality within equation.

Keywords: algebraic notation, letter, misconception, symbols, variable.

INTRODUCTION

Almost all children know algebra after arithmetic in learning mathematics at school. This happens for historical reasons, algebra was created long after the discovery of arithmetic (Carraher et al., 2006). Although, arithmetic is considered a prerequisite for algebra because the basis for algebra manipulation uses four arithmetic operations and maintains its meaning, but algebra cannot be considered as an extension of arithmetic because the problem-solving approach is different (Dettori, Garuti & Lemut, 2006). Knowledge of arithmetic rules that have worked well, to some point does not apply. For example, the equation of the form of $Ax + B = Cx + D$ does not apply to arithmetic ideas because it involves operations with 'unknown' which is outside the arithmetic domain. Herscovics & Linchevski (1994) revealed the cognitive gap between arithmetic and algebra, which can be characterized as the inability of students to work spontaneously with or in unknown. This inability is because students see literal symbols as static positions and operational aspects can only be understood when letters are replaced by numbers (Linchevski &

Herscovics, 1996: 41). To operate on unknown or in general quantities in general (for example variables or parameters), one must think analytically, that is one must consider the uncertain amount as if they are something known or as if it is a specific amount. Student understanding of arithmetic is the prerequisite ability but is not enough to deliver students to understand algebra. Therefore, a smoother bridge is needed to support the transition of students from arithmetic to algebra.

The transition from arithmetic thinking to algebraic thought has become a topic of substantial interest in mathematical education research throughout the world for more than the last two decades (for example, Anniban, et al., 2014; Ann van Amerom, 2003; Herscovics & Kieran, 1980; Kiziltoprak &Kose, 2017; Malisani & Spagnolo, 2008; Onal, 2023; Panorkou, 2013). The main difficulty mentioned in their report is a significant difference between arithmetic and algebra. Arithmetic is a systematic process in mathematics about addition, reduction, multiplication, and division in its primitive form (Akkan, et al., 2011; Mason, 1996; NCTM, 1991). In general, many researchers agree that arithmetic is a procedural and concrete system that produces numerical answers in certain numbers, manipulation of fixed numbers, letters are measurement labels or abbreviations of an object, symbolic expressions represent the process, and the same sign as a signal to calculate (Christou & Vosniadou, 2012; Kieran, 1992; Linchevski & Herscovics, 1996; Stacey and MacGregor, 2000; Ann van Amerom, 2002). Therefore, thinking arithmetic is done with a known quantity.

Meanwhile, algebra requires reasoning about unknown or variable amounts and recognizes differences between certain and general situations. There are differences regarding the interpretation of letters, symbols, expressions, and concepts of equations. For example, in arithmetic letters are usually abbreviations or units, while algebra letters are standing for variable or unknown. Nathan &Koellner (2007) states that algebra has two core concepts, namely equations, and variables. Usiskin, (1999) states that an understanding of "letters" (variables) and operations must be owned by students in studying school algebra. Students tend to believe that the variables are always in the form of letters and that letters always represent numbers. In fact, the values taken by variables are not always numbers. This is because variables have many definitions, referrals, and symbols. The use of variables is determined by or related to the conception of algebra and correlates with different interests. Variables can be interpreted as the generalization of fundamental patterns in mathematical modeling, unknown or constants, arguments (i.e., abbreviated domain values of function), or parameters (i.e., abbreviations of numbers that depend on other numbers). According to Radford (2006) using letters is not the same as algebra because not all symbolizations are algebra, as well as all patterns of pattern lead to algebraic thoughts. Therefore, algebraic thinking can be defined as an approach to quantitative situations that emphasize general relational aspects of tools that are not always symbolized by letters, but which ultimately can be used as cognitive support to introduce and maintain a more traditional school algebraic discourse (Barerjee, 2011; Kieran, 2004).

The body of mathematical knowledge is seen as a result of a long historical construction process, formulation, and clarification so that it cannot be fully understood through its formal dimensions

(Gallardo, 2002). According to Booker & Windsor (2010), algebra does not begin with symbolic reasoning but has been separated into three phases, namely: 1) Rhetorical Algebra which involves the use of words and sentences, 2) Algebra The form of syncopation in which words and actions are expressed in form The abbreviation means the words and sentences used previously, 3) Symbolic algebra, namely modern conception involving special symbols, functions, and structures. It seems that many students follow the same sequence of developments as the historical development of algebra. Therefore, the historical foundation needs to be discussed as a basis for understanding the development of students in algebra and compiling the right steps in helping students according to the development stage.

Historical-critical analysis provides facilities to construct teaching and learning sequences in which students and teachers are involved in reflecting progress in theoretical investigation. The concept and use of symbolization marks the difference between arithmetic thinking and algebra. Viète's work entitled Analitic Art and Development of Experimental Teaching Sequences according to Gallardo (2002A) marked the existence of didactic discipline in the algebraic historical evolution line in connection with the symbolic representation of 'Unknown' and the possibility of operating on "unknown". Ely & Adams (2012) has given a good explanation of the development of variable ideas, which starts from Unknown and Placeholder. Unknowns had been used for thousands of years before placeholders appeared. The beginning of the symbolic algebra was marked by the emergence of Placeholder in 1591 and opened the way for the development of the complete ideas of variables in 1637. When representing and manipulating unknown, the Babylonians and Greece generally used words rather than symbols such as those carried out by Islamic and Indian mathematicians In the Middle Ages. The quantity of unknown is usually referred to as "thing" or "number," or "root". The use of symbols as a placeholder, requires changes in thinking that allows symbols to refer to more common types of objects. This is what allows modern mathematicians to place letters as a substitute for quantity. Manipulation carried out on these letters will work the same way for certain numbers that can represent them. FranciscusVieta (François Viète) in his work in 1591 proposed a new practice representing the values given in problems with letters that could represent them. Therefore, it can be concluded that the general idea in the heart of the placeholder (a letter can represent a set of unknowns quantity) and covariational reasoning (how to represent and measure the way of change of one quantity to another) are two important things for the development of variable ideas.

METHOD

Research Design

This research uses qualitative research with a case study approach because it is considered appropriate to the purpose of this study, namely to explore students' misunderstandings through an in-depth approach to find out the causes of these misunderstandings in order to prevent, reduce or correct misconceptions about interpreting letters in algebra. There are three research questions used, namely "how do students in grade 7 junior high school understand the letters in algebra?"; "what are the misconceptions experienced by grade 7 junior high school students in understanding letters in algebra?"; and "what kind of learning experiences do grade 7 junior high school students have in understanding letters in algebra?".

Setting and Participants

This study took place at a junior high school in Bandung, West Java, Indonesia. Purposive sampling technique was used to determine participants in this study because of its suitability in advancing research objectives. Grade 7 students of junior high school semester 1 were chosen because they are in the transition stage from arithmetic to algebra where students at this level have just been introduced to algebraic material so they are prone to misconceptions. There were 35 students who agreed to become participants in this study.

Instruments

Data collection techniques used were written tests, interviews, and documentation. Written tests are used to identify students' misconceptions. The written test questions consist of five description questions as shown in Table 1.

No	Question Description
1	Ani has a basket of fruit in which there are 4 apples and 3 bananas so that all of Ani's fruits are 7. Can the sentence be written as "$4a + 3b = 7ab$"?
2	Based on question number 1, what is the meaning of $4a$, $3b$, and $7ab$?
3	Budi bought 1 box containing several strawberries, 2 boxes containing several pears and 5 melons. Can the sentence be written as "S+2P+5M"?
4	Based on question number 2, can S+2P+5M also be written as □+2△+5⬠?
5	The price of a watermelon is three times the price of a mango. While the price of 3 watermelons and 2 mangoes is Rp. 55,000. How much does a watermelon cost?

Table 1: Written Test Guidelines

The interview was conducted after the students' responses in the written test were analyzed. Students who became informants in interviews were selected based on written test answers that were considered to be able to provide relevant information. The interview was conducted using a semi-structured interview guide with three standard questions, namely: (a) What do you think about the letters? Why?; (b) Does an algebraic form have to have letters in it? Why?; and (c) How is the process of teaching and learning algebra in class? While other questions were developed based on the responses given by students during the interview process. Each question asked aims to confirm students' answers, the conceptions they have, and their learning experiences. The duration of the interview was between 10 and 15 minutes. Document analysis was carried out by analyzing the mathematics textbook used in the teaching-learning process in class and reading interview transcripts. During the process of collecting, storing and analyzing data, researchers maintain the confidentiality of sources and anonymity.

RESULTS

Student performance in completing the written test is described based on each question number as follows.

Question 1

Question number 1 is a question specifically designed to see how students in class 7th junior high school's translation understanding of letters as symbols (variables) in algebra. Understanding translation according to Sudjana (1995) is the ability of students to understand an idea that is expressed in another way from a known or previously known original statement or sentence in terms of translating sentences in word problems in mathematical form, for example mentioning known or asked variables, the ability to translate symbols, as well as the ability to translate into symbolic forms and vice versa. Usiskin (1997) states that the adaptation of arithmetic thinking to algebraic thinking can be done by algebraic representation of a variable where numbers can be represented by words, blank marks such as "____" or "……", boxes, question marks, or letters. Chick (2009) argues that it is very important for students to understand that letters represent numbers in algebra either as common numbers, unknown numbers, or variables.

There were 23 students in this study who agreed that "$4a + 3b = 7ab$" is a representation of the sentence "Ani has a fruit basket in which there are 4 apples and 3 bananas so that all of Ani's fruits are 7". One of the students interviewed read the equation "$4a + 3b = 7ab$" as "four apples plus three bananas is 7 apples and bananas". This indicates a misunderstanding known as the "letter-as-object misunderstanding" which is explained by Küchemann (1981) as a misunderstanding whereby students view letters as objects derived from abbreviated words such as a for apple rather than as representing a number.

Meanwhile, 12 other students did not agree that $7ab$ was the result of $4a + 3b$ because they could not add apples and bananas, but they agreed that $4a + 3b$ was the sum. Letters as a misunderstanding of objects according to Chick (2009) can also be strengthened by applying letters in formulas such as $L = p \times l$, where L is the area. According to Herscovics & Kieran, (1980) "non-conservator" students, namely students who do not realize that an unknown value does not depend on the letters used, have problems performing arithmetic operations on algebraic expressions. The difficulty they have is thinking of letters representing numbers. This shows that some students failed to develop the symbolism and notation used in the equation.

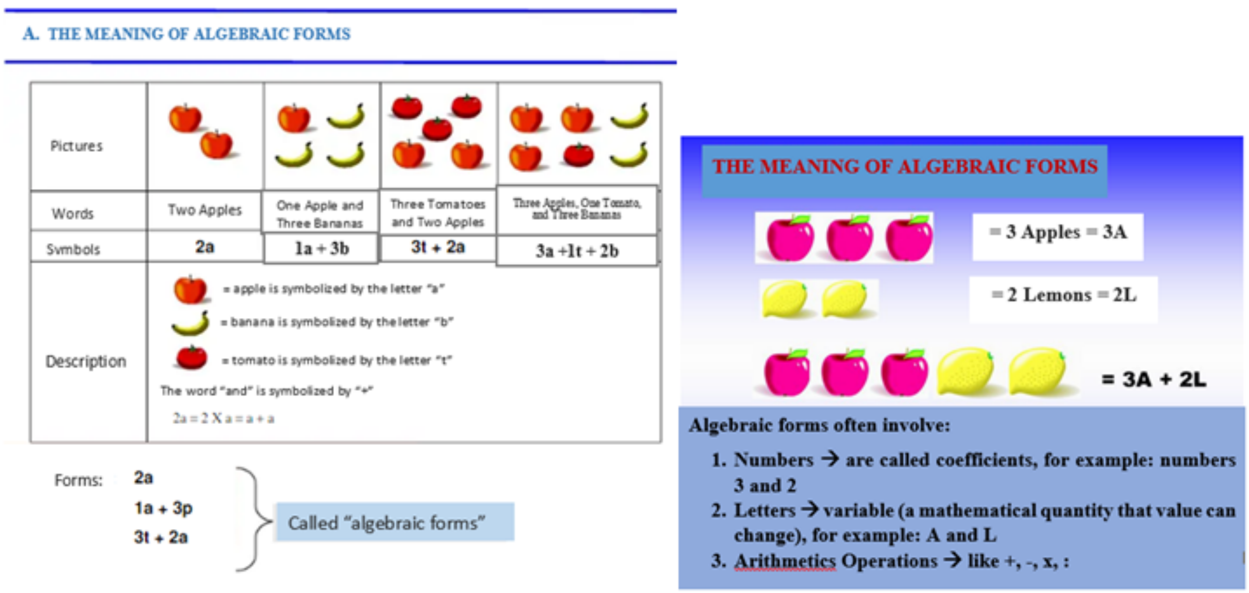


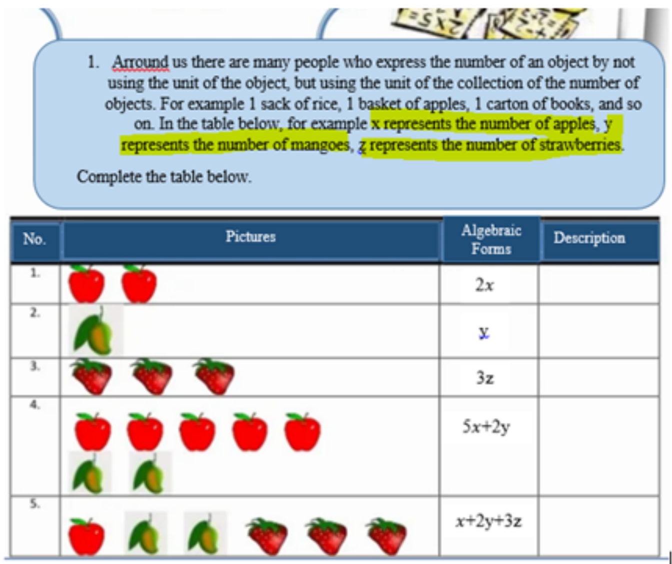
Figure 1: The "Fruit Salad Algebra " where the Letters are Object Name Abbreviation

Based on the results of an analysis of the mathematics textbooks used by students during the teaching and learning process in the classroom, there is evidence that the "fruit salad algebra" approach is present in textbooks as shown in Figure 1. If textbooks are considered as authorities in mathematics, then this evidence gives legitimacy to the meaning of variables in algebra, namely "letters stand for objects" where letters stand for named objects, such as the letter "a" for apple, "L" for lemon, "b" for banana, and so on. This strategy according to Thomas & Tall (2004) can provide short-term success, but is misleading in the future. Like adding $3a + 2b$ with $4a + 3b$ to get $7a + 5b$ by imagining apples and bananas put together, then how to explain an expression like $3ab$ being used? Is it three apples and a banana? Definitely not 3 times applying bananas. This shows that many fail to give a meaning that is in accordance with the meaning of mathematics that should be.

Question 2

All students in this study stated that $4a$, $3b$, and $7ab$ were "four apples", "three bananas" and "seven apples and bananas" respectively. The shift from arithmetic in everyday situations to synthetic arithmetic and algebraic symbolism involves more complicated expressions that cause difficult transitions for many students. This transition is made more difficult by the change in the meaning of the symbolism. In arithmetic, the expression $4 + 3$ is an operational rule in the sense that it has a calculation procedure that shows the result. One of the difficulties found in the context of problem number 2 is that $4a$ in algebra does not represent 4 apples, but four times the number of unknowns. Algebraic complexity is related to syntactic inconsistencies in arithmetic, for example: the invisible multiplication sign such as $4a$ which is $4 \times a$, a variable can simultaneously represent many numbers, letters can be chosen freely, the equals sign as an equality relationship, different concepts and rules exist in arithmetic and algebra (Breiteig & Grevholm, 2006). In algebra, however, the symbol $4 \times a$ is the first expression for the evaluation process, which cannot be executed until the value of a is known. This is one of the hardest things for some seventh graders to deal with, which is "but how can I multiply 4 by a, when I don't know what a is?". The difficulty of imagining algebraic expressions as solutions to problems has been described as a category closure misconception by Thomas & Tall (2004).

Student textbooks not only display the "fruit salad algebra" method which gives the meaning of a letter or variable in algebra which stands for the named object, such as the letter "a" for apple, but also to refer to a quantity or represent a certain number (value) such as shown in Figure 2.



1. Arround us there are many people who express the number of an object by not using the unit of the object, but using the unit of the collection of the number of objects. For example 1 sack of rice, 1 basket of apples, 1 carton of books, and so on. In the table below, for example x represents the number of apples, y represents the number of mangoes, z represents the number of strawberries.

Complete the table below.

No.	Pictures	Algebraic Forms	Description
1.		$2x$	
2.		y	
3.		$3z$	
4.		$5x+2y$	
5.		$x+2y+3z$	

Figure 2: The "Fruit Salad Algebra " where Letters Represent Numbers

There is a practice question page in the textbook that begins with examples of problems and their solutions. In this example, a contextual problem is given which reads "Around us many people express the number of an object by not using the units of the object, but using the unit of the collection of the number of objects. For example 1 sack of rice, 1 basket of apples, 1 box of books, and so on. In the table below, for example x represents the number of apples, y represents the number of mangoes, z represents the number of strawberries. Complete the table below."

After the problem description is given, there is a table consisting of four columns where the first column shows the serial number, the second column contains contextual images, the third column contains the algebraic form of the image shown in the second column, and the fourth column contains information that describes the image and algebraic form in columns two and three. There is an ambiguous concept between the statements given, for example in the introductory sentence it is stated that "...declare the amount of an object with not the unit of the object, but using the unit of the group of the number of objects...for example 1 basket of apples,...", then proceed with the statement which starts the problem, namely "...x represents the number of apples..", while the images and algebraic forms displayed in the table do not reflect the given statement. It can be seen that the picture presented is in the form of fruit in seed quantity, not in "basket" size quantity as stated in the initial statement. Also, if "...x represents the number of apples...", wouldn't it make more sense if the algebraic form shown was "$x = 2$" instead of "$2x$"?. It can be seen that this "fruit salad algebra" approach provides easy access to students' misconceptions.

Question 3

The third question aims to evaluate the effectiveness of introducing letters as"unknown" in a way that often appears in students' math books on algebraic forms material in grade 7 junior high school. In the context of question number three, this adopts the problems in textbooks that are often taught to students as shown by Figure 3.

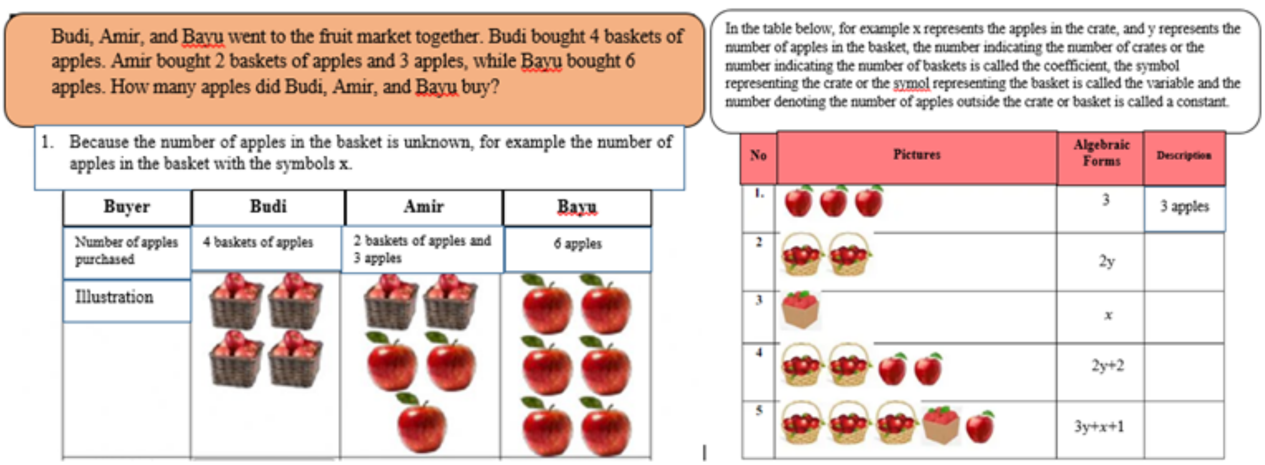


Figure 3: The "Fruit Salad Algebra " where Letters are Unknown

The algebraic representation of the sentence "Budi bought 1 box containing several strawberries, 2 boxes containing several pears and 5 melons" namely "$S + 2P + 5$" where the number of strawberries and pears is unknown, while the number of melons is known so letters are generally used, for example "S" and "P", to represent unknown values. However, the number of "several" as "unknown values" and the number of "units" refer to known values is a logically ambiguous concept when it comes to the concept of equality. The word "some" used in the context of the amount of fruit in a basket, box or bag does not have an equivalent standard unit reference, in contrast to the context referring to "price" which can be used through a "barter" approach which has been reported in Streefland's experimental research (1995) shows the steps in the conceptual development of variables. In this case the letters "S" and "P" represent the fruit itself, not its price value.

One example of the difficulty in the ambiguity of the unknown concept is through the "number of fruit in a basket" approach, for example in the algebraic form in number 4 in Figure 3 namely "$2y + 2$" refers to "the number of apples in two baskets and 2 apples". Thus, if we want to find out the number of apples (each fruit) in each basket using the algebraic form, we get: $2y = -2$ then $y = -1$. Even though y is defined as "the number of apples in the basket", this makes no sense. This can make the letters in algebra (variables) meaningless so that it can cause students to base their interpretation of algebraic letters and expressions on intuition and guesswork, on analogies with other symbol systems they know, or on false foundations made by misleading teaching materials. They are often unaware of the general consistency of mathematical notation and the power it exerts. Their misinterpretation causes difficulties in understanding algebra and can persist for several years if not recognized and corrected.

This ambiguity is proven to cause misconceptions among most students in this study, namely 17 students believe that letters represent objects (abbreviations of object names), such as the letter "a" which stands for the word "apple" represents the apple itself either in a basket or standing alone (outside the basket). So according to these students $S + 2P + 5M$ is a correct or reasonable representation of "1 box of strawberries, 2 boxes of pears and 5 melons".

Question 4

The fourth question aims to explore students' misconceptions that tend to believe that variables are always letters. All students in this study agreed that the symbols $\square$, $\triangle$, and ⬠ are not variables because they are not letters, so $S + 2P + 5M$ cannot be written as $\square$+2$\triangle$ +5⬠. They are also not used to working with symbols like $3 + 5 =$$\triangle$ or $5 +$⬠=15 while studying arithmetic in elementary school. This view is supported by many textbooks and reinforced by many educators as shown in Figure 1 where algebraic forms are described as “combinations of letters and numbers separated by arithmetic operations” such as $2a$, $1a + 3p$, $3t + 2a$, or $3A + 2L$. It also gives a definition that "numbers in algebraic form are called coefficients" with an example of the coefficients in $3A + 2L$ being 3 and 2. In addition, it also states that letters in algebra are variables, namely "a quantity in mathematics whose value can change" with examples of letters A and L is a variable of $3A + 2L$.

Based on the definitions and examples of these variables, it can be concluded that variables are "letters that represent numbers". Though the values that variables take aren't always numbers, even in high school math. Usiskin (1999) clarifies this concept by mentioning the variety of variables in several areas of mathematics such as: variables in geometry often represent points where the variables A, B, and C are used when we write "if $AB = BC$, then ΔABC is isosceles"; in logic, the variables p and q often represent propositions; in real analysis, the variable f often represents a function; in linear algebra, the variable A can represent a matrix or the variable v for vectors;

and in higher algebra, the variable can represent an operation where the variable is not necessary represented by letters, because variables have many definitions, references, and symbols.

Thus, if you refer to the definitions and examples of algebraic forms in the textbook, $5 + x = 8$ is usually considered algebraic, while $5 + \_\_ = 8$; $4 + \Delta = 7$; $3 + ? = 6$ is not considered algebraic even though the blank, triangle and question mark in this context want a solution to an equation that is logically equivalent to x. The definitions given in the textbooks seem to try to fit the notion of variables into a single conception by simplifying the ideas and in turn actually change the goal of algebra.

Question 5

The problem in the fifth item adopts the concept of the Chinese barter problem that inspired Streefland (1995) as a naturally and historically formed starting point for the teaching of linear equations, claimed by Streefland to represent the steps in the conceptual development of variables. Streefland (1995) found in his teaching experiments that literal symbol meanings are important constituents of students' progressive formalization. Furthermore, Streefland reported that students need to be aware of the changes in meaning experienced by letters because in this way the level of students' mathematical thinking can develop. In the concept of this barter problem, according to Ann van Amerom (2003) students are required to be able to compose not only the form of an equation (from the amount of fruit to money) but also the meaning of the unknown (from the object related to the quality of the related object). These considerations show the steps in the conceptual development of variables.

The first step that must be taken by students to solve problem number five is to represent the sentence "the price of a watermelon is three times the price of a mango" and "the price of 3 watermelons and 2 mangoes is Rp. 55,000" into algebraic form. There were 4 students who failed at this stage representing "the price of a watermelon is three times the price of a mango", as shown in Figure 4 below.

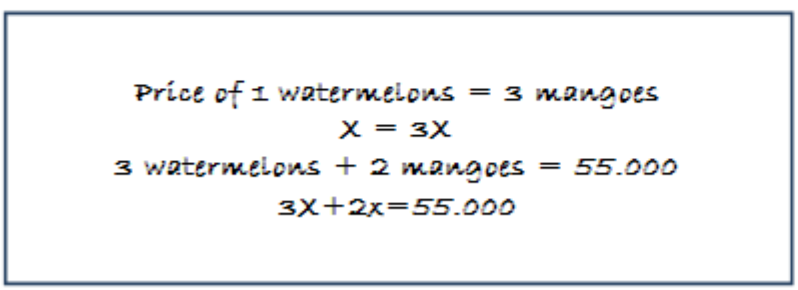


Figure 4: Example of Student Answers in Question 5

As many as 27 other students have been able to make the correct representation "the price of a watermelon is three times the price of a mango" but failed in the next process, namely substituting the first equation (i.e., $s = 3m$ where s refers to the price of a watermelon and m refers to the price of a mango) to the second equation which is $3s + 2m = 55{,}000$ becomes $3(3m) + 2m = 55{,}000$. Figure 5 shows an example of student failure in solving problem number 5.

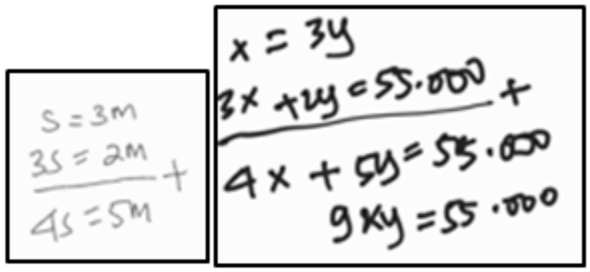


Figure 5: Examples of Student Misconceptions in Question 5

Student errors in solving question number 5 support Ann van Amerom's (2003) statement that if students continue to interpret letters as objects or labels or abbreviations, not variables, they experience difficulties both in dynamic (procedural) conceptions and in static conceptions of ideas algebra known as 'reversal error' when converting verbal descriptions into formulas for example to translate word problems into equations.

DISCUSSION

Algebraic abstraction is one of the biggest problems for students in learning mathematics at the high school and college levels. Students' difficulties in algebra are generally caused by the use of algebraic notation, the meaning of letters and variables as well as the types of relationships or methods used. The generality of algebraic ideas makes semantics weak so that there are deadlocks experienced by students regarding the use of algebraic notation. During the process of learning mathematics in elementary schools, those who had so far seen arithmetic symbolism as a representation of processes that could be carried out by arithmetic procedures suddenly discovered that this "universal law" did not apply. For example, there are 2 apples and 3 bananas on the table. They think of $2a + 3b$ as "2 apples and 3 bananas", then think of it as "5 apples and bananas" and write 5ab what makes sense to them. However, this does not apply to algebra. Expressions with letters cannot be worked out unless the values are known and if the values are known why use algebra? According to algebra students is an unnecessary and irrelevant difficulty.

Breiteig and Grevholm (2006) explained that algebraic complexity is associated with syntactic inconsistencies with arithmetic, such as: a variable can represent many numbers simultaneously, letters can be chosen freely, there is no positional value, equality as an equivalence relationship (equivalence), multiplication sign is not visible, priority rules and the use of brackets. Many students cannot connect arithmetic and algebra because classroom learning treats these two topics as if they were completely different from one another. The concept of equivalence is difficult for a student because they see statements as arithmetic problems such as $2 + 7$ being interpreted as adding 7 to 2 yields 9. This makes expressions such as $a + b$ unintelligible. If a or b is not known, then it is impossible to calculate in $a + b$. So finding the sum of a and b rather than calculating a to b is a more meaningful emphasis. For example, in the fraction 4/7, the numbers 4 and 7 cannot be seen as separate numbers, because 4/7 itself is a number.

Misinterpreting algebraic letters as object names (e.g. interpreting the letter "b" as "banana", so "$5b$" means "five bananas") is agreed upon by researchers (e.g., Ann van Amerom, 2003; Chick, 2009; MacGregor and Stacey, 1997; Malisani & Spagnolo, 2009; Usiskin, 1999) as a well-known and serious obstacle when writing expressions and equations in certain contexts. Also, the conveyance of concepts in applied mathematics is usually denoted by the initial letter of their name (such as A for area, m for mass, t for time, etc.). It is quite possible that this use of letters reinforces the belief that letters in mathematical expressions and formulas stand for words or things, not for numbers. The results of research conducted by Edo & Tasik (2022) show that students tend to interpret variables as "labels" and as "objects" rather than numbers. The use of letters as abbreviations for words or labels such as the "fruit salad algebra" approach is still very much found in student math textbooks or student worksheets compiled by teachers.

If students are properly taught in early grades about some of the important parts of algebra such as equivalence, patterns, expressions, and functions, they will not experience much difficulty in transitioning from arithmetic to algebra or in understanding algebraic notation. Onal (2023) states that students must learn that there are many meanings associated with arithmetic symbols, this is because certain interpretations will suit different contexts and solving procedures. Equality is a relationship that expresses the idea that two mathematical expressions have the same value and must be well understood by students so that it does not become a major stumbling block for them moving from arithmetic to algebra (Oksuz, 2007). Students need an understanding of the equals sign to be able to see the relationships expressed by a number of sentences.

CONCLUSION

The use of the "fruit salad algebra" approach has proven to be still a favorite "menu" in introducing variables in algebraic form in junior high schools. However, despite being a favorite, this approach was reported as a "misguided early presentation" of developing algebraic thinking. Some researchers such as (Edo & Tasik, 2022; Gunawardena, 2011; Widodo et al., 2018) also report that misinterpreting letters as labels is a fundamental misunderstanding that will lead to many other errors in algebra. Different interpretations in different contexts in interpreting letters can cause students to be confused and misinterpret the use of variables (Edo & Tasik, 2022). The student misconceptions found in this study support the report. Algebraic reasoning involving variables and symbolic notation appears to be a cognitive barrier for students learning algebra at school. Students have difficulty recognizing the structure of the problem when they try to represent the problem symbolically. They can recognize the solution procedure (e.g., reverse computation) but they cannot give reasons for the unknown itself.

Responses from discussions with teachers indicated that the “fruit salad algebra” approach was rooted in the teaching culture, reinforced by several textbooks, and influenced by how teachers themselves were taught in the past. Its continued use occurred for a number of reasons, including that the "apple and banana" analogy was perceived by teachers as easy to understand for students, teachers lacked understanding of the long-term dangers of the approach, teachers lacked knowledge of alternative strategies, or that they did not believe that there are other alternatives that can be "helpful" and accessible to students.

Instead of using the “fruit salad algebra” approach to introduce variables to algebra in junior high schools, teachers could consider using the notion of equality of equals sign approach through problems that use students' common knowledge and informal strategies. The following are some examples of approaches that can be used. Ardiansari & Wahyudin (2020) show the steps for expanding the meaning of the equal sign by using the distributive law which can be used to introduce variables as follows. The teacher can explain the steps made by Ardiansari & Wahyudin (2020) through the question-and-answer method in class. Symbols □, ■, and △ represent hidden numbers, then one by one these symbols are replaced with letters such as $a, b, c, \ldots$etc. to represent hidden numbers. The letter that is closely related to the idea of hidden numbers is then referred to as an unknown term.

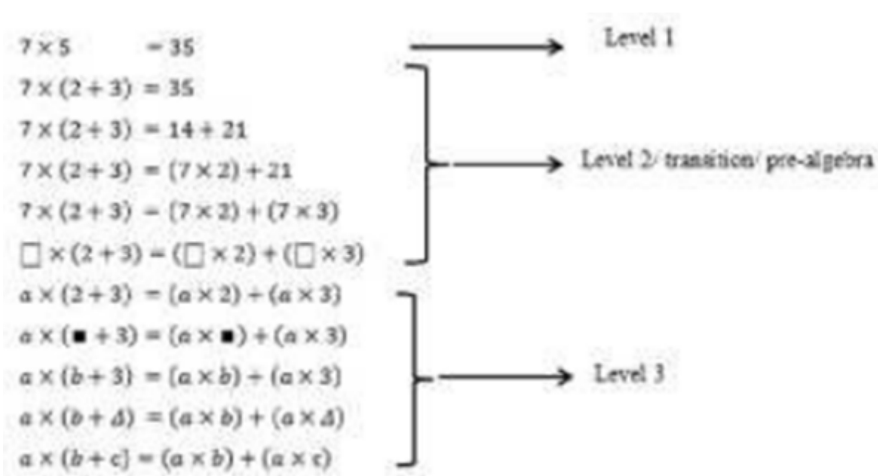


Figure 6: The Steps for Expanding the Meaning of the Equal Sign by using Distributive Law

Another example is through problems in real situations as illustrated by Suryadi (2013) below which can be considered for use in introducing variables. There are three glasses containing Rp. 1000,00 and three other glasses containing Rp. 5000, 00 as shown in Figure 7. Students are asked to find at least three different ways to find the total value of the money in the six glasses.

Figure 7: Problem Illustration

Through class discussion, a number of questions are then asked which encourage students to explain the relationship between the three mathematical representations. Then a further problem is given, namely there are three white glasses where each contains money with the same nominal but it is not yet known how much and three black glasses containing money with the same nominal but it is not known how much it is as shown in Figure 8 to introduce variables.

Figure 8: Further Problem Illustration

Students are asked to find three different ways to determine the total value of the money in the six glasses where the amount of money in the white glass group and the black glass group is not the same.

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Student *Commognitive* Analysis in Solving Algebraic Problems

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Abstract: This study aims to describe students' commognitive in solving algebraic problems. This research is a qualitative research with descriptive approach. The subjects of this research were fifth semester students of the mathematics education study program at the College of Taman Siswa Teacher Training and Education. The research subjects were 9 students and 3 people were selected each as representatives of subjects who answered questions without solving Polya problems (S1), subjects answered with Polya problem solving but were incomplete (S2) and subjects who used Polya problem solving and were correct. (S3). The research method consists of four steps, namely: preparation, research subjects and locations, data collection, and data analysis. The research instruments were algebraic test questions and interviews. The results of the Research showed that the S1 subject only raises word use in solving problems. For the S2 subject, besides raising the word use, it also uses an exploratory routine in solving questions, namely using the necessary but wrong procedure. S2 also experienced an error when determining the time from East Indonesia Time (WIT) to West Indonesia Time (WIB) where S2 added up instead of subtracting 2 hours so the result was wrong. Then for the S3 subject, bring up word used, symbolic mediators, exploratory and ritualistic routines and endorsed narratives in solving problems. The research findings showed that of the 9 research subjects, only 1 subject has all four commognitive components with Polya problem solving, the other 8 are still incomplete. Thus, for further research it is recommended that to see students commognitive it is necessary to use Polya's problem solving with positioning in group discussions.

Keywords: commognitive, students, solving algebraic problems, word use, visual mediators, routines, endorsed narratives

INTRODUCTION

Communication is an important part of mathematics in general and mathematics education. Through communication, ideas become objects of reflection, improvement, discussion, and change. The communication process also helps to build the understanding. When students are challenged to think and make reasons about mathematics and communicate the results of their thoughts to others either orally or in writing, they learn to explain and convince (NCTM, 2000). Brodie agrees that an understanding of mathematics is assumed through mathematical communication (Brodie et al., 2010). It can be said that communication is an inseparable part of the understanding. Brodie further stated that because communication is an important part of understanding, communication is used by students to discuss their understanding with others (Brodie et al., 2010).

The Ontario Ministry of Education explains that communication is the process of expressing mathematical ideas and understanding orally, visually and in writing, using numbers, symbols, pictures, graphs, diagrams, and words (Education, 2005). Students communicate for a variety of different purposes and opponents, such as communicating with teachers, peers, a group of students, or an entire class. Communication is an important process in learning mathematics. Through communication students can contemplate and reflect on their ideas, their understanding of mathematical relationships and their mathematical arguments.

Mathematical communication is an important process in learning mathematics (Cohen et al., 2015; Daher, 2012; Kieran, 2001; Kosko, 2014; Lestari et al., 2019; Thinwiangthong et al., 2012; Umar, 2012). Sfard further stated that mathematical communication is a process of conveying mathematical ideas both in writing and orally by each individual (Sfard, 2001, 2008, 2015). Communication is an important skill in mathematics because it is used to express mathematical ideas to oneself or others either in writing in the form of diagrams, symbols or orally. Mathematical communication is a process of conveying messages, ideas, ideas or opinions in mathematical terms both in writing and orally.

When coming up with an idea to solve the problem at hand, there is information processing that occurs. Information processing is a mental process known as cognitive process (Campos et al., 2013; Iglesias-Sarmiento and Deaño, 2011; Sánchez et al., 2013). Cognitive processes are mental processes in individuals (Montague et al., 2014). Cognitive processes that occur within one's self includes: 1) the process of obtaining new information, 2) the process of transformation information received, 3) the process of testing or evaluating the relevance and accuracy knowledge (Sutarto, 2017). Cognitive processes can be understood as a process of getting new information in memory to be digested and understood into a knowledge.

Communication and cognition are known as *commognitive* (Caspi and Sfard, 2012; Kim et al., 2017; Sfard, 2001, 2006, 2008, 2015; Sriraman, 2009; Viirman, 2015). *Commognitive* can be interpreted as a mental process and the delivery of information to oneself or others that is carried out verbally or non-verbally. *Commognitive* consists of four main components, namely *word use, visual mediators, endorsed narratives* and *routines. Word use* is the use of words in learning mathematics. *Visual mediator* is the media used in learning mathematics. *Visual mediators* can also be in the form of graphs, diagrams and symbols, as well as physical objects used as media/ props. *Routine* is a process of rules, steps that describe a pattern in learning mathematics. The steps in learning are defined as defining, estimating, proving, and generalizing. *Narrative* is a mindset used in learning mathematics about definitions, theorems, principles and facts (Viirman, 2015).

Commognitive research on students' as prospective teachers in teaching and learning has been carried out before, Berger (2013); Heyd-Metzuyanim & Tabach (2018); Ho et al. (2019); Nardi et al. (2014); Tababaru (2016); Tuset (2018); Viirman (2015); Zayyadi et al. (2019, 2020). Berger (2013) used *commognitive* theory to examine the activities of a pair of mathematics teachers in South Africa in giving mathematics assignments using *Geogebra*. Nardi, et al. (2014) investigated the effective communication through analysis of the use of words and visual mediators in the context of problem solving in small groups, analyzing variations in defining routines and *commognitive conflicts* in the transition from school to university. Viirman (2015) conducted research on explanations, motivations and asking questions in teaching from a *commognitive perspective*. In his research, Viirman investigated the learning practices carried out by seven mathematics teachers which were presented in three categories namely, giving explanations, motivation and asking questions. Explanatory routines including known mathematical facts, summaries and repetitions, different representations, everyday language; motivational routines including use of references, mathematical traits, humor; and routines in asking questions including questions about facts understood by students, controlling questions and rhetorical questions.

Research conducted by Heyd-Metzuyanim & Tabach (2018) explains the implications of the *commognitive theoretical framework* in four areas of practice: pre-service teacher preparation, in-service professional development, introduction to mathematics texts for secondary school students, and diagnosis of learning difficulties in mathematics, and ends with discussions about affordability and the challenges of linking *commognitive* with practice. One of the results of research conducted by Heyd-Metzuyanim & Tabach (2018) is that *commognitive* can be used as a tool for sharing learning experiences owned by a teacher to be given to students in the learning process. Research conducted by Tasara (2017) investigated mathematics teachers teaching basic differentials. In his research, Tasara revealed that the inconsistent use of the word "gradient" in material "gradient" can make it difficult for students to understand when "derivative" is used to mean gradient. This shows that the differences in the use of words in learning must really be considered. In addition, the *commognitive framework* provides a powerful conceptual lens to examine how teachers teach mathematics at the micro level. In this case, the research conducted by Tasara places more emphasis on the components of the use of words and *narratives* in investigating the teaching of the mathematics teacher.

Commognitive frameworks that can provide pre-service teacher teaching information in achieving mathematics learning goals was conducted by Tuset (2018). Tuset showed that a *commognitive framework* can provide an analysis of prospective teachers in the learning process. First, to describe in detail the components of the geometric discourse carried out by prospective teachers to achieve learning objectives. Second, to identify and explain the use of learning tools by prospective teachers provided in educational programs.

Commognitive research involving solving IDEAL problems (I- *Identify problems and opportunities*, D- *Define goals*, E- *Explore possible strategies*, A- *Anticipate outcomes and act*, and L- *Look back and learn*) conducted by Zayyadi, et al. (2019) shows that students are still more focused on the end result than on the IDEAL problem solving process and many students do not look back at the IDEAL stage in doing their work. From the *commognitive framework*, the subject tends to use mathematical words and visual mediators at the stage of understanding the problem, and narrative and routine at the stage of exploring and implementing strategies. This research provided initial insight into how students' describe mathematic problems from a *cognitive perspective*. Then Zayyadi, et al. (2020) conducted research with the aim of describing the content and pedagogic knowledge skills of prospective teachers in learning mathematics from a *commognitive perspective*. In this reserach, there are fundamental differences in the *commognitive components* of content knowledge and pedagogical knowledge of prospective teachers. The findings of this study related to the *commognitive pedagogical ability of prospective teachers can be referred to as commognitive* pedagogical. Student *Commognitive* can be measured by solving math problems.

Problem solving is the essence of learning mathematics (Subanji, 2013). Problem solving is a type of learning to think at higher level, so mathematics often requires problem solving skills in cultivating students' creative minds (Chong & Shahrill, 2016; Yassin & Shahrill, 2016). One of the frameworks for thinking about problem solving was proposed by (Polya, 1973) where the strategy is recognized by many researchers as the steps used in solving mathematical problems. Polya suggests four stages for problem solving, namely: 1) Understanding *the* problem; 2) Planning the problem (*Devising a plan*); 3) Carry out problem solving (*Carrying out the plan*); and 4) Looking back *(*Lederman, 2009; Lee, 2017; Okafor, 2019; Simpol, et al., 2018; Tohir, et al., 2020) The stages of solving the Polya problem can be presented in Table 1 below.

Problem Solving Stages	Description
Understanding the problem	It should be clear what the question means, what you are looking for the answer. Need to first realize the key point and context of the problem, then be able to find the answer
Devising a plan	Clearly knowing the relationship between problem points, choosing the appropriate approach and drawing up a plan to solve the problem, which is the most important task in the problem solving process
Carrying out the plan	Follow Steps 1 and 2, and practically calculate alone / in groups and find the answer
Looking back	Looking back at the entire troubleshooting process; check calculations and answers; discuss the meaning of the problem

Table 1: Polya's Problem Solving Stages

Polya's stages are closely related to open problem solving. Open problem solving is one way to reveal students' *cognitive* components, namely by giving questions that are non-routine in nature so that they can dig in depth and not rely on just one answer. An example of a non-routine question is an *open ended* question. Subanji (2013) states that an *open ended question* is a question that has a non-single answer or way of solving it. Furthermore (Yee, 2009) also argues that *open ended* questions are questions that have more than one way to solve them, or have various possible correct answers. *Open ended* questions can give freedom to students in expressing answers, encouraging students to generate various kinds of different thoughts according to their abilities as Cifarelli & Cai (2005) stated that problems in *open ended* are directed to guide students in understanding problems that can be solved with a different and correct point of view.

Commognitive research related to solving algebraic problems with *open ended* questions has never been done by other researchers. In this research, it is necessary to conduct a study of student *commognitive* from the point of view of *open-ended algebraic problem solving*. Therefore, this study aims to describe student *commognitive based on open ended algebraic problem solving*.

METHOD

This research aims to describe students' *commognitive* when solving algebraic problems. The research is qualitative, with a descriptive approach. The four important steps in this research are: (1) preparation, (2) research subjects and locations, (3) data collection, and (4) data analysis.

Preparation

In the preparation stage, the researcher developed test and interview instruments that enabled students to be involved in the process of solving algebraic problems. The test instrument involves math questions in Figure 1, which provide opportunities for students to demonstrate the use of *commognitive components* (*words use, visual mediators, routines* and *endorsed narratives*) in solving algebraic questions. In table 2. Furthermore, interviews are designed to articulate students' thought processes when they solve algebraic problems. The following are algebra problem solving test instruments:

> Two tourists departed on different airplanes from Jakarta to Jayapura. The first plane took off from Jakarta airport at 20.30 West Indonesia Time (WIB) local time and the second plane one hour later, the first plane landed at Jayapura airport at 08.30 East Indonesia Time (WIT) local time and the difference in time for the second plane arrived 1.5 hours afterwards. If during the flight the first plane stopped at Surabaya and Makassar airports for 30 minutes each, and the second plane stopped at Surabaya, Makassar and Timika airports for 30 minutes each, except for Timika with a 30 minute delay.
>
> a. Write down everything that is known from the problem above!
> b. How many hours does each tourist travel from Jakarta to Jayapura without stopping?
> c. Is there a difference in the travel time of the two tourists without stopping?
> d. What can you conclude?

Figure 1: Algebraic Problem-Solving Problems

***Commognitive* Component**		**Indicator**
Visual Mediators	*Symbolic Mediators*	Presenting mathematical information with symbols or algebra in solving problems
	Iconic Mediators	Make representations in the form of graphs, tables, diagrams and pictures in solving problems
	Concrete Mediators	Using real objects as media in solving problems
Word Use	*Mathematical Terms*	Using keywords or mathematical terms contained in the problem in solving the problem
	Mathematical and Non-Mathematical Terms	Using keywords or mathematical terms and not mathematical in solving problems
Routines	*Ritualised*	Can use the necessary procedures in solving problems
	Exploratory	Can provide explanations or reasons for how to solve the problem at hand and can convey when the selection procedure is used
	Applicability	Judging from how to solve the given problem, such as using symbols, making pictures, or directly calculating
	Corrigbility	Examining explanations or narratives of the reasons for using certain procedures or ways of solving problems by making conclusions and checking again
	Flexibility	The use of more than one way to solve a given problem, the general formula or form used and can be seen from the visual media used
Narratives	*Remember and explain*	Can explain reasons and relate objects, relationships with previous material and processes, such as definitions, theorems and proofs in solving problems.

(Adapted from Sfard, 2008; Mpofu & Pournara, 2018)
Table 2: Indicators *Commognitive* Components Used in Solving Algebraic Problems

Research Subjects and Setting.

The research was conducted at Teacher, Training and Education of Taman Siswa Bima NTB Indonesia with research subjects in semester V. it is consisted of 20 members, they were chosen because they had taken school mathematics course. Research subjects were selected using a *purposive sampling technique,* so that out of 20 people, 9 people were selected as research subjects.

Data Collection

The data collection process began with giving math questions based on algebraic problem solving to 9 research subjects to be solved individually. Of the 9 subjects, 1 subject answered using Polya's problem solving and the answer was complete and correct, 4 subjects answered using Polya's problem solving but the answer was incomplete, and 4 subjects answered not using Polya's problem solving and the answer was wrong. After that, the researcher conducted data analysis on 3 subjects, each of whom represented the three categories that the researcher mentioned above.

Data Analysis

In the problem-solving process, students are asked to express their thoughts out loud. Students are given the opportunity to explore, record and express all their thoughts and ideas. Researcher observed and recorded all behaviors including students' verbal thoughts while they were solving open problems. After the students completed the questions given, the researcher then transcribed the data, and the students who were the research subjects were interviewed individually to find out and explore their *cognitive components in solving open problems.* After that, the researcher carried out data reduction by removing elements that were considered unimportant from all data (observations, interviews and field notes) to be examined in data analysis.

RESULTS AND DISCUSSION

In this study, 9 subjects were given questions about algebra and asked to answer by writing all the steps according to the question accompanied by more detailed reasons. Based on the data obtained, there are different strategies used by students in solving the problems. There are students who start with examples so that the solution to the end is clear, while there are others who go straight without examples. Some of the strategies used by students include Polya's problem solving steps. Based on the data collected, there were 9 research subjects as presented in Table 3, consisting of 1 subject who answered correctly with the Polya problem solving steps, 4 subjects who answered with Polya problem solving but were not perfect, and 4 other subjects without using problem solving Polya and wrong. After that, the researcher analyzed the data by looking at the tendency of the answers made by the subject , namely the *commognitive* component based on the stages of solving the Polya problem used .

Using Polya Troubleshooter	No Troubleshooting Polya (*)
5 subjects	4 Subjects who answered did not use Polya's problem solving strategy and their answers were incorrect
1 who answered with Polya's problem solving strategy and the answer was correct	
4 which answered with Polya's problem-solving strategy but was incomplete	

Table 3: Research Subjects

Information:

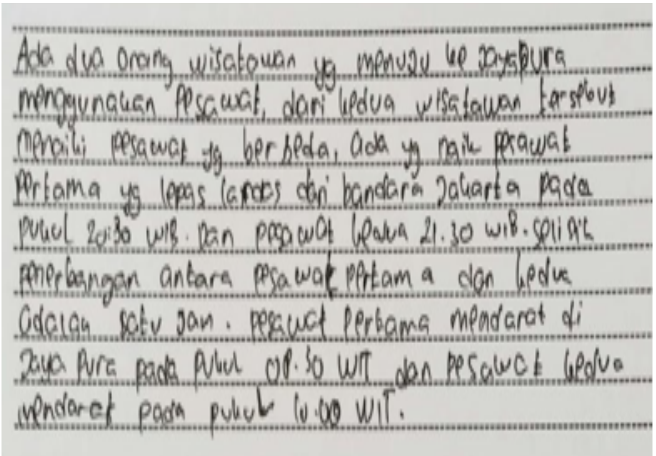
S1 : The subject answered the question without solving the problem and the answer was wrong

S2 : The subject answered the question by solving the Polya problem but it was not complete

S3 : The subject answers the questions by solving the Polya problem and is correct

Subjects without Using Polya Problem Solving (S1)

One of the 4 subjects (*) who answered the question without solving Polya's problem, without paying close attention to the problem and not following the questions asked, so the answer was wrong. For examples of answers from this subject can be seen in Figure 2 below:



Ada dua orang wisatawan yg menuju ke Jayapura
mengunakan pesawat, dari kedua wisatawan tersebut
memiliki pesawat yg berbeda, ada yg naik pesawat
pertama yg lepas landas dari bandara Jakarta pada
pukul 20.30 WIB. dan pesawat kedua 21.30 WIB. selisih
perbedaan antara pesawat pertama dan kedua
adalah satu jam. pesawat pertama mendarat di
Jaya pura pada pukul 08.30 WIT dan pesawat kedua
mendarat pada pukul 10.00 WIT.

Tehere were two tourist who went to jayapura by plane, they used different planes, firsts used the first plane which take off from Jakarta at 20.30 WIB. Andwhile the second one take off at 21.30 WIB so the different time of the two planes is 08:30 so the the first plane landed at Hayapura at 08:30 while the second landed at 10 WIT

Figure 2: Example of one answer from 4 subjects (S1)

Information:
WIB : Western Indonesian Time
WITA : Central Indonesian Time
WIT : East Indonesia Time

Based on the data above, S1 does not use Polya's problem solving stages in finding solutions to the problems given. The first subject (S1) immediately rewrote the existing questions without paying attention to the intent of the questions. S1 writes the second plane's arrival time at 10.00 WIT because the time difference between the first and second planes is 1.5 hours. S1 did not pay attention to the time changes from WIB to WITA and from WITA to WIT, so that S1 came to the wrong conclusion. In this case the S1 *commognitive component* that appears is only *word use*, so the researcher tries to explore it by interviewing. The following is a transcript of the results of the researcher's interview with S1:

R : Do you understand the questions given?
S1 : Hmmm, I understand sir (a bit doubtful about the answer)
R : How do you write what is known from the questions above?
S1 : By writing everything in the question sir
R : Oh, I see... How many hours did each tourist travel I and second traveler from Jakarta to Jayapura?
S1 : Tourist I 12 hours of travel because it starts at 20.30 WIB and arrives at 08.30

WIT, for tourists II because it starts at 21.30 WIB and arrives at 10.00 WIT, then the time needed is 12 hours 30 minutes sir.

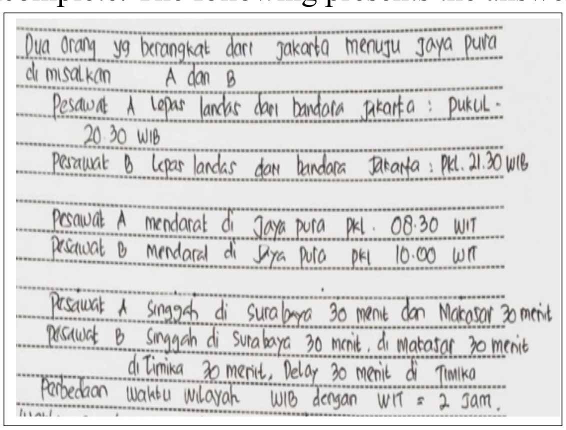
R : Have you not noticed the time difference between WIB, WITA and WIT?

S1 : Oh yes sir, I didn't pay attention to that. (smile)

Based on the researcher's interview with the S1, it can be said that the S1 did not understand the questions well even though the S1 should have understood the differences between WIB, WITA and WIT because this material had previously been obtained at school. It appears that S1 does not yet have sufficient mathematical communication as Uptegrove (2015) states that the influencing factor for effective communication to occur is the students' understanding. When S1 writes everything that is known from the problem, S1 writes everything without choosing which one is important to write. Because S1 also writes the time without paying attention to the time difference from WIB to WITA and from WITA to WIT, S1 assumes that the time at all destinations is the same so that S1 immediately concludes that the travel time for tourists I is 12 hours and the travel time for tourists II is 12 hours 30 minutes, even though the difference between WIB and WIT should have been 2 hours. S1 does not explore strategies that might provide solutions and does not re-check. From this it can be understood that S1 only uses *word use in the commognitive* component in solving questions, namely using the terms difference, star, until, arrive. The terms used by S1 are not only used in mathematics, but these terms are also used in everyday life, namely the use of words (math and non-mathematics) as Yenmez and Özpinar (2017) reveal that solving mathematical problems does not rule out the possibility of using a combination of terms. mathematics and terms in everyday life.

Subject Using Polya Problem Solving but the Answer is Incomplete (S2)

In addition to the types of answers written by the 4 subjects above, there were 5 subjects who provided answers by solving the Polya problem, 4 of which were subjects whose answers were incomplete. The following presents the answers from the Subject (S2):


Dua orang yg berangkat dari Jakarta menuju Jaya pura di misalkan A dan B
Pesawat A lepas landas dari bandara Jakarta : pukul - 20.30 WIB
Pesawat B lepas landas dari bandara Jakarta : Pkl. 21.30 WIB
Pesawat A mendarat di Jaya pura Pkl. 08.30 WIT
Pesawat B mendarat di Jaya pura Pkl 10.00 WIT
Pesawat A singgah di Surabaya 30 menit dan Makasar 30 menit
Pesawat B singgah di Surabaya 30 menit, di Makasar 30 menit di Timika 30 menit, Delay 30 menit di Timika
Perbedaan waktu wilayah WIB dengan WIT = 2 jam.

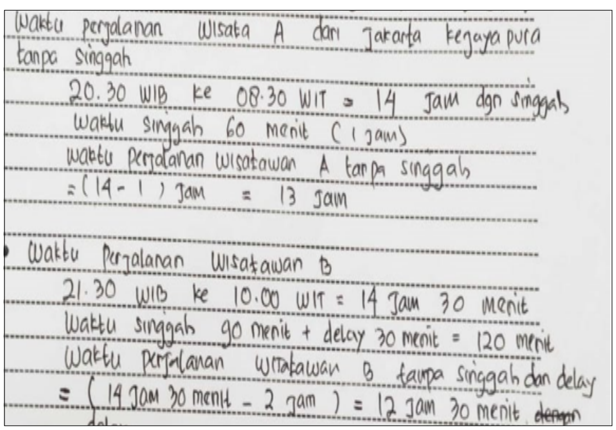
two people departing from Jakarta for Jayapura are A and B
Plane A takes off from Jakarta airport: 20.30 WIB
Plane B takes off from Jakarta airport: 21.30 WIB
Plane A landed in Jayapura at 08.30 WIT
Plane B landed in Jayapura at 10.00 WIT
Plane A stops in Surabaya for 30 minutes and Makassar for 30 minutesPlane B stops in Surabaya for 30 minutes, in Makassar 30 minutes and Timika 30, delays 30 minutes in Timika
The time difference between WIB and WIT = 2 hours

Figure 3: Stages of Understanding & Problem-Solving Plans for S2

At this stage, S2 tries to understand the problem and applies *symbolic visual mediators*, namely for example tourists I and airplane I with symbol A and tourists II and airplane II with symbols B. Then S2 writes down everything that is known by sorting it starting from tourists I and II star from Jakarta until arriving in Jayapura. To dig deeper about *symbolic visual mediators*, the researcher interviewed S2 according to the results of the following transcription:

R : Do you understand the questions given?
S2 : Yes sir, I understand that
R : Why use symbols A and B to compare tourists and aircraft?
S2 : Oh that sir…? To make it easier for me to complete the next question Sir
R : Why is the example not separated between tourists and planes?
S2 : Oh he, it could also be like that sir, but I didn't separate it
R : Are tourists the same as planes?
S2 : Oh that's right, it's different sir

Based on the results of the interview, it is illustrated that S2 does not differentiate between tourists and planes, so the symbols used are the same, namely tourists I with plane I and tourists II with plane II, although this makes it easier for S2 to solve the next problem as (Sfard, 2008) states that *visual mediators* are important in establishing effective communication as they help create a general focal point. Then, the problem-solving stage is shown in Figure 4 below:


Waktu perjalanan wisata A dari Jakarta kejaya pura
tanpa singgah
20.30 WIB ke 08.30 WIT = 14 jam dgn singgah
waktu singgah 60 menit (1 jam)
waktu perjalanan wisatawan A tanpa singgah
= (14 - 1) jam = 13 jam
Waktu perjalanan wisatawan B
21.30 WIB ke 10.00 WIT = 14 jam 30 menit
Waktu singgah 90 menit + delay 30 menit = 120 menit
Waktu perjalanan wisatawan B tanpa singgah dan delay
= (14 jam 30 menit - 2 jam) = 12 jam 30 menit dengan

Travel time for traveler A from Jakarta to Jayapura without stopping:20.30 WIB to 08.30 WIT = 14 hours with a stopover, a layover time of 60 minutes (1 hour) Traveler A's travel time without a stopover = (14 – 1) hours = 13 hours
Tourist travel time B 21.30 WIB to 10.00 WIT = 14 hours 30 minutes Lay time 90 minutes + delay 30 minutes = 120 minutes Travel time for tourist B without stops and delays = (14 hours 30 minutes – 2 hours) = 12 hours 30 minutes

Figure 4: S2 Problem Solving Stages

At the problem solving stage, even though S2 used *the exploratory routine* , namely using the necessary procedures to solve the problem (Mpofu and Pournara, 2018), S2 was wrong in determining the travel times of tourists I and II. S2 already knows that the difference between WIB and WIT is 2 hours, but S2 mistakenly places it so that the narration is wrong. The following are excerpts from the results of interviews with S2 researchers:

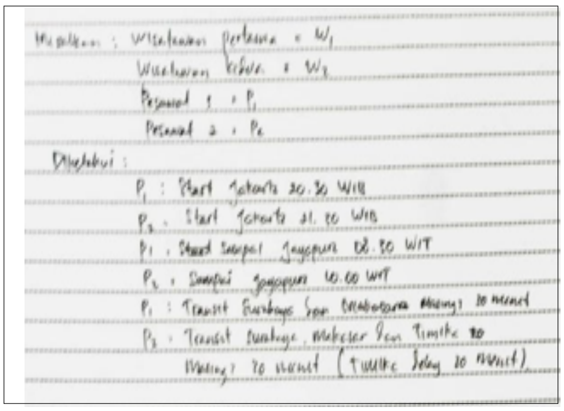
R : Do you know the difference between WIB and WIT?
S2 : Yes sir, I know
R : How much is the difference?
S2 : 2 hours sir
R : Which time is faster, WIB or WIT?
S2 : WIT Sir
R : Pay close attention to your answers, if you say that WIB is 2 hours faster than WIT, but why is your answer like that?
S2 : Which one sir?
R : This (while pointing to S2's answer which wrote 14 hours and 14 hours 30 minutes), try to count first, start at 20.30 WIB and arrive at 08.30 WIT, how many hours before changing to WIB?
S2 : 12 hours sir
R : Well, because the stars from WIB and WIB are 2 hours faster than WIT, plus or reduced?
S2 : Oh yes sir, I was wrong, it should have been reduced, not added. Means not 14 hours sir but 10 hours minus 1 hour layover time to 9 hours. So are for tourists II sir, it must be reduced by 2 hours not added by 2 hours. So 10 hours 30 minutes reduced again by 2 hours layover and delay, so 8 hours 30 minutes
R : Got it now
S2 : Yes Sir…thank you for reminding me

From the excerpts of the interview results above, it can be seen that S2 mistakenly determined the time difference from WIB to WIT, where S2 added 2 hours instead of subtracting 2 hours, so that the travel time for tourist I from Jakarta to Jayapura was 9 hours without a stopover, calculated by S2 13 hours, as well as for tourist II, the journey time should have been 8 hours 30 minutes because the stop and delay time of 120 minutes or 2 hours was calculated by S2 12 hours 30 minutes so that S2's answer was automatically wrong. This is as the expert stated that *routine* describes a person's activity patterns such as calculating, proving and abstracting (Tasara, 2017). Because S2 only knows that the difference between WIB and WIT is 2 hours, and does not know which time comes first, S2 has difficulty synthesizing or integrating his knowledge about real life with solving his mathematical problems. S2 experienced differences *in word use* between mathematics and non-mathematics, but S2 could not correctly determine how this led to a ritualized solution-activity so that S2's explanation was incomplete.

After conducting the interviews, S2 proposes a good narrative that includes all the information required by the researcher, but S2 reverses the procedure when synthesizing real life into a mathematical problem. After the researcher kept digging with questions, S2 realized that he was using real life knowledge incorrectly so he knew what the correct answer looked like. Therefore, S2 internalizes or adapts new information and adapts flexibility although at a lower level than presented. In contrast to S1 who ignores real life knowledge because it is too difficult for him to process and integrate it into problem solving so S1 ignores it. Based on this opinion, S2's activity pattern is wrong, because at the problem-solving stage it is wrong, then automatically S2's conclusion is wrong.

Subject Using Polya Problem Solving and Answer Correct (S3)

Of the 5 subjects who answered with Polya's problem solving steps, only 1 subject gave the correct answer and all four *commognitive components* appeared, namely *visual mediators, word use, routines* and *narratives*. The subject is named as S3. At the stage of understanding the problem, S3 tries to exemplify tourists I and II with symbols W1 and W2, then planes I and II with symbols P1 and P2. This indicates that S3 understands the question well, shown in Figure 4 below:



Misalkan : Wisatawan pertama = W_1
Wisatawan kedua = W_2
Pesawat 1 = P_1
Pesawat 2 = P_2
Diketahui :
P_1 : Start Jakarta 20.30 WIB
P_2 : Start Jakarta 21.30 WIB
P_1 : Sampai Jayapura 08.30 WIT
P_2 : Sampai Jayapura 10.00 WIT
P_1 : Transit Surabaya dan Makassar masing2 30 menit
P_2 : Transit Surabaya, Makassar dan Timika 30 menit masing2 (Timika delay 30 menit)

First travelers: W1
Second traveler: W2
Plane 1: P1
Plane 2: P2
Is known:
P1: start Jakarta 20.30 WIB
P2: start Jakarta 21.30 WIB
P1: until Jayapura 08.30 WIT
P2: until Jayapura 10.00 WIT
P1: transit Surabaya and Makassar 30 minutes each
P2: transit Surabaya, Makassar and Timika 30 minutes each (Timika delay 30 minutes)

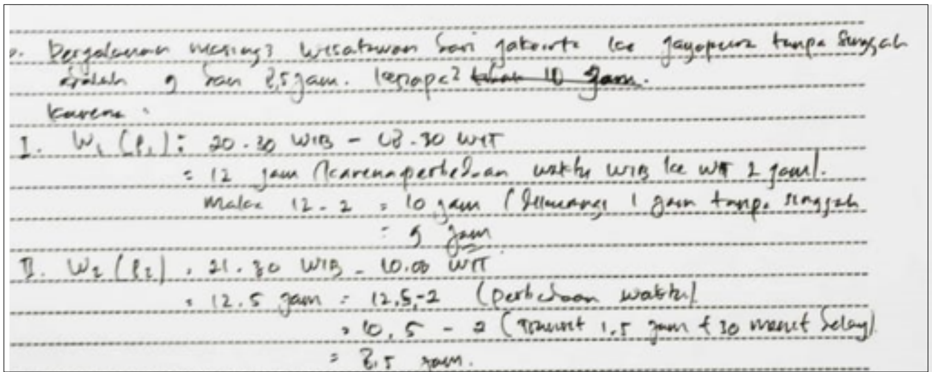
Figure 5: Stages of Understanding & S3 Problem Solving Plan

At the stage of understanding the problem and planning the problem, it appears that S3 uses *word use* well, where S3 after assuming tourists with symbols W1 and W2 and airplanes with symbols P1 and P2. Then S3 determines the starting time for P1 and P2 from Jakarta and arriving at Jayapura, including the transit time. After that, at the problem-solving planning stage, *visual mediator symbolic* S3 shows how to present mathematical information with symbols or algebra in solving problems. The following is a transcript of the results of the researcher's interview with S3:

R : Do you understand the questions given?
S3 : Yes sir, I really understand it
R : How do you understand it?
S3 : First of all we assume the first tourists with W1 and the second tourist with W2, as well as the plane, the first plane with P1 and the second plane with P2
R : What is your goal for example like that?
S3 : To make it easier for me to complete the following questions Sir, because if it's not like that, then I will be in trouble finish it.

Based on the interview above, it is illustrated that S3 performs *visual mediators* with *symbolics* with the aim of making the subject more effective in solving the problems given, as Ryve et al.

(2013) revealed that one of the factors that causes effective communication is the use of *visual mediators*. Then the next step, S3 solves the problem properly and correctly as shown in Figure 5 below.


Perjalanan masing2 wisatawan dari jakarta ke jayapura tanpa singgah adalah 9 jam & 8,5 jam. kenapa? ~~karena 10 jam.~~
karena :
I. W1 (P1) : 20.30 WIB – 08.30 WIT
= 12 jam (karena perbedaan waktu WIB ke WIT 2 jam).
Maka 12 - 2 = 10 jam (dikurangi 1 jam tanpa singgah
= 9 jam
II. W2 (P2) . 21.30 WIB – 10.00 WIT
= 12.5 jam = 12,5-2 (perbedaan waktu)
= 10,5 - 2 (transit 1,5 jam + 30 menit delay)
= 8,5 jam.

The journey of each tourist from Jakarta to Jayapura without stopping is 9 and 8.5 hours. Why?
Because:
I. W1 (P1) = 20.30 WIB – 18.30 WIT
= 12 hours (because the time difference between WIB and WIT is 2 hours)
Then 12 – 2 = 10 hours (minus 1 hour without stopping)
= 9 hours
II. W2 (P2) = 21.30 WIB – 10.00 WIT
= 12.5 hours = 12.5 – 2 (time difference)
= 10.5 – 2 (1.5 hours transit + 30 minutes delay)
= 8.5 hours

Figure 6: S3 Troubleshooting Stages

Observing the solution to the problems carried out by S3 above, S3 uses *exploratory routines* and *ritualized routines*. Following are the results of the researcher's interview with S3.

R : Are you sure that your work is correct? (while pointing at sheet answer S3)

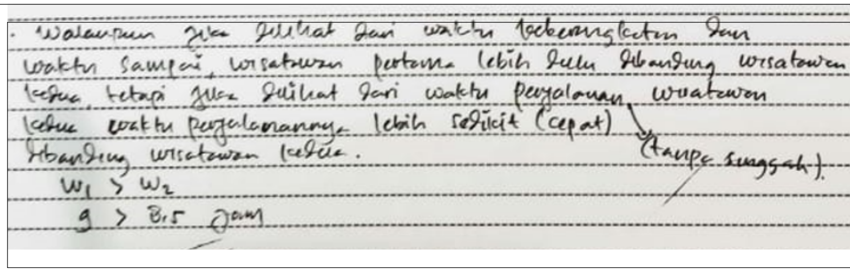
S3 : Very sure sir (answer enthusiastically)

R : Why are you so sure?

S3 : Because I determine the length of the first tourist's trip with the second traveler, where W1 travels for 12 hours because the difference between WIB and WIT is 2 hours so the time becomes 10 hours, then reduced the layover time by 1 hour, so 9 hours. Then for W2, In the same way, the travel time is calculated first, we find the number 12.5 hours or 12 hours 30 minutes minus 2 hours to 10 hours 30 minutes, then subtracting again the 2 hour layover and delay time, so the result is 8 hours 30 minutes.

Based on the results of the doctoral work reinforced by the interviews, it can be understood that the doctoral in solving questions uses an *exploratory routine* where the doctoral student provides an explanation of how to solve the problem at hand and can convey when the selection procedure is used. The explanation given by S3 is correct, namely S3 determines in advance the time used from Jakarta to Jayapura, then S3 explains the change in time from WIB to WIT, then reduces it with layovers and delays. By exploring this, S3 demonstrates its ability to solve a given problem. At the same time, S3 also performs *ritualized routines*, namely using the necessary procedures to solve problems. So in this regard, Thoma and Nardi (2016) explain that *routine* can be said to be a description of the subject's pattern of activity when solving a given problem.

The final stage of solving Polya's problem is looking back. At this stage S3 checks again to ascertain whether the work is correct or needs to be repaired again, after which S3 gives a conclusion. The following is the conclusion as the last step of S3 solving the questions given, as shown in Figure 7 below:


Walaupun jika dilihat dari waktu keberangkatan dan waktu sampai, wisatawan pertama lebih dulu dibanding wisatawan kedua, tetapi jika dilihat dari waktu perjalanan, wisatawan kedua waktu perjalanannya lebih sedikit (cepat) (tanpa singgah) dibanding wisatawan kedua.
$W_1 > W_2$
9 > 8,5 jam

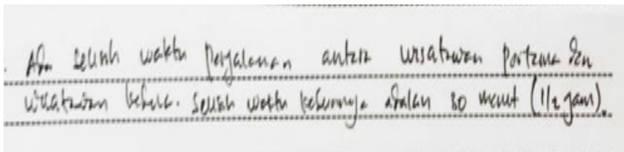
Although when viewed from the time of departure and arrival time, the first tourist is ahead of the second tourist, but when viewed from the travel time (without a stopover) the second tourist travels less (faster) than the second tourist.
W1 > W2
9 > 8.5 hours

Figure 7: Stages of Looking Back at S3

The following is a transcript of the results of the researcher's interview with S3:

R : Why do you conclude that?
S3 : Because based on my description from above Sir, it was found that the time used by W1 is more than the time used by W2, although W1 starts earlier and arrives faster
R : Why is it like that?
S3 : Because W2 has more layover time and delay than W1, where W1 the layover time is only 1 hour while the W2 has a 1.5 hour layover time Plus the delay is 1 hour so it's 2 hours. So there is a time difference between the two of them 30 minutes W1 is faster, like this sir (S3 shows the difference in arrival time both of

them).



There is a difference in travel time between the first tourists and the second tourists. The time difference between the two is 30 minutes (1/2 hour)

R : Oh I see...

S3 : Yes sir.

From the results of the interview above, S3 seems to use *endorsing narratives* to conclude his work, where S3 provides the argument that the second traveler's travel time is less than the time used by the first traveler even though the first traveler is 1 hour earlier than the second traveler. Then S3 mentions the difference in the travel time of the two. Thus, S3 analyzes the questions well so that they arrive at the correct conclusion, namely linking the answers written previously with the rules so that the conclusion of S3 is W1 > W2 because 9 hours is more than 8 hours 30 minutes. In line with (Sfard, 2007, 2008) which states that *endorsed narratives* are descriptions or descriptions so that they can be judged as true or false.

CONCLUSION

This Research reveals the *commognitive* of students in solving Polya problems, namely mental processes and conveying information to themselves or others in the process of expressing ideas to solve open problems which are carried out verbally and non-verbally. Students' *cognitive* in this study consisted of four components, namely *word use*, *visual mediators*, *routines* and *endorsed narratives.* The research results show that:

Subject 1 (S1), at the lowest level, it is not seem to use *symbolic visual mediators* effectively, he also fails to recognize the importance of non-mathematical terms in problem situations, namely *word use* and the first stage of Polya, the stage of understanding the problem. So, S1 performed math routines in the right way but the problem solving planning stage is flawed because he fails to realize the need to integrate routines that involve non-mathematical problem information. As such, the narrative is not complete, and while S1 understands that it is important when pointed out by the researcher the verification step of solving Polya's problem, S1 cannot easily adapted the narrative due to lack of flexibility. The researcher concluded that weaker problem solvers have difficulty understanding the importance of non-mathematical terms and failed to formulate correct problem-solving strategies. S1 also experience flexibility in adapting or accommodating information presented by researchers who show errors, a slow internalization process.

Subject 2 (S2), the problem solver is showing good use of *visual mediators symbolic* in solving the Polya stage 1 problem. S2 formulates the right strategy and gives a good narrative. However, the terms synthesis of real life and unreal life are wrong so the routine is wrong. With the assistance of researchers, S2 seemed to be aware of his mistakes like S1 but also adapted and fixed them using the verification step in Polya's troubleshooting. Subject 3 (S3) is a subject who is able to complete all stages of Polya's problem solving correctly, using effective mediators, relevant and accurate routines and presenting good narratives for their activities. The researcher speculates that peer interaction will help students easily recognize their mistakes when adapting. Due to their flexibility, low level students may need more assistance, otherwise they may find it very difficult to work in groups. We suggest this for future research using positioning in group discussions.

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Students' Mathematics Achievement Based on Performance Assessment through Problem Solving-Posing and Metacognition Level

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Abstract: Performance assessment through problem-solving or problem-posing provides benefits in learning mathematics. This study aims to obtain empirical evidence about the effects of performance assessment and metacognition on senior high school students' mathematics achievement. A quasi-experimental method was employed to engage 163 students in four classes selected through cluster random sampling. In addition, data were collected via an achievement test and a metacognition scale and analyzed using a two-way analysis of variance. The study results indicated a statistically significant discrepancy in the mathematics scores of students who were given performance assessments utilizing problem-solving and problem posing. Depending on the students' levels of metacognition, performance assessment had varying effects on the students' mathematical achievement. Students with a high level of metacognition and performance assessments through problem-solving had more effective mathematics achievement than those with performance assessments through problem-posing. In contrast, in students with medium and low metacognition, the performance assessment through problem-solving and problem-posing did not differ significantly and were classified as having low scores. This study suggested that using performance assessment and considering the level of metacognition support further efforts to enhance students' mathematics achievement.

Keywords: mathematics achievement, metacognition, performance assessment, problem-solving, problem-posing

INTRODUCTION

Performance assessment has a prominent role in assessing the students' progress in learning mathematics. Meanwhile, problem-solving and posing tasks are integral parts of learning and serve as the core of the performance assessment. The tasks help students develop mathematical thinking skills, such as modeling, pattern recognition, building logical arguments, studying, and developing creative thinking. One of the most important aspects of applying successful problem-solving and posing tasks is developing students' metacognition of the tasks.

The term "metacognition" has been defined in numerous ways, but its primary components are knowledge and the control of cognition (Boekaerts, 1997; Fernandez-Duque et al., 2000; Flavell, 1979; Sperling et al., 2004). In addition, metacognition refers to the degree to which students are self-aware in terms of their own memory, cognitive monitoring, and the learning processes themselves. The term "regulation of cognition" is used to describe to what degree students have command over their own mental processes during the learning process. For instance, matching to goal setting, executing strategies, and being aware of the problem they face. Metacognitive activities include planning how to learn something, checking to see if you understand it, and judging your progress toward finishing a task.

Several studies reported that there was a significant correlation between academic achievement and learning-related metacognition. Learners with high metacognitive skills perform much better than those with low metacognitive skills in mathematics classes (Boekaerts, 1997; Jaafar & Ayub, 2010). Also Özsoy (2011) shows that there is a significant and positive relationship ($r = .648$, $p < .01$) between metacognition and mathematics achievement. Furthermore, research results showed that 42% of total variance of mathematics achievement could be explained with metacognitive knowledge and skills.

Mathematics achievement is still relatively low and has shown a decreasing trend in the last ten years. The Program for International Student Assessment (PISA) in 2018 showed that students' mathematics abilities in Indonesia rank 72 out of 78 countries (OECD, 2019). The PISA test divided students' mathematical abilities into 6 levels, where level 1 was the lowest and level 6 was the highest in the higher-order thinking ability. The ability to answer the level 5 and 6 tests of Indonesian students is still low. In addition, the low achievement levels suggest fundamental problems in the process of mathematical learning at school.

An alternative to enhance the students' mathematics achievement is to use performance assessment for teaching, an approach based on problem-solving and posing tasks. Performance in mathematics can be evaluated based on various stages and the quality of students' problem-posing ability towards a problem. According to Nitko and Brookhart (2014), a performance assessment is an authentic procedure in which students are tasked to obtain information on how well they have learned. The rubric comprises two distinct components, namely the assignment and the criteria for assessment of the students' performance. In addition, a performance task is given that aims to show the learning target. The rubric for scoring is a set of guidelines that are used to assess the quality of student performance.

This study proposes that the current low achievement levels are related to the lack of opportunities to explore problem-solving and posing in learning. Therefore, learning mathematics should encourage students to apply solving problems. It is believed that students should develop the capability to attain novel mathematical understanding through the process of problem-solving. This requires them to effectively employ, modify, and adapt a variety of strategies, as well as continuously evaluate and reflect upon their problem-solving process in mathematics.

A multitude of scholars in the field of mathematics education have examined the teaching and learning of mathematics utilizing a problem-solving approach. Their research endeavors have primarily focused on the elaboration of novel instructional techniques and the design of innovative assessment instruments. These studies have made substantial contributions to the advancement of our knowledge on the most effective methods for supporting students' mathematical learning through problem-solving (Charles et al., 1987; Fuchs et al., 1999; Malloy & Jones, 1998; Schoenfeld, 1992). In addition, some concepts were applied to frameworks for evaluating a student's comprehension of problem-posing tasks (Chen et al., 2011; Leung & Silver, 1997; Yuan & Sriraman, 2011).

Brown and Walter (2005) confirmed that problem-posing involves two cognitive aspects such as accepting and challenging. Accepting is an activity where the students obtain the task or problem. Meanwhile, challenging is an activity that questions or tests the task through the formulation of problems. Silver (1994) asserted that challenging is channeled through generating new problems and questions aimed at exploring a given situation and the reformulation during the process of solving it. These situations require students to formulate questions like: (1) proposing solvable math questions within the existing context without providing additional information on core tasks, and (2) formulating math questions that are solvable by creatively adding new information.

This study offers the chance to investigate the effect of performance assessment in the problem-solving and problem-posing approach on the mathematical achievement of students by involving them in the classroom. Integrating students' level of metacognition in performance assessments could improve students' mathematical achievement. Through tasks and rubrics as the core of performance assessment, students solve various mathematical problems and simultaneously assess the quality of processes and results.

Performance assessment as a learning intervention or teaching model is a good empirical study with little related research. Problem-solving and problem-posing studies have not used performance assessment as a learning intervention.

Research Questions

This study uses a quasi-experimental approach to investigate the impact of performance assessment through 'problem-solving and problem-posing' considering the level of metacognitive awareness and its impact on students' mathematical performance. A Split Plot Design was used to test the role of students' metacognition level in performance assessment interventions through 'problem-solving and problem-posing.' The main research questions provided below.

1. Is there a difference in mathematics achievement scores between students who receive performance assessments through problem-solving as opposed to those who receive problem-posing?
2. Is there an effect of performance assessment on mathematics achievement for each level of students' metacognition?

Theoretical Framework

Metacognition and Mathematics Achievement

According to the National Council of Teachers of Mathematics (NCTM, 2000), the mathematical disposition of students can be described as exhibiting confidence in the application of mathematical concepts, having elevated self-expectations, displaying attentiveness during lectures, demonstrating persistence in resolving mathematical problems, possessing a keen sense of curiosity, exhibiting the ability and willingness to articulate mathematical ideas to others, and exhibiting a strong awareness of their own thought processes. Kadir and Sappaile (2019) stated that students' metacognition on a mathematics assignment is a state in which they begin thinking and using what they know and applying the knowledge prior to beginning the assignment itself. In terms of metacognition, it is important to note that possessing a significant amount of knowledge and skills is not sufficient without the ability to make informed decisions, manage, and regulate what has been learned effectively, and apply them to solve mathematical problems. Thus, the capacity for metacognition encompasses the development of executive, managerial, and self-regulatory skills, as they pertain to the acquisition and application of mathematical knowledge.

Students with developed metacognitive skills possess the ability to identify limitations and weaknesses in their thinking process, which includes recognizing others' perspectives, monitoring their progress, and making distinctions between comprehended and misunderstood information. According to Marzano et al. (1988), metacognition is a competency that can be broken down into several different categories. These categories encompass the following: (1) The cultivation of self-regulatory capacities, including the demonstration of a persistent dedication to academic tasks, the adoption of a positive student mindset, and the regulation of one's attentional focus in response to the requirements of academic tasks; (2) the integration of various forms of knowledge, including factual knowledge, step-by-step knowledge, and knowledge based on conditions; and (3) the application of executive control abilities, such as the formation of plans, ongoing progress monitoring, and the systematic evaluation of procedures.

Schoenfeld (1992) proposed that the process of finding a solution to the issue required highly developed organizational skills, as well as control and monitoring mechanisms. It is imperative for educators to highlight the significance of these processes within the pedagogical framework that adopts a problem-solving approach. It is of utmost importance to integrate these processes into the educational curriculum, given their paramount significance in fostering the development of metacognitive abilities. The definition of metacognition encompasses concepts such as self-regulation, monitoring, and controlling of one's own cognitive processes. According to Cohors-Fresenborg et al. (2010), it is crucial to regulate the use of appropriate mathematical tools in the field of school algebra. To correctly measure metacognitive development, it is crucial to monitor its utilization. Numerous scientific investigations have established that monitoring procedures are indicative of achievement. However, there was no connection between monitoring reports and actual behavior or outcomes. Therefore, efforts in learning mathematics must be focused on monitoring changes in student behavior.

Several studies (Hasbullah, 2015; Listiani et al., 2014; Smith, 2007; Suriyon et al., 2013), have indicated that practices or learning activities based on metacognitive strategies have a significant impact on students' learning and mathematics learning achievement through the application of several approaches introduced as part of 21st-century learning. Furthermore, Veenman et al. (2006), several studies reveal that there is a correlation between mathematical performance and metacognitive skills. These studies place metacognition as an independent variable for intellectual ability. Menz and Xin (2016) indicated that students' mastery and proficiency in mathematics are linked to their metacognitive skills. When metacognitive abilities are robust, their performance in mathematics will be outstanding.

Recently, there has been extensive research on the pivotal role of self-regulation in academic achievement. For instance, Zee and de Bree (2017) Studies have revealed positive correlations between self-regulation and academic achievement in mathematics and reading among elementary students in the Netherlands. Kaur et al. (2018) showed that Punjabi secondary school students' academic success was positively impacted by metacognition and self-regulation. A study conducted by Dradeka (2018) in Saudi Arabia revealed a substantial disparity in student self-regulation, with students demonstrating higher academic achievement displaying a greater degree of self-regulation. The results of the study indicated that male students displayed a greater propensity towards academic self-regulation when compared to their female counterparts. The results also showed that male students, on average, reported higher levels of academic self-regulation compared to female students. In addition, Annalakshmi (2019) posits that adolescent girls from low-income rural families in Tamil Nadu who engage in self-regulation demonstrate higher levels of resilience and academic success. Zhou and Wang (2019) also found evidence supporting a positive correlation between academic achievement, self-regulation, and learning motivation, aligning with the findings of the present study.

Performance Assessment

The Principles and Standards for School Mathematics published by the National Council of Teachers of Mathematics (NCTM, 2000) encourage teachers to make use of real-world mathematical problems in the classroom to make the learning process more interactive and engaging. According to Nitko and Brookhart (2014), performance assessment is a kind of alternative assessment or authentic assessment. A performance assessment is a procedure in which students are tasked to obtain information on how well a student has learned. Unlike multiple-choice questions, performance assessment tasks require students to demonstrate mastery of a learning target by integrating knowledge and skills from a variety of subject areas.

According to Nitko and Brookhart (2014) a performance assessment is made up of two parts: the assignment itself, and the rubric that will be used to evaluate the students' work. Performance tasks contain student activity that aims to show the performance of a learning target. Some examples of performance assessment tasks in the study of mathematics are mathematical writing, the practical use of three-dimensional space props, a research project, measuring the height of an object, ‘problem-solving, problem posing, and mathematical modeling.’ A rubric for scoring is a set of guidelines that are used to assess the quality of student performance. The rubric, therefore, is used to provide guidelines for assessors to ensure consistency of assessment results. The grading rubric should be coherent in order to evaluate the performance quality of students. The rubric may take the form of a rating scale or a check list, and it addresses two aspects of student performance: achievement and processes.

According to Kulm (1994) a holistic rubric can be used to evaluate both problem-solving and problem-posing abilities. This hybrid of analytic and holistic approaches to grading includes criteria for evaluating students' grasp of key concepts and skills in application. The Anaholistics Rubric is a great way to evaluate a student's progress in mathematics because it provides a holistic score across multiple areas of study. Thus, the implementation of performance assessment referred to in this research is a measurement and assessment activity that requires students to display their performance through the task of problem-solving and problem-posing in writing related to the processes and outcomes of learning mathematics.

Problem-Solving Task

The study of mathematics education at schools raises two main questions: (1) How to instruct students in how to solve problems, and (2) how to evaluate the performance of students based on their ability to solve problems. Problem-solving activities should be used in the classroom as a means for teachers to evaluate the students' complex thought processes. These activities should require students to comprehend, create, implement, and evaluate their plans based on some theoretical arguments related to problem-solving strategies (NCTM, 2000; Pólya & Conway, 2004). Pólya and Conway (2004) developed a set of questions to be asked at each stage of the process in order to guide the students through it and check the results of problem resolution. Through the use of the following questions, this procedure can be carried out at each stage of the problem-solving process.

First stage: Understanding the problem (it is crucial to comprehend the issue), it is essential to first understand its nature. This requires consideration of the following questions: (1) What is the unknown? (2) Where can the data be located? (3) Can you clearly define the problem at hand? (4) Can the condition be fulfilled? (5) Does the condition fully determine the unknown, or is it inadequate, redundant, or incompatible with other information? (6) Can you create a diagram and use appropriate symbols to represent the information? (7) Can you break down the different parts of the condition? (8) Have you thought about documenting the information and conditions?

Second stage: Devising a plan[1] which consist of: (1) Have you encountered a similar problem before or in a slightly different form? (2) Are you aware of a similar issue or a potentially helpful theorem? (3) Take into account the unknown and compare it to a familiar problem with a similar or identical unknown. (4) Here's a previously solved problem that is similar to yours, could it be helpful in your situation? (5) Can you utilize the outcome of the similar problem? (6) Can you apply the technique used in the similar problem to your situation? (7) Do you need to add any additional elements to make it work? (8) Could you rephrase the problem? (9) Is it possible to

[1] Note that a plan may identify the relationship between the data and the unidentified. If an immediate connection cannot be found, you may be required to consider auxiliary issues. You should eventually obtain a solution plan.

restate the issue in a different way? Revisit the definitions and try solving a related problem if you can't solve the original one. (10) Can you visualize a simpler related problem? A broader or a more specific problem? A similar problem? (11) Is it possible to solve a part of the issue? By retaining only a part of the conditions and eliminating the rest, to what extent is the unknown then determined and how can it change? (12) Are there any useful insights that can be gained from the available data? (13) Can you think of any additional data that would be helpful in finding the unknown? (14) Would it be possible to modify either the unknown or the data, or both, to bring the unknown and data closer together? (15) Have you used all the available data? (16) Have you considered all the conditions? (17) Have you taken into account all the important concepts related to the problem?

Note that a plan may identify the relationship between the data and the unidentified. If an immediate connection cannot be found, you may be required to consider auxiliary issues. You should eventually obtain a solution plan.

Third stage: Carrying out the plan which comprises of: (1) While putting your plan for the solution into action, make sure to check each step. (2) Can you confidently affirm that the step taken is appropriate? (3) Can you provide evidence that it is accurate?

Fourth stage: Looking back (test the result obtained), such as: (1) Can you confirm the result? (2) Would it be achievable the argument? (3) Is there another way you can arrive at the solution? (4) Is it obvious to you at first glance? (5) Is it possible for you to apply the solution or the method to an additional issue?

Problem Posing Task

According to Silver (1994), problem posing involves two cognitive aspects, namely accepting and challenging. Accepting stage, namely a situation such as pictures, manipulation of kids' tools, game, theorem, or concept, equipment, problem, or solution of a problem. Challenging is done by coming up with new problems and questions that aim to learn more about a given situation, or by rewriting a problem as you try to solve it. Silver and Cai (1996) conducted a study with a large group of sixth and seventh grade students and developed a problem-posing scheme based on the type and complexity of student responses, as shown in Figure 1.

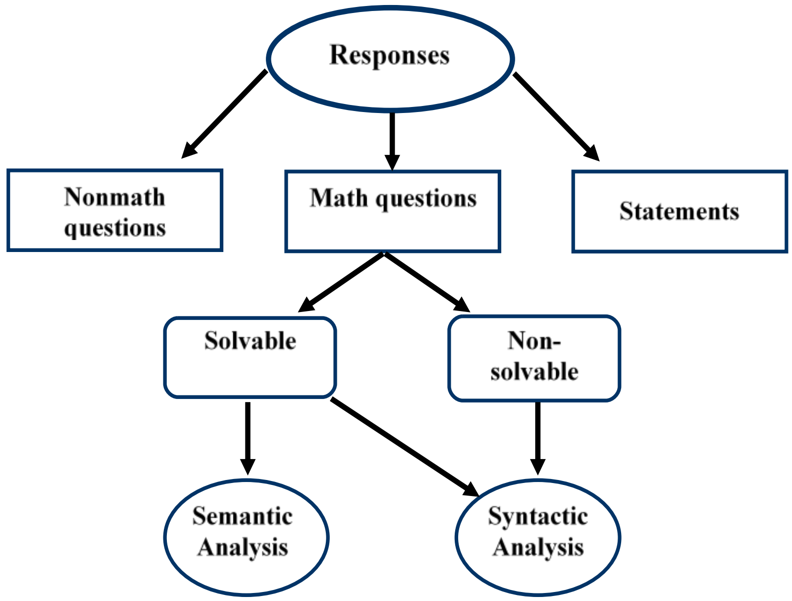


Figure 1: Problem Posing Response (Silver and Cai, 1996)

In Figure 1, the student-formulated responses were classified into three categories: (1) mathematical questions, (2) non-mathematical questions, and (3) statements. Mathematical questions refer to those that involve mathematical problems. These questions are further divided into two types: (1) solvable mathematical questions and (2) non-solvable mathematical questions. Solvable mathematical questions are those that contain sufficient information or conditions from the task. Solvable questions can be further classified into two subcategories: (1) questions that only include information provided in the task, and (2) questions that introduce new information beyond the task. Non-solvable questions are those that lack sufficient information from the task to be solved. Non-mathematical questions, on the other hand, do not involve mathematical problems and are unrelated to the task.

Furthermore, the complexity of the problem generated by the students can be classified into two types: (1) complexity related to the language structure (syntax), and (2) complexity related to the mathematical structure (semantics). The level of syntactic complexity is represented in the form of propositions, such as assignment, relationship, and hypothetical statements. Meanwhile, the level of semantic complexity includes categories such as transforming, grouping, comparing, and stating.

METHOD

Participants

This was a randomized study with a quasi-experimental approach, and class X (tenth) students in the Senior High Schools in Jakarta Indonesia, which consisted of 12 classes with similar characteristics (XA, XB, XC, XD, XE, XF, XG, XH, XJ, XK, XL). Furthermore, four (4) out of 12 classes (XA, XH, XL, XG) were selected at random using the cluster random sampling technique. The study involved 163 students, 99 (60.7%) women, and 64 (39.3%) men, with the following details: class XA 40 students 23 women (57.5%) and men 17 (42.5%), XH 42 students 27 women (64.3%) and men 15 (35.7%), XL 40 students 24 women (60.0%) and 16 men (40.0%), and XG 41 students 25 women (61.0%) and 16 (39.0%).

Measures

The independent and dependent variable used was performance assessment (treatments, namely problem-solving and problem-posing tasks), and mathematics achievement was measured by using a test. The validity of the contents was determined by the "Quantification of Content Validity" method from (Gregory, 2014), involving 10 expert raters, and obtained 47 valid-content items with ranges (0.700 to 0.930) and inter-rater reliability of 0.901. Additionally, the empirical results yielded 40 valid items with a validity range of (0.342 to 0.720), and the reliability coefficient was determined to be 0.917. The test materials include rational exponent and root forms, quadratic equations, inequalities, comparisons and trigonometric functions, logarithms, rules of 'sine and cosine, and the area of triangles.'

Furthermore, another independent variable includes metacognition as a moderator (categorical) variable that was measured using a scale and had been performed before the treatment was conducted. Measurement of student metacognition uses a scale developed by Kadir and Sappaile (2019), and the results was validated through consultation employed a panel of specialists, with 'the inter-rater reliability coefficient' among the panelists having been determined to be 0.830. Empirical test results of the scale consisted of 46 items with a validity range (of 0.197 to 0.804), and the construct reliability coefficient is about 0.938. The Confirmatory Factor Analysis (CFA) reached construct reliability scores of 0.990 for self-regulation skills, 0.980 for type of knowledge, and 0.982 for executive control skills.

Design and Procedure

Based on the selected class sample, XL, XA, XG, and XH, two classes (XA and XH) were randomly assigned to performance assessment through problem-solving while two classes (XL and XG) were randomly assigned through problem-posing. The experimental design used was Group within Treatment (GWT) design and is considered the heterogeneity of the treatment groups/classes. Through the proper grouping of classes, this design can reduce treatment errors (Kadir, 2022). Furthermore, mathematics lessons were delivered for all classes involved for a full semester (5 months) following the national curriculum of Indonesia. There were 2 meetings each week which consisted of 2 X 45 minutes, and at the end of the semester, mathematics achievement was recorded for each participating student.

Student metacognition data were obtained before treatment. Each item of the metacognition scale consists of a Likert scale from 1 to 5. For 46 items, the total metacognition scores for each range from 46 – 230. The higher the scores, the better the metacognition. The results of the metacognition level scale in each class were divided into 3 major categories, namely the high (70th quantile and higher), medium (between 30th and 70th quantiles), and low level (less than or equal to 30th quantile). The metacognition classifications of the high, medium, and low-level students were not conducted in the treatment conditions, but the classification was needed for data analysis. The study design is presented in Figure 2.

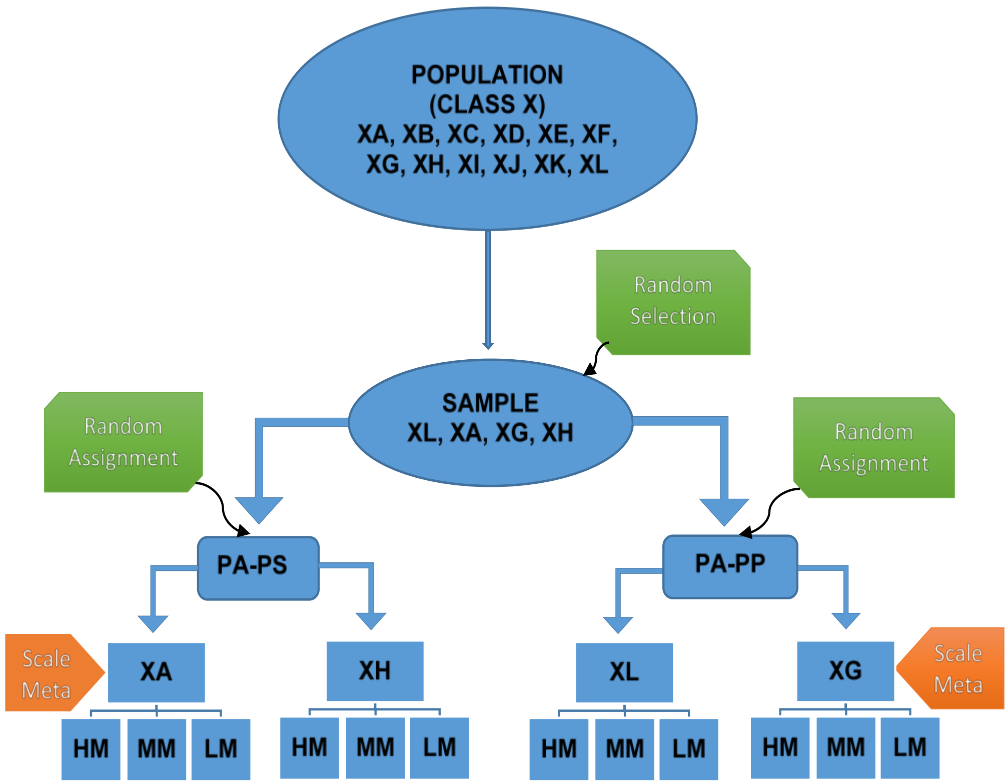


Figure 2: Research Design

Notes:
PA-PS = Performance Assessment-Problem Solving
PA-PP = Performance Assessment-Problem Posing
HM = High Metacognition; MM = Medium Metacognition; LM = Low Metacognition
XA = Class XA 40 Students, HM = 13; MM = 15; LM = 12
XH = Class XH 42 Students, HM = 13; MM = 17; LM = 12
XL = Class XL 40 Students, HM = 13; MM = 15; LM = 12
XG = Class XG 41 Students, HM = 13; MM = 16; LM = 12

Experiment 1: Performance Assessment through Problem-solving

Treatment through performance assessment with problem-solving was conducted in some stages of the task as presented in Table 1.

Stages	Activities
Understanding the problem	Students were given 'a problem-solving task' Students 'were guided to understand the tasks' (e.g., to know what was the known fact and what was asked) Students were guided to see the data fulfillment of the tasks
Devising a plan	Students were guided to identify the problem of the tasks. Students changed the task into more simple language Students were guided to memorize the similar tasks Students were guided to connect mathematical concepts to the tasks Students developed a strategic design or solution method
Carrying out the plan	Students were guided to apply the plan in appropriate steps Students checked the technique of solving problems
Looking back	Students checked the correct process of solving problems Students checked the correct results of solving problems

Table 1: Guide to doing problem-solving tasks

The rubric for assessing students' levels of success in solving problems is in Table 2.

Stages	Score	Descriptors
Understanding the problem	: **2**	Understanding the problems correctly
	: **1**	Misinterpreting in partial/neglecting the condition of the task.
	: **0**	Misinterpretation to all.
Devising a plan	: **4**	Select the procedure which leads to the correct solution
	: **3**	Select some strategies but incomplete
	: **2**	Select a strategy but unsuccessful/not trying another technique
	: **1**	Select a plan which undoable to be implemented
	: **0**	Select a plan which irrelevant/no strategy at all
Carrying out the plan	: **4**	Doing the procedure correctly and there is a correct solution
	: **3**	Using correct strategy but less incorrect calculation
	: **2**	Doing correct procedure which may be giving a correct answer but incorrect in structuring and calculating
	: **1**	Using a part of procedures that is correct but leads to the incorrect answer
	: **0**	Using inappropriate plan and pause/cannot use plan or correct algorithm
Looking back	: **2**	Checking is conducted to the results and the process
	: **1**	There is checking but incomplete
	: **0**	There is no check given or no check at all

Table 2: Problem-solving Rubric

Experiment 2: Performance Assessment through Problem Posing

Treatment of performance assessment problem-posing was conducted in some stages of the task in Table 3.

Stages	Activities
Accepting	▪ Gave the students problem-posing tasks ▪ Students were guided to be familiar with the tasks (observing, writing the data of the tasks) ▪ Students were guided to think about the concepts, formulas, patterns, or samples which connected to the tasks
Challenging	▪ Students were guided to make solvable math questions based on the situation of the tasks ▪ Students made Math questions based on the tasks by adding new information from the main tasks ▪ Students made conditional questions to enrich the context of the main tasks ▪ Students combined other situations with the main tasks ▪ Students were guided to change the previous tasks ▪ Students were guided to make Math questions with new data and new context out of the main tasks.

Table 3: Guide to doing problem-posing task

Rubric for assessing students' response and proposition type in posing, in Table 4.

Response Type	Code	Score	Proposition Type
Statement	(Q0)	: 0	
Non-math questions	(Q1)	: 0	
Math questions non- solvable	(Q2)	: 1	A/R/S
Math questions solvable:			
(a) Without new information	(Q3)	: 2	A/R/S
(b) With new information	(Q4)	: 3	A/R/S

Table 4: Problem Posing Rubric

Note:
Q0 = Statement did not contain math question;
Q1 = Question had no math problem and was unrelated to tasks;
Q2 = Math question had no adequate information to be solved;
Q3 = Math question based on the information provided on the tasks;
Q4 = Math question using additional information out of the main tasks;
A = Assignment (task proposition, namely the question must be solved);
R = Relationship (Relationship proposition, namely the question to compare):
S = Supposition (Conditional proposition, namely the question using a conditional sentence)

Data Analysis

Metacognition is a moderating variable integrated into this study to form a factorial design and was classified into three categories, namely high-level, medium-level, and low-level students. In reality, the design was a split plot with a completely randomized setting. The analysis technique used was a Two Way Analysis of Variance (ANOVA) with a split-plot (Montgomery, 2013). Furthermore, the main and subplot were the class and metacognition levels respectively. Analysis for the split-plot design and the corresponding contrasts were conducted using IBM SPSS Statistics 23.

RESULTS AND DISCUSSION

Students' Metacognition

Baseline information on students' metacognition for each class is presented in Table 5.

Metacognition	Class/Group				Total
	XA (N=40)	XH (N=42)	XL (N=40)	XG (N=41)	(N=163)
Mean	166.9	139.3	144.2	150.3	150.0
Median	166.5	145.0	145.0	152.0	152.0
Mode	158.0	120.0	132.0	154.0	148.0
Std. Deviation	9.97	17.64	11.22	11.12	16.49
Range	152.0 - 191.0	110.0 - 166.0	128.0 - 169.0	130.0 - 176.0	110.0 - 191.0
Q1-Q3	158.0 - 174.0	120.0 - 155.0	133.0 - 151.0	140.0 - 157.0	138.0 - 161.0

Table 5: Baseline metacognition for all students in 4 different classes

Table 5 shows that the median (range) of students' metacognition level was 166.5 (152-191), 145 (110-166), 145 (128-169), and 152 (110-191) for class XA, XH, X, L, and XG respectively. Furthermore, the mean and the mode indicate that the classes XA (166.9 > 158.0), XH (139.3 > 120.0), XL (144.2 > 132.0), XG (150.3 > 154.0,) and X total (150.0 > 148.0). When the theoretical average score is set at 138 (46x3), then the mean metacognition before being given the intervention with performance assessments has exceeded the average theoretical and empirical scores above the mode of each class. Therefore, students' average metacognition before being given a performance assessment is in a good category.

Students' Mathematics Achievement

The mathematics achievement average scores are expressed in percentages unless otherwise specified. The scores are classified by treatment group (problem-solving and problem-posing) and according to students' metacognition level which is presented in Table 6.

Performance Assessment	Metacognition Level	Class	Mean	Std. Deviation	N
Problem-Solving	High	XA	78.62	5.867	13
		XH	78.85	8.143	13
		Total	78.73	6.954	26
	Medium	XA	73.20	3.895	15
		XH	68.24	9.890	17
		Total	70.56	7.980	32
	Low	XA	67.42	8.174	12
		XH	68.00	9.863	12
		Total	67.71	8.864	24
	Total	XA	73.23	7.413	40
		XH	71.45	10.430	42
		Total	72.32	9.073	82
Problem Posing	High	XL	68.31	9.313	13
		XG	70.85	10.431	13
		Total	69.58	9.774	26
	Medium	XL	70.33	6.705	15
		XG	69.38	6.752	16
		Total	69.84	6.634	31
	Low	XL	67.75	8.864	12
		XG	68.08	9.080	12
		Total	67.92	8.777	24
	Total	XL	68.90	8.142	40
		XG	69.46	8.579	41
		Total	69.19	8.319	81

Table 6: Students' mathematics achievement

Table 6 showed that the average scores for students receiving performance assessment through problem-solving were higher (73.32 and std. deviation 9.07) than those based on problem posing (69.19 and std. deviation 8.32). Furthermore, looking at a simple effect of treatment within each level of metacognition, shows that in students with high metacognition, the average mathematics achievement score based on problem-solving was higher than problem-posing (78.73 vs. 69.58 with a standard deviation of 6.95 and 9.77). However, this is not the case for those in the low and medium metacognition, where the mathematics achievement was relatively similar (70.56 vs 69.84 and 67.71 vs. 67.92).

This descriptive finding showed that students with a high level of metacognition given performance assessments both using problem-solving and problem-posing could achieve relatively good mathematics scores. These findings are consistent with the results of Abdellah (2015), that positive correlation was identified between the Metacognitive Awareness Inventory (MAI), academic achievement, and teaching performance. Moreover, the deployment of metacognitive skills was found to have a marked and positive influence on both academic achievement and teaching performance.

Performance of problem-solving

The description of the students' performance of problem-solving according to various stages is in Table 7.

Stages	Max Score	Mean	%
Understanding the problem	20	12.290	61.45
Devising a plan	30	21.060	70.20
Carrying out the plan	30	22.485	74.95
Looking back	20	13.540	67.70
Total	100	69.375	69.38

Table 7: Problem-solving scores according to the stages (N = 82)

Table 7 showed that the average score for problem-solving skills was 69.40%. This value was an average from various stages of problem-solving; conducting the plan contributes to the highest percentages (75.00%) relative to the rest of the stages (understanding the problem (61.50%), devising a plan (70.20%), and looking back (67.70%). Assessment of stages of problem-solving showed that under a maximum score of 100%, students' score was 69.40%. Therefore, the students have shown fairly good problem-solving performance in every step of Polya. This finding aligns with the results of the study conducted by Lee and Chen (2015) that the geometric reasoning performance of students that received Polya question-based learning was superior to those that received direct presentations. In addition, students that received instruction based on Polya's questions expressed a stronger sense of participation than in direct presentations. The recent finding resemble with the results of Aguilar and Telese (2018) on a sample of elementary pre-service teachers, who reported significant advancements in their procedural fluency, conceptual understanding, and strategic problem-solving competencies following their participation in a series of non-routine problem-solving tasks. The participants, as prospective teachers, demonstrated an improvement in their proficiency in employing procedures and exhibited a sufficient level of conceptual knowledge in relation to problem-solving.

Performance assessment problem-solving in teaching-learning started with giving mathematical tasks to students. An example of a problem-solving task is shown in Figure 3.

<table>
<tr><td colspan="2">Tugas pemecahan masalah
“ Siswa kelas X mengikuti penggalangan dana untuk mengumpulkan sejumlah Rp960.000,- kegiatan lomba seni antar kelas. Setiap siswa harus menyumbang jumlah uang yang sama, tetapi empat siswa tidak dapat membayar. Untuk mengatasi kekurangan itu, siswa lainnya harus menambah uang masing-masing sebesar Rp20.000,-. Berapa banyak siswa yang mengumpulkan dana?”</td></tr>
<tr><td>Tahap</td><td>Contoh kinerja siswa</td></tr>
<tr><td>Memahami masalah</td><td>Apa yang diketahui ?
(i) Jumlah dana yg akan dikumpulkan = Rp 960.000,-
(ii) Jumlah siswa yg tidak dapat membayar : 4 orang.
(iii) Uang tambahan yg harus ditanggungulangi setiap siswa : Rp 20.000,-
Apa yang ditanyakan?
Jumlah siswa yg berpartisipasi dalam pengumpulan dana.</td></tr>
<tr><td>Menyusun rencana</td><td>(i) Misalkan jumlah siswa pengumpulan dana : x maka setiap siswa membayar : (Rp 960.000 / x)
(ii) 4 siswa tidak membayar, jadi pembayar (x−4).
(iii) Total dana yg terkumpul dengan 4 siswa tak bayar : (x−4) (Rp 960.000 / x)
(iv) Jumlah dana tambahan untuk mengatasi 4 siswa tidak bisa bayar (x−4). (Rp 20.000)
(v) Jumlah dana yg terkumpul, dinyatakan dengan model matematika (x−4) (960.000 / x) + (x−4) (20.000) = 960.000</td></tr>
<tr><td>Melaksanakan rencana</td><td>Dari persamaan : (x−4) (960.000 / x) + (x−4) (20.000) = 960.000 ... (i)
⇔ $(x-4)\left(\frac{48}{x}\right) + (x-4) = 48$ (kedua ruas bagi 20.000)
⇔ $48 - \frac{192}{x} + x - 4 - 48 = 0$ ⇔ $x - \frac{192}{x} - 4 = 0$
⇔ $x^2 - 4x - 192 = 0$ ⇔ $(x-16)(x+12) = 0$
⇔ x = 16 atau x = −12</td></tr>
<tr><td>Memeriksa proses dan hasil</td><td>Nilai x = 16 (memenuhi) dan x = −12 (tidak memenuhi)
Jadi jumlah siswa yg membayar sebanyak 16.
Total dana terkumpul = Jumlah dengan 4 siswa tidak membayar + total dana tambahan.
⇔ (16−4) (Rp 960.000 / 16) + (16−4). Rp 20.000 = Rp 960.000
⇔ $\frac{12}{16}$ (Rp 960.000) + 12. Rp 20.000 = Rp 960.000
⇔ Rp 720.000 + Rp 240.000 = Rp 960.000 (benar).</td></tr>
</table>

Translate in English:

Problem-solving task

"Grade X students participate in fundraising to collect an amount of Rp960.000,- for the inter-classes arts competition. Each student should contribute an equal amount, but there are four students who could not pay. To overcome the shortness, the rest of the students must add an amount of Rp20.000,- each. How many students who pay/participate in collecting the budget?"

Stage	Example of the students' performance
Understanding the problem	What is known? (i) The amount of fund to be collected: Rp960.000,- (ii) The number of students who could not pay are 4 students. (iii) Additional money of each student must pay is Rp20.000,- What is asked? "The number of students who participate in the fundraising"
Devising a plan	Develop mathematical model (i) Suppose the number of students who raise funds is x, then each student must pay: $\frac{\text{Rp960.000}}{x}$ (ii) There are four students who do not pay, so the number of payer only: $(x-4)$. (iii) The total funds collected taking into account four non-payer students: $(x-4)\left(\frac{\text{Rp960.000}}{x}\right)$ (iv) The amount of additional funds to cope with four non-payer students: (x - 4). Rp20.000,- (v) All funds collected expressed by the equation: $(x-4)\left(\frac{\text{Rp960.000}}{x}\right)+(x-4)(\text{Rp20.000})=\text{Rp960.000}$
Carrying out the plan	From $(x-4)\left(\frac{\text{Rp960.000}}{x}\right)+(x-4)(\text{Rp20.000})=\text{Rp960.000}$ $\Leftrightarrow (x-4)\left(\frac{48}{x}\right)+(x-4)=48$ $\Leftrightarrow 48-\left(\frac{192}{x}\right)+x-52=0$ $\Leftrightarrow x-\left(\frac{192}{x}\right)-4=0 \Leftrightarrow x^2-4x-192=0$ $\Leftrightarrow (x-16)(x+12)=0 \Leftrightarrow x=16$ or $x=-12$
Looking back	Retrieve: x = 16 (satisfied) or x = -12 (is not satisfied). So the number of payers is 16 students. Total funds collected = the amount of fund + 4 non-payers + the total of additional fund: $\Leftrightarrow (16-4)\left(\frac{\text{Rp960.000}}{16}\right)+(16-4)(\text{Rp20.000})=\text{Rp960.000}$ $\Leftrightarrow (12)\left(\frac{\text{Rp960.000}}{16}\right)+(12)(\text{Rp20.000})=\text{Rp960.000}$ $\Leftrightarrow$ Rp720.000, - + Rp240.000, - = Rp960.000,- (correct)

Figure 3: Performance of Problem-Solving

Performance of problem posing

The description of the students' problem-posing ability according to the response and proposition after receiving a performance assessment is presented in Table 8.

Response Type	Score	%	Proposition Type	Score	%
Statement or Non-math questions (Q0/Q1)	0	0	Assignment (A)	0	0
Math questions non-solvable (Q2)	44	0.94	Assignment (A)	3393	45.60
Math questions solvable (Q3)					
a. Without new information	5639	73.54	Relationship (R)	2259	27.27
b. With new information	2029	26.46	Supposition (S)	2019	27.13
Total	7668	100	Total	7441	100

Table 8: Students' problem-posing ability by the response and proposition types (N = 81)

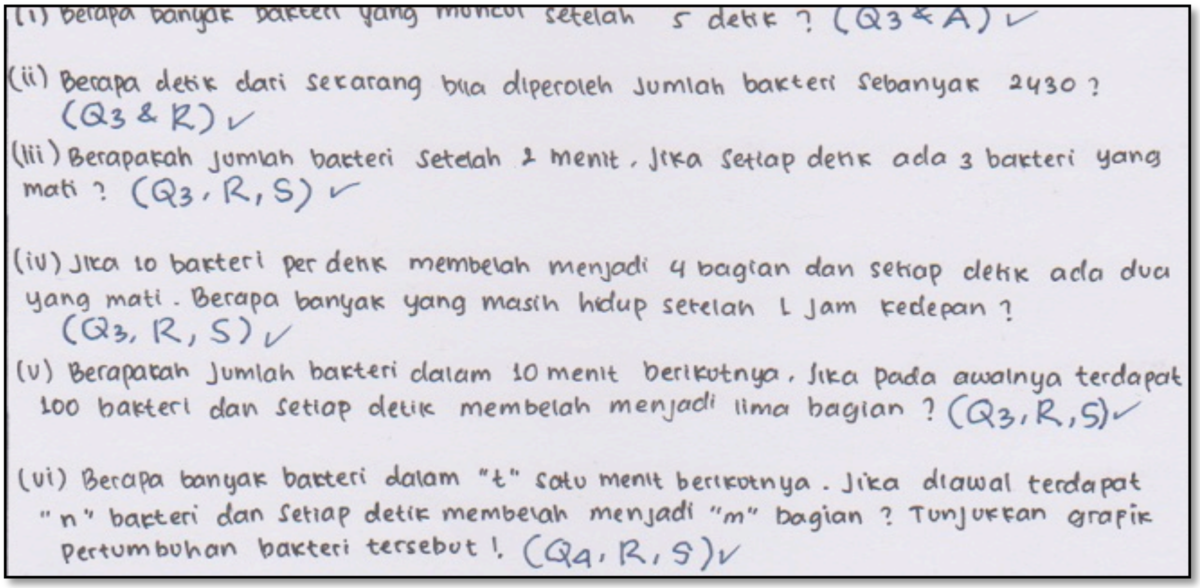
Table 8, showed that out of 7.668 problem-posing responses, all are mathematical questions, of which 0.94% are non-solvable, and 99% solvable mathematical questions (73.54% with new information and 26.46% without new information). The finding is slightly different from the report of Silver and Cai (1996) where out of 1.465 responses, more than 70% were mathematical questions, 20% were of the statement, and 10% were of non-mathematical questions. Currently, responses to statements, non-mathematical responses, and non-solvable mathematical questions were almost non-existent. Most responses are concentrated on solvable mathematical questions. This shows that applying performance assessment based on problem-posing to some extent, facilitated students' metacognition to control their learning activities in posing questions relevant to mathematics both with and without new information.

This finding is consistent with the results found by Cai et al. (2013) that the students generated many Mathematical problems that can be solved, including those that have a complex syntax and meaning. Almost half of 509 middle school students developed sets of related problems, and also eight fairly complex problems were solved. The nexus between problem-solving performance and problem-generating abilities revealed that individuals classified as "good" problem solvers generated more mathematical and intricate problems compared to those classified as "poor" problem solvers.

This finding is also similar to the features of problem-posing performance assessment task-based, which contained two specific stages: (1) accepting, where students are trained to establish their concept, explore prior knowledge, and connect to the problems given by the teachers and (2) challenging, where students are challenged to ask questions through changing the initial problems, obtaining data from the questions and at the end they can change the objectives and solve it to increase their higher-order thinking. This result can be interpreted as a performance assessment based on problem-posing meant for students with low metacognition in order to progress their mathematics learning.

Intervention performance assessment problems-posing in teaching-learning in the class started with giving mathematical tasks to students. An example of problem-posing tasks and student performance is in Figure 4.

Menerima (Problem-posing task)
“Sebuah penelitian tentang pembiakan koloni bakteri melaporkan bahwa satu bekteri setiap detiknya terbagi menjadi tiga begian. Pada awalnya, ada sepuluh bakteri di koloni tersebut”
Menantang


(i) Berapa banyak bakteri yang muncul setelah 5 detik ? (Q3 & A) ✓
(ii) Berapa detik dari sekarang bila diperoleh Jumlah bakteri sebanyak 2430 ? (Q3 & R) ✓
(iii) Berapakah jumlah bakteri setelah 2 menit, Jika setiap detik ada 3 bakteri yang mati ? (Q3, R, S) ✓
(iv) Jika 10 bakteri per detik membelah menjadi 4 bagian dan setiap detik ada dua yang mati. Berapa banyak yang masih hidup setelah 1 Jam kedepan ? (Q3, R, S) ✓
(v) Berapakah Jumlah bakteri dalam 10 menit berikutnya, Jika pada awalnya terdapat 100 bakteri dan setiap detik membelah menjadi lima bagian ? (Q3, R, S) ✓
(vi) Berapa banyak bakteri dalam "t" satu menit berikutnya. Jika diawal terdapat "n" bakteri dan setiap detik membelah menjadi "m" bagian ? Tunjukkan grafik pertumbuhan bakteri tersebut ! (Q4, R, S) ✓

Translate in English:

Accepting (Problem-posing task)
“A research about breeding the colony of bacteria reported that a bacteria every second is divided become three parts. In the beginning, in the colony, there are ten bacteria”
Challenging Example of the students’ performance
(i) How many bacteria come after 5 seconds? (Q3 & A) (ii) How many seconds from now when the obtained amount of bacteria as much as in 2430? (Q3 & R) (iii) What is the number of bacteria after 2 minutes, if in every second there are 3 dead bacteria? (Q3, R & S) (iv) If the 10 bacteria per second splitting into 4 parts and every second there are two dead. How many bacteria still alive after 1 hour ahead? (Q3, R & S) (v) What is the amount of bacteria in the next 10 minutes, if at beginning, there are 100 bacteria and every second splitting into five parts? (Q3, R & S) (vi) How many bacteria "t" in the next minute, if at beginning there are "n" bacteria and every second splitting into "m" part? Show the growth of the bacteria graphically! (Q4, R & S).

Figure 4: Performance of Problem Posing

Hypothesis testing

The ANOVA GWT table according to the split-plot design is displayed in Table 9.

Source		Type III Sum of Squares	df	Mean Square	F	Sig.
Intercept	Hypothesis	804630.95	1	804630.95	24649.73	.000
	Error	68.717	2.105	32.643[a]		
Perform-Assess (A)	Hypothesis	423.055	1	423.055	12,960	.046
	Error	68.717	2.105	32.643[b]		
Metacognition (B)	Hypothesis	1034.124	2	517.062	7.702	.001
	Error	10405.896	155	67.135[c]		
Perform.Assess* Metcog (A*B)	Hypothesis	691.892	2	345.946	5.153	.007
	Error	10405.896	155	67.135[c]		
Class-G(A)	Hypothesis	64.426	2	32.213	.480	.620
	Error	10405.896	155	67.135[c]		

[a] .988 MS(G(A)) + .012 MS(Error); [b] .988 MS(G(A)) + .012 MS(Error); [c] MS(Error)

Table 9: The Summary of two-way ANOVA GWT Split-Plot on Students' Achievement

Table 9 showed that the interaction performance assessment and metacognition were significant on students' mathematics achievement (F = 5.15; *p*-value = 0.007). This result suggests that performance assessment has significant effect on students' mathematics achievement depending on the metacognition level. The simple effects, i.e., the treatment of Performance Assessment (Problem Solving versus Problem Posing) should then be further investigated within each metacognition level.

The study finding showed that there is an interaction effect between performance assessment and metacognition on students' mathematics achievement. These findings are also in line with the study conducted by Özcan and Erktin (2015), where they found a statistically significant difference in the mathematics scores of students who were assigned homework tasks that incorporated metacognitive questions, compared to those who were not. The findings differ from the study conducted by Gul and Shehzad (2012) which took the subject of public and private university students as a population. The study reported a moderate nexus among metacognition, goal orientation, as well as academic achievement. In contrast, a weak relationship was observed between metacognition and achievement.

The difference in the mean score of mathematics achievement between the performance assessment treatments for each metacognition level was conducted using the t-test. The result (F = 1.358; *p*-value = 0.198) was obtained based on the homogeneity test of variance using Levene's test of Equality of Error Variances. Therefore, the variance distribution of data between treatment groups and the level of metacognition is assumed to be equal (homogeneous). The related contrasts (t-tests) of these simple effects assuming equal variances are presented in Table 10.

	F = 1.358 Sig. =0.198	Contrast	Value of Contrast	Std. Error	t	df	Sig. (2-tailed)
Math Achiev	Assume equal variances	PA-PS*HM x PA-PP*HM	5.29[a]	1.457	3.632	151	.000
		PA-PS*MM x PA-PP*MM	1.62[a]	1.602	1.012	151	.313
		PA-PS*LM x PA-PP*LM	-.60[a]	1.400	-.427	151	.670
		PA-PS*HM x PA-PS*LM	10.30[a]	1.420	7.253	151	.000
		PA-PS*MM x PA-PS*LM	15.42[a]	1.502	10.27	151	.000
		PA-PP*HM x PA-PP*LM	4.41[a]	1.437	3.070	151	.003
		PA-PP*MM x PA-PP*LM	13.20[a]	1.507	8.759	151	.000

Table 10: Contrast Tests for Simple Effects ([a] The sum of the contrast coefficients is not zero)

The contrast tests in Table 10 showed the following: (1) for students with high metacognition level (HM), the mathematics achievement scores of those receiving performance assessment through problem-solving (PA-PS) were significantly higher than those receiving problem-posing (PA-PP) (t = 3.63; df = 151; *p*-value = 0.000), (2) for students with medium and low metacognition (MM & LM), there was no significant difference between mathematics achievement scores in those receiving performance assessment through problem-solving (PA-PS) and those receiving problem-posing (PA-PP) (t =1,01; df = 151; *p*-value = 0.31 & t = -0.43; df = 151; *p*-value = 0.67), (3) for students receiving performance assessment through problem-solving (PA-PS), mathematics achievement scores in those with high and medium metacognition (HM & MM) were significantly higher than those with low metacognition (LM) (t = 7.25; df = 151 *p*-value = 0.000 & t = 10.27; df =151; *p*-value = 0.000), and (4) for students receiving performance assessment through problem-posing (PA-PP), mathematics achievement scores in those with high and medium metacognition (HM & MM) were significantly higher than those with low metacognition (LM) (t = 3.07; df = 151; *p*-value = 0.003 & t = 8.76; df = 151; *p*-value = 0.000).

These results showed that students with high metacognition and performance assessments through problem problem-solving are better or more effective in increasing students' mathematics achievement compared to performance assessments through problem-posing. Meanwhile, for students with medium and low metacognition, the class given the performance assessment through problem-solving and problem-posing did not show a distinction in math scores and was classified as having low scores. These findings showed that students' level of metacognition in the performance assessment intervention determines their mathematics achievement.

The result is supported by students' metacognition, such as focusing and monitoring cognitive processes when analyzing and planning until an appropriate solution is achieved. Furthermore, students' skill in applying the mathematical concepts concisely in solving the problem is meant to train students and enhance their skill on the higher cognitive level which in turn may support improvement in mathematics achievement. This result is accordance with Chong et al. (2019), who reported that the senior high school learners in Brunei demonstrated positive attitudes and beliefs towards problem-solving in mathematics, which are not commonly observed in routine mathematics learning. The meaningful activities designed by teachers were found to facilitate the development of both cognitive-metacognitive abilities and student affect, which aligns with the findings of Tachie (2019) that the application of metacognitive skills and tactics was advantageous for solving mathematics problems. These elements encompass breaking down the task, planning, monitoring, reviewing, and reflecting, both individual and group monitoring capabilities, competency in reading and writing, and self-regulatory capacities.

Mathematics achievement scores in students with high metacognition were better than those with low metacognition. These findings indicate that performance assessment through problem-solving requires students to use metacognition skills optimally to understand problems, plan solution models, choose the right model, control, and provide solutions to problems through existing resources. In line with the study conducted by Jacinto and Carreira (2021), it was stated that the resources used during solving and disclosure activities affect the depth of the conceptual model developed in the progressive mathematical process.

Furthermore, the finding expresses that, for the students with high metacognition, mathematics achievement scores and those receiving performance assessment through problem-solving are higher than those with problem-posing. This implies that applying performance assessment through problem-solving for students with high metacognition can improve their mathematics achievement. In addition, this phenomenon was not observed in students with low metacognition, where the performance assessment through problem-solving and problem-posing did not improve students' mathematics achievement. This result is consistent with a prior study carried out by Du Toit and Du Toit (2013), on metacognition and achievement of students in class XI. Findings from the study shows that metacognitive behavior is consistent with the first three stages of Polya, problem recognition, strategy formulation, and implementation but not with the fourth stage (reflection). A similar study carried out at the elementary school level in Singapore by N. H. Lee et al. (2014) reported that the approach focused on metacognition had a significant impact on students' comprehension of the issue, their ability to plan solutions, their confidence, and their control over their actions and emotions during problem-solving.

Another finding is that in students with medium and low metacognition, the mathematics achievement score of those receiving performance assessments through problem-solving was not significantly different from those receiving problem-posing. This finding does not support the hypothesis that performance assessment through problem-posing will be a better option in students with medium and low metacognition. Therefore, students with medium and low metacognition struggle to produce math problems, understand, plan, and solve problems optimally. On the contrary, problem-posing requires students to use their metacognition skills to generate new and quality math questions by creatively adding new information to the task. This is consistent with Van Harpen and Sriraman (2013) which involved participants from one location in the USA and two locations in China. Nevertheless, according to their findings, high school students have difficulty formulating original and challenging mathematical problems.

CONCLUSION

The assessment of student performance through problem-solving and problem-posing, in conjunction with their level of metacognition, has a significant impact on their achievement in mathematics. Specifically, students who receive performance assessments through problem-solving tend to achieve higher scores compared to those who only receive problem-posing assessments. Teachers who adopt a teaching style that includes a diverse set of questions during problem-solving assessments facilitate opportunities for students to provide more accurate and comprehensive responses, given the provision of focused questions at each stage of problem-solving. Conversely, when students undergo performance assessments through problem-posing, they tend to generate a higher number of mathematical questions that they may not have sufficient time to answer. The influence of performance assessment on mathematics achievement is further moderated by the student's metacognition level. Students with high metacognition levels perform better on problem-solving assessments than those who receive problem-posing assessments. Metacognition plays a crucial role in students' ability to regulate their learning activities and ask pertinent questions during different stages of problem-solving. Conversely, for students with low and medium metacognition levels, the impact of performance assessment on their mathematics achievement is not significant.

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Students' Proceptual Thinking Outcomes in Learning Differentiability Using Desmos Classroom Activities Based on The Three Worlds of Mathematics Framework

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Abstract: Studying students' performance in learning differentiability is essential since it requires them to perform proceptual thinking. This case study research investigates students' proceptual thinking outcomes in learning the differentiability concept using Desmos Classroom Activities (DCA) based on the Three Worlds of Mathematics (TWM) framework. In this study, DCA facilitates students' learning of differentiability with a sensible approach and exposes them to the graphical exploration of non-differentiable functions. Students succeeded in concluding that continuity does not imply differentiability. With its graphical, computational, and symbol manipulation capabilities, DCA has facilitated students to perform proceptual thinking in learning differentiability through a richer learning experience. Many students (76%) could perform at the procept level when solving the differentiability problem. Also, twenty-two students (about 88%) had no limitations in the graphical representation. The result of this study confirms that DCA based on the TWM as a generic organizer can contribute to students' proceptual thinking outcomes in learning differentiability.

Keywords: Proceptual, Thinking, Differentiability, DCA, TWM

INTRODUCTION

Calculus is based on the fundamental concepts of limits, functions, derivatives, and integrals, which are essential mathematics skills students need for life. As a result, it is critical to investigate how students learn the derivative of a function and the difficulties they encounter. One issue student frequently has when learning the derivative of a function is limited mental pictures of functions, which needs more attention from calculus lecturers. In fact, when students are given examples slightly different from their experience, such as finding *a* and *b* such that $f(x) = \begin{cases} ax, & x \leq 1 \\ bx^2 + x + 1, & x > 1 \end{cases}$ is differentiable at 1, it can cause problems (Selden, Mason & Selden, 1989). Students receive poor grades as a result. To avoid such problems, students should make sense of the idea of functions in a broader context.

Various ideas have been put forward about how to help students learn the derivative of a function, including the Three Worlds of Mathematics (TWM) framework developed by Tall (2012). The TWM framework describes three different worlds when students learn the derivative of a function. It concerns students' cognitive development, which involves the simultaneous development of conceptual embodiment (the complementary use of human perception and action) and proceptual symbolism (the manipulation of symbols resulting from operations). The term "procept" refers to a symbol representing a process and a concept as the result of that process; e.g., differentiation is the process of finding the derivative of a function. (Gray & Tall, 1994).

Moreover, Tall (1993) also defined the ability to switch between different symbolisms for the same mathematical object and to alter symbols flexibly as a process and a concept as proceptual thinking. The TWM-based learning activities should start by letting students experience embodied dynamic changes. After that, they should be able to do enough arithmetic to get a numerical approximation and finally find the appropriate symbolic expression for the derivative of a function.

Tall (1989) explained that a "generic organizer" is a computer environment where students can explore different examples and non-examples of mathematical concepts or systems of related concepts, such as calculus. Tall (2012) suggests using a computer program as a generic organizer that can dynamically plot graphs to make the display of function graphs bigger so that each part of the function graph can be seen very clearly. Tall (2010) claims that computer programs help demonstrate graphical visualization dynamically and provide accurate numerical and symbolic computations for learning derivatives of functions. Users are granted the ability to create objects (points, lines, graphs of functions), perform measurements (angles, areas, slopes of lines), perform transformations (rotations, reflections, symmetries, magnifications), and carry out other manipulations of selected or constructed objects using dynamic computer programs. Additionally, many features of the dynamic computer program, such as dragging and magnifying, can facilitate a learning environment for discovery, experimentation, viewing patterns, generating, and testing conjectures, and visualizing mathematical objects (Gonzalez & Rodriguez, 2011; Herceg, 2010). Dynamic computer programs allow students to visualize mathematical concepts in multiple representations (algebraic, numerical, and graphical). Students, for instance, can generate tables of values, enter equations, and graph various functions. Because mathematical objects are represented dynamically, students can see mathematical problems or processes in a way that is impossible with paper and pencil (Sacristán et al., 2010).

Desmos Classroom Activities (DCA) is a website-based dynamic program with numerous advantages over other programs or applications, including being free, easy to use, intuitive, powerful in graph creation, and encouraging students to participate in mathematics learning (Ebert, 2014), actively. In this study, the DCA facilitates students' learning of differentiability based on the TWM framework with a sensible approach (human perception). A sensible approach is based on natural continuity (where a graph 'pulls flat') and local straightness (to give the concept of derivative). Based on this explanation, the researcher intends to investigate students' proceptual thinking outcomes in learning the differentiability concept using Desmos Classroom Activities (DCA) based on the TWM framework. For that reason, this study aims to answer the following questions:

1. How is the description of students' proceptual thinking outcomes in learning differentiability using Desmos Classroom Activities based on the TWM framework?
2. Can the learning activities using Desmos Classroom Activities based on the TWM framework contribute to students' proceptual thinking outcomes?

This study contributes to the development of the mathematics education field by illustrating students' proceptual thinking outcomes while studying the differentiability of Desmos Classroom Activities (DCA) as a generic organizer. In this study, the DCA is developed based on the Three Worlds of Mathematics (TWM) framework, which emphasizes a sensible approach for students to comprehend calculus derivatives. DCA is dynamic software with powerful graphical and computational features that facilitate a sensible approach and allow students to actively share mathematical ideas during a teaching-learning session. This study will inspire other researchers and lecturers to change the way calculus is taught by leveraging technology and emphasizing activities that help students make sense of differentiability to reduce cognitive barriers.

LITERATURE

Students' Proceptual Thinking Outcomes According to The Three Worlds of Mathematics

Tall (2008) developed the Three Worlds of Mathematics (TWM) framework to examine students' cognitive development as they learn calculus concepts. Each world operates and evolves differently, distinguishing them as three mathematical mental realms (Tall, 2004):

1. Conceptual embodiment is built on students' perceptions and actions, verbalized mental representations with increasing sophistication, and the development of an appropriate mental entity in the student's imagination.
2. Proceptual symbolism evolves from physical actions into a mathematical procedure that is symbolically and conceptually conceived as both operations to do and a symbol that can be operated independently for computation and manipulation (procept).
3. Axiomatic formalism builds formal knowledge in axiomatic systems inside a suitable fundamental framework whose attributes are deduced by mathematical proof.

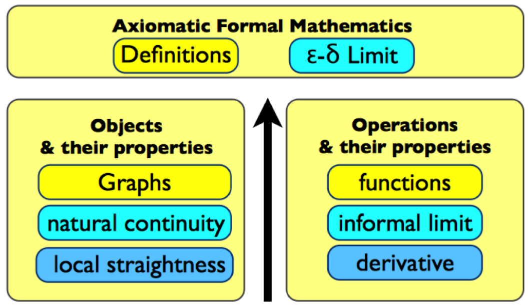

Figure 1: Calculus in The Three Worlds of Mathematics (Tall, 2012)

The derivative calculus develops naturally in a parallel world of embodiment and symbols. It employs a visual and practical way to transform the concepts of slope change (as shown by graphs) and rate of change (the slope of a function) into symbolic manipulation of functions and derivatives. The embodied approach to derivatives has the potential to offer a robust cognitive basis for the proceptual, symbolic, and axiomatic formal worlds. This method differs from one that starts with the formal definition of derivatives. On the other hand, the embodied approach to the derivative focuses on developing derivative notions through bodily encounters with the function graph. It allows students to experience a dynamic change, construct numerical estimation, and finally obtain the appropriate symbolic representations for a function's derivative. Students can construct the tangent line as a slope of the graph by dragging the graph to see and feel it as an object and by sensing the slope changes (see Figure 2).

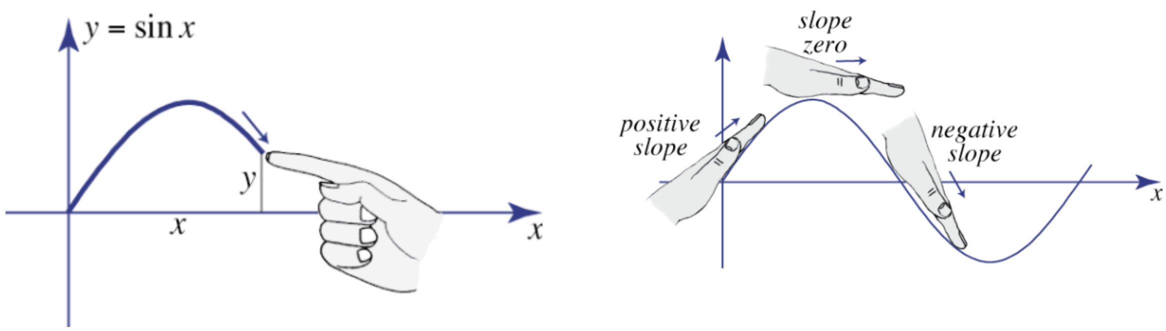

Figure 2: An Embodied Approach to Derivatives (Tall, 2009).

Tall (2002) defines local straightness as the cognitive foundation of derivatives. Local straightness is a simple idea that helps students comprehend that it will appear straight when they zoom in on a little portion of a graph. Students can sense the slope of the entire graph using this cognitive root. Furthermore, Mills and Tall (1989) asserted that when students learn about differentiability, they should be able to examine more complex functions than simple polynomials or smooth combinations of standard functions. It will provide a more comprehensive learning experience for indistinguishable functions at a specific point, as well as other essential concepts that build on earlier experiences to boost analytical insight.

To provide symbolic processes for determining the slope of a function, an embodied approach to differentiability should be connected to the related realm of symbolism, with its numerical calculations and algebraic manipulations. The symbolic concept of a function and its derivatives belong to proceptual symbolism (Tall & Mejia-Ramos, 2004). According to Gray and Tall (1994), a procept is a mental object consisting of a process to perform and a concept generated by that process. Tall (1993) described proceptual thinking as the ability to flexibly change symbols as a process or a concept and to switch between different symbolisms for the same mathematics object. The flexible and ambiguous use of symbolism to represent the duality of processes to be performed and concepts to be understood using the same notation gives proceptual thinking considerable strength. As the students' progress, the number procept increases in internal depth, or "interiority" (Lunzer & Skemp, 1980). It provides greater versatility in manipulation.

There is a significant performance disparity between successful and unsuccessful students when faced with questions that demand them to perform proceptual thinking. Successful students are proceptual thinkers. According to Gray and Tall (1991), proceptual thinkers have a flexible relational understanding of mathematics that is regarded as a meaningful relationship between same-level conceptions. According to Thurston (1990), condensed mathematical concepts make it easier for successful students to grasp them. As the conceptual complexity of proceptual thought rises, procepts can be controlled at a higher level as simple symbols, opened to computations, dismantled at will, and decomposed arbitrarily.

In contrast, the unsuccessful students are procedural thinkers. They must coordinate consecutive processes to solve problems at the next level. For them, this is a complicated procedure. If they meet a problem two layers above, the structure may be too hefty for them to accomplish (Sfard, 1991). The unsuccessful students must climb a more difficult hierarchical ladder. When procedural thinkers fail to establish a sophisticated perceptual framework, such modes of cognition become inaccessible to them. The ambiguity between process and consequence indicated by a procept allows for more natural cognitive development, which bestows immense power on successful students. It exemplifies the proceptual chasm confronted by unsuccessful students striving to comprehend the concept's rising complexity.

Figure 3 displays the proceptual thinking outcomes that Tall et al. (1999) developed at various levels of sophistication. The spectrum consists of pre-procedure, procedure, process, and procept levels. It means that problems requiring typical procedural answers will only differentiate students who fail the transition from those who succeed. Several process paths provide alternative methods for detecting potential execution faults, including an unconscious feeling that something is wrong when an error occurs (Crowley & Tall, 1999). The procept level progresses to a higher level in which symbols function as both a process and a concept, allowing students to investigate symbol linkages in ways that go beyond the processes themselves.

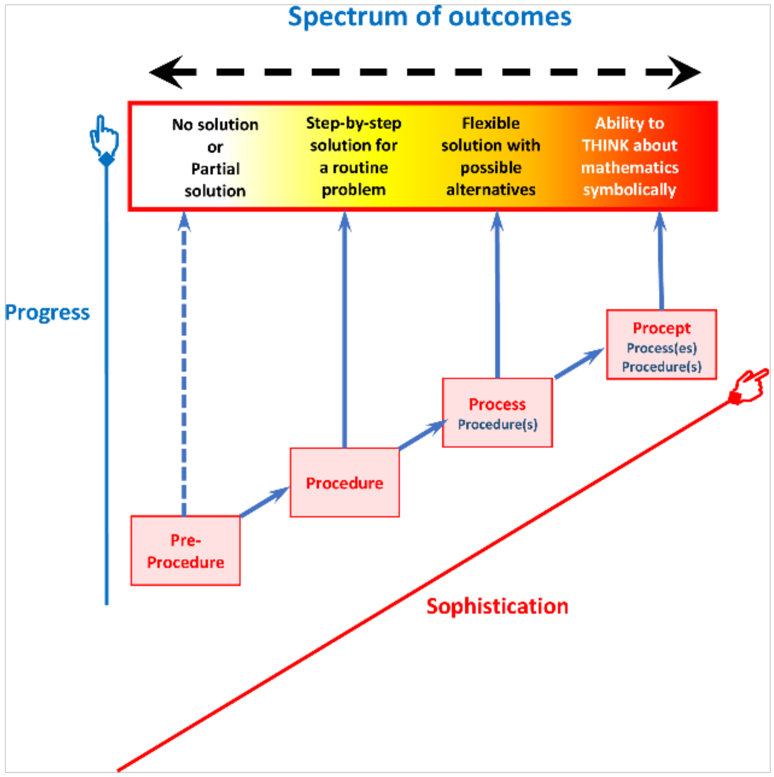

Figure 3: The Spectrum of Performance in Carrying Out Mathematical Processes (Tall et al., 1999)

Desmos Classroom Activities as The Generic Organizer for Learning Differentiability

The physical exploration of a function's graph serves as the foundation for an embodied approach to derivatives. The computer environment has an interface that lets students interact with the objects on the screen by pointing, clicking, dragging, and zooming in. This gives the concept of differentiability a fully embodied mode. Tall (2009) suggested utilizing a dynamic computer program to incorporate a graph's local straightness (a tiny section of the graph that appears as straight lines under high magnification), which is a cognitive underpinning for differentiation. A generic organizer is a computer environment that allows students to manipulate examples and non-examples of certain mathematical concepts or systems of related concepts (Tall, 1989). The generic organizer can be used to create a variety of learning experiences that lead to a better understanding of derivatives.

Desmos is well-known for providing a free online graphing calculator. Currently, it has a new dynamic computer application, Desmos Classroom Activities (DCA), that can generate engaging digital mathematics learning activities. DCA is easy to use, intuitive, and powerful in its graphical features (Ebert, 2014). DCA can motivate students to build and discuss mathematical ideas actively. Students will learn through interacting with mathematical representations and sharing their evolving mathematical ideas. Previous research (Tall, 1986) shows that using the generic organizer in a setting where the lecturer presents ideas, discusses them with the students, and then investigates them with the students has many advantages over conventional methods. Following this, DCA has a dashboard feature that allows the lecturer to monitor and assess each student's learning progress in real-time. The lecturer can quickly identify if students are experiencing difficulties or misconceptions, allowing them to provide immediate assistance. The given learning exercises minimize misunderstandings, encourage class discussion, and increase student comprehension.

DCA also allows students to zoom in on the graph at high magnification while dragging the point or tangent line along the graph to examine locally straight or non-locally straight examples, which is essential for learning differentiability. According to Tall (1986), introducing a function that is not differentiable at a given point through computer activity might provide extra visual insights that assist advanced conceptions in analysis previously regarded as visually complex to students. Based on the TWM framework, DCA enables students to perceive a graph's shifting slope. Finally, DCA provides a context for visual and symbolic concepts that can be easily linked to the concept of differentiability and further investigation.

METHOD

The researcher used a constructivist paradigm to examine students' proceptual thinking outcomes in learning differentiability using DCA based on the TWM framework. Stake (1978) describes constructivism as the concept that knowledge primarily consists of social interpretations as opposed to knowledge of actual realities. This study is based on the researcher's interpretation as a lecturer who interacts directly with the students (research participants). Students' responses to DCA-based learning activities demonstrate constructivist epistemology. This study includes 25 students who took Calculus 1 at Sampoerna University during the 2022–2023 academic year. The case study design is suitable for research that seeks to answer "how" questions (Heck, 2011; Punch, 2009). To answer the research questions, students' proceptual thinking outcomes will be analyzed according to the spectrum of performance in learning differentiability.

This study's primary research instrument is Desmos Classroom Activities (DCA), which is based on the Three Worlds of Mathematics (TWM) paradigm and has been validated by three experts in three distinct fields: (1) mathematics; (2) mathematics education; and (3) technology in education. The DCA, as a generic organizer based on the TWM framework, is aimed at facilitating students' making sense of the differentiability concept. Students' responses in the DCA are used to investigate students' proceptual thinking outcomes while learning differentiability.

Assessment Criteria
Computer Environments for Cognitive Development According to The Three Worlds of Mathematics (Tall, 2002)
A generic organizer: is an environment in which the students can manipulate examples and non-examples of a given mathematical concept or a related set of concepts. a) Under active control, zoom in to sense the diminishing curvature and establish local straightness by sensing it 'happen.' b) Move a magnification window along a locally straight graph to sense the slope changing. c) Under high magnification, move the 'corners' (with different left and right slopes) and more general 'wrinkled' curves to gain a sense that not all graphs are locally straight. d) Draw the slope function to provide a visual relationship between a locally straight function and its slope symbolically. an environment in which the students can manipulate examples and non-examples of a given mathematical concept or a related set of concepts.

Table 1: Expert Validation Test- DCA Assessment Criteria

Other research instruments are used in this study. These include the observation sheet, the test question (see Table 3) and the student worksheet. Observation of teaching-learning activities using DCA will refer to the Three Worlds of Mathematics framework as follows:

Three Worlds of Mathematics	Sample of activity
The conceptual-embodied world	Students move the two points crossed by the secant line so that the two points are very close to each other and then students see the visualization of the tangent line. Students zoom in on a graph near a particular point to see it looks like a straight line. Students draw the slope function to visualize the link between a locally straight function and its slope. Students study 'corners' (with different left and right slopes) and more general 'wrinkled' curves to learn that not all graphs are locally straight.
The proceptual-symbolic world	Determine the continuity of a function at a point. Determine the differentiability of a function at a point.

Table 2: Observation Guidelines

In terms of analysis, an interpretive theoretical approach provides a framework for comprehending students' responses while learning differentiability using DCA based on the TWM framework. The approach is inductive, whereas the outcomes are descriptive (Merriam, 2002). The analysis of data begins with a review of students' responses. Students' responses were evaluated for emerging trends or themes for each item. The topics were then classified according to the included descriptions (Rasslan & Tall, 2002). The students' proceptual thinking outcomes were determined by their responses to the DCA and their solutions to the test problem. The following test question was used to examine students' proceptual thinking outcomes after learning differentiability:

Problem	Expected Proceptual Thinking Outcomes
Show that $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$ is continuous but NOT differentiable at $x = 1$! (You must visualize the problem and perform analytical analysis)	Proceptual & The graphical representation

Table 3: The Test Problem

All data are examined using the proceptual thinking outcomes spectrum constructed by Tall et al. (2001). The spectrum includes pre-procedure, procedure, process, and procept levels. Gray and Tall (1994) emphasized that students are expected to achieve the procept level when studying the derivatives. Procept theory focuses primarily on symbols indicating "to do" activities, "to think about" concepts, and the essential ability to perform dual processes and concepts. The cognitive development spectrum depicted in Figure 3 is described as follows:

- Procedure: a limited series of acts and decisions constructed logically. The steps in a procedure might be considered as leading to the next phase.
- Process: the term process is used when a procedure is understood as its whole, with an emphasis on inputs and outputs rather than the precise processes required to execute a process. A process that can be accomplished using *n* procedures and permits the selection of the best effective solution for a particular circumstance.
- Procept: a procept requires a symbol to be interpreted both as a process to be carried out and thought to consider. This adaptability enables enhanced mental manipulation and introspection for developing novel notions.

Additionally, validity in qualitative research corresponds to trustworthiness, transferability, dependability, and confirmation (Creswell, 2013). The validity of this case study research is achieved by integrating methodologies or data sources that provide a comprehensive picture of the examined issue (Cohen et al., 2011). *Triangulation* is a data collection strategy that employs two or more methodologies (Cohen et al., 2011).

RESULTS

Students' Learning Process Using Desmos Classroom Activities (DCA) Based on the TWM Framework

The learning activity begins with students moving two points along the secant line such that both become very close to one another (the distance between the two points is almost zero), which triggers students to make sense of the tangent line concept. The lecturer asks the question, "*What do you notice?.*" Students respond that "*the tangent line can be formed if the two points coincide.*" Then the lecturer asks, "*What is the linkage between the secant line and the tangent line* *in this graph?*." *"The slope of the tangent line is the limit of the slope of the secant's line, Miss!"* students responded. Students open the next screen, which shows the formal definition of a function's derivative and the concept of differentiability. HR read out loud the formal definition of a function's derivative, which has been discussed before. The lecturer asks students to remember the continuity at one point. Then, VIN directly mentioned the three conditions that should be satisfied for f to be continuous at c: (1) $f(c)$ is defined; (2) $\lim_{x\to c} f(c)$ exists; and (3) $f(c) = \lim_{x\to c} f(x)$.

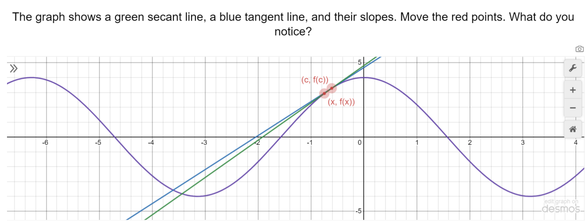

Figure 4: A capture of HR’s screen when moving the two points to construct the tangent line

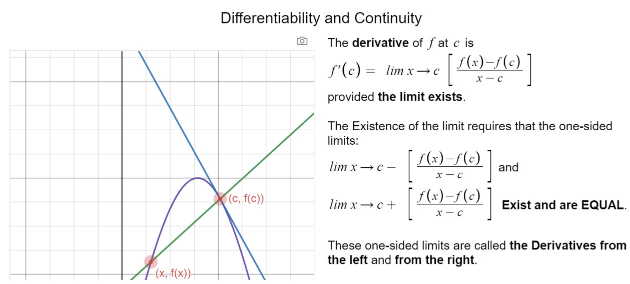

Figure 5: An Explanation of Differentiability

Furthermore, students recall that the limit exists when the left-hand limit equals the right-hand limit. They relate the existence of the limit to the concept of differentiability. Then, they discover that the function $f(x)$ is differentiable at $x = c$ when $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists. The lecturer confirmed that the condition that must be satisfied for a function to have a derivative at point c is that $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exist. In other words, the values of the left-hand limit $\lim_{x\to c^-} \frac{f(x)-f(c)}{x-c}$ must be equal to the right-hand limit $\lim_{x\to c^+} \frac{f(x)-f(c)}{x-c}$.

As for the core learning activities, the students did a graphical exploration of $f(x) = |x - 2|$. Students move the tangent line along the graph of $f(x)$ back and forth to sense the slope changing. The lecturer asked students to zoom in on the graph and identify whether the function is continuous at $x = 2$. IKH answered that the function is continuous at that point, visually as x approaches 2 both from the left and the right directions, the values of f getting close to 0. Since the left-hand limit agrees with the right-hand limit values, there is no discontinuities at that point. Moreover, the lecturer asked students to zoom in on the graph and drag the tangent line toward $x = 2$. HR said that it is impossible to locate the tangent line precisely at $x = 2$. Thus, there is no tangent line at $x = 2$. ORY also said that according to the DCA output, the slope of the tangent line is undefined at $x = 2$. The lecturer then asked students to answer some related questions to $f(x) = |x - 2|$, which were given on the next screen.

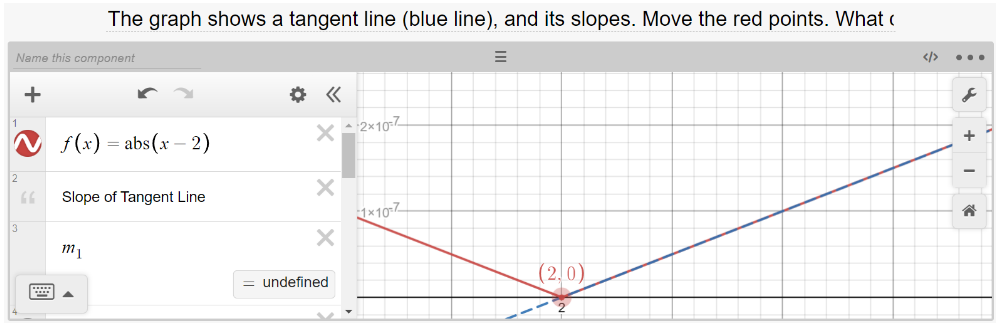

Figure 6: A Graphical Exploration of $f(x) = |x - 2|$

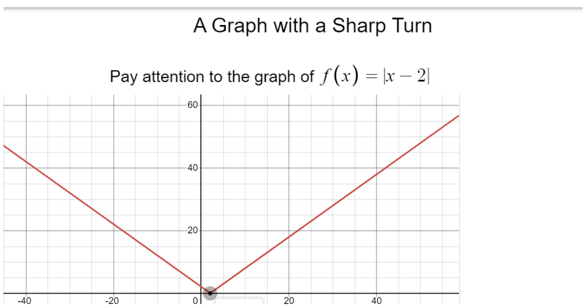

Figure 7: A graph of $f(x) = |x - 2|$

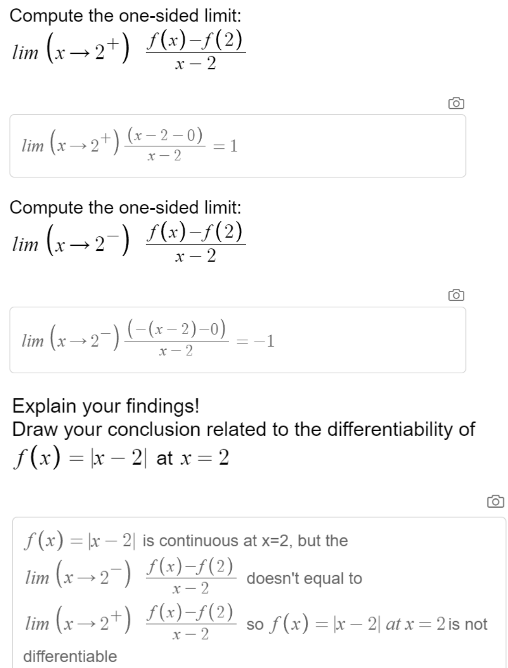

Figure 8: BAT's response

As a sample of students' answers, BAT shows that $f(x) = |x - 2|$ continuous at $x = 2$ because the requirements for continuity at a point are satisfied: $f(2)$ is defined, $\lim_{x\to2}|x - 2|$ exists, and $f(2) = \lim_{x\to2}|x - 2|$. BAT then shows that the left-hand limit $\lim_{x\to2^-}\frac{|x-2|-f(2)}{x-2} = \lim_{x\to2^-}\frac{-(x-2)-0}{x-2} = -1$, and the right-hand limit $\lim_{x\to2^+}\frac{|x-2|-f(2)}{x-2} = \lim_{x\to2^+}\frac{(x-2)-0}{x-2} = 1$. The students concluded that the function is not differentiable at *x=2* because it does not fulfill the existence of the limit according to the different values of the left-hand and the right-hand limits.

To expose students to various contexts of differentiability, they continue to explore another graph with a sharp turn, $f(x) = x^{1/3}$. Students move the tangent line along the graph of $f(x) = x^{1/3}$ back and forth to see and feel the slope changing. Then, they magnify the graph and check whether the function is continuous at $x = 0$. VIN stated that the function is continuous at $x = 0$ because it meets the three conditions for continuity at a point. Furthermore, students zoom in on the graph and drag the tangent line toward $x = 0$. ANG indicated that the slope of the tangent is undefined at $x = 0$, given by the DCA output.

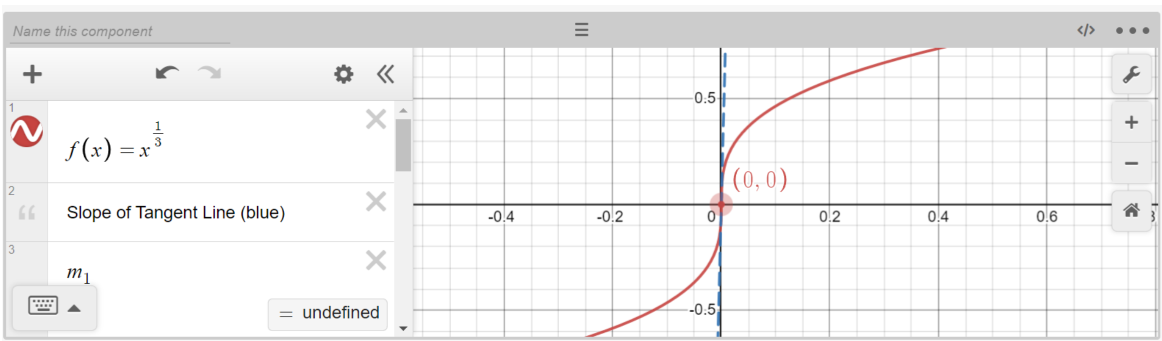

Figure 9: A Graphical Exploration of $f(x) = x^{\frac{1}{3}}$

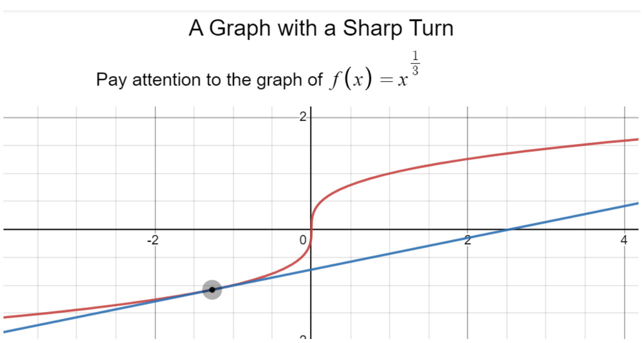

Figure 10: A graph of $f(x) = x^{1/3}$

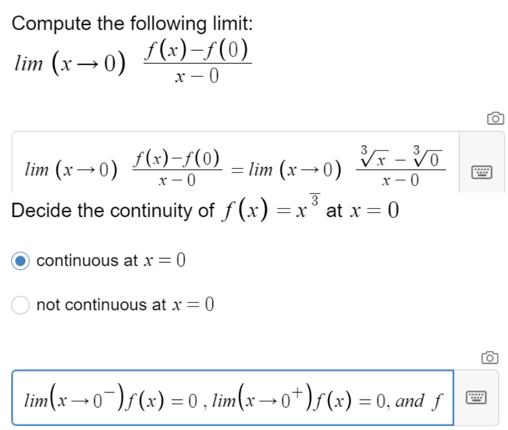

Figure 11: NIS's responses

Students then work on the second problem, whose primary objective is nearly identical to the first problem, i.e., to show that the function $f(x) = x^{1/3}$ is continuous at $x = 0$, but not differentiable at that point. According to NIS's responses, she could verify that $f(x) = x^{1/3}$ is continuous at $x = 0$, since $f(0)$ is defined and $f(0) = \lim_{x\to 0} x^{1/3} = \lim_{x\to 0^+} x^{1/3} = \lim_{x\to 0} x^{1/3} = 0$. Additionally, the students also showed that $\lim_{x\to 0} \frac{x^{1/3} - f(0)}{x-0}$ does not exist (DNE). Therefore, they draw the following conclusion: although the function is continuous at $x = 0$, it does not guarantee that it also differentiable at that point. Students conclude that continuity does not imply differentiability.

Figure 13 captures the sample of students' conclusions regarding the relationship between continuity and differentiability (AUR and BAT). At the end of the learning activities, students were asked to share their learning reflections. Students said they had learned about differentiability and concluded the relationship between continuity and differentiability.

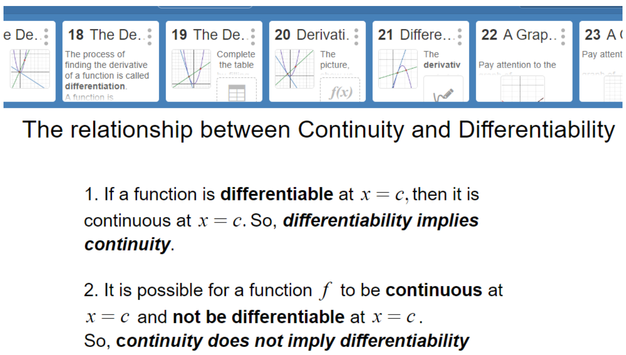

Figure 12: The relationship between continuity and differentiability

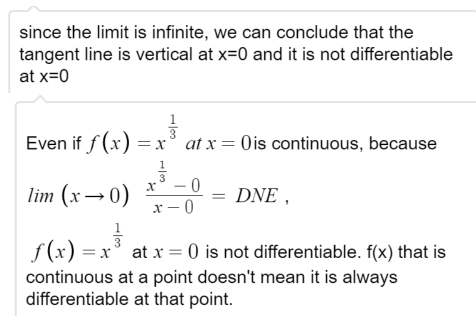

Figure 13: Sample of students' conclusion (AUR and BAT)

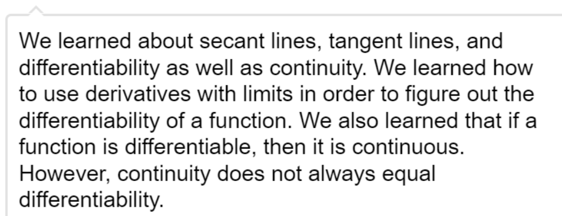

Figure 14: A Sample of JOS's learning reflection

Students' Proceptual Thinking Outcomes

Students' proceptual thinking outcomes after learning the differentiability were analyzed through their answers to the test question (see Table 2). Based on the students' answers to the given question, it was found that nineteen students (76%) reached the procept spectrum. For example, according to BRY's answer, he succeeded in showing that the piecewise-defined function is continuous at $\boldsymbol{f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}}$ is continuous at $\boldsymbol{x = 1}$, i.e., by showing that: $\boldsymbol{f(1)}$ defined, $\boldsymbol{\lim_{x \to 1} f(x)}$ exists, and $\boldsymbol{f(1) = \lim_{x \to 1} f(x)}$. After that, BRY succeeded in showing that $\boldsymbol{\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} \neq \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1}}$. Finally, he concluded that the piecewise-defined function is not differentiable at $\boldsymbol{x = 1}$.

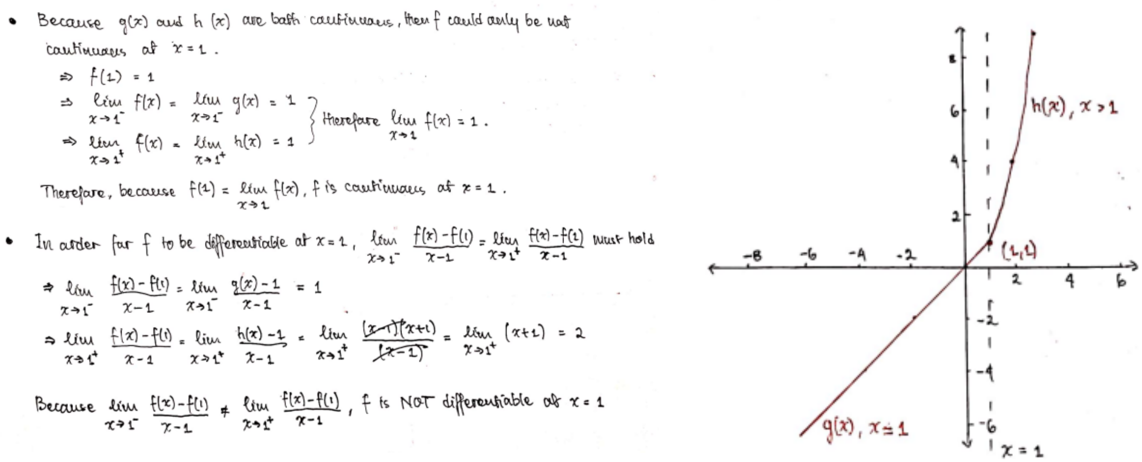

Figure 15: BRY's answer

Sixteen students (64%), including BRY, who successfully achieved the procept level, also had no limitations in a graphical representation. Unfortunately, the remaining three students (SHAH, MAR, and AUR), who succeeded in performing procept level, had limited graphical representations of the given problem. Thus, success in performing proceptual thinking does not guarantee success in the graphical representation of a given problem.

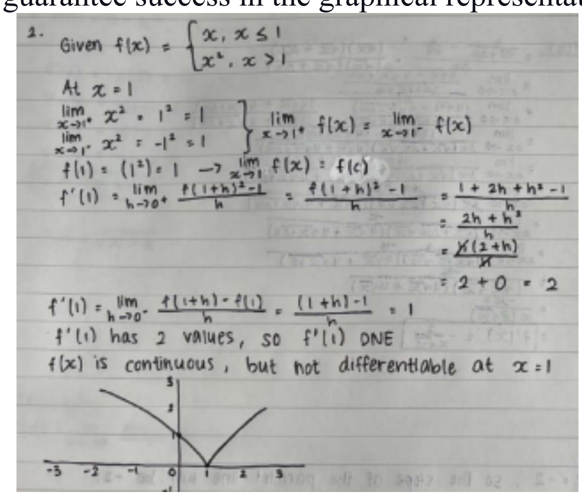

Figure 16: MAR's answer

Furthermore, six of the 25 students, or about 24%, failed to perform proceptual thinking. They faced different difficulties according to their answers. An interesting finding is that they all had no limitations in the graphical representation, including RUB and YUN, who did not attempt to answer the question. The two students also succeeded in sketching the graph of the piecewise-defined function. They reasoned that by sketching the graph, they believed that it could help generate mathematical ideas to solve the given problem. However, at last, they failed to get the solution. These findings indicate that students (twenty two of 25) are accustomed to making sense of the differentiability concept through a graphical representation. When they have difficulties understanding the problem, they choose to visualize it by drawing the function's graph to help them generate mathematical ideas that they expect will lead to the correct solution.

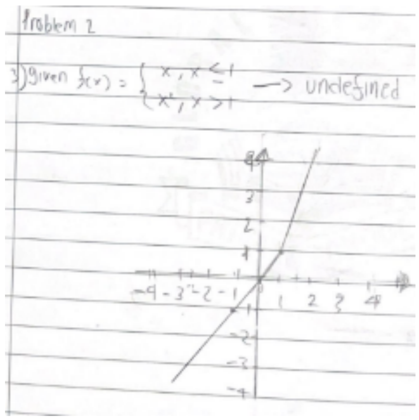

Figure 17: A graphical representation by YUN

The other two students who also failed to reach the proceptual level were KEN and KIR. They have demonstrated that they understand the concept of differentiability. However, when determining the left-hand limit $\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}$ and the right-hand $\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$, they computed the two limits by direct substitution, concluding that both limits do not exist. These procedural errors lead to incorrect solutions.

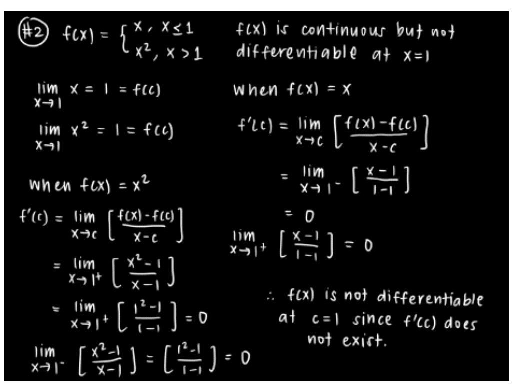

Figure 18: KIR's answer

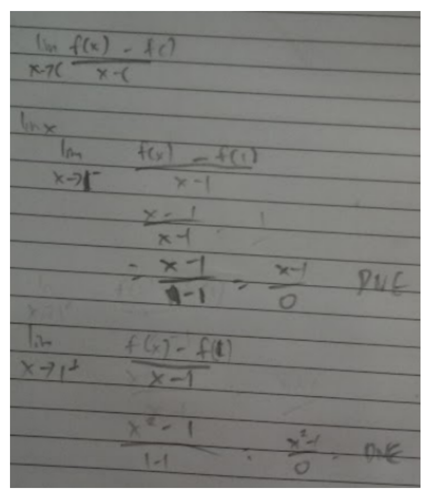

Figure 19: KEN's answer

Additionally, FAK did not employ the differentiability concept properly ($f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$, provided the limit exists). Instead, FAK directly used the basic differentiation rules for the polynomial function, which led to the wrong conclusion. FAK responded that the derivative of f

is 1 for x near one from the left direction (all x less than 1). Further, she exclaimed that the derivative of f is 2 for x near one from the right direction (all x greater than 1). Her answer is an example of a misconception of the differentiability concept.

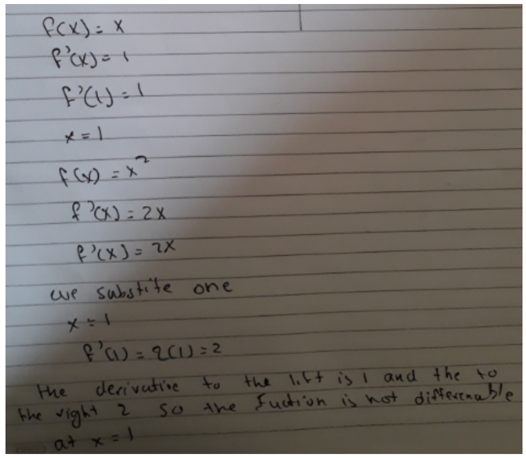

Figure 20: FAK's answer

ORY is a student who also failed to perform the proceptual thinking, but she tried verifying that the piecewise-defined function $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$ is continuous at $x = 1$. She also figured out that the piecewise-defined function is not differentiable because it makes a sharp turn at $x = 1$, which means there is no tangent line at that point. According to her explanation, she has a good graphical representation of the problem and uses it for reasons why the piecewise defined is not differentiable at $x = 1$.

Based on the above explanation, the following findings are obtained:

- 19 of 25 students (76%) successfully achieved the procept level, but unfortunately, three students failed to show the graphical representation of the given problem.
- 6 of 25 students (24%) failed to perform proceptual thinking; surprisingly, they all had no limitation in the graphical representation. Some students confirmed that they sketched the graph to gain mathematical insights into the given problem.
- 16 of 24 students (64%) successfully achieved the proceptual level and had no limitations in the graphical representation.
- Twenty two of 25 students (88%) had no limitation in graphical representation, although some (6 of 22) failed to perform perceptual thinking.

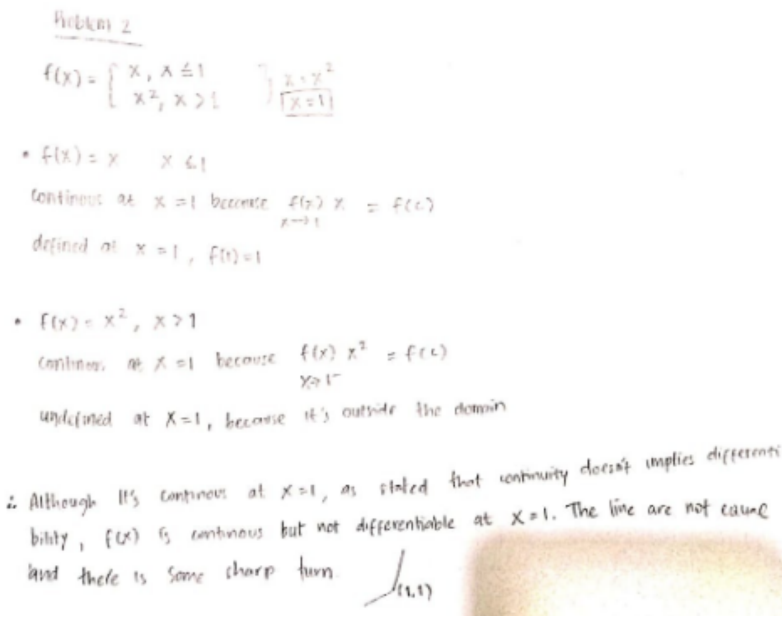

Figure 21: ORY's answer

In summary, the students' proceptual thinking outcomes after learning the differentiability can be depicted by the following Venn Chart:

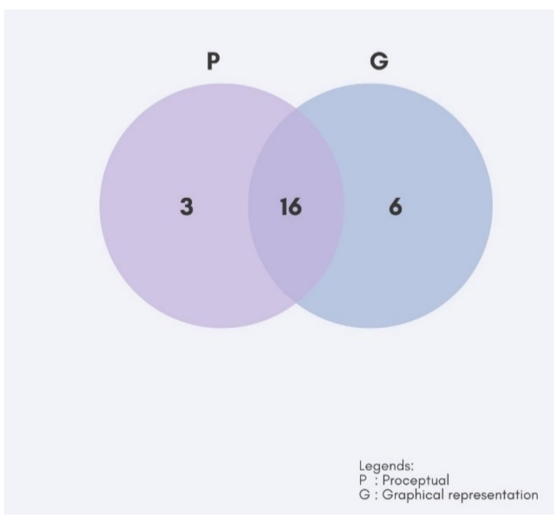

Figure 22: Students' Proceptual Thinking Outcomes

Overall, many students in this study could perform proceptual thinking by showing that the piecewise-defined function $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$ is continuous but not differentiable at $x = 1$. The majority had no limitations in the graphical representation. Few students failed to perform the proceptual thinking, but surprisingly, they all had no limitations in the graphical representation. In fact, they believed that a graphical representation could generate mathematical ideas that can help them solve the given problem.

DISCUSSION

How is the description of students' proceptual thinking outcomes in learning differentiability using Desmos Classroom Activities based on the TWM framework?

The learning process using DCA is based on the TWM framework, focusing on building students' perceptions of differentiability. The initial activities facilitated students to experience the embodied approach to make sense of the tangent line and the derivative of a function as follows:

- Students move two points on the graph, which are passed by the secant line so that the two points are closer to each other. As the distance between the two points is almost zero, the students immediately see the visualization of the tangent line and then move it along the function's graph.
- Students zoom in on the graph while moving the tangent line along the function graph to visualize the local straightness and finally sense the changing slope.

The above-embodied learning activities were connected to build ideas of continuity and differentiability. This was followed by symbolizing embodiment activities for continuity and differentiation. In other words, students were thinking in a proceptual world to make sense of differentiability and determine the relationship between continuity and differentiability. The following activities demonstrated how students used proceptual thinking while making sense of differentiability:

- Students recall the existence of limits: a limit exists when both the left-hand and right-hand limits exist and have equal values. Students relate this concept to the conditions that must be satisfied for a function to be differentiable at point *c.*
- Students discover that the function $f(x)$ is differentiable at $x = c$ when $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists. In other words, $f(x)$ is differentiable at $x = c$ when $\lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} = \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c}$.
- Students recall the three conditions that should be satisfied for f to be continuous at c: (1) $f(c)$ is defined; (2) $\lim_{x \to c} f(c)$ exists; and (3) $f(c) = \lim_{x \to c} f(x)$.
- Students move the tangent line along the graph of $f(x) = |x - 2|$ back and forth to sense the slope changing (embodying symbolism).
- Students zoom in on the graph and see as x approaches 2 both from the left and the right directions, the values of f getting close to 0 (embodying symbolism). Students conclude that graphically the function is continuous at $x = 2$.

- Students zoom in on the graph and drag the tangent line approaching $x = 2$. Students sense that there is no tangent line when he/she locate the tangent line precisely at $x = 2$. Students identify the DCA output for the slope of the tangent line when $x = 2$, which is undefined.
- Students show that $f(x) = |x - 2|$ continuous at $x = 2$ because the three requirements for continuity are satisfied: $f(2)$ is defined, $f(2) = \lim_{x\to 2}|x - 2| = \lim_{x\to 2^+}|x - 2| = \lim_{x\to 2^-}|x - 2| = 0$.
- Students show that $\lim_{x\to 2^-}\frac{|x-2|-f(2)}{x-2} = \lim_{x\to 2^-}\frac{-(x-2)-0}{x-2} = -1 \neq \lim_{x\to 2^+}\frac{|x-2|-f(2)}{x-2} = \lim_{x\to 2^+}\frac{(x-2)-0}{x-2} = 1$.
- Students conclude that although the function is continuous at $x = 2$, it is not differentiable at $x = 2$
- Similarly, students did a graphical exploration for another graph with a sharp turn, $f(x) = x^{1/3}$, then followed by showing that $f(x) = x^{1/3}$ is continuous but not differentiable at $x = 0$ because $\lim_{x\to 0}\frac{x^{1/3}-f(0)}{x-0}$ *does not exist* (DNE).
- Students conclude that continuity does not imply differentiability.

Since the beginning of the learning process, using DCA based on the TWM framework has allowed students to experience dynamic slope changes along the graph. As a generic organizer, DCA has a powerful graphical feature that provides enactive experiences of moving points and lines while magnifying a function's graph. As magnification increases, the graph becomes less curved and viewed straight (local straightness). This activity may bring attention to the fact that what is seen only represents mental thought. When it is possible to flatten it by tugging on it, the students will be able to recognize that the function is continuous (Mills & Tall 1989). Thus, it helps students verify the continuity of the function at a certain point.

Moreover, students in this study were exposed to non-examples of differentiability problems: i.e., $f(x) = |x - 2|$ and $f(x) = x^{1/3}$. Students zoom in on the associated graph and drag the tangent line approaching $x = c$. Students found that there is no tangent line when they locate the tangent line precisely at $x = c$. Then they received confirmation for the slope of the tangent line when $x = c$, which is undefined, from the DCA output. Thus, finally they can conclude that although the function is continuous at $x = c$, it is possible for the function to be not differentiable at $x = c$. This exploration has provided students with a richer learning experience of the differentiability concept. It is possible to give students a problem that is significantly more complex than simple polynomials or smooth combinations of standard functions and to provide a richer learning experience of why things can "go wrong" as well as surprise cases in which they "go well" (Mills & Tall 1989).

According to Tall (2009), forming numerical and symbolic representations includes a fascinating form of cognitive development. The DCA based on the TWM provides recurring cycles of activity in which a process, such as determining the left-hand limit $\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}$ and the right-hand limit $\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}$, and verifying whether the left-hand limit agrees with the right-hand limit, is turned into the differentiability concept. Using DCA with its graphical, computation, and symbol manipulation capabilities has supported students in performing proceptual thinking while learning the differentiability concept. Students' mathematical thinking can improve using technology to learn mathematics. It expands learning, makes it more feasible for students to gain mathematical insights, encourages them to take responsibility for their learning so they can learn more deeply, and finally, perform essential tasks.

Can the learning activities using Desmos Classroom Activities based on the TWM framework contribute to students' proceptual thinking outcomes?

Based on the results and findings related to students' proceptual thinking outcomes in solving the differentiability problem, the researcher suggested the following results:

Most students (19 of 25 or around 76%) reached the procept spectrum when asked to show that piecewise-defined function $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$ is continuous; however, it is not differentiable at $x = 1$. Students carried out complex processes and mastered or produced concepts related to continuity and differentiability (see Figure 23). They flexibly manipulated the symbolism in solving the differentiability problem. Finally, they concluded that continuity does not imply differentiability.

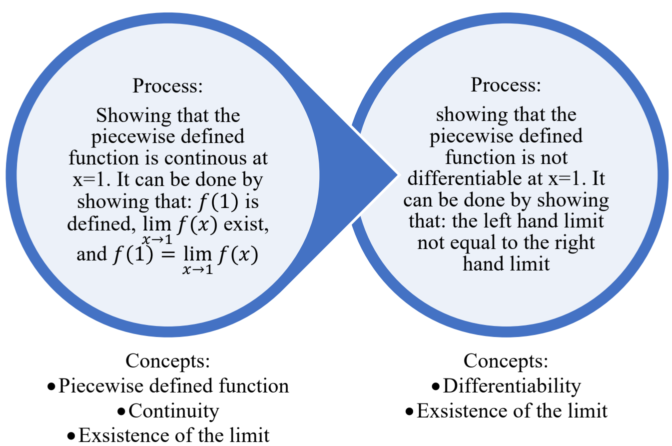

Figure 23: Proceptual Thinking for Solving the Differentiability Problem

Unfortunately, six other students (24%) have not yet achieved the procept spectrum when solving the given question. According to their answers, the difficulties faced were quite diverse. Two students, KEN and KIR, have demonstrated their understanding of differentiability. However, when they determined the left-hand limit $\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1}$ and the right-hand limit $\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$, they used direct substitution, which leads to the wrong conclusion that the two limits do not exists.

FAK showed a misconception of differentiability, in which she immediately used the basic differentiation rule for the polynomial function. Consequently, she concluded that the derivative of the function is 1 for all x close to 1 from the left direction (less than 1), and the derivative of the function is 2 for all x close to 1 from the right direction (greater than 1).

Furthermore, ORY was a student who succeeded in showing that the piecewise-defined function $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$ is continuous at $x = 1$. However, she has not yet showed analytically that the function is not differentiable at $x = 1$. She tried to explain why the function is not differentiable through the graphical representation. She reasoned that the function is not differentiable because it has a sharp turn at $x = 1$, then she implies there is no tangent line at that point. Two other students, RUB and YUN, did not attempt to answer the question. Nonetheless, they also succeeded in sketching the graph of the piecewise-defined function. They believed there was a possibility to generate ideas by sketching the graph. These findings indicate that these students are accustomed to making sense of the differentiability through a graphical representation. When they have difficulty understanding the problem, they choose to sketch the graph to help them generate mathematical ideas that can hopefully guide them to solve the problem.

In this study, most students reached the procept level when solving the differentiability problem. Few students were unable to perform proceptual thinking. It because the problem requires students to be flexible in processing and using or producing mathematical concepts related to differentiability, such as algebraic manipulation, a piecewise-defined function, the continuity of a function at a point, the existence of the limit, the left-hand limit, the right-hand limit, and differentiability (see Figure 23). However, most students (twenty two of 25 or around 88%) succeed in the graphical representation of the problem, including all students who failed in proceptual thinking. Using DCA based on the TWM framework in the learning process familiarized them with making sense of the mathematical problem through a graphical representation. They even believed sketching the graph could aid them in constructing the meaning of the problem, and hopefully, they could solve it correctly. Even so, there were still three students who experienced difficulties in producing graphical representations. They were students who succeeded in performing proceptual thinking. As a result, success in proceptual thinking does not imply success in graphical representation.

Prior knowledge, skills, and motivations of students enrolled in calculus classes vary. The average student may need to learn how to switch from one representation to another, which seems to be a trait of flexible thinkers.Some students could shift from one representation to another, but not the other way around. Successful students use adaptable strategies to generate new facts from old ones, resulting in an internal feedback loop that serves as an independent knowledge generator. If they realize they can develop more ideas independently, students who perform well tend to assume they do not need to remember as much. This will further alleviate mental strain. In contrast, failing students who strive to infer facts may use a creative but more convoluted route that requires enormous effort to be successful. Even most individuals revert to more extensive procedural calculations, which pose significant cognitive barriers. Lecturers must include multiple representations of the derivatives to develop students' conceptual understanding in the future (Prihandhika et al., 2022).

A solution with promise would provide students with access to more efficient methods without increasing cognitive hurdles. This may necessitate using the most appropriate technologies for the job. Using dynamic mathematics software, students can actively develop their mathematical understanding (Dahal, 2022). This study has shown that the DCA based on the TWM framework can be used as a generic organizer to build students' perceptual meaning of continuity and differentiability, as well as their proceptual thinking and graphical representations. The result of this study confirms that DCA based on the TWM as a generic organizer can contribute to students' proceptual thinking outcomes while learning differentiability. In this study, many students (76%) succeeded in performing proceptual thinking when solving the differentiability problem. Also, twenty-two of 25 students (or about 88%) had no limitations in the graphical representations of the given problem. These findings were consistent with those reported in Tall's (1986) study, which also discovered that students' capacity to sketch gradients of supplied graphs considerably increased and that their conceptualizations were extended while learning through a computer program. A robust computer environment that allows students to apply their mental concepts may enable them to avoid some tiny obstacles that hamper future progress.

CONCLUSIONS

Since the beginning of the learning process, using DCA based on the TWM framework has enabled students to verify the continuity of a function at a particular point by pulling it flat. Students were exposed to non-examples of differentiability problems and discovered that there is no tangent line when it is precisely located at a sharp turning point. The DCA data afterward confirmed that the slope of the tangent line is undefined. Thus, they can deduce that continuity does not imply differentiability. Most students in this study solved the differentiability problem at the procept level. Also, most students (about 88%, or twenty-two of 25) had no limitation in the graphical representation. This study concluded that the DCA based on the TWM framework could be used as a generic organizer to build students' perceptual meaning of continuity and differentiability, as well as their proceptual thinking and graphical representation.

This case study involved only 25 students in a calculus class as research respondents. Thus, the researchers suggested further research with a bigger sample size for the possibility of reaching the generalization of the research results. Furthermore, using DCA as a generic organizer based on the TWM framework has the potential to be developed and investigated more on other calculus topics, such as the basic differentiation rules, the tangent line problems, limits, and integrals. Thus, the extent to which the use of DCA as a generic organization based on the TWM framework can facilitate students in making sense the perception of various calculus topics (embodied), gradual exploration in the proceptual world, and ultimately being able to understand these topics symbolically (procept) can be further studied holistically.

Acknowledgments

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The Problem Corner

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The Purpose of **The Problem Corner** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem its solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Greetings, fellow problem solvers!

I am delighted to announce that I have received solutions to both Problem 14 and Problem 15. I am thrilled to report that all of them were not only correct but also truly fascinating and innovative. By showcasing what I consider to be the most outstanding solutions, my primary goal is to enrich and elevate the mathematical understanding of our global community.

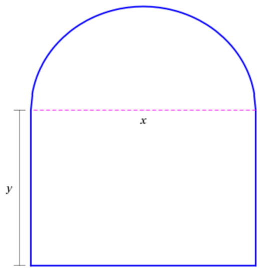
Solutions to **Problems** from the Previous Issue.

Interesting "Optimal configuration" problem.

Problem 14

Proposed by Ivan Retamoso, BMCC, USA.

Let's imagine a scenario where a corral is being enclosed using 130 ft of fencing. The corral is in the shape of a rectangle, and it has a semicircle attached to one of its sides. The diameter of the semicircle aligns with the length of the rectangle, as depicted in the figure provided.

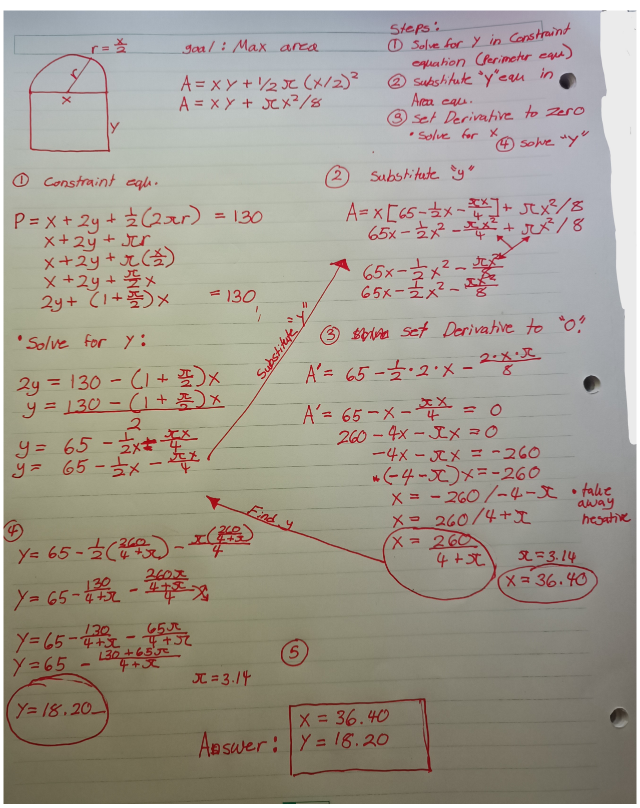


Determine the values of x and y that will result in the corral having the largest possible area.

Solution to problem 14

By Jack Powell, Borough of Manhattan Community College, USA.

In this initial solution, our solver succinctly presents the objective function and expertly transforms it into a single-variable expression using the constraint equation. Moreover, the application of calculus plays a pivotal role as our solver skillfully sets the derivative of the objective function to zero and proceeds to resolve for the variable.

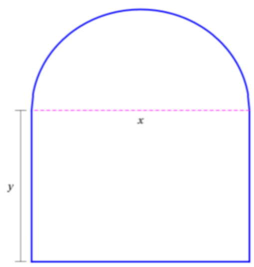


Second Solution to problem 14

By Aradhana Kumari, Borough of Manhattan Community College, USA.

This alternative approach follows a similar strategy, but with a meticulous attention to detail, ensuring no steps are skipped. Our solver presents the solutions in exact form, offering valuable insights into the structure of the final solution. In other words, it reveals the precise proportions of the dimensions x and y that lead to the maximum area of the corral. This level of accuracy enables a better understanding of the corral's optimal configuration.

Solution: Consider the below diagram



As per question we have

Area of the Corral = Area of the rectangle + Area of the Semicircle

$$= \quad xy \quad + \quad \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$$

Area = $xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$ ......(1)

Perimeter of the Corral = $x + 2y + \frac{1}{2}2\pi\left(\frac{x}{2}\right) = 130$

Hence, we have

$x + 2y + \pi\left(\frac{x}{2}\right) = 130$

$y = 65 - \frac{x}{2} - \pi\left(\frac{x}{4}\right) = 65 - x\left(\frac{2+\pi}{4}\right)$ ......(2)

Substituting the value of $y = 65 - x\left(\frac{2+\pi}{4}\right)$ in equation given by (1) we get

Area = $xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$

$= x\left\{65 - x\left(\frac{2+\pi}{4}\right)\right\} + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$

$$= 65\,x - x^2\left(\frac{2+\pi}{4}\right) + \pi\,\frac{x^2}{8}$$

$$= -x^2\,(\frac{4+\pi}{8}) + 65\,x$$

Notice: Area above is a function of x and it represents parabola opening downwards. The vertex of the parabola is the point of maxima. The x-coordinate of the vertex of the parabola is $\frac{-65}{-2\left(\frac{4+\pi}{8}\right)}$ $= \frac{65}{2\left(\frac{4+\pi}{8}\right)} =$

$\frac{65\times4}{4+\pi} = \frac{260}{4+\pi}$

(The general equation of a parabola is given as $Ax^2 + \mathrm{B}\,x + \mathrm{C} = 0$, the x-coordinate of the vertex is given by ($\frac{-B}{2A}$)).

Substituting the value of x in the equation given by (2) we get

$$y\ = 65\ -\ x\,\left(\frac{2+\pi}{4}\right)$$

$$= 65 - [\,\frac{260}{(4+\pi)} \times \left(\frac{2+\pi}{4}\right)] = \ 65 - [\,\frac{65}{(4+\pi)} \times (\,2 + \pi\,)\,]$$

$$= \frac{[65\,(4+\pi)]-[65\times(2+\pi\,)]}{(4+\pi)}$$

$$= \frac{(65\times4)+(65\times\pi)-(65\times2)-(65\times\pi)}{(4+\pi)}$$

$$= \frac{(65\times4)-(65\times2)}{(4+\pi)}$$

$$= \frac{(65\times2)}{(4+\pi)}$$

$$= \frac{(130)}{(4+\pi)}$$

The Corral will have the largest possible area when $x = \frac{260}{(4+\pi)}$ and $y = \frac{130}{(4+\pi)}$

Tricky algebra problem.

Problem 15

Proposed by Ivan Retamoso, BMCC, USA.

x, y, and z are real numbers such that $x + y + z = 17$ and $\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} = \frac{4}{15}$ find the exact value of $\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$.

Solution to problem 15

By Aradhana Kumari, Borough of Manhattan Community College, USA.

Initially, we encounter a challenging system of equations with two equations and three variables, seemingly impossible to solve directly. However, using an ingenious algebraic trick, our solver managed to compute the value of the desired mathematical expression. I invite you to witness this remarkable solution for yourself.

Solution: Consider the equation given in the problem

$$\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} = \frac{4}{15}$$

Multiply above equation by $(x + y + z)$ we get

$$\left\{\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} = \frac{4}{15}\right\} \times \ (x + y + z)$$

$$\frac{x+y+z}{x+y} + \frac{x+y+z}{y+z} + \frac{x+y+z}{z+x} = \frac{4\,(x+y+z)}{15}$$

After rearranging the terms, we get

$$\frac{x+y}{x+y} + \frac{z}{x+y} + \frac{y+z}{y+z} + \frac{x}{y+z} + \frac{z+x}{z+x} + \frac{y}{z+x} = \frac{4\,(x+y+z)}{15}$$

$$1 + \frac{z}{x+y} + 1 + \frac{x}{y+z} + 1 + \frac{y}{z+x} = \frac{4\,(x+y+z)}{15}$$

$$3 + \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} = \frac{4\,(x+y+z)}{15}$$

$$\frac{z}{x+y}+\frac{x}{y+z}+\frac{y}{z+x}=\frac{4\,(x+y+z)}{15}-3$$

Substituting the value of $x+y+z=17$ in the above equation and rearranging the terms we get

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}=\frac{4\times 17}{15}-3$$

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}=\frac{68}{15}-3=\frac{68-45}{15}=\frac{23}{15}$$

hence

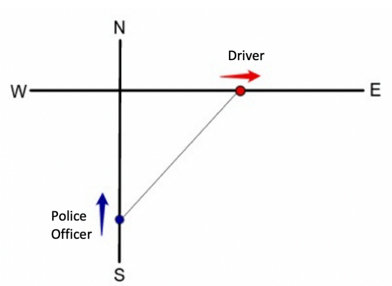
$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}=\frac{23}{15}$$

Dear fellow problem solvers,

I trust that tackling problems 14 and 15 not only brought you enjoyment but also provided valuable insights. Now, let's proceed to the next two problems to continue this rewarding journey of exploration and learning.

Problem 16

Proposed by Ivan Retamoso, BMCC, USA.



Let's consider a situation where a police officer is situated $\frac{1}{2}$ mile to the south of an intersection. This officer is driving northwards towards the intersection at a speed of 35 mph. At the exact same time, there is another car located $\frac{1}{2}$ mile to the east of the intersection, and it is moving eastward, away from the intersection.

a) Let's assume that the officer's radar gun displays a speed of $20\ mph$ when aimed at the other car. This reading indicates that the straight-line distance between the officer and the other car is increasing at a rate of $20\ mph$. What, then, is the speed of the other car?
b) Now, let's consider a different scenario where the officer's radar gun displays $-20\ mph$ instead. This indicates that the straight-line distance between the officer and the other car is decreasing at a rate of $20\ mph$. What is the speed of the other car in this situation?

Note: Round yours answers to three decimals places.

Problem 17

Proposed by Christopher Ingrassia, Kingsborough Community College (CUNY)

Brooklyn, NY, USA

Suppose $n \times n$ matrix A and $n \times 1$ vector x are defined as follows:

$$A_{i,j} = \begin{cases} 1, & i \geq j \\ 0, & otherwise \end{cases}$$

$$x_i = 1$$

Describe, in words, the vector $A^k x$, where $k \geq 0$.

Find an expression for the quantity $x^T A^k x$ in terms of n and k (x^T is the transpose of vector x).

Problem-solving: Growth of Students' Mathematical Understanding in Producing Original Solutions

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Abstract: Students' deep understanding of problem-solving can stimulate the presence of original solutions. This statement led this present study to explore the growth of students' mathematical understanding in producing original solutions through problem-solving. Fifty-five students, from two different junior high schools, are solving a two-dimensional figures problem. Students who produce original solutions are interviewed to investigate their growth in mathematical understanding when generating their solutions. The original solution indicates that the answer fulfills the three aspects: different, unique, and correct. The study observes students' activities, both 'acting' and 'expressing', which refers to layers of understanding of Pirie and Kieren's Model. 'Acting' and 'expressing' were observed to investigate the movement of students' understanding in solving-problem. Students whose understanding grows to the layers of 'image making' and 'image having' will come up with original ideas. The ideas become more complex as students' understanding grows to the 'property noticing' layer. Besides, the original ideas combined with conceptual and procedural knowledge can formally support the presence of an original solution. Students produce original solutions when their understanding has reached the layers of 'formalizing' and 'observing'.

Keywords: growth, mathematical understanding, original idea, original solution, problem-solving

INTRODUCTION

Problem-solving has been a long-standing concern for both learning and research in mathematics education (Hidayah, Sa'dijah, Subanji, & Sudirman, 2020). Previous studies have shown that mathematical problem-solving activities can encourage a deeper and more meaningful understanding (Kotsopoulos & Lee, 2012; Plaxco & Wawro, 2015). Students are given the opportunity to implement their knowledge in problem-solving (Li et al., 2020). In fact, the problem faced by the students requires the implementation of knowledge in new situations as well as high-level thinking skills (Kotsopoulos & Lee, 2012). The various knowledge combinations and the involvement of understanding are crucial in this skill to support students' mathematical competence (Edo & Tasik, 2022; Spencer, Fuchs, & Fuchs, 2019; Stadler, Herborn, Mustafić, & Greiff, 2020; Torbeyns, Schneider, Xin, & Siegler, 2015). Therefore, mathematics problem-solving activities not only encourage students to apply their understanding but also have the potential to form a deep understanding to improve their mathematical competency.

Understanding measures the quality and quantity of the relationship between new knowledge and prior knowledge that is already owned. In other words, the more relationships in the knowledge network, the better the understanding (Walle, Karp, & Bay-Williams, 2019). The level of mathematical understanding is determined by the number and strength of the network of connections between mathematical concepts, procedures, and facts. Understanding can be comprehended thoroughly as long as it is associated with a network of connections that are more numerous or stronger (Hiebert & Carpenter, 1992). Furthermore, this deep mathematical understanding can encourage flexibility of thinking and connection of ideas in problem-solving (Martínez-Planell, Trigueros Gaismán, & McGee, 2017; Musgrave & Carlson, 2017; Weber, 2009). The more ideas linked in problem-solving, the more likely it is to produce original solutions (Agnoli, Franchin, Rubaltelli, & Corazza, 2015).

Original solutions in problem-solving emphasize unique and different ideas (Sidi, Torgovitsky, Soibelman, Miron-Spektor, & Ackerman, 2020). The originality of the solution can be assessed based on objective and subjective perspectives. An original solution objectively refers to various ideas considered from a whole subject in a specific group. Meanwhile, the subject's perspective emphasizes the assessor's point of view. The assessment of originality by the researcher, however, has progressed from subjective to objective, i.e., referring to the only subject in the group which has generated different ideas (Dumas & Dunbar, 2014; Mones & Massonnié, 2022). However, an original solution in solving a problem not only focuses on differences in ideas but also must prioritize the uniqueness and correctness of these ideas (Silver, 1997).

The original solutions do not appear suddenly but through a series of thought processes involving understanding (Munahefi, Kartono, Waluya, & Dwijanto, 2020; Wessels, 2014). Likewise, the results of the preliminary study by the researcher showed that students involved in understanding by linking knowledge to produce original solutions. Students also use various methods in solving problems, and one of these methods is original, resulting in a new, deeper understanding. This initial finding relates to understanding which is not a static learning point but an evolving mental activity (National Research Council, 2002). In other words, mathematical understanding is a dynamic process that shifts from informal actions to more formal abstractions, which can be observed through *acting* and *expressing* (Pirie & Kieren, 1994). It can be communicated through the students' representation in mathematical problem-solving activities (Quintanilla & Gallardo, 2022).

The theory of growth in mathematical understanding was first developed by Pirie and Kieren (1994). This theory explains eight potential layers that describe a person's level of understanding of a particular concept. These layers, from the innermost to the outermost circle, include *primitive knowing, image making, image having, property noticing, formalizing, observing, structuring,* and *inventising*. *Primitive knowing* is the initial point of understanding observed as the whole thing that has been known and done by the students. *Image making* and *image having* are related to creating new knowledge that is different from initial knowledge. However, *image making* involves triggers when creating new knowledge, whereas *image having* does not involve triggers to help understand images. After the image is owned and well understood, students are ready to connect prior and new knowledge at *property noticing* (Gulkilik, Moyer-Packenham, Ugurlu, & Yuruk, 2020; Pirie & Kieren, 1994). *Formalizing* is related to students' ability to use formal mathematical definitions or algorithms (Bobis & Way, 2018). In *observing*, students reflect and coordinate their formalization activities. For example, after formalizing the procedure, then do reasoning (Yao & Manouchehri, 2022). At *structuring*, students realize that a collection of theorems are interrelated and ask for verification through logical thinking. Then, students bring new understanding to create a new concept as an *inventising* achievement (Pirie & Kieren, 1994). The growth of understanding occurs through continuous reciprocating movements through layers of understanding, as illustrated in Figure 1.

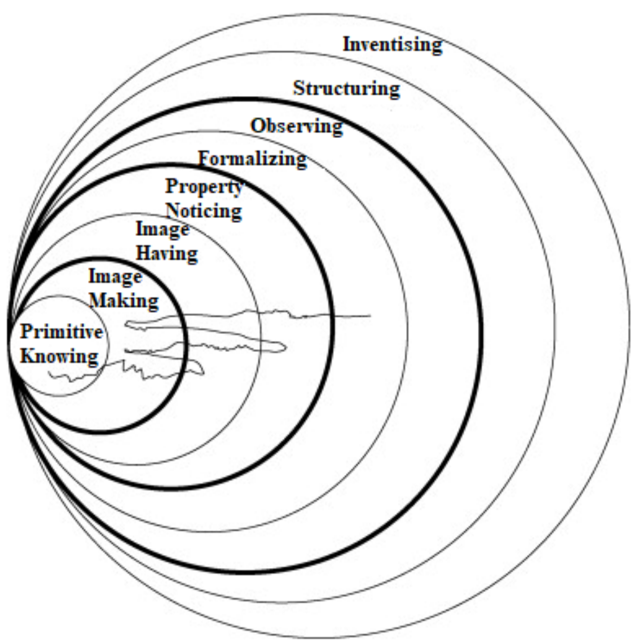

Figure 1: Growth in Mathematical Understanding by Pirie and Kieren (1994)

The eight circles in Figure 1 are depicted in a nested form which represents each layer that contains all the layers in it. Each layer, apart from *primitive knowing* and *inventising*, contains *acting* and *expressing,* which complement each other as activities that can be observed in each layer of understanding. *Acting* contains mental and physical activities, while *expressing* is related to conveying these activities to others and oneself. The pairs of *acting* and expressing can be observed in *image making*, *image having*, *property noticing*, *formalizing*, *observing*, and *structuring,* respectively called *image doing* and *image reviewing*, *image seeing* and *image saying*, *property predicting* and *property recording*, *method applying* and *method justifying*, *featuring identifying* and *featuring prescribing*, and *theorem conjecturing* and *theorem proving*. Pirie and Kieren proposed a model of growth in mathematical understanding with key features: *don't need boundaries, folding back,* and the complementarities of *acting* and *expressing* (Pirie & Kieren, 1994). Moreover, the study focuses on *acting* and *expressing* to observe the movement of the level of understanding.

Mathematical understanding can grow when a person learns new things (Gulkilik et al., 2020). Mathematical understanding also grows when students carry out mathematical problem-solving activities (Patmaniar, Amin, & Sulaiman, 2021). The formation of new understandings and a deeper understanding of mathematics can be facilitated through the assignment of problem-solving that emphasizes the productivity of ideas (Bajwa & Perry, 2021). Hence, the growth of students' mathematical understanding can be triggered through these tasks.

Previous studies have applied Pirie and Kieren's model to investigate the thinking process in solving mathematical problems. These studies focused on *don't need boundaries* (Rahayuningsih, Sa'dijah, Sukoriyanto, & Qohar, 2022), *folding back* (Patmaniar et al., 2021; Risley, Hodkowski, & Tzur, 2015), *primitive knowing* (Putri & Susiswo, 2020), and low-skilled students' understanding growth (Sengul & Yildiz, 2016). However, the studies showed its limitations in dealing with the theory of growth in mathematical understanding in problem-solving that focused on achieving an original solution, while such understanding is closely related to the presence of originality (Paulin, Roquet, Kenett, Savage, & Irish, 2020).

Pirie and Kieren's model has the potential to be the reference for observing mathematical growth and understanding in producing original solutions to problem-solving activities. Some people hold the view that mathematical understanding is a dynamic process that can build connected knowledge and flexible thinking (Martin & Towers, 2016), thus encouraging students to be able to generate new, unique, and useful ideas (Sitorus & Masrayati, 2016). Therefore, this study aims to explore the growth of students' mathematical understanding in producing original solutions to problem-solving activities.

METHOD

This research is an exploratory study with a qualitative approach. There were fifty-five students from two different Junior High Schools (JHS-A and JHS-B) in Malang-Indonesia as participants. Researchers prepared instruments in the form of worksheets and interview guidelines to obtain data on the growth of students' understanding of producing original solutions. The tasks sheet contained problems in the area of two-dimensional figures adapted from Siswono (2010). Students are asked to design any two-dimensional figures that represent a park that has exactly 1200 m^2 as shown in Figure 2.

Mr. Dani, the rural village head of Maju Jaya, plans to design a creative park covering an area of 1200 m^2. He needs others to help him make the design. Assist Mr. Dani based on the following conditions:

a. Create as many creative parks design as possible, and the length of the sides according to the known area!
b. Pat attention to the creative park shape that you think is the most unique one! Tell us in detail how you determine the length of the sides!

Figure 2: Problem-Solving Task

The results of problem-solving are analyzed based on originality in an objective perspective, namely the different ideas of a particular group of people (Dumas & Dunbar, 2014; Mones & Massonnié, 2022) and uniqueness and correctness in problem-solving (Silver, 1997). All student answers were classified based on the solution's types, characteristics, and accuracies, as shown in Table 1.

Types	Characteristics	Accuracies	Code	Category
Similar	Ordinary	Incorrect or Correct	SOI or SOC	Unoriginal
Different	Ordinary	Incorrect or Correct	DOI or DOC	Unoriginal
	Unique	Incorrect	DUI	Unoriginal
		Correct	DUC	Original

Table 1: Classification of Solutions in Problem-Solving

Based on Table 1, the researcher identified the problem-solving solutions of fifty-five participants shown in Figure 3.

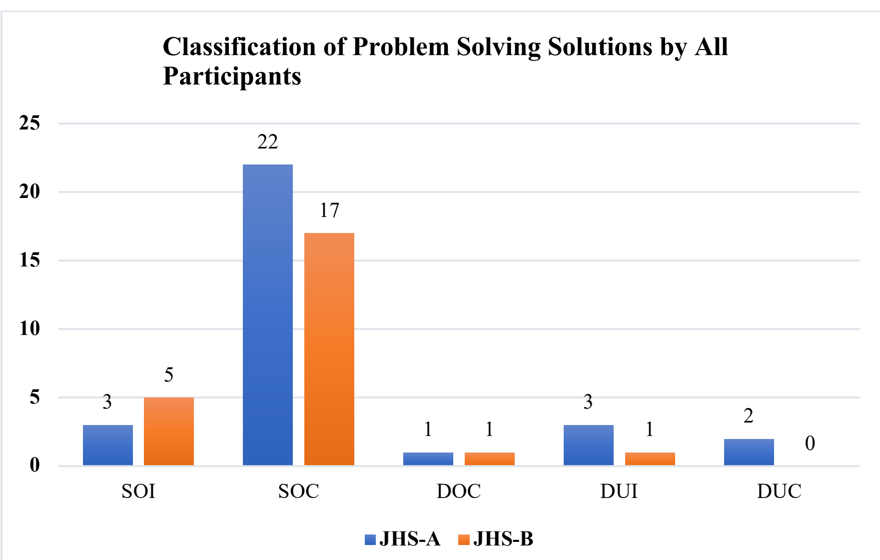

Figure 3: Frequency graph of Problem-Solving Solutions by Fifty-Five Participants

Figure 3 shows that two students provided original solutions from JHS-A. The researcher summarizes the many original solutions for the two selected students presented in Table 2.

No	School	Name	Gender	Number of solutions	
				Original	Unoriginal
1	JHS-A	A[1]	Female	1	4
2	JHS-A	A[2]	Male	1	1

Table 2: The Number of Solutions Generated by A[1] and A[2]

Table 2 shows that even though A[1] produces more solutions than A[2], both of them can produce one original solution. Furthermore, A[1] and A[2] were interviewed to observe the growth of mathematical understanding in producing original solutions. The interview guideline refers to the descriptors of each layer of mathematical understanding. The descriptor of each layer was developed from Pirie and Kieren's model presented in Table 3.

Mathematical Understanding Layers	Descriptors by Pirie and Kieren	Descriptors in This Research
Primitive knowing	Preliminary knowledge is needed to build certain concepts.	Initial knowledge is needed to solve the problem.
Image making	Creating a new image as a differentiator from previous knowledge through mental or physical activity.	Making new knowledge different from prior knowledge by involving triggers such as simple examples.
Image having	Understanding certain concepts without acting on objects.	Modifying knowledge to acquire new knowledge without involving triggers.
Property noticing	Combining images to form certain specific properties.	Combining knowledge to form certain specific properties.
Formalizing	Making generalizations and developing formal mathematical ideas.	Building problem-solving steps in accordance with formal mathematical procedures.
Observing	Thinking of the latest formal ideas and use them to create algorithms.	Reflecting knowledge to be applied in various problem-solving situations.
Structuring	Being aware of the interrelationships between theorems.	Linking between theorems by involving logical arguments as a form of verification in solving problems.
Inventising	Bringing new understanding to create new concepts.	Having a new understanding that can be known through a conclusion statement.

Table 3: Descriptors for Each Layer of Mathematical Understanding

Data were analyzed using data condensation, data display, and conclusion drawing (Miles, Huberman, & Saldana, 2014). Data condensation was done by providing the codes of *acting* and *expressing* on the achievement of each layer of students' mathematical understanding (Pirie & Kieren, 1994), as shown in Table 4.

Mathematical Understanding Layers	Observed Aspects			
	Acting	Acting Code	Expressing	Expressing Code
Primitive knowing	-	-	-	-
Image making	*Image doing*	*Ac-IM*	*Image reviewing*	*Ex-IM*
Image having	*Image seeing*	*Ac-IH*	*Image saying*	*Ex-IH*
Property noticing	*Property predicting*	*Ac-PN*	*Property recording*	*Ex-PN*
Formalizing	*Method applying*	*Ac-F*	*Method justifying*	*Ex-F*
Observing	*Featuring identifying*	*Ac-O*	*Featuring prescribing*	*Ex-O*
Structuring	*Theorem conjecturing*	*Ac-S*	*Theorem proving*	*Ex-S*
Inventising	-	-	-	-

Table 4: *Acting* and *Expressing* in The Mathematical Understanding Layers

Furthermore, data display was done by providing examples of the presence of *acting* and *expressing* at each level of student understanding. Data is also displayed visually to illustrate the presence of original solutions in the mathematical understanding layer. Finally, the researcher provides conclusions regarding the growth of students' mathematical understanding in producing original solutions through problem-solving.

During the data collection and analysis process, researchers conducted member checking and peer debriefing to obtain credible data (Nowell, Norris, White, & Moules, 2017). Member checking is done through interviews with students who have produced original solutions to clarify the results of student problem-solving. In-depth interviews are used to explore the growth of students' mathematical understanding in producing original solutions. Peer debriefing is carried out by researchers through discussions with colleagues of doctoral students and mathematics education lecturers other than the research team to get suggestions regarding the data that has been obtained.

RESULTS

Two students were chosen as research subjects, A[1] and A[2], because both produced original solutions and provided clear information on each *acting* and *expressing,* observed from each mathematical understanding layer achievement. The two subjects have different characteristics in the movement of the growth of mathematical understanding in producing original solutions, but both achieve original solutions at the same level of understanding.

The growth of A[1]'s understanding of producing original solutions

A[1] produces five solutions, and only one reflects originality. The original solution is a composite two-dimensional figure combining two rectangles, as seen in Figure 4 (E).

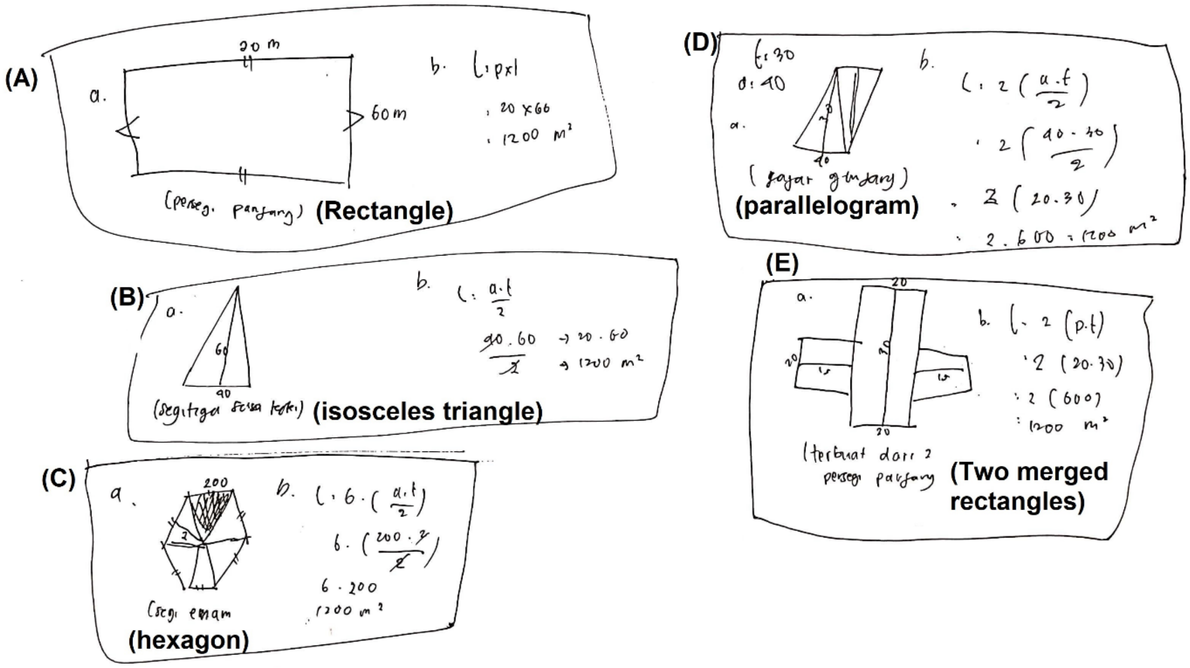

Figure 4: A[1]'s Problem-Solving Task Results

Figure 4 shows that most students in A[1]'s class created similar solutions, as shown in (A) and (B). Meanwhile, solution (C) reflects different solutions from all students and the idea can be said to be unique, but this solution is incorrect. Solution (D) is also different from all of the students' solutions in class A[1], but this idea is quite ordinary because parallelograms have been studied before. Solution (E) is a different solution from all student solutions in class A[1], the idea can be said to be unique, and conceptually the truth can be accepted mathematically. Therefore, this study distinguishes between original ideas and solutions. An original solution is obtained from an original idea equipped with the accuracy of the solution according to formal mathematical concepts.

Furthermore, the researcher conducted interviews with A[1] to explore her *primitive knowing*. A[1] had studied basic two-dimensional figures before and mentioned several examples, such as triangles, rectangles, parallelograms, trapezoids, and circles. However, she can only mention the formula of triangle and rectangle areas. A[1] said that the problem-solving task completed by A[1] was a new learning experience for her. A[1] stated that she never made composite two-dimensional figures of the same area.

In the early stages of solving the problem, A[1] represents a park by drawing a basic two-dimensional figure in the form of a rectangle (A) and a triangle (B) as a trigger for the presence of a composite two-dimensional figure. A[1] also gives the lengths of the sides of rectangles and triangles. Then, A[1] creates a composite two-dimensional figure which is a combination of triangles that form a hexagon, as shown in Figure 2 part (C) ***(Ac-IM)***. A[1] said that this idea came from combining triangles to become a composite two-dimensional figure ***(Ex-IM)***. This *acting* and *expressing* reflect *image making*, namely making new knowledge that is different from *primitive knowing*, so as to produce original ideas in solving problems. A[1] draws a composite two-dimensional figure, which is a hexagon, by combining six triangles ***(Ac-IH)***. A[1] explained that this aims to make it easier for him to divide the area of each forming triangle ***(Ex-IH)***. This *acting* and *expressing* reflects how A[1] 's knowledge grows into *image having*. A[1] also said that he chose as many as six triangles to make it easier to divide 1200 into six parts for each area of the triangle ***(Ac-PN)***. Now A[1] has new knowledge that in creating a composite two-dimensional figure, it is necessary to pay attention to many trigger basic two-dimensional figures to make it easier to determine each area and side, so this is an achievement *property noticing*. A[1] states that if the area of each basic two-dimensional figure has been obtained, then A[1] can easily determine the length of the sides in the form of integers ***(Ex-PN)***.

In *formalizing*, A[1] uses the formula for the area of a triangle, as shown in Figure 2 part (C), to be able to determine the length of the base and height of the triangle. A[1] uses a similar method to generate the lengths of the sides of other two-dimensional figures ***(Ac-F)***. A[1] states that the length of the sides of the composite two-dimensional figure can be found by using the formula of the area of each basic two-dimensional figure ***(Ex-F)***.

A[1] always involves a trigger two-dimensional figure to produce a composite two-dimensional figure. A[1] combines triangles to get a hexagon or parallelogram and also combines rectangles to form other composite two-dimensional figures ***(Ac-O)***. A[1] conveys the steps for determining the sides of a composite two-dimensional figure, namely: drawing a composite two-dimensional figure from the same type of basic two-dimensional figure, dividing the area of the two-dimensional figure by the number of the basic two-dimensional figure, and using the formula for the area of the basic two-dimensional figure to determine the length of the sides of the composite two-dimensional figure ***(Ex-O)***. A[1] has made a schematic in her knowledge that to produce a composite two-dimensional figure, A[1] needs to combine several identic basic two-dimensional figures. This solution step is used to come up with an original solution. It means that A[1] has reached *observing*, even though A[1] always needs triggers to draw a composite two-dimensional figure.

Whenever A[1] needs a trigger to produce an original solution, then A[1] returns to *image making* and expands understanding. This phenomenon of returning to a deeper layer of understanding is referred to as *folding back*. In the end, at *formalizing* and *observing*, A[1] produced an original solution: a composite two-dimensional figure of two identical rectangles with the correct side length. The researcher summarizes the results of observations on *acting* and *expressing* by A[1], which are presented in Table 5.

Mathematical Understanding Layers	*Acting*	*Expressing*
Image Making	The student draws a triangle and a rectangle as triggers, then draws other composite two-dimensional figures built by combining multiple triangles or rectangles.	The student stated that she could create a composite two-dimensional figure by combining identical basic two-dimensional figures.
Image Having	The student draws composite two-dimensional figures by combining congruent basic two-dimensional figures to make it easier to divide the area.	The student stated that if all basic two-dimensional figures that form composite two-dimensional figures are the same, then it is easy to determine the area and length of each side.
Property Noticing	The student pays attention to the many congruent basic two-dimensional figures to make it easier to determine the sides' area and length.	The student stated that each basic two-dimensional figure's area must be considered to produce the length of the sides in the form of integers.
Formalizing	The student determines the length of the sides of the composite shape by writing the area formula of the basic two-dimensional figures.	The student stated that the sides of a composite two-dimensional figure could be found by paying attention to the area formula of each basic two-dimensional figure.
Observing	The student uses a similar pattern of the solution steps to create any composite two-dimensional figures and determine the lengths of the sides.	The student stated that the steps for determining the sides of a composite two-dimensional figure, namely: drawing a composite two-dimensional figure from several identical basic two-dimensional figures, dividing the area that is known by the number of basic two-dimensional figures, and using the area formula of the basic two-dimensional figure to determine the length of the sides of the composite two-dimensional figure.

Table 5: A[1]'s *acting* and *expressing* in producing original solutions

The growth of A[2]'s understanding of producing original solutions

A[2] produces two solutions, and only one reflects originality, which creates three rectangles combinations, as shown in Figure 5 (F).

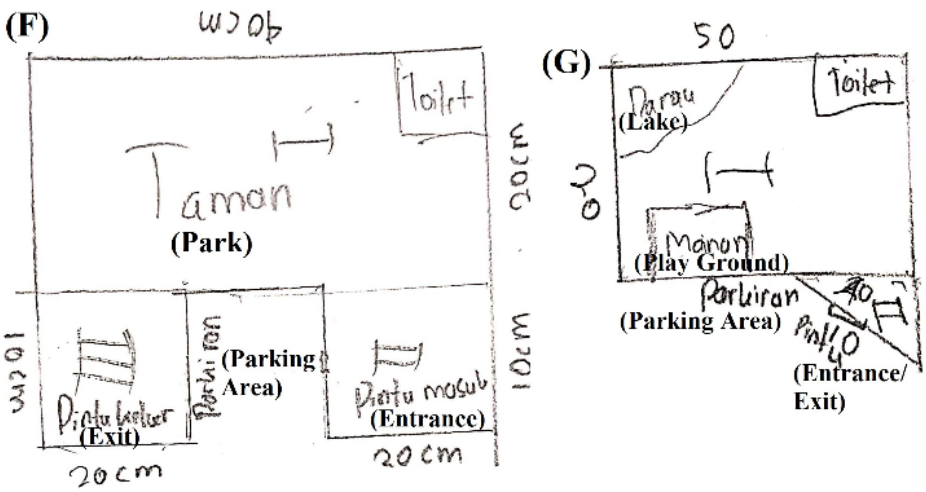

Figure 5: A[2]'s Problem-Solving Task Results

Figure 5 shows that A[2] creates two composite two two-dimensional figures. Figure 5 (F) includes three different rectangles. Figure 5 (G) includes rectangles and triangles, but the length of the base and height of the triangle does not fit the right triangle concept.

Through interviews, the researcher explores A[2] 's understanding of producing original solutions. A[2] already has *primitive knowing,* which is indicated by the statement that A[2] has studied basic two-dimensional figures such as trapezoids, parallelograms, rectangles, squares, triangles, and circles at school. A[2] has also studied pentagons and hexagons with his parents at home. A[2] can state the formula for the area of a basic two-dimensional figure and states that it is difficult to remember the formula for the area of a trapezoid, pentagon, hexagon, and circle. A[2] is used to solve the problem of determining the area of a basic two-dimensional figure by knowing the length of the sides. On the other hand, solving the composite two-dimensional figure problem is a new learning experience for him.

Based on the results of solutions by A[2], as shown in Figure 5, it shows that A[2] has created new knowledge that is different from previous learning experiences. Without using a trigger, A[2] can represent the garden by creating composite two-dimensional figures, as explained in Figure 5 (F) and (G) ***(Ac-IH)***. A[2] also states that the shape of a garden can be a combination of squares, rectangles, or triangles ***(Ex-IH)***. This *acting* and *expressing* indicate that A[2] has reached *image having* in the growth of his understanding. A[2] does not need a trigger to create a composite two-dimensional figure. The researcher conducted further interviews related to A[2] 's understanding of determining the length of the sides of the composite two-dimensional figure he had made. The following is an excerpt from the interview of the researcher (R) and A[2].

Interview Excerpt (1)

R : *Do you depend on formulas when solving problems?*

A[2] : *No, I don't. The problem is that the area of the garden is 1200* m^2*. So I determine the area of each basic flat shape (points to the division of three parts of the flat shape area) is 200, 200, and 800.*

Interview Excerpt (2)

R : *How can you determine the side length of the composite plane shape?*

A[2] : *I divide the area into several parts, then I determine the length of each side (points to the length of the sides).*

Interview Excerpt (1) showed that A[2] determines the length of the sides of the composite two-dimensional figure, starting with dividing the area into three parts, namely for rectangles I, II, and III, as shown in Figure 5 part (F). In Figure 5 (F), this division is based on the proportions of the size of basic two-dimensional figures made, namely rectangle I is given an area of 800, while rectangles II and III are each given an area of 200 ***(Ac-PN)***. A[2] states that A[2] must consider the division of this area based on the proportion of the size of rectangle I, which is larger than rectangles II and III ***(Ex-PN)***. This shows that A[2]'s understanding has reached *property noticing*. A[2] combines knowledge regarding image proportions and their area to get each basic two-dimensional figure's area.

The achievement of the *formalizing* layer can be seen in the Interview Excerpt (2). A[2] determines the length of the sides of the composite two-dimensional figure by determining the length of the sides of each rectangle by multiplying two integers. The concept of the area of a rectangle seems to stick to A[2] 's memory, so A[2] can quickly determine the length of the sides of the rectangle without writing down the formula ***(Ac-F)***. A[2] says that the area of a rectangle can be found by multiplying the lengths of the two adjacent sides ***(Ex-F)***. However, it appears that there are disproportionate side lengths, as shown in Figure 5 (F). The researcher conducted further interviews with A[2], as disclosed in the following interview excerpt.

Interview Excerpt (3)

R : What is the length that you determine from this side? Is it 40? (points to the length of the rectangle I). What about this? (points to rectangles II and III)

A[2] : Yes 40. The length of this side is 20 because it is a rectangle, a square, and a square. Oh, sorry, these are all rectangles. If so, I change the length to 10 (while writing). The length of the side of rectangle I is 40 (points to the length of rectangle I), if the length of side 20 plus 20 (points to the length of rectangles II and III), then the length of both is the same as rectangle I. So I change the length.

Based on the Interview Excerpt (3), A[2] realized that the side lengths that had been made were disproportionate, as shown in Figure 5 (F). Rectangles II and III each have a length of 20 and are separated by some sides of the rectangle I. A[2] then revised it by giving the lengths of the sides of rectangles II and III, respectively, the length and width were originally 20 and 10 to 10 and 20. The revision of the idea that has been carried out by A[2] forms a new understanding by A[2];

namely, A[2] must pay more attention to the details between the given figure and the given side length so that it looks more proportional.

The researcher summarizes the results of observations on *acting* and *expressing* A[2], which are presented in Table 6.

Mathematical Understanding Layers	***Acting***	***Expressing***
Image Having	The student draws composite two-dimensional figures by combining several types of basic two-dimensional figures without involving triggers.	The student stated that basic two-dimensional figures that are made could be any basic two-dimensional figures.
Property Noticing	The student pays attention to the proportion of the size of each basic two-dimensional figure to make it easier to determine the area and length of each side.	The student stated that they had to consider the proportion of the size of each of the basic two-dimensional figures and the area determined.
Formalizing	The student directly determines the length of the composite two-dimensional figures' sides without writing down the formula for the area of the basic two-dimensional figures.	Students stated that they did not depend on the formula, but it had been used to determine the side length of basic two-dimensional figures.
Observing	The student uses the solution steps in a similar pattern to create any composite two-dimensional figures and determine the lengths of the sides.	The student stated the steps for determining the sides of a composite two-dimensional figure, namely: drawing a composite two-dimensional figure, giving the area of each basic two-dimensional figure based on the proportions of the figure and the known area, using the area formula of the basic two-dimensional figure to determine the length of the sides of the composite two-dimensional figure.

Table 6: A[2]'s *Acting* and *Expressing* in Producing Original Solutions

Furthermore, the researcher conducted further interviews to explore A[2]'s idea of making a composite two-dimensional figure. A[2] states that A[2] can directly create a composite two-dimensional figure without involving any triggers ***(Ac-O)***. A[2] states that similar solving steps are used to create other composite two-dimensional figures along with determining the length of the sides, namely: drawing the composite two-dimensional figure, giving the area of each basic two-dimensional figure based on the proportions of the drawing and the known area, using the area procedure of the basic two-dimensional figure to determine the lengths of the sides of the composite two-dimensional figure ***(Ex-O)***. The completion steps that have been used by A[2] make it easier for him to solve the problem given by the researcher. However, A[2] does not pay attention to the concept of triangles in this second solution but only emphasizes procedural knowledge in determining the length of a triangle's sides. As a result, the length of the height and base of the triangle in Figure 5 part (G) does not match the right triangle made by A[2]. In accordance with the right triangle in Figure 5 part (G), conceptually, if the base length is 10 and the height is 40, it will form an acute triangle, not a right triangle, as described in A[2]. Thus, A[2]'s understanding can be said to achieve *observing* but not achieving *structuring*. Even so, A[2] stated that he had to be more careful in making a composite two-dimensional figure and the length of their sides by paying attention to the properties of the basic two-dimensional figure.

The achievement of original solutions in the understanding layer

This observation, namely *acting* and *expressing*, clarifies A[1] and A[2] achievement at each layer. The movements between layers indicate a growth in understanding from the emergence of original ideas to original solutions. Although their growth movements are different, both produce original ideas and solutions at the same level of understanding. The relationship between achieving the original solution in problem-solving and the understanding layer is presented in Figure 6.

Figure 6 explains that *primitive knowing* provides primitive ideas to support original ideas. The original ideas emerge as the student's understanding grows to *image making*, *image having*, and *property noticing*. When students have reached the *formalizing* layer, they formally involve a mathematical procedure in solving the problem to activate the original solution. Meanwhile, when they reach *observing*, they will become more consistent in using the same procedures to come up with another solution. Therefore, students' growth in understanding begins with primitive ideas that develop into original ones that are different from what they produce at *primitive knowing*. The original ideas grow into original solutions when students' understanding grows to reach *formalizing* and *observing*.

Moreover, A[1] and A[2] understanding does not reach the *structuring*. They do not relate to concepts of the basic two-dimensional figure and the composite two-dimensional figure. A[1] does not pay attention to the triangle conditions that can form a hexagon. A[1] Arranging six triangles into a hexagon is an original idea. However, if A[1] combines six identical triangles with a base length of 200 and a height of 2, it will not form a regular hexagon with a side length of 200. Therefore, A[1] cannot reach *structuring*. Similarly, A[2] only focuses on the broad procedures of the basic two-dimensional figure. A[2] does not notice that a triangle with a base length of 10 and a height of 40 cannot form a right triangle.

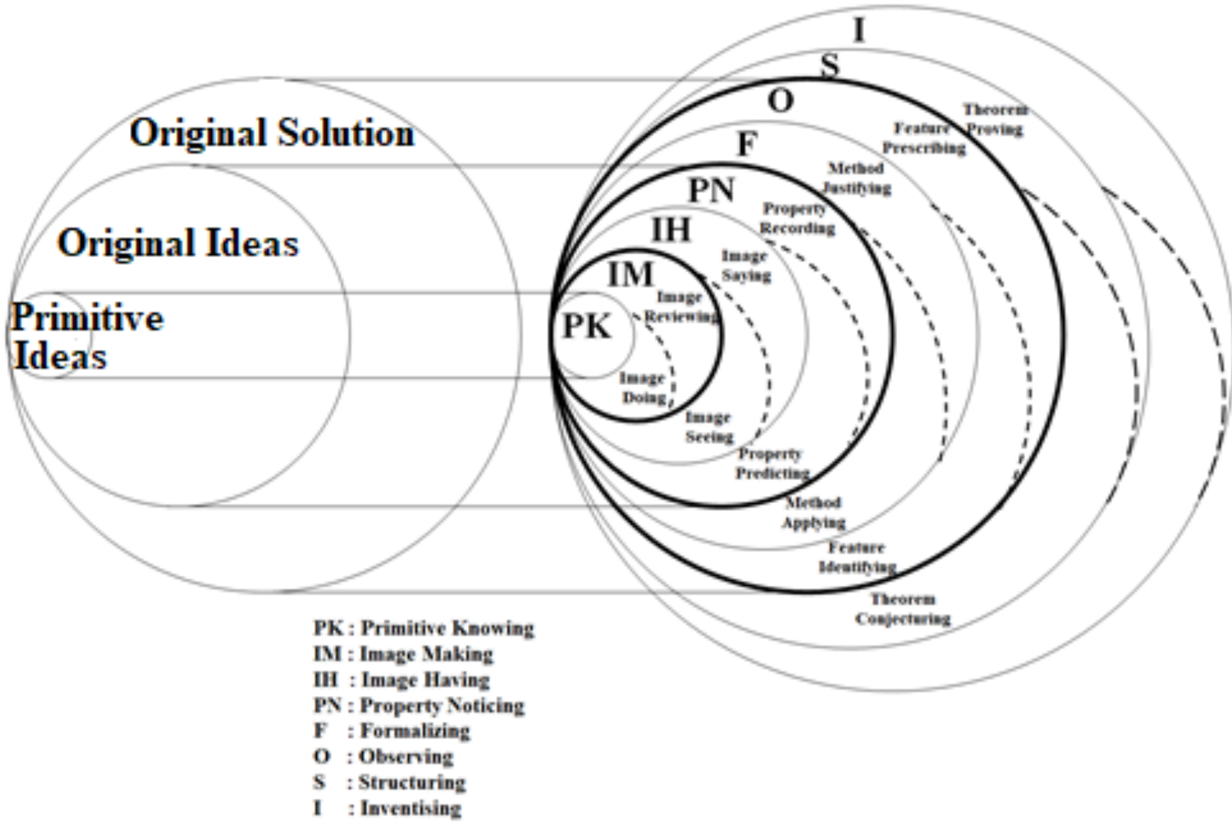

Figure 6: The Relationship between the Achievement of Original Solutions and Understanding Layers in Problem-Solving

DISCUSSION

In the activity of solving two-dimensional figure problems, students can generate original ideas such as forming new composite two-dimensional figures from a combination of rectangles and triangles. The composite two-dimensional figure formed by students is evidence of the productivity of ideas that are unique and appear to be different from other students. According to Dumas & Dunbar (2014), many ideas generate from problem-solving activities. A different idea from the ideas of all subjects in a particular group can be referred to as the original idea. (Sidi et al., 2020) stated that original ideas must emphasize their uniqueness. However, the original idea does not merely become original solutions in solving problems. It has to be applied equally with the procedural and conceptual understandings to produce one. According to (Silver, 1997), original solutions in problem-solving are not only emphasized original ideas but also must pay attention to the accuracy of the solution. Therefore, an original idea can grow into an original solution balanced by the accuracy of the solution according to the context of the problem being solved.

The solution's accuracy is related to one's understanding capacity in connecting mathematical concepts and procedures in solving problems. The accuracy of the solution is not only assessed based on the uniqueness of the composite two-dimensional figures made by students but also must pay attention to the length of the sides of the composite two-dimensional figure that is made so that the area matches the problem being solved. Likewise, a student creates a regular hexagon from a combination of six isosceles triangles with a base length and height of 200 and 2. It certainly does not allow the area of a regular hexagon with a side length of 200 to have an area equal to the sum of the area of six triangles with a base length of 200 and high of 2. Thus, the appearance of unique ideas has to balance with an excellent mathematical understanding. It is relevant to the opinion of the researchers that without good understanding, one cannot make the right decision (Ellis, Ozgur, Kulow, Dogan, & Amidon, 2016; Rupalestari, Juandi, & Jupri, 2021). A deep mathematical understanding is necessary to solve problems by connecting mathematical concepts, which aims to support the productivity of new ideas in problem-solving (Beaty et al., 2020; Paulin et al., 2020; Syahrin, Dawud, Suwignyo, & Priyatni, 2019; Xu, Geng, & Wang, 2022). Not only connecting concepts but also relating concepts with the mathematical procedure to gain mathematical understanding. Legesse, Luneta, & Ejigu (2020) mentioned that the involvement of conceptual and procedural understandings in problem-solving is interdependent, which illustrates the linkage between mathematical concepts and problem-solving procedures. Therefore, mathematical understanding plays a vital role in producing the accuracy of solutions to problem-solving.

Moreover, mathematical understanding highly supports problem-solving. Still, each layer-achieved skill in students' mathematical understanding depends on their *primitive knowing*, in which this layer of understanding can significantly impact the original solution achievement. The area formula for the basic two-dimensional figure, such as rectangles and triangles, is attached to the students to produce a composite two-dimensional figure. Previous studies stated that the critical point for this to succeed is *primitive knowing* (Putri & Susiswo, 2020). However, it should be noted that *primitive knowing* is not the lowest level of understanding but rather background knowledge as an initial basis for the growth of mathematical understanding (Pirie & Kieren, 1994). This background knowledge students obtain from their previous learning experience can be used as the basis of growth in mathematical understanding (MacDonald, 2022). It can also benefit the students to solve problems by emphasizing constructive ideas. So, it can deepen their understanding (Husband, 2021), which they can use to produce the original ideas (Auliasari, Sujadi, & Siswanto, 2021; Beaty et al., 2020; Kao, 2022; Lee & Therriault, 2013).

The original ideas are present as the students form a composite two-dimensional figure from identical and different basic two-dimensional figures. It is where *image making* or *image having* achieved, that is, creating different knowledge from *primitive knowing*. New knowledge occurs when one performs mental or physical actions, creating a new image while processing it (Gulkilik et al., 2020). In this state, students engage in activities that help them to develop mathematical ideas through certain representations to get an idea of a concept. This image background is not merely visual but a verbally expressed idea or action (Martin & Towers, 2016). Similarly, students who reach the initial level of abstraction do not need a trigger to understand the image (Bobis & Way, 2018). Original ideas are also present when the students watch several basic two-dimensional figures that are identical or not to make them easier to figure out each area and the length of the side. It relates to a person's ability to manipulate or combine the image's aspects to form specific characteristics (Pirie & Kieren, 1994). As students reach the image, they will be ready to connect and differentiate between previous and present understandings (Bobis & Way, 2018).

Besides, original ideas must balance with excellent conceptual and procedural knowledge to produce original solutions to a problem. Many original ideas appear when the students are asked to design the park. Nevertheless, students have applied the procedure of the basic two-dimensional figure to achieve a *formalizing* layer of understanding. Accordingly, students might use formal mathematical procedures and coordinate formal activities. Similarly, in *observing*, students could predict another solution concerning the procedure being formed (Pirie & Kieren, 1994). It will trigger the conceptual and procedural understandings to solve the problem and produce the right solution. Therefore, the presence of original solutions can result from the presence of original ideas balanced by good procedural and conceptual knowledge.

Students' mathematical understanding to produce original solutions can shift and grow. A[2] does not require a trigger to produce them, but he can directly achieve *image having*. This is due to the involvement of the *don't need boundaries* phenomenon, which grows the students' understanding not to fasten to the previous layer. Rahayuningsih et al. (2022) mention that this phenomenon can be between *image making* and *image having*. As students reach *image making,* they can go through it and arrive at *image having*. Similarly, A[1]'s mathematical understanding while producing original solutions involved *folding back* in. *Folding back* occurs each time A[1] makes a composite two-dimensional figure, and he needs a trigger to form it. Previous studies found that the *folding back* phenomenon is regarded as the way students expand their understanding and connect it conceptually. Since their background knowledge is insufficient to solve new problems, so they must return to the deeper layer to expand it (Martin & Towers, 2016). *Folding back* is not only to remember but also to view the former understanding from a new perspective (Palha, Dekker, Gravemeijer, & van Hout-Wolters, 2013).

CONCLUSIONS

Mathematical understanding is crucial to producing an original solution in problem-solving activities. Besides, observing students' understanding by *acting* and *expressing* themselves can help clarify the students' growth movement. The presence of original solutions while solving a problem is not sufficiently observed based on the occurrence of the original idea. It should emphasize the accuracy of ideas as well. The accuracy has to come from a person's understanding capacity to link between conceptual and procedural knowledge so thus the original solutions in the problem-solving activities appear.

Pirie and Kieren's theory can be relied upon as a good reference for investigating students' mathematical understanding in producing original solutions to problem-solving activities. In comparison, *primitive knowing* provides a great primitive idea that can trigger students to think of original ideas while solving problems. Originality ideas arise when they start to understand a new concept that differs from the previous ones. It grows an initial understanding of the layers of *image making* and *image having*. And it grows more complex as it reaches *property noticing*, where students make the details of original ideas by noticing specific characteristics. Furthermore, original solutions are shown when the students formally connect their original ideas with the procedure – this skill achieved grows at the *formalizing* layer. A similar procedure the students used is to produce another solution that makes them grow to the level of *observing.* Nevertheless, these students are unaware of any relationship between the theorems that can be applied in solving the problem, so their understanding does not reach the *structuring* level. Therefore, further researchers are expected to investigate the growth of students' understanding of problem-solving activities by considering another characteristic of the research subject and setting.

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Exploring Students' Work in Solving Mathematics Problem through Problem-Solving Phases

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Abstract: This study explores students' use of mathematical objects in each problem-solving phase based on an onto-semiotic perspective. The subjects of this research are students who solve problems in different ways but all with the correct result. The first student uses organizational data by applying the concept of permutations. In contrast, the second student uses visual representation (making pictures) by applying the concept of filling in slots or multiplication rules. The results showed variations in the formation of mathematical objects in each problem-solving phase and indicators of activity using mathematical objects. The detail of students' work will be discussed comprehensively.

Keywords: mathematical object; onto-semiotic approach; problem-solving; combinatorics.

INTRODUCTION

One of the essential activities in learning mathematics is problem-solving. The main goal of learning mathematics is to develop the ability to solve various complex mathematical problems (Baykul & Antalya, 2011; Csachová, 2021; Prayitno et al., 2020). Godino and Batanero (2020) state that problem-solving activities are central to constructing mathematical knowledge. Problem-solving has always been part of the mathematics curriculum (Bien et al., 2020). Furthermore, Piñeiro et al. (2019) suggest that problem-solving is the primary indicator in demonstrating one's mathematical competence, evaluating the quality of the education system, and being an essential aspect of teacher learning and knowledge. However, although problem-solving is very essential, there is no formal activity for using mathematical objects explicitly in the problem-solving process. For example, indicators that are commonly used in the understanding phase of the problem are writing down what is known and what is asked without being explained explicitly about the use of mathematical objects in the activity. Docktor and Heller (2009) explain that despite many efforts to improve problem-solving across the education system, there is no standard way to evaluate written problem-solving that is valid, reliable, and easy to use. It underlies the need to study problem-solving processes using a theoretical perspective emphasizing adherence to mathematical objects. Compliance with the explicit use of mathematical objects in each problem-solving phase is expected to make students successful problem-solvers. In his article, Starikova (2010) states that the choice of representation of abstract objects can lead to breakthroughs and significant concept development. Furthermore, Godino et al. (2011) say that getting a better analysis of mathematical activities requires the introduction of mathematical objects.

Several theoretical perspectives can be used in mathematics education research, including APOS (Action, Process, Object, and Scheme), semiotics, and onto-semiotics. Each theoretical perspective has advantages over other theoretical perspectives. The onto-semiotic approach (OSA) was conceived to complement the APOS theory (Font et al., 2010). The onto-semiotic approach is a lens that can provide more complex descriptions and extensions of observations that focus on conceptual and procedural understanding. It is as the description that an onto-semiotic approach is a configuration tool that facilitates a detailed description of the mathematical practice involved in solving a problem (Godino et al., 2021). Based on the description of the advantages and several studies that have used the onto-semiotic approach, no research results have been found that specifically integrate problem-solving theory with the use of mathematical objects in the onto-semiotic approach. The onto-semiotic approach has been used in analyzing combinatoric problems (Godino et al., 2005) but has not been specifically linked to theories related to the stages of problem-solving. In addition, in this study, the subjects observed were students in high school. Furthermore, several studies in Indonesia have used the onto-semiotic approach to analyze students' understanding of algebra (Amin et al., 2018). However, no research has been found that analyzes secondary entities or cognitive dualities in the onto-semiotic approach.

The object of mathematics in the onto-semiotic approach is anything that can be used, recommended, or directed when doing, communicating, or learning mathematics (Montiel et al., 2012). The results of studies related to the use of mathematical objects in each phase of problem-solving can provide an overview of the direction of completion that allows students to arrive at the result or the correct solution to the problem. Teachers need to know the use of a conceptual framework or methodological approach to plan, implement, and assess the mathematics learning process (Burgos et al., 2019; Giacomone, Godino, & Beltrán-Pellicer, 2018).

This research is to answer the question: How does the appearance of mathematical objects in the onto-semiotic approach on each problem-solving phase by students? The study of the use of mathematical objects can show the process that can be done to arrive at the correct solution to a mathematical problem and can show the difficulties experienced in solving a problem. It is like the statement that one of the dilemmas that most often arise in learning is regarding the introduction of mathematical objects (Nachlieli & Tabach, 2012). In line with that, previous research revealed that difficulties in understanding the meaning of mathematical objects are related to semiotic representations occurring at every level of education (Font et al., 2015).

Empirical evidence showcased that the students' habit of giving short answers has an impact on the lack of use of mathematical objects in the structure of the answer to a problem. Then the lack of use of mathematical objects causes students to produce wrong solutions. Furthermore, the habit of overly believing in the completeness and correctness of answers without carrying out activities to check the suitability of the result with the problem situation can also impact the wrong solution. It underlies the need for research to obtain information about using mathematical objects based on the onto-semiotic approach in each problem-solving phase carried out by students to produce solutions. These findings, regarding the formation of the use of mathematical objects in each phase of problem-solving, become an essential framework or guide to be applied in the mathematics learning process.

LITERATURE REVIEW

Onto-Semiotic Approach

The development of the onto-semiotic approach is based on several theories. Godino (2014) states that four groups of models underlie the development of the onto-semiotic approach, namely: (1) mathematical epistemology; (2) semiotic-based mathematical cognition; (3) an instructional model on a socio-constructivist basis; and (4) a systemic ecological model. The onto-semiotic approach includes an explicit typology of mathematical objects, which facilitates the description and analysis of mathematical activity (Giacomone, Díaz-Levicoy, et al., 2018). The onto-semiotic approach introduces the notion of the observable as an entity or object that can be identified by the subject or observer using a particular theory by reference (Bencomo et al., 2004). This approach is named onto-semiotic because of its essential role in language and the categorization of various objects that appear in mathematical activity (Wilhelmi et al., 2021). The onto-semiotic approach can be used as an analytical tool for the processes and objects involved in mathematical activities or practices, a tool for analyzing the learning process in the classroom, as well as for meta-didactic reflection and reflection on the suitability of learning (Godino, 2017). Thus, it can be stated that the onto-semiotic approach is a theoretical perspective or lens that can be used as an analytical tool to use mathematical objects and the position of each part of the object in mathematical activities.

There are two primary components of the onto-semiotic approach, known as entities. The primary entity consists of six aspects, namely: language, situations, concepts, procedures, propositions, and arguments (Burgos et al., 2019; Burgos & Godino, 2020; D'Amore & Godino, 2006; Font et al., 2007; Giacomone, Godino, & Beltrán-Pellicer, 2018; Godino, 2002; Godino et al., 2005, 2006, 2007; Godino, 2019). Furthermore, primary entities can be viewed according to five pairs of points of view known as secondary entities or cognitive dualities. A complete description of the components of the onto-semiotic approach, as developed by Godino et al. (2007), can be seen in Figure 1.

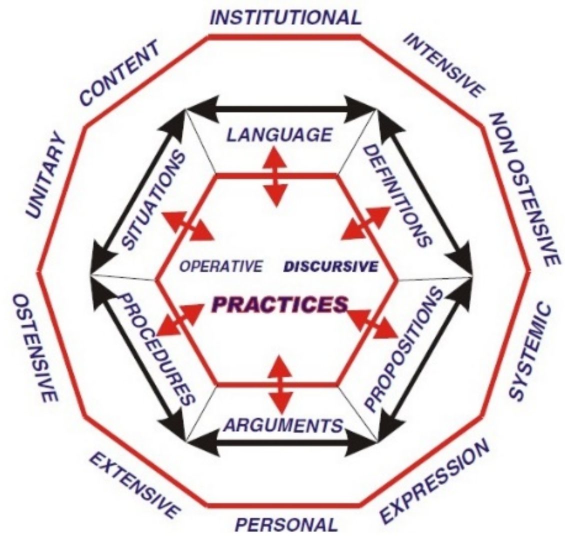


Figure 3: The components of the onto-semiotic approach developed by Godino et al. (2007)

Problem Solving

Several experts have given opinions on the definition of the problem. Avcu and Avcu (2010) concluded that a problem is a situation faced by a person with several obstacles. Mathematical problems are an instrument for developing thinking and problem-solving skills, including those related to everyday life (Arfiana & Wijaya, 2018; Pimta et al., 2009). Polya (1957) stated that problems are broadly divided into two types, namely: (1) routine problems and (2) non-routine problems.

According to Polya (1957) a problem is routine if a problem can be solved by substituting specific data into the problem or by following steps from similar problems that have been solved. Meanwhile, non-routine problems arise when a person faces a particular situation and intends to achieve the critical situation but needs to know how to achieve the goal (Arfiana & Wijaya, 2018; Elia et al., 2009). Nonetheless, Avcu and Avcu (2010) explain that routine problem-solving is essential in developing computational skills.

There are eight classifications of problems that Johnstone developed in 1993, where problems are divided into eight types based on three components, namely the data provided, the methods that can be used, and the objectives (see Table 14.1 page 306 in the work of Bien et al. (2020)). In this study, the first task is a question that is a problem with complete data, non-routine methods, and closed objectives (type 2). At the same time, the second task is a question that is a problem with complete data, routine methods, and closed objectives (type 1).

There are four phases of problem-solving, according to Polya (1957). Several experts have proposed modifications related to the problem-solving phase developed by Polya, such as Zelaso's team, as cited by Kotsopoulos and Lee (2012). Carson (2007) develops problem-solving phases into five steps based on Krulik and Rudnick's modification work for these phases. According to

John Dewey, George Polya, Stephen Krulik, and Jesse Rudnick, the different problem-solving phases can be seen in Carson's article (2007). However, several theories about the problem-solving phase are used in this study, such as Polya's four phases of problem-solving.

Furthermore, Kılıç (2017) and Bien et al. (2020) have described each problem-solving phase. The indicators of problem-solving activities used to classify student activities in this study can be seen in Table 1.

Phase of Problem Solving	The Indicators of Activity
Understanding the Problem	• write or state what is known • write or state what develops
Devising a Plan	• state the method or concept that will be used to solve the problem • state or write down certain rules that will be used to solve the problem • state the reason for using a concept
Carrying out the Plan	• carry out procedures by applying the planned concept
Looking Back	• re-check the procedures that have been carried out • check the final result with the problem situation

Table 1: The Activity Indicators in Each Problem-Solving Phase

METHOD

This qualitative research describes the use of mathematical objects used in the problem-solving process by students based on an onto-semiotic theoretical perspective.

Instrument

The main instrument in this research is the researcher himself. Qualitative researchers, as human instruments, determine the research focus, select informants, analyze and interpret and make conclusions based on research findings. The supporting instruments used are assignments in two number description questions and semi-structured interview guidelines. However, the first question to apply the concept of combination or addition needs to be answered correctly by all prospective research subjects. So, the answer analyzed is the answer to the second question, which is a question that can be solved by applying the concept of permutation or multiplication. Prior to use, the supporting instruments were validated. Mathematical tasks that can be answered correctly by prospective subjects are shown in Figure 2.

Ali wants to change his email account password. To create a password that is difficult for others to guess, Ali created a password that is 8 characters long, consisting of 4 different letters taken from the letters a, b, c, d, and e and followed by 4 different numbers taken from the number 0, 1, 2, 3, 4, and 5. Specify as many possible new passwords for Ali's email account!

Figure 4: Math Assignment

Research Subject

The subjects of this study were twelve-grade students who had obtained the material on the rules of enumeration. Of the 28 students, there were 24 students whose answers got the correct result. By paying attention to the order of completion of the 24 students' answers, there are two groups of correct answers. The first group comprises 17 students who make two subsections: letter and number permutations. The second group consists of 7 students who use multiplication rules to count the number of possible numbers and letters. The selected subject is one person from each group who has similar answers. In administering the test, the researcher does not suggest that students use permutations or filling slots. Students are left to determine how to solve according to their compassionate nature. Students are given this test precisely one week after the teacher provides the material about permutations. It was first asked of the teacher. Students selected from both groups have complete, correct answers among friends who use permutations or filling slots.

Data Collection

The data collected were students' written answers in the form of photos of student answer sheets sent via a google form, student explanations regarding answers in the form of videos which are also sent via a google form, and interview result data. Data collection begins with giving math assignments to students. Furthermore, semi-structured interviews were conducted based on the results of student work. The interview emphasized confirming the use of mathematical objects in problem-solving based on data from answer sheets and explanation videos.

Data Analysis

The steps taken in the data analysis of each subject were: (1) transcribing the explanation video data related to the answers; (2) transcribing the recorded interview data; (3) reviewing the results of written answers, explanatory video transcripts, and interview recordings; (4) perform data reduction; (5) tabulating the use of mathematical objects; (6) describes the position of mathematical objects; (7) interpreting and meaning of the data; and (8) draw conclusions. Furthermore, the units and categories are arranged for making drawings to interpret and describe the data. The units and categories used in this study were adapted from the onto-semiotic perspective developed by Godino, et al. (2007). It is described in detail as follows.

1. Arrange Units

The arrangement of the units in this study is based on the component approach onto-semiotic. The use of color is intended to facilitate differentiating each component students raise. Units' use of mathematical objects can be seen in Table 2.

Category	Descriptor	Code
Language	Written statements (words, symbols, signs, and pictures) and verbal statements used in solving problems	Red trapezoid image
Situations	The information contained in the given problem	Orange trapezoid image
Definitions/ Concepts	Statements relating to certain concepts	Yellow trapezoid image
Procedures	The steps taken in implementing a concept or strategy	Green trapezoid image
Propositions	A statement about the principle or formula used in solving the problem	Blue trapezoid image
Arguments	A statement used to justify a proposition or procedure	Indigo trapezoid image
Personal	Student point of view which can be in the form of student answer (personal side)	Black line
Institutional	The institutional point of view is a reference for understanding and evaluating the teaching and learning process (in this case, including problem-solving)	Red line
Ostensive	Objects that appear to be used explicitly or can be observed directly because they are written on the answer sheet such as symbols and pictures	Brown line
Non-ostensive	The object is only stated verbally without being accompanied by a written form that can be observed on the answer sheet (for example, a multiplication sign that is not written but there is a multiplication process)	Yellow line
Extensive	Object is used as a specific case, e.g., $P_4^6 = \frac{6!}{(6-4)!}$	Dark green line
Intensive	Object used as a more general form, e.g., $P_r^n = \frac{n!}{(n-r)!}$	Light green line
Unitary	Mathematical object used as a unified entity (object that should have been known beforehand)	Dark blue line
Systemic	The system being studied. For example, in teaching the general formula for permutations, factorial, division, and subtraction are considered as something known (unitary). While the same object (factorial, division, and subtraction) in a particular class or on a particular occasion, must be treated as a systemic and complex object to be studied	Light blue line
Expression	A symbol that represents a certain meaning	Purple line
Content	The meaning represented by a symbol	Light purple line

Table 2*:* Units of Use of Mathematical Objects

2. Organize Categories

Subanji, quoted by Sukoriyanto (2017), states that the arrangement of categories is carried out to facilitate data interpretation, simplify problems, and facilitate the thought analysis process of research subjects. Categorization or coding is distinguished by color coding and image coding, presented in Table 3.

Object Name	Color Code			Image Code
	Red	*Green*	*Blue*	
Language	255	115	115	Trapezoid
Situations	255	185	115	
Definitions/ Concepts	255	255	125	
Procedures	125	255	125	
Propositions	145	255	255	
Arguments	255	115	255	
Personal	0	0	0	Line
Institutional	255	0	0	
Ostensive	128	64	0	
Non-ostensive	255	255	0	
Extensive	128	128	64	
Intensive	128	255	0	
Unitary	0	0	255	
Systemic	0	255	255	
Expression	128	0	255	
Content	255	0	255	

Table 3*:* Coding of Primary Entity and Secondary Entities Components

The subsections of language components and definitions/concepts are coded using letters followed by an index number based on their occurrence. The description and code of each language component subsection and concept are provided in Table 4.

Category	Descriptor	Code
Language	Write words or sentences (in the form of text), for example known, asked, numbers, letters, possibilities, etc.	T
	Sate a word or sentence (in the form of an oral statement)	L
	Using symbols, for example: 1, 2, 3, 4, etc.; *a*, *b*, *c*, *d*, and *e*; P (permutation symbol)	S
	Using signs, for example: Operation signs +, -, x, or.,: atau /, (), ^, and √; Relation signs (=, <, >, ≤, ≥, ≠, etc.); and Factorial sign (!)	Ta
	Creating images, for example: lines, flat shapes, building spaces, etc.	G
Concepts	Permutation	C1
	Factorial	C2
	Distribution	C3
	Subtraction	C4
	Multiplication	C5

Table 4*:* Coding Subsection of Language and Concept Components

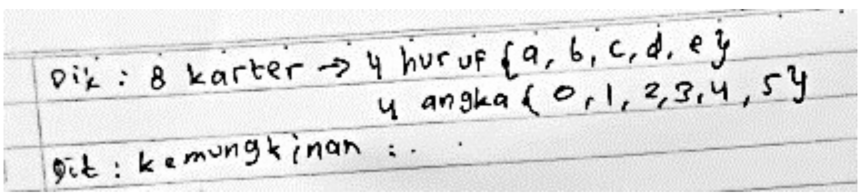
RESULTS AND DISCUSSION

This section contains exposure and analysis of data and research findings. The problem-solving process carried out by each subject is described in four phases, namely: understanding the problem, devising a plan, carrying out the plan, and looking back. Furthermore, in each phase of problem-solving, all mathematical objects raised by the subject are presented, which can be categorized as primary entities in an onto-semiotic theoretical perspective: language, situation, definitions/concepts, procedures, propositions, and arguments. Furthermore, the position of the primary entity used by the subject when viewed from a second entity or cognitive duality is presented, including personal-institutional, ostensive-non-ostensive, extensive-intensive, unitary-systemic, and expression-content.

Data Exposure of Subject 1 (S1)

Understanding the Problem

After reading about the given problem, S1 writes down the known elements and the asked elements of the problem. The snippet from the S1 answer sheet, which can be interpreted as part of the phase of understanding the problem, can be seen in Figure 3.



Translating in English:
Given: 8 characters→ 4 letters {a, b, c, d, e}
4 numbers {0, 1, 2, 3, 4, 5}
Asked: possibly: …….

Figure 3: S1's Part Answer

Before specifying all the mathematical objects used by S1, it is also observed in the section that shows the Phase of understanding the problem in the explanatory video transcript. The explanatory video fragment that corresponds to the written answer by S1 is:

> *It is known that Ali wants to create an email password that consists of eight characters. It consists of four letters colon (in this case it means "of") a, b, c, d, and e. It is followed by four different numbers, namely-from 0, 1, 2, 3, 4, and 5. Four different letters and four different numbers. Keep asking for possibilities. I asked if it was possible, with many possible passwords.*

From Figure 3. and the transcript of the explanatory video, S1 uses two primary entity components: language and situation. The object implied in the practice of mathematics by S1 is the concept of a set. The concept of this set was confirmed through the interview session as one of the objects used by S1. The transcript of the interview with S1 related to the object is as follows:

P: What do the curly braces mean here? (While showing the first line written in curly braces)
S1: The set, the set of letters.

From the analysis results related to the use of primary entities and the position of each sub-section of each primary entity when viewed from the five pairs of cognitive dualities by S1 at the Phase of understanding the problem, it can be seen in Figure 4.

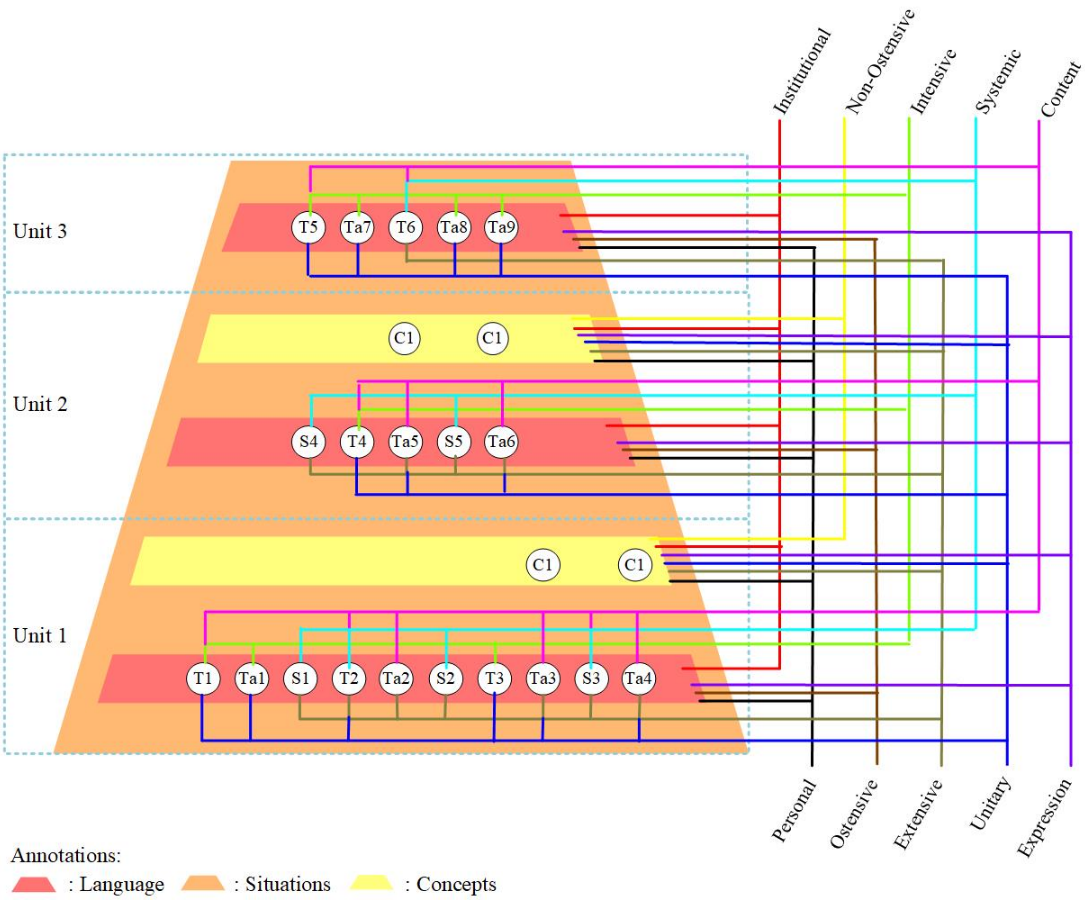


Figure 4: The Use of Mathematical Objects by S1 at the Understanding the Problem Phase

Devising a Plan

After writing down the information that is known and asked from the problem, S1 determines a problem-solving strategy using permutations. However, no specific section on the answer sheet shows this section. The part of the problem-solving process that shows the phase of planning a solution is contained in the explanatory video recording. This work is as in the S1 statement fragment in the following video transcript:

> *Continue to solve it using permutations because he says different, four different letters and four different numbers. This is a permutation.*

This was also revealed in the interview session. Excerpts of interview transcripts that show S1 is planning a solution by stating the use of the permutation concept, namely:

> *P: Is this your answer? (While displaying the answer sheet S1)*
> *S1 : Yes*
> *P: Please explain why it is answered like this!*
> *S1 : That's a permutation of four out of five because there are five letters and four are taken. So, I use permutations.*

From this statement, it can be stated that the mathematical objects used by S1 in the planning phase of completion are situations, language, concepts, and arguments. The use of mathematical objects by S1 in planning solutions can be seen in Figure 5.

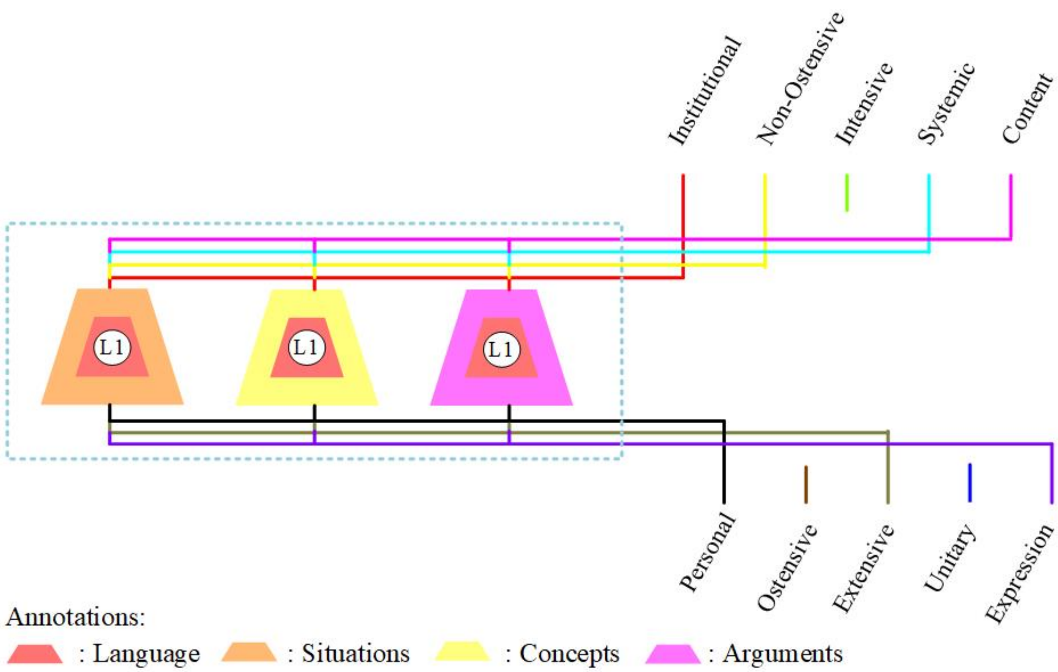
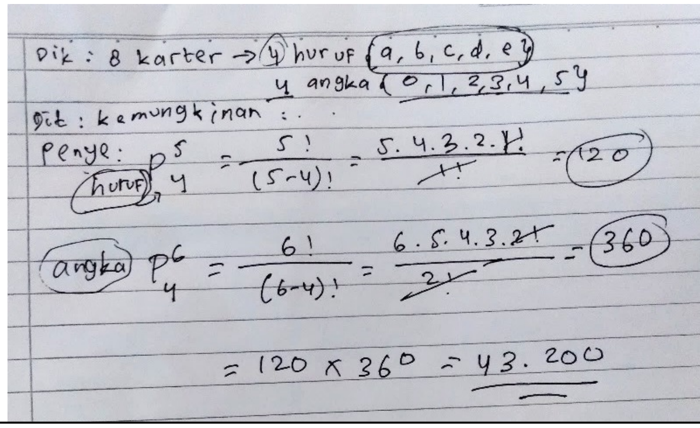


Figure 5: The Use of Mathematics Objects by S1 at Devising a Plan Phase

Carrying out the Plan

After determining the strategy that can be used, S1 calculates the permutation of letters, calculates the permutations of numbers, then multiplies the results of the permutations of letters and numbers. A piece of the answer sheet that shows the process of carrying out the completion by S1 can be seen in Figure 6.



Translating in English:
Given: 8 characters→ 4 letters {a, b, c, d, e}
4 numbers {0, 1, 2, 3, 4, 5}
Asked: possibly: …….
Completion:
Letter
Number

Figure 6: S1's Part Answer

Furthermore, it is observed simultaneously with the explanatory video fragment by S1, who corresponds to the written answer. The video snippet that shows the phases of carrying out the completion by S1 are:

> *First permutation four out of five. Because four are chosen, and there are five letters (then circle the number 4 and the letter in curly brackets in the first line of the answer part about the information that is known from the question). So, five factorials per five minus four factorials equal five times four times three times two factorials per one factorial equal one twenty. So, this is a letter. It is a letter (circling the letter word that has been written and making an arrow pointing to the form of four out of five permutations).*
>
> *Continue to number (while writing the number of words and circling the number word). Also, use permutations because they do not repeat. So, this is four out of six because four numbers are chosen from six numbers (then make a line under the number 4 and the numbers 0, 1, 2, 3, 4, and 5 in the second line of the answer section about the information that is known from the question). Six factorials by six minus four factorials. By two factorials. Six five four three two factorials. This crossed-out equals the result three sixty.*
>
> *Continue this (while circling the number 120) and this (while circling the number 360) multiplied. So, one twenty times three sixty equals four thousand, forty-three thousand two hundred. This way (while drawing a line twice).*

From Figure 6. and the video transcript, there are several mathematical objects used by undergraduate students: language, situation, concept, procedure, and argument. The statement regarding the concept of permutations and the formula for calculating permutations was confirmed in the interview, and the results showed that S1 used propositions, but it was not stated in the video. The part of the interview that shows the use of propositions is:

P: Why don't you write down the general form of the permutation first?
S1: to keep it short.
P: Please re-explain your answer!
S1 : P(5,4) because 4 letters will be chosen from 5 letters (a, b, c, d, and e)
P(6,4) because 4 numbers will be selected from 6 numbers (0, 1, 2, 3, 4, and 5)
P: Have you ever received a similar question?
S1 : ever
P: Is there any other way to solve this problem?
S1 : that is my way, ma'am.

The interview excerpt found that before starting the procedure in completing, S1 gave arguments and stated the propositions related to the formula for calculating permutations. Mathematics objects by S1 can be seen in Figure 7.

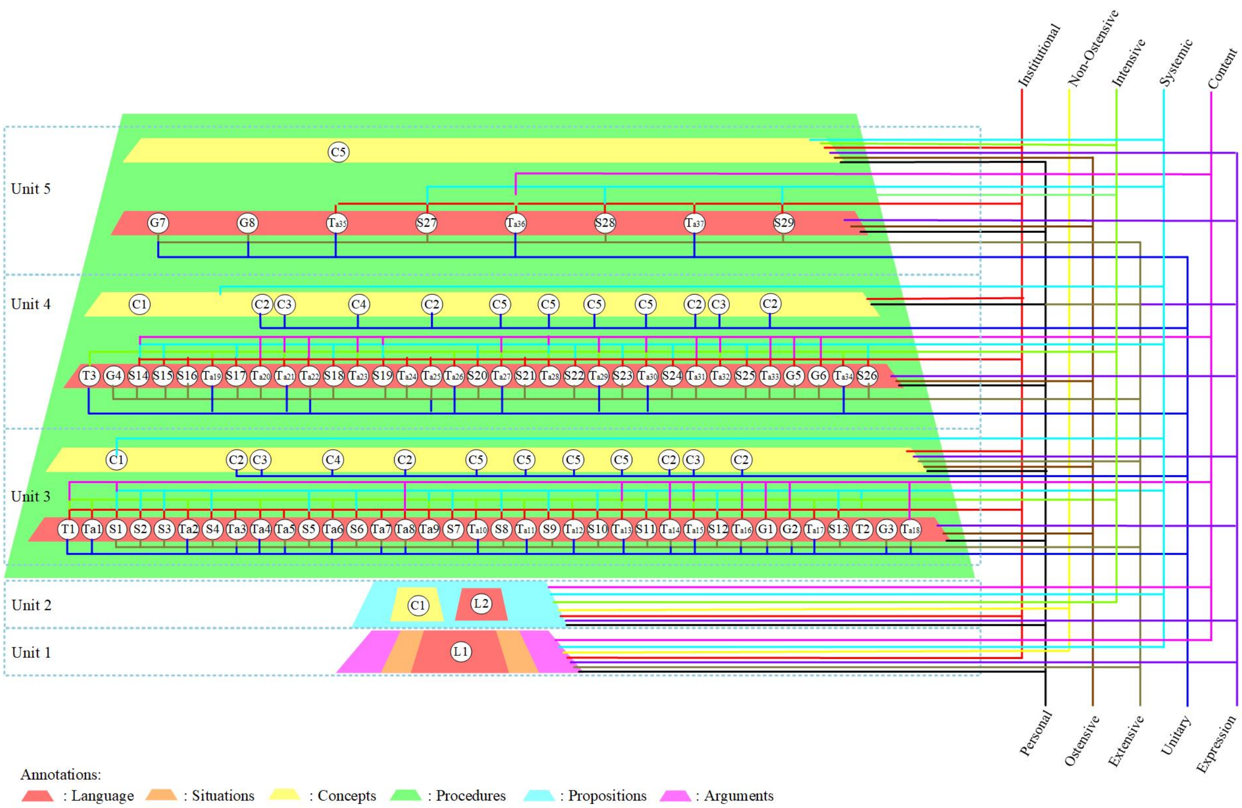


Figure 7: The Use of Mathematical Objects by S1 at the Phase of Carrying out the Plan

Looking Back

Part of the problem-solving process carried out by S1, which can be considered part of the re-examination phase, occurs when S1 carries out the completion phase. The activity of checking procedures carried out previously by circling and underlining certain sections indicates that the checking activity is carried out by S1. The part which is the looking back phase is marked in Figure 8.

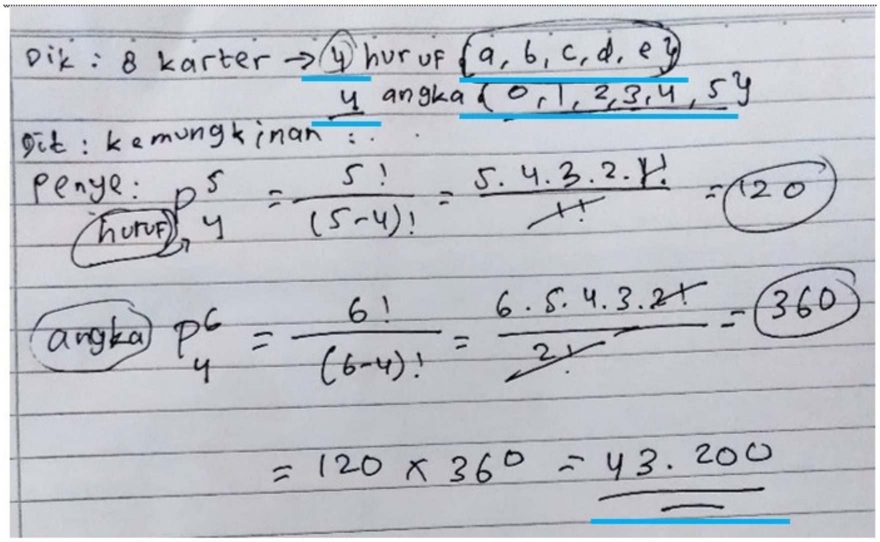


Figure 8: S1’s Part Answer

The use of objects by S1 at the looking back phase cannot be observed directly from the answer sheet. An explanatory video transcript was showing the activity at the re-examination phase co-occurred at the phase of completing (in the image, it is marked with a blue underline). S1 performs the check-back phase back and forth. After performing specific procedures, S1 checks again by verifying the parts that support the procedures’ justification. This is done repeatedly every time the procedure has been carried out until the result is obtained as an answer to the existing problems. The mathematical objects used by S1 are language, situations, procedures, and arguments. The use of mathematical objects by S1 can be seen in Figure 9.

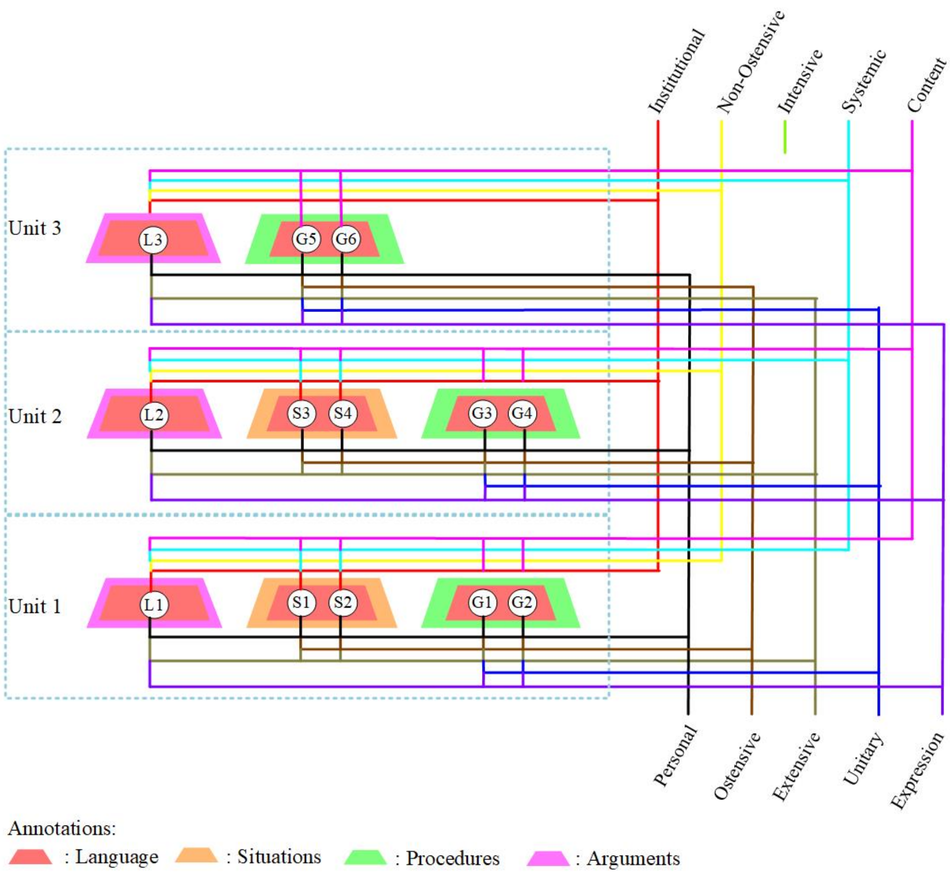


Figure 9: The Use of Mathematics Objects by S1 at the Looking Back Phase

Data Exposure of Subject 2 (S2)

Understanding the Problem

No part of S2's answer sheet can be determined directly as an activity to understand the problem. Explanation fragments from S2 that show the phase of understanding the problem are:

> *So, Ali made a password that was eight characters long, consisting of four different letters taken from the letters a, b, c, d, e and followed by four different numbers zero one two three four five. Determine the number of possible passwords?*

The S2 statement confirming the given task's information shows the use of mathematical objects. The mathematical objects used by S2 at the phase of understanding the problem are language and situations. An illustration of the use of mathematical objects by S2 can be seen in Figure 10.

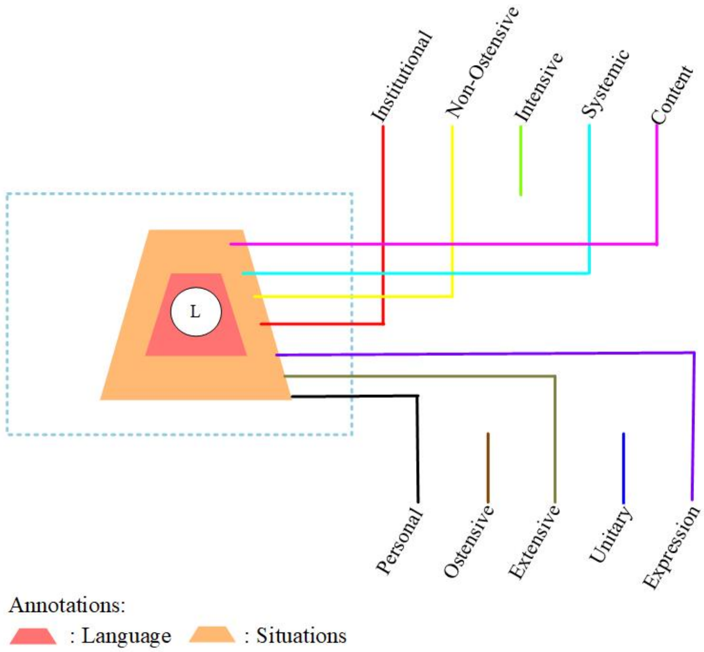


Figure 10: Use of Mathematical Objects by S2 at the Understanding the Problem Phase

Devising a Plan

What S2 does as part of the planning phase for completion is to draw eight lines. It is adapted to the existing problem situation. The pieces of S2 answers that show the activities in the planning phase of completion can be seen in Figure 11.

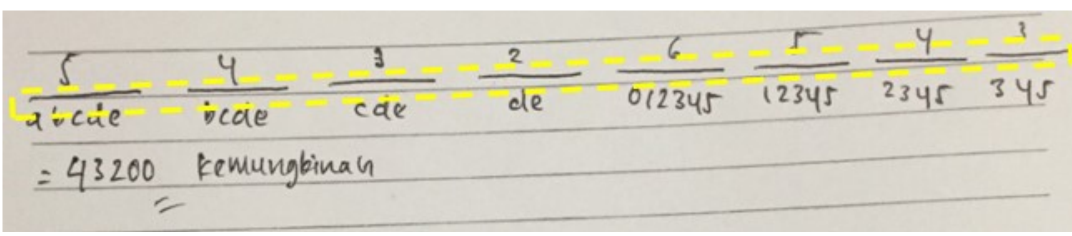


Figure 11: S2's Part Answer

Oral statements showing part of the planning phase for completion can also be obtained from the explanation video by S2. The fragment that contains the part planning for completion by S2 is as follows:

> *So, a, b, c, d, e are five letters so five per abcde, then it can't be the same so bcde has four letters, then it can't be the same anymore so cde becomes three letters and de is two.*

Based on data from written answers and explanation videos, it can be stated that S2 uses two mathematical objects. The mathematical objects used by S2 at this phase are language, concepts, procedures, and arguments. From the personal side, S2 is planning a solution using language objects and arguments. In addition, during the interview session, it was also clarified that the phases of planning the completion were carried out. The excerpts of the interview related to the Phase of planning this settlement are as follows:

P: Here's the answer. Tell me about the process!
S2 : Starting from the line, because it's eight characters, what I learned at the tutoring center was first to draw the line.

From the results of interviews related to the process carried out by S2, there are statements indicating the use of the concept. It shows the use of mathematical objects in the form of concepts in devising a planning phase by S2. The use of objects by S2 can be seen in Figure 12.

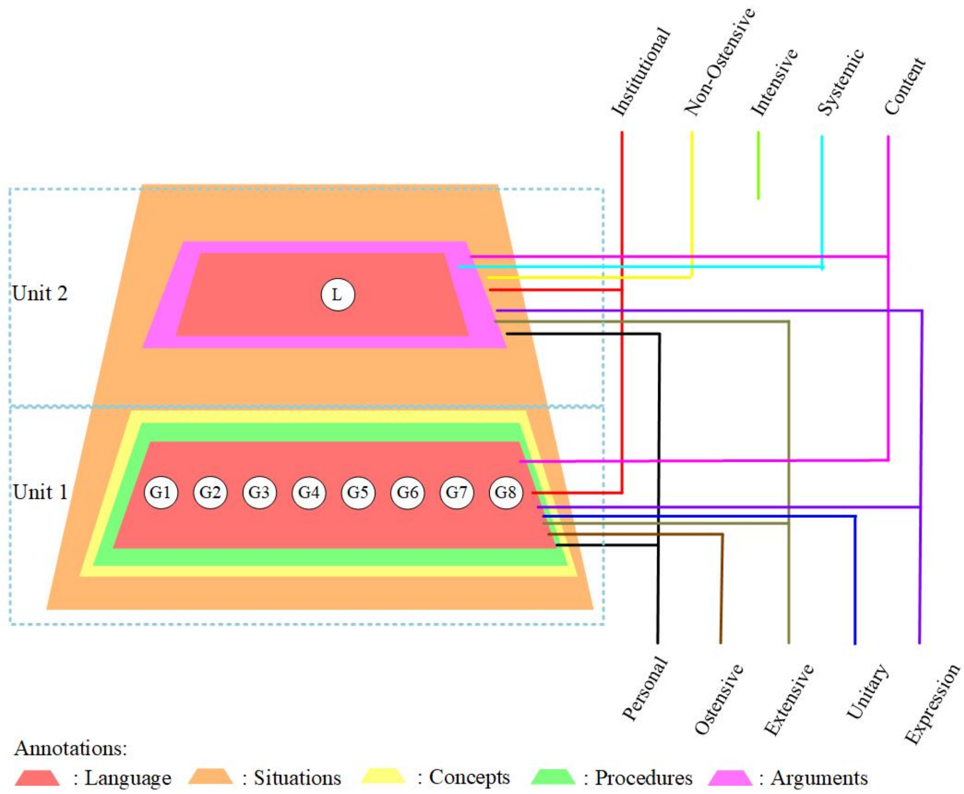
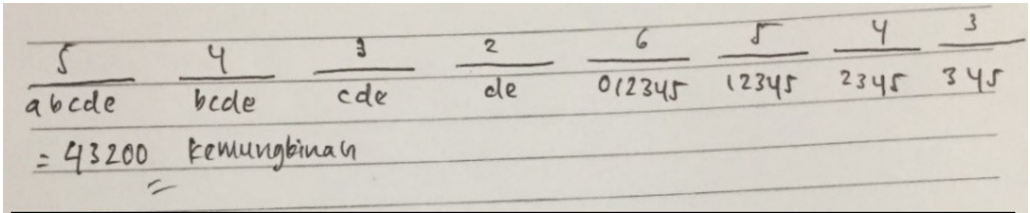


Figure 12: Use of Mathematics Objects by S2 at the Devising a Plan Phase

Carrying out the Plan

After planning the solution, S2 applies the rules for filling in places to count the number of letters and numbers that can occupy certain positions according to the problem. The answer sheet that shows the use of mathematical objects in the process of carrying out the completion by S2 can be seen in Figure 13.



Translating in English:
= 43200 possibilities

Figure 13: S2 Answer Pieces

From the video explanation of the answers by S2, it can also be seen about the activities in the phase of carrying out the completion. The transcript showing the phases of carrying out the completion by S2 are:

> *So, a, b, c, d, e are five letters so five per abcde, then it can't be the same so bcde has four letters, then it can't be the same anymore so cde becomes three letters and de is two. Then the numbers can't be the same, so zero one two three four five is six letters, one two three four five is also five, two three four five four letters, three four five three letters to four. All times.*

From Figure 13 and the transcript of the explanatory video, the mathematical objects used by S2 in the completion phase are language, situations, concepts, procedures, propositions, and arguments. In the content definition/concept object, the concept of subtraction and the concept of multiplication are used as objects from the non-ostensive side.

> *P: So here it was reduced?*
> *S2: Yes.*
> *P : O. So, after finishing this one, did you start writing again six five four three?*
> *S2: Yes*
> *P: Only six, it keeps decreasing again?*
> *S2: It's reduced.*
> *P: So how do you get this? (While circling the result on the answer sheet.*
> *S2 : Multiplied by the top number, the top number is five four three two six five four three.*

The mathematical objects used by S2 can be seen in Figure 14.

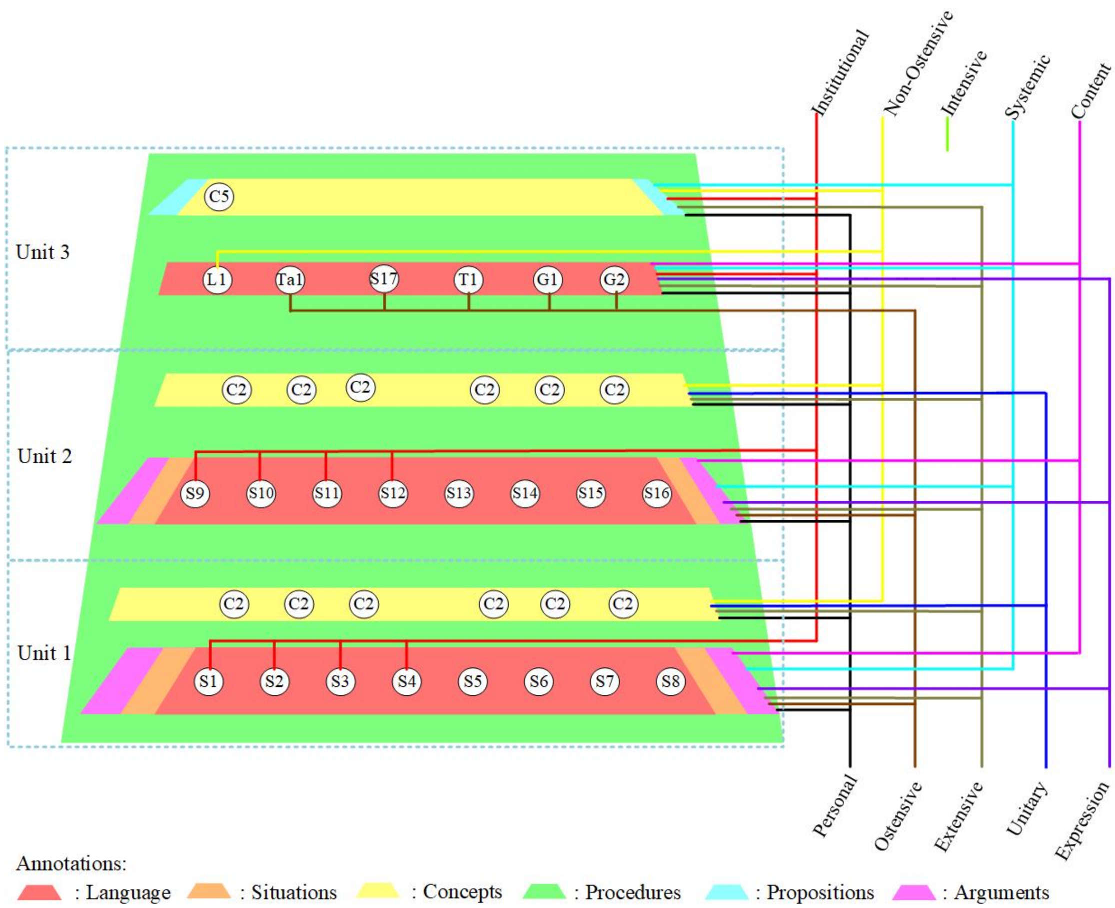


Figure 14: Use of Mathematical Objects by S2 at the Phase of Carrying out the Plan

Looking Back

On the S2 answer sheet, it cannot be directly determined which activities are part of the looking back phase. Data regarding the looking back activity was obtained while simultaneously observing the answer sheet and the video along with the explanation video transcript (not separate from the completion activity). From the video transcript, as in the implementing section, it can be stated that the looking back phase of S2 uses language objects and arguments. The use of mathematical objects by S2 can be seen in Figure 15.

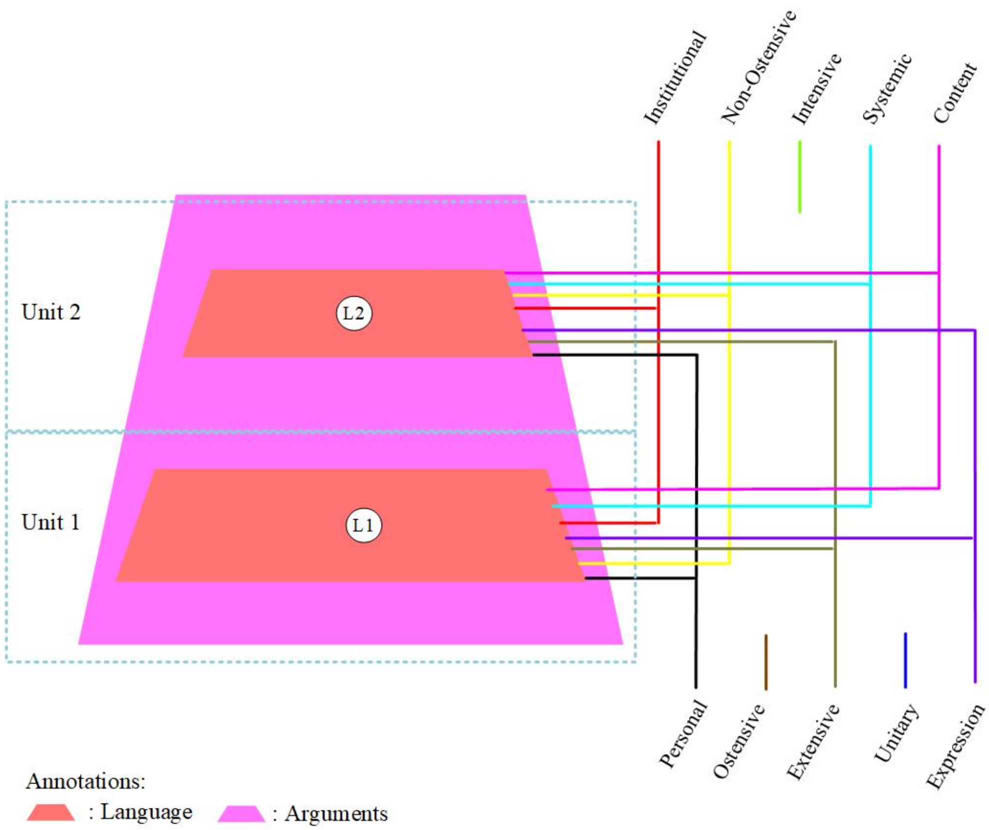


Figure 15: Use of Mathematics Objects by S2 in the Looking Back Phase

Based on the previous data exposure, a summary of the use of mathematical objects from each subject is made to make it easier to see the slices or combinations of the use of mathematical objects between subjects. The summary of the use of mathematical objects by Subject 1 is presented in Figure 16.

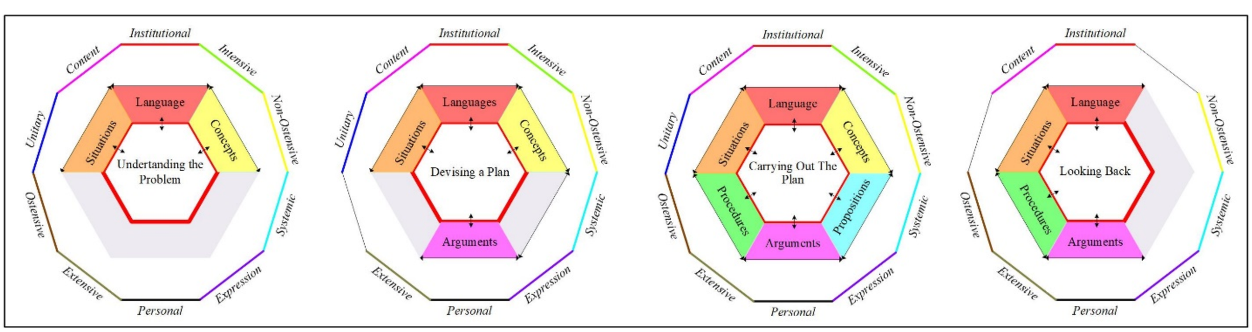


Figure 16: Summary of the Use of Mathematical Objects of Subject 1

Furthermore, a summary of the use of mathematical objects in each phase of problem-solving by Subject 2 can be seen in Figure 17.

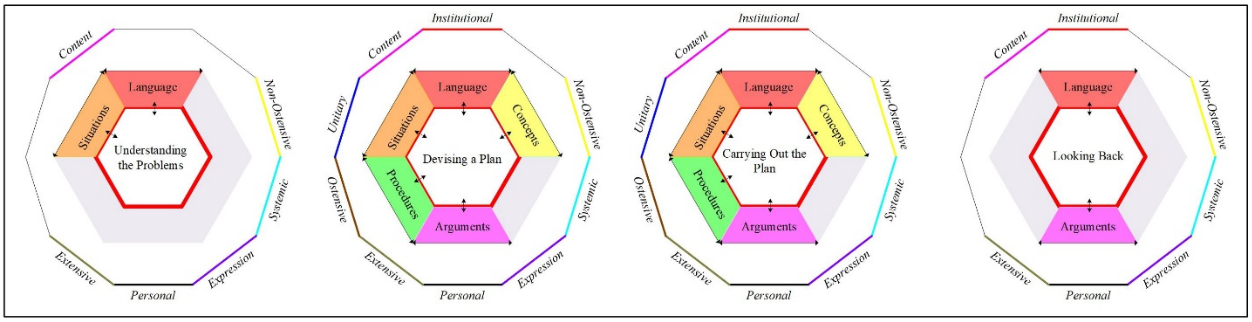


Figure 17: Summary of the Use of Mathematical Objects of Subject 2

Finding

Based on the analysis results and the use of objects between subjects, it can be stated that there are variations in the formation of the use of mathematical objects between the phases of problem-solving. The use of objects in each phase can be seen in Figure 18.

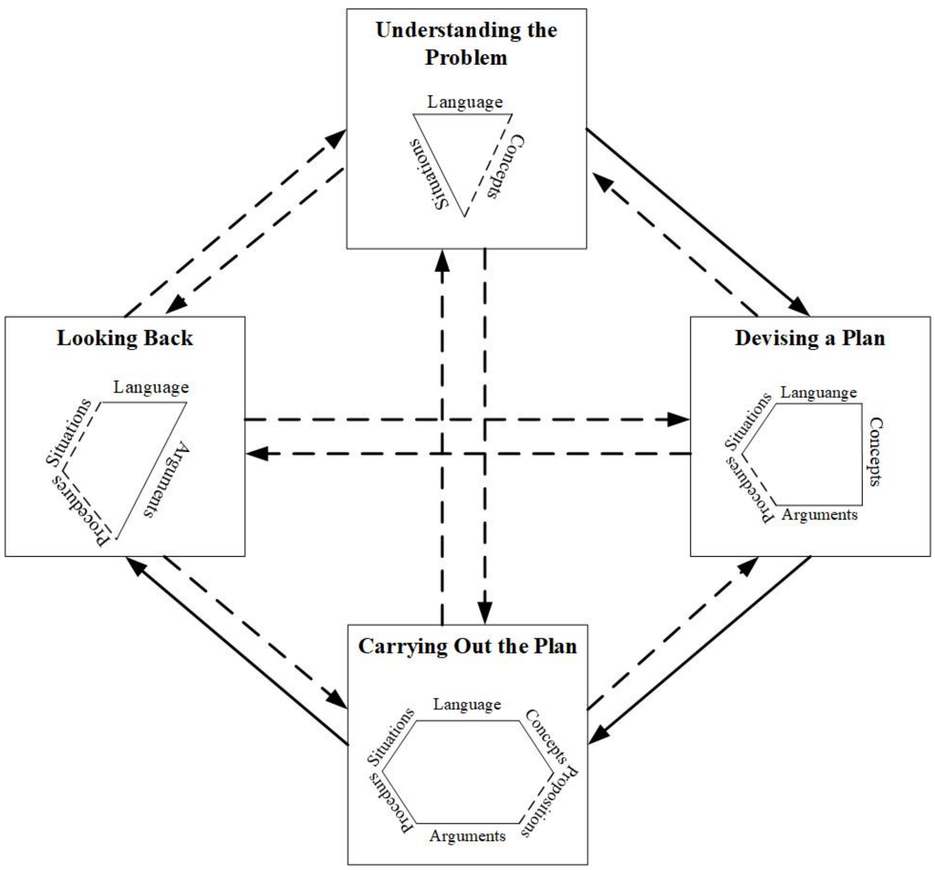


Figure 18: The Use of Mathematical Objects in Each Problem-Solving Phase

Figure 18 shows activity indicators formulated in each problem-solving phase because of integrating mathematical objects with an onto-semiotic approach. In detail, the indicators referred to can be seen in Table 5.

Phase of Problem Solving	Indicators of Mathematical Objects
Understanding the Problem	Using written language objects (words, symbols, signs, and picture) or spoken language in presenting problem information (problem situations)
Devising a Plan	• State the concept that will be used to solve the problem • Provide arguments against selected concepts
Carrying out the Plan	• State a proposition or write a general formula. • Perform procedures involving language objects, situations, and concepts.
Looking Back	• Examine procedures using language objects, other procedures, and arguments. • Check the suitability of the result with the problem situation (can use language objects, situations, procedures, and arguments)

Table 5*:* Indicators of Using Mathematical Objects in the Problem-Solving Phase

By integrating the theory of problem-solving and the onto-semiotic approach, the minimum indicators of activity that can be used in each problem-solving phase can be seen.

DISCUSSION

This section discusses the research findings, namely the variations in the formation of the use of mathematical objects in each problem-solving phase. Furthermore, the mathematical objects that appear in each phase of problem-solving need references in their use. The reference in question is given as indicators for using mathematical objects. The use of mathematical objects in each problem-solving phase is described as follows.

Understanding the Problem

The research findings show that the mathematical objects used by students in the problem-understanding phase are language, situations, and concepts. The results of this study indicate something new to complement the existing theory about understanding the problem, where it is obtained to produce the correct solution. It is enough to use two or three of the six primary entity components in the onto-semiotic approach. This result is in line with Kılıç (2017) research, which states that this phase is significant for the right solution and involves understanding the problem situation and determining and deciding facts and goals. Every student needs to understand the problem to have a chance to come up with a solution. Understanding the problem is a phase that involves determining what is needed regarding problems related to mathematical concepts and procedures, accessing prior knowledge, and isolating relevant information from irrelevant contexts (Kotsopoulos & Lee, 2012).

The results showed that, in general, students understanding the problem used mathematical objects as part of understanding on the personal side. Students use mathematical objects that are not by the institutional side. Godino (2018) states that students are expected to adjust progressively to institutional meaning by participating in relevant practices to achieve a combination of initial personal and institutional meaning in the learning process.

The results showed a variation between the use of objects that could be observed directly on the answer sheet (ostensive) and using objects that were only used in students' minds (no ostensive) in the phase of understanding the problem. The use of this ostensive object is very influential in assessing the competence results of students, especially if the assessment is only in the form of a written test. Students who are more dominant in using non-ostensive objects need confirmation to get used to writing parts that have the potential to be elements that the teacher assesses. As Hough and Gluck (2019) state, performance-based measurement is the most popular technique for assessing human understanding. However, the measurement process needs to be modified to assess understanding. It is necessary that more complex cognitive models can be broken down into components for appropriate empirical testing.

In understanding the problem, students are more dominant in using objects specifically (extensive) than objects in general form (intensive). In this phase, students use the object being studied (systemic) more than objects that have been understood previously (unitary). Some expressions have linguistic meanings, situations, and concepts in this phase. The use of mathematical objects marks the students' understanding of the information on the given task. Students classify the information in the test given as one indication of an understanding of the test passed. Fuentes (1998) describes that part of the expected results in the learning process is that students can understand text math problems because this is the right way to solve problems. Reading comprehension is one of the cognitive factors that can play an essential role in solving high school students' issues (Öztürk et al., 2020).The basic understanding possessed by students plays an essential role in the problem-solving process (Lee, 2011). It is difficult for a person to have a good idea about problem-solving. It is even impossible to have an idea if he does not have knowledge and ideas related to problem-solving based on previously acquired knowledge or experience (Polya, 1986).

Devising a plan

The research findings indicate that in the planning phase for completion, all students use mathematical objects in the form of language, situations, concepts, and arguments. In addition, there are mathematical objects in the form of procedures by students. These results provide a specific answer compared to the previous theory about planning completion. In planning completion, students can use four or five of the six primary entity components in the onto-semiotic approach. Kotsopoulos & Lee (2012) stated that designing a settlement plan involves selecting the mathematical processes and operations appropriate to the problem and establishing the procedures to be applied to solve the problem.

The results showed that in the planning phase of completion, students used mathematical objects as part of their understanding but institutional ones. In planning the completion of mathematical objects, non-ostensive objects are dominated by ostensive objects, students are more dominant in using extensive objects than intensive objects, and students use systemic objects more than unitary objects. Furthermore, some expressions mean situations, concepts, and arguments. Students' use of mathematical objects in the planning phase shows an understanding of concepts that can be applied based on problem situations. The problem given is a problem with the context of the email password with the consideration that high school-age children in their daily lives have a lot to do with the use of passwords. The choice of the context of the problem was also chosen to make it easier to understand and more accessible to plan the steps for solving it. As stated by Batanero et al. (2021), context dramatically influences children's strategies, so it is necessary to provide different contexts, including situations in children's lives and different media such as dice, chips, and coins. Vásquez et al. (2021) also describe that problem-solving is one of the keys to demonstrating mathematical competence, including managing knowledge, skills, and emotions to achieve goals that are more towards practical situations and in the context of everyday life.

Carrying out the plan

The research findings show that in the phase of implementing the completion, students use all components of the primary entity in an onto-semiotic approach. Students use mathematical objects in language, situations, concepts, procedures, propositions, and arguments. This result adds to the details of the previous theory that carrying out the solution is a part that requires a mathematical process, including operations, to produce a solution (Kotsopoulos & Lee, 2012). This study shows that the mathematical process of carrying out the solution involves all components of the primary entity from the onto-semiotic approach, not only procedures that involve mathematical operations.

The results show that students generally use mathematical objects corresponding to institutional objects in the completion phase. In completing mathematical objects, ostensive objects are dominated by non-ostensive objects. Students first use intensive and extensive objects and use systemic objects more than unitary objects. Some expressions mean situations, concepts, propositions, and arguments. Differences in the use of mathematical objects also occur in this phase, even though they both lead to the same correct result due to differences in the formulas or procedures used. Giacomone et al. (2018) state that each procedure used to solve problems can mobilize different mathematical objects and lead to necessary consequences in mathematical activities. Under certain conditions, the aspects of planning and implementing a settlement must be separated. In line with this, preparing and implementing plans are two aspects that become an integrated whole (Nurkaeti, 2018).

Looking back

From the research findings, the mathematical objects students use in looking back are dominated by language and arguments. However, there is also the use of situation objects and procedures. These results provide more operational information about the mathematical practice carried out in the review phase to complement the previous theory that in reviewing and evaluating, the key idea is to explore the problem solution by evaluating whether the results are reasonable and the reliability and validity of the results (Kotsopoulos & Lee, 2012). The practice of mathematics in the looking back phase is carried out by using two or four of the six primary entity components in the onto-semiotic approach, namely language and arguments or language, situations, procedures, and arguments.

The results show that in the looking back phase, the mathematical objects used as personal understanding still need to be per the mathematical objects on the institutional side. The same thing has previously been part of the conclusion of the study by Kazemi et al. (2010) that the most common student difficulty in solving combinatoric problems is the inability to ensure the correctness of the answers they find. Students need help looking back at the truth of the answers to questions caused by incorrect understanding, planning, and implementation of problem-solving (Nurkaeti, 2018).

In looking back phase, mathematical objects are dominated by non-ostensive objects rather than ostensive objects. These results, as in the study by Giacomone et al. (2018), found that justifications, arguments, or explanations do not appear explicitly in test books in general, nor specifically in the tasks given. These results are in line with some of the conclusions from the research of Moguel et al. (2020), which stated that at secondary-level learning in Mexico, mathematics usually does not have official evidence, which implies that teachers are not accustomed to providing evidence to justify the solution of a problem. The same thing was also concluded by Arfiana and Wijaya (2018) that the lowest result for middle and high school was the phase of re-checking the answers.

In this phase, all objects used by students are extensive objects; intensive objects are not used. As in the previous phase, all objects used by students are systemic objects; there is no use of unitary objects. Mathematical objects used by students are more dominant in expressions that have the meaning of arguments. However, some students use situational objects and procedures. The lack of meaningful expressions of situations and procedures is evidence that, in general, students need to check the results concerning the problem situation to verify the solutions obtained. Similar results were obtained in the study of Moreno et al. (2021), where the study showed that none of the students interpreted the solutions found about the actual situation.

CONCLUSIONS

The results of the research and discussion show that students' onto-semiotics in solving combinatoric problems provide insight into the variations in the formation of the use of mathematical objects in each problem-solving phase. Integrating problem-solving theory with an onto-semiotic perspective provides the basis for using mathematical objects in each problem-solving phase. The variations in the use of mathematical objects in the problem-solving process as a result of integration with the onto-semiotic approach, namely

1. In the phase of understanding the problem, students use written language objects (words, symbols, signs, and pictures) or spoken language in presenting problem situations or problem information.

2. In the devising a planning phase of completion, students state the concept that will be used to solve the problem and provide arguments against the chosen concept.
3. In the carrying out the planning phase, students state propositions or write general formulas and perform procedures involving language objects, situations, and concepts; and
4. In the looking back phase, students examine the procedure using language objects, other procedures, and arguments and check the suitability of the final result with the problem situation using language objects, situations, procedures, and arguments.

The process suggests several things, results, and discussion: (1) in each problem-solving phase, there is the use of non-ostensive mathematical objects. Hence, teachers need to conduct an authentic assessment by assessing the mathematical objects that are not explicitly written on student answer sheets; (2) language objects and arguments dominate the use of mathematical objects in the looking back phase by students. Thus practically, the checking activities need to be familiarized teachers and students with the mathematics learning process; (3) in order to improve students' problem-solving skills, it is recommended for teachers to emphasize the components of mathematical objects that must be raised in each phase of problem-solving as indicators of the activity of using mathematical objects as found in this study; (4) further researchers are advised to further explore the differences in the proportions of the area of each trapezoid (use of mathematical objects) by using specific indices such as how many words/ sentences, time students use/ spend in the area of situations, language, concepts, etc. and (5) further research can be done to see variations in the use of mathematical objects specifically in terms of differences in problem-solving strategies. In this case, the instrument in the form of a mathematical task is confirmed to be a task that can be solved in several different strategies.

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