Schema development in solving systems of linear equations using the triad framework

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Abstract: Solving systems of linear equations is a core concept in linear algebra and a wide variety of problems found in the sciences and engineering can be formulated as linear equations. This study sought to explore undergraduate students' development of the schema for solving systems of linear equations. The triad framework was used to describe the schema development in general and the system of linear equations was used as an example. A case study of an undergraduate class doing a linear algebra course in 2020 was considered in this study, where fifteen students participated in the study whereby they responded to a task with three question on solving a system of linear equations. The findings revealed that albeit minor manipulation errors, students were able to solve given systems of linear equations using the Cramer's rule and Gaussian elimination. However, students could not adequately go beyond the algorithmic computations to attain appropriate mathematical reasoning and establish underlying relations required to solve systems of linear equations, as no-one attained the trans-stage of conceptualization. It follows from this study that the identifications of the challenges that students encounter when solving systems of linear equations empowers course instructors on how to overcome the challenges.

Keywords: schema development; system of linear equations; triad theory; matrices; linear algebra

INTRODUCTION

Linear algebra is a branch of mathematics concerned with solving systems that are modelled with multiple linear functions. Modelling in linear algebra is essential in the study of systems of equations and is based on the description of the properties that characterize them and it is an activity that cuts across all disciplines (Barragan, Aya & Soro, 2023). The term systems indicates that the linear equations are not considered individually, but collectively. Many problems found in natural sciences and engineering can be formulated as multiple linear equations. Systems of linear equations (SLEs) play an important and motivating role in the subject of linear algebra. In the introductory stages of linear algebra, undergraduate students learn SLEs and how to solve them using each of the methods of Cramer, inverse, Gaussian elimination and Gaussian-Jordan elimination. However, Gauss-Jordan is an extension of Gaussian elimination, hence regarded as unitary. There is no guaranteed method which is regarded as the best in solving SLEs. Students are often left to determine for themselves the method which is efficient and easier. Efficiency in solving SLE entails determining the values of the unknowns in less time and complicated procedures (Gharib, Ali, Khan & Munir, 2015). For instance, the inverse and Cramer methods are not usable in cases where the coefficient matrix is non-square or the determinant is zero. In addition, the inverse and Cramer's methods cannot be used to evaluate solution beyond the unique solutions. By not relying on determinants, the Gaussian elimination method can be used to solve almost all types of SLEs.

Students' learning of linear algebra has not been without difficulties. Linear algebra is considered to be a difficult subject by students (Possani, Trigueros, Preciado & Lozano, 2009) and Kazunga and Bansilal (2017) note that the teaching and learning of undergraduate linear algebra is regarded as a daunting experience. This affirms that linear algebra is difficult for both instructors and students (Salgado & Trigeuros, 2015). Jupri and Gozali (2021) no doubt posit that prospective teachers encounter difficulties in mathematics concepts but these should be overcome. One way to overcome difficulties is to strengthen students' conceptual understanding and procedural fluency. In all the methods of solving SLEs mentioned above, memorizing algorithms without understanding the basic concepts of linear algebra is rife among students. Kazunga and Bansilal (2017) note that undergraduate students cope with procedural aspects of linear algebra but have problems of conceptual understanding of the same. Students' preferably procedural successes coming from repeating algorithms to some extent masks possible underlying conceptual gaps (de Villiers & Jugmohan, 2012).

The use of non-matrix methods is common in high schools mathematics but wanes as students transition to undergraduate mathematics. Notably, as the number of equations and unknowns increase, the foundational methods of substitution, elimination and graphical become cumbersome. Undergraduate students are introduced to matrices and matrix operations to develop procedures that are suitable for solving SLE of any size, that is, of order $m \times n$ where $m, n \in \mathbb{N}$ (Mandal, Cherukuru & Rani, 2021). Thus, the matrix method is useful and convenient to solve SLEs of many equations and unknowns effectively.

In Skemp's (1962) seminal work on schema, he defined it as an organized body of knowledge that students develop in the process of learning a particular concept. According to Skemp, a schema connects students' past learning to the current and future as follows: "more efficient current learning, preparation for future learning, and automatic revision of past learning" (p. 140). A schema is a mental construction that enable students to solve problems and this disposition plays a central role in formal learning and teaching (Kirschner & Hendrick, 2020). Students rely on developed schemas to organize knowledge and solve problems (Powell, 2011). If the schema is broad, it is more likely that students will recognize connections between strategies that have been taught and the implied problems that make use the same strategies (Fuchs, et al., 2006). According to Soderstrom and Bjork (2015), this forms part of knowledge transfer whose attainment is regarded as one of the most important goals of instruction. Success in solving problems creates full mental representations of the schema in students, which in turn facilitates the recall of information as needed to solve related problems (Skinner & Cuevas, 2023). Schemas are key determinants of the progress of students' learning of mathematics (Ndlovu & Brijlall, 2019) and having fully developed schemas would provide opportunities for students to make connections to mathematics concepts in the same and unfamiliar contexts.

Problem statement

Solving SLEs is a core concept in introductory linear algebra (Liu, 2015) and having a deep understanding of it assists in comprehending many applications and topics in linear algebra (Karunakaran & Higgins, 2021). The choice of the method to use in solving a SLE is dependent on the make-up of the system. As part of schema development in solving SLEs, undergraduate students ought to demonstrate knowledge to determine the appropriate method to use in the given circumstances. It is important to study how students learn solving SLEs since there are many subtleties involved in students' understanding of SLEs (Borji, Martínez-Planell & Trigueros, 2023). Hence, the purpose of this study was to explore the nature of students' development of the schema for solving SLEs. The objective of this study was the use of the Triad stages to assess undergraduate students' schema development in learning linear algebra, particularly in SLEs. By seeking to understand how students think when engaged in solving mathematics problems, course instructors might be able to improve the teaching and learning of solving SLEs by making it more meaningful so that the schema for the given concept is attained. Research on students' schema development of solving SLEs through the lens of the triad theory is scarce (Parraguez & Oktaç, 2010).

LITERATURE REVIEW AND THEORY

Students are initially introduced to matrix theory in linear algebra then to vector calculus in order to be able to solve SLEs of any order. Studies on students' understanding of matrices and solving SLEs mostly used the Action-Process-Object-Schema (APOS) theory. Kazunga and Bansilal (2017) explored the students' mental conceptions of matrix operations using the APOS theory. Students in that study managed to interiorise matrix addition, subtraction, multiplication and scalar multiplication concepts. But the same students did not develop the schema for multiplication of a row and column matrices. Students' lack of conceptual understanding of essential concepts in SLEs might negatively affect their achievement in linear algebra. Dis-fragmented knowledge and reliance on procedures is an indication of non-schematic learning according to Skemp (1962). To alleviate this, student-developed worksheets were used as a strategy to make sense of new concepts in linear algebra in the study by Arnawa, Yerizon, Nita and Putra (2019). The use of APOS-based approach improved students' achievements in learning matrices and vector spaces, particularly in SLEs. Modelling problems were also used in the worksheets, where modelling SLEs represents one of the applications of linear algebra to real life situations.

Without full schema development, students oftentimes have difficulties in understanding and integrating new mathematics knowledge. A study by Berger and Stewart (2020) used the concept of topology proofs to investigate how students' schema develop and the effect of interactions with peers and instructors on that development. They discussed the definition of schema according to Skemp (1962) and Dubinsky & McDonald (2001) and how these align to the Piaget and Garcia's (1989) triad framework. The content analysis of students' responses to a final examination revealed that the majority of students did not develop the full schema of topology proofs but operated in the initial stages by the end of the semester. Similarly, a study by Hannah et al. (2016) sought to develop students' conceptual understanding in linear algebra by combining the frameworks of Tall's (2004) three worlds and APOS theory to analyse students' resulting levels of cognition. After the analysis of students' tests and examination scripts and interview transcriptions, Hannah et al. found that students did not rely on rote to learn basic concepts in linear algebra. The students could explain precisely the concepts and were able to use the acquired knowledge to solve given problems. Moreover, a study by Stewart and Thomas (2010) also combined the APOS and Three worlds of mathematics thinking frameworks to describe all possibilities of understanding in linear algebra concepts of linear independence, basis and span. After analysing students' responses using this dual framework, Stewart and Thomas found that students in traditional courses were proficient in matrix manipulation as a process or symbolic conception but were inferior in establishing connections between matrix manipulations and the associated concepts. This normally happens when students regard linear algebra concepts as a collection of definitions and procedures to be learnt by rote (Martin et al., 2010). Kazunga and Bansilal (2020) assert that "mathematics instructors need to focus on their students' understanding of interrelationships between concepts, rather than carrying out procedures" (p. 341).

The choice of a method to solve SLEs depends on the spontaneity, speed and accuracy. Maharaj (2018) investigated students' choice of a method as a relation to the students' level of understanding; the majority chose to solve SLEs using the Cramer's rule and the Gauss elimination method. His results revealed that students prefer the Cramer's rule relative to the Gaussian elimination, even though it requires longer steps. The Gaussian method is a systematic elimination of variables using elementary row operations which should be spontaneous for systems involving large number of unknowns. Students require guidance in order to initiate the Gaussian elimination method. To solve large systems of equations, a good method hinges on precision and speed because the computations involved are sometimes immense (Mandal, Cherukuru & Rani, 2021; Gharib et al., 2015). The goal of solving SLEs using the Gaussian elimination is to reduce computational time and complexity.

The development of the level mathematics cognition of the students culminates in the schema for that mathematical concept. To determine the possible mathematical understanding of the students in this study, the Piaget and Garcia's (1989) triad framework was used. The triad theory focuses on the hierarchical mental constructions that goes on in the mind of students when trying to learn a mathematical concept. According to the triad theory, before a schema becomes coherent, it must go through the three stages; the intra-, inter- and trans-operational stages. The intra-stage is the preliminary level of conceptualisation whereby a concept is conceived in an isolated manner. The students' understanding is localised and relationships between processes are not perceived. At this lowest stage of schema development, students would have a collection of rules for solving SLEs using Gaussian elimination, inverse or Cramer's methods, but would not recognise the relationships between them. For example, in the schema development of solving SLEs by the Gaussian elimination, performing step-by-step elementary row operations is at the intra-stage. The student realises that some facts and principles in the schema are connected but is not able to justify and explain the connection (Borji & Martinez-Planell, 2020). Once connections are established between concepts and other previously-held schemas, the individual is at the inter-stage. The schema enters the inter-stage when students make connections between the nature of SLE (facts) and the method that corresponds to type of solution (principles). This includes more complex examples and links to the geometrical representation of solutions to SLEs. The student can explain and justify how changes in one structure leads to changes in the other.

The trans-stage outlines a coherent structure that underlie the relationships constructed in the inter-stage. It is further characterised by the construction of coherent structures underlying some of the connections discovered in the inter-stage of development (Clarke et al., 1997). Being able to take appropriate decisions to effectively solve SLEs depicts the trans-stage (Possani et al., 2009). At this stage, students can modify their knowledge to solve related situations. The triad framework is not linear but represents a continuous spectrum for developing a schema of a mathematics concept. The APOS theory's description of schema coincides with the trans-stage of triad since that is where a coherent structure emerge. The isolated (intra-) and connected (inter-) objects constitute a pre-schema. Students at trans-stage can coordinate two or more different interpretations of solving SLEs to mean the same thing.

Conceptual learning of mathematics should enable students to see relationships between facts and principles, instead of relying on procedures (Dewi et al., 2021). The triad theory is used in this study as a mechanism to describe schema development in general and the solution of SLEs is used as a particular instance. It provides the structure for interpreting the students' understanding of the SLEs and classifying their responses according to the three stages of the triad theory.

METHODOLOGY

To address the research problem on solving SLEs, a descriptive-exploratory study design was adopted on a case study of 160 B.Ed. degree students majoring in mathematics. After a traditional instructional in linear algebra, the author conducted semi-structured interviews with selected students to determine the extent of their schema development. This allowed the author to obtain information about students' understanding of solution of SLEs. The following task formed part of the interview:

1. Find the solution to the 3×3 system: $x + y - z = 6$; $3x - 2y + z = -5$; $x + 3y - 2z = 14$.
2. Solve this system of linear equations: $2x + 3y + 3z - u = 3$; $x + y - 2z + 3u = 4$; $5x + 7y + 4z + u = 5$

3. If $A = \begin{pmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{pmatrix}$, find the products AB and BA and hence solve $x - y + z = 4$; $x - 2y - 2z = 9$; $2x + y + 3z = 1$.

In item 3, participants were expected to see the underlying role played by the identity matrix in showing that two matrices are inverses of each other. The relationship was further complicated by the presence of the constant 8, which led to the intricate relationship $B^{-1} = \frac{1}{8}A$. The second part of the question required participants to identify that matrix B was given, hence the need to solve the SLE using the inverse method. But in place of B^{-1} on the right-hand side, they need to pre-multiply by $\frac{1}{8}A$. The solution for this item represent a coherent connection among many concepts in matrix algebra, an attainment of such is evidence of full schema development in solving SLEs. The expected full solution for this question is given in Figure 1.

We find $AB = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8I$

and $BA = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} = \begin{bmatrix} -4+7+5 & 4-1-3 & 4-3-1 \\ -4+14-10 & 4-2+6 & 4-6+2 \\ -8-7+15 & 8+1-9 & 8+3-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8I.$

So, we get $AB = BA = 8I$. That is, $\left(\frac{1}{8}A\right)B = B\left(\frac{1}{8}A\right) = I$. Hence, $B^{-1} = \frac{1}{8}A$.

Writing the given system of equations in matrix form, we get

$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$. That is, $B\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$.

So, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} = \left(\frac{1}{8}A\right)\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$

Hence, the solution is $x = 3, y = -2, z = -1$

Figure 1: The expected solution for question 3

These items were carefully chosen to appeal to each of the cognitive operations identified in the triad theory.

The author used the purposive sampling technique to select 15 students to participate in semi-structured interviews covering solving of SLEs. To strike a balance in the distribution, five above average, five average and five below students made up the sample, based on their performance in final examination of the previous year. This enabled the researcher to observe a wide range of student responses according to the principle of maximum variation sampling technique (Doruk, 2019). The author conducted individual task-based interviews to each of the participants. The participants were assigned pseudonyms T1, T2 and so on up to T15 for ease of reference while maintaining the real identify of participants confidential. The limitation of this study was the impossibility to deduce the schema of another individual, thus, the best the researcher could do was to conjecture about participants' schema development based on analysis of their responses. Nonetheless, the task-based interviews provided data that acts as proxy of how students think and reason about the mathematical concept at hand (Plaxco & Wawro, 2015).

Data for this study were generated from the written and verbal responses to the three questions on the task on solving SLEs. The verbal responses were audio-recorded and transcribed. Data collection provided evidence of how students made the mental relationships when learning the matrix method to solving SLEs. Data analysis was two-fold: the author performed content analysis to the participants' responses to the three items and ascribed categories to describe observations according to the intra-, inter- and trans-stages for all questions, as well as to describe any unexpected observations (Borji, Martínez-Planell & Trigueros, 2023). The categories were introduced as observations were made and included re-analysing previously categorised data to make sure no prior occurrences of categories had been missed by mistake.

FINDINGS

The analysis of data was according to the intra-, inter- and trans-operation stages for all the questions in the interview.

Question 1 analysis

All the students' development of the solving SLEs schema commences at the intra-stage. Item 1 could be solved by any of the Cramer, inverse, Gaussian elimination methods. The frequencies of the method used are given in Table 1.

Method	Gaussian	Cramer	Inverse	Other	Total
Frequency	8	4	0	3	15
Frequency of correct solution	6	4	0	3	11

Table 1: The frequencies of the usage of methods to solve SLEs

The Gaussian elimination had the highest frequency as participants realised that this method can practically solve any given SLE, unlike the Cramer and inverse. The latter two only work for square matrices and are inconclusive if the solutions to the SLE are infinitely many or do not exist. Upon probing why they preferred the Gaussian elimination method, T1, T6 and T8 respectively said: "*because the question was not specific so I picked the easy one for me*", "*in most of the question papers I used to practice with Gaussian; it was mentioned that I should use the Cramer's rule on certain questions. So I thought if it wasn't specified that we use the Cramer's rule I should always use the Gaussian elimination*" and "*Yes, I can use the Cramer's rule it's fine but for* 3×3, *I can also use this one. It's the one that's much easier for me*". Figure 2 illustrates the correct solution by T12 using Gaussian elimination.

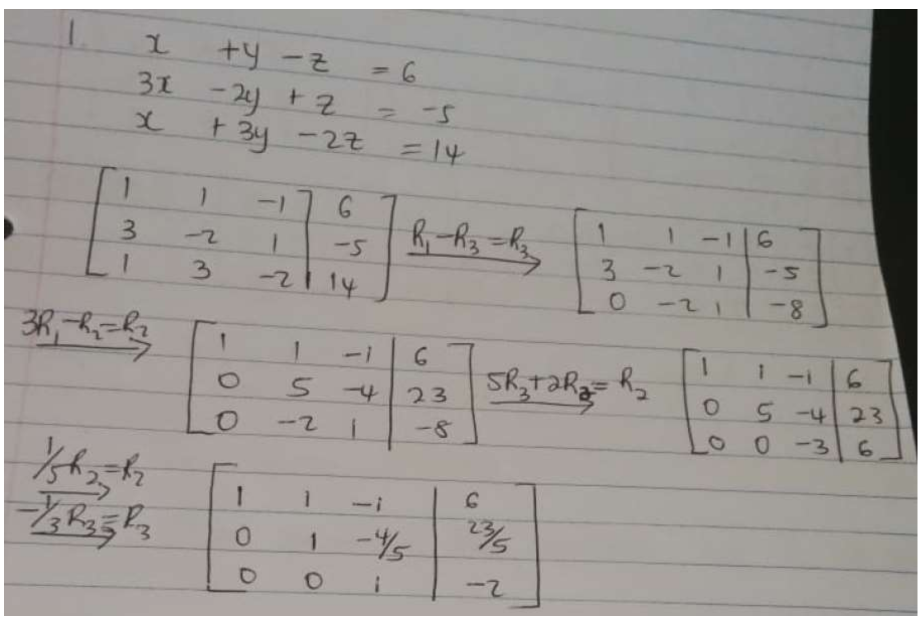

Figure 2: Correct steps in the Gaussian elimination method by T12

After T12 obtains $z = -2$, the rest of the values were obtained by back-substitution. T7 commented that, "*Since there were no instructions I thought it would be easier to use the Gauss-Jordan*" but did not get the correct final solutions due to multiplicity of steps involved. T2 also used the Gauss-Jordan method and managed to get the correct solutions despite doing a few more steps.

Three-quarters of those who chose to use the Gaussian got the correct solutions. The two who got incorrect solutions erred in executing elementary row operations. Four participants, T4, T9, T14 and T15 chose to use the Cramer's rule and they all solved it flawlessly. The four found the Cramer's rule easier to use, as shown in the dialogue with T9. R stands for the interviewer.

R: Why did you prefer the Cramer's rule in question 1?

T9: The question says find the solution; that's what makes me use Cramer.

R: Why didn't you use the inverse method in question 1?

T9: I dont know the inverse method; I only know the Cramer's rule. I don't know how to find the solution using any method other than the Cramer's rule.

None of the participants chose the inverse method, and T9 commented that he does not know it. After having been taught the topic of linear algebra, three participants did not use matrices to solve the problem in item 1. Instead, the three used the elimination of variables method to solve the SLE and obtained the expected solutions (shown in Figure 3).

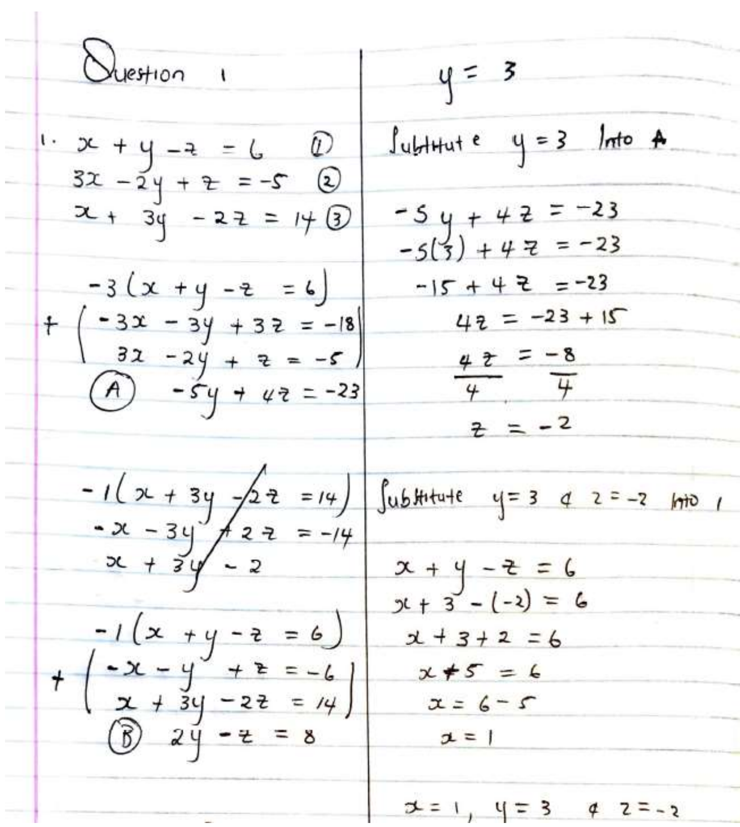

Figure 3: A correct elimination of variables method by T11

T3 also used the elimination of variables method, but due to the lengthy process, he made an error, which spoiled the entire solution. In the interview, T3 said the following concerning the elimination of variables.

R: I can see you used the substitution method where you made x the subject, y the subject and then you substituted into z to get your z. Is it the best method to use if we are given such a scenario?

T3: No Sir. But I tried.

R: You got all your answer correct. But can we rely on it to solve a system of linear equations in linear algebra?

T3: I don't want to say I can rely on it. But there are many ways to kill a cat.

R: Yes as long as the cat is dead. It's not important 'how'? But suppose I am going to give you a system with five equations and five unknowns. Do you think you can rely on the substitution method?

T3: No sir.

However, this represented a pre-intra-stage of solving SLEs in linear algebra. With an intra-conception, participants had high chances of doing the step-by-step processes to solve the SLE using the Gaussian elimination, Cramer's rule or inverse method but they do not perceive the connection of the relationships. For example, all the participants who used the Cramer's rule could not explain why it works.

Question 2 analysis

This item appealed to both the participants' intra- and inter-operational conceptions of solving SLEs. Participants were expected to establish the connection between the appropriate method to solve a SLE and the nature of the solutions. It was important for participants to identify that all SLEs with more variables than equations lead to free variables, hence yield either infinitely many solutions or no solution. The unique solution can never be possible when free variables are involved. In this item, only eight participants were able to apply the Gaussian elimination method, which was the only possible method in the given scenario, as shown in Figure 4.

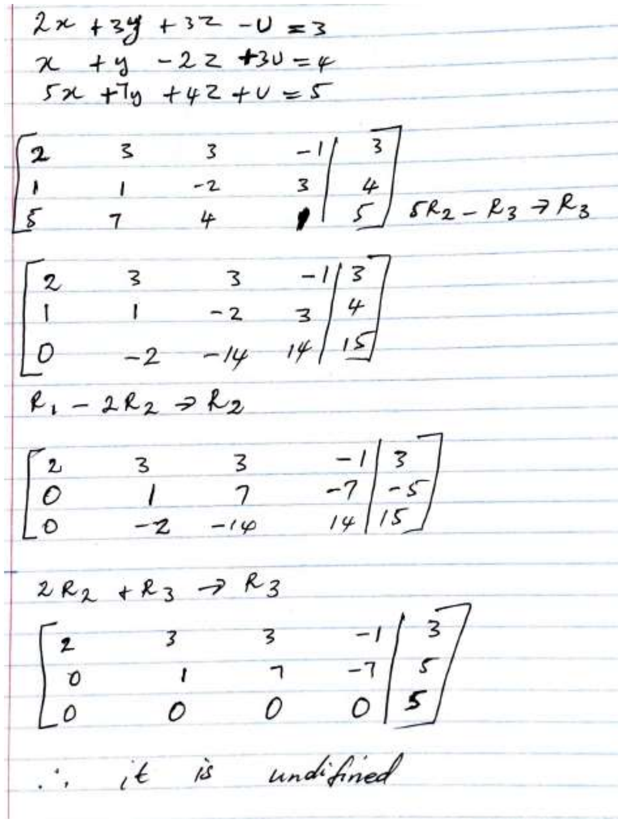

Figure 4: Correct use of the Gaussian elimination method on free variables by T4

Almost all the participants swopped row 1 and row 2 as the initial row operation in this question. Applying the rule to swop rows was correct, but some participants still had challenges to reduce the augmented matrix to upper triangle, as depicted in the dialogue below.

R: Why did you reverse row 1 and row 2?

T9: Because they say the leading number in row 1 must be 1, so I thought if I exchange the rows it makes things easier for me.

R: Perfect. However, why didn't you finish this question up to the point of finding the values of x, y, z and u?

T9: But didn't they say we must have that main diagonal composed of one's.

R: I get your point [interrupts]

T9: That's where I got stuck because I don't understand where to go to from here. So I just left it here.

Inadvertently, T4 still solved the problem without swopping rows. This implored the inter-operational skill. Five of the eight who chose the Gaussian elimination went ahead and performed the elementary row operations and obtained four zeros on the last row. This was obviously concluded as *no solution* since the element after the vertical line was a 5 or -5 in some cases. Performing the elementary row operations constitute inter-operational conception. The remaining three participants were lost in the row-reductions, culminating in incorrect conclusions.

T7 only managed to transform the equations to augmented form but aborted the solution as shown in Figure 5.

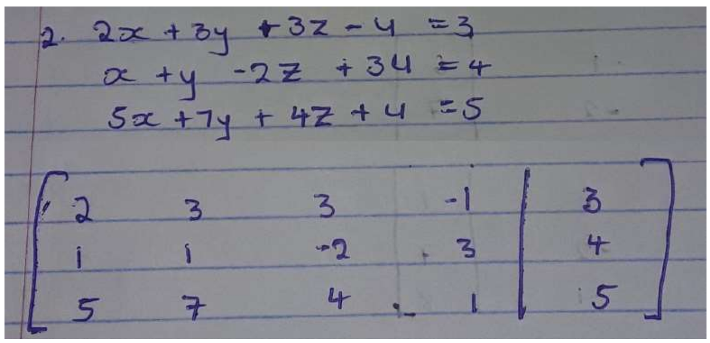

Figure 5: An incomplete solution by T7

On being asked why he didn't complete the solution, T7 replied "*Oh yes I was wondering how my entries would be there. I know that for a 3x3 matrix its 1-0-0, 0-1-0 then 0-0-1. I wasn't quite sure what I was going to do with the last row.*" He did not realise that reducing to upper triangle can be done for matrices of any order. For this item, six participants left it unanswered. T3 was one of them and when probed, he had this to say:

R: Why didn't you attempt this question?

T3: Because I don't know it Sir.

This was an evidence of lack of inter-operational conception of SLEs with more variables than equations. Also, none of the participants attempted to use the Cramer's or inverse methods as these were totally inapplicable. The dialogue below depicts T8's attempts to exclude the other methods.

R: You still prefer the Gaussian elimination method. But do you think the other two methods are applicable; the Cramer's rule and the inverse method in this particular case?

T8: It's going to be very difficult. Yes, they can be applicable, but to find the determinant of a 3x4 matrix is very difficult.

R: Is it difficult or impossible?

T8: No it's not impossible; you can find it. You can also find the determinant for this 3x4 matrix.

R: Is that so? We talk of determinant of a square matrix, don't we? And this is not a square matrix, is it?

T8: No we can't. I remember now. Because this is a 3x4. This one it's not possible.

Finally, one participant circumvented the matrix approach and pursued the elimination of variables method. Interestingly, he managed to get the *no solution* after all the variables were eliminated as shown in Figure 6.

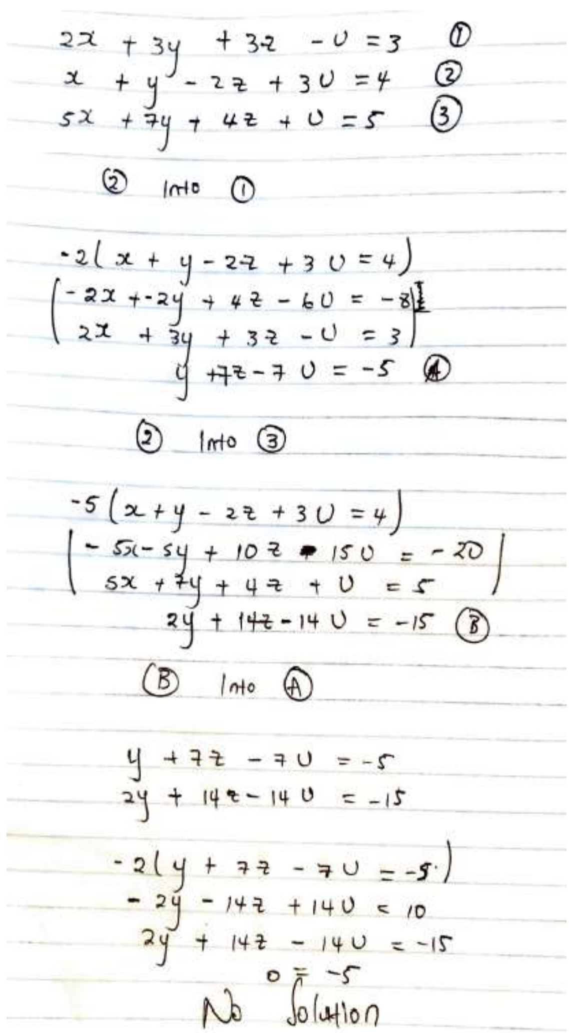

Figure 6: The correct use of the elimination of variables method by T11

However, these did not depict the schema development for linear algebra, even though the answer was correct.

Question 3 analysis

The analyses of data revealed that three participants, T3, T8 and T10, skipped the question altogether, which could be an indication of lack of inter- and trans-operational conceptions. Disregarding the products of the matrices A and B and B and A in the first part of the question, the rest of the participants attempted to solve the SLE in the second part using one of the methods they have learnt. Seven participants went straight ahead to solve the SLE using Gaussian elimination in the second part. These seven failed to use the fact that the question had two parts, whereby the first part imply the second. In-advertently, all of them were lost in the elementary row operations so that none of them got all the solutions right. T15 and T4 only managed to get the first solution $z = 1$. This on its own represent inadequate intra-conception of solving SLEs. Similarly, four students chose to use the inverse method of solving the SLE to the second part of the question. To find the inverse of the coefficient matrix, the participants were faced with monstrous task of computing the determinant, matrix of co-factors and the adjoint. Since finding the inverse using that way was involving, T13 aborted the process whilst T12 got lost in evaluating the adjoint and obtained the incorrect inverse. Finally, as was the case in question 1 and 2, T11 used the substitution and circumvented the matrix method to solve the SLE. This high school method is cumbersome to SLEs with three or more variables. Thus, his attempt did not yield correct solutions due minor manipulation errors. Figure 7 illustrates the simple error in T11's substitution method.

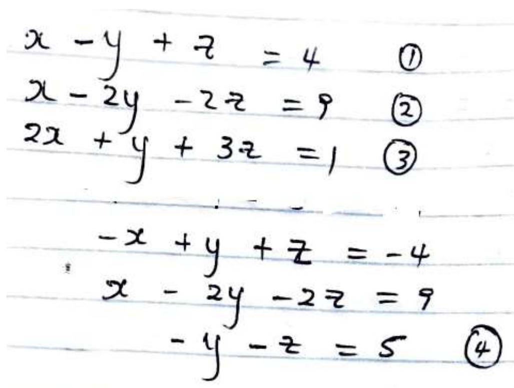

Figure 7: Minor error in the substitution method.

From Figure 7, equation 4 should be $-y - 3z = 5$, which caused the final result be wrong.

Ten participants followed the instructions to evaluate AB and BA but due to computational errors, they could not get $8I$ in either case. Figure 8 illustrates an attempt by T6. In the case of T1, he admitted to his errors by saying "*I was just lazy to multiply those two matrices*". This normally happens when participants feel that the steps are cumbersome.

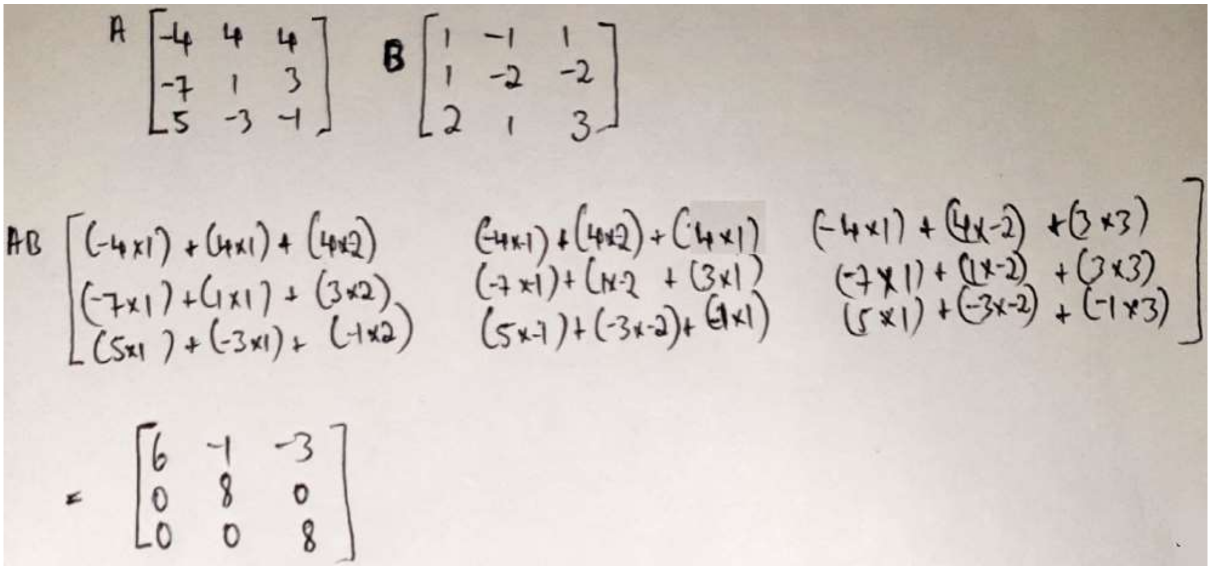

Figure 8: Computational error in matrix multiplication of A and B

The very essence of matrix multiplication was also a challenge to some participants. Figure 9 depicts an instance of incorrect multiplication. This direct multiplication of corresponding elements led to the unintended result $AB = BA$, but the participants did not take note of that.

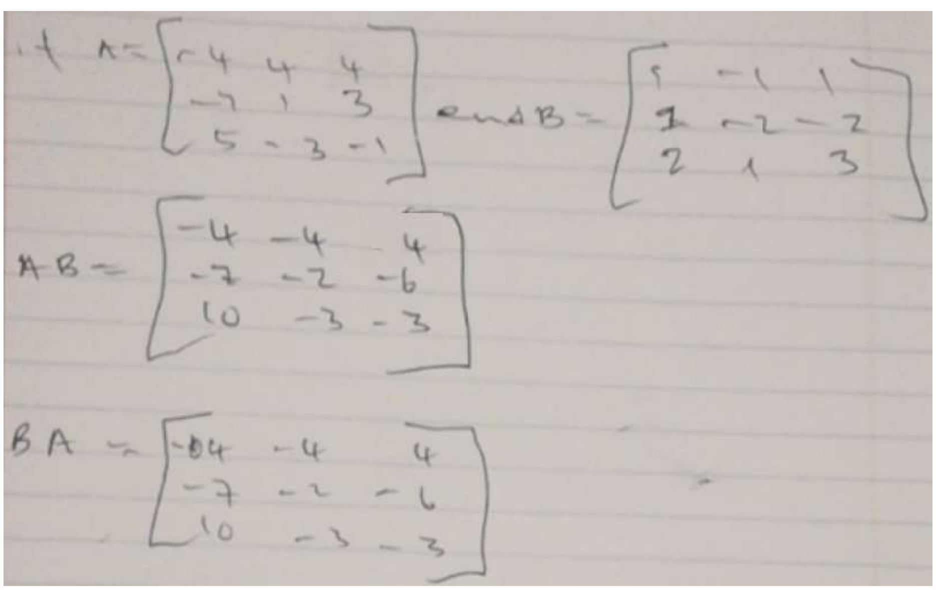

Figure 9: Wrong conception of matrix multiplication by T9

As such, the inverse relationship between A and B could not be established nor could they make use the fact that the coefficient matrix of the SLE is matrix B. Thus, the underlying connection of the inverse and identity matrices and inverse method could not be realised. On the other hand, some participants avoided the steps in matrix multiplication by skipping it. They went ahead to solve the SLE in the second part of the question using the other methods without using the concept of inverses arising from AB and BA. Many participants managed to get the correct solution using the other methods. T15 said, "*I used a different method. I used the Cramer's rule and got the correct answers as yours*". T12 seemingly used the inverse method since the question indicated the inverse arising from AB and BA (shown in the dialogue below).

R: Why did you the inverse method?

T12: It's in the question.

T5 also tried to use the inverse method but aborted it due to the cumbersomeness of the work (shown in Figure 10). She just stopped at the stage of computing the matrix of co-factors. A better way would have been to evaluate the inverse based on the elementary row operations on the augmented matrix $\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & -2 & -2 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right)$. Regardless, this solution would still be unacceptable as the question required a connection to the AB and BA relationship.

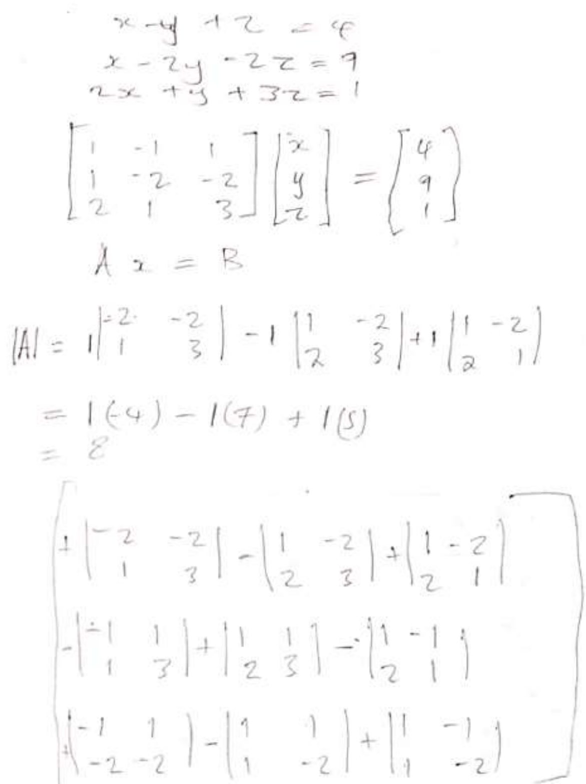

Figure 10: Attempt to solve the SLE using the inverse method by T5

DISCUSSION

Undergraduate students' level of understanding of linear algebra determines their universal approach to deal with situations in diverse mathematical tasks in linear algebra (Trigueros, 2019). In this study, comparison of students' written and verbal responses revealed critical differences in understanding that can be attributed to the stage schema development that they evidence (Arnon et al., 2014). Students have a tendency to go for an easy method for solving a SLE, thus the high number of students who used the Gaussian elimination and Cramer's rule. Mandal, Cherukuru and Rani (2021) concur that Gaussian elimination is the easy and best to solve an arbitrary SLE. Oftentimes, many students' approach to Cramer's rule is rule-bound and procedural (Ndlovu & Brijlall, 2019) thus in most cases, they cannot explain why the Cramer's rule works. This procedural approach indeed works better for the lowest level of conceptualisation, the intra-stage. Hence, the majority of the students who used the Gaussian elimination and Cramer's rule managed to attain the full solution. Hanna et al. (2016) argued that students can cope with the manipulation of numbers involved when solving systems of equations, but they do not develop a deep understanding of these concepts and cannot apply them in new contexts. Maullina and Setyaningsih (2023) posit that students' understanding of concepts should go beyond developing abilities for step-by-step routine computations. The only thing which students needs to strengthen is the manipulation of the each of the methods, which are the execution elementary row operations for Gaussian elimination and inverse methods, and computation of determinants for the Cramer's rule and inverse methods. Findings revealed that students are affected by manipulation errors in the process of solving SLEs. This was more pronounced on T7 who chose to use the Gauss-Jordan method in question 1; the extra steps of reducing the augmented to both upper and lower triangles led to many errors. Gaussian elimination is faster than the Gauss-Jordan method since the latter is an extension of the former (Gharib, Ali, Khan & Munor, 2015). The inverse method was not popular with students in question 1 and T9 confessed ignorance of this method. Moreover, when the inverse method was required as part of the solution in question 3, none of the students was able to apply it.

Oftentimes students focus on the methods which they find comfortable with at the detriment of alternative methods. At the inter-stage, students are expected to be decisive about the appropriate method to use based on the underlying relationships called upon in the question. Non-matrix method also featured amongst the methods used by students but the users admitted that for higher orders, these methods are cumbersome. That means that some students did not apply the Gaussian elimination spontaneously, even for systems with relatively large number of equations and unknowns (Harel, 2017). Though there is no best method yet proposed to solve SLEs, consensus point to the need for accuracy and speed as the determining factors (Gharib, Ali, Khan & Munor, 2015).

The findings also revealed that some students stick to a sole method of solving SLEs despite changes in problem-contexts. T1, T4, T6 and T9 used the Gaussian elimination to solve all the three items and T11 resorted to the elimination of variables technique for all the three questions. Students lack the flexibility in the application of methods of solving SLEs, leading to incorrect solutions in some cases. Especially the trans-stage require students to make decisions for the technique to solve SLEs by taking into consideration the implicit underlying structures. Learning occurs when experiences produce changes to connections in the brain (Gallistel & Matzel, 2013). Students had to recognise the structures in order to adequately respond to the questions (Maullina & Setyaningsih, 2023).

The schema development according to the Triad is hierarchical hence item 2 had both intra- and inter-conceptions and question 3 had all the three stages of intra-, inter- and trans-conceptions. In item 2, only 50 percent of the participants recognised the much-connection between the 3×5 augmented matrix and the Gaussian elimination method. If the matrix is non-square, the solution can only be either no solution or infinitely-many solutions and requires the Gaussian elimination method. The rest of the participants tried other unsuitable methods to this question and this represented inadequate inter-conception skills. As for intra-stage, based on the chosen approach to solving SLEs, most of the students proceeded to perform the correct procedures. However, this was only effective if the selected approach was the correct one.

Students are said to attain schema development in solving SLEs if they can demonstrate the trans-stage of understanding. In question 3, students were expected to establish the underlying relationship of the inverse of matrix A and B and its use in the solution of the SLE. To achieve this, students must have established that linkage of the identity matrix and A as a partial inverse of B. This pivotal connection represents the inter-conception of solving SLEs. The isolated relations of matrix multiplication and the use of the inverse method, Gaussian elimination or Cramer's rule to solve SLEs represent the intra-conception. According to Donevska-Todorova (2016), performing row reductions and computing determinants and inverses is regarded as procedural, whilst if an individual intends to coherently link row reductions, determinants and inverses to a particular method of solving SLEs, then such an understanding is regarded as trans-conceptual. It is important for students to be able to determine the relationship between facts and principles, instead of memorising algorithms (Dewi et al., 2021). In addition, students tend to perform better on questions that require procedural than conceptual understanding (Bouhjar et al., 2018), hence the frequency of correct responses honed as students progressed from question 1 to 3.

The coherence of these intra-, inter- and trans-conceptions typify a student who has attained the schema for solving SLEs. The intra-stage is necessary for inter-stage to take effect whilst the coordination of both the intra- and inter-stages precedes the trans-stage. Students' good decision-making skills are required to unpack some of the implicit and explicit global nuances of solving SLEs. However, the findings revealed that none of the participating students achieved the schema for solving SLEs, yet it is the climax of students' learning of linear algebra. The majority of students operated at the lower echelons of schema developments for solving SLEs, which is the intra-stage (Berger & Stewart, 2020). The nature of matrices and the thinking required to understand them pose challenges in linear algebra (Dorier & Sierpinska, 2001), most commonly to students who take linear algebra for the very first.

CONCLUSION AND IMPLICATIONS

The three questions in this study were carefully designed to bring out students' conceptualisations in solving SLE at the intra-, inter- and trans-stages. The final stage of trans-conceptualisation was not achieved due to manipulation errors, un-connected relations and bad decisions on the method to use. Students were limited in identifying and selecting the suitable methods that were applicable to the specific contexts of solving SLEs. An individual's level of cognition of a mathematical concept determines the individual's general tendency to deal with problems in diverse mathematical tasks of the concept. The complexity and depth of students' understanding of a topic depends on their ability to establish relations among the corresponding components of the schema. Instruction in linear algebra should appeal more to the trans-stage of understanding since the intra- and inter-stages represent pre-schema. Based on the findings, the majority of the students attained the intra-stage of conception as they used diverse methods to solve the SLE and no relationships were extrapolated therefrom. However, only fifty percent managed to reduce the 3×4 augmented matrix to echelon and conclude that there was no solution to the SLE. Furthermore, none of the students managed to establish the connection among matrix multiplication, the inverse relationship and the solution of the SLE. This classification of data according to the triad theory represents an incomplete development of the schema for solving SLEs. The study by Ndlovu and Brijlall (2019) also classified students' mental constructions in using the Cramer's rule to solve SLEs as predominantly at the action stage of the APOS theory. Using both the APOS and triad theories, Trigueros (2019) revealed the notion of schema development, which gives important information about students' progress in a course.

By patterning students' cognition of solving SLEs using the triad, this study fills the literature gap of improving the understanding of schema development (Borji & Martínez-Planell, 2020). The implication of this study is the need to develop activities that address higher stages of the triad in the students' minds. These activities foster coordination, accommodation and assimilation of new relationships into their developing schema and the construction of those relationships strengthens instruction of concepts through promoting coherence. It is necessary to continue studying the teaching and learning of linear algebra as this provides new insights into better ways for students to understand the subject (Stewart et al., 2022).

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