Problematic and Supportive Aspects of Indirect Proof in Afghan Undergraduate Students' Proofs of the Irrationality of $\sqrt{3}$ and $\sqrt{5/8}$

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Abstract: *In this study, we aimed to find problematic and supportive issues in Afghan undergraduate students' proofs of the irrationality of* $\sqrt{3}$ *and* $\sqrt{5/8}$ *while using the indirect proof method. Collecting and analyzing produced proofs of 30 sophomore and 48 senior undergraduate students on the irrationality of* $\sqrt{3}$ *and* $\sqrt{5/8}$*, respectively, revealed that the majority were not able to extend indirect proof beyond showing the irrationality of* $\sqrt{2}$ *due to not being able to apply the supportive 'multiples of an integer' notion in their reasoning process of the irrationality of* $\sqrt{3}$*, and this notion was not supportive in the case of showing irrationality of* $\sqrt{5/8}$*, and finally we propose the method of finding nonzero integral solutions to the resulted Diophantine equation and using the rational roots theorem to prove that* $\sqrt{p/q}$ *is irrational.*

INTRODUCTION

Theoretical Background: We trace back how the concept of irrational numbers is understood in high school considering the works of prominent researchers to form a theoretical background for the current study and determine how this mental image of irrational numbers may affect students' understanding of irrationality at the university level. The important notions that may influence students' understanding of rational and irrational numbers are repeating and nonrepeating decimal representations. We start with a decimal representation of some rational numbers at the school level, usually starting beyond grade 7 in Afghanistan. For example, there are rational numbers that their decimal representation is finite or ends such as $\frac{1}{2} = 0.5, \frac{2}{5} = 0.4, \frac{1}{8} = 0.0125$ and so on.

There are some rational numbers that their decimal representations does not end, rather it periodically repeats infinitely, such as $\frac{1}{3} = 0.333\cdots, \frac{1}{6} = 0.16666\cdots, \frac{5}{11} = 0.454545\cdots$, $\frac{3079}{9900} = 0.31282828\cdots$ and so on. Conversely, if we let $x = 0.333\cdots$, compute $10x = 3.333\cdots$, and subtract both sides of the former equality from the latter, we obtain $9x = 3$ and solving for x yields $x = \frac{1}{3}$. This observation indicates that there are two types of rational numbers: rational numbers with finite/ending decimal representations and those with repeating non-ending decimal representations. At some point in school, the teacher puts the results of this observation into the following proposition without providing a general proof: every rational $\frac{a}{b}$, provided $b \neq 0$ and a are integers, can be written as a finite/ending decimal representation or repeating infinite/non-ending decimal representation; conversely, every finite or periodically infinite decimal expansion is equal to a rational number.

There is another interesting observation while dealing with $\frac{1}{3} = 0.333\cdots$; that is, multiplying both sides of $\frac{1}{3} = 0.333\cdots$ gives $1 = 0.999\cdots$, which is counterintuitive for some students. This counterintuitive observation is actually correct, but why? Suppose we have the decimal $0.99999\cdots$ and is a rational number. Suppose $x = 0.99999\cdots$ and multiplying both sides by 10, gives $10x = 9.99999\cdots$and subtraction gives $9x = 9$ or $x = 1$. As a final result of these few above observations, we knew two types of rational numbers $\frac{a}{b}$ in its reduced form: those $\frac{a}{b}$ in which the only factors of b are 2 and 5 (which can be written as finite or infinite decimal representation such as in $\frac{1}{2} = 0.5 = 0.499999\cdots$) and the rest of rational numbers that can only be written as infinite decimal representations such as in $\frac{1}{3} = 0.33333\cdots$ (Niven, 1961).

According to Tall (2013) theory of 'three worlds,' all of us learn mathematics by building on our previous knowledge and experiences that may be supportive and facilitate generalization of ideas in new contexts or may be problematic and impede our understanding of mathematical ideas. A met-before is "a mental structure we have now as a result of experiences we have met before." Supportive met-befores give the pleasure to learn more, whereas problematic met-befores cause frustration and force us to quit learning. For instance, the met-before "take away leaves less" is true while working in the set of positive integers and finite sets, but this met-before is no longer true in the set of negative integers and infinite sets. Students who are capable of making sense of new ideas will develop confidence in responding to problematic met-befores, address difficulties in learning mathematical concepts, and build increasingly knowledge structures latter on, whereas those students who are unable to deal with new situations becoming increasingly sophisticated may feel detached from learning new ideas, which in turn may lead to mathematical anxiety, causing them to quit learning new ideas just because they do not cope with problematic met-befores. In addition, a student's success in learning mathematical concepts at one stage may be impeded by problematic met-before in future learning. According to the goal-oriented theory of Skemp (1979), this problematic met-before deviates students from the goal of understanding a mathematical concept to the goal of learning procedures to solve routine problems. Learning procedures for solving standard problems is not a bad thing itself because it can enable a student to be successful initially, but overemphasizing it may not facilitate learning in new contexts.

Students' mental images of repeating and non-repeating decimals learned in school may be supportive in high school, but problematic met-before at university level mathematics, which in turn may pose difficulties for students in the process of transitioning from rational to irrational numbers. Tall (2013), for instance, points out that though students encounter irrational numbers such as $\sqrt{2}$, π and e in school being on the number line filling the gaps between rational numbers, they do not know what really these irrational numbers are. However, infinite decimals are conceived as one of its finite approximations, in which the precision is improved by adding one more digit after the decimal point. This perception is characterized by Kidron and Vinner (1983) as "the dynamic perception of the infinite decimal" (p. 306). On the other hand, Durand-Guerrier and Tanguay (2016) contend that students entering university have limited knowledge of real numbers and their conception of a number is dependent on how it is written; that is, they sometimes say π is not a "true" number but rather seen as a "sign," and only those numbers have a real number status when they are written in the decimal representation. Furthermore, Fischbein et al. (1995) claimed that little attention has been paid to clarifying and discussing irrational numbers in school mathematics, as well as not conveying mathematics as a coherent and structurally organized body of knowledge to students. One reason for this shortcoming may be the complex epistemological process of extending rational numbers to real numbers. For instance, in modern algebra, extending integers to rational numbers is easier to understand than extending rational numbers to real numbers.

Method of Indirect Proof: The method of indirect proof is prevalent in all mathematics. Indirect proof includes both methods of proof by contraposition and proof by contradiction. Proof by contraposition, put in simple terms, constitutes a process in which we first negate the hypothesis H and conclusion C and then prove the implication $\neg C \rightarrow \neg H$ for a mathematical claim or theorem in the form of $H \rightarrow C$ whenever it seems difficult or impossible to be proved by direct proof; the main reason behind allowing us to carry out such a process is that in the language of propositional logic we have $H \rightarrow C \equiv \neg C \rightarrow \neg H$ (Koshy, 2007). Proof by contradiction includes a process in which we first negate the conclusion C of a theorem $H \rightarrow C$ while imposing and maintaining specific conditions in the process and reaching a contradiction at the end, and this contradiction at the end of careful reasoning process proves that the theorem is correct; again the laws of propositional logic allows us to carry out this process because we have $H \rightarrow C \equiv [H \wedge (\neg C)] \rightarrow F$. Sometimes, both are needed in an indirect proof, and sometimes one is sufficient, depending on the nature of the theorem or mathematical claim.

Although the processes of indirect proof, proof by contraposition, and proof by contradiction may seem explicit according to the laws given above, it has been stated in the literature that indirect proof is difficult and problematic for students to carry out in proving mathematical claims (Harel & Sowder, 1998; Leron, 1985; Robert & Schwarzenberger, 1991; & Tall, 1979). The literature reports several reasons why students have difficulty with indirect proof. For example, difficulty in negating statements (Wu Yu et al., 2003) influenced by natural daily language in the form of being opposite of the given statement such as increasing function is seen as the negation of decreasing function, considering several possibilities for the given statement such as negating 'f is strictly decreasing' as 'g increasing, g is constant, or g is decreasing but not strictly decreasing' (Antonini, 2001), emotional barriers (Reid & Dobbin, 1998), and disliking indirect proof (Antonini & Mariotti, 2008) are some to name. On the other hand, the logical representation of proof by contradiction, that is, $H \to C \equiv [H \wedge (\neg C)] \to F$, is problematic mode of proving because one must simultaneously stick to the falsehood of the conclusion to be true and argue why it is false under a tremendous state of stress (Tall, 1979).

To reduce such difficulties and stress, we may consider Leron's (1985) suggestion of preferring to reorganize proof by contradiction such that it initially involves some steps of construction in the process of proving and postpone the contradiction to the end. For example, in proving the infinity of prime numbers, one may initially consider constructing some new primes from old ones in few construction steps for $N = p_1 p_2 \cdots p_k + 1$ as $2 \times 3 + 1, 2 \times 3 \times 5 + 1, 2 \times 3 \times 5 \times 7 + 1, \cdots$ and then deducing that if there were a finite number of primes, then we would always get another prime or a composite involving a new prime, leading to a contradiction. An alternative technique that has been employed is the use of generic proof (a generic example or particular case that is neither trivial nor very complicated) in which the irrationality of $\sqrt{2}$ is proved by first proving that if one squares a rational number where the denominator and numerator are factorized into different primes, then its square has an even number for each prime factor in the numerator or denominator. Thus, one can deduce that $\sqrt{2}$ cannot be rational because its square is 2, which only has an odd number of occurrences of prime 2. In similar approach, Tall (1979) presented the generic proof that $\sqrt{5/8}$ is irrational because 5/8 contains an odd number of 5s and also an odd number of 2s in its prime factorization, while the square $\frac{5}{8} = \frac{r^2}{s^2} = \frac{(p_1 \cdots p_i)^2}{(q_1 \cdots q_j)^2}$ has even number of each prime factor in the numerator or denominator, leading to a contradiction. He contends that this generic proof has explanatory power and can be generalized easily, whereas proof by contradiction is both problematic and not easily generalized by students. Malek and Movshovitz-Hadar (2011) also conducted an extensive study with first-year Israeli students taking linear algebra to investigate the use of generic proofs and found that their students benefitted from generic proof by "transformable cognitive structures related to proof and proving" (Male and Movshovitz-Hadar, 2009, p. 71). Looking at the benefits of using generic proofs as a starting point to proof by contradiction, the author also recalls his class experiences, showing that generalizing proof by contradiction is problematic for students beyond showing the irrationality of the square root of square free positive integers and fractions. The author asks: Why is this generic proof still unable to penetrate into textbooks so that students have the opportunity to easily access the generalizability of proof by contradiction, at least in the case of irrationality of the square root of non-zero rational numbers?

The alternative to proof by contradiction in the form of $H \rightarrow C \equiv [H \wedge (\neg C)] \rightarrow F$ may seem promising, but turning all contradiction proofs to more direct proofs of the nature explained earlier may not always be helpful because proof by contradiction in its form of $H \rightarrow C \equiv [H \wedge (\neg C)] \rightarrow F$ is central in all mathematics, whether hypothesis H is explicitly or implicitly stated. Dreyfus and Eisenberg (1986) contended that mathematicians prefer the proof of irrationality of $\sqrt{2}$ by contradiction proof over alternative proofs because proof by contradiction of the form $H \rightarrow C \equiv [H \wedge (\neg C)] \rightarrow F$ does not require complex prerequisite knowledge and is very appropriate for teaching.

Although the literature focuses on many aspects of students' difficulties in indirect proof, there is little discussion on how students can extend indirect proof beyond proving the irrationality of $\sqrt{2}$ in the context of Afghanistan. Thus, the purpose of this study is to address this problem; it specifically focuses on addressing the following research questions:

- Which methods of proof do Afghan undergraduate students employ to prove the irrationality of $\sqrt{3}$ and $\sqrt{5/8}$?
- How supportive or problematic is the indirect proof for Afghan undergraduate students when employing it beyond proving the irrationality of $\sqrt{2}$?

METHOD

The main objective of this research was to determine whether indirect proof can become supportive or problematic for undergraduate Afghan students once they know what indirect proof is and learn to carry out the method considering its logical form of $H \rightarrow C \equiv [H \wedge (\neg C)] \rightarrow F$ on the specific example showing $\sqrt{2}$ is irrational, and then apply the same method to prove the irrationality of $\sqrt{3}$ and $\sqrt{5/8}$. The research method used in this research is qualitative in nature, seeking to find supportive and problematic aspects of employing indirect proof in proving the irrationality of $\sqrt{3}$ and $\sqrt{5/8}$. The main data source for the current research was students' written works about proving the irrationality of $\sqrt{3}$ and $\sqrt{5/8}$. The sample consisted of 30 second-year and 48 fourth-year undergraduate Afghan students who took a modern algebra1 course with the researcher as their fulfillment of their degree in mathematics at the Mathematics Department, School of Education, Balkh University, Afghanistan. Second-year students were asked to prove the irrationality of $\sqrt{3}$ and fourth-year students were asked to prove the irrationality of $\sqrt{5/8}$. As embedded in the curriculum of modern algebra, the course must discuss the methods of proofs in mathematics. The main methods of proofs to be discussed in this course are direct and indirect proofs as well as the method of proof by mathematical induction. This is required because the later concepts discussed in modern algebra require these proof methods to deduce further results in group theory and ring theory; thus the foundations must be laid so that students can easily understand the process of deducing results from axioms related to concepts in group and ring theory.

RESULTS

The main objective of this research was to determine whether indirect proof can become supportive or problematic for undergraduate Afghan students once they know what indirect proof is and learn to carry out the method considering its logical form of $H \to C \equiv [H \wedge (\neg C)] \to F$ on the specific example showing $\sqrt{2}$ is irrational and then apply the same method to prove the irrationality of $\sqrt{3}$ and $\sqrt{5/8}$. These questions were asked to see whether students could apply reasoning similar to that discussed for the case of $\sqrt{2}$ to prove that $\sqrt{3}$ and $\sqrt{5/8}$ are irrational.

The written works of 30 students proving the irrationality of $\sqrt{3}$ were analyzed. Eleven out of the 30 students did not provide any answers. The remaining 19 students produced proofs that differed; however, their various proofs can be categorized into two categories. The first category included 17 students who could only provide the following reasoning, were unable to go beyond this stage, and simply concluded that $\sqrt{3}$ is irrational. Their reasoning includes the following steps:

Suppose $\sqrt{3}$ is rational. Then we will have $\sqrt{3} = \frac{m}{n}, n \neq 0$ and $\frac{m}{n}$ is in irreducible terms. Hence, $3 = \frac{m^2}{n^2}$ or $m^2 = 3n^2$, so $\sqrt{3}$ is irrational.

This category students' produced partial proofs show that they did not had any problem with negating the conclusion C of the theorem in the way mentioned in the literature, however, they were not able to continue the process by showing that if the square of the integer a is a multiple of 3, then a must be a multiple of 3. This could be done easily if these students followed the process of reasoning applied in the case of irrationality of $\sqrt{2}$, using contrapositive proof to show that if the square of an integer is even, then the integer must be even. When asked why some of these students could not go beyond this stage, students mentioned that the main barrier that hindered their reasoning to go beyond this stage was that they did not know what to call the right-hand side of $m^2 = 3n^2$ and forgot the definition of 'multiples of an integer,' because everywhere the word 'even' is used in the case of proving the irrationality of $\sqrt{2}$. The author believes that the main problem in the case of proving the irrationality of $\sqrt{n}$, n a positive and square free integer, is knowing how to deal with the equation $a^n = cb^n$ for a fixed integer c to find nonzero solutions $a, b \in \boldsymbol{Z}$ or knowing how to use the definition of multiple of an integer and contraposition; which of these two methods can facilitate students' reasoning process can be determined in a comparative study, which is not the concern of this study.

On the other hand, only two students could provide more evidence beyond the above reasoning to show that $\sqrt{3}$ is irrational, but none of them proved that if the square of integer a is a multiple of 3, then a must be a multiple of 3. They simply took this for granted from the proof of the irrationality of $\sqrt{2}$.Their reasoning is as follows:

> Suppose $\sqrt{3}$ is rational. Then we will have $\sqrt{3} = \frac{m}{n}, n \neq 0$ and $\frac{m}{n}$ is in irreducible terms. Hence, $3 = \frac{m^2}{n^2}$ or $m^2 = 3n^2$. Now $m = 3r$. Substituting it in $m^2 = 3n^2$ gives $m^2 = (3r)^2 = 9r^2 = 3n^2$ or $n^2 = 3r^2$ which implies that$n = 3s$ which contradicts the irreducibility of $\frac{m}{n}$. Thus, $\sqrt{3}$ is irrational.

In addition, the written works of 48 students proving the irrationality of $\sqrt{5/8}$ were analyzed. None of them provided complete proof of the irrationality of$\sqrt{5/8}$, but their written works can be categorized as follows. 34 students were able to negate the conclusion of the theorem and follow up to find the equality $8a^2 = 5b^2$ and simply concluded that it was irrational. Their work can be summarized as follows:

> Suppose $\sqrt{5/8}$ is rational, that is $\sqrt{5/8} = \frac{a}{b}$ where the numerator and denominator have no common factor. Then squaring both sides gives $\frac{5}{8} = \frac{a^2}{b^2}$, from which we have $8a^2 = 5b^2$.

Although all of them presented their partial proofs, as above, they had different types of reasoning. For instance, some of these students wrote that no non-zero integer values for a and b satisfy $8a^2 = 5b^2$ by giving specific values.

> Solution: suppose $\sqrt{5/8}$ is rational. Then we have:
>
> $$\sqrt{5/8} = \frac{a}{b}$$
>
> $$\left(\sqrt{5/8}\right)^2 = \left(\frac{a}{b}\right)^2$$
>
> $$\frac{5}{8} = \frac{a^2}{b^2}$$
>
> $$5b^2 = 8a^2$$
>
> $$b^2 = \frac{8}{5}a^2$$
>
> $$b = \pm\sqrt{8/5}\,a$$

Figure 1. A student's written work on showing the irrationality of $\sqrt{5/8}$

(translated from Dari to English)

And in the second category of 14 students, 2 students computed $\sqrt{5/8} \approx 0.790569415\cdots$ and said that since this decimal number neither ends nor repeats at some decimal place, then $\sqrt{5/8}$ must be irrational, and the rest just simplified $\sqrt{5/8}$ algebraically using laws of radicals.

$$\sqrt{5/8} = 0.790569415\cdots$$

If it is in this state, it is rational. Then

$$\sqrt{5/8} = \frac{a}{b}$$

$$\frac{5}{8} = \frac{a^2}{b^2}$$

$$8a^2 - 5b^2 = 0$$

Since Δ is not defined in the field of real numbers, then $\sqrt{5/8}$ is not rational. Therefore, $\sqrt{5/8}$ is irrational.

Figure 2. A student's written work on showing the irrationality of $\sqrt{5/8}$

(translated from Dari to English)

This indicates that using the definition of multiples of an integer, as in the case of $\sqrt{2}$ or $\sqrt{3}$, could not help students decide on $8a^2 = 5b^2$ to show that $\sqrt{5/8}$ is irrational. They recalled a met-before that a fraction either ends or repeats in decimal representation, but none of them showed whether it ends or not.

DISCUSSION

This study aimed to investigate the supportive and problematic aspects of indirect proof and any other met-before concept that may influence students ability to determine the irrationality of $\sqrt{3}$ and $\sqrt{5/8}$ using their knowledge of the irrationality of $\sqrt{2}$ as proved by the indirect proof method. The results showed that the majority of participants were not able to extend the method of indirect proof beyond proving the irrationality of $\sqrt{2}$. Their written works showed that while some students were able to provide partial proof of the irrationality of $\sqrt{3}$ and $\sqrt{5/8}$, most of them were unable to reach a contradiction and prove the irrationality using the method of indirect proof. The main problem reported by students not being able to apply indirect proof was that they forgot or were unable to connect the definition of multiples of 3 in deducing by contraposition that if a^2 is a multiple of 3, then so is a.

The study also found an important problematic aspect that hindered students application of indirect proof, as they did not know how to go beyond step $a^2 = 3b^2$ in proving the irrationality of $\sqrt{3}$ and reaching a contradiction. One thing that can be drawn from this is that the most problematic step in the process of indirect proof for proving the irrationality of $\sqrt{n}$, for positive nonzero and square free integer n, is to reach a contradiction from $a^2 = nb^2$. Indeed, once the student understands well the supportive notions of multiples of an integer and contrapositive proof, he/she will have little difficulty showing that the square root of any positive non-zero and square free integer is irrational.

However, latter in this study it was found that this supportive notions of 'multiples of an integer and proof by contrapositive' in the case of$\sqrt{n}$, n is positive nonzero and square free integer, may not help students in other situations such as in proving the irrationality of $\sqrt{5/8}$ or generally in $\sqrt{p/q}$, where p/q is irreducible. The study found that most of the senior undergraduate students who attempted to prove the irrationality of $\sqrt{5/8}$, using multiples of an integer, were unable to reach a contradiction using indirect proof. This was because of the complexity of the equation and the presence of different multiples on both sides of the equality.

The study also found two other interesting issues in the students' written works on the proof of the irrationality of$\sqrt{5/8}$. The first issue was related to the infinite and non-repetitive decimal expansion of $\sqrt{5/8}$, which some students approximated using a calculator and then deduced that is irrational because the decimal expansion is non-repetitive, this finding is consistent with Patel and Varma's (2018) study (as cited in Rafiepour et al., 2022). However, it is unclear how the students knew that the decimal expansion did not end, and whether they used any other previous knowledge such as the following theorem: "every rational a/b, provided $b \neq 0$ and a are integers, can be written as a finite(ending) decimal representation or repeating infinite (non-ending) decimal representation; conversely, every finite or periodically infinite decimal expansion is equal to a rational number." Whatever previous knowledge is used, there is still a challenge in calculating all decimal digits of an irrational number because hand calculators cannot compute it beyond 11th digit, as found in Rafiepour et al.'s (2022) study.

The second issue is related to the theoretical analysis of finding nonzero solutions to Diophantine equations as an alternative to indirect proof, which may help undergraduate students continue until the last step is reached to show the irrationality of $\sqrt{p/q}$ where p/q is irreducible. There is one legitimate question to be asked: what would cost both the teacher and student to search for nonzero integer solutions of $8a^2 = 5b^2$ or generally of $qa^2 = pb^2$ using university mathematics? The empirical search for answering this question requires working with students and many Diophantine equations, but working with the polynomial of two variables $f(x, y) = 8x^2 - 5y^2 \in \boldsymbol{Z}[x, y]$ and finding its nonzero integral solutions will tell us whether $8a^2 = 5b^2$ has nonzero integral solutions, which in turn will determines whether $\sqrt{5/8}$ can be written as a rational p/q. Thus, theoretical analysis of finding nonzero solutions to such equations as an alternative to indirect proof may shed light and help undergraduate students continue until the last step is reached to show the irrationality of $\sqrt{p/q}$ where the rational p/q is irreducible. There are also advanced methods to show that $f(x, y) = 8x^2 - 5y^2 \in \boldsymbol{Z}[x, y]$ has no nonzero integer solution such as solving the Diophantine equations in number theory.

Finally, this study proposed a simple and clever method to determine the irrationality of such numbers. That is, if we set $x = \sqrt{2}$, then squaring both sides of the equality yields $x^2 = 2$ or $f(x) = x^2 - 2$. Now, we know that $f(x) = x^2 - 2 \in \boldsymbol{Q}[x]$ is a polynomial in the ring of polynomials over the field of rational numbers $\boldsymbol{Q}$. Suppose $f(x)$ is reducible in $\boldsymbol{Q}[x]$. Then $x = x_0$ must be rational; otherwise, it is irrational. The irreducibility of the polynomial can be verified using the rational root theorem. The rational root theorem states that:

> Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be a polynomial in $\mathbf{Z}[x]$, where $a_0 \neq 0$ and $a_n \neq 0$. Then every rational root of $f(x)$ has the form $\frac{c}{d}$, where $c|a_0$ and $d|a_n$(for proof, see Nicholson, 2012, p. 211).

We now apply the rational root theorem. Because $a_0 = -2$ and its factors are $\pm 1, \pm 2$; $a_n = 1$ and its factors are ± 1. Then the values for $\frac{c}{d}$ are: $\pm 1, \pm 2$. Now, we can see that $f(\pm 1) \neq 0$ and $f(\pm 2) \neq 0$. Thus, according to the rational root theorem, $f(x) = x^2 - 2$ is irreducible over $\boldsymbol{Q}$. Therefore, $\sqrt{2}$ is irrational. The same method can be applied to $\sqrt{3}$, $\sqrt{5/8}$ and any other number in the form $x_0 = \sqrt[n]{t}$, where $n \in \boldsymbol{Z}^+_{>1}$ and $t > 0 \in \boldsymbol{Q}$, to prove its irrationality.

CONCLUSION

In conclusion, the study provided valuable insights into the supportive and problematic aspects of indirect proof and related concepts that may impact students' ability to determine the irrationality of square roots. This study highlighted the challenges faced by undergraduate students in extending the method of indirect proof beyond proving the irrationality of $\sqrt{2}$. The study identified the main problem faced by students in applying indirect proof as forgetting or being unable to connect the definition of multiples of an integer in deducing by contraposition. The study also highlighted the complexity of the equation and the presence of different multiples on both sides of equality as a significant challenge in proving the irrationality of $\sqrt{5/8}$ and other similar square roots. The study proposed alternative methods to indirect proof, such as the theoretical analysis of finding nonzero solutions to Diophantine equations; and a simple and clever way to determine the irrationality of numbers in the form of $\sqrt[n]{t}$. This study acknowledges the limitations of the sample size and calls for further research to investigate the generalizability of the findings to other populations and to explore other instructional strategies suggested here for teaching indirect proof.

Based on the findings of this study, there are several pedagogical and methodological approaches that educators can adopt to effectively teach the topic of proving the irrationality of square roots using indirect proof. First, educators should focus on developing students' understanding of the supportive concepts of multiples of an integer and proof by contrapositive, which are crucial for applying indirect proof. Second, educators should adopt a problem-solving approach to teach this topic. This involves presenting students with challenging problems and guiding them through a problem-solving process, emphasizing the importance of logical reasoning, critical thinking, and perseverance. Finally, educators should consider using alternative methods for indirect proof, such as the rational root theorem. These methods can provide students with alternative strategies to approach complex problems and enhance their understanding of underlying concepts.

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