

The Theory on Loops and Spaces – Part 2

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Abstract: The study of loops and spaces in mathematics has been the subject of much interest among researchers. In Part 1 of "The Theory on Loops and Spaces," published in the "Mathematics Teaching Research Journal," introduced the concept and the basic underlying idea of this theory. This article continues the exploration of this topic and aims to advance the understanding of the theory through observation and analysis of patterns. A systematic examination of intersection points and their numerical Sum is conducted, and the effect of the order of numbering on the analysis is analyzed. Furthermore, the physical implications of the theory are discussed, and the validity of the theory in the third dimension is confirmed through analysis. This article provides a solid foundation for the understanding of the elementary principles of Graph Theory and paves the way for the development of more advanced theorems in the field. Additionally, the article demonstrates how patterns in nature can be analyzed and expressed mathematically, offering a unique perspective on the interplay between mathematics and the natural world. The work is inspired by the video posted by mathematician Dr. James Tanton on his YouTube channel on September 27th, 2021.

INTRODUCTION

The world of loops is a captivating one, from our childhood days of doodling simple shapes to the complex celestial orbits that surround us. In mathematics, space is defined as the region surrounded by a loop, with basic loops such as circles enclosing only one space within their boundaries. But what happens when a loop intersects with itself, creating an intersection point? This simple change in the structure of the loop now creates two distinct spaces, opening up a whole new realm of exploration.

In Part 1 of our investigation, we ventured into exploring the properties of loops, including the remarkable ability to trace the entire loop without lifting the pencil and the possibility of coloring the loop with two colors such that no two same-colored spaces share a common boundary. In this article, we continue our journey by exploring the Number of intersection points in a loop and the

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significance of their numerical Sum. We also delve into the application of these findings in the realm of three dimensions and consider the potential physical implications of our results.

DEFINITIONS AND NOTATIONS

Before we begin, we will discuss a few definitions and notations which will be used in this article.

Definitions:

- Loop: A closed curve in a space whose starting and ending point is the same.
- Directed Graph: The kind of loops we study in this article are formally known as a directed graph, also called a digraph, in graph theory. A digraph is a graph in which the edges have a direction.
- Space: The region enclosed within the loop, excluding the boundary of the loop.
- Ray: A segment of the loop that either begins or ends with a fixed intersection point P and contains no intersection point other than P.
- Intersection Point: A point created when the loop cuts itself. An Intersection point produces a minimum of 4 rays either going in or out of the point.
- Single intersection point: A point through which the loop passes only once. It contains only 4 rays either going into the point or emerging out of the point.
- Multiple intersection point: A point through which the loop passes more than once. It contains more than 4 rays either going in or emerging out of the point.
- Value of a Point, $V(P)$: If we consider a point as a source of two or more rays emanating from it in opposite directions. For instance, if we take a point on a line, then we have two rays emanating from that point. Similarly, a point with two intersecting lines will have four rays. Now the Value of a Point, P, will be evaluated as:

  $$V(P) = 1 + \frac{n-4}{2} = \frac{n-2}{2} \quad (1)$$

  Where 'n' is the Number of rays through that intersection point.
- Numerical Value ($\mathcal{N}$): It is the Number assigned to each intersection point created while tracing the loop as 1, 2, 3, and so on in a definite order.
- I_P: Set of all intersection points passed when loop L is traced starting from a fixed point P.
- Number of Crossings: Number of times an intersection point is crossed while tracing the loop. It is equal to half the Number of rays that emerge from an intersection point.

Notation:

- If we follow the direction of the loop, and we choose two consecutive points, P and Q, on the loop. If we pass through P before passing through Q, then we write $P \rightarrow Q$.

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ANALYSING INTERSECTION POINTS

Suppose we trace a loop starting at a fixed point. Let P be an intersection point. In this section, we will obtain a formula for the Number of intersection points passed before reaching P.
Before we begin, we will motivate our result using the following two examples:

Example 1:

An important point to note is that we are tracing along the arrows starting from 1→ 2 → 3… and so on.

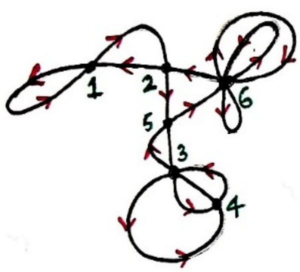

Figure 1: Example 1

Here in Figure 1, we see that we have 2 multiple points, the point labeled 3 with 6 rays and the point labeled 6 with 8 rays, and 4 single intersection points labeled 1, 2, 4, and 5.

Let us observe either of the single intersection points labeled 1, 2, 4, and 5. For instance, let us take point labeled 1, then we will pass through 13 intersection points in the order 2 → 5 → 3 → 4 → 3 → 4 → 3 → 5 → 6 → 6 → 6 → 6 → 2.

However, if we observe 3, we will pass through a total of 12 intersection points in the order 4 → 4 and 5 → 6 → 6 → 6 → 6 → 2 → 1 → 1 → 2 → 5, which gives a total of 1+1+10 = 12, or if we observe 6, then we will pass through a total of 11 intersection points in order 2 → 1 → 1 → 2 → 5 → 3 → 4 → 3 → 4 → 3 → 5.

Let us analyze it further in terms of V_P, Number of rays, and Number of crossings

Point Label	Value	Rays	Crossings (Points Passed)
1	1	4	2
2	1	4	2
3	2	6	3
4	1	4	2
5	1	4	2
6	3	8	4

Table 1: The table shows the Value of each point labeled in Figure 1, along with the Number of rays and crossings at each point.

From table 1 above, the Number of crossings is exactly half of the Number of rays. Now, we will use this observation for further results.

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Example 2:

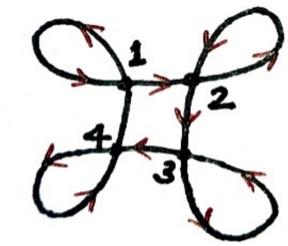

Figure 2: Example 2

In figure 2, we see that we have 4 intersection points labeled 1, 2, 3, and 4, all of which are single intersection points. If we observe either of these points, for instance, let us take point labeled 2, then we will pass through 6 intersection points before reaching 2 again in the order: 3 → 3 → 4 → 4 → 1 → 1.

In fact, we can show that the Number of intersection points passed on tracing the whole loop containing single intersection points will be even.

Theorem 1: If there are only single intersection points in the whole loop, then on tracing the loop starting from a fixed point P, the Number of intersection points passed before reaching P will always be even.

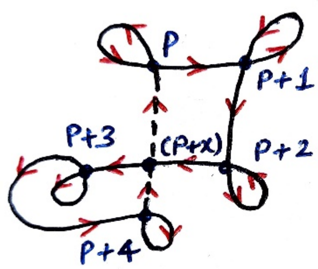

Figure 3: Theorem 1

Proof: Let us take a loop consisting of only single intersection points and fix a point P, then $P + x$ is the x^{th} point from P and the dashed line indicates that there can be some extended loop that is not shown for simplicity.

If we trace the loop starting from P in the order: $P \rightarrow (P + 1) \rightarrow (P + 1) \rightarrow (P + 2) \rightarrow (P + 2) \rightarrow (P + x) \rightarrow (P + 3) \rightarrow (P + 3) \rightarrow (P + 4) \rightarrow (P + 4) \rightarrow (P + x) \rightarrow P$.

From above, we observe that we have crossed every point twice. So, the total Number of intersection points passed by tracing the entire loop once is 2 N, where N is the total Number of Intersection points.

Hence, the Number of Intersection points passed before reaching P will be 2N – 2 = 2 (N – 1). Thus, we have the result when there are only single intersection points.

Thus, by the above examples, we make the following observations:

Observations

1. After passing through an arbitrary intersection point P, the total Number of intersection points passed before reaching P is solely governed by the Value of intersection points.
2. If we trace the loop and pass through an intersection P, then the Number of Intersection points passed before reaching P depends on the 'Total Number of Intersection Points" present in the loop.

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The above observations give us the following results:

Lemma 1: In any given loop L, every multiple intersection point (if it exists) must satisfy the condition of a single intersection point; that is, it must have a minimum of 4 rays.

Proof: We know that by the definition of intersection points, every intersection point must have a minimum of 4 rays. Moreover, a single intersection point contains exactly 4 rays either going into the point or emerging out of it. Hence by the definition of multiple intersection point, every multiple intersection point (if it exists) must satisfy the condition of the single intersection point.

Theorem 2: If we have a loop L having N intersection points. Suppose there are S single intersection points and N – S multiple intersection points. Let P be the point on the loop L. Then, starting from P, if we trace the loop, the Number of intersection points passed before returning to P is given by:

$$2 \times (N-1) + \sum_{\substack{Q \in I_P \\ Q \neq P}} (V_Q - 1) \tag{2}$$

Where, I_P: Set of all intersection points passed when loop L is traced starting from a fixed point P, and V_Q is the Value of point Q.

Proof:

We will consider two cases:

Case 1**:** Considering the loop, L consists of only single intersection points. N = S:

Proof, in this case, follows from Theorem 1, and we have: starting from fixed point P, on tracing the loop, the Number of intersection points passed before reaching P is given by $2 \times (N-1)$.

Case 2**:** Considering the loop, L consists of both single and multiple intersection points N ≠ S:

Consider a loop L with N intersection points labeled as P, $P+1$, $P+2$, … $P+m$ in such a way that P has r_0 rays, $P+1$ has r_1 rays, $P+2$ has r_2 rays,… $P+m$ has r_m rays, and $r_i \geq 4$ ∀ i ∈ {0, 1, 2, … m }. Now, assuming that we are tracing the whole loop starting from P, then,

By lemma 1, from each intersection point, 4 rays are used in making single intersection points, giving us $2 \times (N-1)$ intersection points (case 1).
The remaining ($r_i - 4$, ∀ i ∈ {1, 2, … m}) rays will account for the multiple intersection points.
Now, the Number of crossings given by each ($r_i - 4$) rays is equal to $\frac{r_i - 4}{2}$ (3)
This can be rewritten as:

$$\text{Number of crossings} = \frac{r_i - 2 - 2}{2} = \left(\frac{r_i - 2}{2} - \frac{2}{2}\right) = V_{P+i} - 1, \ \forall \text{ i} \in \{1, 2, \dots m\}) \tag{4}$$

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Summing over the whole loop (excluding P), we get: $\sum_{\substack{Q \in I_P \\ Q \neq P}} (V_Q - 1)$, where $V_Q = V_{P+i}$

Hence, combining both the results,
Total Number of Intersection points that belong to I_P will be given by:

$$2 \times (N-1) + \sum_{\substack{Q \in I_P \\ Q \neq P}} (V_Q - 1) \quad (5)$$

Remark:

In case 1 of single intersection points, the Value V_Q of all the points is 1. Hence, $\sum_{\substack{Q \in I_P \\ Q \neq P}} (V_Q - 1)$ will be equal to 0, and we get the Total Number of Intersection points that belongs to I_P as $2 \times (N-1)$.

SUM OF INTERSECTION POINTS

This section will find the relation to calculate the Sum of the 'Numerical Value' of intersection points as we trace through the loop once.

Before we begin, let us assume that while tracing the loop, the total Sum of Intersection points passed with respect to a fixed-point P is defined based on the Numerical Value given to the intersection points (in a definite order). That means that if we change the order of numerical values of the points, the total Sum of the intersection point will change, and further analysis will be affected.

Now, let us consider the following example:

Example 3:

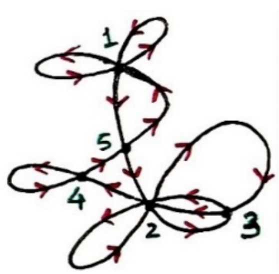

Figure 4: Example 3

In Figure 4, we will find the sequence of intersection points as we trace the loop starting from 1. The sequence is unique to every loop and depends on the numbering of intersection points.

So, the sequence for the Loop in Figure 4 will be:

$$1 \to 1 \to 1 \to 5 \to 2 \to 3 \to 2 \to 2 \to 3 \to 2 \to 4 \to 4 \to 5$$

By observing the sequence, we can have the following lemma:

Lemma 2: While tracing the loop, we will pass through an intersection point one more than the Value of that intersection point.

Proof: By the definition of the Value $V(P)$ of the intersection point, we have,

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For an intersection point with 'n' rays, the Value of the intersection point is given by:

$$V(P) = 1 + \frac{n-4}{2} = \frac{n-2}{2} \quad (6)$$

Simplifying equation 6, we get,

$$n = 2 \times V(P) + 2 \quad (7)$$

Now, by definition, the Number of crossings is always equal to half the Number of rays hence,

The Number of crossings will be equal to $\frac{n}{2} = \frac{1}{2}(2 \times V(P) + 2)$ (8)

Or, Number of crossings $= V(P) + 1$ (9)

Hence, while tracing the loop, we will pass through an intersection point one more than the Value of that intersection point.

Theorem 3: While tracing the loop and defining the sequence with respect to a fixed-point P, the Sum (S) of all the numbers appearing in the sequence is given by:

$$S = \sum_{Q \in I_P} (1 + V(Q)) \times \mathcal{N}_Q \quad (10)$$

Proof: From lemma 2, we know that the Number of crossings for any fixed point Q is given by: Number of crossings = $V(Q) + 1$

So, the Number of times point P appears in the sequence is equal to $V(Q) + 1$, which gives the Sum of the Number of times point P appears in the sequence as: $(V(Q) + 1) \times \mathcal{N}_Q$

Hence, summing over the whole sequence, we have,

$$S = \sum_{Q \in I_P} (1 + V(Q)) \times \mathcal{N}_Q \quad (11)$$

Now, returning to Example 3,

So, the Sum of the sequence corresponding to Figure 4 will be:

$$S = (1 + 2) \times 1 + (1 + 3) \times 2 + (1 + 1) \times 3 + (1 + 1) \times 4 + (1 + 1) \times 5 = 35$$

Let us now analyze that; if we trace the loop and pass through an intersection P then the Sum of the numerical Value of Intersection points passed before reaching P, For that, we will refer to the table below made by referring to Figure 4:

Point	Intersection points passed	Sum
1	5 → 2 → 3 → 2 → 2 → 3 → 2 → 4 → 4 → 5	32
2	4 → 4 → 5 → 1 → 1 → 1 → 5; 3 and 3	27
3	2 → 2 and 2 → 4 → 4 → 5 → 1 → 1 → 1 → 5 → 2	29
4	5 → 1 → 1 → 1 → 5 → 2 → 3 → 2 → 2 → 3 → 2	27
5	1 → 1 → 1 and 2 → 3 → 2 → 2 → 3 → 2 → 4 → 4	25

Table 2: Sum of Intersection Points passed

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From the above table, we can relate the total Sum of the sequence, and the Sum of numerical values of intersection points passed before reaching P.

Corollary 1: Given a loop 'L,' if we trace the loop and pass through an intersection P, then the Sum of the numerical Value of Intersection points passed before reaching P will be given by:

$$S_P = S - (1 + V(P)) \times \mathcal{N}_P \tag{12}$$

Where 'S' is the total Sum of intersection points given by equation 10.

Proof: From theorem 3, we have the total Sum of intersection points given as:

$$S = \sum_{Q \in I_P} (1 + V(Q)) \times \mathcal{N}_Q \tag{13}$$

Now, since we are excluding P in our analysis, so the Sum corresponding to point P will be subtracted from the total Sum and will be given as:

$$\sum_{Q \in I_P} (1 + V(Q)) \times \mathcal{N}_Q - \{(1 + V(P) \times \mathcal{N}_P\} \tag{14}$$

Which, on simplification, gives:

$$S_P = S - (1 + V(P)) \times \mathcal{N}_P \tag{15}$$

Hence, the Sum of the numerical Value of Intersection points passed before reaching P will be given as $S - \{(1 + V(P)) \times \mathcal{N}_P\}$

Now, again returning to Example 3,

If we apply the above corollary 1 for point 3 (Figure 4), then the Sum of the numerical Value of Intersection points passed before reaching 3 will be given as $35 - \{(1 + 1) \times 3\} = 29$

Remark: Special case of a simple loop:

If we have a loop with all the intersection points of 'value 1', starting from any arbitrary point P, the Sum of the numerical Value of intersection points passed as we trace the loop before returning P will decrease by 2 for each consecutive point. So, if we have the total Sum of numbers in the sequence of intersection points equal to 'S,' then starting from point' 1,' the Sum of the numerical Value of intersection points passed while tracing the loop before returning to '1' will be 'S – 2', for '2' will be '(S – 2) – 2', for '3' will be '{(S – 2) – 2} – 2' and so on. This happens so because, while we are starting from '1', we are subtracting (2 × 1) from S, while we are starting from '2', we are adding (2 × 1) but subtracting (2 × 2), similarly when we start from '3', we are adding (2 × 2) but subtracting (2 × 3), so overall we are subtracting '2' in every consecutive step.

Also, in this case, if we have total 'N' intersection points, each having the value 1, then the Sum of numbers of the sequence generated while tracing the loop is given by:

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$$S = 2 \times \frac{N \times (N+1)}{2} = N \times (N+1) \tag{16}$$

This happens because, in this case, the Value of each intersection point is '1' implies that every intersection point occurs twice (1+1) while tracing the loop. So, if we have 'N' intersection points, then the Sum of numbers of the sequence generated while tracing the loop is two times the Sum of natural numbers from 1 to N. then, from arithmetic progression, we have the Sum of natural numbers from 1 to N is given by {N × (N+1)} / 2. So, we get the Sum of the sequence of the numerical Value of intersection points as N × (N+1)

PHYSICAL SIGNIFICANCE OF LOOP THEORY – MOTIVATION FOR HIGHER DIMENSION

The study of loops and spaces has important implications in the field of astronomy, particularly in the analysis of orbits. The theory developed in "The Theory on Loops and Spaces" provides valuable insights when dealing with the intersection points of multiple orbits or when celestial bodies undergo periodic changes in their orbits (Figure 5).

This theory is especially useful in predicting the movement of celestial bodies and creating mathematical models of their behavior. For example, it can be applied to the study of space debris, a complex network of abandoned satellites and other objects, to determine the probability of collisions between these objects. This information can be used to quickly estimate the likelihood of collisions between celestial bodies (Figure 6).

The theory developed in "The Theory on Loops and Spaces" represents a valuable contribution to the field of astronomy and has the potential to revolutionize our understanding of the movements and interactions of celestial objects in the cosmos.

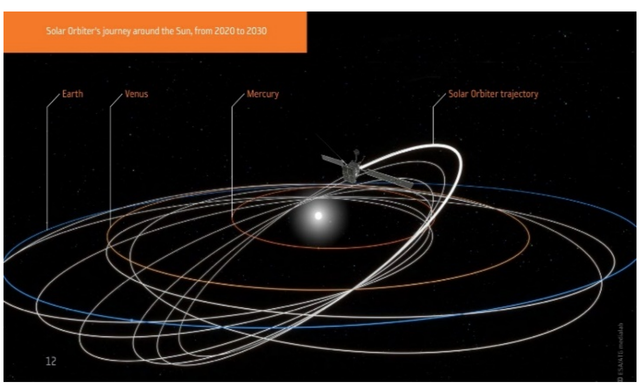

Figure 5: Solar Orbiter Trajectory intersecting with Planetary Orbits
Image Source: ESA Operations on Twitter

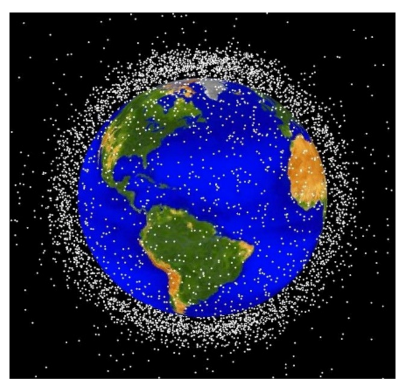

Figure 6: Space Debris Network Around Earth
Image Source: Space Debris | NASA

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ILLUSTRATION IN THREE DIMENSIONS

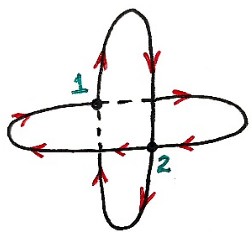

Figure 7: Example 4

The results that we have obtained for the two-dimensional space could be extended for the three-dimensional space in some cases. To verify that, let us consider the following example.

Example 4

On a Blank Piece of paper, we slowly start drawing a self-intersecting loop in 3 – D, as shown in Figure 7. Here dotted lines represent the loop behind the planes.

If we first draw the vertical loop, after completing one vertical loop, we draw a horizontal loop that cuts the vertical loop at point 2; then, we count the Number of spaces created in the following table given below:

Piece	Intersection Point	Region/Space	The Number assigned to that Intersection Point	Value
-	+1	+1	1	0
-	+1	+1	2	+1
+1	-	+1	1	+1

Table 3: Tally of Number of Regions, Intersection Points, Loops and Value of Intersection Points in 3 – D.

Here, when we complete one Vertical Loop and return to point 1, the Value of that point (1) does not increase to +1 because it formed one closed curve and did not form any intersection point, so the Value depends on the intersection point. After completing one closed vertical loop, we have formed one vertical space. Then, we draw the horizontal loop, which intersects the vertical loop at point 2; another space is created as the right part of the horizontal loop, followed by line joining points (1) and (2). When we complete the remaining horizontal loop, 1 Piece of combination loops is completed, and one more extra space is created by the left part of the horizontal loop, followed by line joining points (1) and (2). So, we have a total of 3 Spaces created with 2 intersection points and 1 Piece, and the Value of each intersection point is also preserved. Hence, this shows that our theory could also be extended to 3 – dimensional space.

As a future direction in this area, it would be beneficial to extend this work to include proof of the above results in a general 3 – dimensional space or even a general n – dimensional space.

ACKNOWLEDGEMENT

This article would not have been possible without the invaluable guidance and inspiration of *Prof. James Tanton*. I am deeply grateful for the thought-provoking video (Link) he uploaded on his YouTube channel on September 27th, 2021, which served as the catalyst for my exploration of the theories on loops and spaces.

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I would also like to extend my sincerest thanks to *Dr. Nikita Agarwal* from the Department of Mathematics at the *Indian Institute of Science Education and Research, Bhopal (India)*. Her expertise in Graph Theory and her unwavering support in refining my ideas and the structure of the paper were invaluable in bringing this piece of work to fruition.
I am truly thankful for their support and guidance, and I hope that this article will inspire others to delve further into the fascinating world of loops and spaces.

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