Investigation of Students' Mathematical Thinking Processes in Solving Non-routine Number Pattern Problems: A Hermeneutics Phenomenological Study

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Abstract: This study aimed to interpret and describe students' mathematical thinking processes of non-routine mathematical problems that were solved based on didactic situation theory. This study uses a qualitative method, a phenomenological hermeneutics study for grade 8 students at a junior high school in Banda Aceh in the 2021-2022 academic year. Research data obtained through data were collected using instruments, namely written tests based on the didactic mathematical situation theory framework, structured observation, documentation, and clinical interviews carried out after the action. The results of the study show that the students' mathematical thinking processes in the critical reflection category can reach the convincing stage with algebraic arguments in validation situations. Subjects in the explicit reflection category can reach the convincing stage by providing arithmetic arguments in validation situations. Meanwhile, the category of students who cannot solve problems and can only specialize by giving examples of what is being asked. Students in this category have difficulty identifying relevant patterns and formulating the mathematical models needed to solve the problems. To support students in developing the level of mathematical thinking, the teacher can present contextual problems that are in accordance with the level of student thinking, can predict possible responses or ways of thinking of students to the problems given and present problems according to the structure of the concept sequence and the functional order of students' thinking. To support students in algebraic thinking category, teachers can start learning by presenting contextual problems that are easily recognized by students, then expand that context in symbolic form.

Keywords: Mathematical Thinking, Didactical Situations, Problem Solving, Number Patterns

INTRODUCTION

To improve student learning activities, it is necessary to understand the status of developing their thinking and reasoning. The more information we have about what they know and think, the more opportunities we can provide for student success in the classroom (Pellegrino et al., 2001). To determine the level of students' thinking and reasoning in learning mathematics can be observed by examining the use of problem-solving strategies, display of mathematical domain knowledge, representation of the completion process, and justification of mathematical reasoning based on problem situations (Cai, 2003). Therefore, differences in students' thinking in solving mathematical problems are important to understand in order to help students learn mathematics.

Developing students' mathematical thinking has recently become a research focus (Carpenter et al., 2017; Breen & O'Shea, 2010; Fraivillig et al., 1999; Schoenfeld, 2016). Schoenfeld (2016) says that the mathematical thinking found in the process of learning mathematics means (a) developing a mathematical perspective, appreciating the process of mathematization and abstraction and having a tendency to apply it, and (b) developing competence with mathematical tools, and use them to understand the structure and build mathematical understanding. Schoenfeld says that mathematics learning needs to be directed to 1) find solutions, not just memorize procedures; 2) explore patterns, not just memorize formulas; and 3) formulate conjectures, not just do exercises. There are two references in mathematics education to define mathematical thinking. One perspective focuses on mathematical processes (Burton, 1984; Mason et al., 2010a; Polya, 1985; Schoenfeld, 1992). This perspective focuses on the problem of how mathematical thinking is realized. Another philosophy is based on conceptual improvement (Dreyfus, 1991; Freudenthal, 1973; Tall, 2002). This view is related to how individuals construct mathematical concepts in their minds.

There are three essential goals for thinking of mathematics taught in schools, first as an important goal of learning mathematics at school, second as a way of learning mathematics, and third for teaching mathematics (Stacey, 2006). Stacey said that thinking mathematically to solve problems is essential to learning mathematics at school. In a broader scope, mathematical thinking will support the development of science, technology, and economy in a country. Currently, more and more countries in the world are realizing that the economic welfare of a country is greatly influenced by a strong level of mathematical thinking, which is called mathematical literacy. Therefore, students' mathematical thinking in learning mathematics needs to get more attention from mathematics educators to equip students with the ability to think mathematically.

Number patterns are mathematical concepts that students need to solve broader problems. As a general skill, distinguishing a pattern is the foundation of the ability to generalize and abstract

(Burton, 1982; Threlfall, 1999). Threlfall (1999) says that often through the use of patterns, a teacher can unlock truths in mathematical theorems and proofs. With the experience of learning number patterns, students can explore new ideas for solving problems. This shows that the experience of identifying patterns in solving mathematical problems will enrich students' experiences in being able to solve broader problems that cannot be solved in the usual way but can be solved using patterns.

In order to obtain more complete data on mathematical thinking processes, this study will use the didactical theory framework of learning mathematics from (Brousseau, 2002), namely action situations, formulation situations and validation situations. This is as stated (Vygotsky, 1978) that learning can generate various stored mental processes that can only be operated when a person interacts with adults or collaborates with fellow friends. The interaction between students or students and teachers is expected to occur in the exchange of different learning experiences so that mental action can continue as expected. Meanwhile, scaffolding techniques can be used not only to direct the thinking process, but also to provide further challenges so that the desired mental action can occur properly. Nickels & Cullen (2017) reported on increasing the learning activities and mathematical thinking of critically ill children by using robotics within the framework of Brousseau's mathematical didactical situation theory.

Many previous researchers have examined students' mathematical thinking processes in solving mathematical problems (e.g., Gereti & Savioli, 2015; Lane & Harkness, 2012; L. Burton, 1984; Uyangör, 2019; Yıldırım & Köse, 2018). All the research that has been done only considers students' abilities in mathematical thinking that are mature in solving problems or actual development, without considering the skills of students who are still in the process of maturation or potential development. As stated by (Vygotsky, 1978), to see the effect of student learning outcomes in addition to seeing results that are already mature, actual developmental, it is also necessary to consider the abilities of students still in the process of maturing potential development.

This paper explores and makes sense of students' mathematical thinking processes in solving number pattern mathematical problems. Consistent with this aim, we will answer the following research questions. What are the students' mathematical thinking processes in solving non-routine number pattern problems within didactical situations?

Literature Review

The most common sense is that thinking mathematical can be defined as using mathematical techniques, concepts, and methods, directly or indirectly, in solving problems. Sumarmo (2010) Mathematical thinking is processing information in drawing specific conclusions based on arguments that can be justified based on mathematics. Mathematical thinking is a way of thinking about mathematical processes or methods of solving simple and complex mathematical tasks. Mason et al. (2010) say that mathematical thinking is a dynamic process that allows us to increase the complexity of ideas and broaden our understanding of mathematics. Burton (1984) argues that mathematics is not about the subject matter of mathematics but a style of thinking which is a function of certain operations, processes, and identifiable dynamics of mathematics.

Many researchers use indicators of mathematical thinking, namely specialization, generalization, conjecture, and convincing (such as Aiyub, 2023; Burton, 1984; Mason et al., 2010; Stacey, 2006; Uyangör, 2019). Tall (2002) states that mathematical thinking includes components such as abstraction, synthesis, generalization, modeling, problem-solving, and proof. (Uyangör, 2019) says specialization means choosing clear or systematic examples and testing examples of problems to understand and interpret the status of the problem. Arslan & Yildiz (2010) say specialization is completion, demonstration, explanation, and selecting one or more examples is relevant. Nihayatus et al. (2023) uses three steps in the mathematical thinking process, namely abstraction, representation, and verification.

Based on the definitions and components of the mathematical thinking process presented by the experts above, the indicators of students' mathematical thinking processes in solving number pattern problems in this study are 1) specializing; 2) making generalizations; 3) making conjectures, and 4) convincing a statement based on facts from general conclusions. The descriptions of the four indicators of mathematical thinking processes are listed in table 1 below:

Num	Mathematical Thinking process indicator	Description of Mathematical Thinking Process Indicator
1	Specialization	Choose clear or systematic examples and test samples of problems to understand and interpret problem status
2	Generalizations	Set examples, and define relationships in linguistics or mathematics
3	Conjectures	Predict relationships and outcomes in linguistic or mathematical terms
4	Convincing	Showing and communicating the reasons why something is true in arithmetic or algebraic form

Table 1 Description of the indicators of students' mathematical thinking processes in solving the problem of being patterns

METHODS

Types and Research Subjects

This type of qualitative research uses the Interpretive Phenomenological Analysis (IPA) approach, which aims to interpret and interpret a phenomenon based on human experience (Eatough & Smith, 2017). The study of meaning is closely related to phenomenology and hermeneutics, which focus on one's experience. As said (Ricoeur, 1986), it is necessary to combine the study of experience and the study of meaning and meaning with that experience because they complement each other. This was chosen to reveal the various meanings of students' mathematical thinking processes in solving non-routine mathematical problems in number pattern material. The framework for this study is based on the Theory of Didactical Situations in Mathematical (Brousseau, 2002). Hausberger (2020) combined a phenomenological-hermeneutic approach and didactic situation theory would result in a fruitful interaction between philosophy and mathematics education.

The research design used to reveal this phenomenon refers to the Indonesian Didactical Design Research (DDR), which contains three stages of analysis: prospective analysis, metapedadidactical analysis, and retrospective analysis (Suryadi, 2013). As for the study participants, there was 32 grade 8 students from a junior high school in Banda Aceh for the 2021/2022 academic year. In addition, two research subjects will be selected for each category of student groups to explore and interpret mathematical thinking processes in solving number pattern problems from the results of essay tests within the TDSM framework (Brousseau, 2002) and interviewed in depth.

Instruments and Materials

The instruments and materials used in data collection and analysis include (1) the instrument for testing the mathematical thinking process of the number pattern material, (2) the student's Didactical and Pedagogical Anticipation (DPA) instrument for learning obstacles in solving problems, (3) semi-structured interview guidelines, aims to find out students' mathematical thinking processes in the category of solving mathematical problems related to numbers, (4) a digital voice recorder used to record interviews. (5) Ethical considerations. The college, school, mathematics teacher, participant, and parent or guardian granted permission to conduct this research. Before the study, each participant signed a consent form, and after the interview transcripts were completed, each participant read and confirmed the accuracy of the results of each interview. In addition, the participant's initials were also used to disguise the student's identity.

Based on the results of data analysis on the results of research instrument trials on 16 participants who participated in this study, potential learning obstacles, scaffolding, and predictions of student responses in solving non-routine number pattern problems were found. The following is a summary of data on potential learning obstacles, scaffolding, and forecasts of student responses in solving non-routine number pattern problems.

Num	Potential Learning Obstacles	Provided scaffolding	Responses Given by Students
1	Do not know how to show proof of a statement.	1. Is it sufficient to show proof of all palindromes divisible by 11 by giving some examples of palindromes divisible by 11?	1. (Students did not answer immediately, still thinking about possible answers.) A few moments

Num	Potential Learning Obstacles	Provided scaffolding	Responses Given by Students
		2. If not enough, how can we show all 4-digit palindromes divisible by 11?	later, the students answered, "Not enough, sir!" 2. Students do not answer directly, still thinking about how). How many moments later, they replied, "Maybe by looking at the pattern, sir!"
2	Difficulty identifying relevant patterns or rules	3. What are the observable patterns of 4-number palindromes? 4. Yes, what other patterns can be observed? 5. OK, which is the smallest palindrome?	3. The difference between one palindrome and the next is 110 packs! 4. Silent students have not been able to identify other patterns. 5. 1001 sir!
3	Difficulty constructing number patterns/rules in mathematical sentences or statements	6. What is the relationship between the smallest palindrome 1001 and the smallest palindrome for the next thousand, namely 2002 or 3003? 7. Yes, that's right; then what is the relationship between 110 and 220, and 330? 8. Then how do we write the palindrome 3220 in forms 1001 and 110?	6. 2002 equals 2 x 1001 and 3003 equals 3 x 1001; 7. 220 equals 2 x 110, and 330 equals 3 x 110. 8. 3. 3223 = 3003 + 220 = 3(1001) + 2(110)
4	Do not have the awareness to examine the results of his work	9. 1. Is it necessary to re-examine the results of our work? 10. What is the need to re-examine our work?	9. Need sir! 10. Yes, you need to check, sir, because if you make a mistake, you can fix it again.
5	Difficulty identifying alternative settlement strategies	11. How can we check the results that have been obtained? 12. For example, ABBA is a 4-digit palindrome with the first and second A being thousands and units each and the first and second B being hundreds and tens each. Can it be shown that ABBA is divisible by 11?	11. Partisans are silent about what can be done to be able to check the results of the work. 12. Participants show ABBA = (10001) A + (110) B = (11.91) A + (11.10) B = 11(91A+ 10B)

Table 2: Didactical and Pedagogical Anticipation Instrument (DPA) for students in solving non-routine number pattern problems

Procedure

The research procedure refers to the research design used in the Indonesian Didactical Design Research (DDR), which contains three stages of analysis: prospective analysis, metapedadidactical analysis, and retrospective analysis (Suryadi, 2013). In the future analysis stage, analyzing the phenomena that underlie the didactic design process for solving hypothetical problems is found from the results of testing the test instrument on students in the research target schools. In particular, the phenomena disclosed include 1) the didactic situation presented in the implementation of research instrument trials from didactic designs; 2) learning obstacles experienced by students in solving given mathematical problems. 3) Students gave scaffolding and responses in solving mathematical problems.

Metapedadidactic analysis is an analysis that aims to see the ability of researchers to identify and analyze student responses as a result of the didactic and pedagogic actions taken and the ability of researchers to carry out further didactic and pedagogic actions based on the results of the response analysis towards achieving the problem-solving target (Suryadi, 2013). In the retrospective analysis stage, the Researcher reflects and evaluates the situation of the hypothetical problem-solving design by analyzing the relationship between the results of the prospective analysis and metapedadidactical analysis. More specifically, at this stage, the Researcher conducted a suitability analysis between the hypothetical didactical situation and the didactical situation during implementation. This reflection and evaluation suggest improvements to the design of didactic situations for solving hypothetical number pattern problems in students' mathematical thinking processes in solving non-routine number pattern problems.

Data analysis

The results of data collection in the form of recordings of the problem-solving process, answer sheets and student scratch paper, student and teacher interview documents, as well as observation data during the study, were analysed based on the stages developed by Creswell (2007), namely data managing, reading-memoing, describing-classifying-interpreting, and representing-visualizing. Data managing, namely organizing data into computer files for analysis, transcribing recorded data and student interviews, and typing observation notes. Reading-memoing is reading and interpreting the collected data and giving letters or memos in the margins of field notes, transcripts, or under photographs to assist in the initial data exploration process. Describing-classifying-interpreting, namely forming codes or categories representing the essence of data analysis. Researchers construct detailed descriptions, categorize themes, and provide interpretations based on their views or perspectives in the literature. Representing visualizing, namely representing the results of data analysis in text, tables, or images.

RESULTS

Students' Mathematical Thinking Process in Solving Nonroutine Number Pattern Problems

Based on the results of research data analysis of the 32 subjects who participated in this study, students' mathematical thinking processes in solving non-routine pattern problems can be grouped into three categories, namely the first category of subjects who solve problems with new strategies (critical reflection), the second category of subjects solving problems with the help of scaffolding (explicit reflection), and finally the category of subjects who cannot solve non-routine number pattern problems. The following is a description of the data on the mathematical thinking processes of the three subject categories in solving the problem of non-routine number patterns described based on indicators of mathematical thinking processes from Mason et al. (2010), namely specialization, generalizing, guessing, and convincing.

The non-routine problems of number patterns given to research subjects are as follows:

> "A number like 1221 is called a palindrome because it reads the same both forward and backward. A friend of yours says that all palindromes with four digits are divisible by 11. Is your friend's statement true?"

Specialization

In the number pattern problem above, the three categories of subjects in this study were able to specialize well in the action-situation phase. S1 subject (critical reflection category) specializes by showing four examples of 4-digit palindromes that are divisible by 11 independently. The following is S1's response when specializing in solving non-routine number pattern problems in action situations.

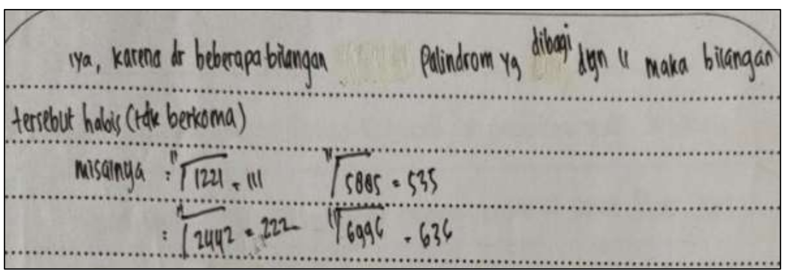


Figure 1: S1 response in specialization to solve number pattern problems in action situations

Based on the response given by S1 in Figure 1 above, it shows that subject S1 can specialize by choosing four examples of 4-digit palindromes, such as 1221, 2442, 5885, and 6996, to check whether the selected 4-digit palindrome is divisible by 11? Based on the results shown by S1, the selected examples show that the four palindromes are divisible by 11. The results of this specialization can be carried out by S1 subjects in action situations or working independently.

Like the S1 subject, the S2 subject (from the explicit reflection category) can also specialize independently or in an action situation. The following is the response given by S2 when specializing in solving non-routine number pattern problems in action situations.

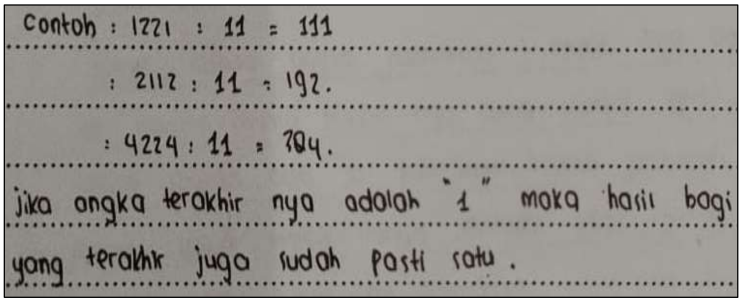

Contoh : 1221 : 11 = 111
: 2112 : 11 = 192.
: 4224 : 11 = 384.
jika angka terakhir nya adalah "1" maka hasil bagi yang terakhir juga sudah pasti satu.

Figure 2: S2 response in specialization to solve number pattern problems in action situations

Based on the response given by S2 in figure 2 above, it shows that the subject can specialize by giving three examples of 4-digit palindromes, such as 1221, 2112, and 4224. Next, the subject tries to look at the four selected 3-digit palindromes. Is it divisible by 11? Based on the selected examples, S2 shows that the three palindromes are divisible by 11. This result can be given S2 in the action situation phase.

Like Subjects S1 and S2, Subject S3 (Subjects from the category that do not solve the 4-digit palindrome problem) can also specialize well independently or in action situations. The following is the response given by S3 when specializing in solving non-routine number pattern problems in action situations.

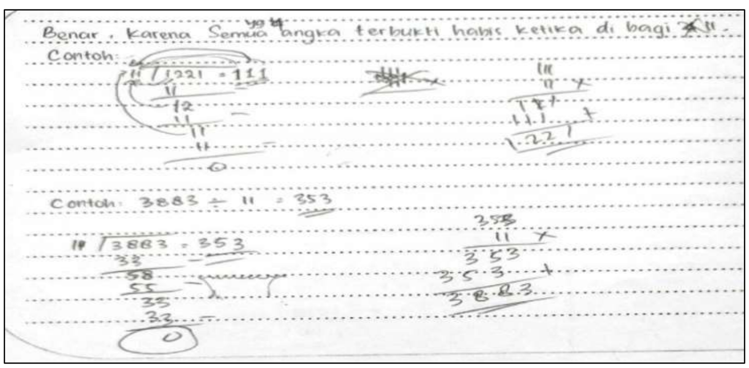

Benar, karena semua angka terbukti habis ketika di bagi 11.
Contoh: 11 / 1221 = 111
Contoh: 3883 ÷ 11 = 353

Figure 3*:* S3 response in specialization to solve number pattern problems in action situations

Based on the response given by S3 in Figure 3 above, it shows that the subject can specialize by choosing two examples of 4-digit palindromes, namely 1221 and 3883, to check whether the two selected 4-digit palindromes are divisible by 11. Based on the examples chosen, S2 shows that the two palindromes are divisible by 11. Subject S3 can carry out this result in the action situation phase.

The description of specialization data on the number pattern problem above shows that the three categories of subjects can carry out specialization properly in solving non-routine number pattern problems.

Recapitulation of the specialization phase of the mathematical thinking process of the three categories of students in solving non-routine number pattern problems is listed in Table 3 below.

Numb	Research Subject Categories	Specialized Response
1	Critical reflection subject	Give examples properly in action situations
2	Explicit reflection subject	Give examples properly in action situations
3	The subject cannot solve the problem	Give examples properly in action situations

Table 3 Data specialization of the three subject categories in solving number pattern problems

Generalization

Generalization is estimating a broader situation by acting on several examples, or it can be expressed as searching for patterns/relationships. Departing from a certain number of operations, a decision is attempted to be made about the claim, suggesting the specific procedures performed during the generalization (Arslan & Yildiz, 2010). Pilten (2008) said several strategies that students can use in this process could set more examples for determining relationships, collect as many samples as possible, and test conjectures (Uyangör, 2019). Student responses at this stage are categorized into three types: linguistic/mathematical expressions of exact relationships, linguistic/mathematical expression of the right relationship with the help of scaffolding, or the linguistic/mathematical term of the relationship is imprecise/ left blank.

S1 subjects can generalize in solving given problems by responding uniquely to linguistic and mathematical expressions or critically reflecting on formulation situations. The following is a generalization of the mathematical form provided by S1 subjects in solving number pattern problems in formulation situations.

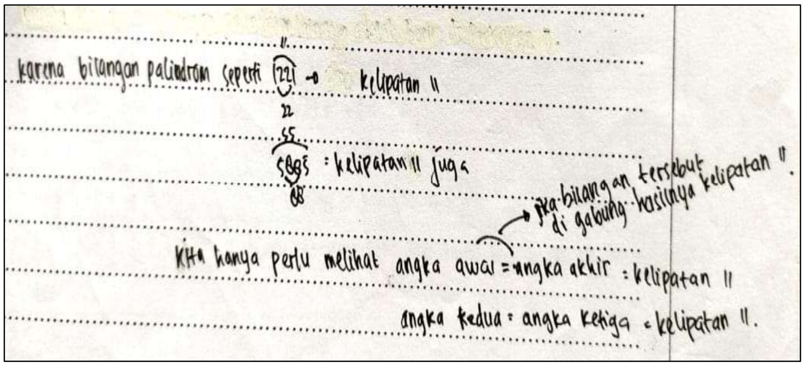


Figure 4: S1's generalization form response to the number pattern problem in a formulation situation

Based on the responses in Figure 4 above, subject S1 determines the generalization of the number palindrome problem in a mathematical form, namely two pairs of numbers multiples of 11. The first pair of numbers, namely the first number, is equal to the fourth number, and the second is the second number, the same as the third number, which is a multiple of 11. The results of this generalization can be given in S1 in the formulation situation.

The following is an excerpt from the interview of Researcher (R) with S1 (Subject from the critical reflection category) to gather information about generalizations on number pattern problems.

R : "What rules or patterns apply to the 4-digit palindromes you found?"

S1 : "Yes, here I see that in a 4-digit palindrome, the first and fourth numbers are the same, so this is a multiple of 11, so the second number is the same as the third number, so this is also a multiple of 11."

The responses and interview excerpts from S1 show that the subject can generalize to the problem of number patterns in the form of linguistic/mathematical expressions with precise and unique relationships.

While S2 can generalize from 4-digit palindromes by identifying patterns and relationships with the help of scaffolding from researchers, he can formulate patterns or rules from a 4-digit palindrome where the difference between a palindrome and the next palindrome is 110. Following is the response of subject S2 when generalizing in solving number pattern problems after being given a scaffolding or validation situation.

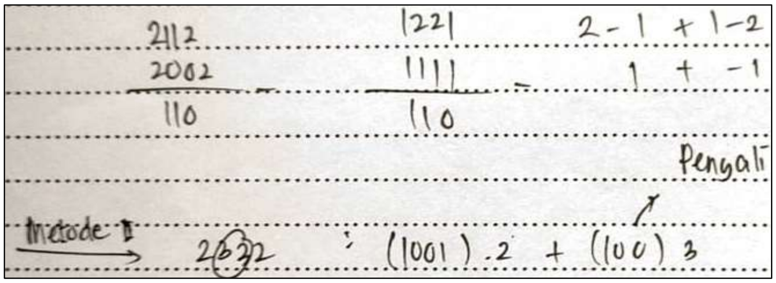


Figure 5: S2's generalization form response to the number pattern problem in a validation situation

Based on the response of subject S2, as listed in Figure 5 above, it shows that the subject can identify patterns or rules contained in a 4-digit palindrome, namely the difference between one palindrome and another palindrome is 110 so that the second and third digits of each number palindrome are multiples of 110. In addition, a palindrome's first and fourth digits are multiples of the smallest palindrome, namely 1001. S2 gives an example: the palindrome 2332 equals (1001)2 plus (110)3. S2 can carry out this form of mathematical generalization in validation situations.

The following is an excerpt of the Researcher's (R) interview with the subject of S2 to dig up complete information in generalizing the problem of number patterns.

R : "How can the relationship between the 2332 palindromes be arranged into (1001)2 and (110)3?"

S2 : "The distance of one palindrome from the next in the same thousand is 110, so the second and third digits of a 4-digit palindrome are multiples of 110". Then the first and fourth digits of a palindrome are multiples of the smallest palindrome, namely 1001, so 2002 can be written as (1001)2."

Based on the responses and excerpts from the S2 subject's interview, it was shown that the subject could generalize in solving number pattern problems in the form of mathematical expressions with the right relationships. S2 subjects can do this with the help of scaffolding from researchers or validation situations.

On the other hand, S3 has difficulty generalizing in identifying patterns or rules that apply to 4-digit palindromes independently or with the help of scaffolding. The following is an excerpt from the interaction between the Researcher and the S3 subjects in providing scaffolding assistance to generalize the 4-digit palindrome problem.

R : "What patterns or rules did S3 find?"

S3 : "The smallest palindrome is 1001, and the thousands and units of other palindromes can be added from this smallest palindrome."

R : "That's right... then which pattern can you see?"

S3 : "The difference of one palindrome from the next palindrome in the same thousand is 110, so the second and third digits of a palindrome can be added from 110."

R : "Do all palindromes have a difference of 110?"

S3 : S3: "It seems yes, sir, the difference is all 110."

R : "The correct palindrome in one thousand is 1991, and the smallest of the two thousand palindromes is 2002. What is the difference between the palindromes of 2002 and 1991?"

S3 : Subject S3 calculated the difference between the 2002 and 1991 palindromes, "the difference is 11, sir!"

R : "Yes, then how can we construct a 4-digit palindrome so that we can show it is divisible by 11?"

S3 : Stop thinking about the shapes that can be arranged. "I don't know, sir!"

Based on the quote from the Researcher's interaction with the S3 subject in providing scaffolding assistance, it shows that the subject has not been able to make generalizations in solving number pattern problems, whether in language or mathematical expressions with the right relationships.

Data recapitulation of students' mathematical thinking processes in generalizing to solve non-routine number pattern problems as shown in Table 4 below.

Numb	Research Subject Categories	Generalization Response
1	Critical reflection subject	Linguistic/mathematical expression of the exact relationship in the formulation situation
2	Explicit reflection subject	The mathematical expression of the appropriate relationship in a validation situation
3	The subject cannot solve the problem	Haven't been able to give a linguistic/mathematical expression of the right relationship.

Table 4: Recapitulation of data generalization of the three subject categories in solving the 4-digit palindrome problem

Conjecture

Conjecture arises in the process of specialization and generalization and is the process of researching the accuracy of a hypothesis by predicting that it is likely to be true. Mason et al., (2010b) said conjecture recognizes developing generalizations. Actions such as making linguistic or mathematical conjectures, formulating mathematical claims, generating results from hypotheses, and establishing and testing ideas can be relevant to this process (Arslan & Yildiz, 2010). Student responses at the guessing stage were categorized/grouped into three forms: linguistic conjecture, mathematical conjecture, and; incorrect math/linguistic assumption/left blank.

In the formulation situation phase, S1 can make linguistic and mathematical conjectures by using patterns or rules already identified at the generalization stage. The following is the response given by S1 in making mathematical conjectures based on the regulations or patterns specified in the previous generalizations.

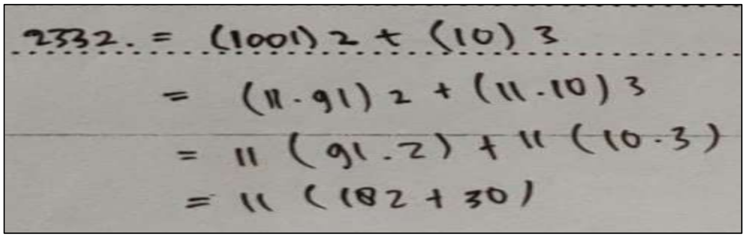


Figure 6: S1's response in making mathematical conjectures in solving problems in formulation situations

The response given in the figure above shows that subject S1 can make mathematical conjectures with one example of a 4-digit palindrome, namely 2332. S1 translates the form 2332 into 2002 + 330 and factors into (1001)2 + (110)3 so that the condition can be shown as a multiple of 11. Subject S1 can make this conjecture in the formulation situation phase.

The following is an excerpt from an interview with S1 to identify the assumptions made in solving the 4-digit palindrome problem.

R : "Why can the 2332 palindrome be translated into (1001)2 + (110)3?
S1 : "Yes sir 2332 can be written as 2002 + 330, so it can be translated into (1001)2 + (110)3
R : "Why does it need to be written as (1001)2 + (110)3?"
S1 : "Because it's easier to count multiples of 11."
R : "Why does using this form of proof conclude that all 4-digit palindromes are divisible by 11?"
S1 : "All 4-digit palindromes can be factored by 1001 and 110 packs, while 1001 and 110 can be divided by 11."

Based on the responses and excerpts from the results of the interviews, it was shown that S1 subjects could make mathematical conjectures in solving number pattern problems in formulation situations.

Meanwhile, S2 subjects can make conjectures using patterns or rules already identified at the generalization stage in validation situations. Based on the results of interviews with S2, it can be determined that the subject can make conjectures in mathematical form.

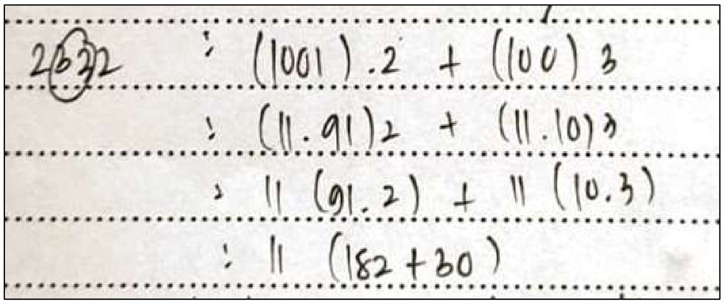


Figure 7: S2's response in making mathematical conjectures in solving problems in validation situations

Based on the response of subject S2, as listed in Figure 7 above, it shows that the subject can formulate mathematical conjectures through the example of the 2332 palindrome, which is translated into (1001) 2 plus (110)3. Using the 2332 palindrome, the S2 subject can show that the 4-digit palindrome is divisible by 11. This conjecture can be given S2 with the help of scaffolding from the Researcher or in a validation situation.

The following is an excerpt from an interview with S2 when exploring the conjecture stages carried out by S2 in solving the 4-digit palindrome problem.

R : "What is the meaning of the pattern or rule found? What can be concluded?"

S2 : "Because the smallest palindrome is 1001, and the distance of one palindrome from the next palindrome in the same thousand is 110, and the distance of the largest palindrome from a certain thousand to the smallest palindrome of the next thousand is 11. Because 1001,110 and 11 are divisible by 11, all 4-digit palindromes are divisible by 11."

The excerpt from the interview with the S2 subject above shows that the subject can guess the correct mathematical form in solving the 4-digit palindrome problem in validation situations.

Recapitulation of the conjectures stage data on the process of thinking mathematically in solving non-routine mathematical problems as listed in Table 5 below.

Numb	Research Subject Categories	Conjectures Response
1	Critical reflection subject	Make appropriate language predictions and relate the rules/patterns identified in the formulation situations
2	Explicit reflection subject	Perform appropriate language guesses and relate rules/patterns identified in the validation situation
3	The subject cannot solve the problem	Haven't been able to make precise language/math guesses yet

Table 5: Alleged data of the three subject categories in solving the 4-digit palindrome problem

Convincing

Convincing or giving evidence is an essential concept in learning mathematics (Knuth, 2002), it is also crucial for mathematical thinking. While creating evidence, actions such as explaining a hypothesis, saying why it is true or false, and selecting and using different ways of logical thinking (inductive and deductive) and types of proof become relevant (Uyangör, 2019). Student responses at this stage are categorized in code 3: arithmetic proof, algebraic proof, and left incomplete/incorrect/blank.

S1 subjects can provide proofs of algebraic proofs with the help of scaffolding from researchers or in validation situations. Evidence of the algebraic form of subject S1 can be done by assuming a 4-digit palindrome as naan, where the first and second n are the thousands and one's digits, and the first and second a are the hundreds and tens, respectively. The following is the response given by S1 at the convincing stage in solving the 4-digit palindrome problem in validation situations.

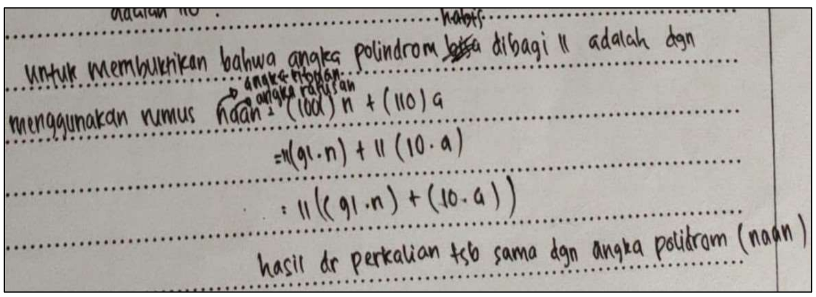


Figure 8: S1's response in convincing the 4-digit palindrome problem in a formulation situation

Based on the response given by the subject in the picture above, S1 proves the algebraic argument by assuming the general form of the palindrome, namely naan. The first letter n has the value as thousands, and n in the fourth digit has the value as the units digit. Then the first a is worth hundreds, and the second a is worth tens. The form of naan can be described according to the condition that has been identified so that it can be shown to be a multiple of 11. As for this algebraic proof, S1 can be carried out with the help of the Researcher's scaffolding.

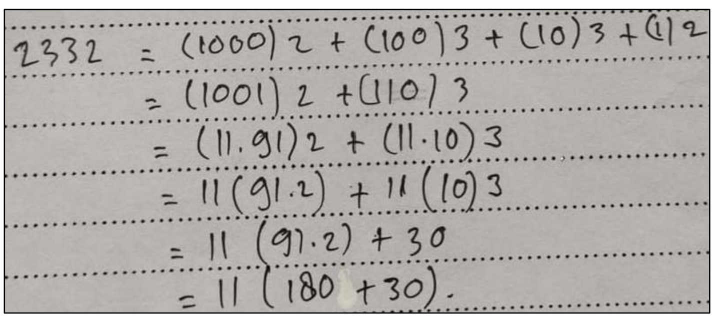
The following is an excerpt from the Researcher's interaction with the S1 subject to provide scaffolding assistance in solving the 4-digit palindrome problem.

R : "Using this pattern, how do you show that all 4-digit palindromes are divisible by 11?"
S1 : S1 silently thinks about the strategies that can be used.
R : "Suppose anna is the general form of a 4-digit palindrome. Using the pattern above, can it be shown that anna is divisible by 11?"

S1 : "Yes sir, if naan is a 4-digit palindrome, then the first n has the value as thousands, and n in the fourth digit has the value as the units digit. Then the first a is worth hundreds, and the second a is worth tens".
R : "Yes, can the form of anna's palindrome be shown to have a factor of 11"
S1 : "Yes, sir, the result is 11 times 91a plus 10n."
R : "Then, what conclusions can be drawn using the general form of this palindrome?"
S1 : "Yes sir, by using this general form, it can be shown that the general form of the naan palindrome also has a factor of 11, so it can be concluded that all 4-digit palindromes are divisible by 11."

The responses and excerpts from the interview results show that S1 subjects can be convincing in solving number pattern problems in algebraic form. This algebraic proof shows that S1 is a new or unique way of validating situations.

While S2 can provide algebraic proof in solving the second problem with the help of scaffolding from researchers, S2 can show all 4-digit palindromes divisible by 11 using the place value of a 4-digit palindrome. The following is the response of the S2 subject in providing evidence for solving number pattern problems.



$$
\begin{aligned}
2332 &= (1000)2 + (100)3 + (10)3 + (1)2 \\
&= (1001)2 + (110)3 \\
&= (11.91)2 + (11.10)3 \\
&= 11(91.2) + 11(10)3 \\
&= 11\ (91.2) + 30 \\
&= 11\ (180 + 30).
\end{aligned}
$$

Figure 9: S2's response in convincing in arithmetic form using place value

Based on the response given by S2 in Figure 9 above, it shows that S2 can carry out the convincing stage by using inductive arguments or arithmetic proofs, namely by conducting all palindromes divisible by 11 using the place value of a 4-digit palindrome. Using the concept of place value, the subject can show the general form of a 4-digit palindrome having a factor of 11. S2 subjects, with the help of scaffolding from researchers or in validation situations, can carry out this solution.

The following is an excerpt of the Researcher's interaction with the subject S2 when providing scaffolding in the stage of compiling evidence in solving number pattern problems.

R : "Can you explain the value of each digit position of the 2332 palindrome?"

S2 : "The first 2nd position is worth thousands, the first 3's is worth hundreds, and the second 3's is tens, while the second 2nd is units."
R : "That's right... What can you write down if you describe it?"
S2 : "Trying to translate the 2332 palindrome into (1000)2 + (100)3 + (10)3 + (1)2
R : "OK.. Now how does it look if those with the same multiplication number are combined?"
S2 : Try concatenating to "(1001)2 + (110)3"
R : "Can you translate the numbers 1001 and 110 into multiples of 11?"
S2 : "Try dividing the numbers 1001 and 110 by 11 each".
R : Based on the results obtained, what conclusions can be drawn?
S2 : "All 4-digit palindromes can be shown to have a factor of 11, and this indicates that all 4-digit palindromes are divisible by 11

The responses and quotes from the interaction with the subject show that 'S2 can be convincing in solving number pattern problems with inductive arguments in arithmetic form. S2 subjects can do this through scaffolding assistance from researchers or in validation situations.

Recapitulation of students' mathematical thinking process data in convincing according to the research subject category in solving non-routine number pattern questions as listed in Table 6 below.

Numb	Research Subject Categories	Convincing Response
1	Critical reflection subject	Can reason algebraic forms in validation situations using proper rules
2	Explicit reflection subject	Can reason arithmetic forms in validation situations using place value concepts
3	The subject cannot solve the problem	Not yet able to give reasons to convince

Table 6: The response data convinced the three subject categories to solve the 4-digit palindrome problem

Discussion

Recapitulation of students' mathematical thinking process data for the three categories of research subjects in solving non-routine number pattern questions is listed in table 7 below.

Mathematical Thinking Process	Subject Categories Research Results		
	Critical Reflection Subject	Explicit Reflection Subject	The subject cannot solve the problem
Specialization	Shows some examples of 4-digit palindromes divisible by 11 in action situations	Shows some examples of 4-digit palindromes divisible by 11 in action situations	Shows some examples of 4-digit palindromes divisible by 11 in action situations
Generalizations	The linguistic and mathematical expression of the exact relationship in the formulation situation;	The mathematical expression of the exact relationship in the formulation situation;	Not yet able to provide linguistic/mathematica l expressions with proper relationships
Conjectures	Gives linguistic and mathematical conjectures of the exact relations in the formulation situation	It gives mathematical conjectures of the exact relations in the validation situation	Haven't been able to make precise language/math guesses yet
Convincing	Can provide reasons in an algebraic form in validation situations	Can reason in arithmetic with place value concepts in validation situations	Can't give a proper reason

Table 7: Recapitulation of mathematical thinking process data for the three subject categories in solving non-routine number pattern problems

The data from this study, as listed in table 7 above, shows that the three categories of student groups in solving number pattern problems can specialize well in action situations. This indicates that the three categories of research subjects can understand the problem using their knowledge or actual development. This result follows the results of previously reported studies (e.g., Arslan & Yildiz, 2010; Keskin et al., 2013; Uyangör, 2019; Yıldırım & Köse, 2018) that students can quickly fulfill the specialization process. Other studies report that specialization does not occur naturally (Lane & Harkness, 2012) unless given instructions. Based on research results, Lane & Harkness (2012) said that many students did not carry out the specialization process but immediately jumped to the guessing stage and even directly to the generalization stage.

In the generalization stage, the subject from the critical reflection category can correctly generalize with linguistic and mathematical expressions in formulation situations. This shows that the subject of the critical reflection category can be generalized in uniquely solving number patterns or achieving critical reflection (Suryadi, 2019b). At the same time, students in the explicit reflection category or solving problems with the help of scaffolding can generalize in the form of mathematical expressions using scaffolding in validation situations and achieve explicit reflection (Suryadi, 2019b). The results showed that students were able to write relationships linguistically in the process of generalizing problems but had difficulty writing them algebraically (Arslan &

Yildiz, 2010; Yıldırım & Köse, 2018; Keskin et al., 2013). However, it appears that students in this study, with the help of scaffolding provided in the form of questions, helped students make abstractions and reveal relationships between variables. This shows students' knowledge in generalizing in solving number pattern problems in the Zone of Proximal Development (ZPD). The ZPD area consists of actions that children can understand but are not capable of performing. In other words, the ZPD area is a zone where children act with understanding and awareness with the help of adults.

While students from the third category group have not been able to identify patterns and mathematical models in solving number pattern problems both in formulation situations and validation situations. This shows that the knowledge of this category of students in making generalizations on 4-digit palindrome problems experience technical difficulties or instrumental learning obstacles (Suryadi, 2019a) or are in the zone of student difficulties that cannot be overcome (Zaretskii, 2009). To be able to overcome this difficulty, it can be overcome by presenting the problem given to students in accordance with the student's level of thinking and predicting the possible responses given so that the scaffolding assistance provided can be useful for students. This is as stated by Suryadi (2019a) that students will experience learning obstacles due to too high or too low thinking demands they face. Apart from that, to overcome this problem it is necessary to pay attention that when presenting the problem you must pay attention to the structural order of the material, namely the relationship between concepts and functional order to see the continuity of the thinking process which has an impact on the student learning process. The presentation stages can also be interpreted based on certain theoretical perspectives, for example the theory of didactic situations in mathematics including action situations, formulation situations, validation situations, and institutional situations (Brousseau, 2002). The stages of presentation according to the philosophical-pedagogical view include a series of mental actions that form ways of thinking (WoT) and produce ways of understanding (WoU) (Harel, 2008).

Conjectures arise in the process of specialization and generalization by conjecturing that they may be true. Mason et al. (2010b) said conjecture is recognizing developing inferences. Actions such as making linguistic or mathematical conjectures, formulating mathematical claims, generating results from the thesis, and establishing and testing hypotheses can be relevant in this process (Arslan & Yildiz, 2010). The results of this study indicate that students from the category of being able to solve problems with a unique strategy or critical reflection can make predictions of linguistic and mathematical expressions in formulation situations. These results indicate that the subject of the critical reflection category can provide predictions in a new way or achieve critical reflection. Whereas students from the explicit reflection category can make conjectures with mathematical expressions with the help of scaffolding or validation situations. This suggests that the scaffolding assistance provided by the researchers from identifiable patterns can be used to denote all 4-digit palindromes divisible by 11. This shows that students in the explicit reflection category make conjectures with expressions or mathematical models in solving 4-digit palindrome problems using their potential knowledge or are in the Zone of Proximal Development (ZPD).

Convincing or proof is an important concept in learning mathematics (Knuth, 2002) and is also important for mathematical thinking. While creating evidence, actions such as explaining a hypothesis, saying why it is true or false, and choosing and using different ways of logical thinking (inductive and deductive thinking) and types of proof become relevant (Uyangör, 2019). Harel & Sowder (1998) define verification as the process used by a person to remove doubts about the truth of a statement. A distinction is made between confirming oneself and convincing others. One person's evidentiary scheme consists of what constitutes confirming and persuading others. The results of this study indicate that the subject of the critical reflection category can provide evidence or reasons in algebraic form by using rules or patterns from generalization forms that can be identified from a 4-digit palindrome. Meanwhile, the subject of the explicit reflection category can provide evidence or reasons in arithmetic form by using place value. This is by Harel & Sowder (1998) found that the most common proof scheme found by students is an inductive proof scheme, in which students ensure themselves and persuade others about the truth of the conjecture by direct measurement of quantities, numerical calculations, the substitution of specific numbers in algebraic expressions. And others. At the same time, Uyangör (2019), based on the results of his research, said that students preferred arithmetic proofs where of the five correct answers given at this stage, only one was an algebraic proof. The results of this study are also by the research of Lee et al. (2011) that proficiency in patterns predicts ability in algebra. Proficiency in patterning tasks is, in turn, expected to renew children's capacities. These findings suggest that providing algebraic proofs may be difficult for students who have difficulty recognizing and generalizing rules about patterns. To support students in algebraic thinking, teachers must design learning that begins by presenting real or contextual problem designs that are easily recognized by students, then expanding the context in symbolic form. This is as stated by (Tall, 2008) that the transition from arithmetic to formal axiomatic thinking can be built through concrete and symbolic experiences.

Future research is urgently needed to explore how mathematical thinking can be used to address modern problems in work and life. As stated by Goos & Kaya (2020a) that studying mathematical thinking in real-world contexts can produce insights into the nature of critical mathematical thinking in the workplace, the role of digital technology in providing problem-solving and reasoning strategies, and new approaches to dealing with interdisciplinary problems that require synthesis of mathematical thinking in knowledge domains in the fields of Science, Technology, Engineering, and Mathematics (STEM).

CONCLUSIONS

The results of the research show that the mathematical thinking process of students in the critical reflection category specializes by providing examples that are asked in action situations, making generalizations and conjectures in linguistic and mathematical form in formulation situations, and convincingly by giving reasons in algebraic form in validation situations. Meanwhile, students in the explicit reflection category specialize by providing specific examples that are asked about in action situations, generalize in linguistic form in formulation situations, make conjectures in mathematical form in validation situations, and convince by giving reasons in arithmetic using the concept of place value in validation situation. Meanwhile, students in the category who cannot solve problems can only specialize by providing specific examples of what is being asked in action situations. To support students in developing their level of mathematical thinking, teachers can present contextual problems that are appropriate to students' level of thinking, can predict possible responses or ways of thinking of students to problems given and present problems in accordance with the conceptual and functional sequence structure of students' thinking. To support students in algebraic thinking, teachers must design learning that begins by presenting real or contextual problem designs that are easily recognized by students, then expanding the context of the problem in symbolic form. Future research is urgently needed to explore how mathematical thinking can be used to address modern problems in work and life.

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