

The Problem Corner

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The Purpose of **The Problem Corner** is to give Students and Instructors working independently or together a chance to step out of their “comfort zone” and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem and its solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Greetings, fellow problem solvers!

I'm happy to share that I've obtained answers for both Problem 18 and Problem 19. I'm pleased to report that not only were all the solutions accurate, but they also demonstrated the application of effective strategies. My primary objective is to present what I consider to be the best solutions to contribute to the enhancement and elevation of mathematical knowledge within our global community.

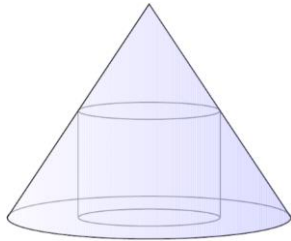
SOLUTIONS TO PROBLEMS FROM THE PREVIOUS ISSUE

Interesting “Cylinder inside Cone” problem

Problem 18

Proposed by Ivan Retamoso, BMCC, USA.

What are the dimensions of the cylinder that can be placed inside a right circular cone measuring 5.5 feet in height and having a base radius of 2 feet to maximize its volume?

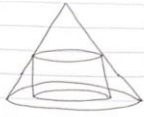


Note: Round your answers to three decimal places.

First solution to problem 18

By Manvinder Singh, Borough of Manhattan Community College, India.

Our solver skillfully applies the essential relationship between the radius and height of the cylinder, along with the corresponding dimensions of the cone. This proportional connection is vital for placing the cylinder correctly inside the cone. Subsequently, our solver uses the derivative from Calculus to optimize the cylinder's volume.



Height of the cone $h_{\text{cone}} = 5.5$ feet

Base radius of the cone $r_{\text{cone}} = 2$ feet

Objective \Rightarrow To find the dimensions of a cylinder (radius r_{cyl} and height h_{cyl}) that can be placed inside the cone to maximize its volume.

Relationship b/h cone & cylinder \Rightarrow

For the cylinder to fit inside the cone its dimensions must be scale with the cone's dimensions. The cylinder's height decreases as its radius increases, maintaining the slope of

the cone's side.
Therefore,

$$h_{\text{cyl}} = h_{\text{cone}} - \frac{h_{\text{cone}}}{r_{\text{cone}}} \cdot r_{\text{cyl}}$$

Volume of the cylinder \Rightarrow

The volume of the cylinder is given by the formula $V = \pi r_{\text{cyl}}^2 h_{\text{cyl}}$.

Substituting the expression for h_{cyl} gives:

$$V = \pi r_{\text{cyl}}^2 \left(h_{\text{cone}} - \frac{h_{\text{cone}}}{r_{\text{cone}}} r_{\text{cyl}} \right)$$

$$V = \pi r_{\text{cyl}}^2 \left(5.5 - \frac{5.5}{2} r_{\text{cyl}} \right)$$

$$V = \pi r_{\text{cyl}}^2 (5.5 - 2.75 r_{\text{cyl}})$$

Maximizing the Volume \Rightarrow

To find the maximum volume, we differentiate V with respect to r_{cyl} and set the derivative equal to zero.

$$\frac{dV}{dr_{\text{cyl}}} = \frac{d}{dr_{\text{cyl}}} [\pi r_{\text{cyl}}^2 (5.5 - 2.75 r_{\text{cyl}})]$$

using this in the height equation:

$$h_{\text{cyl}} = 5.5 - 2.75 \times 1.333$$

$$\frac{dV}{dr_{\text{cyl}}} = \pi (2r_{\text{cyl}}(5.5 - 2.75 r_{\text{cyl}}) - 2.75 r_{\text{cyl}}^2)$$

$$h_{\text{cyl}} = 1.83425$$

$$\frac{dV}{dr_{\text{cyl}}} = \pi (11r_{\text{cyl}} - 8.25r_{\text{cyl}}^2)$$

$$h_{\text{cyl}} = 1.833 \text{ feet}$$

$$\text{Set } \frac{dV}{dr_{\text{cyl}}} = 0:$$

Conclusion \Rightarrow

$$\pi (11r_{\text{cyl}} - 8.25r_{\text{cyl}}^2) = 0$$

$$11r_{\text{cyl}} - 8.25r_{\text{cyl}}^2 = 0$$

$$r_{\text{cyl}} (11 - 8.25r_{\text{cyl}}) = 0$$

This gives us two solutions:

$$r_{\text{cyl}} = 0 \quad | \quad r_{\text{cyl}} = \frac{11}{8.25}$$

Solving for r_{cyl} \Rightarrow

$$r_{\text{cyl}} = \frac{11}{8.25} = 1.333 \text{ feet}$$

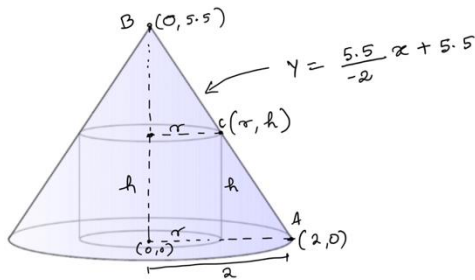
The cylinder that maximizes volume inside the given cone has a radius of approximately 1.333 feet and a height of approximately 1.833 feet.

Second solution to problem 18

By Aradhana Kumari, Borough of Manhattan Community College, USA.

Our alternate solution is characterized by a meticulous attention to detail, a strong organizational structure, and a comprehensive justification for every step taken towards the ultimate solution. The sign of the second derivative is utilized to demonstrate that the volume of the cylinder reaches its maximum at the critical point.

Solution: Consider the picture below.



The equation of the line passing through the points

A (2,0) and B (0,5.5) is given as $y = \frac{5.5}{-2} x + 5.5$

Since the point C (r, h) lies in the above line we have:

$$h = \frac{5.5}{-2} r + 5.5$$

$$h = -2.75 r + 5.5 \dots\dots\dots(\text{eq 1})$$

The Volume V of a cylinder with radius r and height h is given as

$$V = \pi r^2 h$$

Substituting the value of h in the formula for volume of cylinder we get

$$V = \pi r^2 (-2.75 r + 5.5)$$

$$V = -2.75 \pi r^3 + 5.5 \pi r^2 \dots\dots\dots(\text{eq 2})$$

We differentiate equation given by (eq 2) with respect to r we get

$$\frac{dv}{dr} = -2.75 \pi (3r^2) + 5.5 \pi (2r) \dots(\text{eq 3})$$

For Maxima or minima, we have $\frac{dv}{dr} = 0$

$$\text{i.e } -2.75 \pi (3r^2) + 5.5 \pi (2r) = 0$$

$$\pi r [-2.75 3r + 11] = 0$$

Therefore, we have $\pi r = 0$ or $[-2.75 3r + 11] = 0$

$$\text{Since } r \neq 0 \text{ we have } r = \frac{11}{8.25} \approx 1.33$$

We differentiate equation given by (eq 3) we get

$$\frac{d^2v}{dr^2} = -2.75 \pi (6r) + 5.5 \pi (2) \dots\dots(\text{eq 4})$$

Substituting the value of $r = \frac{11}{8.25}$ in the equation given by (eq 4) we get

$$\begin{aligned} \frac{d^2v}{dr^2} &= -2.75 \pi (6 \times \frac{11}{8.25}) + 5.5 \pi (2) \\ &= -22\pi + 11 \pi \\ &= -11\pi < 0 \end{aligned}$$

Hence $r = \frac{11}{8.25}$ is a point of maxima.

Substituting the value of $r = \frac{11}{8.25}$ in the equation given by $h = -2.75 r + 5.5$

$$\text{We get } h = -2.75 \left(\frac{11}{8.25}\right) + 5.5 \approx 1.83$$

Therefore, radius of the required cylinder is $r = \frac{11}{8.25} \approx 1.33 \text{ ft}$

Height of the required cylinder is $h = -2.75 \left(\frac{11}{8.25}\right) + 5.5 \approx 1.83 \text{ ft}$

“Largest cord in a circle” problem.

Problem 19

Proposed by Dr. Michael W. Ecker, (retired) Pennsylvania State University, USA.

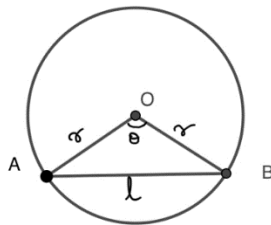
Prove that the diameter of a circle is the largest possible size of a chord of said circle.

First solution to problem 19

By Aradhana Kumari, Borough of Manhattan Community College, USA.

Without loss of generality, our solver cleverly positions a cord of the circle with arbitrary length “horizontally” and finds its length in terms of a central angle and the radius of the circle. Finally, using the derivative from Calculus our solver maximizes the length of the cord, showing that it is indeed equal to the diameter of the circle.

Solution: Consider below Circle C with center O and radius r. Let AB be a chord of length l.



Let $\angle AOB$ be θ

Using Cosine rule we have:

$$l^2 = r^2 + r^2 - 2r^2 \cos \theta, \quad 0 < \theta < 360^\circ$$

$$l = \sqrt{r^2 + r^2 - 2r^2 \cos \theta} = \sqrt{2r^2 - 2r^2 \cos \theta} = \sqrt{2r^2(1 - \cos \theta)} = \sqrt{2r^2} \sqrt{(1 - \cos \theta)}$$

Differentiate both side with respect to θ we get

$$\frac{dl}{d\theta} = \sqrt{2r^2} \times \frac{1}{2} \times (1 - \cos \theta)^{-1/2} \sin \theta$$

For maxima and minima, we equate $\frac{dl}{d\theta} = 0$

$$\sqrt{2r^2} \times \frac{1}{2} \times (1 - \cos \theta)^{-\frac{1}{2}} \sin \theta = 0$$

$$\sqrt{2r^2} \times \frac{1}{2} \frac{\sin \theta}{\sqrt{(1 - \cos \theta)}} = 0$$

Hence $\sin \theta = 0$,

$$\theta = 180^\circ$$

$$\frac{d^2l}{d\theta^2} = \frac{d}{d\theta} \left(\sqrt{2r^2} \times \frac{1}{2} \times (1 - \cos \theta)^{-\frac{1}{2}} \sin \theta \right)$$

$$= \frac{\sqrt{2r^2}}{2} \left((1 - \cos \theta)^{-\frac{1}{2}} \cos \theta + \left(\sin \theta \left(\frac{-1}{2} (1 - \cos \theta)^{-3/2} \sin \theta \right) \right) \right)$$

When $\theta = 180^\circ$ we get

$$\begin{aligned} \frac{d^2l}{d\theta^2} &= \frac{\sqrt{2r^2}}{2} \left((1 - \cos 180^\circ)^{-\frac{1}{2}} \cos 180^\circ \right. \\ &\quad \left. + \left(\sin 180^\circ \left(\frac{-1}{2} (1 - \cos 180^\circ)^{-3/2} \sin 180^\circ \right) \right) \right) \\ &= \frac{\sqrt{2r^2}}{2} \left((1 - \cos 180^\circ)^{-\frac{1}{2}} \cos 180^\circ \right) \\ &= \frac{\sqrt{2r^2}}{2} 2^{-1/2} (-1) < 0 \quad (r > 0) \\ \frac{d^2l}{d\theta^2} &< 0 \end{aligned}$$

Hence $\theta = 180^\circ$ is a point of maxima.

Substituting the value $\theta = 180^\circ$ in below equation we get

$$\begin{aligned} l &= \sqrt{r^2 + r^2 - 2r^2 \cos 180^\circ} \\ l &= \sqrt{r^2 + r^2 + 2r^2} \\ &= \sqrt{4r^2} \\ &= 2r \\ &= \text{diameter of the Circle C.} \end{aligned}$$

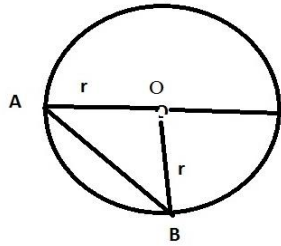
Hence diameter of the circle is the largest possible chord of said circle.

Second solution to problem 19

By Dr. Michael W. Ecker (The proposer)
(retired) Pennsylvania State University, USA.

The proposer's solution takes a distinct approach, omitting the use of Calculus. Instead, it capitalizes on an essential condition regarding the lengths of a triangle's sides, specifically, that the length of one side cannot be greater than the sum of the lengths of the other two sides.

Typical chord AB is shown in circle O . If AB is not a diameter, then drawing radii AO and BO results in a triangle, AOB . The length of AB then is smaller than the sum of the lengths of the two other sides of triangle AOB . Those two sides have a total length of twice the radius, or $2r = d$.



Hence, $AB < d$, as claimed. (Note: It does not matter how you draw AB. It's the argument, the proof, that matters here.)

NEW PROBLEMS

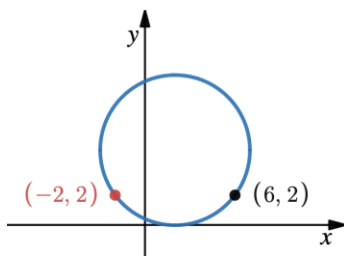
Dear fellow problem solvers,

I am confident that the resolution of problems 18 and 19 not only provided you with enjoyment but also granted valuable insights. Now, let's progress to the next two problems to continue this journey of exploration and learning.

Problem 20

Proposed by Ivan Retamoso, BMCC, USA.

Find the radius and the equation of the circle shown below.



Problem 21

Proposed by Ivan Retamoso, BMCC, USA.

Solve the equation below to find all real numbers x that satisfy:

$$\frac{8^x + 27^x}{12^x + 18^x} = \frac{7}{6}$$