The Problem Corner

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The Purpose of **The Problem Corner** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem and its solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Greetings, fellow problem solvers!

As the editor of **The Problem Corner**, I'm delighted to announce that I've successfully obtained answers for both Problem 20 and Problem 21. I'm pleased to report that all solutions were not only accurate but also showcased the effective application of strategies. My main goal is to present what I consider the best solutions to contribute to the enhancement and elevation of mathematical knowledge within our global community.

Solutions to **Problems** from the Previous Issue.

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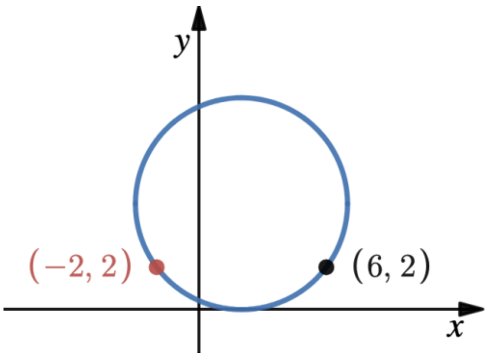
Engaging "Circle" puzzle.

Problem 20

Proposed by Ivan Retamoso, BMCC, USA.

Find the radius and the equation of the circle shown below.

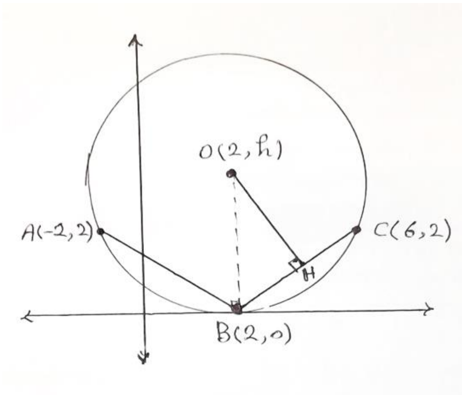


First solution to problem 20

By Dr. Hosseinali Gholami, University Putra Malaysia, Serdang, Malaysia.

This solution stands out for its remarkable simplicity and elegance, enjoy it.

We consider the following shape based on the given information and some geometric theorems.



$H(4, 1)$ is the middle point of BC.

$$a_{BC} = \frac{y_C - y_B}{x_C - x_B} = \frac{2-0}{6-2} = \frac{1}{2} \rightarrow a_{OH} = -2$$

Therefore, the equation of OH is as below.

$$y = ax + b \rightarrow y = -2x + b$$

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The coordinates of $H(4,1)$ satisfy in the equation OH, hence we have:

$$1 = -2(4) + b \to b = 9 \to y = -2x + 9$$

The coordinates of the center $O(2,h)$ satisfy in the equation OH.

$$h = -2(2) + 9 \to h = 5$$

It means $O(2,5)$ is the center of the circle.

The value of radius is calculated as follows:

$$r = OC = \sqrt{(x_C - x_O)^2 + (y_C - y_O)^2} = \sqrt{(6-2)^2 + (2-5)^2} = 5$$

Therefore, according to the standard form of circle equation, the equation of the given circle is as follows:

$(x-2)^2 + (y-5)^2 = 25.$

Second solution to problem 20

By Abir Mahmood, Borough of Manhattan Community College, Queens, USA.

Our solver employs symmetry to its fullest extent, employing various but equivalent methods to derive the equation of the circle. Notably, it offers both algebraic and geometric solutions in parallel, thereby enhancing the overall depth and richness of the solution.

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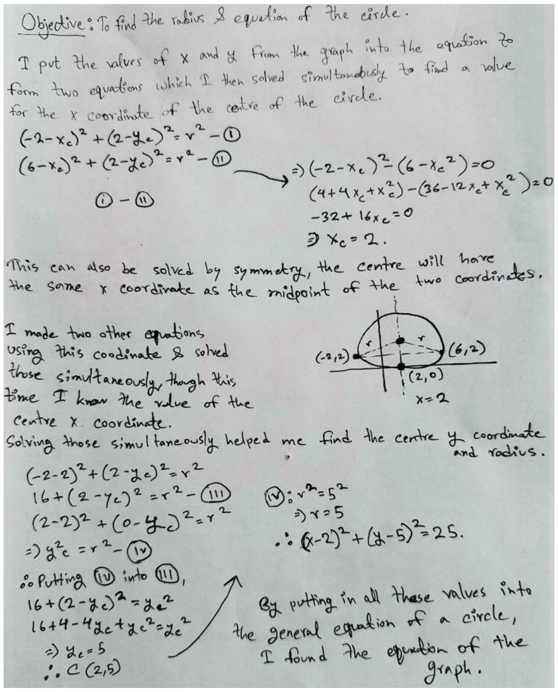


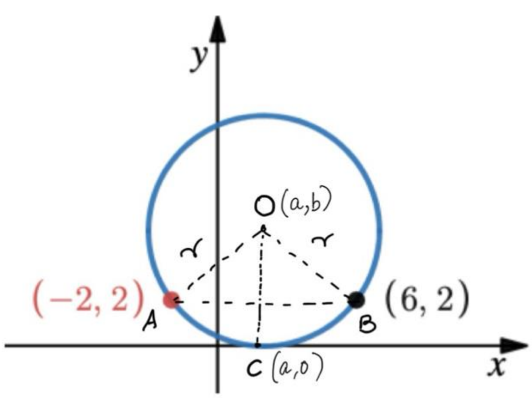
Third solution to problem 20

By Dr. Aradhana Kumari, Borough of Manhattan Community College, USA.

Our alternate solution ingeniously employs the circle equation and the distance formula to first derive the coordinates of the circle's center and its radius. Subsequently, the circle's equation falls into place effortlessly, well done!

Solution: Consider the Circle with center O, radius r as shown below. Let (a, b) be the coordinates of the center.

Hence the equation of the circle is $(x-a)^2 + (y-b)^2 = r^2$



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Since Point A (-2, 2) and point B (6,2) lie on the circle hence we have

$(-2-a)^2 + (2-b)^2 = r^2 = (6-a)^2 + (2-b)^2$

Hence $(-2-a)^2 + (2-b)^2 = (6-a)^2 + (2-b)^2$

$(-2-a)^2 = (6-a)^2$

$4 + a^2 + 4a = 36 + a^2 - 12a$

$4 + 4a = 36 - 12a$

$4a + 12a = 32$

$16a = 32$

$a = \frac{32}{16} = 2$

Hence the x −coordinate of the center is 2.

Draw a line passing from the center $0(2, b)$ of the circle and perpendicular to $x - axis$, this line will meet $x - axis$ at a point, call this point C. The coordinate of point C is (2,0).

Point C (2,0) lies on the circle hence

$(2-2)^2 + (0-b)^2 = r^2 = (-2-a)^2 + (2-b)^2 = (-2-2)^2 + (2-b)^2$

$(2-2)^2 + (0-b)^2 = (-2-2)^2 + (2-b)^2$

$b^2 = 16 + 4 + b^2 - 4b$

$4b = 16 + 4 = 20$

$4b = 20$

$b = 5$

$r = \sqrt{(6-2)^2 + (5-2)^2} = \sqrt{16+9} = 5$

Hence the coordinate of the center of the circle is (2, 5), the radius of the circle is 5 and the equation of the circle is $(x-2)^2 + (y-5)^2 = 5^2$.

Problem 20 was also solved correctly by **Jahidul Islam, Borough of Manhattan Community College, Bangladesh.**

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"Tricky" algebraic problem.

Problem 21

Proposed by Ivan Retamoso, BMCC, USA.

Solve the equation below to find all real numbers x that satisfy:

$$\frac{8^x + 27^x}{12^x + 18^x} = \frac{7}{6}$$

First solution to problem 21

By Dr. Aradhana Kumari, Borough of Manhattan Community College, USA.

This well-structured solution first identifies potential solutions through careful inspection. Subsequently, it employs a rigorous methodology, including a clever variable transformation, to systematically derive the final two solutions.

Solution: Consider the below equation

$\frac{8^x+27^x}{12^x+18^x} = \frac{7}{6}$ ....(1)

Just as a trail let's substitute $x = 1$ in the left-hand side of the above equation we get

$\frac{8^1+27^1}{12^1+18^1} = \frac{35}{30} = \frac{7}{6}$

Hence $x = 1$ is a solution for the above equation.

Let's substitute $x = -1$ in the left-hand side of the above equation we get

$\frac{(8)^{-1}+(27)^{-1}}{(12)^{-1}+(18)^{-1}} = \frac{\frac{1}{8}+\frac{1}{27}}{\frac{1}{12}+\frac{1}{18}} = \frac{\frac{(27+8)}{(8\times27)}}{\frac{18+12)}{(12\times18)}} = \frac{\frac{35}{216}}{\frac{30}{216}} = \frac{35}{30} = \frac{7}{6}$

Hence $x = -1$ is a solution for the above equation.

Next, we will prove that $x = 1$ or $x = -1$ are the only solutions for the above equation given by (1).

$$\frac{8^x + 27^x}{12^x + 18^x} = \frac{7}{6}$$

$$\frac{(2^3)^x + (3^3)^x}{6^x[2^x + 3^x]} = \frac{7}{6}$$

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$\frac{(2^3)^x+(3^3)^x}{(2^x3^x)[2^x+3^x]} = \frac{7}{6}$ ....(2)

Let $2^x = A$, $3^x = B$

Hence equation given by (2) becomes

$\frac{A^3+B^3}{AB[A+B]} = \frac{7}{6}$

$$\frac{(A+B)^3 - 3AB(A+B)}{AB[A+B]} = \frac{7}{6}$$

$$\frac{(A+B)[(A+B)^2 - 3AB]}{AB[(A+B)]} = \frac{7}{6}$$

$$\frac{[(A+B)^2 - 3AB]}{AB} = \frac{7}{6}$$

$$\frac{A^2 + B^2 + 2AB - 3AB}{AB} = \frac{7}{6}$$

$$\frac{A^2 + B^2 - AB}{AB} = \frac{7}{6}$$

$6(A^2 + B^2 - AB) = 7AB$

$6A^2 + 6B^2 - 6AB = 7AB$

$6A^2 + 6B^2 - 6AB - 7AB = 0$

$6A^2 + 6B^2 - 13AB = 0$

$6A^2 - 13AB + 6B^2 = 0$

$6A^2 - 9AB - 4AB + 6B^2 = 0$

$3A(2A - 3B) - 2B(2A - 3B) = 0$

$(3A - 2B)(2A - 3B) = 0$

$(3A - 2B) = 0$ or $(2A - 3B) = 0$

$3A = 2B$ or $2A = 3B$

$3\ 2^x = 2\ 3^x$ or $2\ 2^x = 3\ 3^x$

$\frac{3}{2} = \left(\frac{3}{2}\right)^x$ or $\left(\frac{2}{3}\right)^x = \frac{3}{2} = \left(\frac{2}{3}\right)^{-1}$

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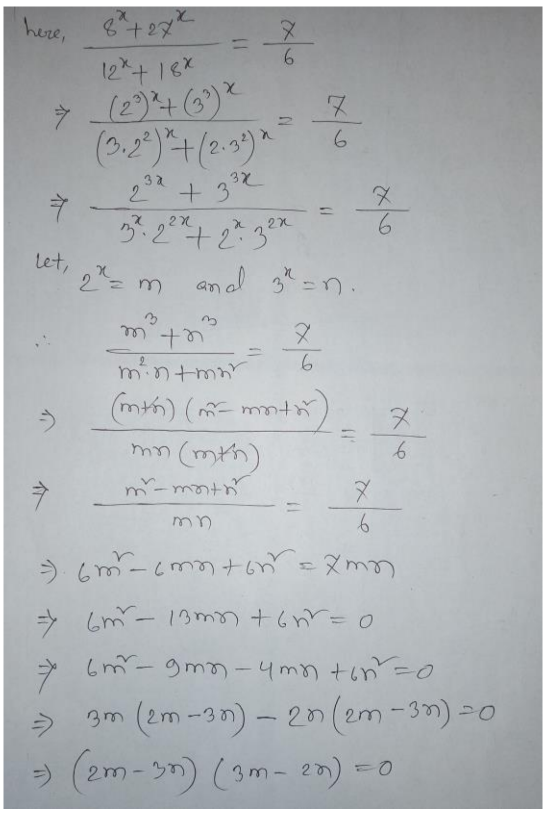
Hence $x = 1$ or $x = -1$

Second solution to problem 21

By Jahidul Islam, Borough of Manhattan Community College, Bangladesh.

The alternative solution avoids finding solutions through inspection and instead, skillfully rewrites the powers of 2 and 3 before proceeding directly to a change of variable. This strategic maneuver ultimately reveals the two possible solutions.



here, $\frac{8^x+27^x}{12^x+18^x} = \frac{7}{6}$

$\Rightarrow \frac{(2^3)^x+(3^3)^x}{(3\cdot 2^2)^x+(2\cdot 3^2)^x} = \frac{7}{6}$

$\Rightarrow \frac{2^{3x}+3^{3x}}{3^x\cdot 2^{2x}+2^x\cdot 3^{2x}} = \frac{7}{6}$

Let, $2^x = m$ and $3^x = n$.

$\therefore \frac{m^3+n^3}{m^2\cdot n+mn^2} = \frac{7}{6}$

$\Rightarrow \frac{(m+n)(m^2-mn+n^2)}{mn(m+n)} = \frac{7}{6}$

$\Rightarrow \frac{m^2-mn+n^2}{mn} = \frac{7}{6}$

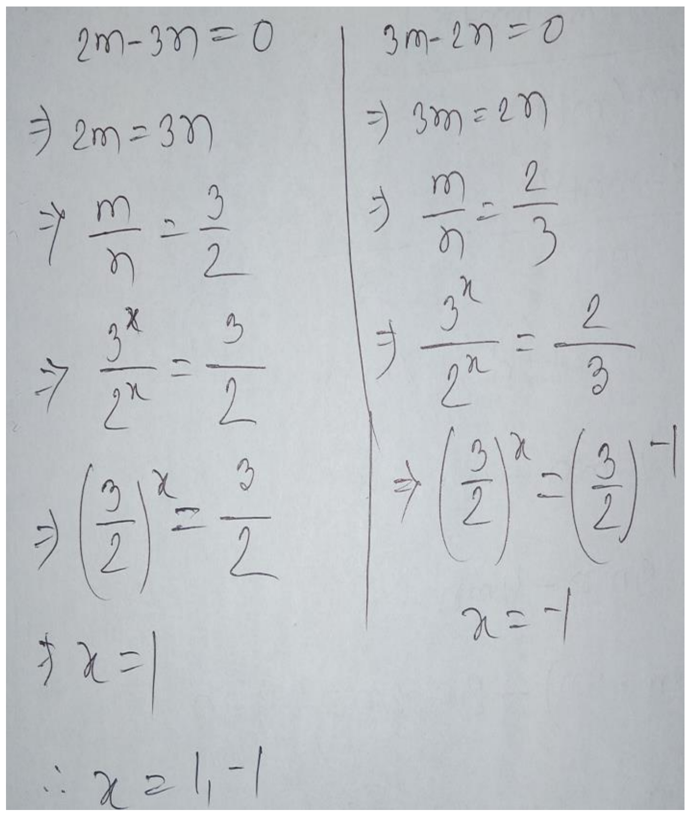
$\Rightarrow 6m^2-6mn+6n^2 = 7mn$

$\Rightarrow 6m^2-13mn+6n^2=0$

$\Rightarrow 6m^2-9mn-4mn+6n^2=0$

$\Rightarrow 3m(2m-3n)-2n(2m-3n)=0$

$\Rightarrow (2m-3n)(3m-2n)=0$



$2m-3n=0$

$\Rightarrow 2m=3n$

$\Rightarrow \frac{m}{n}=\frac{3}{2}$

$\Rightarrow \frac{3^x}{2^x}=\frac{3}{2}$

$\Rightarrow \left(\frac{3}{2}\right)^x=\frac{3}{2}$

$\Rightarrow x=1$

$3m-2n=0$

$\Rightarrow 3m=2n$

$\Rightarrow \frac{m}{n}=\frac{2}{3}$

$\Rightarrow \frac{3^x}{2^x}=\frac{2}{3}$

$\Rightarrow \left(\frac{3}{2}\right)^x=\left(\frac{3}{2}\right)^{-1}$

$x=-1$

$\therefore x=1,-1$

Dear fellow problem solvers,

I'm confident that you had an enjoyable experience solving problems 20 and 21, and you've gained new strategies for your mathematical repertoire. Now, it's time to move on to our next two problems to keep improving.

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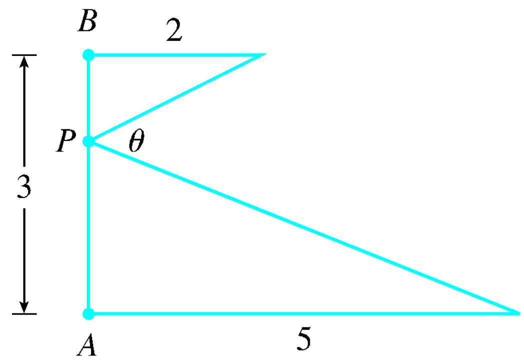
Problem 22

Proposed by Ivan Retamoso, BMCC, USA.

In the illustration below, at what distance from B should point P be positioned to maximize the angle θ?



Problem 23

Proposed by Ivan Retamoso, BMCC, USA.

Calculate the radius of the circle in which an isosceles triangle, with a base of 24 inches and legs each measuring 15 inches, is inscribed.

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