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Enhancing the Learning of Limits of Functions Using Multiple Representations

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Abstract: This is an exploratory study considering instructional methods in mathematics education. The study examined misconceptions arising from selected assignments undertaken by sixty-five students. The students were attending a calculus mathematical course unit on limits of functions, in different academic areas offered at selected tertiary institutions in Western Uganda. In this study, forms of expressing ideas on limits of functions are considered as far as they facilitate access to underlying mathematical principles. We explored the use and application of tools for representing mathematical ideas to enhance students' conceptual understanding and problem-solving skills. A small-scale pilot of interventions using readily available GeoGebra dynamical computer software was applied. The diagnostic assignment on limits of functions was used for data collection. The main objective was to examine whether or not students preferred the application of multiple representations to the analytic approach. The semi-structured interview protocol was also conducted to probe further and correlate students' responses and their problem-solving abilities. The results showed that multiple representations-based instructions significantly changed students' understanding of the limits of functions. The results promise better access and understanding of more abstract mathematical concepts and may support students' problem-solving abilities. The semi-structured interviews conducted indicated that multiple representations supported and, therefore, enhanced students' understanding and solving of the limits of functions. This study highly recommends that mathematics educators should adapt multiple representations-based instructions to enhance students' critical thinking, and problem-solving.

INTRODUCTION

Research in the 21st-century education system on learning mathematics requires that educators should adapt or adopt effective learning approaches. This is aimed at ensuring that students appreciate the usefulness of mathematics and subsequently apply it in new contexts. Sometimes and most often the learning of mathematics is taking a differential trend. Mathematics teaching practices require that educators explicitly use and connect mathematical representations to learning challenging and more complex concepts. Accordingly, "effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem-solving" (NCTM, 2014 p.10). Thus, different representations help students to communicate mathematically, reason insightfully and build procedural fluency from conceptual understanding through the process of problem-solving (Mainali, 2021). The learning of mathematics should be structured in such a way that students are involved in making sense of mathematics tasks. To do this, educators should apply varied strategies and representations, justifying solutions, making connections to prior conceptual knowledge and understanding, and linking mathematical representations to underlying ideas and other representations. This may help to evaluate students' mathematical reasoning and explanations. Educators can, therefore, select tasks that promote students' reasoning and problem-solving.

Representations are inevitably unique and inherent in supporting the learning of mathematics. They are intended to visualize presumably harder scientific tasks. They help to simplify complex and abstract concepts so that they can be concretized, and make mathematics more attractive and interesting. According to Ainsworth et al. (2006), multiple external representations support different ideas and processes, constrain interpretations and, promote a deeper understanding of the subject-specific domain. Generally, representations are aimed at mitigating and simplifying challenging mathematical concepts. Representations may take two forms: internal or external(Mainali, 2021). The former is all about cognitive configurations of mathematical thinking and problem solving perceived as mental images while the latter refers to structured physical situations that can be seen as embodying mathematical ideas. External representations are therefore used to demonstrate and communicate mathematical relationships visually. Representational modes include verbal descriptions, videos, tabular forms, dynamic graphical representations, and the building of models (equations), animations, and simulations (Ainsworth, 2008). These representational modes are applied to help learners understand and solve complex forms of mathematical concepts. The objective is to develop learners' understanding of basic mathematical ideas, concepts, or principles and use them to support problem-solving strategies. External representations are the main focus of this study.

Multiple representation learning practices can effectively be used to support specific mathematics content (NCTM, 2014). This involves educators' competencies in delivering the conceptual approach, relational understanding, and adaptive reasoning of the subject matter (Kathirveloo & Marzita, 2014). This knowledge component is what Hill et al. (2008) referred to as mathematical knowledge for teaching (MKT) and the mathematical quality of instruction (MQI), the unique knowledge that intersects with the specific subject teacher characteristics to produce effective and meaningful instruction. According to Hill, "teachers with weak MKT would have teaching characterized by few affordances and many deficits". Hill further noted elements for MQI as those that involve dealing with students' mathematical errors, responding to students appropriately, connecting classroom practice to mathematics in real life, mathematical language, and richness of mathematics (p. 437).

Unfortunately, learners often fail to exploit benefits accruing from the application of appropriate combinations of multiple representations. Janvier (1987) was among the pioneers in exploring problems of multiple representations in teaching and learning mathematics. The author reiterates the common tendency of educators in underestimating the teacher's role in the representation system in the standard curriculum. Mainali (2021) supports this claim since multiple representations would enhance learning, hence supporting students' understanding of mathematical concepts and constructing mental relationships. This is vital in communicating mathematical concepts, providing arguments, critical thinking, and a sign of understanding. Consequently, learners apply mathematical concepts in solving societal realistic problems situations through the process of modeling. Thus, the translation modes of multiple representations (e.g., symbols, signs, characters, diagrams, objects, pictures, or graphs) are important for learners in developing their cognitive skills to be more proficient in learning the limits of functions. To adequately understand these concepts, Arnal-palacián and Claros-Mellado (2022) highlight the significance of the teachers' specialized content knowledge and advanced mathematical thinking. The teacher's role is to carefully help learners to apply them successfully and effectively.

Some empirical studies conducted in different settings and contexts have demonstrated the significance of multiple representations in enhancing students' understanding of science and mathematics (Adadan, 2013; Ainsworth, 2006, 1999; Ainsworth et al., 2006; Desai & Bush, 2021; Dreher et al., 2015; Kozma et al., 2000; Kuntze et al., 2018; Mainali, 2021; Meij & Jong, 2006; Rosengrant et al., 2005; Vogt et al., 2020). In understanding the learning of mathematics and the limits of functions, in particular, some studies (e.g., Liang, 2016; Tall & Vinner, 1981) show that the topic is challenging and that students have limited conceptual understanding. Some studies (e.g., Arnal-palacián & Claros-mellado, 2022) report on limits and infinite limits in particular. In their study, prospective teachers failed to understand the notion of infinite limits and its algorithmic procedures. Thus, they applied the wrong graphical representation system. Some of the factors that account for students' challenges in this topic include those ranging from analytic to graphical. Tall and Vinner (1981) investigated students' concept images and their cognitive structure regarding the limits of functions. The author noted that students' differing concept images from the formal definitions of a mathematical theory cause cognitive conflict since mathematical concepts, rules, and principles are defined accurately. Multiple representations are likely to help boost their ability to explicitly hold presumably harder concepts in their mind and to mentally retrieve and manipulate them to suit any context.

Thus, the limits of some functions are best evaluated using graphical methods for better visualization. The causes of students' learning challenges in evaluating the limits of functions are enormous and mainly stem from students' preconceptions of solving equations, functions, and inequalities. Students, on one hand, fail to understand the relationship between equations, functions, and inequalities, while educators, on the other hand, have not adequately applied students' flawed conceptions with suitable approaches to address the causes and sources of students' learning challenges. Thus, this study aims to use multiple external representations using limits of various functions to compare and contrast analytical approaches of evaluating the limits of a function, and graphical representations to visually examine the convergence of limits.

The Conceptual Framework and Literature Review

This research is situated on the PCK conceptual framework based on Shulman (1986). Shulman conceptualized that "pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons" (p. 9). According to Shulman, effective learning strategies involve teachers' integration of students' preconceptions and misconceptions held previously and how these preconceptions relate to subsequent learning. In supporting students' mathematical thinking and understanding, Taşdan & Çelik (2016) developed a framework for examining mathematics teachers' PKC. The framework is important in enhancing teachers' PCK (e.g., the use of graphics, and manipulatives) with the main objective of understanding students' mathematical thinking.

Indeed, the above theoretical framework aligns with the five strands of mathematical proficiency. Kilpatrick, Swafford, and Findell (2001) proposed a multidimensional five interwoven and interdependent strands of mathematical proficiency teachers should target during classroom instruction. These strands are conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (NCTM, 2014). Kilpatrick, Swafford, and Findell have argued: "that proficiency should enable them to cope with the mathematical challenges of daily life and enable them to continue their study of mathematics in high school and beyond" (p. 116). These five interrelated strands are inevitable for learning mathematics in the sense that they support, foster, and promote students' identification and acquisition of conceptual knowledge, procedural knowledge, and problem-solving abilities, which are all supported by the cognitive load theory. This is what is referred to as conceptual change, and is aimed at understanding and connecting previous knowledge to new knowledge (Merenluoto & Lehtinen, 2002; Trumper, 2006; Vamvakoussi et al., 2007; Vosniadou, 2007).

However, educators should ask themselves the effective ways learners can be motivated to represent and connect prior knowledge and understanding and effectively use it deeply and broadly during problem-solving. According to NCTM (2014), students' effective learning "depends fundamentally on what happens inside the classroom as teachers and learners interact over the curriculum" (p. 8).To support this theoretical stance, "the cognitive linking of representations creates a whole that is more than the sum of its parts … Cognitive flexibility theory highlights the ability to construct and switch between multiple perspectives of a domain as fundamental to successful learning" (Ainsworth, 2008 p.198). Other studies conducted by Adelabu et al. (2022) Yimer (2022) and Yimer and Feza (2019) show that students' attitude, conceptual knowledge, and understanding is influenced by varying classroom instructional methods integrated with multiple representations.

The Concept of a Limit of Functions

Definition (Limit)
From Contemporary calculus 1 (Hoffman, 2012), the limit of a function is defined below.
Let $S \subseteq \Re$ *and* $f : S \to \Re$ be a function. A real number L is said to be a limit point of *f* at point $a \in S$ if given any $\in > 0\ \exists\ \ a\ \ \delta > 0$ such that if $x \in S$,
$\lim_{x \to a} f(x) = L$: For every given number $\varepsilon > 0$ there is a number $\delta > 0$ so that if x is within δ units of a (and $\neq a$) then $f(x)$ is within ε units of L. The symbol " $\to$ " means "approaches" or "gets very close to."
Equivalently: $\left|f(x) - L\right| < \varepsilon$ Whenever $\left|x - a\right| < \delta$ for $0 < |x - a| < \delta$.
Note that *f* may or may not be defined as $x = a$
We say that $f(x) \to L$ *as* $x \to a$ and write

$$\lim_{x \to a} f(x) = L. \tag{1}$$

This above definition does not apply to the one side (left and right) limits. For the left limit, as x approaches α of $f(x)$ is L if the values of $f(x)$ get as close to L as possible when x is very close to and left of α, $x < \alpha$: $\lim_{x \to \alpha^-} f(x) = L$. Conversely, the right limit, written with $x \to \alpha^+$, requires that x lies to the right of α, $x > \alpha$. Hence, the **One–Sided Limit Theorem states that:**
$\boldsymbol{\lim_{x \to \alpha} f(x) = L}$ **iff** $\lim_{x \to \alpha^-} f(x) = \lim_{x \to \alpha^+} f(x) = L$.

Properties of limits

a) If $\lim_{x \to a} f(x)$ exists then it is unique. (1)
b) If $\lim_{x \to a} f(x) = L_1$, and $\lim_{x \to a} g(x) = L_2$ then (2)
c) $\lim_{x \to a}[f(x) \pm g(x)] = L_1 \pm L_2$ (3)
d) $\lim_{x \to a}[f(x)\, g(x)] = L_1\, L_2$ (4)
e) $\lim_{x \to a} kf(x) = k\lim_{x \to a} f(x) = kL$ (5)
f) $\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{L_1}{L_{2_1}}$ provided $g(x) \neq 0\ \forall x$ and $L_2 \neq 0$ (6)
g) $\lim_{x \to a}\{f(x)\}^n = \left\{\lim_{x \to a} f(x)\right\}^n = L^n$ (7)
h) $\lim_{x \to a}\left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\left[\lim_{x \to a} f(x)\right]} = \sqrt[n]{L}$ (If L > 0 when n is even). (8)
i) $\lim_{x \to a} k = k$ (9)
j) $\lim_{x \to a} x = a$ (10)

k) For polynomial and rational functions, If $P(x)$ and $Q(x)$ are polynomials, and a is any number, then $\lim_{x \to a} P(x) = P(a)$ and $\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$ provided $Q(a) \neq 0$ (11)

All the above properties can be proved. However, this is outside the scope of this study.

Calculus has played a significant role in the history of mathematics and mathematical analysis in particular. The subject of calculus has been integrated with learning science, technology, and engineering (Rasmussen et al., 2014). The concept of limits has been investigated for the last twenty decades (Tall & Vinner, 1981), and it is the most important topic in learning calculus. The definition and application of limits provide prerequisite knowledge for learning other advanced mathematics topics (e.g., differentiation, integration, and sequences and series) (Palacián et al., 2020). Understanding the concept of limits may guarantee a further grasp of other concepts like functions. However, as Juter (2005a) and Juter (2005b) note, many mathematicians have found learning the concept of limits and related concepts challenging with multiple misconceptions. Moreover, other empirical findings (e.g., Juter, 2003) on the learning of limits at university mathematics support this claim. The learning at the university level is structured and formally presented in textbooks and as lectures. The concepts in these textbooks need thorough conceptual understanding to minimize misconceptions and errors.

In this research, multiple representations were applied to examine students' understanding of the limits of functions. Our own experience, as university educators have shown that students' learning of limits of functions is demanding in terms of time and conceptual understanding as compared to other course units. We examined the significance of external multiple representations in enhancing students' understanding of the limits of functions. External multiple representations are those used to symbolize, describe and refer to the same mathematical entity. They are used to understand, develop, and communicate different mathematical features of the same object or operation, as well as connections between different mathematical properties and principles. This research provides insight and adds knowledge to other empirical findings on students' conceptual understanding of the limits of functions. The findings will also provide additional knowledge on the usefulness of external multiple representations to both learners and educators aimed at enhancing the learning of mathematics generally. This study aims to answer the research question of whether or not external multiple representations enhance students' mental representations of limits of functions.

METHODOLOGY

The Sample

The sample consisted of 65-year one university students (21 female and 44 male). The students' average age was aged 19.45 (S.D=0.95). The students had been admitted to sampled universities in western Uganda to pursue a Bachelor of Science with education and were in their first year, the first semester in a mathematics class with calculus 1 as a course unit. The duration for this course was16 weeks, and all students were in a full-time program. Two lectures in calculus were conducted weekly each with a duration of two hours. The total teaching time was 48 hours. Limits of functions were taught to provide prerequisite knowledge to other related courses (derivatives, integrals, sequences, and series). The students were taught both analytically and later used external multiple representations. Specifically, the two approaches were compared and contrasted. Finally, a written follow-up test was administered and students who presented ambiguous solution sketches were interviewed to examine their understanding and preference of the two approaches.

Diagnostic Tasks

The students were given two tasks on limits of functions to be solved analytically. Later, the same tasks were solved by the use of external multiple representations. Students received the questions as an assignment immediately after the topic was fully covered. The questionnaire contained tasks about evaluating the limits of functions (questions 1 and question 2). Based on the theory of constructivism (Czarnocha, 2020), The students were also asked to explain the necessary prior knowledge for learning concepts of limits of functions since it was their first year, first-semester university course. After limits had been fully covered, as a course unit, a second questionnaire with similar tasks on limits of functions at different levels of difficulty was administered to check their conceptual understanding and problem-solving abilities. The main objective was to examine students' preference for the two methods of computing the limit of a function, and their ability to explain what they did. The students consented before participating in a focus group and individual interviews. Of the 65 students, 52 students consented to participate in a semi-structured interview. By taking into consideration gender differences, 20 students out of 52 were systematically selected for individual interviews. Each interview session per student lasted for 20 minutes. Students were asked about definitions of a limit of a function and solved tasks on the limit of a function with various levels of difficulty. They were expected to reveal specific and general knowledge on their solution sketches to clarify and/or justify their answers to the given tasks. Students' responses to the questions were analyzed. The interviews were transcribed verbatim. We specifically examined how students presented solutions to the tasks.

Instruments

The questionnaire contained two tasks on the limits of functions. They were intended to test students' understanding and misconceptions when evaluating the limits of functions using L'Hopitals' rule and by rationalization. The solutions to these tasks can be obtained graphically. The tasks were solved analytically. Later, students' solution sketches were compared with the graphical solutions to challenge students to come up with alternative solutions. These tasks were challenging and, therefore, some students did not present substantial solutions. Others left these tasks unsolved. The following tasks were given to the students as an assignment. The students solved the tasks about the limits of functions and later submitted them for marking. They were asked to evaluate the limits of the following functions:

(a) $\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2}$

(b) $\lim_{x\to 16} \frac{\sqrt{x}-4}{x-16}$

The above tasks were designed to explore students' understanding of the limits of functions (including students' misconceptions) for analytical or external multiple representations.

Procedure

Students were individually interviewed to examine their understanding of the limits of functions. Interviews were conducted at their respective colleges or faculties and lasted for one hour during regular school working hours. Each interview was designed to cover specific aspects of the limits of functions to correlate with students' preconceptions. We are particularly concerned with the evaluation of limits of functions to L' Hopital's rule and rationalization using multiple representations. All interviews conducted were recorded and transcribed verbatim to support multiple representations in fulfilling the purpose of the present study. To do this, each student was provided with a paper-and-pen assignment test, as a questionnaire. Students thought aloud and justified their solutions to the stated questions.

RESULTS

There are three methods for evaluating the limit of a function. These are the algebraic method, the tabular method, and the graphical method. The purpose of this research is to use visual representation to compare and contrast these methods. The algebraic method involves the simplification of algebraic functions before evaluating their limit. This may take the form of factoring and dividing, although often more complicated algebraic and/or serious trigonometric functions with inherent steps are needed. Normally, the steps are difficult to handle algebraically or the algebraic properties of such functions are not known to the learners. The tabular and graphical methods are used to evaluate a limit of a function f(x) as x approaches a given value say α. This method involves calculating the values of f(x) for many values of x very close to α so that we can algebraically determine which value f(x) approaches α. If f(x) converges very first, we may not need many values of x. Important to note is that this method may be used to evaluate the limits of some complicated functions, mainly those that require learners to rationalize. This is done by evaluating f(x) for many values of x.

The graphical method is closely related to the tabular method. However, a graph of the given function is drawn, and then the graph is used to determine which value f(x) approaches α. (see Table 1 and Table 2). The choice of the method to apply depends on the difficulties inherent in the question. However, each of these methods serves as an alternative to the other. Moreover, graphing the function or evaluating it at a few points using tabular form provides learners with the skills to visualize and verify the solutions obtained algebraically. We now visualize the solutions to tasks (a) and (b) and together with the students' challenges.

(a) For $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}$, some students computed $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}$ directly and obtained either 0 or ∞. That is to say,

$$\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}=\lim_{x\to 4}\frac{4-4}{\sqrt{4}-2} \qquad (6)$$
$$=\lim_{x\to 4}\frac{0}{0}$$
$$=\infty$$

Some students stopped here and submitted their work for marking. However, those who had grasped the concept of limit applied L'Hopital's rule to evaluate the limit of the above function since $\frac{0}{0}$ is the prerequisite step for using L'Hopital's rule.

$$=\lim_{x\to 4}\frac{2(1-0)}{-\sqrt{x}-0} \qquad (7)$$
$$=\lim_{x\to 4} 2\sqrt{x}$$
$$=2\sqrt{4}$$
$$=2(2)$$
$$=4$$

Hence, $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}=4$

Differentiation was also another hurdle for some students especially functions with fractional indices. Some students did not differentiate correctly from which the limit could be computed.

We need to investigate the values of $f(x)$ when x is close to 4. If the $f(x)$ values get arbitrarily close to or even equal to some number L, then L will be the limit. One way to keep track of both the x and the $f(x)$ values is to set up a table and pick several x values which are closer and closer (but not equal) to 4. We can pick some values of x that approach 4 from the left, say x = 3.91, 3.9997, 3.999993, and 3.9999999, and some values of x which approach 4 from the right, say x = 4.1, 4.004, 4.0001, and 4.000002. The only thing important about these particular values for x is that they get closer and closer to 4 without equaling 4. This is illustrated in the table below to confirm that the limit is convergent whenever $x \to 4$.

x	$x-4$	$\sqrt{x}-2$	$f(x)=\frac{x-4}{\sqrt{x}-2}$	x	$x-4$	$\sqrt{x}-2$	$f(x)=\frac{x-4}{\sqrt{x}-2}$
3.9100000	-0.0900000	-0.0226280	3.9773720	4.100000	0.100000	0.024846	4.024846
3.9997000	-0.0003000	-0.0000750	3.9999250	4.004000	0.004000	0.001000	4.001000
3.9999930	-0.0000070	-0.0000018	3.9999983	4.000100	0.000100	0.000025	4.000025
3.9999999	-0.0000001	0.0000000	**4.0000000**	4.000002	0.000002	0.000000	**4.000001**

Table 1: Values of $f(x)=\frac{x-4}{\sqrt{x}-2}$ as values of x tends closer and closer to 4 (from – and +).

As the x values get closer and closer to 4, the $f(x)$ values are getting closer and closer to 4. We can get $f(x)$ as arbitrarily close to 4 as we want by taking the values of x sufficiently close to 4. Hence, $\lim_{x\to4}\frac{x-4}{\sqrt{x}-2}=4$. This answer is the same as that obtained by the graphical method in Fig. 2.

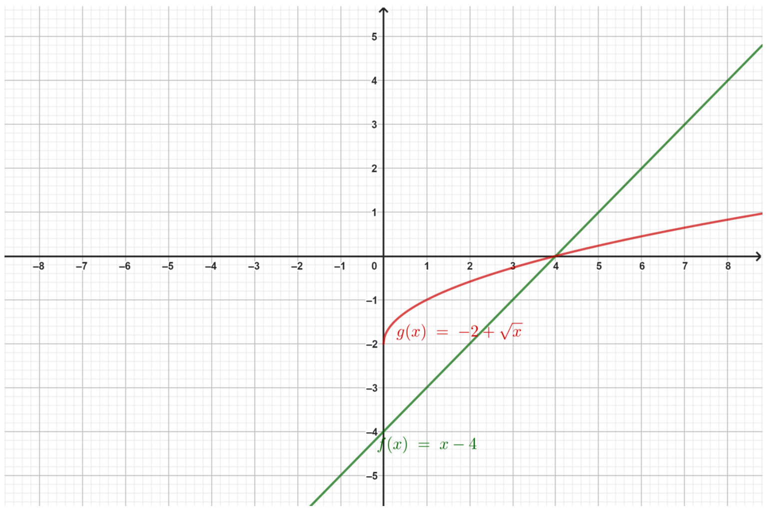

Figure 1: The graph of $f(x)=\frac{x-4}{\sqrt{x}-2}$

From Figure 1 and Table 1, it can be visualized that $\lim_{x\to4}\frac{x-4}{\sqrt{x}-2}=4$. The same result is obtained when the graph of $f(x)=\frac{x-4}{\sqrt{x}-2}$ and $2\sqrt{x}$ (by recognizing that $\lim_{x\to4}\frac{x-4}{\sqrt{x}-2}=\frac{0}{0}$ and applying L' Hopital's rule i.e. $\frac{f'(x)}{g'(x)}$) are plotted. This is visualized in Figure 2 below.

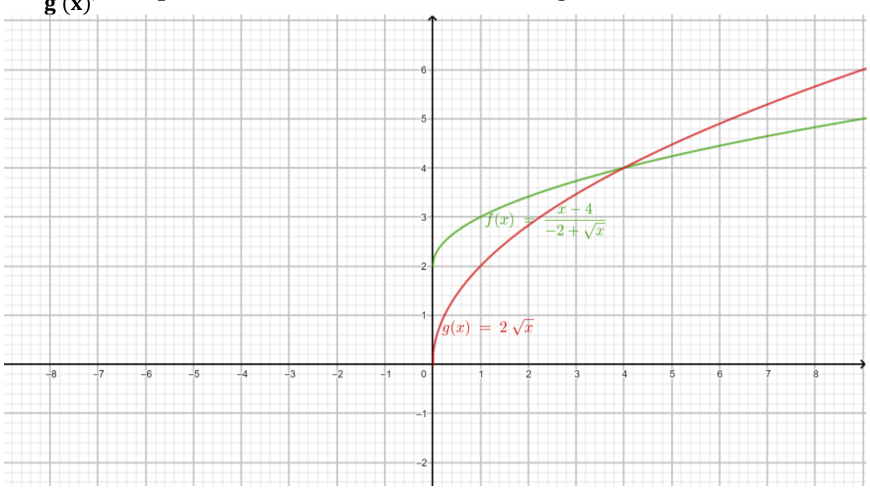

Figure 2: The graph of $\frac{f(x)}{g(x)}=\frac{x-4}{\sqrt{x}-2}$ and $\frac{f'(x)}{g'(x)}=2\sqrt{x}$

(b) $\lim_{x\to16}\frac{(\sqrt{x}-4)}{(x-16)}$

$=\lim_{x\to16}\frac{(\sqrt{16}-4)}{(16-16)}$

$= \lim_{x\to 16} \frac{(4-4)}{(16-16)}$
$= \frac{0}{0}$
$= \infty$

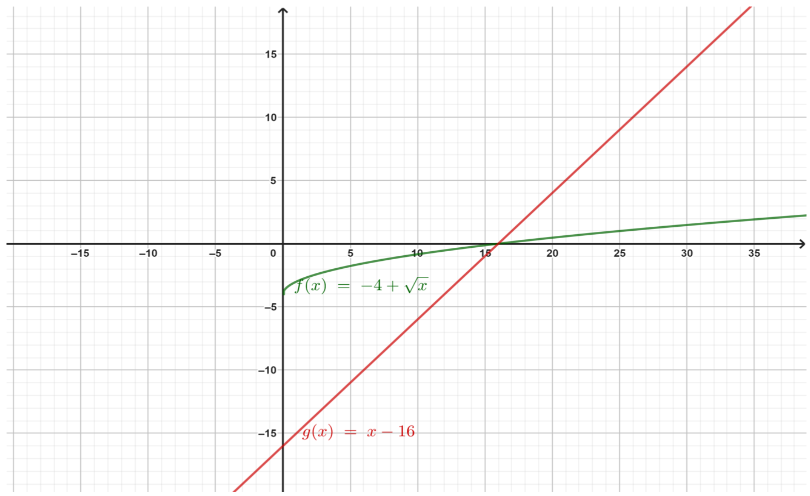

Figure 3: The graph of $\boldsymbol{f(x)} = \frac{(\sqrt{x}-4)}{(x-16)}$

The above graph of $\boldsymbol{f(x)} = \frac{(\sqrt{x}-4)}{(x-16)}$ leads to incorrect solution ($\lim_{x\to 16} \frac{(\sqrt{x}-4)}{(x-16)} = 16$). Yet, $\lim_{x\to 16} \frac{(\sqrt{x}-4)}{(x-16)}$, the rationalized function yields the correct solution (see Figure 4).

Recognizing $(\sqrt{x} - 4)$, $\lim_{x\to 16} \frac{(\sqrt{x}-4)}{(x-16)}$ can be evaluated by rationalization. To do this, we introduce an innocent 1 by multiplying the numerator and denominator by $(\sqrt{x} + 4)$, the conjugate of the numerator $(\sqrt{x} - 4)$.

$= \lim_{x\to 16} \frac{(\sqrt{x}-4)*(\sqrt{x}+4)}{(x-16)*(\sqrt{x}+4)}$

$= \lim_{x\to 16} \frac{(x-16)}{(x-16)*(\sqrt{x}+4)}$

Clearly, $\frac{f(x)}{g(x)} = \frac{1}{\sqrt{x}+4}$ for $x \neq 16$

$= \lim_{x\to 16} \frac{1}{(\sqrt{x}+4)}$

$= \lim_{x\to 16} \frac{1}{(\sqrt{16}+4)}$

$= \lim_{x\to 16} \frac{1}{(\sqrt{x}+4)}$

$= \lim_{x\to 16} \frac{1}{(4+4)}$

$= \frac{1}{8}$

Hence, $\lim_{x \to 16} \frac{(\sqrt{x}-4)}{(x-16)} = 0.125$

The same wrong answer is obtained when the graph of $f(x) = \frac{1}{(\sqrt{x}+4)}$ is plotted (Figure 4). This means the rationalized function does not converge.

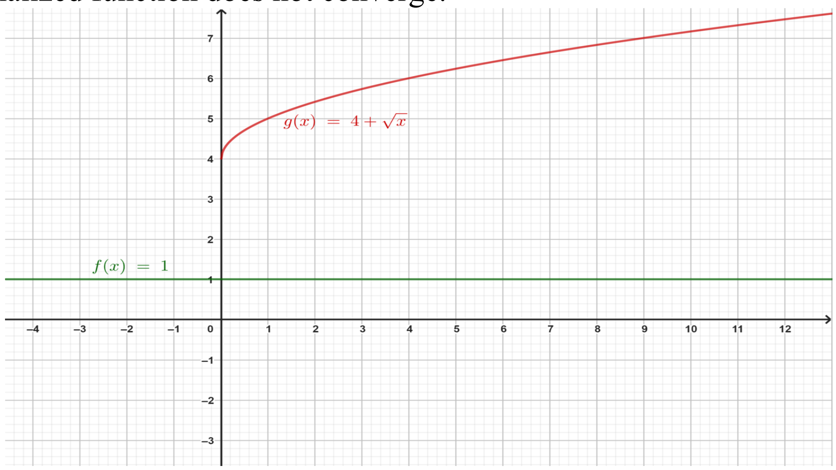

Figure 4: The graph of $f(x) = \frac{1}{(\sqrt{x}+4)}$

However, when the graphs of $f(x) = \frac{(\sqrt{x}-4)}{(x-16)}$ and $g(x) = \frac{\mathbf{f'(x)}}{\mathbf{g'(x)}} = \frac{1}{(2\sqrt{x})}$ were plotted, the limit was easily visualized. From Figure 5, $\lim_{x \to 16} \frac{(\sqrt{x}-4)}{(x-16)} = 0.125$ (visualized as 0.1).

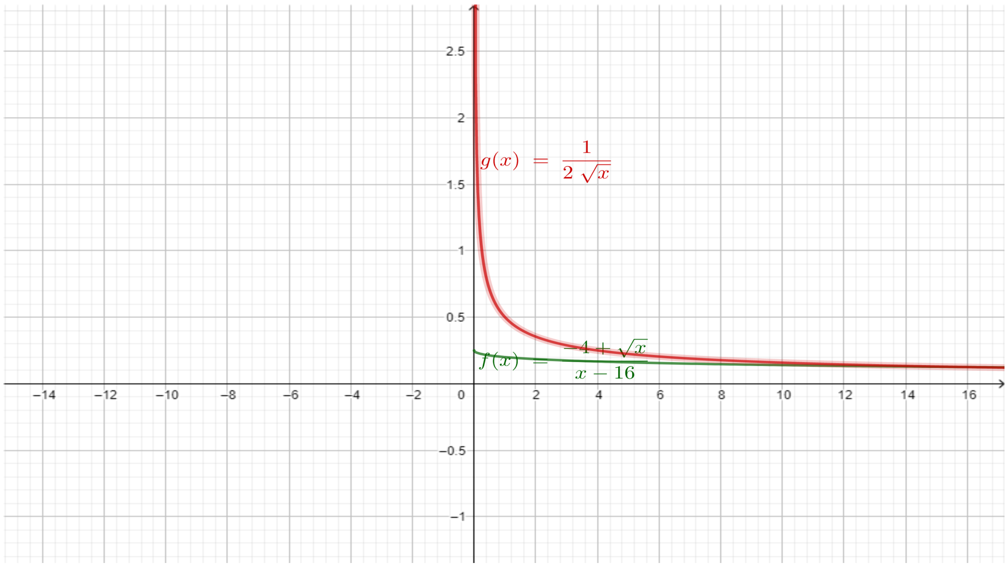

Figure 5: Graph of $f(x) = \frac{(\sqrt{x}-4)}{(x-16)}$ and $g(x) = \frac{\mathbf{f'(x)}}{\mathbf{g'(x)}} = \frac{1}{(2\sqrt{x})}$

Similarly, using the tabular method for (b) $\lim_{x\to 16} \frac{(\sqrt{x}-4)}{(x-16)}$,

x	$(\sqrt{x}-4)$	$(x-16)$	$f(x) = \frac{(\sqrt{x}-4)}{(x-16)}$	x	$(\sqrt{x}-4)$	$(x-16)$	$f(x) = \frac{(\sqrt{x}-4)}{(x-16)}$
15.910000	-0.011266	-0.090000	0.125176	16.10000	0.01248	0.10000	0.12481
15.991000	-0.001125	-0.009000	0.125018	16.00500	0.00062	0.00500	0.12499
15.999930	-0.000009	-0.000070	0.125000	16.00020	0.00002	0.00020	0.12500
15.999999	0.000000	-0.000001	0.125000	16.00002	0.00000	0.00002	0.12500

Table 2: Values of $f(x) = \frac{(\sqrt{x}-4)}{(x-16)}$ as values of x tends closer and closer to 16 (from – and +). As the x values get closer and closer to 16, the $f(x)$ values are getting closer and closer to 0.125. In fact, we can get $f(x)$ as arbitrarily close to 16 as we want by taking more values of x sufficiently close to 16. Hence, $\lim_{x\to 16} \frac{(\sqrt{x}-4)}{(x-16)} = 0.125$. This answer is approximately the same as that obtained by graphical method in Fig. 5.

Students' conceptualization of Task a and Task b

Table 2 below shows percentages of students who solved the tasks correctly and partially. There were 65 students who participated in this study.

Tasks	Correct	Partially Correct	Incorrect
Task 1	25	31	52
Task 2	34	44	69

Table 1: Percentage of students who solved tasks correctly, partially correct and incorrectly.

The students presented solutions to the tasks with lots of misconceptions and errors. This explains why most students obtained incorrect solutions (52% in task 1 and 69% in task 2 respectively). One very prominent misconception peculiar in task 2 was the concept of $\frac{0}{0}$ and where most students got 0 instead of ∞. Others got ∞ but could not justify it by going ahead to rationalize the expression correctly. Below are students' incorrect vignettes of task 1 and task 2:

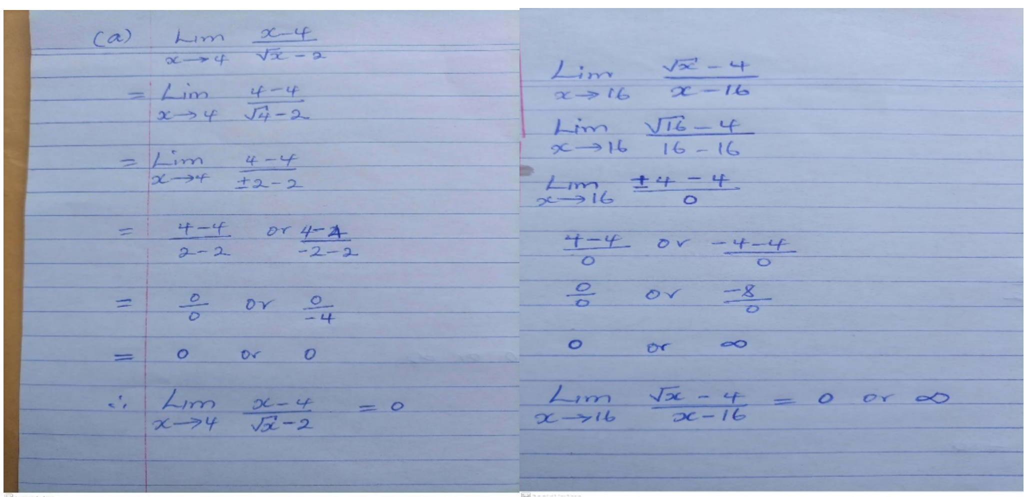

Figure 1: Students' wrong vignettes for task 1 and task 2

From figure 1 above, It appears that students had not fully understood how to evaluate $\frac{\mathbf{0}}{\mathbf{0}}$ or $\frac{\boldsymbol{a}}{\mathbf{0}}$. Many students obtain ∞ as a prerequisite step for using L' Hopital's rule. Yet, a prerequisite condition for applying L'Hopital's rule is to get $\frac{\mathbf{0}}{\mathbf{0}}$ analytically when evaluating the limit of a function. This was obtained by differentiating or multiplying by "the conjugate" of the numerator (also called the rationalization process). This was similarly done in task 2 confirming that most students lacked prior conceptual knowledge and understanding of basic concepts relating to the learning or limits of functions.

As indicated in Table 2 and Figure 1, Task 2 was very difficult and perhaps more challenging for students to conceptualize and solve. Analytical method was the most commonly used method meaning that the students had not fully grasped the solution (s) of limits of functions using L'

Hopitals' rule and by rationalization. The results further show students' confidence and capability in answering the two tasks.

Tasks	Confident	Partially Confident	Unconfident
Task 1	25(31)	44(51)	59(64)
Task 2	22(36)	51(62)	63(75)

Table 2: Number and (%) of correct solutions to Task 1 and Task 2.

In Table 2 above, students' confidence in the two tasks has been compared and contrasted. Most students were partially confident (44(51), and 51(62)) in task 1 and task 2 respectively, and unconfident (59(64) and 63(75)) in task 1 and task 2 respectively. This shows again that the students were more confident in answering task 1 compared to task 2. Worth noting is that some students (5(3.25)) did not solve any of the above tasks while others solved just one ((16(24.62)) for task 1 and (34(22.1)). Generally, the overall students' confidence in evaluating the limits of functions in calculus was weak.

Tasks	Task 1	Task 2	Both
Number (%)	16(24.62)	34(22.1)	5(3.25)

Table 3: Number of students who did not solve Task 1 or Task 2 or both

Comparing and contrasting the results of Table 2 and Table 3, task 2 was seemingly harder for most students to conceptualize. This has an implication on the learning process. Educators should devise suitable learning strategies to enhance the learning of the limits of functions.

The Interview Notes

The interviews conducted with twenty students revealed that the students generally had a weak conceptual understanding and perhaps had developed a negative attitude towards the limits of functions. The observed lecture sessions in their small discussion groups further revealed that students' might not have fully understood the concepts of limits which might subsequently hinder their understanding of other topics. The students were fully engaged in task-solving during lectures. They solved from the blackboard as other peers observed and critiqued or paper and pen in their small groups. However, due to the complexity of task 2, some students lacked prior conceptual and procedural understanding and were unable to complete the tasks. They instead kept on requesting fellow students to complete them or their lecturers to solve the would-be students' problems. Students were asked whether or not they found evaluating the limit of functions harder than other tasks in calculus, and the response was in affirmative.

When students were asked if they found specific concepts on limits challenging, the students agreed and asked if there were alternative approaches to evaluating limits of functions than the analytical approach. In this case, students sought other approaches to the analytical method. Specifically, the application of L' Hopital's rule and rationalization were presumably harder procedures for answering the two tasks involving multiple calculations. To visualize and evaluate the limit of functions of the two tasks, external multiple representations were used. Students were amazed and excited to observe and realized the same answers that were obtained through the analytical approach.

DISCUSSION

The present study investigated the significance of external multiple representations in enhancing the learning of limits of functions. The above results support the conceptual and theoretical framework. The results of the present study confirmed our hypothesis that students face learning challenges in the limits of functions. The results, thus, address the stated research question of whether or not multiple representations may support students' conceptual understanding of the limits of functions. There is a connection between students' prior conceptual understanding, positive attitude, and confidence in learning mathematics. Juter (2005b) investigated students' attitude towards solving the limit of functions. The results revealed a positive relationship between students' confidence and attitudes towards mathematics and their ability to solve tasks on limits of functions. Indeed, students with positive attitudes performed better in solving tasks on limits. The author further noted the importance of a favorable student learning environment as this may offer and support varied opportunities for discussion and problem-solving.

Some students, however, applied previous conceptual knowledge and understanding to consolidate their knowledge thereby enhancing their problem-solving skills. Some students claimed they worked excessively hard to understand concepts of limits of functions that they had not grasped previously. When multiple graphical representations were applied, students were able to visualize the limits of functions and compare and contrast the solutions. Some students who had solved the two tasks analytically in their small discussion groups or individually during problem-solving sessions quickly noted that the external graphical representations supported the problem-solving strategies. This helped to demystify students' fear that the limits of functions were hard to conceptualize. Consequently, students' confidence and abilities were enhanced.

The fact that many students had weak conceptual mappings, and answered partially or eluded one or all the two tasks in the present study indicates that understanding mathematical concepts require several approaches and not just one. In this study, the analytical approach seemed to have yielded negative results. Multiple representations perhaps enhanced the learning of limits of functions. This is also emphasized in Ainsworth's (2008) recommendations on the application of multiple representations as diagrams, graphs and equations may bring unique benefits. This is because mathematics is not all about solving problems analytically.

The interviews conducted with selected students revealed that some students did not solve the two tasks. The tasks were either too difficult for them to solve or the reasons for not solving the two tasks fully might be attributed to a lack of interest and confidence. The problem-solving sessions confirm this claim as some students were observed using the wrong approaches or failing to solve completely. Students who solved tasks in class or in their small groups showed a lack of confidence and positive attitudes towards the limits of functions. This was further revealed through face-to-face interactions with students' peers. The students were aware that the new approach worked. Although many of them still lacked confidence in applying information and telecommunications technology (ICT) gadgets. This means that the effective application of available gadgets in computer laboratories may boost the application of multiple representations to adequately answer tasks on limits of functions.

About half of the students were revealed to have learned by rote, meaning that multiple representations consequently enhanced their conceptual understanding. If students are learning the limits of functions by rote, educators may not guarantee specific concepts they may remember since many students find the limits of functions difficult to understand (Desai & Bush, 2021; Dreher et al., 2015). "If students are unable to understand the concept's critical features, then they do not know what to learn by heart" (Juter, 2005b). Multiple representations may help students to learn and remember, for example, the application of rationalization of functions and L' Hopital's rule in evaluating limits of functions. This is because the understanding of the limits of functions requires time and effort for most students to fully understand. Table 1, Table 2, and Table 3 indicate that the limits of functions are hard for students to understand. Therefore, educators should try as much as possible to vary approaches that cultivate and develop students' positive attitude towards the limits of functions and mathematics generally. If students successfully construct their cognitive representations, the use of external multiple representations and a positive attitude towards the graphical representation of limits of functions may be guaranteed (Liang, 2016; Juter, 2005b).

CONCLUSIONS

This study investigated the significance of external multiple representations in enhancing the learning of limits of functions. To answer the research question of whether or not multiple representations enhanced students' understanding of the limits of functions, it was observed that most students learned by rote. Indeed, multiple representations enhanced students' critical thinking and problem-solving strategies. Limits of functions were regarded to be one of the most difficult topics to understand. The implication is that most students either worked hard or applied cram work to answer tasks on the limits of functions. Yet, the limits of functions provide prerequisite knowledge for understanding differentiation, integration, sequences, and series. Therefore, the integration of new previous knowledge and understanding of the existing concept images is significant in ensuring students' conceptual understanding.

The study conducted by Aguilar and Telese (2018) revealed that procedural fluency, conceptual understanding, and problem-solving strategies enhances students' understanding of non-routine mathematical tasks. The fact that limits of functions are important in learning subsequent topics (e.g., differentiation, integration, and sequences and series), students are encouraged to apply several approaches including multiple representations to confirm, compare and contrast the solutions to several tasks. In so doing, students' conceptual understanding, procedural fluency, and attitude towards the limits of functions can be enhanced. The results from the present study also provide educators with evidence of students' flawed concepts. This points to the importance of prior knowledge and understanding and applying it in subsequent learning. This is vital in conceptualizing the limits of functions and may trigger their attitudes towards the topic.

We, therefore, recommend future studies in different or similar settings and contexts, and in different mathematics topics with the diversity of methods of multiple representations to compare and contrast our findings, and to gain deeper and broader insights into students' understanding and their attitude towards limits of functions and multiple representations generally. Students' attitudes point to issues related to their latent constructs for learning mathematics. Specifically, to gain more insight, this research recommends that future researchers should apply multiple representations to investigate other properties of limits not covered in this study (e.g., tangent lines as limits, use of squeezing theorem to compare limits of functions, and functions whose limit does not exist). This is a potential area for a further investigation aimed at improving the instructional strategies, the teachers' pedagogical content knowledge, and mathematical knowledge for teaching. To achieve this, the teachers' may routinely come together to hold their professional development programs aimed at emphasizing content knowledge and pedagogical content knowledge of learning LP.

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